An economic order quantity model for imperfect and deteriorating items with freshness and inventory level-dependent demand

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Abstract

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by

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Abstract

Consumer purchasing behaviour is influenced by many factors. Depending on the circumstances, these factors may become relevant drivers of important supply chain decisions. Expiration dates have an influence on the purchasing decision of consumers for perishable goods. Another behavioural influence that stimulates demand is the volume of goods that are available on display as part of the purchase transaction. Furthermore, the fact that certain goods deteriorate over time must also be evaluated within the context of the study of perishable goods. The market is increasingly seeking goods that have no inherent defects or imperfections. This investigation seeks to determine the impact of imperfect quality, deterioration, freshness and inventory level and also, how those issues can be improved upon in workable situations. This paper proposes an inventory model that stipulates the demand as a function of freshness and the inventory level. In addition, the inventory depletes through both deterioration and demand, and the product quality is not always perfect. The objective of the inventory model is to maximise the system's profit, hence the study has developed a theoretical mathematical model for imperfect and deteriorating items with freshness and inventory level-dependent demand. A numerical example was used to illustrate the practical application of the model in a real life environment. Sensitivity studies were conducted to determine the impact of changes or variations to the inputs that are used in the model. The findings were that the date of expiry, the elasticity of demand and the selling price of the perfect products are the main constituents that affect the profitability.

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Acronyms

DEL Dynamic Economic Lot

EOQ Economic Order Quantity

EPQ Economic Production Quantity

MPC Manufacturing, planning and control

TCF Total Cost Function

TPF Total Profit Function

TPU Total Profit per Unit of Time

TRF Total Revenue Function

Chapter 1

Introduction

1.1 Background

The increasingly complex modern world demands better management of an ever increasing and diverse range of goods that needs to be supplied to the markets. This range of products comprises various attributes including the availability of a specific range of products for a specific purpose, but also importantly, those goods that are made available need to have increasingly complex attributes. Attributes such as shelf-life and quality of the goods thus become increasingly significant in the overall supply chain process. Most businesses and organisations rely on inventory planning and control as one of the key elements to enhance that supply chain. A significant portion of the capital of a company can expected to be tied up in inventory. This makes inventory planning and control a very important consideration for such decisions.

The earliest research studies of these issues sought to build a single, or relatively few, models to portray this situation. As the complexities of modern life continue to evolve, it is becoming increasingly apparent that supply chain models need to replicate the complexity of the real world and hence they need to evolve in tandem with these changing developments. Being able to keep up-to-date with emerging trends in supply chain management makes it a critical input in any planning and production situation and it is important to be able to remain efficient in our increasingly competitive and complex modern world.

Appropriate planning is a complex issue. It has to take into account the increasing number of permutations that the market desires. There are many types of products and many of those products can also be sold in different forms, with different levels of processing having been applied thereto. This is particularly applicable in a situation where freshness of the product would be an important driver of the demand for products. The increasing trend of specific consumer tastes also means that quality control is becoming more specific and rigid to address the requirements of the modern consumer market. Product deterioration also needs to be carefully managed to minimise adverse outcomes of inferior products being provided to the market.

In summary, product freshness, product quality and product deterioration would be attributes within the range of characteristics that need to be investigated within the supply chain. This study focuses specifically on these matters within the supply chain management discipline.

1.2 Research methodology

Various methodologies have been proposed to conduct quantitative research in operations and research management. Bertrand and Fransoo (2002) have been instrumental in this investigation. The methodology outlined here was adapted from Bertrand and Fransoo (2002) and utilised in the research presented herewith. It proposes a theoretical of the problem that has been based on real-life situations. Thereafter, a mathematical or scientific model of the problem is formulated based on those specific issues. As such the model seeks to emulate real life situations but never-the-less, it is necessary to make certain assumptions that bridge the real situation with the mathematical formulation thereof. The outcome should be a set of solutions that are arrived at from resolving that mathematical model. Those solutions are thereafter applied to numerical examples to test their use in resolving the original issue. Thereby, it can be assumed that unique solutions to the mathematical models that represent the inventory systems are available. This is finalised with sensitivity analysis to evaluate the most important inputs. All of this culminates in a recommendation about a real-life inventory scenario situation.

1.3 Dissertation outline

The research project is categorised into four separate sections. It commences with an introductory chapter and is followed by an additional three chapters that comprise the investigatory output. Chapter 2 consists of an investigation into already published models and available research. Chapter 2 is a literature overview of the development of the main models and research that is applicable to inventory theory. Following the review of these models, the salient features thereof are used as a basis for further investigation to incorporate imperfect, deteriorating and freshness inventory level-dependent demand scenarios. Such a review encompasses a discussion of the mathematical logic that underpins each model. Chapter 2 is the review of relevant inventory models in the literature for this study. This provides the mathematical basis upon which the detailed study in Chapter 3 is based. The primary objective of this dissertation is realised in Chapter 3 through the development of an economic order quantity model for imperfect and deteriorating items with demand that depends of the freshness and inventory level of the product. Chapter 3 culminates in a generalised inventory equation which seeks to maximise profit. Thereafter, it is subjected to sensitivity testing by applying numerical analysis. The study concludes in Chapter 4 with a discussion of the findings from this submission and suggests possible areas for further study and research.

Chapter 2

Literature review

2.1 Introduction

Manufacturing, planning and control (MPC) creates an environment within which the supply chain operates. It is necessary to adjust or modify the MPC methods as appropriate, to ensure that the desired outputs are attained. MPC encompasses a variety of manufacturing aspects, which range from materials management, the scheduling of people and machines and also being able to optimise various aspects of the supply chain. Because of the inconsistent requirements and inherent variability of the markets, there is an ongoing effort to improve on the ideas and concepts within the MPC sector to reflect that reality. The supply chain environment is competitive, which means that there is a constant need to improve and optimise on aspects thereof to ensure that it functions optimally. Jacobs et al. (2018) is of the opinion that supply chains need to adapt their strategies, as the environment is always dynamic. The primary task of the MPC system is to efficiently manage the use of materials and make better use of resources and also to be able to respond to any changes in customer requirements. This would include influencing the adaptability of those suppliers to respond to market-driven situations.

A significant portion of the assets of various companies are tied up in their respective inventories. This makes inventory planning and control an important component of the supply chain mechanism. In 1913 inventory planning and control was first conceptualised and studied by Harris (1913). The early models created the foundation upon which inventory planning and control has subsequently evolved. Solving problems relating to inventory planning and control to determine the optimal amount to order, based on various parameters, such as a specific item at the optimal time can be determined. However, to align that which is evident with those concepts developed by Harris (1913) becomes much more challenging as many interconnected factors have to be considered. Malakooti (2014) has proposed that the primary driver for inventory planning would be to be able to forecast the demand. Two errors may occur when the demand has not been properly determined. Underestimating demand results in not having enough inventory to satisfy your customers (and loss of profit may occur), while overestimating the demand may result in the capital of the company being tied up in unused or underutilised inventory. Bad planning may result in detrimental outcomes along the entire supply chain process. To avoid the errors of understating and overstating demand there is a move towards research that creates models that are more reflective of real-world and real-time situations.

This means that decisions have to be made about the quality of stock on order and the method of replenishing stock that has been sold or discarded. Mathematical models do assist in evaluating order quantities and frequencies of the order process. This is referred to as the Economic Order Quantity (EOQ) model concept that had originally been proposed by Harris (1913).

Published and available case studies were investigated in the literature to provide background information.

2.2 The primary EOQ model

The initial EOQ model was conceptualised by Harris (1913). It sought to determine the quantity by optimising the set-up costs, the purchase costs and the inventory holding costs. This became the foundation of what has now become accepted as the EOQ model. However, it had been based on certain impractical assumptions. For example, those assumptions were restrictive in respect of real-life inventory scenarios from a practical perspective.

This concept had its foundation in 1913 with the publication of an initial determination of economic order quantities. It has since then become the point of departure for further research, initiatives and attempts to extend this work. The outcome has led to the publication of more research initiatives that have refined and taken the development of these prior concepts further.

Prominent amongst these is the need to determine an optimal order size which keeps inventory related costs at the lowest possible values. Inventory costs vary depending on the size and the frequency with which the ordering had taken place. The primary EOQ model records the costs required to place an order and cost associated to keep it in storage.

Earlier work from Harris (1913) needs to be made more realistic with introduction of new variables and the application of a less rigid structure.

Wilson (1934) elaborated on what Harris (1913) had done prior to that.

2.2.1 The classic EOQ model: The first version of the EOQ model

The model by Harris (1913) is the most basic inventory control model. It computes a non-varying order quantity that minimises the holding costs and also those associated costs of the order.

The acquisition cost is generally (but not always) omitted because it has no impact on the order size unless discounts become applicable. This means that a balance needs to be established between the holding and the ordering costs, because as order quantities get larger, the cost of holding stock increases and the cost of ordering stock declines. The Figure 2.1 provides a graphical depiction of such a scenario.

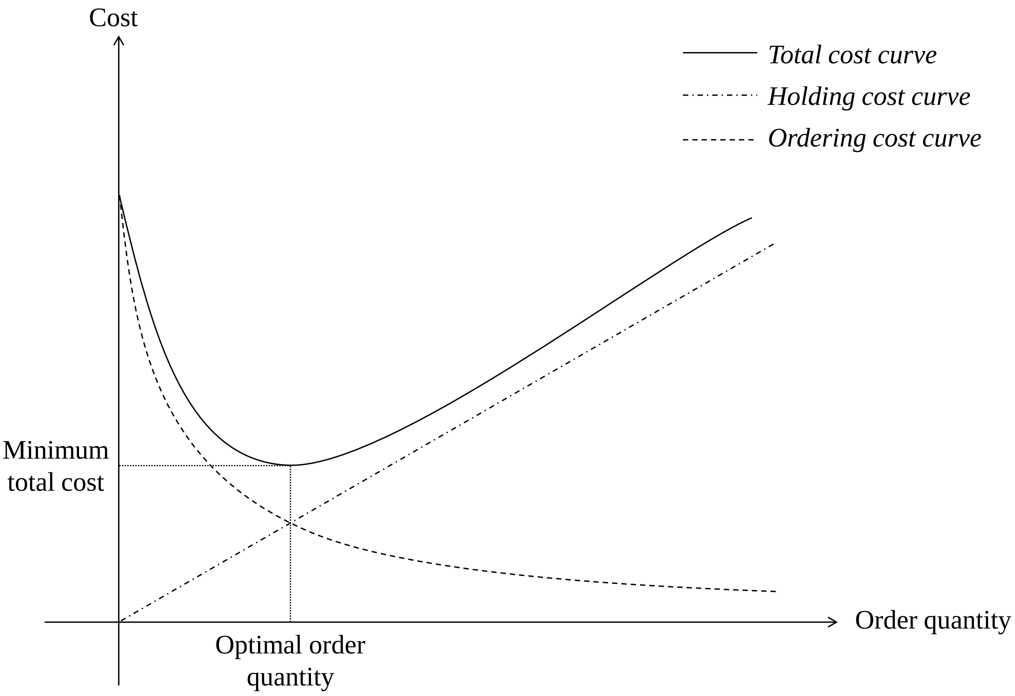
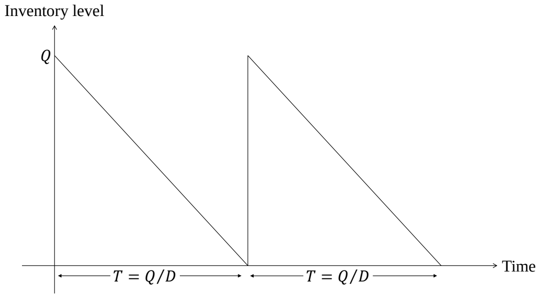


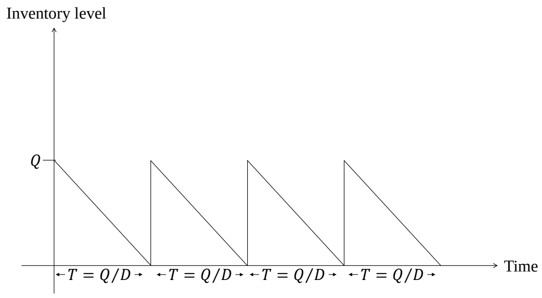
Figure 2.1: The quantity ordered expressed as a function of the ordering cost the holding cost and the total cost scenario.

However, the reverse is also true, since the quantity ordered decreases, the cost of holding similarly declines and the cost of ordering increase.

This concept can also be illustrated graphically with level/s of inventory during the cycle period. This is shown in Figure 2.2.



(a) Fewer large orders result in greater inventory holding costs and lower set-up costs.



(b) Numerous small orders result in lower inventory holding costs and higher set-up costs.

Figure 2.2: An idealised inventory scenario for the initial Harris (1913) EOQ model.

Essentially, this graphic evaluates a single item that is not constrained by any lead-time. Q items would be ordered and is received at the commencement of each cycle for that inventory. When the order for items (defined as Q) is initially received at the start of each planning cycle for that inventory. When the request for the Q items has been placed, an ordering cost of K would be charged. The rate of consumption of these items is assumed to have been constant (defined as D) until they have been depleted after a time-frame T. A replacement order is then received for the Q items. There is an annual holding cost of h for those items that are held as inventory. The total cost per unit of time TCU is given by Equation (2.1).

$$TCU = h\left(\frac{Q}{2}\right) + K\left(\frac{D}{Q}\right) \tag{2.1}$$

In this situation no discounts for larger quantities are considered and shortages are not incorporated. The value of Q (detonated by Q^*) that minimises T is found by equating the first derivative of Equation (2.1) to zero (and noting that the second derivative will be positive) to solve for Q^* shown in Equation (2.2).

$$Q^* = \sqrt{\frac{2KD}{h}} \tag{2.2}$$

2.2.2 Subsequent developments that follow from the original classic EOQ model

The original work of Harris (1913) prompted further research and study of that original model. From the Andriolo et al. (2014) study, a timeline of further developments in respect of the EOQ concept can be constructed. Table 2.1 shows a timeline of the evolution of EOQ inventory management models. However, this process is open-ended in that as the supply chain evolves, the scope for EOQ models will also evolve in tandem with that process. This further development of the EOQ models is the primary focus of this study. The classic EOQ model has been around for more than a century. Through the development of this model, various assumptions have been modified to enable the model to be more realistic for a particular operating environment. Table 2.1 shows the evolution of the basic EOQ model and in more recent years, it has included environmental and social sustainability issues, which have become a more topical feature of the society today.

Table 2.1: A timeline illustrating the evolution of inventory theory.

Date	Name	Researcher	Focus
1913	EOQ	Harris	Balancing the costs of ordering and holding inventory.
1918	EPQ	Taft	Finite production rate.
1958	DEL	Wagner and Whitten	Time-varying demand.
1963	Quantity Discounts	Hadley and Whitten	Vendor offers discounts for larger orders.
1963	Shortages	Hadley and Whitten	Back orders are permitted.
1963	Deterioration	Ghare and Schrader	Stocked items deteriorate during replenishment cycle.
1975	Inflation	Buzacott	Time-varying costs.
1985	Trade credit financing	Goyal	Vendor grants buyer grace period to settle payment.
2000	Imperfect quality	Salameh and Jaber	Certain items in each order would be of inferior quality.
2011	Environmental sustainability	Hau, Cheng and Wang	An additional cost is levied for carbon emissions.
2014	Growing items	Rezaei	Stocked items grow during replenishment cycle.

For example, the first extension to the EOQ model is described as the Economic Production Quantity (EPQ) extension thereof. Two major events happen in the EPQ model simultaneously. These events are ongoing use and occasional production. For this extension, the rate of production is estimated to exceed the rate at which those goods are consumed. This EPQ extension is the result of work done by Taft (1918).

Earlier work had the limitation that it always assumed a static rate of demand. Thereafter, work by Wagner and Whitin (1958) permitted the model to incorporate varying demand over a number of periods. This is known as the Dynamic Economic Lot (DEL) model.

With the development of better road and transportation infrastructure it made sense to increase the volumes of deliveries through larger delivery loads. Hadley and Whitin (1963) sought to address this matter. They developed a model that enabled discounts to be applied for larger quantities of goods that had been ordered. Their work resulted in two types of discount being applicable. The first, allowed for a single discount to be applied per delivery. The other variation permitted ever increasing discounts depending on the size of that delivery. In summary, this catered for price breaks which resulted from step-changes in the prices of the goods ordered.

In another analysis Hadley and Whitin (1963) studied supply chain issues where shortages were permissible. In inventory theory where shortages are permitted they are either partially or fully back-ordered. With comprehensive back-ordering of shortages consumers can wait until the forthcoming delivery arrives. With partially fulfilled orders some customers are willing to wait to complete those orders but others may not be willing to do so and this can result in lost sales.

The work that Harris (1913) initially did assumed that stored goods will not change or become damaged if stored for an indefinite period. However, in real situations items can deteriorate over time. Examples of such items would be the deterioration of dairy and poultry products or fruits and vegetables. These are known as perishable goods or deteriorating items. Palanivel and Uthayakumar (2016) categorised deterioration as being a broad term that encompasses all kinds of damage, theft, evaporation, or loss of use of an item. This can occur while goods are kept as stock items. Ghare and Schrader (1963) modelled deterioration rates with a decaying exponential function. Their work pioneered the development of exponentially decaying items. So, inventory can get used up, not only through demand, but also by decay. This can also happen through the combined effect of decay and demand.

A feature of the mid 1970's was the emergence of inflation. In this context, the prior inventory models had assumed that costs remain constant over time. In reality, the value of money became eroded through inflation and this was adversely affecting costs. Buzacott (1975) was investigating the impact that inflation would have had on costs. The work of Buzacott (1975) sought to consider the impact of inflation on the various cost inputs. Using a discounted cash flow analysis, Gurnani (1983) studied some inventory models to quantify the impact of declining monatary value on total cost and EOQ.

The issue where delays in payment between the parties was investigated by Goyal (1985). The driver for this model was the fact that in many situations the items that had been ordered were not paid for at the instant that they are actually delivered to the consumer or customer. They delay in payment could have resulted because the buyer needed to inspect that order to be able to ascertain weather it was of an acceptable condition or weather it could have resulted from suppliers permitting buyers some leeway with which to settle their debts.

Salameh and Jaber (2000) removed the assumption that all goods are of adequate quality in the basic EOQ model. Their inventory system had assumed that a some of the items were inferior. The imperfect quality items were sold in conjunction with the perfect quality items but the former, had been sold at a discount.

Emerging environmental concerns in certain jurisdictions required a reduction in carbon emissions. Hua et al. (2011) incorporated the concept of carbon emissions into the original EOQ model and to do this they suggested that a supplier would have incurred an associated cost with the emissions.

By 2014, Rezaei (2014) had considered the feeding habits and growth rate of chickens to develop an additional category of inventory items which they called growing items for the EOQ model. These items increase in mass as part of the replenishment cycle. All of this requires that models reflect reality. Certain related features are combined or added with certain specific attributes.

From the developments in the various inventory models, this project will focus on imperfect quality, deterioration, freshness and inventory level-dependent demand. It is necessary therefore, to have a more in depth appreciation of these aspects as the study progresses. These are practical issues which should be expected to arise in food supply chain mechanism.

2.3 Imperfect Items

2.3.1 The original EOQ approach to imperfect quality items

Salameh and Jaber (2000) investigated the notion that items that were delivered may not necessarily have been of perfect quality for the recipient. They proposed a model or a scenario to cater for such situations. The basis for this was that imperfect quality items, may not necessarily be defective. In other words, they could partially be reused. This study also includes the classic EOQ model, but it takes away an unrealistic assumption, namely that all goods are of perfect quality. Their proposal suggested an inventory situation where a fraction of those ordered goods would realistically be of inferior quality. A screening process would be applied prior to the sale of the goods to separate those of poor quality from the good saleable products. Both the good products and the the poor products are separately sold. The good quality products are sold on an ongoing basis within the replenishment cycle of the inventory but the still saleable poor quality items are salvaged and sold as a single entity at the end of the screening exercise. The optimal solution of the model was established to have been in closed form.

In the paper the authors propose the idea that after the inspection procedure, the items that are not of the required standard are put on sale at a discount. A statistical model was created to illustrate such a scenario.

Figure 2.3 diagrammatically illustrates this scenario. This scenario incorporates a number of assumptions to explain Figure 2.3 and those assumptions have been itemised herewith.

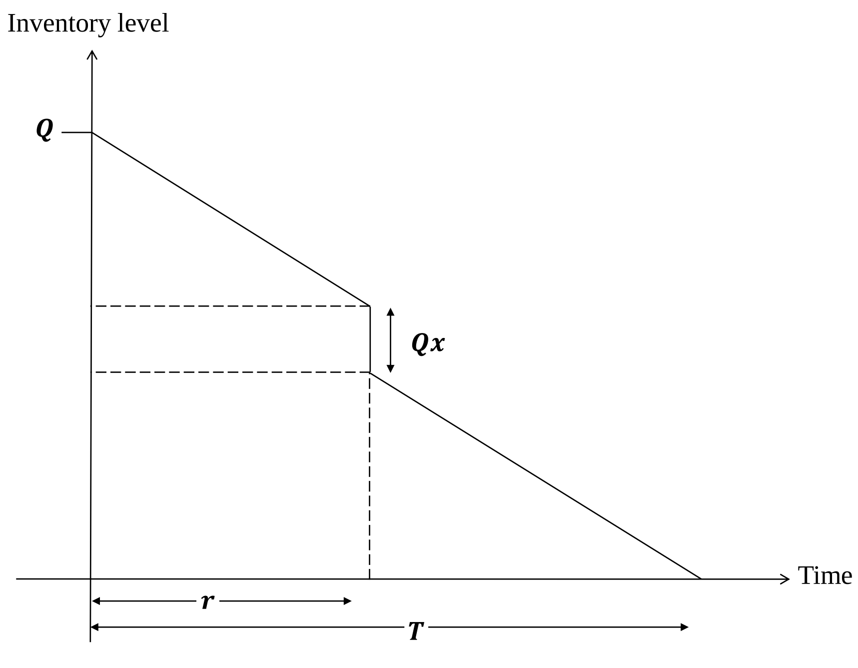


Figure 2.3: The behaviour of imperfect quality items in an inventory system.

- Q is the total number of items ordered for each lot size.
- K, is a fixed ordering cost is charged with every order that the company places.
- Items are ordered per lot size at the cost of p_v per item.
- The ordered items that have been received are of varying quality. Some of those

items x are of inferior quality. That fraction of inferior quality items x, is assumed to be random.

- The recipient company keeps all these items in stock. It costs h per item per unit of time to keep it stored.
- Prior to the sale of the items, a thorough screening process occurs. This separates the items into good and poor quality products. The sale of the good quality products takes place only after proper inspection thereof. That sale transaction occurs at a rate of z items per unit for the time period r.
- Good quality items, $Q(1-x)$, are sold for a price p_r per unit during the cycle.
- D is the demand rate per unit of time.
- Inferior quality items, Qx, are sold in single batches at a price of p_q per unit. This is less than the price of the good quality items following the screening process.
- v is the cost incurred to differentiate the poor and acceptable quality items into different categories.

This implies that there are two streams of revenue for the company, one being from the sale of quality goods and another from selling the product of inferior quality. The total profit expected, $E[TPU]$, for the company per unit of time would be given by Equation (2.3).

$$E[TPU] = p_r D + \frac{p_q D E[x]}{(1-E[x])} - \frac{p_v D}{(1-E[x])} - \frac{KD}{Q(1-E[x])} - \frac{vD}{(1-E[x])} - h\left[\frac{Q(1-E[x])}{2} + \frac{QDE[x]}{z(1-E[x])}\right] \tag{2.3}$$

To optimise the value of Q the Equation (2.3) would have to be maximised. This would correspond to the optimal order quantity Q^* and given as Equation (2.4).

$$Q^* = \sqrt{\frac{2KD}{h(1-E[x])^2 + \frac{2E[x]D}{z}}} \tag{2.4}$$

2.3.2 The EOQ model with extensions to cater for items of imperfect quality

Goyal and Cárdenas-Barrón (2002) used other investigation by Salameh and Jaber (2000), but made use of an easier methodology to ascertain the EPQ in conjunction with the inferior quality of an item. The anticipated profit value per unit of time was achieved by amending the formula of Salameh and Jaber (2000). Salameh and Jaber (2000) first formulated an expression of the total profit prior to consideration of the projected quantum that was required. Goyal and Cárdenas-Barrón (2002) did this differently by projecting revenue values and total costs as separate entities. The result was that the work of Goyal and Cárdenas-Barrón (2002) had less mathematical computation than the method deployed by Salameh and Jaber (2000). Like Salameh and Jaber (2000), the solution came from addressing the problem within a closed or restricted form. The difference between these respective models was minimal and not statistically significant. In essence, Goyal and Cárdenas-Barrón (2002) found a model that was less complicated to compute and furthermore, it yielded similar results without significant further literature review about imperfect quality items for inventory models. In summary, it was apparent that in the results that the initial and the more basic approach, versus the approach presented by

Salameh and Jaber (2000), had been that the two approaches actually yielded similar results for a near optimal batch size.

Huang (2002) broadened the study of Salameh and Jaber (2000) by investigating the cooperative relationship that may exist between supplier and customer in a supply chain. This involved an optimisation of costs for both the supplier and the buyer. The model assumes that items are screened by the buyer prior to making them saleable. Only the good quality items are made available to third parties for onward sale by the buyer. A cost is allocated for that fraction of inferior quality items that have been provided to the buyer. The vendor incurs this associated cost.

Chan et al. (2003) furthermore investigated an inventory model with reduced pricing scope, other rework and reject some of that product. The Chan et al. (2003) investigation elaborated on the work of Salameh and Jaber (2000) to incorporate a screening procedure which separated the items into three different groups, whereas the screening process of the Salameh and Jaber (2000) model separated the products into two groups. The Chan et al. (2003) models three groups comprising the acceptable quality items, the inferior quality items and the defective items.

Another approach by Chang (2004), investigated the theory of fuzzy sets for the basic EOQ model. This Chang (2004) study also incorporated defective items. The intention had been to ascertain the best order amount that results in the greatest total profit. It does this by acknowledging that certain inputs for the model demonstrated fuzzy model behavioural attributes. Chang (2004) considered two scenarios in applying the fuzzy model. Firstly, fuzzy variable model attributes had assumed that certain items were of imperfect quality. Secondly, fuzzy variable model attributes were also assumed for the rate of demand together with the fraction of the imperfect quality items.

Subsequent work by Yu et al. (2005) elaborated on the Salameh and Jaber (2000) approach by incorporating item deterioration together with the inclusion of some back-ordering. Their assumption was that the inventory deteriorates while in stock and also that shortages would be partially back-ordered. This applies to consumers who would be able to wait for stock. A charge for lost sales was allocated for consumers who were unwilling to wait for back-ordered stock.

The initial model by Salameh and Jaber (2000), does not include shortages. Eroglu and Ozdemir (2007) created an EOQ model that for each order, that will have some defective items and shortages in the back order. The model accepts that the distribution of faulty items is uniformly spread. The defective items are disposed of at a reduced price or classified as scrap. The salient findings of the paper were that the when scrap (or unusable) rates increase, a decline in the optimal total profit per unit is recorded.

The imperfect quality concept was extended by Jaber et al. (2008). In this respect, the EOQ model was extrapolated by taking into account learning effects, in that the efficiencies thereof increase over time, as the benefits of that learning effect become apparent during the process. The only variation between the Jaber et al. (2008) model and that of Salameh and Jaber (2000) model was that the Jaber et al. (2008) model assumed that the amount of inferior quality items would decrease in accordance with a learning curve to identify the defects. It assumes that for repetitive operations the cost of producing an item reduces as the quantity of those items that are produced doubles. It was assumed that the learning capabilities cold be modelled in accordance with the mathematically derived S-shaped logistics learning curve.

An extension was made to the imperfect quality items EOQ model by incorporating two factors namely, imperfect quality items, together with capacity constraint of two warehouses in the basic EOQ model. In many everyday circumstances, there are many considerations like price discounts, or enticing retailers to purchase goods on consignment that exceeds the capacity of their own warehouse facility. Retailers may rent other premises under such circumstances. Many investigations studied inventory models with dual warehouses. Studies that incorporated imperfect product quality under this scenario, were modelled by Chung et al. (2009). The mathematical model by Chung et al. (2009) aimed to maximise the total profit. The model makes the assumption that the one warehouse with limited capacity is owned, while the other warehouse, with an assumption of unlimited capacity would be rented. By keeping one of the items in stock in the second warehouse, more costs are incurred. Therefore, items from the supplementary warehouse are sold first. Prior to putting the goods up for sale the items in both the warehouses undergo screening to remove the poor quality items.

The impact of EOQ models that have learning capabilities was studied by Khan et al. (2010). This investigation noted that a cost for a loss in sales would be allocated as the result of back-ordered shortages and from the effects of the learning process. For three different learning scenarios, an EOQ model was derived. The scenarios that were studied incorporated full, partial and zero learning transfer. This corresponds to scenarios where inspection would be complete, partial or absent by the inspector during the screening experience.

A model refinement for goods of imperfect quality with storage was derived by Chang and Ho (2010). This model was a subsequent model to evaluate shortages. The significant differences from this and earlier versions for inferior quality items with shortages, was that Chang and Ho (2010) had not applied differential calculus methodology. They resolved their model algebraically instead.

A system that includes imperfect quality items within vendor-buyer framework was researched by Chen and Kang (2010). Their model allows the purchaser trade credit finance by letting the purchaser receive stock but only pay for it sometime thereafter. This facility incurs interest costs which the seller would be obliged to pay. The model sought to keep costs at a minimum for both the parties as part of the transaction.

A model for inventory and an influential buyer for items with imperfect quality was developed by Lin (2010). This model assumed that the buyer had negotiation supremacy over the supplier. It meant that discounts could have been made to the buyer. These discounts were available only to selected an influential customers. The amount of the discount was structured and based on the quality of the goods that have been ordered.

Random supply to the EOQ model was introduced by Maddah et al. (2010a) for items of inferior quality. They assumed a production process of the provider that followed a two-state Markov process. The Markov process is a stochastic model which would describe the sequence of a possible set of events in which the probability of that event relies entirely on the state that had been previously attained. Furthermore, the Maddah et al. (2010a) model examined duel scenarios of relevance to the transport and shipping of items that are of imperfect quality. The first scenario, separates the inferior items from stock without any further costs. Thereafter, those items get aggregated into a single consignment which is sold at a discounted price, later on.

An alternative approach was used by Maddah et al. (2010b) to ensure that shortages were considered. A screening procedure was developed to determine the imperfect items when an order is received. These imperfect items would be salvaged as part of that assessment process. To remove the possibility of shortages, orders are placed where there is just sufficient to satisfy the demand while the sorting process occurs. With sorting, the demand for the order would be combined with the inventory for the previous order. This improves the overall level of customer service delivery. However, it is costly to the supplier, even if it does address the customer demand aspect.

The model for inventory that have imperfect quality in the EOQ study was extended by Hu et al. (2010). More specifically it introduced the concept of fuzzy variables to items with imperfect quality. Fuzzy variables are measurements that emanate from two types of uncertainties. Shortages and fully back-ordered inventory were permitted in the model. According to the model, the rate of demand and the fraction of quality items had been fuzzy. Levels of customer service were also addressed by the model. The levels of service were measured by customer demand having been satisfied with the back order items.

The EOQ model assumed that all of the items are of perfect quality in the order lot. This would only be attainable under idealistic situations but in practice it may not be applicable. For such situations inspection of the lots becomes vital for the process. Furthermore, this becomes more of an issue when products deteriorate. In the Jaggi and Mittal (2011) study, the aim has been to resolve an inventory model for imperfect items that deteriorate.

Wahab et al. (2011) elaborated on the investigation of Salameh and Jaber (2000) with a vendor-buyer supply chain model. This differed from other models in that it studied three practical scenarios. Firstly, it assumed both participants originated from the same country which was a new dimension in inventory management study. Secondly, the vendor and the buyer were assumed to have been from different countries. The rate of exchange of the currency between the counties was assumed to have been stochastic. Finally, the model was studied under a scenario for different countries assuming emission costs are charged for logistics activities and production of the goods when orders are executed.

The assumption by Salameh and Jaber (2000) that the screening process would be perfect is not practical in the real world. Further work by Khan et al. (2011) did away with the assumption of perfect screening by developing a stock model that noted inconsistencies in the process of screening. The likelihood that a screening inspector would commit an error was assumed to have been known.

In the Salameh and Jaber (2000) model the assumption was always made that the screening process happens at the customer, even though it is the supplier that provides the imperfect product. Rezaei and Salimi (2012) derived a model for inventory model where the management of the screening process switches from the customer to the supplier. This is a departure from the prior accepted norm.

A version of the EPQ model was investigated by Yassine et al. (2012). It evaluated two scenarios from the delivery of inferior quality products namely, through aggregation and disaggregation. For aggregation, inferior quality products are collected during various production runs and then they are shipped on a per consignment basis. For disaggregation, inferior quality items are deemed to have been sold throughout each production cycle.

The demand rate is impacted through pricing and marketing considerations according to Lee and Kim (1993). However, these factors are seldom considered when such inventory models are developed. This resulted in an EPQ model by Sadjadi et al. (2012) for goods of inferior quality that takes into consideration the effect of the marketing plans for a company. The model considered variables such as the marketing budget allocation, the costs of maintenance, the costs of production, warehousing availability and the machine hours available. These were deemed to be constraints or limiting factors in the overall process.

Yadav et al. (2012) produced a fuzzy demand model for items of imperfect quality and fully back-ordered shortages. Demand was projected to depend on the funds that had been spent on advertising together with a learning curve that would become increasingly efficient with the screening process.

Certain suppliers permit the return of goods if there is not complete satisfaction therewith. The reality is that there would be a greater probability of customers returning goods in an environment with more imperfect quality items. Hsu and Hsu (2013) suggested an inventory model that incorporated sales returns for items of imperfect quality. That model also assumed errors in the screening process with an increased probability of items being returned if more items were found to be defective. Returning an imperfect product incurred additional charges for the retailer. Such charges include the costs of the item, the refund cost and the associated logistical costs.

Lin and Hou (2015) investigated an inventory model with overlapping and an advanced receiving attributes, wherein the supplier would provide a discount factor to that procurement cost. This discount factor would compensate that buyer for any further holding costs and maintain a collaborative business relationship.

An extension to this two-warehousing scenario was examined by Jaggi et al. (2015). To assume that the items that are produced are of perfect quality is unrealistic because virtually all inventory comprises a small number of items that are of sub-standard or inferior quality due to defective production or the mistreatment of those goods. It may also be incorrect to assume that goods maintain their physical characteristics while being held. To mitigate against financial losses due to this, the consumer may be forced to rent other warehouses with more appropriate storage facilities which may not be readily available. This model develops an inventory scenario with two-warehouses, one being for goods that have some imperfect quality and the other for goods that will deteriorate.

Khan et al. (2016) considered a more recent trend in supply chain management for a vendor-buyer system of inventory. The model ensures that the buyer is provided with items which are not all of perfect quality. An agreement with the vendor keeps the stocks at the warehouse of the buyer who would be responsible for managing that stock. There is an increasing number of manufacturers who elect to have retailers oversee that sort of inventory.

A model has been developed by De et al. (2018) for imperfect quality items where certain of those items can be reprocessed and the remainder are disposed of at a lower price. Added to that environmental regulations were applied to the study through a carbon tax that is charged subject to the production processes of the manufacturer producing a prior specified quantity of carbon emissions.

There is an emerging trend to use these EOQ models to ensure sustainability and economic benefit. Tiwari et al. (2018) considered carbon emission items, in the framework of a vendor-buyer integrated one supplier, one customer inventory model, for deteriorating items or items of imperfect quality. The aim of the study had been to enable policy makers with the appropriate knowledge, to collectively determine the order frequency and quantity of the product required with minimal pollution and inventory costs. The costs of carbon emissions, from the production process, the logistics and warehousing functions were also considered as part of the overall cost. The findings of the investigation by Tiwari et al. (2018) suggested that the combined model would be an improvement, in terms of inventory cost and carbon emission reductions, with a cost minimisation objective of the total costs to be incurred by both parties.

There is ongoing study to improve on existing models by making them more applicable to real-world situations. The study by Nobil et al. (2020) initiated a significant trend towards the optimal economic lot-size model for sets of imperfect items. Research is now able to calculate the best re-order points for the Salameh and Jaber (2000) inventory model and this constitutes a key threshold for managers to determine the continuity that would be relevant for order scheduling.

According to Sebatjane and Adetunji (2020a) quality control would also be a factor in food production systems. This would encompass farming, processing and would end up with use. This investigation examines a control model for inventory within a supply chain that encompasses agricultural farming, agricultural processing, agricultural screening and agricultural-related retail operations. It becomes a four-echelon supply chain system. The processor exchanges the processed inferior quality products at a lower price and as a uniform component to the secondary market.

Hauck et al. (2021) assumed that the speed at which the screening is done would be a decision making variable together with the order quantity. Furthermore, other screening cost and defect detection concepts are introduced.

2.4 Deterioration

2.4.1 The original deterioration model

Prior inventory models such as Harris (1913) had assumed that products could be stored indefinitely without deterioration. This is not realistic as many perishable items such as fruit, vegetables, medication and certain liquids can deteriorate or degrade over time. This means that inventory can get used up or depleted by both decay of the item, or changes in the demand. Deteriorating items can be categorised into items with a fixed-span lifetime (for example, medication with an expiry date) or items that deteriorate over time (for example, perishable goods) or alternatively, items where the deterioration is age dependent (for example, changing fashion styles).

The reality of modelling this in the supply chain management assumes that certain items do deteriorate. In most practical real-life situations, this would be reflected in situations whereby for example, volatile liquids would evaporate, batteries could fail with age and perishable items would degenerate. The manner in which this deterioration takes place, is not the same in each instance for each category of items.

The initial approach by Harris (1913) had been to examine the problem assuming a constant demand profile and a constant rate of deterioration. Practical observations referred to here, has established that these rates of deterioration do however, not adequately reflect reality. It means that a lot of scope exists to investigate and incorporate the impact of items that deteriorate over time. Much more work and study can still be undertaken to improve upon research that is available so far.

The concept of having items in an inventory that deteriorate was first examined by Whitin (1957). He investigated the notion of deterioration of those items in storage. Subsequently, Ghare and Schrader (1963) incorporated the probability that inventory may deteriorate through a modelling process. The basic EOQ model which had considered deterioration was first done by Ghare and Schrader (1963) in the context of an exponentially decaying inventory. Raafat (1991) was able to illustrate that Ghare and Schrader (1963) did consider the impact of decay in inventory by noting the potential for cost saving measures and improvements in the inventory reordering cycle. The Ghare and Schrader (1963) model formulated the change in inventory level $I(t)$ in the Equation (2.5).

$$\frac{dI}{dt} + \theta I(t) = -D(t) \tag{2.5}$$

The concept of decaying inventory was graphically depicted by Ghare and Schrader (1963) and is shown in Figure 2.4.

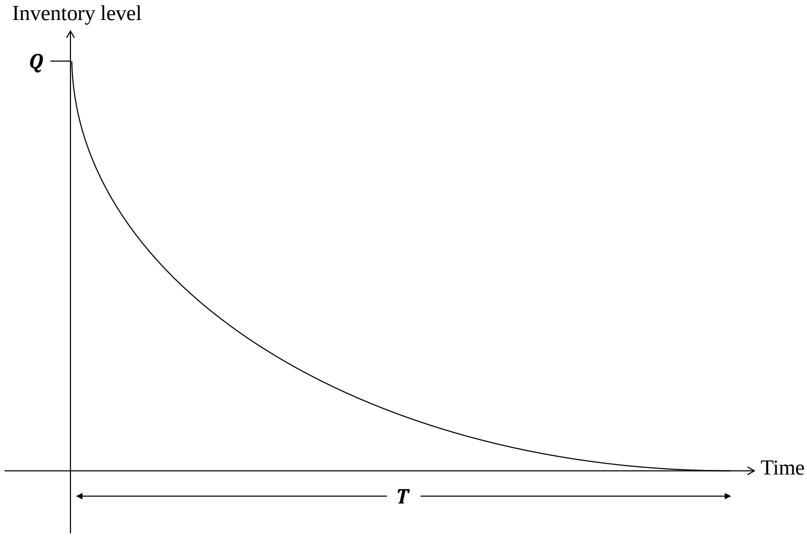


Figure 2.4: Inventory profile for items that deteriorate.

By examining the graph the following can be illustrated.

- $D(t)$ is the demand per unit of time.
- T is the time of the inventory cycle.
- Q is the order quantity.
- θ is the constant rate of decay.
- C is the cost per unit of the inventory item.
- K is the ordering cost per batch.
- h is the inventory holding cost expressed as a fraction of the maximum permissible inventory.
- $I(0)$ is the initial inventory, which the total sales during the inventory cycle plus the loss due to decay.
- $D(t)T$ is the total sales during the inventory cycle.
- $\frac{D(t)\theta T^2}{2}$ is the loss due to decay for the cycle.
- t is a representative of time.
- $I(t)$ is the inventory level at time t.
- dt is a small change in time.

Using the parameters of Ghare and Schrader (1963) it has been established that the total cost per cycle C_T will be the aggregate of the purchase costs, the costs of ordering together with the holding costs. This is illustrated in the Equation (2.6) from Ghare and Schrader (1963).

$$C_T = C\left[D(t)T + \frac{D(t)\theta T^2}{2}\right] + K + hC\left[D(t)T + \frac{D(t)\theta T^2}{2}\right] \tag{2.6}$$

Ghare and Schrader (1963) were able to derive an equation for the optimum order quantity Q^* in Equation (2.7).

$$Q^* = D(t)T + \frac{D(t)\theta T^2}{2} \tag{2.7}$$

This is only an approximation on the part of Ghare and Schrader (1963), but it is sufficiently accurate since T would most likely be rounded off in any event.

2.4.2 Subsequent deterioration models

Additional work was done to the basic model to incorporate varying rates of deterioration. Extensions to the first model of Ghare and Schrader (1963) assume that the deterioration rates need not necessarily be constant.

It became apparent that deterioration needed further investigation. There is substantial additional work in this regard that has already been completed. The work of Raafat (1991) collated various of the prior studies and this output has resulted in a comprehensive synopsis of deterioration, before the publication of the Raafat (1991) study. Some of the salient literature that is available in this respect, is referred to here.

Covert and Philip (1973) used a two parameter Weibull distribution for an EOQ model that has a fluctuating deterioration rate. It assumed that the rate of decay was constant which would have implied that the inherent deterioration was an exponential distribution with instantaneous delivery and no shortages. This is a specific case of a Weibull distribution analysis.

The investigation of Covert and Philip (1973) was further developed by Philip (1974) to a Weibull distribution involving three parameters. It took into consideration the impact that deteriorated items would have in an inventory system together with items that commence with deterioration in the future.

Extension of the Philip (1974) model, without any back-ordering and with a gamma distribution assumption for the inventory deterioration was investigated by Tadikamalla (1978). This findings noted that where two such distributions have similar shapes their deterioration rates are different at any specific point in time. It means that it is relevant to have a thorough knowledge of the deterioration and its attributes to fully understand the cost implications.

Shah and Jaiswal (1977a) produced an inventory model that is deterministic for items that have deterioration with back-ordering and instantaneous replenishment permitted.

Work has also been done to assess finite rate production models that have deterioration. A model with both a changing deterioration rate and also a constant rate of deterioration for a production lot size had been developed by Misra (1975). Under a scenario of differing rates of deterioration, the expression of the production lot size without back-ordering was computed. It appears not to be possible to obtain a basic general expression as an alternative for a production lot size. This means that as an alternative, a numerical method would have been appropriate. In constant rates of deterioration scenarios an estimated expression was determined for the size of that production lot.

Furthermore, Shah and Jaiswal (1976) have found results that are close to those of Misra (1975) and extended the model to include back-ordering for a constant deterioration rate. They assume the average inventory on-hand would be about half of its maximum size and derived a production lot size as a function of the given inventory time of the cycle.

Further probabilistic models with deterioration assumed immediate delivery and constant deterioration rate. Shah and Jaiswal (1977a) provided an inventory order level model with immediate delivery and deterioration at a constant rate. It was then extended to include a stochastic demand profile. Their work also considered the average carrying inventory to be a linear function estimate and derived a order level function subject to the cost of the product.

Aggarwal (1978) excluded the linearity presumption in favour of precise statements for the inventory attributes of this model.

Shah and Jaiswal (1977b) proposed a probabilistic review model for an inventory system with deteriorating items. The model had been derived for all general rates of deterioration. Here, both the uniform and varying deterioration rate had been considered.

In the Rengarajan and Vartak (1983) investigation a generalisation of the model was set out to find a general solution for deteriorating items. Results were found for the optimal lot size and order level, incorporating a constant deterministic demand throughout scheduling of the product. The optimal environment for the best opening stock-level does not rely on the nature of the demand, for a fluctuating demand schedule that is known.

Deterioration is also a factor that has been studied in pricing and financial inventory models. As an example, Cohen (1977) evaluated the issue of concurrently permitting the selling price and the order quantity under demand that is known for a product that decays exponentially. Such products have been found to experience a proportional loss in usefulness to all the stock on hand. This depends on the price at which the unit is sold.

Models can be considered in terms of the best control theory to derive the optimum replenishment scenario, assuming algebraic cost functions that are quadratic for inventory models and changing rates of deterioration. These issues were considered by the work of Bensoussan et al. (1975).

Deteriorating inventory models also feature in studies with varying rates of demand. Goel and Aggarwal (1981) considered an inventory model with an exponential demand as well as a constant rate of deterioration. Their work incorporated demand studies with or without any shortages.

Where the demand rate scenario would be a function of the inventory level was examined by Padmanabhan and Vrat (1990). They categorised this as the stock dependent consumption rate. For the model an inventory scenario is suggested, that is based on the opening stock dependent consumption rate and the exponential rate of item decay. It looks at the cost of material that has been lost to deterioration, together with the ordering costs, the carrying costs and the cost of the materiel. Ongoing replacement is permitted without any back-ordering. This produces an optimal ordering quantity.

Further trends became apparent in respect of being able to model items or inventory that deteriorates. In broad terms, this emphasis focuses more on the shelf-life characteristics of those deteriorating items. Goyal and Giri (2001) extrapolated the earlier work of Raafat (1991) to cover subsequent developments in their analysis of deterioration. The salient features of the subsequent study of deterioration are elaborated on in the following literature.

The Raafat et al. (1991) study examined an inventory model that deteriorates in a uniform rate of demand and with a specific replacement rate. The precise cost expression of the average total cost for the production lot-size of the model comprising sustained deterioration is arrived at.

For many items the presumption of a constant rate of demand may not always be relevant and so the concept of time varying demand with deterioration becomes significant.

A model of inventory policy for items that deteriorate is presented by Xu and Wang (1990). This happens by assuming that the demand rate is deterministic, linear and modifies over time with a constant deterioration rate. The planning horizon would be known and finite. It was assumed that the replenishment cycles were unequal. The result is applicable to cases where the demand would be increasing or decreasing.

Inventory scenarios with sales rates that were stock level-dependent had been investigated by Padmanabhan and Vrat (1995). The rate of the sale was assumed to be dependent on the inventory level in a situation of constant stock deterioration. With no lead time and immediate refilling the model included situations such as complete, partial and even zero back-ordering scenarios. For each of the situations the system is configured to maximise the return as reflected in the total profit.

The concept with respect to deteriorating stock with pre-agreed or permissible delay in payment has also been investigated. Usually suppliers offer credit or some form of beneficial incentive to retailers so that they can also contribute to stimulating demand for the product. Interest would generally not be levied in such instances and this assists retailers to enhance revenue for the specific period of delay in the payment. Haley and Higgins (1973) investigated this aspect. Later researches such as Jaggi and Aggarwal (1994) also looked at optimal inventory replacement scenarios under various conditions. Furthermore researches have examined the impact of trade credit and the stock replacement policy in subsequent practical studies.

Traditional inventory models are applied within the realm of a single warehouse facility. Practically however, limited capacity of the facility may mean that additional storage space is required and this leads to a deteriorating inventory scenario with two warehouses. The notion of a dual-warehouse stock model was investigated by Pakkala and Achary (1991). They developed models for this scenario of deteriorating items with shortages and finite replacement rates.

Since 2001, inventory systems that deteriorate have been complied into a research report and documented by Bakker et al. (2012). This report deals with specific categories that would be applicable to the model of deterioration.

The authors Bai et al. (2010), suggest an EOQ event with general economies of scale, cost functions that would be applicable to decaying inventory. The investigation pertains to a uniformly distributed fixed lifetime inventory. Back-ordering is permitted and optimal solutions together with certain properties are examined.

For fixed lifetime stock level-dependent demand and also for age dependent decaying rate, the work of Akkerman et al. (2010) reviewed qualitative operations management. They considered perishability in the supply chain. The focus of their work concentrated on food quality, food safety and overall sustainability. They investigated the impact of the strategic design of networks, tactical network plans and operational transport planning.

A model by Alamri (2011) investigated a general uniform stock-level scenario for the integrated production of new items, within a longer term planning horizon. The first facility produces new items while the second one re-works and recycles the return goods so that they appear new. Another facility collects the items to be sent back for reprocessing as in the initial step.

The dynamic pricing issue that a firm faces when it sells the first inventories of many perishable and substitutable products over a given time horizon was studied by Akçay et al. (2010). They account for interchangeable and complimentary items in a multi-item inventory model under situations of uniformly distributed age dependent deterioration.

Balkhi and Tadj (2008) investigated an age dependent deterioration rate with time varying deterministic demand. Furthermore cost parameters are also presumed to be universal variables of time.

In recent years authors such as Blackburn and Scudder (2009) have modelled the costs and benefits of using preservation technology in the supply chain process relating to the storage of fresh produce. This would have an impact on the rate at which normal deterioration occurs. It can be considered as an extension to the concept of deterioration. The study shows that the correct model to minimise the value lost in a supply chain would be a compensate of the time taken from harvesting to refrigeration, with an efficient preservation in the supply chain at the end.

Various assumptions that may be required to justify using the EOQ model may not always be met. According to the Moussawi-Haidar et al. (2014) model that closely replicate real-life situations may need to have certain assumptions relaxed. Research investigated a modified inventory EOQ model for a deteriorating item with erratic supply. It means that a certain percentage could be lost to deterioration of that inventory on hand.

Subsequently, the investigation by Jaggi et al. (2017) arrived at a dual-warehouse model for inventory simultaneously taking into account imperfect quality items deterioration and certain trade credits. Total profit is maximised in this study through optimisation of the order quantity over time.

In many issues related to deterioration in a real world situation some items may be impacted on by other items in proximity. Studies by Khakzad and Gholamian (2020) made proposals that considered the effect of inspection times when replenishment was taking place on the average deterioration of those items. In this examination the supplier is required to implement certain prepayments onto the retailer.

More recently the inspection process is important to monitor the quality of the items. Despite good planning certain defective items may be delivered to retailers in each delivery lot. The EOQ paper by Jayaswal and Mittal (2022) arrives at a set of models that demonstrate learning, or assimilates the impact of defective and decaying items in a scenario of inflation. The purpose is to evaluate the impact of learning on the best order quantity and the corresponding total profit in a scenario of inflation.

Work by Rahman et al. (2022) evaluated a model with interval based parameters for items that deteriorate. Two situations were considered namely, those where shortages are applicable and those without any shortages in a discounted environment. The rate of deterioration is considered as the interval value while the carrying cost would be a function of the time that it had been stored together with the purchase cost.

As the review of the literature progresses the focus will increasing shift towards one of the anticipated outcomes, which is an investigation of perishable items with expiry dates.

The literature shows that researchers had previously focused their studies on the effects of perishability as viewed from the perspective of a supplier of goods. In other words, the impact of perishability from the perspective of the consumer needed further attention. More recently, it has been found that the impact of freshness is one of the most critical considerations that would affect the decision of consumers when purchasing goods.

The work by Fujiwara and Perera (1993) considers ongoing deterioration of product usefulness together with an exponential cost function penalty as the measurement of that deterioration. This enables the impact of item freshness on the customer demand to be taken into account.

The impact of inventory policies on demand for perishable products is negatively affected by the age of the available stocks according to Sarker et al. (1997). The research describes such a model where the demand is considered as a composite function incorporating both consistent and variable components that would be directly related to the inventory level.

The Bai and Kendall (2008) model investigated inventory control issues that are associated with fresh produce. It considers deteriorating inventory control for perishable goods by extending the demand to incorporate freshness and shelf-space dependencies. Many perishable goods have very limited shelf-life and demand tends to zero as the expiry dates are approached.

The date of expiry was examined by Wu et al. (2016). That work configured the best replacement cycle time for the retailer and also the stock-end level when the demand is based on the freshness of the product with the stock level prominently displayed.

Chen et al. (2016) expanded this concept to find the optimal lot size, ending-stock levels and shelf space for a retailer by considering the aspect of shelf-space size as an additional variable for consideration.

Work by Wang et al. (2014) looked at the maximum life time span as an additional concept for study. Certain perishable products deteriorate continuously but they may also have expiry dates. To increase sales sellers may offer purchases a credit period to settle the purchase amount.

According to Giuseppe et al. (2014) the food supply chain can be affected by the loss of products as the expiry date approaches. In order to manage expiring products such as food at the retail stage this work investigated the optimal time that would be required to withdraw that food from the shelf of retail supplier.

The work of Giuseppe et al. (2014) was extended by Muriana (2015) to consider products that could be adversely impacted on by uncertainties in shelf-life. Furthermore, loss of profits and a deterioration costs are included in this study.

In another study by Aiello et al. (2015) alternatives of food recovery for humanitarian non-profit organisations was discussed. Consumer demand for perishable products is more appropriately stochastically modelled than through deterministic models. The paper presents a mathematical model for the coordination of the supply chain process by operating a food recovery policy. It determines the optimal time for withdrawing products from the shelves and passing that product to other users thereof.

Additional work was also carried out by Muriana (2016) with the development of an EOQ model for goods that are perishable under the stochastic demand scenario. This permits amendments to consumer behaviour that can be weather related and where purchasing power can be a factor.

It is found that retailers may discount their products if they have expiry dates, as that date of expiry approaches. According to the work of Banerjee and Agrawal (2017) it was possible to create a model to optimise the ordering, discounting and pricing policies for an inventory system that will expire if it has a demand that depends on the sales price. As a perishable good begins to deteriorate it is often discounted to be able to boost sales. This may benefit the retailer if appropriately applied.

The Feng et al. (2017) zero-level inventory model (similar to the Wu et al. (2016) investigation), assumptions were relaxed. A profit maximisation model for inventory, with expiration dates, price-dependent demand and freshness was investigated. An additional consideration permitted for a postponement in the payment which would have been an inducement for the supplier to deliver goods to the seller without immediate payment settlement.

An extension of the work of Wu et al. (2016) is presented in Wu et al. (2018) by considering allowable payment delays as an incentive where the supplier is able to deliver the order but does not require immediate settlement. Rather a specific amount of time is grated for the settlement of the bill to be implemented. This is of particular significance in respect of perishable products in a situation where the buyer is offered a interest free payback period over the short term on the purchasing cost.

Trade credit was incorporated into the Wu et al. (2016) model by Li et al. (2019) in a manner whereby the period of trade credit was treated as an added variable in the overall decision making process. The optimal credit term, order size and selling price are derived simultaneously for the retailer to achieve maximum profit.

The issue of expiry dates when shortages are permitted for deteriorating items had previously not been considered. Khan et al. (2019) suggested an EOQ model to cater for this in a situation, where the end user demand relies on the sales price of the products.

Perishable food products may be influenced by the selling price and the age of those items. As perishable products have little brand identification, factors such as freshness and age become significant determinants of demand. Sebatjane and Adetunji (2020b) used this concept to investigate perishable inventory. However, it is increasingly important to do this investigation in a collaborative sense with other participants in the supply chain process rather than studying it merely from the perspective of the retailer. This has resulted in a model to manage perishable goods within the supply chain that commences with farming operations and ends with the consumption of that inventory at its destination. In effect there are three echelons in this specific supply chain comprising production (or farming), processing and finally retail distribution. Jointly optimising pricing and inventory policies is investigated for this study.

2.5 Inventory level-dependent demand

2.5.1 The original inventory level-dependent demand model

It has been observed that larger display volumes can influence demand for a product. The configuration and volume of product that is displayed thus becomes an additional attraction for end users of that product. The Baker and Urban (1988) analysis sought to develop a scenario where the demand for the product of a company would be a function of the inventory on-hand of that particular product. A typical inventory profile for such a situation is illustrated in Figure 2.5. This depiction is based on consumer demand and marketing theory which notes an increase in consumer demand with increased levels of stock on display according to the work of Levin et al. (1972).

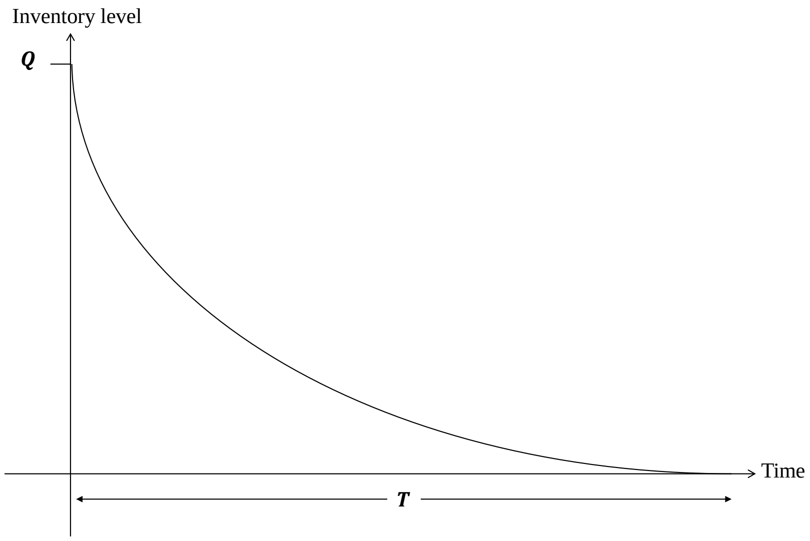


Figure 2.5: Inventory level-dependent demand rate graph.

The variables used in this analysis are defined below;

- D is the demand rate.
- $I(t)$ is the inventory level over time.
- δ would be the scaling parameter (the maximum attainable demand) for that demand rate.
- ψ the elasticity of demand with respect to inventory level displayed (a shape parameter for the graph).
- p_v purchasing price for each item in an order.
- p_r selling price for each item that is sold.

- h_r is the holding cost for the company per unit of time to keep each item in stock.
- K_r is the fixed ordering cost for the company.
- Q is the lot size ordered.

The function Equation (2.8) of Baker and Urban (1988) was applied as follows.

$$D = \delta[I(t)]^{\psi}, \qquad 0 \leq t \leq T \tag{2.8}$$

Inventory levels would change with time according to the differential equation as shown in Equation (2.9).

$$\frac{dI(t)}{dt} = -D = -\delta[I(t)]^{\psi}, \qquad 0 \leq t \leq T \tag{2.9}$$

The solution to the Equation (2.9) would produce an expression for $I(t)$ (in terms of Q) which would be used to calculate the holding costs for the company per unit of time. This means that if the company sells the items for p_r then the total profit in each unit of time, TPU is in Equation (2.10).

$$TPU = p_r\delta(1-\psi)Q^{\psi} - \frac{K_r\delta(1-\psi)}{Q^{(1-\psi)}} - p_v\delta(1-\psi)Q^{\psi} - \frac{h_r(1-\psi)Q}{2-\psi} \tag{2.10}$$

It is difficult to obtain a closed form solution to Equation (2.10) so various other techniques are applied to determine the optimal lot-size Q^*.

2.5.2 Extensions to the inventory level-dependent demand model

Research by Baker and Urban (1988) has confirmed that large exhibits or quantities of inventory are able to stimulate market demand. Baker and Urban (1988) devised an EOQ model for items with demand which would depend on inventory levels that are available on-hand. This applies inventory with continuous deterministic demand illustrated by means of a polynomial function that would rely on the inventory level. Their model was expressed as an exponential function that represented the rate of demand of inventory dependency levels and it had suggested that a new inventory series would commence when the level of the inventory in the current series had reduced to nothing.

Mandal and Phaujdar (1989) extended the work of Baker and Urban (1988) with a linear function that illustrates the rate of the demand of the inventory level dependency. With deteriorating items produced by constant production rates with stock-dependent demand an order-level inventory model had been produced. Shortages had been permitted and surplus demand was back-ordered.

An extension was provided by Urban (1992) that illustrated the advantage of applying a policy whereby the level of the inventory had been a positive, at the conclusion of the cycle. The nil policy of inventory at the end of the cycle is not necessary for demand rates in deterministic models and hence it is limitation that is relaxed in this study.

Complimentary studies by Urban and Baker (1997) and Teng and Chang (2005) produced research where the rate of demand would be dependent on the selling price and the level of inventory. Urban and Baker (1997) investigated a positive or non-zero final inventory level together with the reduction in price at the termination of the replacement cycle. The model of Teng and Chang (2005) applied to items that deteriorate under this scenario.

The demand of retail items that have been displayed has been recognised as being dependent on the quantity of inventory that was on the display. Work by Urban (2005) where the rate of demand for an item would be based on the initial inventory level and furthermore, that the demand relies on the specific amount of stock at any moment in time.

Another approach that used a discounted cash flow model to reflect the impact of inflation on an inventory system was investigated by Hou and Lin (2006). This model, together with the time-value of money relies on a stock level-dependent demand scenario, with the amount of the stock items and their selling prices.

The Wu et al. (2006) study investigated the issue of ascertaining the best replacement strategy for items that deteriorate with stock-dependent demand. These items would not deteriorate instantaneously in the model. Shortages are permitted with a variable back-ordering rate, but it requires a waiting time for future replenishment. The work identifies the optimal circumstances in such situations, benchmarked against the minimum total respective cost per unit of time.

Work by Goyal and Chang (2009) examined an inventory model that determines the best order quantity for the retailer and the rate of transfer per order from the warehouse to the final selling show premises. The display space would be constrained and the demand relies on the level of inventory on exhibit. The aim had sought to maximise the mean profit over time that the customer would be able to realise.

Two inventory control models were considered by Duan et al. (2012). One of the models considered the back order shortages, while the other model did not. The model examined items that deteriorate at a rate of demand that relies on the actual stock level.

Mishra et al. (2013) evaluated a model of deterministic stock with demand that is time-dependent and also with time-varying storage costs in a situation where the deterioration would be time proportional. This model tolerates shortages with partially back-ordered demand. It seeks to find the best inventory costs for the business where the deterioration rates and the holding costs are time-dependent.

Many retailers increase product offerings to enhance their market share. By offering a number of like products means that they are substitutable by the consumer. The work by Krommyda et al. (2015) investigated inventory control in a situation where demand could be addressed through substitution. The demand for the specific product is affected by the stock levels of each of those products at specific points in time.

A study of production capabilities by Sargut and Işık (2017) produced a dynamic economic order quantity issue for a single item that perishes. The objective was to identify the production, inventory and back-ordering decisions that would be necessary during the planning horizon. The parameters are deterministic but change over time and the producer has constant production capacity that limits production in each period. Outstanding demand is able to be met at some time in the future.

The Pando et al. (2019) study examined a stock system that sought to establish the maximum profit cost ratio. This assumed that the rate of demand would depend on the amount of the stock. Furthermore, the stock holding cost and the amount of stock would be not be a linear function over time.

A production inventory model for items that deteriorate with non-linear price and linear dependent inventory demand was investigated by Halim et al. (2021). This incorporated a production facility that could produce more goods in overtime situations.

A model where the rate of demand relies on the eventual selling price and that stock level was considered by Pando et al. (2021). Lower prices or higher stock levels did result in increased demand rates. The selling price, the order level and the point at which re-ordering happens were the three decision variables that were evaluated. The objective was to maximise the relationship between the total profit and the total costs that accrue to the inventory system.

Expiry dates on perishable goods impact significantly on purchasing behaviour from consumers. Marketing and inventory display also increases demand by consumers for such items. Furthermore, primary products generally require processing before they can be transformed into a consumable product by the retail sector. Customer demand at the consumer end relies on the stock level and also the date of expiry of the products. Sebatjane and Adetunji (2021) derived a model to ascertain the effectiveness of a mechanism to enhance profit in a situation that removes the earlier zero-end inventory policy in the supply chain. In essence, it permits for shortages. The assumption is that inventory is kept at the retailer and that it is replenished when a certain minimum level has been attained. Clearance sales are held to ensure that the required level of inventory is maintained at an acceptable level of freshness. This enhances profitability and is preferred to the traditional notion of ending with zero inventory level studies.

Palanivel and Suganya (2022) examined an inventory scenario to maximise the profit for an optimally ordered quantity, with back-ordering and holding cost dependent on the storage time period and the market demand. The model assumed to vary as a function of the selling price together with the stock level. It was concluded that total profit can be increased by permitting shortages together with some back-ordering.

Using a fuzzy EOQ model analysis Poswal et al. (2022) investigated deteriorating items that have a price sensitive and stock-dependent function where shortages would have been permitted. By displaying a large quantity of goods sellers invariably attract more customers to purchase more in certain business situations.

2.6 Literature review conclusion

An important feature of this study is to analyse the available literature to be able to place into context any shortcomings within the supply chain framework that has been studied. There is detailed referral to other literature that is available in respect of these issues in Chapter 2. Primarily, studies by Harris (1913) (the classical model), Salameh and Jaber (2000) (the imperfect quality model) and Ghare and Schrader (1963) (the concept of deterioration analysis), forms the basis of supply chain inventory models. With a broader approach, Baker and Urban (1988) expressed demand rate as a power function of the level of the inventory. The concept of freshness was used by Wu et al. (2016) to derive a freshness index that was based on the expiry dates of those goods. However, demand by consumers is increasingly seeking goods that have minimal inherent defects. To extend on this concept Lin and Hou (2015) considered an inventory model that has imperfect quality items with advanced receiving methods. Work by Jaggi and Mittal (2011) elaborated on these studies to integrate deterioration with imperfect quality items. Sebatjane and Adetunji (2021) incorporated the concept of both freshness and inventory level to determine demand.

One of the shortcomings identified from the Lin and Hou (2015) study was the omission of the concept of deterioration. The Jaggi and Mittal (2011) assumed constant demand which may also not be realistic. The Sebatjane and Adetunji (2021) study did not focus on deterioration and imperfect quality items.

This has resulted in a shortcoming (or gap in the research) which is to combine elements of imperfect quality, deterioration and freshness into a single entity model. This has become the central theme of the study as proposed here. The gap is illustrated in Table 2.2.

Table 2.2: Research gap.

Author	Imperfect	Deterioration	Freshness	Inventory-level
Salameh and Jaber (2000)	✓			
Ghare and Schrader (1963)		✓		
Baker and Urban (1988)			✓	
Wu et al. (2016)				✓
Lin and Hou (2015)	✓			
Jaggi and Mittal (2011)	✓	✓		
Sebatjane and Adetunji (2021)			✓	✓
This thesis	✓	✓	✓	✓

The objective of this study is to address a research gap by creating a scenario whereby imperfect quality, deterioration and freshness can be quantitatively assessed. A mathematical model will be constructed to reflect these issues and it will be tested with numeric data to ascertain the viability thereof. Certain sensitivity studies are also to be conducted to determine the effects of changes to the numeric data on the overall outcome of the model. Ultimately this should lead to it being able to create a scenario which will better reflect real-life situations.

The benchmark against which these various components will be adjudicated is the total profit per cycle. The primary objective will be to maximise the profit per cycle in each instance.

Although previous studies have contributed much to the understanding of the effects of stock-levels on demand, most of them have not taken into consideration the combined effects of freshness (which is determined by the expiration date), and stock-level on demand. Imperfect quality and deterioration have been comprehensively addressed in many studies, but the concept of combining this with levels of freshness and stock-levels collectively is something that has not been adequately investigated. The objective is to address a research gap by creating a scenario whereby imperfect quality, deterioration and freshness are combined to maximise profit.

Chapter 3

Mathematical model

3.1 Introduction

Reference has already been made to the fact that the concepts of quality, deterioration and freshness in Chapter 1 have not been adequately investigated within a collective environment, or a situation where these factors are integrated within a single supply chain mechanism. Furthermore the literature review in Chapter 2 had provided a basis on which to identify the shortcomings that manifest themselves with such issues. This has culminated with a proposal to address the shortcomings (or a gap in the research, identified in Chapter 1) in terms of the published models currently available. The purpose of the Chapter 3 is to clearly understand those shortcomings and to develop a quantitative or theoretical background upon which a solution to this issue can be based. After the theoretical model has been proposed the intention is to assess its applicability by testing it with a set of numerical examples that test the viability thereof.

3.2 Problem definition

In a study of imperfect quality, deterioration and freshness, the analysis assumes that the products (Q in Figure 3.1) get delivered and are stored at their destination.

This is based on the assumption that it is a single product-type, but with that product-type being of varying quality. On arrival, the products are screened to determine their quality. The process of screening individually examines each product for defects. The screening process results in two separate categories of product but of differing quality. The first category would comprise the good product, while the second category comprises the goods that are of imperfect quality. The rate at which screening takes place results in the amount of good product always exceeding the demand for that good product. During the screening phase when a product has been classified as an imperfect product, it is separated from the delivered products and is then stored separately. Immediately following the completion of the screening process (t_1 in Figure 3.1), the imperfect quality goods are sold at a price that is lower than the price of the good product.

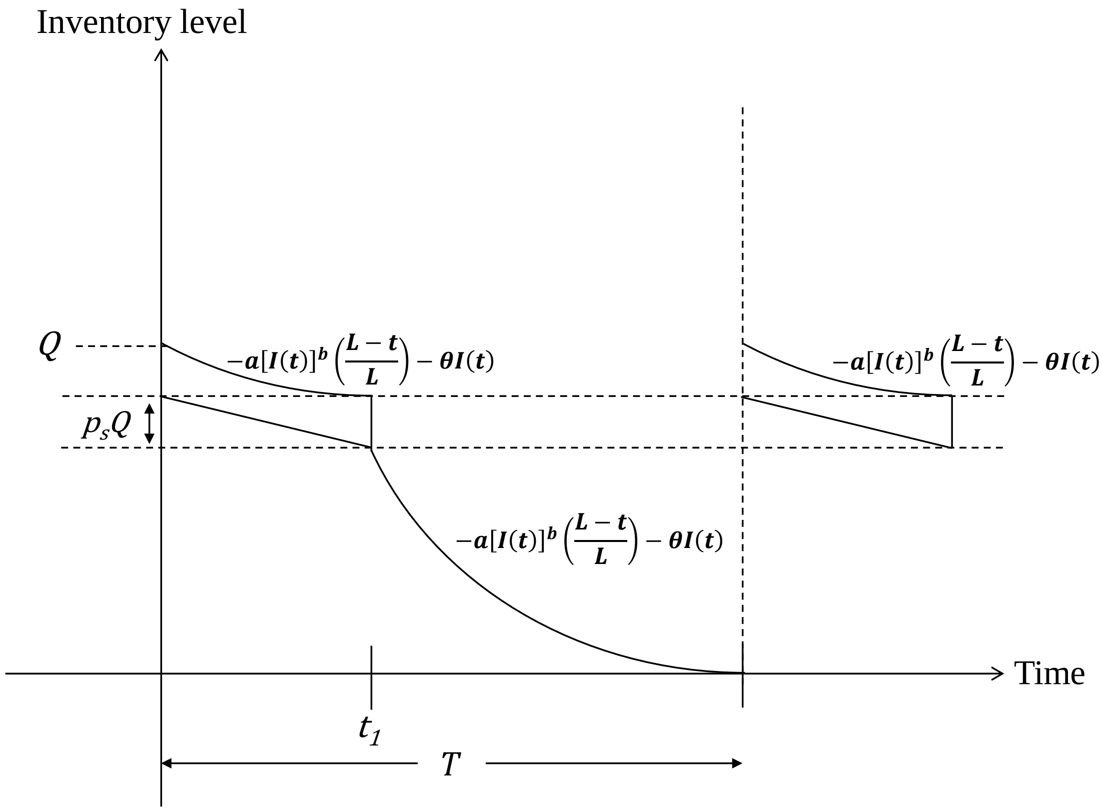


Figure 3.1: A graphical representation of an Economic Order Quantity (EOQ) model for imperfect and deteriorating items with freshness and inventory level-dependent demand.

Assuming that there was no inherent deterioration and no imperfect quality products, the graph would illustrate the demand for the good product. The demand in such instances is a function of the inventory level, the freshness of the goods and a measure of consumer trends. Furthermore, the demand for the product decreases as the expiry date for that product approaches. By introducing deterioration, it impacts adversely on the quantity of goods that are available for sale. However, deterioration does occur and certain intervention is necessary to remove the deteriorated items from the saleable goods. The deteriorated items are sold off at a reduced price as part of this process. This is an additional factor that needs to be considered in the determination of the good product to be sold. All the good products are held until a sale takes place.

At the end of the cycle (T in Figure 3.1) there are no goods left for sale. The process then repeats itself for subsequent goods delivery. The Figure 3.1 illustrates a commodity that would be subject to deterioration and also its freshness. A practical example would be fruit such as a bananas.

3.3 Notations

Table 3.1: Notations used in the formulation of the mathematical model.

Symbol	Description
Q	Order size per cycle.
D	Demand rate.
x	Screening rate.
p_s	Percentage rate of imperfect items in Q.
$I(t)$	The instantaneous state of inventory level at time t.
t_1	Screening time.
T	Cycle length.
$F(t)$	Products freshness index of the inventory at time t, which is a function of the expiration date.
L	The expiration date (or shelf life) of the product.
θ	Rate of deterioration for each unit of time.
a	Scaling parameter for the demand function.
b	The shape parameter representing the elasticity of demand.
h	Holding cost per unit of the perfect product for each unit of time.
h_s	Holding cost per unit of the imperfect product for each unit of time.
K	Fixed ordering cost.
C_d	Deterioration cost.
C_s	Screening cost per unit item screened.
C_g	Cost price per unit of each product.
S_g	Selling price per unit of each perfect product.
S_d	Selling price per unit of each imperfect product.

3.4 Model development

3.4.1 Assumptions

Levin et al. (1972) noted that an important determinant of demand is the inventory level of that item that is on display. It has been found that the greater extent to which inventory is displayed has an effect of being able to induce customers to increase their levels of purchase. For inventory models various functions can be applied to portray this observation. The use of the power function to address this matter is acknowledged as illustrated in Equation (3.1). Baker and Urban (1988) have expressed the rate of demand as being some power function of the level of the inventory.

$$D = a[I(t)]^b \tag{3.1}$$

In this instance a would be the scaling parameter for the rate of demand. Alternatively it is the asymptotic level of demand that would be attainable whenever an inventory level is considered at the most optimal for its consumers. Similarly, b is a shape parameter that represents the elasticity of the demand rate with respect to the inventory level that is being displayed. Furthermore, $0 < a$ and $0 \leq b < 1$. From the Equation (3.1) the relationship shows that the rate of demand for inventory increases with a rising levels of inventory that are displayed at the customer. Similarly, when the inventory level decreases, the demand rate also decreases. The relationship expressed in the power form also illustrates that at the commencement of a replacement cycle (with maximum inventory level), inventory is depleted at a higher rate and with the progression of time the depletion rate of that inventory slows down.

The dependence on the rate of demand on the expiry date is introduced with the concept of a freshness index. In an environment where a product has a finite shelf life the consumers of that product are likely to make their purchases after consideration of the age of that product to be purchased. This means that expiration date on food products are more likely to be purchased when the expiry date is further away than those items which are close to becoming expired. Wu et al. (2016) applied the date of expiry of an item to categorise the freshness index of that product as in Equation (3.2).

$$F(t) = \frac{L - t}{L} \tag{3.2}$$

The expiry date of the product is represented by L. Over time, the product becomes less fresh and is less attractive to consumers. This means that Equation (3.2) will be able to quantify the attributes of freshness just after delivery to the customer. The inventory would be in its most fresh just after it had been delivered to the customer. At that moment of time $t = 0$, with $F(0) = 1$. The product becomes least fresh when it has reached its expiry date. The time to expiry or the maximum shelf-life of the product would then be $t = L$ with $F(L) = 0$. This means that the inventory cycle time of the retailer, T, cannot exceed the expiration date $T < L$.

From studies by Chen et al. (2016) and Feng et al. (2017), Equation (3.1) and Equation (3.2) are integrated to compute the demand rate as being a multiplication function of inventory together with the freshness attributes of that inventory. This is shown in Equation (3.3).

$$D = a[I(t)]^b \left(\frac{L - t}{L} \right) \tag{3.3}$$

During the cycle, deterioration occurs and intervention is initiated to remove the deteriorated items from the goods that are for sale. This deterioration is determined by the size of the inventory level $\theta I(t)$ where $0 < \theta < 1$.

The screening rate x, which is, $p_s(Q/t_1)$. The screening of the goods has to be undertaken so that there would be sufficient goods available for the end-user during the screening period, t_1. In this instance, it is assumed that the screening rate is larger than the demand namely, $D < x$. It is necessary to have sufficient product of adequate quality available to satisfy the demand for that product to the end user.

3.4.2 General inventory equation

The inventory throughout the replenishment cycle would be depleted because of demand and deterioration. The inventory level is dependent on the freshness index of that inventory and the manner in which product deterioration occurs. Hence, the level of the inventory is determined by the differential Equation (3.4).

$$\frac{dI(t)}{dt} = -D - \theta I(t), \qquad 0 \le t \le T \tag{3.4}$$

By substituting Equation (3.3) in Equation (3.4) it becomes Equation (3.5).

$$\frac{dI(t)}{dt} = -a[I(t)]^b \left(\frac{L-t}{L}\right) - \theta I(t), \qquad 0 \le t \le T \tag{3.5}$$

With algebraic rearrangement of the terms, Equation (3.5) can be rewritten as shown in Equation (3.6).

$$dI(t) = \left[-a[I(t)]^b \left(\frac{L-t}{L}\right) - \theta I(t)\right] dt, \qquad 0 \le t \le T \tag{3.6}$$

Integrating both sides of Equation (3.6) results in Equation (3.7).

$$e^{(1-b)\theta t}[I(t)]^{1-b} = \left(ae^{(1-b)\theta t}\right)\left[\left(\frac{t}{L}-1\right)\left(\frac{1}{\theta}\right) - \left(\frac{1}{(1-b)\theta^2}\right)\left(\frac{1}{L}\right)\right] + C \tag{3.7}$$

The amount of the inventory reduces to zero at the completion of each cycle of replenishment (alternatively stated, $I = 0$ at $t = T$). Therefore, the boundary condition $I(T) = 0$ is binding. The boundary condition $I(T) = 0$ is then used to solve for C from Equation (3.7) and the result is Equation (3.8).

$$C = -\left(ae^{(1-b)\theta T}\right)\left[\left(\frac{T}{L}-1\right)\left(\frac{1}{\theta}\right) - \left(\frac{1}{(1-b)\theta^2}\right)\left(\frac{1}{L}\right)\right] \tag{3.8}$$

An expression for the inventory level at any time is determined by substituting Equation (3.8) into Equation (3.7) and rearranging the terms. The result is Equation (3.9).

$$[I(t)] = \left[\frac{a}{\theta L}\left[t - L - \frac{1}{(1-b)\theta} - e^{(1-b)\theta(T-t)}\left[T - L - \frac{1}{(1-b)\theta}\right]\right]\right]^{\dfrac{1}{1-b}} \tag{3.9}$$

When a subsequent replenishment cycle commences, the retailer receives an order of inventory from the supplier of Q items. This implies that the boundary condition $I(0) = Q$ is binding. The inventory lot size or the amount of inventory at the beginning of the cycle is determined by substituting the boundary condition into Equation (3.9). It is illustrated in Equation (3.10).

$$Q = \left[\frac{a}{\theta L}\left[-L - \frac{1}{(1-b)\theta} - e^{(1-b)\theta(T)}\left[T - L - \frac{1}{(1-b)\theta}\right]\right]\right]^{\dfrac{1}{1-b}} \tag{3.10}$$

3.4.3 Total Cost Function (*TCF*)

Inventory models generally have some form of costs associated with them which enables them to reflect real-world situations. The investigation so far provides the background to quantify these costs. In this section, the investigation aims to clarify and quantify the total cost function for the parameters being studied.

Figure 3.2 is a graphical illustration of this.

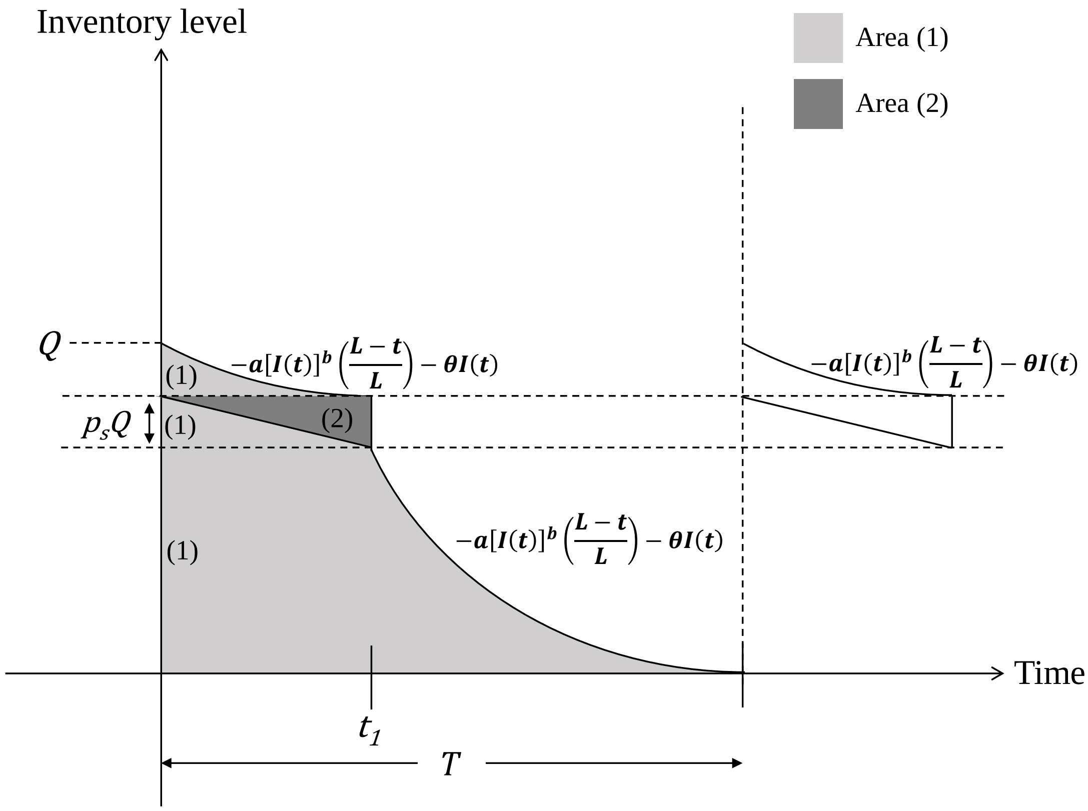


Figure 3.2: A graphical representation of the holding cost for an EOQ model for imperfect and deteriorating items with freshness and inventory level-dependent demand.

The total cost function that is being investigated in this study only focuses on the costs that are defined by the Equation (3.11).

$$\begin{aligned} \textit{Total cost function} = \; & \textit{Holding cost of the good product} \\ & + \textit{Holding cost of the imperfect product} \\ & + \textit{Deterioration cost} \\ & + \textit{Screening cost} \\ & + \textit{Purchasing cost of the product} \\ & + \textit{Ordering cost} \end{aligned} \tag{3.11}$$

Alternatively, the Equation (3.11) can be stated as Equation (3.12).

$$TCF = HC + HC_s + DC + SC + PC + OC \tag{3.12}$$

TCF is the total cost, HC is the holding cost of the good product, HC_s is holding cost of the imperfect product, DC is the deterioration cost, SC is the screening cost, PC

is the purchasing cost of the product and OC is the ordering cost.

Holding cost of the good product

The Area (1) in Figure 3.2 is multiplied by holding cost per unit of the good product for each unit of time, to determine the holding cost of the good product. Stated mathematically the area is reprsented by Equation (3.13).

$$HC = h\int_0^T I(t)dt + h\left(\frac{1}{2}\right)(t_1)(p_sQ) \tag{3.13}$$

Where Equation (3.9) is substituted into Equation (3.13). This is depicted in Equation (3.14).

$$\begin{aligned} HC = h\int_0^T \Bigg[\frac{a}{\theta L}\Bigg[t - L - \frac{1}{(1-b)\theta} \\ - e^{(1-b)\theta(T-t)}\left[T - L - \frac{1}{(1-b)\theta}\right]\Bigg]\Bigg]^{\dfrac{1}{1-b}} dt + h\left(\frac{1}{2}\right)(t_1)(p_sQ) \end{aligned} \tag{3.14}$$

To arrive at a practical solution of the computation that is required to solve the matter, it may be done by applying the Maclaurin expansion theorem. With this theorem it becomes feasible to replace the exponential function of $I(t)$ in Equation (3.9) and Q in Equation (3.10).

Maclaurin expansion of $I(t)$

$$\begin{aligned} e^{\theta(1-b)(T-t)} &= \sum_{i=1}^{\infty}\frac{\theta^i(1-b)^i(T-t)^i}{i!} \\ &= 1 + \frac{\theta(1-b)(T-t)}{1!} + \frac{\theta^2(1-b)^2(T-t)^2}{2!} \\ &\quad + \frac{\theta^3(1-b)^3(T-t)^3}{3!} + \frac{\theta^4(1-b)^4(T-t)^4}{4!} + \ldots \end{aligned} \tag{3.15}$$

In the situation where the values of $(T-t)$ and $(1-b)$ are low, for small values of θ, then the expansion shown in Equation (3.15) is approximated and represented by Equation (3.16).

$$e^{\theta(1-b)(T-t)} \approx 1 + \theta(1-b)(T-t) \tag{3.16}$$

Using Equation (3.16) and substituting this into $I(t)$ from Equation (3.9), Equation (3.17) is obtained.

$$\begin{aligned} I(t) = \Bigg[\frac{a}{\theta L}\Bigg[t - L - \frac{1}{(1-b)\theta} \\ - (1+\theta(1-b)(T-t))\left[T - L - \frac{1}{(1-b)\theta}\right]\Bigg]\Bigg]^{\dfrac{1}{1-b}} \end{aligned} \tag{3.17}$$

Simplification of Equation (3.17) provides a modified $I(t)$ as shown in Equation (3.18).

$$I(t) = \left[-\left(\frac{a}{L}\right)(1-b)(T-t)\,(T-L)\right]^{\dfrac{1}{1-b}} \tag{3.18}$$

Maclaurin expansion of Q

$$\begin{aligned} e^{\theta(1-b)(T)} &= \sum_{i=1}^{\infty} \frac{\theta^i(1-b)^i(T)^i}{i!} \\ &= 1 + \frac{\theta(1-b)(T)}{1!} + \frac{\theta^2(1-b)^2(T)^2}{2!} \\ &\qquad + \frac{\theta^3(1-b)^3(T)^3}{3!} + \frac{\theta^4(1-b)^4(T)^4}{4!} + \dots \end{aligned} \tag{3.19}$$

In the situation where the values of (T) and $(1-b)$ are low, for small values of θ, then the expansion shown in Equation (3.19) is approximated and represented by Equation (3.20).

$$e^{\theta(1-b)(T)} \approx 1 + \theta(1-b)(T) \tag{3.20}$$

Using Equation (3.20) and substituting this into Q from Equation (3.10), Equation (3.21) is obtained.

$$Q = \left[\frac{a}{\theta L}\left[-L - \frac{1}{(1-b)\theta} - (1+\theta(1-b)(T))\left[T - L - \frac{1}{(1-b)\theta}\right]\right]\right]^{\dfrac{1}{1-b}} \tag{3.21}$$

Simplification of Equation (3.21) provides a modified Q as shown in Equation (3.22).

$$Q = \left[-\left(\frac{a}{L}\right)(1-b)(T)\,(T-L)\right]^{\dfrac{1}{1-b}} \tag{3.22}$$

This information from Equation (3.18) and Equation (3.22) can be used to substitute into Equation (3.14) which is the holding cost of good products.

$$\begin{aligned} HC = h\int_0^T \left[-\left(\frac{a}{L}\right)(1-b)(T-t)\,(T-L)\right]^{\dfrac{1}{1-b}} dt \\ + h\left(\frac{1}{2}\right)(t_1)(p_s)\left[-\left(\frac{a}{L}\right)(1-b)(T)\,(T-L)\right]^{\dfrac{1}{1-b}} \end{aligned} \tag{3.23}$$

Using integration and by simplifying Equation (3.23) the holding cost of the good product is obtained, as shown in Equation (3.24).

$$HC = h\left[-\left(\frac{a}{L}\right)(1-b)(T-L)\right]^{\frac{1}{1-b}}\left[\frac{1-b}{2-b}(T)^{\frac{2-b}{1-b}}\right] + h\left(\frac{1}{2}\right)(t_1)(p_s)\left[-\left(\frac{a}{L}\right)(1-b)(T)(T-L)\right]^{\frac{1}{1-b}} \tag{3.24}$$

Holding cost of the imperfect product

The Area (2) in Figure 3.2 is similarly multiplied by holding cost per unit of the imperfect product for each unit of time, to determine the holding cost of the imperfect product. Stated mathematically the area is represented by Equation (3.25).

$$HC_s = h_s\left(\frac{1}{2}\right)(t_1)(p_sQ) \tag{3.25}$$

By substituting Q from Equation (3.22) into Equation (3.25), Equation (3.26) is obtained, which is the holding cost of the imperfect product.

$$HC_s = h_s\left(\frac{1}{2}\right)(t_1)(p_s)\left[-\left(\frac{a}{L}\right)(1-b)(T)(T-L)\right]^{\frac{1}{1-b}} \tag{3.26}$$

Deterioration cost

This is a cost that is attributed to the products that deteriorate. It is derived from the effort applied to identify and remove those deteriorated products from the remaining inventory that is still adequate for sale. The derivation of the cost is based on the level of the inventory. It is the cost of the deterioration multiplied by the rate of deterioration for the cycle as shown in Equation (3.27).

$$DC = C_d\int_0^T \theta I(t)dt \tag{3.27}$$

By substituting the approximation of $I(t)$, given in Equation (3.18), into Equation (3.27) the result becomes Equation (3.28).

$$DC = C_d\int_0^T \theta\left[-\left(\frac{a}{L}\right)(1-b)(T-t)(T-L)\right]^{\frac{1}{1-b}} dt \tag{3.28}$$

With mathematical integration and simplification the result from Equation (3.28) is shown in Equation (3.29).

$$DC = C_d\theta\left[-\left(\frac{a}{L}\right)(1-b)(T-L)\right]^{\frac{1}{1-b}}\left[\frac{1-b}{2-b}(T)^{\frac{2-b}{1-b}}\right] \tag{3.29}$$

Screening cost

A certain component of those items p_s, would be of imperfect quality. The screening procedure for the period t_1, is conducted to separate the items of good quality from those of imperfect quality. The costs of the screening process are C_s, to screen a unit of the product. The cost to screen all of the items in each cycle is presented by Equation (3.30).

$$SC = C_s Q \tag{3.30}$$

By substituting Q from Equation (3.22) into Equation (3.30), Equation (3.31) is obtained, which is the screening cost of the cycle.

$$SC = C_s \left[-\left(\frac{a}{L}\right)(1-b)(T)\,(T-L) \right]^{\dfrac{1}{1-b}} \tag{3.31}$$

Purchasing cost of the product

The cost of the delivered product per unit multiplied by the numerical quantity ordered is shown in Equation (3.32).

$$PC = C_g Q \tag{3.32}$$

By substituting Q from Equation (3.22) into Equation (3.32), Equation (3.33) is obtained, which is the purchasing cost of that product.

$$PC = C_g \left[-\left(\frac{a}{L}\right)(1-b)(T)\,(T-L) \right]^{\dfrac{1}{1-b}} \tag{3.33}$$

Ordering cost

Ordering costs are the expenses incurred to acquire the products that are ordered. This is a fixed cost per order and is shown in Equation (3.34).

$$OC = K \tag{3.34}$$

From *TCF* represented in Equation (3.12) and by substituting in the the holding cost of the good product (HC) from Equation (3.24), the holding cost of the imperfect product (HC_s) Equation (3.26), the deterioration cost (DC) from Equation (3.29), the screening cost (SC) from Equation (3.31), the purchasing cost of the products (PC) from Equation (3.32) and the ordering cost (OC) from Equation (3.34) the following cost function is obtained in Equation (3.35).

$$\begin{aligned} TCF = {} & h\left[-\left(\frac{a}{L}\right)(1-b)(T-L)\right]^{\dfrac{1}{1-b}}\left[\frac{1-b}{2-b}(T)^{\dfrac{2-b}{1-b}}\right] \\ & + h\left(\frac{1}{2}\right)(t_1)(p_s)\left[-\left(\frac{a}{L}\right)(1-b)(T)\,(T-L)\right]^{\dfrac{1}{1-b}} \\ & + h_s\left(\frac{1}{2}\right)(t_1)(p_s)\left[-\left(\frac{a}{L}\right)(1-b)(T)\,(T-L)\right]^{\dfrac{1}{1-b}} \\ & + C_d\theta\left[-\left(\frac{a}{L}\right)(1-b)(T-L)\right]^{\dfrac{1}{1-b}}\left[\frac{1-b}{2-b}(T)^{\dfrac{2-b}{1-b}}\right] \\ & + C_s\left[-\left(\frac{a}{L}\right)(1-b)(T)\,(T-L)\right]^{\dfrac{1}{1-b}} \\ & + C_g\left[-\left(\frac{a}{L}\right)(1-b)(T)\,(T-L)\right]^{\dfrac{1}{1-b}} \\ & + K \end{aligned} \tag{3.35}$$

Simplification of Equation (3.35) to get to the total cost function in Equation (3.36).

$$\begin{aligned} TCF = {} & (h+C_d\theta)\left[-\left(\frac{a}{L}\right)(1-b)(T-L)\right]^{\dfrac{1}{1-b}}\left[\frac{1-b}{2-b}(T)^{\dfrac{2-b}{1-b}}\right] \\ & + \left[(p_s)(h+h_s)\left(\frac{1}{2}\right)(t_1)+C_s+C_g\right]\left[-\left(\frac{a}{L}\right)(1-b)(T)\,(T-L)\right]^{\dfrac{1}{1-b}} \\ & + K \end{aligned} \tag{3.36}$$

3.4.4 Total Revenue Function (*TRF*)

The total revenue comprises the revenue from the good product and the revenue from the imperfect/deteriorated product.

$$\begin{aligned} \textit{Total revenue function} = {} & \textit{Revenue of good product} \\ & + \textit{Revenue of the imperfect/deteriorated product} \end{aligned} \tag{3.37}$$

Alternatively, the Equation (3.37) can be stated as Equation (3.38).

$$TRF = TRG + TRD \tag{3.38}$$

TRF is the total revenue, TRG is the total revenue from the good product and TRD is the revenue from the imperfect/deteriorated product.

The selling price of the good product is multiplied by the number of good products that are sold. The quantity of the good product is made up of the quantity delivered, less the imperfect/deteriorated products that had been removed from the system. This is the revenue of the good product. The goods that have been removed is made up of imperfect/deteriorated product. Those goods are then multiplied by the discounted selling price of the deteriorated/imperfect. This represents the revenue of the imperfect/deteriorated product. These two revenue streams make up the total revenue function in Equation (3.39).

$$TRF = S_g\left[Q - \theta\int_0^T I(t)dt - (p_sQ)\right] + S_d\left[\theta\int_0^T I(t)dt - (p_sQ)\right] \tag{3.39}$$

By substituting $I(t)$ from Equation (3.18) and Q from Equation (3.22) the Equation (3.40) is obtained.

$$\begin{aligned} TRF = S_g\Bigg[&\left[-\left(\frac{a}{L}\right)(1-b)(T)\left(T-L\right)\right]^{\frac{1}{1-b}} \\ &- \theta\int_0^T\left[-\left(\frac{a}{L}\right)(1-b)(T-t)\left(T-L\right)\right]^{\frac{1}{1-b}}dt \\ &- p_s\left[-\left(\frac{a}{L}\right)(1-b)(T)\left(T-L\right)\right]^{\frac{1}{1-b}}\Bigg] \\ + S_d\Bigg[&\theta\int_0^T\left[-\left(\frac{a}{L}\right)(1-b)(T-t)\left(T-L\right)\right]^{\frac{1}{1-b}}dt \\ &- p_s\left[-\left(\frac{a}{L}\right)(1-b)(T)\left(T-L\right)\right]^{\frac{1}{1-b}}\Bigg] \end{aligned} \tag{3.40}$$

Integration and simplification of Equation (3.40) results in Equation (3.41).

$$\begin{aligned} TRF =& (S_g + S_d)\left[-\left(\frac{a}{L}\right)(1-b)(T-L)\right]^{\frac{1}{1-b}}\left[\frac{1-b}{2-b}(T)^{\frac{2-b}{1-b}}\right] \\ &+ (S_g - p_s(S_g - S_d))\left[-\left(\frac{a}{L}\right)(1-b)(T)\left(T-L\right)\right]^{\frac{1}{1-b}} \end{aligned} \tag{3.41}$$

The Equation (3.41) is the final TRF.

3.4.5 Total Profit per Unit of Time (TPU)

Profit is defined as the revenue less the costs. This is stated in Equation (3.42).

$$\textit{Total profit function} = \textit{Total revenue function} - \textit{Total cost function} \tag{3.42}$$

Alternatively, the Equation (3.42) can be stated as Equation (3.43).

$$TPF = TRF - TCF \tag{3.43}$$

Total Profit Function (TPF) is the total profit, TRF is the total revenue and TCF is the total cost.

By substituting TRF from Equation (3.41) and TCF from Equation (3.36) into Equation (3.43) to give the TPF in Equation (3.44).

$$\begin{aligned} TPF &= (S_g + S_d)\left[-\left(\frac{a}{L}\right)(1-b)(T-L)\right]^{\frac{1}{1-b}}\left[\frac{1-b}{2-b}(T)^{\frac{2-b}{1-b}}\right] \\ &+ (S_g - p_s(S_g - S_d))\left[-\left(\frac{a}{L}\right)(1-b)(T)(T-L)\right]^{\frac{1}{1-b}} \\ &- \Bigg[(h + C_d\theta)\left[-\left(\frac{a}{L}\right)(1-b)(T-L)\right]^{\frac{1}{1-b}}\left[\frac{1-b}{2-b}(T)^{\frac{2-b}{1-b}}\right] \\ &+ \left[(p_s)(h+h_s)\left(\frac{1}{2}\right)(t_1) + C_s + C_g\right]\left[-\left(\frac{a}{L}\right)(1-b)(T)(T-L)\right]^{\frac{1}{1-b}} \\ &+ K\Bigg] \end{aligned} \tag{3.44}$$

With simplification Equation (3.44) becomes Equation (3.45).

$$\begin{aligned} TPF &= \left[-h + \theta\left(-S_g + S_d - C_d\right)\right] \\ &\quad \left[-\left(\frac{a}{L}\right)(1-b)(T-L)\right]^{\frac{1}{1-b}}\left[\frac{1-b}{2-b}(T)^{\frac{2-b}{1-b}}\right] \\ &\quad + \left[S_g - C_s - C_g + p_s\left[-S_g + S_d - (h+h_s)\left(\frac{1}{2}\right)(t_1)\right]\right] \\ &\quad \left[-\left(\frac{a}{L}\right)(1-b)(T)(T-L)\right]^{\frac{1}{1-b}} \\ &\quad - K \end{aligned} \tag{3.45}$$

TPF in Equation (3.45) is divided by time T, to arrive at the TPU in Equation (3.46).

$$
\begin{aligned}
TPU &= [-h + \theta(-S_g + S_d - C_d)] \\
&\qquad [T]^{-1}\left[-\left(\frac{a}{L}\right)(1-b)(T-L)\right]^{\dfrac{1}{1-b}}\left[\frac{1-b}{2-b}(T)^{\dfrac{2-b}{1-b}}\right] \\
&\quad + \left[S_g - C_s - C_g + p_s\left[-S_g + S_d - (h+h_s)\left(\frac{1}{2}\right)(t_1)\right]\right] \\
&\qquad [T]^{-1}\left[-\left(\frac{a}{L}\right)(1-b)(T)\,(T-L)\right]^{\dfrac{1}{1-b}} \\
&\quad - K\,[T]^{-1}
\end{aligned}
\tag{3.46}
$$

Through the simplification of Equation (3.46) it becomes Equation (3.47).

$$
\begin{aligned}
TPU &= [-h + \theta(-S_g + S_d - C_d)] \\
&\qquad \left[\frac{1-b}{2-b}\right]\left[-\left(\frac{a}{L}\right)(1-b)(T)(T-L)\right]^{\dfrac{1}{1-b}} \\
&\quad + \left[S_g - C_s - C_g + p_s\left[-S_g + S_d - (h+h_s)\left(\frac{1}{2}\right)(t_1)\right]\right] \\
&\qquad [T]^{\dfrac{b}{1-b}}\left[-\left(\frac{a}{L}\right)(1-b)\,(T-L)\right]^{\dfrac{1}{1-b}} \\
&\quad - \left[\frac{K}{T}\right]
\end{aligned}
\tag{3.47}
$$

In this format the Equation (3.47) can be set-up for testing with numerical data.

3.4.6 Numerical example

The initial priority is to ascertain the optimal cycle time namely, T so that the maximum total profit per unit of time, TPU can be calculated. To illustrate that a unique solution to this exercise exists, it first needs to be proven that the objective function would be concave which would seek to maximise the total profit. To test that the profit is maximised a second derivative of Equation (3.47) would need to be computed to ensure that it is negative. A negative value for the second derivative of Equation (3.47) ensures that it is concave and hence a maximum value can be obtained. Using Equation (3.47) the first mathematical derivative with respect to the T is calculated and shown in Equation (3.48).

$$\begin{aligned}\frac{d(TPU)}{dT} &= \left[-h+\theta\left(-S_g+S_d-C_d\right)\right]\left[\frac{1-b}{2-b}\right] \\ &\qquad \left[-\left(\frac{a}{L}\right)(1-b)(T)(T-L)\right]^{\frac{b}{1-b}}\left[-\left(\frac{a}{L}\right)(2T-L)\right] \\ &+\left[S_g-C_s-C_g+p_s\left[-S_g+S_d-(h+h_s)\left(\frac{1}{2}\right)(t_1)\right]\right] \\ &\qquad \left[\left[\frac{b}{1-b}T^{\frac{2b-1}{1-b}}\right]\left[-\left(\frac{a}{L}\right)(1-b)\left(T-L\right)\right]^{\frac{1}{1-b}}\right. \\ &\qquad\quad \left.+\left[-\frac{a}{L}\right]\left[-\left(\frac{a}{L}\right)(1-b)(T)\left(T-L\right)\right]^{\frac{b}{1-b}}\right] \\ &+\left[\frac{K}{T^2}\right]\end{aligned} \tag{3.48}$$

Thereafter, with the outcome of Equation (3.48) the second mathematical derivative with respect to T is represented in Equation (3.49).

$$
\begin{aligned}
\frac{d^2(TPU)}{dT^2} = {} & [-h+\theta\left(-S_g+S_d-C_d\right)]\left[\frac{1-b}{2-b}\right] \\
& \Bigg[\left[-\left(\frac{a}{L}\right)(1-b)(T)(T-L)\right]^{\dfrac{b}{1-b}}\left[-\frac{2a}{L}\right] \\
& \quad +\left[-\left(\frac{a}{L}\right)(1-b)(T)(T-L)\right]^{\dfrac{2b-1}{1-b}}\left[\left(\frac{a}{L}\right)^2(b)(2T-L)^2\right]\Bigg] \\
& +\left[S_g-C_s-C_g+p_s\left[-S_g+S_d-(h+h_s)\left(\frac{1}{2}\right)(t_1)\right]\right] \\
& \Bigg[\left[\frac{b}{1-b}T^{\dfrac{2b-1}{1-b}}\right]\left[-\frac{a}{L}\right]\left[-\left(\frac{a}{L}\right)(1-b)\left(T-L\right)\right]^{\dfrac{b}{1-b}} \\
& \quad +\left[\frac{(b)(2b-1)}{(1-b)^2}T^{\dfrac{3b-2}{1-b}}\right]\left[-\left(\frac{a}{L}\right)(1-b)\left(T-L\right)\right]^{\dfrac{1}{1-b}} \\
& \quad +\left[\left(\frac{a}{L}\right)^2(b)(2T-L)\right]\left[-\left(\frac{a}{L}\right)(1-b)(T)\left(T-L\right)\right]^{\dfrac{2b-1}{1-b}}\Bigg] \\
& -\left[\frac{2K}{T^3}\right]
\end{aligned}
\tag{3.49}
$$

The complex nature of Equation (3.49) means that it is cumbersome to assimilate and therefore it is very difficult from a practical perspective to prove concavity. It means that an alternate method to do this exercise will be required.

To address this, an iterative process has been applied to determine concavity of the TPU in Equation (3.47). In short, a series of steps are followed. The initial task is to set T=1. Thereafter, the value of T is substituted into Equation (3.47) to compute TPU. By making incremental increases to T, different values of TPU can be ascertained. The expectation is that the TPU ought to increase until a turning point (or a maximum value) is derived. After which the TPU starts to decline.

Figure 3.3 is a graphical representation of this concavity. For this exercise Figure 3.3 is able to provide a maximum value and hence it indicates that the Figure 3.3 is concave in nature and hence a relevant and applicable solution methodology.

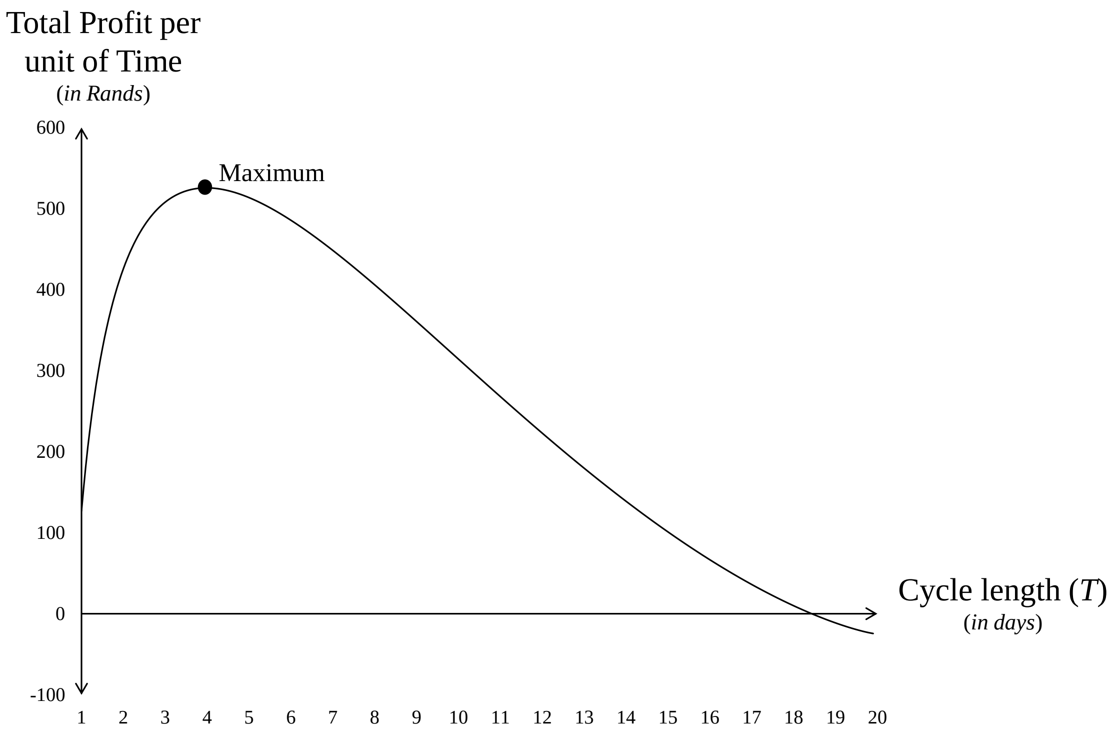


Figure 3.3: Concavity TPU maximisation illustration.

3.5 Sensitivity analysis

A sensitivity analysis can also be applied to verify the optimum cycle length T^* from the *TPU* of Equation (3.47). This has been computed using the Solver function that is available from Microsoft Excel. In the Microsoft Excel solution, the following parameters from Table 3.2 were used as inputs for that computation. These inputs are representation of a product that had deterioration, imperfect quality and inventory and freshness level dependent demand. An example of a product is a banana and used as the determination of the parameter inputs in Table 3.2.

Table 3.2: Sensitivity analysis numerical inputs.

Symbol	**Value**
a	50 kg/day
L	20 Days
b	0.2
h	0.1 Rand/kg/day
h_s	0.05 Rand/kg/day
θ	0.05
S_g	12.00 Rand/kg
S_d	5.00 Rand/kg
C_d	0.01 Rand/kg/day
p_s	0.15
t_1	3 Days
K	500 Rand
C_s	0.05 Rand/kg
C_g	4.00 Rand/kg

The Microsoft Solver enabled the parameters in Table 3.2 to be used. After substitution of those parameters into Equation (3.47), the optimal T^* value is computed. The optimal cycle length, T^* becomes 3.98 days with this method of calculation. Thereafter, substituting the input parameters of Table 3.2 and T^* into Equation (3.22), the optimal quantity, Q^* becomes 428.75 kgs. Using the input parameters from Table 3.2 and by substituting T^* into Equation (3.47) the *TPU* is arrived at, which is Rand 527.85.

The basic methodology of the construction and formulation of the numerical Table 3.3 was derived as follows. Numerical values for each of the variables listed was estimated from Table 3.2. Thereafter, the value of each of those variables from Table 3.2 was increased and/or decreased by a certain percentage as shown in the Table 3.3. The resultant application of those changes are reflected in changes in cycle length T (here expressed as number of days), the quantity Q (here expressed as kilograms) and *TPU* (here expressed as Rands). From Table 3.3 certain trends emerge which are further commented on below.

An analytical overview of Table 3.3 would indicate that certain variables have a greater impact on the overall *TPU* situation than other variables. The salient features that emerge from testing of the model with numeric data are available in Table 3.3. The written comment herewith highlights specific findings and attributes from the Table 3.3 that need to be further elaborated upon. The objective here is to attribute greater focus to understanding those variables that have the greatest overall impact. The change in parameter is calculated by using the base parameter and increasing and decreasing the value by increments of 25 percent.

When the scale parameter of the demand rate a, decreases, the cycle length T increases. This is because fewer products will be required so the goods do not have to be delivered as frequently. Similarly, if a, increases, the cycle length T decreases. However, when the scale parameter of the demand rate a is decreased, there is a corresponding decrease, so that the quantity ordered Q would also decrease. Similarly, for increases in the demand rate a, it is found that Q also increases. If the quantity Q increases, the ordering cost K per item, will actually decrease because more items per shipment will have been delivered and this has a positive impact on the TPU. For a constant value of K the price per unit (or the Rand price per kilogram) of stock ordered has actually declined. However, this also has an adverse impact on the holding cost of both the good h and the imperfect/deteriorated items h_s. It would thus have a negative impact on the TPU. The reverse applies if the quantity ordered for each shipment of the goods reduces. The scale parameter a, is used as a parameter to determine the scale or size of the demand. It implies that a large value for a will have the effect of appropriately increasing demand. With more goods to sell the TPU would increase. The reverse is also true where the value of a declines.

When the expiry date of that inventory L is reduced, the optimal solution of the model would manifest itself through a reduction in the cycle length T. A reduction in the expiration date L implies that the quantity Q is reduced. This leads to a reduced holding costs, because the inventory spends less time as stock on hand. Alternatively, the fixed costs of placing an order K will increase because the frequency of the number of orders has also increased. Since the rate of demand is a function of the freshness index and the stock level of that inventory, reduced expiration dates L will have a negative effect on the rate of demand. When the expiration date L increases, the adverse effects of smaller order sizes Q on the demand, combined with the fixed increased cost K and the reduced holding costs (h and h_s where applicable) has the overall effect of reducing the TPU. The TPU increases when the expiration date L increases, even though the order size K and holding costs (h and h_s) moves in the opposite direction. Practically, management may be able to apply this outcome to increase the TPU by extending the cycle length T of that inventory. Freshness needs to be protected and this can possibly be attained by investing in technology, such as refrigeration and other preservation methods.

When the shape parameter b decreases, which incorporates the elasticity of demand, the optimal solution responds by decreasing the cycle length T, but by a small amount. With a big decrease in the quantity Q ordered, the holding costs (h and h_s) are decreased and this increases the per unit ordering cost K. As the shape parameter b is decreased, the impact would result in a lower demand rate and hence the TPU is decreased. The reverse is also applicable when b is increased. Essentially b has the most significant impact on the TPU and the quantity Q.

There is another tier of variables that have an impact on specific aspects of the TPU calculation. These variables have an impact in terms of how they influence the overall process within the comprehensive TPU scenario.

The impact of the holding cost of the good items h and the holding cost of the imperfect product h_s were of relatively little significance in terms of their impact in the numerical analysis.

The unit rate of deterioration for each item θ, provides a quantitative method to assess deterioration. If the rate of deterioration is low, there will be additional quantity Q available for a longer cycle length T. This has a positive impact by increasing the TPU, even if the holding costs (h and h_s) increase in tandem, because there is more quantity Q. This will also have an associated ordering cost K per unit reduction. Similarly, the reverse applies for increased deterioration rates θ.

With no other change, the greater the selling price S_g of a good product, the greater the influence will be on the TPU. The reverse applies when prices are reduced. However, this might impact on demand because the price may no longer be competitive in the marketplace. It also applies to the selling price S_d of the imperfect/deteriorated items. Attempts should always be made to maximise these prices because that will have a direct influence on the TPU.

The cost of deterioration C_d did not have much of an impact in the numerical analysis.

The percentage rate p_s of imperfect items in Q also has an impact on the TPU. The amount of imperfect product impacts on the profitability because with a higher p_s more products become available for sale at the deteriorated/imperfect price S_d, even if this price is less than the price that the good products are sold for. Low levels of deteriorated/imperfect goods means that there will be more good products to be sold at better prices S_g, even if there is a small cost associated with the storage h of the additional volume of the good product.

The time it takes to screen the products t_1 did not have much of an impact in the numerical analysis.

The ordering cost K has a direct influence on the TPU because it is a fixed cost per order that is placed. If the ordering cost K increases, the unit cost attributable to K will increase. If the ordering cost K decreases the unit cost attributable to K will decrease. Ordering cost K have a marginal impact on cycle length L and quantity ordered Q.

The cost associated to the screening of the products C_s did not have much of an impact in the numerical analysis.

The C_g is the cost price per unit of the goods. The lower C_g, the bigger the TPU purchase scenario outcome would be. As a purchaser goods are purchased from the same supplier at the lowest possible price. Buyers must strive for the lowest possible price to optimise the TPU. Buyers may elect to find alternate suppliers if the C_g can be sourced elsewhere.

Table 3.3: Numerical analysis.

	Parameter		**Cycle length (T)**		**Quantity (Q)**		***TPU***	
Symbol	**Value**	**% change**	**Days**	**% change**	***kgs***	**% change**	**Rands**	**% change**
Base			**3.98**		**428.75**		**527.85**	
	25	-50	5.24	+31.66	229.47	-46.48	159.20	-69.84
	37.5	-25	4.43	+11.16	329.77	-23.09	332.51	-37.01
a	50	0	3.98	0	428.75	0	527.85	0
	62.5	+25	3.70	-7.12	528.18	+23.19	739.54	+40.10
	75	+50	3.50	-12.08	628.78	+46.65	964.38	+82.70
	10	-50	2.81	-29.47	242.23	-43.50	365.63	-30.73
	15	-25	3.46	-13.14	341.95	-20.24	465.75	-11.76
L	20	0	3.98	0	428.75	0	527.85	0
	25	+25	4.41	+10.81	504.69	+17.71	570.96	+8.17
	30	+50	4.77	+19.82	571.32	+33.25	602.92	+14.22
	0.10	-50	4.13	+3.70	256.84	-40.10	251.14	-52.42
	0.15	-25	4.01	+0.60	324.07	-24.42	363.54	-31.13
b	0.20	0	3.98	0	428.75	0	527.85	0
	0.25	25	4.07	+2.20	602.20	+40.45	776.83	+47.17
	0.30	50	4.28	+7.40	908.14	+111.81	1 172.58	+122.14
	0.050	-50	4.06	+1.99	436.74	+1.86	538.67	+2.05
	0.075	-25	4.02	+0.99	432.73	+0.93	533.24	+1.02
h	0.100	0	3.98	0	428.75	0	527.85	0
	0.125	+25	3.94	-0.97	424.82	-0.92	522.5	-1.01
	0.150	+50	3.91	-1.93	420.93	-1.82	517.2	-2.02
	0.0250	-50	3.97	-0.01	428.72	-0.01	528.45	+0.11
	0.0375	-25	3.98	0	428.74	0	528.15	+0.06
h_s	0.0500	0	3.98	0	428.75	0	527.85	0
	0.0625	+25	3.98	0	428.77	0	527.55	-0.06
	0.0750	+50	3.99	+0.01	428.78	+0.01	527.24	-0.11

(continued on the next page)

Table 3.3: Numerical analysis *(continued)*

	Parameter		Cycle length (T)		Quantity (Q)		*TPU*	
Symbol	**Value**	**% change**	**Days**	**% change**	***kgs***	**% change**	**Rands**	**% change**
Base			**3.98**		**428.75**		**527.85**	
	0.0250	-50	4.27	+7.33	457.76	+6.77	562.36	+6.54
	0.0375	-25	4.12	+3.57	443.00	+3.32	544.82	+3.22
θ	0.0500	0	3.98	0	428.75	0	527.85	0
	0.0625	+25	3.85	-3.37	415.08	-3.19	511.42	-3.11
	0.0750	+50	3.72	-6.54	402.00	-6.24	495.51	-6.13
	6	-50	6.11	+53.46	612.74	+42.91	54.29	-89.71
	9	-25	4.53	+13.80	482.41	+12.51	284.71	-46.06
S_g	12	0	3.98	0	428.75	0	527.85	0
	15	+25	3.68	-7.47	398.14	-7.14	775.49	+46.91
	18	+50	3.49	-12.27	377.98	-11.84	1025.39	+94.26
	2.50	-50	3.81	-4.37	410.95	-4.15	464.1	-12.08
	3.75	-25	3.89	-2.20	419.86	-2.07	495.86	-6.06
S_d	5.00	0	3.98	0	428.75	0	527.85	0
	6.25	+25	4.07	+2.21	437.59	+2.06	560.05	+6.10
	7.50	+50	4.16	+4.42	446.36	+4.11	592.47	+12.24
	0.0050	-50	3.99	+0.01	428.79	+0.01	527.90	+0.01
	0.0075	-25	3.98	0	428.77	0	527.87	0
C_d	0.0100	0	3.98	0	428.75	0	527.85	0
	0.0125	+25	3.98	0	428.73	0	527.82	0
	0.0150	+50	3.97	-0.01	428.71	-0.01	527.8	-0.01
	0.0750	-50	3.95	-0.75	425.74	-0.70	586.20	+11.06
	0.1125	-25	3.97	-0.38	427.21	-0.36	557.02	+5.53
p_s	0.1500	0	3.98	0	428.75	0	527.85	0
	0.1875	+25	4.00	+0.40	430.37	+0.38	498.68	-5.53
	0.2250	+50	4.02	+0.83	432.08	+0.78	469.52	-11.05

(continued on the next page)

Table 3.3: Numerical analysis *(continued)*

	Parameter		Cycle length (T)		Quantity (Q)		***TPU***	
Symbol	**Value**	**% change**	**Days**	**% change**	***kgs***	**% change**	**Rands**	**% change**
Base			**3.98**		**428.75**		**527.85**	
	1.50	-50	3.98	-0.02	428.65	-0.02	529.66	+0.34
	2.25	-25	3.98	-0.01	428.70	-0.01	528.76	+0.17
t_1	3.00	0	3.98	0	428.75	0	527.85	0
	3.75	+25	3.98	+0.01	428.80	+0.01	526.94	-0.17
	4.50	+50	3.98	+0.02	428.85	+0.02	526.03	-0.34
	250	-50	3.36	-15.63	363.61	-15.19	595.87	+12.89
	375	-25	3.69	-7.32	398.77	-6.99	560.42	+6.17
K	500	0	3.98	0	428.75	0	527.85	0
	625	+25	4.25	+6.64	455.05	+6.13	497.47	-5.75
	750	+50	4.49	+12.77	478.56	+11.62	468.86	-11.17
	0.0250	-50	3.96	-0.04	428.61	-0.03	530.54	+0.51
	0.0375	-25	3.97	-0.02	428.68	-0.02	529.19	+0.25
C_s	0.0500	0	3.98	0	428.75	0	527.85	0
	0.0625	+25	3.99	+0.02	428.82	+0.02	526.50	-0.25
	0.0750	+50	4.00	+0.04	428.90	+0.03	525.16	-0.51
	2	-50	3.89	-2.43	418.90	-2.30	743.34	+40.82
	3	-25	3.93	-1.32	423.41	-1.24	635.56	+20.41
C_g	4	0	3.98	0	428.75	0	527.85	0
	5	+25	4.05	+1.60	435.18	+1.50	420.24	-20.39
	6	+50	4.13	+3.59	443.10	+3.35	312.75	-40.75

Chapter 4

Conclusion and future work

The study has been completed by following a sequential format which has itemised the issues, set certain study objectives, evaluated the available research in respect of these matters and finally, developed a mathematical model and numeric scenario to address the concerns that had been highlighted.

A research gap in the literature has been identified and it has resulted in the creation of a scenario whereby it has been possible to investigate imperfect quality, deterioration and freshness in a quantitative manner. The mathematical outcome has been tested with numeric data to ascertain its applicability to this situation. Sensitivity studies were conducted to measure the effects of changes to the numeric data on the model that was derived. Ultimately, it has led to a situation whereby more informed real-life situations can be replicated. The benchmark to evaluate this outcome in each instance, has always been to maximise the total profit for each cycle in the logistical space. It is now clear that the nature of the product will have an increasingly important influence so as to be able to satisfy the demands of the end user. From the sensitivity analysis the most important factor affecting the Total Profit per Unit of Time (TPU) of the model was the selling price of the perfect product. Increasing the selling price of the good product however may have adverse affects on demand as the price would also need to be competitive in the market especially for a product that is readily available to consumers. A product that is more rare or specialised there may be an opportunity to increase the price of the perfect product but in the scenario used in this thesis it is unlikely that increasing the selling price of the good product will result in increased profits as demand would likely decrease. The elasticity of demand b has the most significant impact on the TPU and the quantity Q. The price and demand are important factors that affect the output of the model.

As more recent trends and studies indicate, logistical solutions do not apply uniformly across an economic environment. It means that customers are increasingly demanding business solutions that are unique to their own specific operating environments. The concepts of imperfect quality, deterioration and freshness and inventory level-dependent demand clearly illustrate this phenomenon in the marketplace.

It is also true that there is no single solution to the complex realities of the modern supply chain operating environment. While such Economic Order Quantity (EOQ) models offer realistic attempts at addressing the issues, there is always need for updates or enhancements to the processes as technology evolves. This model can be enhanced by further optimising the profit scenarios for both the retailer and the supplier through mutual co-operation between both parties seeking the same set of objectives. If profit maximisation remains the ultimate objective, other efficiencies need to be investigated to see if they can further enhance the functioning of such a supply chain, to the benefit of everyone involved therein. For example, savings may be achievable through bulking-up of orders to reduce the costs of transportation. Such discounts can even be shared through an arrangement between the supplier and the end-user. Furthermore, there is scope to do additional studies of items that are defective and need to be returned back to the supplier. The whole aspect of return logistics needs further investigation, especially if issues such harmful carbon emissions are factored into the overall costing thereof. For example, the function of returning goods to a supplier needs to happen in such a manner that the overall profitability of the entire process is not adversely impacted upon. It is also possible to investigate a range of goods that have more than one category of deterioration, or possible expiry dates. Additionally, the concept of goods being in short supply, can be a realistic impediment to the proper functioning of any supply chain. There is scope to do further study to ensure continuity of supply (with minimum disruptions), to enhance the efficiency of that overall process.

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