UNIVERSITEIT VAN PRETORIA
UNIVERSITY OF PRETORIA
YUNIBESITHI YA PRETORIA

University of Pretoria
Department of Mathematics and Applied Mathematics
Faculty of Natural and Agricultural Science

# A guide to the Rado graph:

exploring structural and logical properties of the Rado graph

by

Michelle Michau

February 14, 2023

Submitted in partial fulfilment of the requirements for the degree of Magister Scientiae in Mathematics
prepared under the supervision of Dr. Ruaan Kellerman

I, Michelle Michau declare that the following thesis/dissertation, which I hereby submit for the degree MSc Mathematics at the University of Pretoria, is my own work and has not previously been submitted by me for a degree at this or any other tertiary institution.

SIGNATURE:

DATE:

# Contents

# List of Symbols

| **Symbol** | **Description** | **Page where defined** |
|---|---|---|
| $\exists$ | The class of all $\exists$-formulas | 27 |
| $\forall$ | The class of all $\forall$-formulas | 27 |
| $\boldsymbol{R}$ | The Rado graph | 3 |
| $E(G)$ | The set of edges of the graph $G$ | 1 |
| $H \circ G$ | The lexicographic product of the graph $H$ by the graph $G$ | 18 |
| $H \diamond G$ | The weak lexicographical product of the graph $H$ by the graph $G$ | 19 |
| $K_m$ | Complete graph on $m$ vertices | 17 |
| $P(\varphi)$ | The (labeled asymptotic) probability of $\varphi$ | 55 |
| $V(G)$ | The set of vertices of the graph $G$ | 1 |
| $x \sim y$ | $\{x, y\} \in E(G)$, there is an edge connecting $x$ and $y$. | 4 |
| **at** | The class of all atomic formulas | 71 |
| **qf** | The class of all quantifier-free formulas | 27 |
| AP | The amalgamation property | 51 |
| $\mathrm{Aut}(\boldsymbol{R})$ | The automorphism group of $\boldsymbol{R}$ | 23 |
| $\mathrm{DLO}_{--}$ | The theory of all dense linear orderings without endpoints | 77 |
| EP | The extension property | 3 |
| HP | The hereditary property | 51 |
| JEP | The joint embedding property | 51 |
| $\mathrm{Th}\boldsymbol{R}$ | The first-order theory of $\boldsymbol{R}$ | 24 |

# Acknowledgements

First and foremost I would like to express my special thanks to Dr. Kellerman, my supervisor, for his guidance, support, encouragement and always useful advice during the course of my studies and in the planning and writing of this thesis. Thank you for staring at the problems with me. And not just staring but also nudging me in the right direction. I appreciate the countless hours you spent working on this project with me and the gentle way in which you pointed out where corrections were needed. You have truly helped me to deliver my best work, and continue to inspire me.

It is a pleasure to extend my thanks also to my family and friends. Words do not describe how much your motivation and support means to me. You helped me stand strong through personal trials, you took care of me when I buried myself in work, and you motivated me when I felt disheartened. Moemmie, thank you for the cheese and coffee. OD, thank you for tech-support and hugs. Boetie, thank you for listening to all my problems, academic or otherwise. You will always be my "beste boetie". Paps, thank you for your continued belief that I can do nothing wrong. To all the grandparents, my thanks will never be enough for your direct and indirect support through lunches, card games and road trips. To all my friends, who I will not attempt to list because there are too many of you, thank you for your prayers, for making me laugh, regularly checking in on me, and reminding me that *I can do it*! Thank you for listening to my sometimes incomprehensible mathematical problems and even attempting to help me figure them out. You are wonderful.

Finally, I praise God for providing me with both the skill and opportunity to complete this thesis. You have helped me finish this race. Thank You.

# Abstract

The Rado graph, denoted $\boldsymbol{R}$, is the unique (up to isomorphism) countably infinite random graph. It satisfies the extension property, that is, for two finite sets $U$ and $V$ of vertices of $\boldsymbol{R}$ there is a vertex outside of both $U$ and $V$ connected to every vertex in $U$ and none in $V$. This property of the Rado graph allows us to prove quite a number of interesting results, such as a 0-1-law for graphs. Amongst other things, the Rado graph is partition regular, non-fractal, ultrahomogeneous, saturated, resplendent, the Fraíssé-limit of the class of finite graphs, a non-standard model of the first-order theory of finite graphs, and has a complete decidable theory.

We classify the definable subgraphs of the Rado graph and prove results for finite graphs that satisfy a restricted version of the extension property. We also mention some parallels between the rationals viewed as a linear order and the Rado graph.

# Introduction

Imagine you have a piece of play dough. It can be any colour. It can be any size. And of course it can be any shape. It's soft enough so you can effortlessly mould it, but hard enough to keep shape. It's just the right amount of sticky, in that it does not stick to your hands while you mould it, but holds firm when you add two pieces together. As you roll a piece of dough between your fingers you realise that the possibilities are endless. You marvel at the fact that such a simple thing can be almost anything. Your heart starts racing at the dough's invitation to play and create and relive a moment of your youth.

You sigh, because you realise that the dough is only imaginary and quite soon this feeling of nostalgia will be ripped away by some serious mathematical discussion. I wish to assure you, however, that the Rado graph is quite like this play dough and it is indeed something to get excited about. You see, the Rado graph is the treasure chest in which every mathematician can find a jewel which is to their liking. The valuables range from graph theory to number theory to algebra to Ramsey theory to logic and the list goes on. It is easy now, to feel a rush, a calling to dive in and see whether there is something for you.

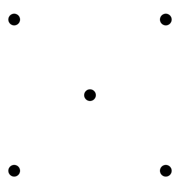
While we are eager to embark on this journey of discovery, we must first make the necessary preparations. Surely, we do not wish to go on a seven day hike without the right equipment. So let us start packing our bags.

Firstly, we cannot study something called the Rado graph if we are not clear on what a graph is. We will take a *graph* to be a pair $G = (V, E)$ consisting of *vertices* ($V$ or $V(G)$), also called *nodes*, and *edges* ($E$ or $E(G)$), where the edges are two-element subsets of $V$. This is often referred to as a *simple* graph in the literature, meaning that there are no loops (an edge connecting a vertex to itself) or multiple edges (sometimes called parallel edges) between the same vertices. The easiest way to picture a graph is to use dots for vertices and line segments, possibly curved, joining dots for edges.

We often identify a graph with its vertices, so instead of writing $x \in V(G)$ we just write $x \in G$. Consequently we identify the *order* of a graph, with the number of vertices in the graph, i.e. $|G| = |V(G)|$. We will also say that a graph is *on* $V$ rather than saying that the graph has vertex set $V$.

Next we need the notion of a random graph. We skip the detail of precise definition for now and play a game instead. Quickly get a coin or a die or a stack of ordinary playing cards.

Have it? Great! Below we have drawn some vertices for a graph. Pick your two favourite vertices. Now flip the coin or roll the die or draw a card. If you got heads, or an even number, or a red card, connect the two vertices. Pick another pair of vertices and repeat. You should flip or roll or draw for each pair of vertices.



There! You have constructed a random graph. For those who want to be a bit more specific, for a given set of vertices $V$, we independently join distinct vertices based on the outcome of some random experiment. We call the resulting graph a *random graph*.

Now, the *Rado graph* is a random graph on a countably infinite number of vertices. As it turns out, it is *the* countably infinite random graph and so we can justify naming it.

Let's pause here for a moment, and think again about how far reaching this flip-a-coin-graph is in the world which we call Mathematics. Can you see now the similarity between the play dough and the Rado graph? It is completely simple, but not at all ordinary.

"Are we ready? Can we start this adventure?" No, we need to familiarise ourselves with the terrain. The beast which we are about to hunt is accustomed to all sorts of different landscapes. We will start by gathering reports of the beast: what previous hunters saw, how they approached it and what tools they used on their expedition. These reports will then be compared and compiled into a bestiary. Taking our completed bestiary we will explore The Grassland of Graph Theory which the beast calls home. Here we will observe more closely under what conditions the beast functions. How does it behave? Does it undergo seasonal changes? How does it interact with other wildlife? We will then track the beast up into The Algebraic Alps and down into The Marshes of Model Theory, all the while unveiling more of its secrets. Finally we will go after it into The 0-1 Forest, going the full circle back to probability.

Mathematically speaking, our journey will start in Chapter 1: The One Graph with the extension property and different definitions of the Rado graph. In Chapter 2: The Grasslands of Graph Theory we will look at the graph theoretical aspects of the Rado graph, like induced subgraphs, colourings and partition properties. Chapter 3: The Algebraic Alps is all about ultrahomogeneity and automorphisms. Some basic concepts of model theory are assumed for Chapter 4: The Marshes of Model Theory, in which we discuss the first-order theory of the Rado graph, definable subsets, types, saturation, elimination, categoricity, Fraïssé limits and resplendence. In Chapter 5: The 0-1 Forest we prove a 0-1 law for graphs and use it to show that there are finite graphs that satisfy a weaker form of the extension property. The original contributions of this thesis include the non-fractal nature of the Rado graph (discussed at the end of Chapter 2), classifying the definable subgraphs of the Rado graph (found in Chapter 4), and work regarding finite graphs with $r$-extension, a weaker form of the extension property (discussed in Chapter 5). We also give a brief overview of a certain similitude between the Rado graph and the rationals viewed as a linear order as part of the appendices.

It is assumed that everyone joining the hunting party has some prior experience with mathematical wildlife. They should know, for instance, what an isomorphism is or how modular arithmetic works. There are appendices available for readers who wish to refresh their memory.

Are you ready to discover the mysteries of the Rado graph? Then let the adventure begin!

# Chapter 1

# The One Graph

> Deep in the land of Mordor, in the Fires of Mount Doom, the Dark Lord Sauron forged a master ring, and into this ring he poured his cruelty, his malice and his will to dominate all life.
> One ring to rule them all.
>
> Galadriel
> *The Lord of the Rings:*
> *The Fellowship of the Ring (movie)*

We begin our adventure by studying, in detail, the Rado graph which we will from now on denote as $\boldsymbol{R}$. We start off with what is probably the most important property of $\boldsymbol{R}$, namely the extension property. Then we take a look at different constructions of graphs which all turn out to be $\boldsymbol{R}$. [Cam13] and [Hen19] both contain nice summaries of the results in this chapter.

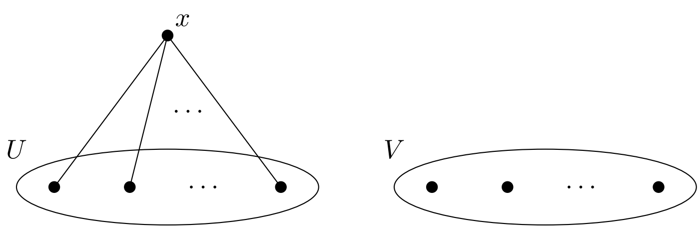
## 1.1 A key property: The extension property

The extension property, which we will state now, is the key with which we will unlock the secrets of $\boldsymbol{R}$.

**Property 1.1.1** (**The Extension Property**, EP for short, see [ER63], Lemma 3, pg. 309)**.** *Let $G$ be a graph. Then $G$ has* the extension property *if for all finite disjoint subsets $U$ and $V$ of $G$ there is an $x \in G \setminus (U \cup V)$ such that $x$ is connected to every vertex in $U$ and not connected to any vertex in $V$.*

*In this case we will call $x$ a* witness *of EP for $U$ and $V$, or if the situation is clear we will just say that $x$ witnesses EP.*

We give an illustration of EP.



**Remarks 1.1.2.** 1. The vertices of $U$ and $V$ may be connected; this will have no effect on EP. All that matters is that $U \cap V = \emptyset$.

2. $U$ and $V$ may be *any* disjoint finite subsets of vertices of $G$, including the empty set.

We will now use our master key to unlock the following result.

**Theorem 1.1.3** (see [ER63], pg. 309)**.** *Any two countably infinite graphs satisfying EP are isomorphic.*

We introduce some notation to shorten writing for the proof and throughout the text. Let $G$ be a graph. If two vertices $x$ and $y$ of $G$ are connected, i.e. $\{x, y\} \in E(G)$, we will write $x \sim y$. In this case we may say any of the following: $x$ is *connected* to $y$, $x$ is *adjacent* to $y$, $x$ is a *neighbour* of $y$. When we are working with more than one graph, we will use $\sim$ to indicate that vertices are connected for all the graphs involved, as no confusion will arise. We may shorten writing even more by taking $x \sim U$, where $U$ is a set of vertices, to mean that $x$ is connected to every vertex from $U$, and $x \not\sim U$ to mean that $x$ is not connected to any vertex from $U$.

*Proof of Theorem 1.1.3.* Let $G$ and $H$ be two countably infinite graphs satisfying EP. We construct an isomorphism between $G$ and $H$ using a back-forth-argument.

Consider enumerations $\{a_1, a_2, \dots\}$ and $\{b_1, b_2, \dots\}$ of the vertices of $G$ and $H$ respectively. We will successively choose sequences $(a'_i : i \in \mathbb{N})$ in $G$ and $(b'_i : i \in \mathbb{N})$ in $H$ such that

$$a'_i \sim a'_j \text{ iff } b'_i \sim b'_j. \tag{1.1}$$

This will ensure that the mapping $f : a'_i \mapsto b'_i$ is a homomorphism.

Suppose we have already chosen $a'_i \in G$ and $b'_i \in H$ satisfying condition (1.1) for all $i < n$ for some $n \in \mathbb{N}$ and let $A := \{a'_1, \dots, a'_{n-1}\}$ and $B := \{b'_1, \dots, b'_{n-1}\}$. So we have $f : G \restriction A \cong H \restriction B$.

<u>If $n$ is even:</u> let $a'_n$ be $a_j$ where $j$ is the smallest index such that $a_j \notin A$. Take $U := A \cap \{x : x \sim a_j\}$ and $V := A \cap \{x : x \not\sim a_j\}$. Clearly $U \cap V = \emptyset$ and so we will have that $f[U]$ and $f[V]$ are finite disjoint subsets of $H$. Since $H$ satisfies EP there is a $b \in H \setminus (f[U] \cup f[V])$ such that $b \sim f[U]$ and $b \not\sim f[V]$. In this case we extend $f$ by stipulating that $f(a'_n) = b$.

<u>If $n$ is odd:</u> let $b'_n$ be $b_j$ where $j$ is the smallest index such that $b_j \notin B$. We take $U := B \cap \{x : x \sim b_j\}$ and $V := B \cap \{x : x \not\sim b_j\}$ and so $U \cap V = \emptyset$. It now follows that $f^{-1}[U]$ and $f^{-1}[V]$ are finite disjoint subsets of $G$ and since $G$ satisfies EP there exists $a \in G \setminus (f^{-1}[U] \cup f^{-1}[V])$ with $a \sim f^{-1}[U]$ and $a \not\sim f^{-1}[V]$. So we can extend $f$ by letting $f^{-1}(b'_n) = a$.

This alternating method of choosing vertices to extend $f$ will exhaust both $G$ and $H$ and so we end up with a bijective map $f : G \to H$, which, by condition (1.1), will be a homomorphism. This shows that $f : G \cong H$ and we conclude that any two countably infinite graphs satisfying EP are isomorphic. □

Up to this point we have not yet seen that $\boldsymbol{R}$ indeed satisfies EP. Hopefully one can still see how powerful the previous result is. If $\boldsymbol{R}$ satsisfies EP, then what Theorem 1.1.3 is telling us, is that any countably infinite graph with EP is (up to isomorphism) the Rado graph. Let's now discuss the full weight of its implications.

## 1.2 Different points of view: Other definitions of the Rado graph

> There is only one way to see things, until someone shows us how to look at them with different eyes.
>
> PABLO PICASSO

We remind ourselves how we defined $\boldsymbol{R}$. Take a set with countably infinitely many vertices, and connect each pair of vertices independently with probability $\frac{1}{2}$. This probabilistic definition was given by Erdős and Rényi in [ER63].

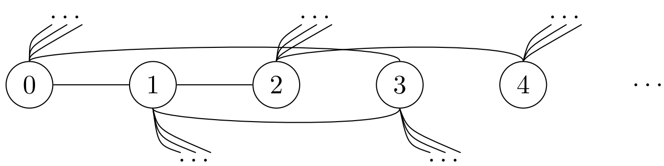
**Proposition 1.2.1** (see [ER63], Lemma 3, pg. 309)**.** *$\boldsymbol{R}$ satisfies EP.*

*Proof.* Let $U$ and $V$ be disjoint finite sets of vertices of $\boldsymbol{R}$. Now for any $x \in \boldsymbol{R} \setminus (U \cup V)$, the probability that $x$ is connected to, or not connected to a given vertex is $\frac{1}{2}$. So the probability that $x \sim U$ is $\frac{1}{2^{|U|}}$ and the probability that $x \nsim V$ is $\frac{1}{2^{|V|}}$. Therefore, the probability of $x$ witnessing EP for $U$ and $V$ is $\frac{1}{2^{|U|}} \times \frac{1}{2^{|V|}} = \frac{1}{2^{|U|+|V|}}$. Since there are infinitely many vertices outside $U$ and $V$ we will surely find such a witness. $\square$

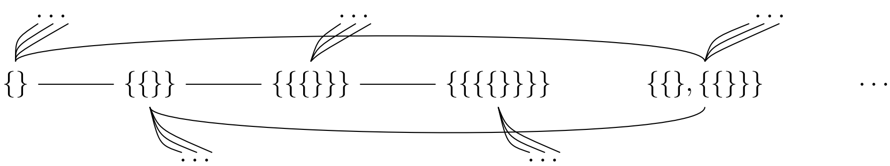
Now we will look at constructions of countably infinite graphs which, at first glance, might not seem very similar to that of $\boldsymbol{R}$. Just for the fun of it, we will depict each of these constructions. Since you already had some practice, you can do the probabilistic one from above. Go on then, grab that coin, or whatever you used before and start connecting some dots.




The first construction we look at was given by Rado in [Rad64] and uses the BIT predicate. Given $x, y \in \mathbb{N}_0$ with $x < y$, BIT$(y, x)$=1 if the $x$-th bit of $y$ is 1 when $y$ is written in binary, $a_k \dots a_1 a_0$, where $a_i$ is the $i$-th bit of $y$, and 0 otherwise. Take $\{0, 1, 2, \dots\}$ to be vertices and connect $i$ and $j$, with $i < j$, if BIT$(j, i)$=1.




For our next construction we use for vertices the class of hereditary finite sets, i.e the class of finite sets whose elements are again finite sets, constructed using only brackets. We connect two sets if one is an element of the other, i.e. for hereditary finite sets $x$ and $y$, we will join $x$ and $y$ if $x \in y$ or $y \in x$.




The final construction uses a bit of number theory. We look to [Pin10] and [Dud09] for the definitions of some basic concepts.

**Definition 1.2.2** (**Congruence modulo n**, see [Pin10] pg. 227)**.** *Let $a$ and $b$ be any two integers and $n$ be any positive integer. When we divide $a$ and $b$ by $n$ and get the same remainder, then we say that $a$* is congruent to *$b$* modulo *$n$. In this case we will write $a \equiv b(\text{mod } n)$.*

We take the set of primes congruent to 1 modulo 4 as vertices. There are infinitely many of these by Dirichlet's Theorem.

> Dirichlet's Theorem (see [Sel49])
>
> Let integers $k$ and $l$ be relatively prime. The arithmetic progression
>
> $$k + l, \qquad k + 2l, \qquad k + 3l, \qquad \ldots$$
>
> has infinitely many primes.

**Definition 1.2.3** (**Least residue modulo n**, see [Dud09] pg. 13)**.** *An integer $b$ with $0 \leq b \leq n-1$ and $a \equiv b(\text{mod } n)$ is called the* least residue of *$a$* modulo *$n$.*

**Definition 1.2.4** (**Solution modulo n**, see [Dud09] pg. 17)**.** *An integer $c$ is a* solution modulo *$n$ of the congruence $ax \equiv b(\text{mod } n)$ if*

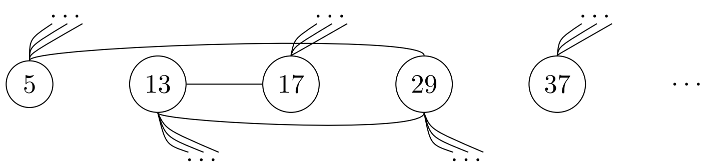
*1. it satisfies the congruence, i.e. $ac \equiv b(\text{mod } n)$ and*

*2. it is a least residue modulo $n$.*

**Definition 1.2.5** (**Quadratic residue**, see [Dud09] pg. 53)**.** *Let $p$ be an odd prime and $a$ be an integer with $1 \leq a \leq p-1$. If the congruence $x^2 \equiv a(\text{mod } p)$ has a solution then we say that $a$ is a* quadratic residue *of $p$ modulo $n$. Otherwise we call $a$ a* quadratic non-residue *of $p$ modulo $n$.*

**Theorem 1.2.6** (**Quadratic reciprocity**, see [Dud09] pg. 61)**.** *Let $p$ and $q$ be distinct odd primes such that neither of $p$ and $q$ is congruent to 3 modulo 4. Then $q$ is a quadratic residue of $p$ iff $p$ is a quadratic residue of $q$.*

Given $p$ and $q$ from the set of primes congruent to 1 modulo 4, we let $p \sim q$ iff $x^2 \equiv p(\text{mod } q)$ is solvable, i.e. iff $p$ is a quadratic residue of $q$. From the statement of quadratic reciprocity (Theorem 1.2.6) we know this happens exactly when $x^2 \equiv q(\text{mod } p)$ is solvable so there is no risk of having more than one edge between any two vertices.




Considering the figures above, these graphs clearly look different. And thinking about how these graphs were constructed in a concrete manner, it is hard to imagine how any of them can give rise to $\boldsymbol{R}$, since $\boldsymbol{R}$ was completely randomly constructed.

It is here that the fun begins. We verify that each one of the above graphs has EP.

**The graph constructed using the BIT predicate has EP.**

Let $G$ be the graph constructed using the BIT predicate, and let $U = \{a_1, \ldots, a_m\}$ and $V = \{b_1, \ldots, b_n\}$ be finite disjoint subsets of $G$. Then for $x = \sum_{i=1}^{m} 2^{a_i}$ we have $\text{BIT}(x, a_i)$=1 for each $i$ and $\text{BIT}(x, b_j) \neq 1$ for each $j$. So $x$ will be connected to every vertex in $U$ and none in $V$, i.e. $x$ witnesses EP.

**The graph constructed using hereditary finite sets has EP.**

Let $H$ be the graph constructed using hereditary finite sets. We will need a bit of set-theory if we want to show that $H$ has EP.

Some ZFC (Zermelo–Fraenkel set theory with the axiom of choice)

We will state only those axioms which we need for our argument.

**Axiom of extent:** Two sets are equal iff they have the same elements.

**Axiom of pairing:** Given any two sets $x$ and $y$ there is a set $z$ whose elements are exactly $x$ and $y$.

The axiom of extent implies that the $z$ in the axiom of pairing is unique, and we may write $z = \{x, y\}$. We also write $\{x\}$ instead of $\{x, x\}$.

**Axiom of regularity**(or sometimes called the axiom of foundation)**:** Every non-empty set $x$ has an element which is disjoint from $x$.

Consider a set $x$. By the axiom of pairing $\{x\}$ is a set. According to the axiom of regularity the set $\{x\}$ must have an element disjoint from itself. This means that $x \cap \{x\} = \emptyset$ which implies that $x \notin x$. So no set can be an element of itself. This is the part that we need.

Back to our graph on hereditary finite sets, let $U = \{a_1, \ldots, a_m\}$ and $V = \{b_1, \ldots, b_n\}$ be finite disjoint subsets of $H$ and let $x = U \cup \{V\}$. Clearly $x$ is hereditary, since it is finite and all elements of $x$ are again finite sets.

First we check that $x \notin U \cup V$. Suppose, on the contrary that $x \in U \cup V$, then $x \in U$ or $x \in V$. If $x \in U \subseteq x$ then $x \in x$, a contradiction. So $x$ must be in $V$. In this case $\{x\} \cap V = x$. Since $V \in x$ this means that $V \in \{x\} \cap V$ which implies that $V \in V$, but we know this is not possible. Hence $x \notin U \cup V$.

Now we need to check that $x$ is connected to all of the vertices in $U$ and none of the vertices in $V$. Note that $a_i \in x$ for all $a_i \in U$ since $U \subseteq x$, so $x$ is connected to every vertex in $U$. We will show via contradiction that $x$ is not connected to any vertex in $V$. We need to have both $b_j \notin x$ and $x \notin b_j$ for each $j$ according to our definition of $H$. Suppose $b_j \in x$ for some $b_j \in V$. Since $x = U \cup \{V\}$ and $b_j \notin U$ this means that $b_j = V$ and hence $V \in V$, which cannot be the case. Therefore $b_j \notin x$ for any $j$. Now suppose that $x \in b_j$ for some $b_j \in V$. Then $\{x\} \cap b_j = x$. Since $V \in x$, this implies that $V \in \{x\} \cap b_j$ which in turn implies that $V = x$. This then gives $x \in x$ which cannot happen. Hence $x \notin b_j$ for each $j$. This shows that $x$ is not connected to any vertex in $V$.

In conclusion, $x = U \cup \{V\}$ serves as a witness to EP.

**The graph constructed using primes congruent to 1 modulo 4 has EP.**

Let $P$ be the graph on primes congruent to 1 modulo 4, and let $U = \{u_1, \ldots, u_m\}$ and $V = \{v_1, \ldots, v_n\}$ be finite disjoint subsets of $P$.

Let $d = 4u_1 \cdots u_m v_1 \cdots v_n$ and consider the system of congruences

$$y \equiv 1 (\text{mod } 4), \quad y \equiv a_1 (\text{mod } u_1), \quad \ldots \quad y \equiv a_m (\text{mod } u_m), \quad y \equiv b_1 (\text{mod } v_1), \quad \ldots \quad y \equiv b_n (\text{mod } v_n)$$

where $a_i$ is a quadratic residue of $u_i$ for each $i$, and $b_j$ is a quadratic non-residue of $v_j$ for each $j$.

The Chinese Remainder Theorem (see [Dud09] pg. 21)

Given integers $n_1, \ldots, n_k$ which are pairwise relatively prime, the system of congruences

$$x \equiv c_1(mod\ n_1), \qquad x \equiv c_2(mod\ n_2), \qquad \ldots \qquad x \equiv c_k(mod\ n_k)$$

has a unique solution modulo $n_1 n_2 \ldots n_k$.

Using the Chinese Remainder Theorem we see that this system has a unique solution $y$, modulo $d$, of the form $y \equiv c(\text{mod } d)$. Now none of 1, $a_1$, $\ldots$, $a_m$, $b_1$, $\ldots$, $b_n$ are zero (by Definition 1.2.5) so $y$, and hence $c$, cannot be a multiple of any of 4, $u_1$, $\ldots$, $u_m$, $v_1$, $\ldots$, $v_n$. This implies that $c$ and $d$ are relatively prime. From Dirichlet's Theorem we have that $c + d$, $c + 2d$, $c + 3d, \ldots$ contains infinitely many primes. We take one of these primes to be $x$, i.e. $x = c + kd$ for some $k$. This implies that $x \equiv c(\text{mod } d)$ and hence $x \equiv y \equiv c(\text{mod } d)$ meaning that $x$ satisfies the system of congruences.

So we have that $x$ is a prime congruent to 1 modulo 4, i.e. $x$ is a vertex in $P$, and it is not one of the $u_i$'s or the $v_i$'s, since it's not a multiple of any of 4, $u_1$, $\ldots$, $u_m$, $v_1$, $\ldots$, $v_n$, i.e. $x \notin U \cup V$.

Also, $x \equiv a_i(\text{mod } u_i)$ for each $i$, meaning it will be a quadratic residue of each $u_i$ which implies that $x \sim u_i$ for all $u_i \in U$. Considering that $x \equiv b_j(\text{mod } v_j)$ for each $j$ we see that $x$ will be a quadratic non-residue of each $v_j$, i.e. $x \nsim v_j$ for all $v_j \in V$. This $x$ will be our witness for EP.

Let's summarise what we have seen. We have a probabilistic definition of $\boldsymbol{R}$. Then we have a countably infinite graph constructed using the BIT predicate. We also have the graph on hereditary finite sets and finally we have a graph with a number theoretical definition. All of these are countably infinite. All of these satisfy EP. Some knowledgeable and enthusiastic members of the hunting party exclaim, "These are all the same graph!", and we name[1] the following result after them.

**Theorem 1.2.7** (**Aficionado**, see [ER63], pg. 309)**.** *Any countably infinite graph satisfying EP is isomorphic to* $\boldsymbol{R}$.

*Proof.* This follows from Proposition 1.2.1 and Theorem 1.1.3. □

Now that we have a better understanding of $\boldsymbol{R}$, and how seemingly different graphs all give rise to it, let's move on to some graph theory to see what secrets lie there.

[1]This is just for ease of referencing, as we will be using this result many times throughout our work.

# Chapter 2

# The Grasslands of Graph Theory

> If being human is not simply a matter of being born flesh and blood...
> If it is instead a way of thinking, acting and feeling, then I am hopeful that one day I will discover my own humanity.
>
> DATA
> *Star Trek: The Next Generation*

For convenience, we restate some basic definitions. Other concepts will be defined when they are needed.

We take a *graph* to be a pair $G = (V, E)$ consisting of *vertices* (which we may refer to as *nodes* or *points*), $V$, and *edges* (or sometimes *lines*), $E$, such that the edges are two-element subsets of $V$. In this case we will say that $G$ is *on* $V$.

As convention we will always write $V(\cdot)$ and $E(\cdot)$ to denote the vertices and edges of a graph, so even if the graph under consideration is $H = (X, Y)$ we will still write $V(H)$ for its set of vertices and $E(H)$ for its set of edges. We will also write $v \in G$ or $e \in G$ instead of $v \in V(G)$ and $e \in E(G)$.

Given a vertex $v$ and edge $e$ with $v \in e$ we say that $v$ is *incident* with $e$. If $e = \{x, y\}$ we will say that $x$ and $y$ are *adjacent* or *connected*. We may also call them *neighbours* and we will write $x \sim y$. For a set of vertices $U$, $x \sim U$ means that $x$ is connected to every vertex in $U$, and $x \nsim U$ means that $x$ is not connected to any vertex in $U$.

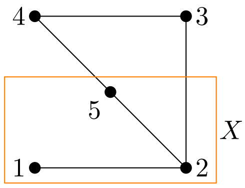
## 2.1 Divide and conquer: Subgraphs of the Rado graph

> Can't cut it out, it will grow right back.
>
> RAFIKI
> *The Lion King*

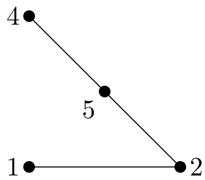
Now that we have caught up on the needed terminology, we continue our quest. In this section we will observe just how robust the beast is. We will stab and slash in an attempt to draw blood.

First we need to sharpen our blades. We define some operations on graphs. For a graph $G$ and vertex $x \in G$, we can *delete* $x$ to get a new graph $H$ with $V(H) = V(G) \setminus \{x\}$ and $E(H) = \{e \in E(G) : x \notin e\}$. *Flipping* an edge will turn an edge into a non-edge. We can similarly flip non-edges to edges. Given a set of vertices

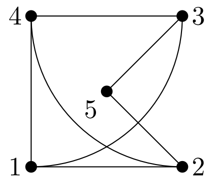
$X \subseteq G$, we *switch* $G$ with respect to $X$ if we flip all the edges and non-edges between vertices in $X$ and vertices in $G \setminus X$. We give an illustration of these operations.




(a) A graph $G$

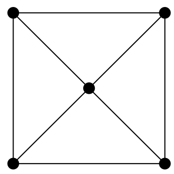



(b) Deleting vertex 3

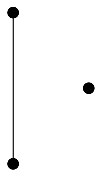



(c) Switching w.r.t. $X$

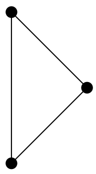
Let $G = (V, E)$ and $H = (V', E')$ be graphs. We say $H$ is a *subgraph* of $G$ if $V' \subseteq V$ and $E' \subseteq E$. We will denote this by $H \subseteq G$. If, in addition, $H$ contains all the edges $\{x, y\} \in E$ for $x, y \in V'$, then $H$ is an *induced* subgraph of $G$. The following illustrates this concept.



(a) A graph $G$



(b) A subgraph of $G$



(c) An induced subgraph of $G$

**Proposition 2.1.1** (see [Cam13], Proposition 1, pg. 4)**.** *Given finite disjoint subsets, $U$ and $V$, of vertices of $\boldsymbol{R}$, and $X$ the set of vertices that witness EP for $U$ and $V$, this $X$ induces a subgraph of $\boldsymbol{R}$ which is isomorphic to $\boldsymbol{R}$.*

*Proof.* Let $G$ be the subgraph of $\boldsymbol{R}$ induced by $X$. First we show that $G$ is countably infinite.

Suppose not, then $|G| = n$ for some $n \in \mathbb{N}$. This means that there are only $n$ witnesses, say $x_1, x_2, \ldots, x_n$ to EP for $U$ and $V$. Now consider the sets $U' := U \cup \{x_1, x_2, \ldots x_n\}$ and $V$. These are finite disjoint subsets of $\boldsymbol{R}$. Since $\boldsymbol{R}$ has EP we will be able to find $x \in \boldsymbol{R} \setminus (U' \cup V)$ such that $x \sim U'$ and $x \not\sim V$. In particular $x$ is connected to every point in $U$ and no point in $V$ and $x \neq x_i$ for any of the $x_i$'s. Hence we have $n+1$ witnesses of EP for $U$ and $V$, contradicting the assumption that there were only $n$ witnesses. So $G$ is countably infinite.

Now we show that $G$ has EP. Let $U'$ and $V'$ be finite disjoint subsets of $X$. Then $U \cup U'$ and $V \cup V'$ are finite disjoint subsets of $\boldsymbol{R}$ and since $\boldsymbol{R}$ has EP we are able to find an $x$ such that $x \in \boldsymbol{R} \setminus (U \cup U' \cup V \cup V')$ and $x \sim U \cup U'$ and $x \not\sim V \cup V'$. This $x$ will be in $X$ since $x \sim U$ and $x \not\sim V$, i.e. since it is a witness for $U$ and $V$. We also have that $x \in X \setminus (U' \cup V')$ such that $x \sim U'$ and $x \not\sim V'$, hence $x$ is a witness to EP for $U'$ and $V'$ in $G$.

In conclusion, $G$ is countably infinite and satisfies EP. Therefore, by Aficionado, $G$ is isomorphic to $\boldsymbol{R}$ as required. □

We can use EP in the same way as above to prove the following two propositions.

**Proposition 2.1.2.** *Every vertex of $\boldsymbol{R}$ has infinitely many neighbours.*

*Proof.* Suppose, on the contrary, that a given $x \in \boldsymbol{R}$ has only $n$ neighbours, for $n \in \mathbb{N}$. Let $U$ be the set consisting of the neighbours of $x$ together with $x$, and let $V$ be a finite subset of $\boldsymbol{R}$ such that $U \cap V = \emptyset$. Then, since $\boldsymbol{R}$ has EP there is a $z$ from $\boldsymbol{R}$, not already in $U$ or $V$ s.t $z \sim U$ and $z \not\sim V$. This means that $z \sim x$ and hence we have $n + 1$ neighbours of $x$, a contradiction.

Therefore $x$ has infinitely many neighbours. □

**Proposition 2.1.3.** *Every vertex of $\boldsymbol{R}$ has infinitely many non-neighbours.*

*Proof.* This is the same as the proof for Proposition 2.1.2, but with the word "neighbours" replaced by "non-neighbours" and the roles of $U$ and $V$ interchanged. □

We are ready for our first attack on $\boldsymbol{R}$. We try to hack away at it...

**Proposition 2.1.4** (see [Cam13], Proposition 2, pg. 5)**.** *$\boldsymbol{R}$ is isomorphic to the resulting graph under each of the following operations:*

1. *deleting a vertex of $\boldsymbol{R}$,*
2. *deleting a finite subset of vertices of $\boldsymbol{R}$,*
3. *flipping an edge (resp. non-edge) of $\boldsymbol{R}$,*
4. *flipping a finite number of edges (resp. non-edges) of $\boldsymbol{R}$,*
5. *and switching with respect to a finite subset of vertices of $\boldsymbol{R}$.*

*Proof.* First note that in any of the above cases, the resulting graph will still be countably infinite. Hence, by Aficionado, it is enough to show that EP is satisfied in each resulting graph. Let $U$ and $V$ be finite disjoint subsets of vertices of the graph obtained under each operation.

1. If $x$ is the required witness, then we can delete any other vertex of $\boldsymbol{R}$ without running into trouble. In the case where we delete $x$ we can just pick one of the infinitely many other witnesses which exists by Proposition 2.1.1.
2. This follows from 1.
3. Let $x$ be the required witness, i.e. $x \sim U$ and $x \not\sim V$. Flipping an edge (or non-edge) not incident with $x$ would not give rise to any problems. In case we do flip an edge (or non-edge) incident with $x$ we might have that $x \not\sim u$ for some $u \in U$ or $x \sim v$ for some $v \in V$, which is not what we want. Again we can use Proposition 2.1.1 to choose some other witness.
4. This follows from 3.
5. Let $X$ be a finite subset of vertices of $\boldsymbol{R}$. We are going to switch $\boldsymbol{R}$ with respect to $X$.
   Consider the sets $A := (U \setminus X) \cup (V \cap X)$ and $B := (V \setminus X) \cup (U \cap X)$. $A$ and $B$ are finite disjoint subsets of $\boldsymbol{R}$, which satisfies EP and so we can find a witness $x$ for $A$ and $B$. After switching, this $x$ will be a witness for $U$ and $V$. We give a sketch for clarity.

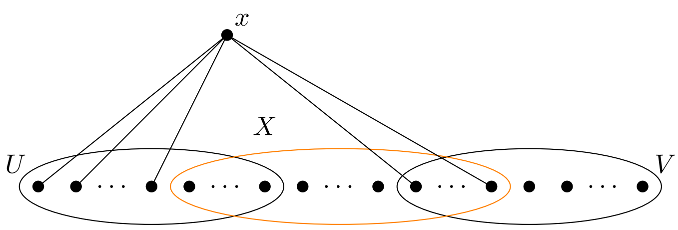



Figure 2.3: Before switching

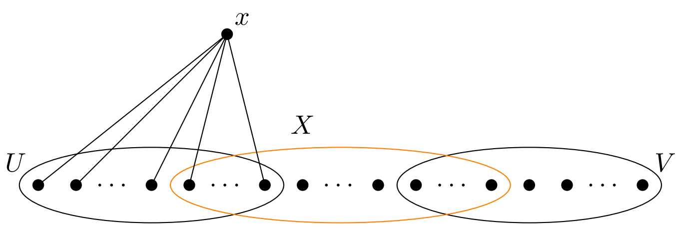



Figure 2.4: After switching

This shows that each resulting graph satisfies EP and is therefore isomorphic to $\boldsymbol{R}$. $\square$

**Remark 2.1.5.** Consider the proof of point 5 above. Switching $\boldsymbol{R}$ with respect to $X$ is not the same as flipping a finite amount of edges. This is because Proposition 2.1.1 implies that there are infinitely many edges and infinitely many non-edges connected to vertices in $X$, meaning there are infinitely many edges and non-edges that are going to be flipped.

... but it is futile. We can see that removing a vertex or edge or finitely many of each, does not have any effect on $\boldsymbol{R}$.

The next two results are not very surprising, but they do hint at the richness of $\boldsymbol{R}$.

**Theorem 2.1.6** (see [Cam13], Proposition 6, pg. 6)**.** *1. Every countably infinite graph and 2. every finite graph is isomorphic to an induced subgraph of* $\boldsymbol{R}$.

*Proof of Theorem 2.1.6 part 1.* Let $G$ be any countably infinite graph and let $\{a_1, a_2, \dots\}$ be an enumeration of the vertices of $G$. We define an embedding $f : G \to \boldsymbol{R}$ inductively.

Let $f_0 = \emptyset$. Now suppose that $f_n : \{a_1, \dots, a_n\} \to \boldsymbol{R}$ has already been chosen such that it is an isomorphism preserving the induced subgraphs of $G$ and $\boldsymbol{R}$.

Next let $U := \{a_1, \dots, a_n\} \cap \{\text{Neighbours of } a_{n+1}\}$ and $V := \{a_1, \dots, a_n\} \cap \{\text{Non-neighbours of } a_{n+1}\}$. Then $f_n(U)$ and $f_n(V)$ are finite disjoint subsets of $\boldsymbol{R}$. Since $\boldsymbol{R}$ has EP there is an $x$ such that $x \sim f_n(U)$ and $x \not\sim f_n(V)$ and we define $f_{n+1}(a_{n+1}) = x$ to extend $f_n$.

Finally take $f := \bigcup_{n \geq 0} f_n$. This is the required embedding, and so we have that $G$ is an induced subgraph of $\boldsymbol{R}$. $\square$

*Proof of Theorem 2.1.6 part 2.* This is can be done in the same way as the proof for part 1, but ending the process after a finite amount of steps. $\square$

Our next move is to split up and divide $\boldsymbol{R}$'s attention. Maybe in this way it will lose its power.

**Theorem 2.1.7** (**Partition-regularity**, see [Cam13], Proposition 3, pg. 5)**.** *The induced subgraph of at least one cell of any finite partition of the vertices of* $\boldsymbol{R}$ *is isomorphic to* $\boldsymbol{R}$.

*Proof.* Suppose on the contrary that there is a partition $P := \{P_1, \ldots, P_n\}$ of $\boldsymbol{R}$ for which this is not true. Then none of the graphs $G_i$, induced by each $P_i$, has EP. In this case we will be able to find, for each $i$, finite disjoint sets $U_i$ and $V_i$ which are counterexamples to EP in $G_i$. So the sets $U := U_1 \cup \cdots \cup U_n$ and $V := V_1 \cup \cdots \cup V_n$ will be a counterexample to EP in $\boldsymbol{R}$, a contradiction.

This proves that for any finite partition $P := \{P_1, \ldots, P_n\}$ of $\boldsymbol{R}$ there is at least one $P_i$ for which the induced subgraph $G_i$ is isomorphic to $\boldsymbol{R}$. In this case $G_i$, and hence $P_i$, is countably infinite. $\square$

We state the following result without proof.

**Theorem 2.1.8** (see [Hen19], Theorem 4, pg. 6)**.** *The only countably infinite graphs with partition regularity are the complete graph (i.e. a graph with all possible edges), the null graph (i.e. a graph with no edges) and $\boldsymbol{R}$.*

Even when our enemy is divided, we are no match. But take heart, for as David struck down Goliath with a simple pebble, so we will bring down $\boldsymbol{R}$ with something as simple as a vertex.

**Proposition 2.1.9.** *The graph obtained from $\boldsymbol{R}$ by adding a vertex and no new edges is not isomorphic to $\boldsymbol{R}$.*

*Proof.* Without loss of generality, suppose $x \notin \boldsymbol{R}$. Let $G$ be the graph obtained from $\boldsymbol{R}$ by adding $x$ as a vertex with no new edges. We give a counterexample to show that EP is not satisfied. This will imply the result.

Consider the sets $U := \{x\}$ and $V := \emptyset$. These are clearly finite and disjoint, but no vertex of $G$ is connected to $x$, and so, there is no witness to EP for $U$ and $V$ in $G$. $\square$

**Proposition 2.1.10.** *Let $G$ be the graph obtained from $\boldsymbol{R}$ by adding a vertex $x$ and letting $x \sim r$ for each $r \in \boldsymbol{R}$. Then $G \ncong \boldsymbol{R}$.*

*Proof.* Again, we only need to show that EP is not satisfied.

Consider the sets $U := \emptyset$ and $V = \{x\}$. $U$ and $V$ are finite disjoint subsets of $G$, but every vertex of $G$ is connected to $x$, hence there is no witness to EP for $U$ and $V$ in $G$. $\square$

We see that $\boldsymbol{R}$ is resilient against removing, but not adding vertices. If we want $\boldsymbol{R}$ to recover from getting an extra vertex, some work needs to be done. If we add too few edges or too many edges (as in the cases above) then EP won't be satisfied. We need to add just the right amount of edges. So now, after breaking $\boldsymbol{R}$ down, we show how to rebuild it.

**Proposition 2.1.11.** *Let $G$ be the graph obtained from $\boldsymbol{R}$ by adding a vertex $x$ and letting $x \sim r$ for each vertex $r \in \boldsymbol{R}$ independently with probability $\frac{1}{2}$. Then $G \cong \boldsymbol{R}$.*

*Proof.* Clearly $G$ is countably infinite and so, by Aficionado, we only need to show that $G$ has EP.

Let $U$ and $V$ be disjoint finite sets of vertices of $G$. Any $y \in G \setminus (U \cup V)$ has probability $\frac{1}{2^{|U|+|V|}}$ of witnessing EP. Since there are infinitely many vertices outside $U$ and $V$ we will be able to find such a witness. $\square$

**Remark 2.1.12.** Note that we cannot add only a finite number of edges to recover $\boldsymbol{R}$ after adding a single vertex $x$ to $\boldsymbol{R}$. If we add only a finitely number of edges, say $x \sim r_i$ for each $r_i \in \{r_1, \ldots, r_n\} \subseteq \boldsymbol{R}$, then we can choose $U := \{x, r_1, \ldots, r_n\}$ and $V := \emptyset$, leaving us with no witness to EP for $U$ and $V$ in the new graph.

We have seen how to take $\boldsymbol{R}$ down, by striking at its core, by aiming directly at EP.

## 2.2 Walking and leaping: The connectedness of the Rado graph

Walking is good exercise and so we must include it to ensure that this dissertation stays healthy. Definitions in this section follow those as in [Die00]. The results from this section and the next are derived from analogous results as found in [Cam13].

Diestel defines a *walk* in a graph as a sequence of vertices and edges of the form $v_0e_0v_1e_1 \dots v_{k-1}e_{k-1}v_k$ and such that $v_i, v_{i+1} \in e_i$ for all $i < k$. We will write this sequence only as $v_0v_1 \dots v_k$, since the edges needed for the walk will be clear. Such a walk is of *length* $k$, i.e. we count the number of edges traversed. In case all the vertices are distinct we say *path* instead of walk.

**Proposition 2.2.1** (see [Cam13] pg. 6)**.** *Any two vertices of $\boldsymbol{R}$ can be connected with a finite walk. In particular any two vertices of $\boldsymbol{R}$ can be connected with a walk of length* 2*.*

*Proof.* Consider any two points, say $x, y \in \boldsymbol{R}$. We want to find a finite walk between these two points. Let $U = \{x, y\}$ and $V = \emptyset$. Since $\boldsymbol{R}$ has EP we will be able to find a vertex $r \in R \setminus (U \cup V)$ with edges connecting $r$ to both $x$ and $y$. We therefore have a walk, consisting of two edges, with vertex sequence $xry$.

Thus, we can find a finite walk between any two points of $\boldsymbol{R}$. In fact, we can find a walk of length 2 between any two points of $\boldsymbol{R}$. □

**Remark 2.2.2.** We can replace the word "walk" with "path" in the above result without changing the outcome. This is because EP allows us to choose vertices for the finite walk in such a way that each of them are distinct, giving us the required path.

Following Diestel again, we take the *distance* between two vertices, say $x$ and $y$, of a graph to be the shortest path beginning in $x$ and ending in $y$. The *diameter* of a graph is then the greatest distance between any two vertices of the graph.

The following result is thus a direct consequence of Proposition 2.2.1.

**Corollary 2.2.3.** *$\boldsymbol{R}$ has diameter 2.*

If we can find a path between any two vertices of a graph, then we call that graph *connected*. We thus have another consequence of Proposition 2.2.1.

**Corollary 2.2.4.** *$\boldsymbol{R}$ is connected.*

We can say more about the connectedness of $\boldsymbol{R}$.

**Definition 2.2.5** (**$k$-connected**, see [Die00] pg. 10)**.** *Let $k \in \mathbb{N}$. A graph $G$ with $|G| > k$ is* $k$-connected *if $G \setminus X$ is connected for all $X \subseteq V(G)$ with $|X| < k$.*

**Proposition 2.2.6.** *$\boldsymbol{R}$ is $k$-connected for every $k \in \mathbb{N}$.*

*Proof.* Fix $k \in \mathbb{N}$ and let $X \subseteq \boldsymbol{R}$ with $|X| < k$.

Note that $\{\boldsymbol{R} \setminus X, X\}$ constitute a partition of the vertices of $\boldsymbol{R}$. Now by the partition-regularity of $\boldsymbol{R}$ (Theorem 2.1.7) the graph induced by one of these will be isomorphic to $\boldsymbol{R}$. Since $X$ is finite, it follows that the graph induced by $\boldsymbol{R} \setminus X$ is isomorphic to $\boldsymbol{R}$.

Finally, since $\boldsymbol{R}$ is connected we will also have that $\boldsymbol{R} \setminus X$ is connected. Hence $\boldsymbol{R}$ is $k$-connected.

This works for any $k \in \mathbb{N}$, giving the result. □

**Definition 2.2.7** (**$k$-edge-connected**, see [Die00] pg. 10)**.** *Let $k \in \mathbb{N}$. A graph $G$ with $|G| > 1$ is* $k$-edge-connected *if $G \setminus X$ is connected for all $X \subseteq E(G)$ with $X$ having fewer than $k$ edges.*

**Proposition 2.2.8.** *$\boldsymbol{R}$ is $k$-edge-connected for every $k \in \mathbb{N}$.*

*Proof.* Let $k \in \mathbb{N}$ and let $X$ be any finite subset of $E(\boldsymbol{R})$.

Note that $\boldsymbol{R} \setminus X$ will be the graph we obtain by flipping all the edges in $X$. We know from Proposition 2.1.4 that the isomorphism type of $\boldsymbol{R}$ is unchanged under flipping with respect to a finite amount of edges.

Hence $\boldsymbol{R} \setminus X$ is isomorphic to $\boldsymbol{R}$ and so it is connected.

This result holds for all $k \in \mathbb{N}$, so we have that $\boldsymbol{R}$ is $k$-edge-connected for every $k \in \mathbb{N}$. □

While connecting the dots (vertices) of $\boldsymbol{R}$, you might be thinking that this will take forever, which technically it will, so we stop trying to connect more dots and try to see if we can find parts of $\boldsymbol{R}$ which are not connected.

**Proposition 2.2.9** (see [Cam13] pg. 6)**.** *For each $n \in \mathbb{N}$ with $n \geq 2$, there is a set of $n$ vertices of $\boldsymbol{R}$ such that none of the vertices in the set are connected.*

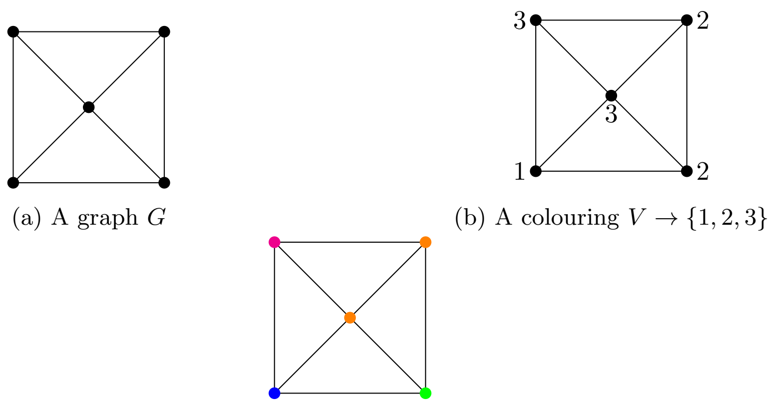
*Proof.* We know from Theorem 2.1.6 that every finite graph is an induced subgraph of $\boldsymbol{R}$. Specifically, for each $n \in \mathbb{N}$, the null graph on $n$ vertices is an induced subgraph of $\boldsymbol{R}$ and hence will constitute a set of $n$ vertices of $\boldsymbol{R}$ such that none of the vertices in the set are connected. □

One might be thinking that Proposition 2.2.9 can be used to oppose the connectedness of $\boldsymbol{R}$, but it is EP that flows like ichor in the veins of $\boldsymbol{R}$ that has given us both the connection and isolation of the vertices of $\boldsymbol{R}$. The fact that these two opposing ideas can co-exsist in $\boldsymbol{R}$ should just be another reminder of how mysterious our beast is.

## 2.3 I see your true colours: Vertex- and edge-colourings of the Rado graph

At this point we have a good rough sketch of our beast. However, going into the wilds with just a rough sketch will be of no use to us. We might end up wandering around aimlessly, losing sight of the great mysteries which we started off looking for. So we stay on the beast's trail, following it closely in an attempt to observe it in more detail.

We are now going to, quite literally, colour our rough sketch. We will take a *colouring* (more specifically a *vertex*-colouring) of a graph to be a mapping that assigns to each vertex of the graph a colour, i.e. $c : V \to C$ is a colouring and $C$ is a set of *colours*. We give an illustration of this concept.




(a) A graph $G$

(b) A colouring $V \to \{1, 2, 3\}$

(c) A colouring $V \to \{$•,•,•,•$\}$

Figure 2.5: Colourings of a graph

Now a *proper* colouring of a graph is a colouring such that no two connected vertices have the same colour. Neither of the colourings in the above illustration are proper, but the one in the next illustration is.

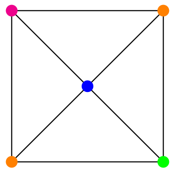


Figure 2.6: A proper colouring $V \to \{\bullet, \bullet, \bullet, \bullet\}$

How many colours do we need to properly colour a graph? More specifically how many colours do we need to properly colour $\boldsymbol{R}$? The easy answer is infinitely many colours, so let's rephrase the question. What is the least amount of colours we need to properly colour a graph? To answer this question we introduce the concept of *$k$-colourability.* A graph $G$ is said to be $k$-colourable if there is a proper colouring $c : V \to C$ with $|C| \leq k$ for some $k \in \mathbb{N}$, i.e. $G$ is $k$-colourable if we can properly colour $G$ with at most $k$ colours.

**Proposition 2.3.1.** *$\boldsymbol{R}$ is not $k$-colourable for any $k \in \mathbb{N}$.*

*Proof.* Let $U$ be a set of $k$ vertices of $R$, with each vertex coloured a different one of $k$ colours and let $V = \{x\}$ where $x$ is any vertex from $R \setminus U$. Since $R$ has EP, we can find a vertex $y \in R \setminus (U \cup V)$ such that $y$ is connected to every vertex in $U$ and not to $x$.

We want a proper colouring of $R$, so we need a new colour, which is not one of the $k$ colours already used, with which to colour $y$.

Hence $R$ is not $k$-colourable. □

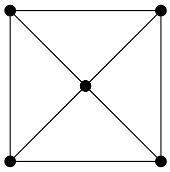
**Remark 2.3.2.** A $k$-colouring of a graph can also be seen as a partition of the vertices of the graph into $k$ sets. We can use this to restate Theorem 2.1.8 in the following way. The complete graph, the null graph and $\boldsymbol{R}$ are the only countably infinite graphs for which any finite colouring contains a monochromatic copy (i.e. consisting of only one colour) of the original graph.

Now that we know we need infinitely many colours to properly colour $\boldsymbol{R}$, and that we can find a monochromatic copy of $\boldsymbol{R}$ in any finite colouring of $\boldsymbol{R}$, the next interesting question is, what other monochromatic configurations can we find in $\boldsymbol{R}$? Assuming we can use infinitely many colours, and supposing we colour each vertex a different colour, then the only configurations we will get are single vertices, which are not very interesting. So let's rephrase this question also. What monochromatic configurations can we find in $\boldsymbol{R}$ using only finitely many colours? As it turns out, we can find a monochromatic copy of any finite or countably infinite graph. This is due to Theorem 2.1.7 and Theorem 2.1.6. Using only finitely many colours is the same as partitioning the vertices of $\boldsymbol{R}$ into finitely many sets. We know from Theorem 2.1.7 that at least one of the cells of the partition will be isomorphic to $\boldsymbol{R}$. This gives us a monochromatic copy of $\boldsymbol{R}$. Next, Theorem 2.1.6 tells us that any finite or countably infinite graph is an induced subgraph of $\boldsymbol{R}$, and so we have monochromatic copies of every finite or countably infinite graph.

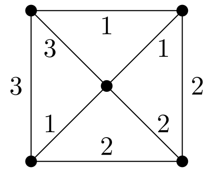
We summarise this discussion with the following proposition.

**Proposition 2.3.3.**
1. *Any $k$-colouring of $\boldsymbol{R}$, with $k \in \mathbb{N}$, will contain a monochromatic copy of $\boldsymbol{R}$ as an induced subgraph.*
2. *Given a finite or countably infinite graph $G$, there is a $k$-colouring of $\boldsymbol{R}$, with $k \in \mathbb{N}$, that contains a monochromatic copy of $G$ as an induced subgraph.*

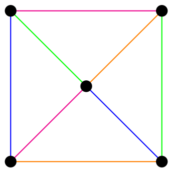
The next thing we consider when colouring a graph is the edges. An *edge*-colouring is a mapping from the edges of the graph to a set of colours, i.e. $c : E \to C$ is an edge-colouring and $C$ is a set of colours. We give an illustration of this concept.



(a) A graph $G$




(b) A colouring $E \to \{1, 2, 3\}$



(c) A colouring $V \to${•,•,•,•}

Figure 2.7: Edge-colourings of a graph

Like for vertices, we will be looking at *proper* edge-colourings of graphs. A proper edge-colouring of a graph will be one where no two edges that share a vertex have the same colour. So Fig. 2.7c is an example of a proper edge-colouring of $G$.

It is natural to ask the same types of questions for edge-colourings, as vertex-colourings. We skip the formulation of these questions and just give the results.

**Proposition 2.3.4.** *$\boldsymbol{R}$ is not $k$-edge-colourable for any $k \in \mathbb{N}$, i.e. it is not possible to properly colour the edges of $\boldsymbol{R}$ using at most $k$ colours.*

*Proof.* Let $U = \{u\}$ and $V = \{v\}$. From Proposition 2.1.1 there are infinitely many witnesses to EP for $U$ and $V$ and for each of these witnesses, the edge connecting $u$ and the witness will need to be a different colour. Hence we need infinitely many colours. □

Now for the monochromatic configurations (configurations with all the edges coloured with the same colour), we use Ramsey theory to show that we have a monochromatic complete graph on $m$ vertices, denoted by $K_m$, for each $m \in \mathbb{N}$.

Ramsey's Theorem (see [Bru10], Theorem 3.3.1, pg. 78)

For all $c, m \geq 2$ there is an $n \geq m$ such that every $c$-edge-colouring of $K_n$ has a monochromatic copy of $K_m$.

Fix an $m \in \mathbb{N}$ with $m \geq 2$. Since $2, m \geq 2$ we now have from Ramsey's Theorem that there is an $n \geq m$ such that every 2-edge-colouring of $K_n$ has a monochromatic copy of $K_m$. In particular we can find, for this $n$, a copy of $K_n$ in $\boldsymbol{R}$ with a specific 2-edge-colouring. And so we will also have a monochromatic copy of $K_m$ in $\boldsymbol{R}$. Since each finite graph on $m$ vertices is an induced subraph of $K_m$, and this holds for each $m \in \mathbb{N}$, we have a monochromatic copy of every finite graph in $\boldsymbol{R}$.

If we consider a colouring of the edges of $\boldsymbol{R}$ with finitely many colours, there need not be a monochromatic copy of $\boldsymbol{R}$ ([EHP75]), but in [PS96] it is shown that there is a copy of $\boldsymbol{R}$ containing at most two of the colours.

The next proposition sums up this discussion.

**Proposition 2.3.5.** *1. (see [PS96], pg. 509) Any $k$-edge-colouring of $\boldsymbol{R}$ contains a copy of $\boldsymbol{R}$ of which the edges are coloured using at most two of the colours, as an induced subgraph.*

2. *Given a finite graph $G$, there is a $k$-edge-colouring of $\boldsymbol{R}$ that contains a monochromatic copy of $G$ as an induced subgraph.*

## 2.4 The never ending story: The non-fractal nature of the Rado graph

> If we find ourselves with a desire that nothing in this world can satisfy, the most probable explanation is that we were made for another world.
>
> C.S. Lewis

Fractals occur everywhere in nature. Yes, even here in the grasslands which we are exploring we will see fractals. By now we know $\boldsymbol{R}$ quite well. We have seen that it is self-similar in nature, and therefore expect it to be fractal. We will show that it is not. We will then define what it means for a graph to be self-similar in order to try and retain this aspect of $\boldsymbol{R}$. Unfortunately, we will see that $\boldsymbol{R}$ is also not self-similar, at least not in the sense that we want it to be.

We need the idea of a *lexicographic product* of graphs. Let $G$ and $H$ be any graphs. Then the lexicographic product of $H$ by $G$, denoted $H \circ G$, is the graph with vertex set $V(H) \times V(G)$ and vertices $(h_1, g_1)$ and $(h_2, g_2)$ are connected iff either

1. $h_1 \sim h_2$ or
2. $h_1 = h_2$ and $g_1 \sim g_2$.

In layman's terms, when taking the lexicographic product of $H$ by $G$ we replace each vertex in $H$ with a copy of $G$ and we add all the possible edges between two copies exactly when the vertices of $H$ were connected. Here we give an illustration.

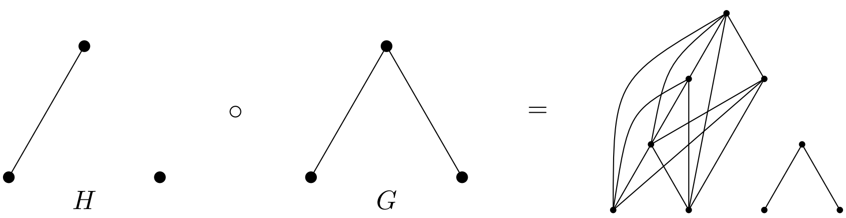



Figure 2.8: The lexicographic product of $H$ by $G$

**Definition 2.4.1** (**Fractal graph**, see [IW19] pg. 53)**.** *A graph $G$ is* fractal *if $G \cong H \circ G$ for some graph $H$ having at least two vertices.*

Let's see if $\boldsymbol{R}$ is fractal. Let $H$ be any graph with at least two vertices and consider $H \circ \boldsymbol{R}$. For this product to be isomorphic to $\boldsymbol{R}$ it must be countably infinite and satisfy EP.

So we pick $H$ such that $H \circ \boldsymbol{R}$ is countably infinite. All we need to do now is check that EP is satisfied and then we will be able to strike again with the mighty Aficionado.

There are a few cases to consider when looking for a witness for sets $U$ and $V$ in this lexicographic product of $H$ by $\boldsymbol{R}$. At this point some excited members of the hunting party have already begun to consider the different cases. We will not be going down each of these paths. Instead we are only going to follow the one which is meaningful to our journey.

If two vertices of $H$ are connected, this means that the corresponding copies of $\boldsymbol{R}$ in $H \circ \boldsymbol{R}$ will be connected. So any $x$ from the one copy of $\boldsymbol{R}$ will be connected to all other vertices from the other copy of $\boldsymbol{R}$. In this case there are too many edges to find a witness for EP.

If two vertices of $H$ are not connected, then the corrsponding copies of $\boldsymbol{R}$ in $H \circ \boldsymbol{R}$ will not be connected. Now any $x$ from the one copy of $\boldsymbol{R}$ will have no edges connecting it to any vertex from the other copy of $\boldsymbol{R}$. In this case there are too few edges to find a witness for EP.

Let's formalise this discussion.

**Proposition 2.4.2.** *$\boldsymbol{R}$ is not a fractal graph.*

*Proof.* Let $H$ be countable. Then $H \circ \boldsymbol{R}$ will be countably infinite.

Given any two vertices $a$ and $b$ of $H$, let $R_1$ and $R_2$ be the copies of $\boldsymbol{R}$ with which we replace the two given vertices. Take $U$ to be the set containing $u_1 \in R_1$ and $u_2 \in R_2$ and take $V$ to be the set containing $v_1 \in R_1 \setminus \{u_1\}$ and $v_2 \in R_2 \setminus \{u_2\}$.

Suppose that $a \sim b$ in $H$: then $R_1$ is connected to $R_2$ meaning that every vertex of $R_1$ is connected to every vertex of $R_2$. If $x$ witnesses EP for $U$ and $V$ in $H \circ \boldsymbol{R}$ then $x \notin R_1$ otherwise $x \sim v_2$. Also, $x \notin R_2$ otherwise $x \sim v_1$. So $x$ must be in some other copy of $\boldsymbol{R}$. This copy needs to be connected to both $R_1$ and $R_2$ to ensure that $x \sim u_1$ and $x \sim u_2$. But then we will again have the problem that $x \sim v_1$ and $x \sim v_2$. In this case we won't be able to find a witness to EP.

Suppose now that $a \not\sim b$ in $H$: then there are no edges between $R_1$ and $R_2$. Let $x$ be the required witness, i.e. $x$ is connected to each vertex of $U$ and no vertex in $V$. If $x \in R_1$ then $x \not\sim u_2$, so this cannot be the case. If $x \in R_2$ then $x \not\sim u_1$, so this is also not the case. If $x$ is in any other copy of $R$ then we want this copy to be connected to both $R_1$ and $R_2$ so that $x \sim u_1$ and $x \sim u_2$. But then we also have that $x \sim v_1$ and $x \sim v_2$. Again we are not able to find a witness to EP.

This shows that EP is not satisfied in $H \circ \boldsymbol{R}$ and hence that $\boldsymbol{R}$ is not fractal. $\square$

Our enemy has managed to elude us once again. We pace up and down, thinking, how did we fall for this trap? Utterly convinced of its self-similar nature, and partly because we like using Aficionado, we try to find a new way of showing that $\boldsymbol{R}$ has fractal properties.

When looking at the proof for Proposition 2.4.2, we see that the original lexicographic product is creating too many edges for our witness. We try to get rid of these undesired edges by defining a weak lexicographical product. For graphs $G$ and $H$ the *weak* lexicographical product, $H \diamond G$, is the graph with vertex set $V(H) \times V(G)$ and vertices $(h_1, g_1)$ and $(h_2, g_2)$ are connected iff either

1. $h_1 \sim h_2$ and $g_1 \sim g_2$ or
2. $h_1 = h_2$ and $g_1 \sim g_2$.

We give an illustration to show how this differs from the lexicographic product.

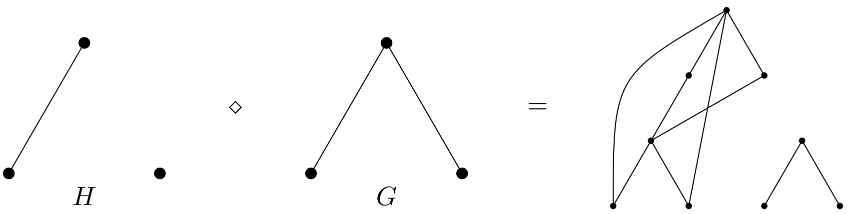



Figure 2.9: The weak lexicographic product of $H$ by $G$

We introduce the notion of a *self-similar* graph.

**Definition 2.4.3 (Self-similar graph).** *A graph $G$ is said to be self-similar if there is a graph $H$ with at least two vertices such that $G \cong H \diamond G$.*

We now have a hope to capture the self-similar nature of $\boldsymbol{R}$.

Let $H$ be countable so that $H \diamond \boldsymbol{R}$ is countable. To check if EP is satisfied there are many cases for $U$ and $V$ which we need to consider. We will only look at those cases which are meaningful.

Let $U = \{(a, r), (b, s)\}$ and $V = \{(b, r)\}$. Now, if $(x, y)$ witnesses EP then it is incident with both $(a, r)$ and $(b, s)$. From this, and the definition of $\diamond$, we are in one of the following cases.

1. Either $x = a$ and $y \sim r$ and $x = b$ and $y \sim s$ or
2. $x = a$ and $y \sim r$ and $x \sim b$ and $y \sim s$ or
3. $x \sim a$ and $y \sim r$ and $x = b$ and $y \sim s$ or
4. $x \sim a$ and $y \sim r$ and $x \sim b$ and $y \sim s$.

In cases 1 and 3 we have $x = b$ and $y \sim r$ which implies that $(x, y)$ is connected to $(b, r)$. In cases 2 and 4 we have $x \sim b$ and $y \sim r$, so $(x, y)$ is connected to $(b, r)$. This shows that there is no witness to EP for $U$ and $V$.

We thus have another woeful result.

**Proposition 2.4.4.** $\boldsymbol{R}$ *is not self-similar.*

And so the beast evades us again. Even with this weaker version of the lexicographical product, the conditions under which we connect vertices are still too strict. An easy way to fix this would be to connect vertices $(h_1, g_1)$ and $(h_2, g_2)$ of the new graph independently with probability $\frac{1}{2}$ whenever $h_1 \sim h_2$, or if $h_1 = h_2$ and $g_1 \sim g_2$. We can then verify that EP is satisfied using probability, as we did in Chapter 1, when we verified that $\boldsymbol{R}$ satisfies EP. This means that if we take a countable graph $H$, and replace each vertex of $H$ with a copy of $\boldsymbol{R}$, and then randomly connect the vertices of the copies corresponding to the connected vertices in $H$, we get a graph which is isomorphic to $\boldsymbol{R}$. This process boils down to the way in which we constructed $\boldsymbol{R}$ to begin with, and so we don't really get anything new from this.

We conclude this section with the following idea. $\boldsymbol{R}$ is "fractal" in the sense that when we zoom in on it, we find copies of $\boldsymbol{R}$. This is demonstrated in Theorem 2.1.7. But $\boldsymbol{R}$ is not fractal (as defined in this section) in that when we zoom out, we don't find copies of $\boldsymbol{R}$.

This brings an end to our journey in the grasslands.

# Chapter 3

# The Algebraic Alps

Our fate lives within us; you only have to be brave enough to see it.

MERIDA
*Brave*

We now enter the realm of algebra. Though there are many things we could possibly discuss, we will only focus on the automorphisms and ultrahomogeneity of $\boldsymbol{R}$.

## 3.1 Another weapon: The ultrahomogeneity of the Rado graph

Before reading any further it is important that we are in agreement about what automorphisms and isomorphisms are. Intuitively, isomorphisms are structure preserving mappings. So if a structure $\mathcal{M}$ has certain properties and gets mapped to $\mathcal{N}$ through an isomorphism, then $\mathcal{N}$ will also have these properties. Now an automorphism is an isomorphism from a structure to itself. In other words, an automorphism scrambles the elements of a structure without having any effect on the properties of the structure. Readers who wish to see a more detailed account of these mappings are referred to Appendix A.4.

We have seen in the previous chapter that $\boldsymbol{R}$ has a certain level of indestructibility. Internal changes do not affect the beast's appearance. In this section we explore these internal changes in more depth.

We introduce the following concept from model theory, stated in terms of graphs.

**Definition 3.1.1** (**Ultrahomogeneous**, see [Rot00] pg. 135)**.** *A graph $G$ is* ultrahomogeneous *if every isomorphism between finite induced subgraphs of $G$ can be extended to an automorphism of $G$.*

We can now state the following, somewhat unsurprising, result.

**Proposition 3.1.2** (see [Cam13], Proposition 9, pg. 7)**.** *$\boldsymbol{R}$ is ultrahomogeneous.*

We have sort of already seen the proof of this. Remember the back-forth-argument we used to prove Theorem 1.1.3? We use the same argument here, with a few minor tweaks.

*Proof.* Consider two enumerations of the vertices of $\boldsymbol{R}$, $\{a_1, a_2, \dots\}$ and $\{b_1, b_2, \dots\}$.

Let $f$ be any isomorphism between finite induced subgraphs $A$ and $B$ of $\boldsymbol{R}$, i.e. $f : A \cong B$ with $V(A) := \{a'_1, \dots, a'_{n-1}\}$ and $V(B) := \{b'_1, \dots, b'_{n-1}\}$ for some $n \in \mathbb{N}$ such that $f(a'_i) = b'_i$. We want to extend $f$ to an automorphism of $\boldsymbol{R}$.

<u>If $n$ is even:</u> let $a'_n$ be $a_j$ where $j$ is the smallest index such that $a_j \notin A$. Take $U := A \cap \{\text{Neighbours of } a_j\}$ and $V := A \cap \{\text{Non-neighbours of } a_j\}$. Clearly $U \cap V = \emptyset$ and so we will have that $f[U]$ and $f[V]$ are finite disjoint subsets of $\boldsymbol{R}$. Since $\boldsymbol{R}$ satisfies EP there is a $b \in \boldsymbol{R} \setminus (f[U] \cup f[V])$ such that $b \sim f[U]$ and $b \nsim f[V]$. So we extend $f$ by letting $f(a'_n) = b$.

<u>If $n$ is odd:</u> let $b'_n$ be $b_j$ with $j$ is the smallest index such that $b_j \notin B$. We take $U := B \cap \{\text{Neighbours of } b_j\}$ and $V := B \cap \{\text{Non-neighbours of } b_j\}$ and so $U \cap V = \emptyset$. It now follows that $f^{-1}[U]$ and $f^{-1}[V]$ are finite disjoint subsets of $\boldsymbol{R}$ and since $\boldsymbol{R}$ satisfies EP there exists $a \in \boldsymbol{R} \setminus (f^{-1}[U] \cup f^{-1}[V])$ such that $a \sim f^{-1}[U]$ and $a \nsim f^{-1}[V]$. So we extend $f$ by letting $f^{-1}(b'_n) = a$.

This alternating method of extending $f$ will exhaust $\boldsymbol{R}$ and so we end up with a bijective map $f : a_i \mapsto b_i$, which will be a homomorphism. This shows that $f : \boldsymbol{R} \cong \boldsymbol{R}$ and so we have the required automorphism. □

The ultrahomogeneity of $\boldsymbol{R}$ will prove to be a very useful weapon for our adventure. The following illustrates how powerful it can be.

**Definition 3.1.3** (**Vertex-transitive**, see [Die00] pg. 50)**.** *A graph, $G$, is* vertex-transitive *if for any $g_1, g_2 \in G$ there is an automorphism that maps $g_1$ to $g_2$.*

**Proposition 3.1.4.** *$\boldsymbol{R}$ is vertex-transitive.*

*Proof.* Let $a$ and $b$ be any two vertices of $\boldsymbol{R}$ and consider the induced subgraphs with vertex sets $\{a\}$ and $\{b\}$ respectively. It is easy to find an isomorphism between these two graphs. It now follows from Proposition 3.1.2 that this isomorphism can be extended to an automorphism of $\boldsymbol{R}$.

This can be done for any two vertices of $\boldsymbol{R}$, and so we have the result. □

## 3.2 Treasures in the mountainside: Some results about the automorphisms of the Rado graph

> We dig up diamonds by the score
> A thousand rubies, sometimes more
>
> DWARFS
> *Snow White and The Seven Dwarfs (movie)*

We have only scratched the surface of the mine in the previous section. We now delve deeper into the mountains and hope to find something valuable. We will state most of the results in this section without proof, since this is not our main focus, and we use [Pin10] as a guide for some basic concepts.

The most common example of a binary operation is perhaps addition. Intuitively we know that a binary operation takes two numbers of a set and combines them into a new number of the same set. We define this in terms of arbitrary sets.

**Definition 3.2.1** (**Binary operation**, see [Pin10] pg. 19)**.** *A* binary operation *on a set $A$ is a rule which assigns to every $(a, b) \in A \times A$ a unique element in $A$ which we denote as $a * b$.*

Binary operations may have various properties.

A binary operation on a set $G$ is called *associative* if

$$a * (b * c) = (a * b) * c$$

for all elements $a, b$ and $c$ from $G$. An element $e \in G$ such that

$$a * e = a = e * a$$

for all $a$ from $G$, is called an *identity element* with respect to $*$. In this case, an element $b \in G$ such that

$$a * b = e = b * a$$

is called an *inverse* of $a$ under $*$. Inverses, if they exist, are unique. We will write $a^{-1}$ to denote the inverse of $a$.

**Definition 3.2.2** (**Group**, see [Pin10] pg. 25)**.** *Let $G$ be a set and $*$ be a binary operation on $G$ such that*

1. *$*$ is associative,*
2. *there is an element of $G$ which is an identity element with respect to $*$ and,*
3. *every element of $G$ has an inverse in $G$ under $*$.*

*Then the pair $\langle G, * \rangle$ is a* group.

We will often leave out the operation when it is clear from the context and just write $G$ instead of $\langle G, * \rangle$.

Consider $\mathrm{Aut}(\boldsymbol{R})$, the set of all automorphism of $\boldsymbol{R}$. This set forms a group under function composition.

**Proposition 3.2.3** (see [Cam13], Proposition 13, pg. 12)**.** $|\mathrm{Aut}(\boldsymbol{R})| = 2^{\aleph_0}$.

**Remark 3.2.4.** The fact that there are continuum many automorphsims of $\boldsymbol{R}$ follows easily from the fact that $\boldsymbol{R}$ is countably infinite and resplendent (see Theorem 4.6.2), but we will only discuss resplendence in the next chapter.

We review some more algebra before going forward.

**Definition 3.2.5** (**Subgroup**, see [Pin10] pg. 44)**.** *Let $\langle G, * \rangle$ be a group and let $S$ be a nonempty subset of $G$. If both*

1. *$a * b \in S$ for all $a, b \in S$ ($S$ is closed w.r.t. $*$) and*
2. *$a^{-1} \in S$ for all $a \in S$ ($S$ is closed w.r.t. inverses under $*$),*

*then $\langle S, * \rangle$ is a* subgroup *of $\langle G, * \rangle$.*

Again, if the context is clear then we will just say that $S$ is a subgroup of $G$ without writing out the binary operation explicitly.

**Definition 3.2.6** (**Normal subgroup**, see [Pin10] pg. 140)**.** *Let $G$ be a group and $H$ be a subgroup of $G$. We say $H$ is a* normal *subgroup of $G$ if for any $a \in H$ and $b \in G$ we have that $b * a * b^{-1} \in H$.*

**Remark 3.2.7.** The element $b * a * b^{-1}$ is called the *conjugate* of $a$ and so we can say that a subgroup is normal if it is closed with respect to conjugates.

**Definition 3.2.8** (**Simple group**, see [Wil09] pg. 17)**.** *A group $G$ is called* simple *if the only normal subgroups of $G$ are the trivial group (consisting only of the identity element), and $G$ itself.*

**Proposition 3.2.9** (see [Cam13], Theorem 8, pg. 12)**.** $\mathrm{Aut}(\boldsymbol{R})$ *is simple.*

We conclude this section with the following nice result. The proof and other content needed for the proof are located too deep in the mine for our adventure to continue there.

**Proposition 3.2.10** (see [Cam05])**.** *Let $G$ be an ultrahomogeneous graph with $|G| < 2^{\aleph_0}$ and $\mathrm{Aut}(\boldsymbol{G}) \cong \mathrm{Aut}(\boldsymbol{R})$. Then $G \cong \boldsymbol{R}$.*

# Chapter 4

# The Marshes of Model Theory

I can't think of any better representation of beauty than someone who is unafraid to be herself.

EMMA STONE

Our journey has brought us to model theory. This will be the longest part of our adventure so far. Sometimes it will feel like we are walking on solid ground, other times we might feel like we are basically swimming through the wetlands, but so long as we don't get stuck in the mud and reeds, we will discover many secrets about our beast. Basic notions of model theory, as presented by [Rot00], can be found in Appendix A, and are stated where needed throughout this chapter.

## 4.1 Basic instincts: The first-order theory of the Rado graph

Consider a signature with no constant symbols, no function symbols and only the relation symbol $\sim$, as introduced in Chapter 1, meaning "is connected to". So $x \sim y$ reads $x$ is connected to $y$. It is quite obvious that $\boldsymbol{R}$ is a structure of this signature, or of the language $L(\sim)$. In due course we will see that our beast, studied as a structure, has many interesting properties.

The first thing we look at is the beast's most basic rules.

**Definition 4.1.1** (**Theory of**, see [Rot00] pg. 33)**.** *Let $\boldsymbol{K}$ be a class of $L$-structures. The $L$-theory, or just* theory of $\boldsymbol{K}$ *is the the set of $L$-sentences which are true in all nonempty structures from $\boldsymbol{K}$, and we denote this set by* $\mathrm{Th}(\boldsymbol{K})$.

We will write $\mathrm{Th}\mathcal{M}$ instead of $\mathrm{Th}(\{\mathcal{M}\})$.

In particular we have that $\mathrm{Th}\boldsymbol{R}$ is the set of all $L(\sim)$-sentences which are true in $\boldsymbol{R}$. So $\mathrm{Th}\boldsymbol{R}$ says everything about $\boldsymbol{R}$ that there is to say in terms of first-order sentences. We already know that there is a lot to say about $\boldsymbol{R}$, that's the whole point of this adventure, but here in model theory we wonder how we can say everything that needs to be said, using as few as possible words. Let's get some terminology out of the way.

**Definition 4.1.2** (**Consequence**, see [Rot00] pg. 28)**.** *An $L$-sentence $\varphi$ is a* consequence *of a set of $L$-sentences $\Sigma$ if every model of $\Sigma$ is also a model of $\varphi$. In this case we write $\Sigma \models \varphi$.*

**Definition 4.1.3** (**Deductive closure**, see [Rot00] pg. 29)**.** *The* deductive closure *of a set of $L$-sentences $\Sigma$ is the set $\Sigma^{\models}$ consisting of all consequences of $\Sigma$.*

*In case* $\Sigma = \Sigma^{\models}$ *we say that* $\Sigma$ *is* deductively closed.

**Definition 4.1.4** (**Contradiction**, see [Rot00] pg. 31)**.** *An $L$-sentence of the form $\varphi \wedge \neg\varphi$ is called a* contradiction.

**Definition 4.1.5** (**Consistency**, see [Rot00] pg.31)**.** *A set of $L$-sentences $\Sigma$ is* consistent *if $\Sigma^{\models}$ contains no contradictions. Otherwise $\Sigma$ is* inconsistent.

**Definition 4.1.6** (**$L$-theory**, see [Rot00] pg. 32)**.** *A deductively closed and consistent set of $L$-sentences is called an $L$-*theory.

You might be asking yourself if Th$\boldsymbol{R}$ is an $L$-theory in the sense of Definition 4.1.6. The answer is, yes. Let's check this. It follows easily enough from the definition of consequence that Th$\boldsymbol{R}$ is deductively closed. Consistency follows from the fact that $\boldsymbol{R}$ cannot model both $\varphi$ and $\neg\varphi$.

**Definition 4.1.7** (**Axiomatize**, see [Rot00] pg. 34)**.** *An $L$-theory $T$ is* axiomatized *by a set of $L$-sentences, $\Sigma$, if $\Sigma \subseteq T \subseteq \Sigma^{\models}$.*

**Remark 4.1.8.** $\Sigma$ axiomatizes $T$ exactly when the class of models of $\Sigma$ and the class of models of $T$ are the same, or equivalently, when $\Sigma^{\models} = T$.

We want to axiomatize Th$\boldsymbol{R}$, thereby "summing up" the theory of $\boldsymbol{R}$. There are, in effect, two major points that we need to cover. We need to say that $\boldsymbol{R}$ "is countably infinite" and "satisfies EP". This guides us as to what $L(\sim)$-sentences we need to include in our axiomatization.

Let's get the technical detail out of the way first. We need

$$(\forall x, y)((x \sim y) \rightarrow (x \neq y))$$

to say that $\sim$ is not reflexive, i.e. there are no loops. Then we need

$$(\forall x, y)((x \sim y) \rightarrow (y \sim x))$$

for the symmetry of $\sim$. Now for the countably infinite part we need, for each $n \in \mathbb{N}$, a sentence

$$\xi_n := (\exists x_1, \ldots, x_n)\left(\bigwedge_{i \neq j} x_i \neq x_j\right)$$

saying that there are at least $n$ distinct points. And finally we need, for each $m, n \in \mathbb{N}$, an extension axiom

$$\varphi_{m,n} := (\forall x_1, \ldots, x_m, y_1, \ldots, y_n)\left(\left(\bigwedge_{\forall i, \forall j} x_i \neq y_j\right) \rightarrow (\exists z)\left(\left(\bigwedge_{\forall i} z \sim x_i\right) \wedge \left(\bigwedge_{\forall j} z \neq y_j \wedge z \not\sim y_j\right)\right)\right)$$

which ensures that EP is satisfied.

We collect all these axioms in the set $\Sigma_{\boldsymbol{R}}$. The question now is, can we get away with fewer axioms? Can we possibly use only a finite number of axioms?

Well, we can do away with the $\xi_n$'s and just use $(\exists x, y)(x \neq y)$. This means that there are at least two distinct vertices. The rest is taken care of by the extension axioms. To see this, take $\varphi_{1,1}$, which implies the existence a $z$ different from both $x$ and $y$. So we have at least three distinct vertices. Now take $\varphi_{2,1}$, which implies the existence of a $z'$, different from $x, y$ and $z$, i.e. we have at least four distinct vertices. We can continue in this way to find at least $n$ vertices for each $n \in \mathbb{N}$, giving a countably infinite amount of vertices.

All that's left for us to possibly get rid of now is some of the extension axioms. Suppose that we could get away with only using finitely many of the $\varphi_{m,n}$'s. In this case we will be able to find the largest indices, say $n_1$ and $m_1$, for which $\varphi_{m_1,n_1}$ is part of our axiomatization. But then we have no guarantee that EP will be satisfied for sets of cardinality $n_1+1$ and $m_1+1$ respectively. It is entirely possible to find a model of $\Sigma_{\boldsymbol{R}} \setminus \{\varphi_{m,n} : m > m_1, n > n_1\}$ that does not model $\varphi_{m_1,n_1}$. The details of this will become clear in Section 5.2. So excluding some of our extension axioms won't do the trick.

The question remains, can we axiomatize Th$\boldsymbol{R}$ with finitely many $L(\sim)$-sentences? You might be thinking, didn't $\Sigma_{\boldsymbol{R}}$ and the fact that we need all the $\varphi_{m,n}$'s answer this question already? Technically, yes, but we need to do a bit more to back this argument. There might be another axiomatization of Th$\boldsymbol{R}$, different to $\Sigma_{\boldsymbol{R}}$, consisting of only finitely many axioms.

We now give the necessary results to see that this is not the case.

**Theorem 4.1.9** (**Compactness Theorem**[1], see [Hod93], Theorem 6.1.1, pg. 265)**.** *Let $T$ be an $L$-theory. $T$ has a model iff every finite subset of $T$ has a model.*

**Corollary 4.1.10** (see [Rot00], Corollary 4.3.3, pg. 47)**.** *Let $\Sigma$ be a set of $L$-sentences. Every consequence of $\Sigma$ is a consequence of some finite subset of $\Sigma$.*

**Definition 4.1.11** (**Finitely axiomatizable**, see [Rot00] pg. 34)**.** *An $L$-theory is* finitely axiomatizable *if it can be axiomatized by a finite set of $L$-sentences.*

**Proposition 4.1.12** (see [Rot00], Exercise 4.3.2, pg. 47)**.** *If a set of $L$-sentences $\Sigma$, axiomatizes a finitely axiomatizable $L$-theory $T$, then $T$ is axiomatized by a finite subset of $\Sigma$.*

*Proof.* Suppose that $T$ is axiomatized by $\Sigma$ and a finite set $\Delta$.

Let $\varphi$ be the conjunction of all the sentences from $\Delta$, and $\mathcal{M}$ be any model of $\Sigma$. Then $\mathcal{M}$ is a model of $T$ which implies that $\mathcal{M}$ would also be a model of $\Delta$ and hence a model of $\varphi$. This shows that every model of $\Sigma$ is also a model of $\varphi$ and we have that $\Sigma \models \varphi$. It now follows from Corollary 4.1.10 that there is some finite subset of $\Sigma$, say $A$, such that $A \models \varphi$. We will show that $A$ and $T$ have the same models, i.e. that $A$ axiomatizes $T$.

To see that all models of $A$ are models of $T$, let $\mathcal{M}$ be any model of $A$. Then $\mathcal{M}$ is a model of $\varphi$ and hence a model of $\Delta$. So $\mathcal{M}$ is a model of $T$.

Next we show that every model of $T$ is a model of $A$. Suppose on the contrary that there is an $\mathcal{M} \models T$ such that $\mathcal{M} \not\models A$. Then there is some sentence $\psi \in A$ such that $\mathcal{M} \not\models \psi$. Since $A \subseteq \Sigma$ we now have $\psi \in \Sigma$ with $\mathcal{M} \not\models \psi$, so $\mathcal{M} \not\models \Sigma$. This is a contradiction since $\mathcal{M} \models T$ and $\Sigma$ axiomatizes $T$.

This shows that $A$ and $T$ have the same models. In conclusion $A$ axiomatizes $T$ and $A$ is a finite subset of $\Sigma$, as required. $\square$

This then answers our question. No, we cannot axiomatize Th$\boldsymbol{R}$ with finitely many $L(\sim)$-sentences. If it were possible, then according to Proposition 4.1.12, we would have been able to do so with a finite subset of $\Sigma_{\boldsymbol{R}}$, but we know having only finitely many of the $\varphi_{m,n}$'s won't do. We bring this discussion together with the following result.

**Proposition 4.1.13.** Th$\boldsymbol{R}$ *is not finitely axiomatizable.*

We might not be able to cut down on the number of axioms we use, but maybe we can say something about the form they take on.

Every $L$-formula $\varphi$, is logically equivalent to a formula in the same free variables of the from $Q_1x_1 \ldots Q_nx_n\psi$, where the $Q_i$'s are quantifiers, i.e. $\forall$ or $\exists$, and $\psi$ is quantifier free. If $\forall$ does not occur in $\psi$ then $\varphi$ is called an

[1] This is sometimes called the Finiteness Theorem.

*existential* or $\exists$-formula. If $\exists$ does not occur in $\psi$ then $\varphi$ is called an *universal* or $\forall$-formula. We denote the classes of such formulas by $\exists$ and $\forall$ respectively.

Now, for an arbitrary set of $L$-formulas, $\Delta$, we say that an $L$-theory $T$ is a $\Delta$-theory if it can be axiomatized using only sentences from $\Delta$.

We investigate whether or not Th$\boldsymbol{R}$ is an $\forall$-theory or an $\exists$-theory. The easiest way to check this is to use preservation theorems. We remind ourselves of some needed model theoretic concepts.

We say that an $L$-theory, $T$, is *preserved in substructures* if $\mathcal{M} \models T$ implies $\mathcal{N} \models T$ for all nonempty $L$-structures $\mathcal{N} \subseteq \mathcal{M}$. If $\mathcal{N} \models T$ implies $\mathcal{M} \models T$ for all nonempty $L$-structures $\mathcal{N} \subseteq \mathcal{M}$ then we say that $T$ is *preserved in extensions*.

**Theorem 4.1.14** (**Łoś-Tarski Preservation Theorem**, see [Rot00] pg. 74)**.** *A theory is an $\forall$-theory iff it is preserved in substructures.*

**Theorem 4.1.15** (**Łoś Preservation Theorem**, see [Rot00] pg. 75)**.** *A theory is an $\exists$-theory iff it is preserved in extensions.*

Let's check now if Th$\boldsymbol{R}$ is an $\forall$-theory or an $\exists$-theory.

We know from Theorem 2.1.6 that any finite graph is a substructure of $\boldsymbol{R}$. Let $\mathcal{M} \subseteq \boldsymbol{R}$ with $|\mathcal{M}| = n$. Then the sentence, say $\xi_{n+1} \in$ Th$\boldsymbol{R}$, expressing that there are at least $n+1$ elements will not be satisfied in $\mathcal{M}$, i.e. we have $\boldsymbol{R} \models \xi_{n+1}$ but $\mathcal{M} \not\models \xi_{n+1}$. This shows that Th$\boldsymbol{R}$ is not preserved in substructures and hence is not an $\forall$-theory.

Consider the graph, $\mathcal{N}$, resulting from adding a vertex and no edges to $\boldsymbol{R}$. Clearly $\boldsymbol{R} \subseteq \mathcal{N}$ and $\mathcal{N}$ will not satisfy EP. This means that the $\varphi_{m,n}$'s are not satisfied in $\mathcal{N}$ and so we have $\boldsymbol{R} \models \varphi_{m,n}$ but $\mathcal{N} \not\models \varphi_{m,n}$ for each $m, n$. This shows that Th$\boldsymbol{R}$ is not preserved in extensions and hence is not an $\exists$-theory.

This gives us the following result.

**Proposition 4.1.16.** Th$\boldsymbol{R}$ *is neither an $\forall$-theory nor an $\exists$-theory.*

The beast's theory is as elusive as the beast. We might not have been able to finitely axiomatize Th$\boldsymbol{R}$ or axiomatize Th$\boldsymbol{R}$ using only $\exists$- or $\forall$-sentences, but we can still say other things about Th$\boldsymbol{R}$. We do this in the next section.

## 4.2 Putting an end to quantifiers: Quantifier elimination and the implications thereof

> To know your enemy, you must become your enemy.
>
> Sun Tzu

**Definition 4.2.1** (**Elimination**, see [Rot00] pg. 127)**.** *Let $\Delta$ be a set of $L$-formulas and $T$ an $L$-theory. Then $T$ admits* elimination *up to formulas in $\Delta$ if for every $L$-formula $\varphi$ there is $\delta \in \Delta$ such that $T \models \forall \bar{x}(\varphi \leftrightarrow \delta)$. In this case we say that $T$ has $\Delta$-elimination.*

When we speak of *quantifier-free* formulas, we mean exactly what we say, a formula without any quantifiers. We denote the class containing all such formulas by **qf**.

Note that any *boolean combination* of atomic formulas, i.e. formed using only $\neg$ and $\wedge$, will be a **qf** formula.

**Remark 4.2.2.** It might seem that $\wedge$ and $\neg$ are too few symbols to work with, but remember that

1. $\varphi \vee \psi \equiv \neg(\neg\varphi \wedge \neg\psi)$,
2. $\varphi \to \psi \equiv \neg\varphi \vee \psi$ and
3. $\varphi \leftrightarrow \psi \equiv (\varphi \to \psi) \wedge (\psi \to \varphi)$.

**Definition 4.2.3** (**Quantifier elimination**, see [Rot00] pg. 130)**.** *An $L$-theory has* quantifier elimination *if it admits elimination up to formulas in* **qf***.*

We want to show that Th$\boldsymbol{R}$ has quantifier elimination. This can be done directly via induction, but we opt for the more interesting indirect way. To this end we introduce some other nice properties of Th$\boldsymbol{R}$.

**Definition 4.2.4** (**Categorical**, see [Rot00] pg. 122)**.** *Let $\kappa$ be a cardinal. An $L$-theory $T$ is $\kappa$-*categorical *if it has, up to isomorphism, exactly one model of power $\kappa$.*

**Proposition 4.2.5.** Th$\boldsymbol{R}$ *is $\aleph_0$-categorical.*

We have actually seen this argument already, without labelling the outcome. We give the argument again.

*Proof.* Let $\mathcal{M}$ and $\mathcal{N}$ be any two countably infinite models of Th$\boldsymbol{R}$. They will both satisfy EP and so by Aficionado we have that both $\mathcal{M} \cong \boldsymbol{R}$ and $\mathcal{N} \cong \boldsymbol{R}$, hence $\mathcal{M} \cong \mathcal{N}$, which is the required result. □

**Definition 4.2.6** (**Complete theory**, see [Rot00] pg. 36)**.** *We call an $L$-theory $T$* complete *if for every $L$-sentence $\varphi$, either $\varphi \in T$ or $\neg\varphi \in T$.*

The next result follows from a more general statement about first-order theories in [Gai64] (see pg. 16).

**Theorem 4.2.7.** Th$\boldsymbol{R}$ *is complete.*

We can show that Th$\boldsymbol{R}$ is complete in a number of ways. We will discuss three proofs for the completeness of Th$\boldsymbol{R}$. We need the following results.

**Definition 4.2.8** (**Elementary substructure**, see [Rot00] pg. 115)**.** *Let $\mathcal{M}$ and $\mathcal{N}$ be $L$-structures and $\Delta$ be an arbitrary set of $L$-formulas. If $M \subseteq N$, and for all $\varphi \in \Delta \cap L$ and matching tuples $\bar{a}$ from $M$ we have that $\mathcal{N} \models \varphi(\bar{a})$ if and only if $\mathcal{M} \models \varphi(\bar{a})$, then we write $\mathcal{M} \preccurlyeq_\Delta \mathcal{N}$.*

*If $L = \Delta$ then we write $\mathcal{M} \preccurlyeq \mathcal{N}$ or $\mathcal{N} \succcurlyeq \mathcal{M}$ and say that $\mathcal{M}$ is an* elementary substructure *of $\mathcal{N}$ or $\mathcal{N}$ is an* elementary extension *of $\mathcal{M}$.*

**Theorem 4.2.9** (**Downward Löwenheim-Skolem Theorem**, see [Rot00], Theorem 8.4.1, pg. 119)**.** *Every infinite $L$-structure has an elementary substructure of power $\leq |L|$.*

*Proof 1 of Theorem 4.2.7.* We know that any model of Th$\boldsymbol{R}$ will have to satisfy each extension axiom and hence will be infinite.[2]

Suppose on the contrary that Th$\boldsymbol{R}$ is not complete. Then there is an $L$-sentence, say $\varphi$, and models $\mathcal{M}$ and $\mathcal{N}$ of Th$\boldsymbol{R}$ with $\mathcal{M} \models \varphi$ and $\mathcal{N} \models \neg\varphi$. Since both $\mathcal{M}$ and $\mathcal{N}$ are infinite, it follows from Theorem 4.2.9 that there are countable models $\mathcal{M}'$ and $\mathcal{N}'$ of Th$\boldsymbol{R}$ such that $\mathcal{M}' \models \varphi$ and $\mathcal{N}' \models \neg\varphi$. But from the $\aleph_0$-categoricity of Th$\boldsymbol{R}$ (Proposition 4.2.5) $\mathcal{M}'$ and $\mathcal{N}'$ must be isomorphic, a contradiction.

Hence Th$\boldsymbol{R}$ must be complete. □

The next way of proving that Th$\boldsymbol{R}$ is complete is similar to what we have just done.

**Theorem 4.2.10** (**Łoś-Vaught Test**, see [Rot00], Theorem 8.5.1, pg. 123)**.** *A categorical theory has no finite models iff it is complete.*

[2]This was discussed in the previous section.

The proof of this uses the Downward Löwenheim-Skolem Theorem. In effect, our first proof was just a specific case of this more general result.

*Proof 2 of Theorem 4.2.7.* Th$\boldsymbol{R}$ is categorical and has no finite models. Hence from Theorem 4.2.10 it is complete. □

Before we give the final proof of the completeness of Th$\boldsymbol{R}$, we return to the subject of quantifier elimination.

**Theorem 4.2.11** (see [Rot00], Exercise 9.2.4, pg. 136)**.** *Let $T$ be a $\kappa$-categorical and complete $L$-theory with $\kappa \geq |L|$. If the (up to isomorphism) model of $T$ of power $\kappa$ is ultrahomogeneous, then $T$ has quantifier elimination.*

We need some additional terminology and results to prove Theorem 4.2.11.

**Definition 4.2.12** (**Diagram**, see [Rot00] pg. 69)**.** *The* diagram *of an $L$-structure $\mathcal{M}$ is the set of all $L(M)$-literals that are also sentences and that are true in $\mathcal{M}$. We denote this set by $D(\mathcal{M})$.*

**Definition 4.2.13** (**Substructure-complete**, see [Rot00] pg. 131)**.** *Let $\boldsymbol{K}$ be the class of all $L$-structures. Then an $L$-theory $T$ is said to be* substructure-complete *if, for every $\mathcal{M} \models T$ and every $\mathcal{N} \subseteq \mathcal{M}$ with $\mathcal{N} \in \boldsymbol{K}$, the deductive closure of $T \cup D(\mathcal{N})$ is a complete $L(N)$-theory.*

**Theorem 4.2.14** (see [Rot00], Theorem 9.2.2, pg. 133)**.** *An $L$-theory admits quantifier elimination iff it is substructure-complete.*

**Theorem 4.2.15** (**Upward Löwenheim-Skolem Theorem**, see [Rot00], Theorem 8.4.3, pg. 120)**.** *Every infinite $L$-structure, $\mathcal{M}$, has an elementary extension of power $\geq |L| + |M|$.*

*Proof of Theorem 4.2.11.* Let $T$ be an $L$-theory which is $\kappa$-categorical and complete with $\kappa \geq |L|$. Let $\mathcal{M}$ and $\mathcal{N}$ be any two models of $T$ with a joint substructure $\mathcal{A}$. If we can show that $(\mathcal{M}, A) \equiv (\mathcal{N}, A)$, this implies that any two models of $T \cup D(\mathcal{A})$ are elementarily equivalent, so we will have that $T$ is substructure complete, by Proposition 4.2.19. Hence $T$ will also have quantifier elimination by Theorem 4.2.14.

Since $T$ is categorical and complete, it follows from the Łoś-Vaught Test (Theorem 4.2.10) that $T$ has no finite models. We therefore have that both $\mathcal{M}$ and $\mathcal{N}$ are infinite $L$-structures. In the case where $|\mathcal{M}| \geq \kappa$ we can use the Downward Löwenheim-Skolem Theorem (Theorem 4.2.9) to find $\mathcal{M}_0 \preccurlyeq \mathcal{M}$ with $|\mathcal{M}_0| = \kappa$. In the case where $|\mathcal{M}| \leq \kappa$ we can use teh Upward Löwenheim-Skolem Theorem (Theorem 4.2.15) to find $\mathcal{M} \preccurlyeq \mathcal{M}_0$ with $|\mathcal{M}_0| = \kappa$. In both these cases we will have that $(\mathcal{M}, A) \equiv (\mathcal{M}_0, A)$. Similarly, we can find $\mathcal{N}_0$ of power $\kappa$ which is an elementary substructure or extension of $\mathcal{N}$ and $(\mathcal{N}, A) \equiv (\mathcal{N}_0, A)$.

Now since $T$ is $\kappa$-categorical we have that $\mathcal{M}_0$ is isomorphic to $\mathcal{N}_0$ and we will have that $(\mathcal{M}_0, A) \equiv (\mathcal{N}_0, A)$ iff $(\mathcal{M}_0, A_0) \equiv (\mathcal{N}_0, A_0)$ for every finite $A_0 \subseteq A$. But we have that the model of $T$ of power $\kappa$ is ultrahomogeneous, i.e. $\mathcal{M}_0 \cong \mathcal{N}_0$ is ultrahomogeneous and so $(\mathcal{M}_0, A_0) \cong (\mathcal{N}_0, A_0)$, which implies the latter.

In conclusion, $(\mathcal{M}, A) \equiv (\mathcal{M}_0, A) \equiv (\mathcal{N}_0, A) \equiv (\mathcal{N}, A)$, which is the required result. □

Gaifman introduced a way to eliminate quantifiers (see [Gai64], pg. 17) from which we can obtain the following result. Our proof, however, does not follow that of Gaifman.

**Theorem 4.2.16.** Th$\boldsymbol{R}$ *admits quantifier elimination.*

*Proof.* We have from Theorem 4.2.7 and Proposition 4.2.5 that Th$\boldsymbol{R}$ is complete and $\aleph_0$-categorical. We also have from Proposition 3.1.2 that $\boldsymbol{R}$ is ultrahomogeneous, hence from Theorem 4.2.11 we have that Th$\boldsymbol{R}$ admits quantifier elimination. □

We are getting closer to the third proof that Th$\boldsymbol{R}$ is complete.

**Definition 4.2.17** (**Elementarily equivalent**, see [Rot00] pg. 69)**.** *Two $L$-structures $\mathcal{M}$ and $\mathcal{N}$ are* elementarily equivalent*, denoted $\mathcal{M} \equiv \mathcal{N}$, if $\mathcal{M} \models \varphi$ iff $\mathcal{N} \models \varphi$ for all $L$-sentences $\varphi$.*

*This is the same as saying that $\mathrm{Th}\mathcal{M} = \mathrm{Th}\mathcal{N}$.*

We use the following lemma without proof. Hunters who wish to see the detail are referred to [Rot00]'s guide.

**Lemma 4.2.18** (see [Rot00], Lemma 3.5.1, pg. 36)**.** *Let $T$ be any $L$-theory. The following are equivalent.*

1. *$T$ is complete.*
2. *$T = \mathrm{Th}\mathcal{M}$ for all $\mathcal{M} \models T$.*

**Proposition 4.2.19** (see [Rot00], Proposition 8.1.2, pg 112)**.** *An $L$-theory $T$ is complete iff all its models are elementarily equivalent.*

*Proof.* For the forward implication, let $T$ be a complete $L$-theory. Then for any $\mathcal{M} \models T$ we have by Lemma 4.2.18 that $T = \mathrm{Th}\mathcal{M}$. Hence for any two models $\mathcal{M}$ and $\mathcal{N}$ of $T$ we have $\mathrm{Th}\mathcal{M} = T = \mathrm{Th}\mathcal{N}$, which means that $\mathcal{M}$ and $\mathcal{N}$ are elementarily equivalent.

For the converse, suppose that $T$ is not complete. Then for any $\mathcal{M} \models T$ we have that $T \subsetneq \mathrm{Th}\mathcal{M}$, so there is a $\varphi \in \mathrm{Th}\mathcal{M} \setminus T$. Then $T \not\models \varphi$ and $T \cup \{\neg\varphi\}$ will be consistent, and hence have a model, say $\mathcal{N}$ of $T$ such that $\mathcal{N} \models \neg\varphi$. So we have two models of $T$ which are not elementarily equivalent.

This proves the result. $\square$

*Proof 3 of Theorem 4.2.7.* Since our language, $L(\sim)$, is a language without constants, one can easily verify that we cannot form any atomic sentences. The only atomic, or quantifier free sentences, by convention, are then $\perp$ and $\top$. Since quantifier free sentences are just boolean combinations of atomic sentences, this means that all quantifier free sentences are logically equivalent to $\perp$ or $\top$. But $\top$ is true in every structure and $\perp$ is true in none.

Now since $\mathrm{Th}\boldsymbol{R}$ admits quantifier elimination, this means that every $L(\sim)$-sentence is $\mathrm{Th}\boldsymbol{R}$-equivalent to a quantifier free sentence which, by the argument above, is logically equivalent to $\perp$ or $\top$. So for any two models, $\mathcal{M}$ and $\mathcal{N}$ of $\mathrm{Th}\boldsymbol{R}$ and any $L(\sim)$-sentence $\varphi$, we have that $\mathcal{M} \models \varphi$ iff $\mathcal{N} \models \varphi$, i.e. $\mathcal{M} \equiv \mathcal{N}$. Hence by Proposition 4.2.19 $\mathrm{Th}\boldsymbol{R}$ is complete. $\square$

This last argument may seem a bit circular, as we originally used the completeness of $\mathrm{Th}\boldsymbol{R}$ to show that it admits quantifier elimination. But this can also be proved directly as mentioned, via induction, so there is no danger of being circular.

We mention one more thing about $\mathrm{Th}\boldsymbol{R}$.

**Definition 4.2.20** (**Decidable**, see [EFT94] pg. 145)**.** *Given an $L$-theory $T$, and an $L$-sentence $\varphi$, if there is an algorithm*[3] *that determines whether or not $\varphi$ belongs to $T$, then $T$ is* decidable*.*

**Theorem 4.2.21** (see [EFT94], Theorem 6.5, pg. 166)**.** *Every complete axiomatizable $L$-theory is decidable.*

**Theorem 4.2.22.** *$\mathrm{Th}\boldsymbol{R}$ is decidable.*

*Proof.* This follows from Theorem 4.2.21 and the fact that $\mathrm{Th}\boldsymbol{R}$ is axiomatizable (discussed in the previous section) and complete (Theorem 4.2.7). $\square$

[3] Refer to Appendix B for more on this.

## 4.3 Rado's type: The $n$-types of the Rado graph

In this section we will discuss the types of $\boldsymbol{R}$, and by types we do not mean love interests. This is probably obvious; that we will be discussing mathematical concepts instead of love, but at some point of a mathematician's life the line between the two does get a bit blurry. One can easily go on about how it first caught your attention and how it drew you in closer with each satisfying solution or each rigorous argument and don't let me mention the marvellous proofs ... Let's not get distracted. We introduce some concepts.

**Definition 4.3.1** (**$n$-type**, see [Rot00] pg. 166)**.** *Let $\mathcal{M}$ be an $L$-structure and $\bar{x}$ and arbitrary $n$-tuple of variables. A set $\Phi(\bar{x}) \subseteq L(M)$ is an* $n$-type *of $\mathcal{M}$ if it is simultaneously satisfied by an $n$-tuple $\bar{c}$ for some $\mathcal{N} \succcurlyeq \mathcal{M}$, that is $\bar{c} \in N^n$ and $\mathcal{N} \models \varphi(\bar{c})$ for all $\varphi \in \Phi$.*

*In this case we call $\bar{c}$ a* realization *of $\Phi$ in $\mathcal{N}$ and we may write $\bar{c} \models_{\mathcal{N}} \Phi$. We also say that $\Phi$ is* realized *in $\mathcal{N}$.*

If all the parameters of $\Phi$ are contained in a set $A \subseteq M$, then we say that $\Phi$ is an $n$-type of $\mathcal{M}$ *over* $A$. We will just say *type* if the arity need not be specified.

We give an example of this concept.

**Example 4.3.2** (see [Rot00] pg. 167)**.** Let $\mathcal{M}$ be an infinite structure. Then $\{x \neq a : a \in M\}$ is a 1-type of $\mathcal{M}$ which is realized in $\mathcal{N} \succcurlyeq \mathcal{M}$ by those elements which are not in $M$. We can guarantee the existence of such an $\mathcal{N}$ by the Upward Łöwenheim-Skolem Theorem (Theorem 4.2.15).

We can now, if we wanted to, talk about the types of $\boldsymbol{R}$, but love is complicated so we avoid this conversation until we have a bit more detail and experience. We will say this however, types describe elements in a way similar to that in which theories describe models.

**Definition 4.3.3** (**Complete type**, see [Rot00] pg. 169)**.** *Let $\mathcal{M}$ be an $L$-structure, $\bar{x}$ be an arbitrary $n$-tuple of variables and $A \subseteq M$. An $n$-type $\Phi$ of $\mathcal{M}$ over $A$ is* complete *if either $\varphi \in \Phi$ or $\neg\varphi \in \Phi$ for all $\varphi(\bar{x}) \in L(A)$.*

*Given an $n$-tuple $\bar{a}$ from $M$ the set $tp^{\mathcal{M}}(\bar{a}/A) := \{\varphi(\bar{x}) \in L(A) : \mathcal{M} \models \varphi(\bar{a})\}$ is the complete type of $\bar{a}$ over $A$ in $\mathcal{M}$.*

If it is clear from the context what structure $\mathcal{M}$ we are considering, we will omit the superscript $^{\mathcal{M}}$. Also, we will write $\mathrm{tp}(\bar{a})$ in case $A = \emptyset$.

Every $n$-type of $\mathcal{M}$ over $A \subseteq M$ is contained in a complete $n$-type of $\mathcal{M}$ over $A$. To see this we just realize the type by a tuple $\bar{a}$ in some $\mathcal{N} \succcurlyeq \mathcal{M}$ and take $\mathrm{tp}^{\mathcal{N}}(\bar{a}/A)$, which will also be a type of $\mathcal{M}$. This means that every complete type is of this form.

**Definition 4.3.4** (**Isolated type**, see [Rot00] pg. 175)**.** *Let $\mathcal{M}$ be an $L$-structure, $\bar{x}$ be an arbitrary $n$-tuple of variables and $A \subseteq M$. An $n$-type $\Phi$ of $\mathcal{M}$ is* isolated *over $A$ (respectively* principal*) if there is $\varphi(\bar{x}) \in L(A)$ (respectively $\varphi \in \Phi$) satisfiable in $\mathcal{M}$ such that $\mathcal{M} \models \forall\bar{x}(\varphi \to \psi)$ for all $\psi \in \Phi$.*

*In this case $\varphi$ is said to* isolate *$\Phi$ and we write $\varphi \leq_{\mathcal{M}} \Phi$.*

**Remark 4.3.5.** Note that every isolated type of $\mathcal{M}$ is realised in $\mathcal{M}$, since $\mathcal{M} \models \exists\bar{x}\varphi$ and $\varphi \leq_{\mathcal{M}} \Phi$ (i.e. $\mathcal{M} \models \forall\bar{x}(\varphi \to \psi)$ for all $\psi \in \Phi$) gives a realization of $\Phi$ in $\mathcal{M}$.

This implies that $\boldsymbol{R}$ will realize all of its isolated types. But what other types do $\boldsymbol{R}$ realise? We don't expect $\boldsymbol{R}$ to realize all of its types, since $\{x \neq a : a \in \boldsymbol{R}\}$ (like in Example 4.3.2) will not be realized in $\boldsymbol{R}$, but maybe we can put some restriction on the amount of parameters we use. This brings us to the next definition.

**Definition 4.3.6** (**Saturated**, see [Rot00] pg. 185)**.** *Let $\kappa$ be an infinite cardinal and $\mathcal{M}$ an $L$-structure. If $\mathcal{M}$ realizes all of its $n$-types, for each $n$, over $A \subseteq M$ with $|A| < \kappa$, then $\mathcal{M}$ is* $\kappa$-saturated.

*We just say* saturated *if $\mathcal{M}$ is $|\mathcal{M}|$-saturated.*

From this we gather that $\boldsymbol{R}$ will be saturated if it realizes all of its $n$-types in finitely many parameters, for each $n$.

For an $L$-theory $T$, we define a type of $T$ to just be a type (in the sense mentioned above) of some model of $T$. We can then use the following result to show that $\boldsymbol{R}$ is saturated.

**Proposition 4.3.7** (see [Rot00] pg. 167)**.** *Let $\mathcal{M} \models T$. Then every type of $\mathcal{M}$ over $A \subseteq M$ can be realised in some model of $T$ with power $\leq |A| + |L|$.*

**Proposition 4.3.8.** *$\boldsymbol{R}$ is saturated.*

*Proof.* Clearly any type of $\boldsymbol{R}$ will also be a type of Th$\boldsymbol{R}$. Now, let $A$ be any finite set of parameters from $\boldsymbol{R}$. Then by Proposition 4.3.7 every $n$-type, say $\Phi$ of $\boldsymbol{R}$ can be realized in some countably infinite model, say $\mathcal{M}$ of Th$\boldsymbol{R}$. That is, there is an $n$-tuple $\bar{c}$ from $\mathcal{M}$ such that $\mathcal{M} \models \varphi(\bar{c})$ for each $\varphi \in \Phi$. But from the $\aleph_0$-categoricity of Th$\boldsymbol{R}$ (Proposition 4.2.5) it follows that $f : \mathcal{M} \cong \boldsymbol{R}$, and since $\mathcal{M} \models \varphi(\bar{c})$ for each $\varphi \in \Phi$ we have that $\boldsymbol{R} \models \varphi(f[\bar{c}])$ for each $\varphi \in \Phi$. In other words, $\Phi$ is realized in $\boldsymbol{R}$.

This holds for all $n$-types over a finite set of parameters, and we can therefore say that $\boldsymbol{R}$ is saturated. □

Now that we know $\boldsymbol{R}$ realizes all of its $n$-types (particularly all its complete $n$-types) in finitely many parameters, we might ask ourselves, how much is this "all"?

To answer this question we need the following result.

**Theorem 4.3.9** (**Homogeneity**, see [Rot00], Theorem 12.1.1, pg. 186)**.** *Let $\mathcal{M}$ be a countably infinite saturated structure and $A \subseteq M$ finite, and $\bar{a}$ and $\bar{b}$ tuples of the same length from $M$. Then $tp^{\mathcal{M}}(\bar{a}/A) = tp^{\mathcal{M}}(\bar{b}/A)$ iff there is an automorphism $f$ of $\mathcal{M}$ such that $f\restriction_A = id_A$ and $f[\bar{a}] = \bar{b}$.*

Since $\boldsymbol{R}$ is countably infinite and saturated we have, from Theorem 4.3.9 that $\mathrm{tp}^{\boldsymbol{R}}(\bar{a}) = \mathrm{tp}^{\boldsymbol{R}}(\bar{b})$ iff there is an $f \in \mathrm{Aut}\boldsymbol{R}$ such that $f(\bar{a}) = \bar{b}$. Note that if we restrict $f$ to $\bar{a}$ this will give rise to an isomorphism between the induced subgraphs of $\bar{a}$ and $\bar{b}$ in $\boldsymbol{R}$. The reverse is also true. From the fact that $\boldsymbol{R}$ is ultrahomogeneous we are able to extend any isomorphism between the subgraps induced by $\bar{a}$ and $\bar{b}$ to an automorphism of $\boldsymbol{R}$ which in turn will ensure the equality of the complete types $\mathrm{tp}^{\boldsymbol{R}}(\bar{a}) = \mathrm{tp}^{\boldsymbol{R}}(\bar{b})$.

So the number of complete $n$-types will be exactly the number of graphs up to isomorphism on $n$ vertices. In other words the number of complete $n$-types depends upon how the vertices in the $n$-tuple are related to one another. This is a good illustration of the fact that types describe elements in the same way that theories describe models.

## 4.4 Defining moments: Classifying the definable subgraphs of the Rado graph

> There is no king who has not had a slave amongst his ancestors, and no slave who has not had a king amongst his.
>
> Helen Keller

In this section we look at what structures we can get from $\boldsymbol{R}$. We have already seen a bit of this in Chapter 2 in that every countable and every finite graph is an induced subgraph of $\boldsymbol{R}$. Now we will look at it from a model theoretic point of view. As usual, we introduce some terminology first.

**Definition 4.4.1** (**Definable set**, see [Rot00] pg. 26)**.** *Let $\mathcal{M}$ be an $L$-structure and $\varphi$ be an $L$-formula in $n$ free variables. The set*

$$\varphi(\mathcal{M}) := \{\bar{a} \in M^n : \mathcal{M} \models \varphi(\bar{a})\}$$

*is the set* defined *by $\varphi$ in $\mathcal{M}$ or the* solution set *of $\varphi$ in $\mathcal{M}$.*

*A set $A \subseteq M^n$ is* definable *in $\mathcal{M}$ if $A = \varphi(\mathcal{M})$ for some $\varphi$.*

**Definition 4.4.2** (**Parametrically definable**, see [Rot00] pg. 27)**.** *Let $\varphi(\bar{x}, \bar{y})$ be an $L$-formula where $\bar{x}$ is an $n$-tuple and $\bar{y}$ an $m$-tuple of variables with no variables in common. For $\bar{c} \in M^m$, the formula $\varphi(\bar{x}, \bar{c})$ is an* instance *of $\varphi(\bar{x}, \bar{y})$. The set*

$$\varphi(\mathcal{M}, \bar{c}) := \{\bar{a} \in M^n : \mathcal{M} \models \varphi(\bar{a}, \bar{c})\}$$

*is the set defined by $\varphi(\bar{x}, \bar{c})$ in $\mathcal{M}$ and we call $\bar{c}$ the* parameter *tuple.*

*A set $A \subseteq M^n$ is* parametrically definable *if $A = \varphi(\mathcal{M}, \bar{c})$ for some $\bar{c} \in M^m$ and some $\varphi(\bar{x}, \bar{y})$.*

We can now ask two questions. Firstly, what sets can we define in $\boldsymbol{R}$ without any parameters? And what sets can we define in $\boldsymbol{R}$ with parameters?

The first, most obvious sets we can define are the empty set, by using the formula $x \neq x$ and $\boldsymbol{R}$ itself, using $x = x$.

Defining the most obvious sets are not the aim of the game, so let's see if we can do something more interesting. Let's consider the formula $x \sim y$. Then we get the set $\{(x, y) : x \sim y \text{ in } \boldsymbol{R}\}$, where $(x, y)$ are pairs of vertices of $\boldsymbol{R}$. This set is, as it says, just a set and not a structure. We might use it as the universe of a structure, but for now we will leave it as it is. If we view the ordered pair as a sequence of vertices, this set can be described as all the walks of length 1 in $\boldsymbol{R}$.

Now let's look at $x \sim y \wedge y \sim z$. In this case we get a set of ordered triples $(x, y, z)$ of vertices from $\boldsymbol{R}$, which we can again view as a sequence of vertices to get all the walks of length 2 in $\boldsymbol{R}$. In a similar fashion, we can use $\varphi := x_0 \sim x_1 \wedge \cdots \wedge x_{n-1} \sim x_n$ to define the set containing all walks of length $n$. Considering $\varphi \wedge x_0 \neq x_n$ we get the set containing all open walks and $\varphi \wedge x_0 = x_n$ gives all closed walks. We can also take $\varphi \wedge \bigwedge_{i \neq j} x_i \neq x_j$ to get all paths of length $n$. As interesting goes, this does not quite hit the mark.

**Remark 4.4.3.** Note that whenever we use a formula in $n$ free variables to define a set in $\boldsymbol{R}$, then this set will contain ordered $n$-tuples. To turn this set into a graph we will first have to define a relation. We are not interested in doing this. For this reason we will from now on only focus on what we can define with formulas in only one free variable.

Before we continue to the case with parameters, let's make it clear what we are looking for. We want to define sets which, given the existing relation on $\boldsymbol{R}$, are induced subgraphs of $\boldsymbol{R}$. As per the above remark we therefore only need to consider $L(\sim)$-formulas in one free variable. This does not mean exactly one variable, but only one of the variables used may be free, for example $\exists x(x \sim y)$. Many $L(\sim)$-formulas will define the same graph. Our true aim thus is to classify the definable induced subgraphs. This means we will have to check all $L(\sim)$-formulas and the graphs which they define. Checking this will be, to say the least, extremely tedious. We introduce some concepts to lessen the amount of work that needs to be done.

**Definition 4.4.4** (**Disjunctive normal form**, see [Rot00] pg. 30)**.** *For an $L$-formula $\varphi$, a* disjunctive normal form *of $\varphi$ is a formula of the form $\bigvee_i \bigwedge_j \varphi_{ij}$ where each $\varphi_{ij}$ is a literal and $\varphi \equiv \bigvee_i \bigwedge_j \varphi_{ij}$.*

**Remark 4.4.5.** We will call the $\bigwedge_j \varphi_{ij}$ part, i.e. a conjunction of literals, of the disjunctive normal form a *conjunctive form.*

Note that every quantifier free formula is logically equivalent to a boolean combination of literals and hence to a disjunctive normal form of such a formula. Since Th$\boldsymbol{R}$ has quantifier elimination, i.e. every $L(\sim)$-formula is Th$\boldsymbol{R}$-equivalent to a quantifier free one, we only have to look at the sets defined by formulas of the form $\bigvee_i \bigwedge_j \varphi_{ij}$, for $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$, with each $\varphi_{ij}$ a literal.

This makes our work less, but still very tedious. Let's see what happens when we use only one parameter, say $a$. Then the literals we can use are $x = a$, $x \sim a$, $x \neq a$, $x \not\sim a$. Using only these literals we can respectively define the trivial graph, $\boldsymbol{R}$ (by Proposition 2.1.1), $\boldsymbol{R}$ (by Proposition 2.1.4) and a countably infinite graph (by Proposition 2.1.3) with vertex $a$ isolated. The last graph mentioned is just $\boldsymbol{R}$ (by Proposition 2.1.1) with an extra vertex $a$ which is not connected to any vertex in $\boldsymbol{R}$.

Since we are only considering disjunctions of conjunctions of literals, i.e. disjunctive normal forms, the above discussion describes all the possible outcomes when using only one literal in the disjunctive normal form. Next we will need to consider the possible outcomes when using two literals and thereafter all the possible outcomes when using three literals and then after that four literals and so on. Having discussions, as the one above, for each of these scenarios (using one, two, three, etc. literals) seems the most natural rout to take from here since the results from these discussions aren't loaded enough to be grouped under the banner "Proposition" or "Lemma". However, following this rout leaves the arguments somewhat unstructured and hence also slightly harder to follow. To provide some structure to our arguments we introduce "Cases", under which we will state the situation to be considered, and "Conclusions", under which we will state the outcomes of the discussion.

We also introduce some notations to shorten writing. We label

$$\varphi_1 : x = a \qquad\qquad \varphi_3 : x \neq a$$
$$\varphi_2 : x \sim a \qquad\qquad \varphi_4 : x \not\sim a$$

and write disjunctive normal form simply as DNF. For the rest of this chapter we will use $\equiv$ to denote logical equivalence and $\equiv_{\boldsymbol{R}}$ for $\boldsymbol{R}$-equivalence, i.e. for two $L(\sim)$-formulas $\varphi(x)$ and $\psi(x)$, $\varphi(x) \equiv_{\boldsymbol{R}} \psi(x)$ if for all $r \in \boldsymbol{R}$ we have $\boldsymbol{R} \models \varphi(r)$ iff $\boldsymbol{R} \models \psi(r)$.

Our first discussion, following the case-conclusion structure, presents as follows.

*Case.* DNF's using one literal in one free variable and one parameter

- $\varphi_1$ defines the trivial graph.
- $\varphi_2$ defines $\boldsymbol{R}$ by Proposition 2.1.1.
- $\varphi_3$ defines $\boldsymbol{R}$ by Proposition 2.1.4.
- $\varphi_4$ defines a graph with vertex set {Non-neighbours of $a$} $\cup \{a\}$. The {Non-neighbours of a} part gives $\boldsymbol{R}$, using Proposition 2.1.1, and the $\{a\}$ part is an extra vertex, isolated in this case.

*Conclusion.* Every one-literal-DNF in one free variable and one parameter is $\boldsymbol{R}$-equivalent to itself and hence defines either the trivial graph, $\boldsymbol{R}$, or a graph consisting of a copy of $\boldsymbol{R}$ and an isolated vertex.

The obvious next case would be DNF's using two literals. We will take the less obvious, but somewhat more streamline approach and first consider only conjunctive forms (CF's) of literals, and once these have all been worked out consider taking disjunctions of the CF's.

*Case.* CF's using two literals in one free variable and one parameter

- $\varphi_1 \wedge \varphi_2 \equiv_{\boldsymbol{R}} \perp$ which does not define a subgraph of $\boldsymbol{R}$. In fact any formula equivalent to $\perp$ won't define a subgraph of $\boldsymbol{R}$ and we will leave it at that.
- $\varphi_1 \wedge \varphi_3 \equiv \perp$
- $\varphi_1 \wedge \varphi_4 \equiv_{\boldsymbol{R}} \varphi_1$ which puts us in the case of one literal.
- $\varphi_2 \wedge \varphi_3 \equiv_{\boldsymbol{R}} \varphi_2$ also putting us in the case of one literal.

- $\varphi_2 \wedge \varphi_4 \equiv \perp$
- $\varphi_3 \wedge \varphi_4$ defines $\boldsymbol{R}$ by Proposition 2.1.1.

*Conclusion.* Every two-literal-CF in one free variable and one parameter is $\boldsymbol{R}$-equivalent to either $\perp$, a one-literal-DNF (specifically $\varphi_1$ or $\varphi_2$), or $\varphi_3 \wedge \varphi_4$ and hence defines either $\boldsymbol{R}$, or a graph already defined using only one literal, or no graph at all.

Next we look at conjunctions of three literals, but we don't have to consider all possible conjunctions. Any conjunction containing $\perp$ will just be logically equivalent to $\perp$. Due to the associativity of $\wedge$, and the equivalence from the case above, any conjunction containing both $\varphi_1$ and $\varphi_4$ can be written without $\varphi_4$ and similarly any conjunction containing $\varphi_2$ and $\varphi_3$ can be written without $\varphi_3$. So the only possible new graphs would come from conjunctions containing $\varphi_3 \wedge \varphi_4$.

*Case.* CF's using three literals in one free variable and one parameter

- $\varphi_1 \wedge \varphi_3 \wedge \varphi_4 \equiv_{\boldsymbol{R}} \varphi_1 \wedge \varphi_3 \equiv \perp$
- $\varphi_2 \wedge \varphi_3 \wedge \varphi_4 \equiv_{\boldsymbol{R}} \varphi_2 \wedge \varphi_4 \equiv \perp$

*Conclusion.* Each of the above three-literal-CF's in one free variable and one parameter is $\boldsymbol{R}$-equivalent to $\perp$ and hence defines no graph at all.

This covers all possible CF's we need to consider. Technically we can continue to take conjunctions with four and five and so forth literals, but these will all just be $\boldsymbol{R}$-equivalent to either $\perp$, $\varphi_1$, $\varphi_2$ or $\varphi_3 \wedge \varphi_4$. We therefore continue to cases concerning disjunctions of the above one-literal-DNF's and CF's.

*Case.* DNF's using two literals in one free variable and one parameter

- $\varphi_1 \vee \varphi_2$ defines a graph with vertex set $\{\text{Neighbours of } a\} \cup \{a\}$. The $\{\text{Neighbours of } a\}$ part gives $\boldsymbol{R}$ by Proposition 2.1.1 and $\{a\}$ is an extra vertex connected to every vertex in the copy of $\boldsymbol{R}$.
- $\varphi_1 \vee \varphi_3 \equiv \top$ and will define $\boldsymbol{R}$. In fact, any formula equivalent to $\top$ will define $\boldsymbol{R}$ and we will leave it at that.
- $\varphi_1 \vee \varphi_4 \equiv_{\boldsymbol{R}} \varphi_4$ taking us back to the case of one literal.
- $\varphi_2 \vee \varphi_3 \equiv_{\boldsymbol{R}} \varphi_3$ also taking us back to the case of one literal.
- $\varphi_2 \vee \varphi_4 \equiv \top$
- $\varphi_3 \vee \varphi_4 \equiv_{\boldsymbol{R}} \top$

*Conclusion.* Each of the above two-literal-DNF's in one free variable and one parameter is $\boldsymbol{R}$-equivalent to either $\varphi_1 \vee \varphi_2$, a one-literal-DNF (specifically $\varphi_3$ or $\varphi_4$), or $\top$, and therefore defines either a graph consisting of a copy of $\boldsymbol{R}$ and an extra vertex connected to every vertex in the copy of $\boldsymbol{R}$, or $\boldsymbol{R}$, or a graph already defined using only one literal.

For the DNF's to follow, any disjunction containing $\perp$ can be written without $\perp$ and any disjunction containing $\top$ is logically equivalent to $\top$. Also, since $\varphi_1 \wedge \varphi_4 \equiv_{\boldsymbol{R}} \varphi_1$ and $\varphi_2 \wedge \varphi_3 \equiv_{\boldsymbol{R}} \varphi_2$ we only have to consider disjunctions of $\varphi_1$ and $\varphi_2$ with $\varphi_3 \wedge \varphi_4$.

*Case.* DNF's using three literals in one free variable and one parameter

- $\varphi_1 \vee (\varphi_3 \wedge \varphi_4) \equiv (\varphi_1 \vee \varphi_3) \wedge (\varphi_1 \vee \varphi_4) \equiv_{\boldsymbol{R}} \top \wedge \varphi_4 \equiv \varphi_4$ defining a graph as in the one-literal case above.
- $\varphi_2 \vee (\varphi_3 \wedge \varphi_4) \equiv (\varphi_2 \vee \varphi_3) \wedge (\varphi_2 \vee \varphi_4) \equiv_{\boldsymbol{R}} \varphi_3 \wedge \top \equiv \varphi_3$ also defining a graph as in the one-literal case above.

*Conclusion.* Each of the above three-literal-DNF's in one free variable and one parameter is $\boldsymbol{R}$-equivalent to either $\top$, or a one-literal-DNF (viz. $\varphi_3$ or $\varphi_4$) and defines a graph already defined in a previous case.

This covers all possible outcomes of disjunctions of conjunctions of literals in the case of one parameter. Again, technically we can continue taking disjunctions, but these will just be Th$\boldsymbol{R}$-equivalent to one of the disjunctions above. We can summarize the above conclusions with the following result.

**Proposition 4.4.6.** *The graphs definable in $\boldsymbol{R}$ using one parameter are*

1. *the trivial graph,*
2. $\boldsymbol{R}$ *itself,*
3. *the countably infinite graph $G_{a\sim}$ consisting of a copy of $\boldsymbol{R}$ and one vertex connected to every vertex in the copy of $\boldsymbol{R}$,*
4. *and the countably infinite graph $G_{a\not\sim}$ consisting of a copy of $\boldsymbol{R}$ and an isolated vertex.*

We discuss the case of two parameters before looking at a general finite number of parameters just to get a really good grip on the types of definable graphs. We will structure the arguments as before, using cases and conclusions. We take two parameters, say $a$ and $b$ with $a \neq b$, and label the literals in one free variable[4] as follows:

$$\begin{aligned} &\varphi_1 : x = a, && \psi_1 : x = b, \\ &\varphi_2 : x \sim a, && \psi_2 : x \sim b, \\ &\varphi_3 : x \neq a, && \psi_3 : x \neq b, \\ &\varphi_4 : x \not\sim a, && \psi_4 : x \not\sim b. \end{aligned}$$

There are a great deal of cases to consider. The different definable graphs, using two parameters, are summarized at the end of the considered cases.

Note that a formula containing only $\varphi_i$'s or $\psi_j$'s will define graphs as in the case for one parameter. We therefore only have to consider formulas containing at least one $\varphi_i$ and one $\psi_j$. Also, the roles of $a$ and $b$ are interchangeable, so we have to consider, for example, only one of $\varphi_i \wedge \psi_j$ and $\varphi_j \wedge \psi_i$ for $i \neq j$. Almost all of the following statements can be justified using Proposition 2.1.1, Proposition 2.1.2, Proposition 2.1.3, Proposition 2.1.4 or a combination of them. We will therefore just state the claims, and give additional arguments where necessary.

Like before, we streamline the argument by first considering only CF's.

*Case.* CF's using two literals in one free variable and two parameters

- $\varphi_1 \wedge \psi_1 \equiv_{\boldsymbol{R}} \perp$
- $\varphi_1 \wedge \psi_2$ is $\boldsymbol{R}$-equivalent to $\varphi_1$ if $a \sim b$ and $\perp$ if $a \not\sim b$ defining (in both cases) graphs as in the case with one parameter.
- $\varphi_1 \wedge \psi_3 \equiv_{\boldsymbol{R}} \varphi_1$ defining a graph as in the case with one parameter.
- $\varphi_1 \wedge \psi_4$ is $\boldsymbol{R}$-equivalent to $\perp$ if $a \sim b$ and $\varphi_1$ if $a \not\sim b$ also defining graphs as in the case of one parameter.
- $\varphi_2 \wedge \psi_2$ defines $\boldsymbol{R}$.
- $\varphi_2 \wedge \psi_3 \equiv_{\boldsymbol{R}} \varphi_2$ if $a \not\sim b$. In case $a \sim b$ then the graph defined has vertex set $\{\text{Neighbours of } a\} \setminus \{b\}$. But the neighbours of $a$ is just (up to isomorphism) $\boldsymbol{R}$ and so we can easily delete $b$ and still be left with $\boldsymbol{R}$. In this cas the graph defined by $\varphi_2 \wedge \psi_3$ is isomorphic to the graph defined by just $\varphi_2$.
- $\varphi_2 \wedge \psi_4$ defines a graph with vertex set $\{x : x \sim a \wedge x \not\sim b\} \cup \{b\}$, i.e. a copy of $\boldsymbol{R}$ and isolated vertex $b$ if $a \sim b$, and $\boldsymbol{R}$ if $a \not\sim b$.
- $\varphi_3 \wedge \psi_3$ defines $\boldsymbol{R}$.
- $\varphi_3 \wedge \psi_4 \equiv_{\boldsymbol{R}} \psi_4$ when $a \sim b$. The graph defined in case $a \not\sim b$ will have vertex set

$$(\{\text{Non-neighbours of } b\} \setminus \{a\}) \cup \{b\}.$$

Ignoring $b$ for the moment we just have a copy of $\boldsymbol{R}$. So, the graph defined consists of a copy of $\boldsymbol{R}$ and

[4] Remember, using only one free variable ensures that the set defined can be interpreted as a graph. Using more than one free variable will give rise to sets containing ordered tuples, which is not what we want.

isolated vertex $b$ and hence is isomorphic to the graph defined using just $\psi_4$ as in the case of one literal.

- $\varphi_4 \wedge \psi_4$ defines $\boldsymbol{R}$ if $a \sim b$, and if $a \not\sim b$, defines a graph with vertex set $\{x : x \not\sim a \wedge x \not\sim b\} \cup \{a, b\}$. This is just a copy of $\boldsymbol{R}$ and two isolated vertices $a$ and $b$.

*Conclusion.* Every two-literal-CF in one free variable and two parameters defines either (up to isomorphism) a graph already definable with one parameter, or a graph consisting of a copy of $\boldsymbol{R}$ and two isolated vertices.

From the case of one parameter we know that for the next case we only need to consider CF's containing $\psi_3 \wedge \psi_4$ and one of the $\varphi_i$'s. The remaining CF's will either be $\boldsymbol{R}$-equivalent to $\perp$ or one of the CF's above.

*Case.* CF's using three literals in one free variable and two parameters.

- $\varphi_1 \wedge \psi_3 \wedge \psi_4 \equiv_{\boldsymbol{R}} \varphi_1 \wedge \psi_4$ defining a graph as in the case with two literals.
- $\varphi_2 \wedge \psi_3 \wedge \psi_4$ defines $\boldsymbol{R}$.
- $\varphi_3 \wedge \psi_3 \wedge \psi_4$ defines $\boldsymbol{R}$ for both $a \sim b$ and $a \not\sim b$.
- $\varphi_4 \wedge \psi_3 \wedge \psi_4$ defines $\boldsymbol{R}$ if $a \sim b$ and a graph consisting of a copy of $\boldsymbol{R}$ and isolated vertex $a$ if $a \not\sim b$.

*Conclusion.* Every three-literal-CF in one free variable and two parameters defines (up to isomorphism) a graph already defined in a previous case.

There is one last CF to consider, the remaining CF's will either be equivalent to $\perp$ or one of the conjunctions above.

*Case.* CF's using four literals in one free variable and two parameters

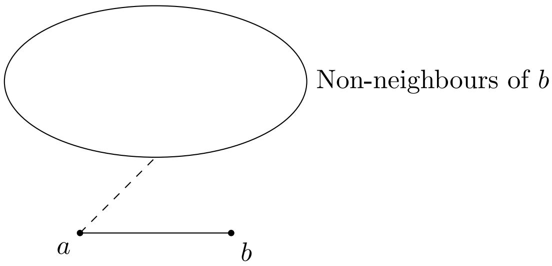
- $\varphi_3 \wedge \varphi_4 \wedge \psi_3 \wedge \psi_4$ which defines $\boldsymbol{R}$.

*Conclusion.* Every four-literal-CF in one free variable and two parameters defines (up to isomorphism) a graph already defined in a previous case.

As in the case with one parameter, we can technically continue looking at conjunctive forms of these literals, but the remaining conjunctions will be $\boldsymbol{R}$-equivalent to either $\perp$ or one of the CF's in the above cases. Next we have to look at disjunctions of the above CF's and one-literal-DNF's. We can reduce the number of combinations of literals to consider, by noting that any disjunction with $\varphi_3$ (similarly $\psi_3$) will just define $\boldsymbol{R}$.

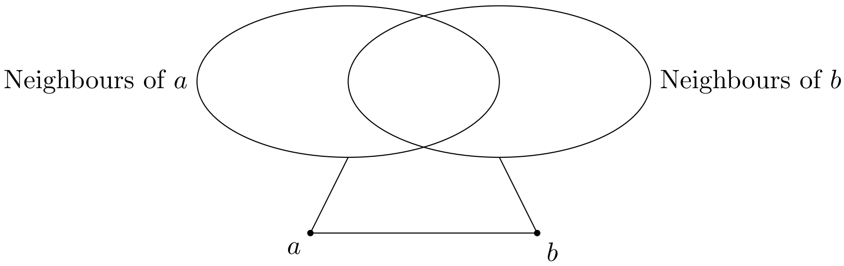
*Case.* DNF's using two literals in one free variable and two parameters

- $\varphi_1 \vee \psi_1$ defines a graph on two vertices. If $a \sim b$ then we get the complete graph on two vertices and if $a \not\sim b$ we get the empty graph on two vertices. Note that these are all possible graphs on two vertices.
- $\varphi_1 \vee \psi_2 \equiv_{\boldsymbol{R}} \psi_2$ if $a \sim b$. For $a \not\sim b$ we get a countably infinite graph with vertex set $\{\text{Neighbours of } b\} \cup \{a\}$. Now EP will be satisfied for all the neighbours of $b$. The only problem might possibly be, finding a witness for sets containing $a$. Consider, for example, the sets $U = \{a\}$ and $V = \{v\}$ for some $v \sim b$. Now, $U \cup \{b\}$ and $V$ will have a witness in $\boldsymbol{R}$, say $w$. This $w$ will be in the newly defined graph and will be a witness to EP for $U$ and $V$. In a similar way, we can find a witness to EP for any sets containing $a$ by just augmenting the first set (to which the witness must be connected) with $b$. So the graph defined is (up to isomorphism) just $\boldsymbol{R}$. We will call this type of argument an *augmentation argument*. The graph defined is thus the same (up to isomorphism) as the one defined with just $\psi_2$.
- $\varphi_1 \vee \psi_4 \equiv_{\boldsymbol{R}} \psi_4$ if $a \not\sim b$. If $a \sim b$ then we get a graph with vertex set $\{\text{Non-neighbours of } b\} \cup \{a, b\}$. Ignoring $a$ and $b$ for the moment, we can show that $\{\text{Non-neighbours of } b\}$ has EP (by Proposition 2.1.1). So the graph defined consists of a copy of $\boldsymbol{R}$ and extra vertices, $a$ and $b$, where $b$ is not connected to any vertex in the copy of $\boldsymbol{R}$. We illustrate the graph below.

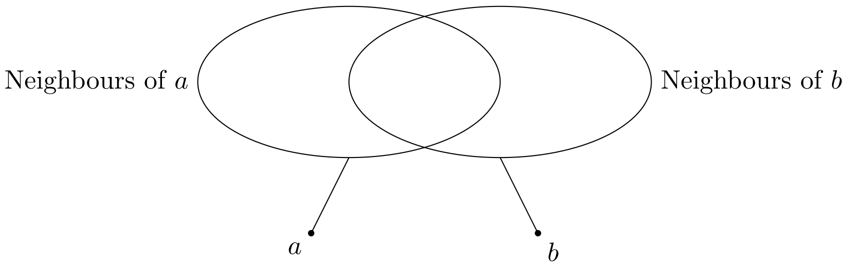



The dashed line between $a$ and the non-neighbours of $b$ indicates that $a$ might be connected to some of the non-neighbours of $b$.

- $\varphi_2 \vee \psi_2$ defines $\boldsymbol{R}$ in case $a \not\sim b$. This is because the sets defined by $\varphi_2 \vee \psi_2$ and $\varphi_4 \wedge \psi_4 \equiv \neg(\varphi_2 \vee \psi_2)$ partition $\boldsymbol{R}$, $\boldsymbol{R}$ is partition regular, and the graph defined by $\varphi_4 \wedge \psi_4$ was not (up to isomorphism) $\boldsymbol{R}$. In case $a \sim b$ we get a countably infinite graph not isomorphic to $\boldsymbol{R}$, since every vertex in the graph will be connected to at least one of $a$ and $b$ and hence $U = \emptyset$ and $V = \{a, b\}$ will have no witness to EP. We depict the graph below.

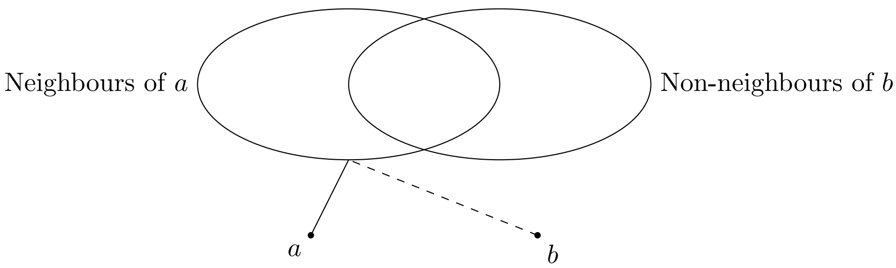



Before moving on to the next disjunction of this case, consider the formula $\varphi_2 \vee \psi_2 \vee \varphi_1 \vee \psi_1$. This formula defines a similar looking graph, in fact the same graph (up to isomorphism) in case $a \sim b$. If $a \not\sim b$ we get the same graph, but with $a$ not connected to $b$, as in the diagram below.



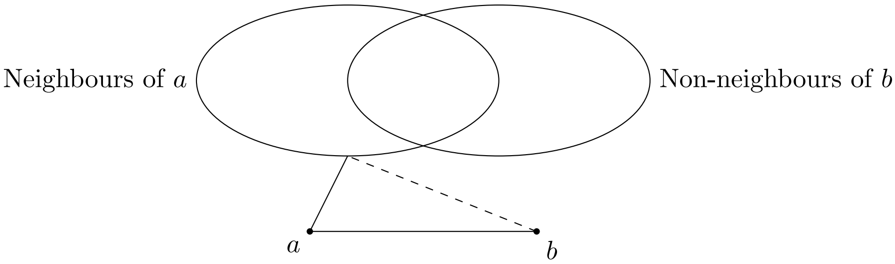

"Why this formula? It's out of nowhere." It will be clear shortly. Let's continue with taking disjunctions.

- $\varphi_2 \vee \psi_4$ defines $\boldsymbol{R}$ if $a \sim b$. This is also due to partition regularity like above. If $a \not\sim b$ then we get a countably infinite graph, not isomorphic to $\boldsymbol{R}$, since the sets $U = \{b\}$ and $V = \{a\}$ won't have a witness to EP. We draw the graph below.



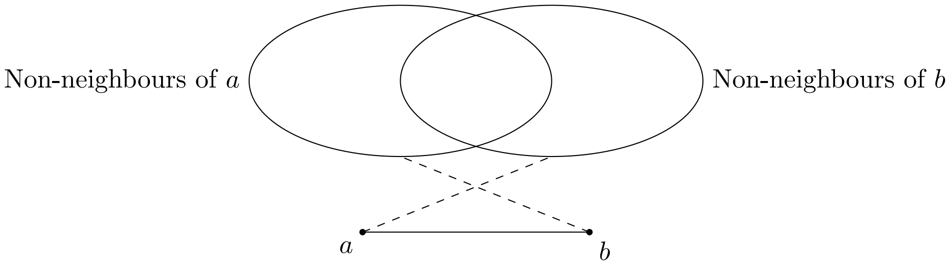

The dashed line indicates that $b$ might have some neighbours in common with $a$.

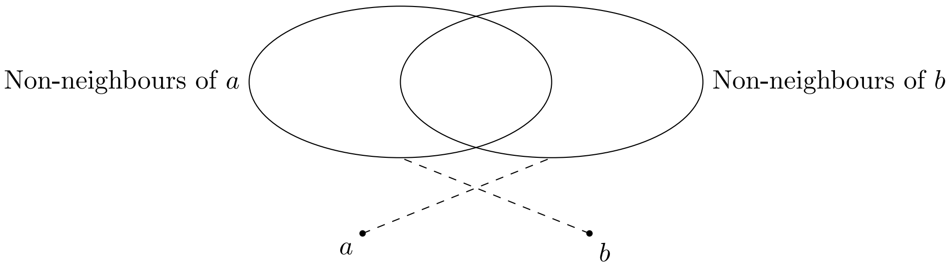
We return for a brief moment to "nowhere" and consider the formula $\varphi_2 \vee \psi_4 \vee \varphi_1$ in case $a \sim b$. This defines a similar graph to the one above, but with $a \sim b$.




Note that in all four of the graphs drawn above, if we ignore the vertices $a$ and $b$, the remaining vertices constitute a graph isomorphic to $\boldsymbol{R}$. This can be shown using a relevant augmentation argument. We can see a pattern arising; a copy of $\boldsymbol{R}$ and some extra vertices connected to the points of the copy of $\boldsymbol{R}$ in some way.

- $\varphi_4 \vee \psi_4$ defines a countably infinite graph not isomorphic to $\boldsymbol{R}$ because the sets $U = \{a, b\}$ and $V = \emptyset$ will have no witness. This is the case for both $a \sim b$ and $a \not\sim b$. We get the following two graphs.




The dashed lines indicate that $b$ might be connected to some of the non-neighbours of $a$, and similarly $a$ might be connected to some of the non-neighbours of $b$.

*Conclusion.* Every two-literal DNF in one free variable and two parameters is Th$\boldsymbol{R}$-equivalent to either $\varphi_3$, $\varphi_1 \vee \psi_1$, $\varphi_1 \vee \psi_2$, $\varphi_1 \vee \psi_4$, $\varphi_2 \vee \psi_2$, $\varphi_2 \vee \psi_4$, or $\varphi_4 \vee \psi_4$ and defines either a graph on two vertices, or a graph consisting of a copy of $\boldsymbol{R}$ and either one or two extra vertices connected to the copy of $\boldsymbol{R}$ in some way, or a graph already defined (up to isomorphism) as in one of the previous cases.

We are by no means done, there are still many cases of disjunctions of CF's to consider; considering DNF's with three, four, five etc. literals, but by looking at the cases above we can already see a pattern.

Apart from a few exceptions, most of the formulas just define $\boldsymbol{R}$ (up to isomorphism). Using only literals with the $=$ symbol in the DNF is either $\boldsymbol{R}$-equivalent to $\perp$ or defines a finite graph. In fact, for the case of two parameters, we are be able to define all finite graphs of order $\leq 2$. The definable countably infinite graphs, not isomorphic to $\boldsymbol{R}$, all consist of a copy of $\boldsymbol{R}$ and either one or two extra vertices, connected to the copy of $\boldsymbol{R}$ in some specific way. Notice that the ways in which the extra vertices are connected to the copy of $\boldsymbol{R}$ depend only on the combination of the literals $\varphi_2$, $\varphi_4$, $\psi_2$ and $\psi_4$ in the DNF.

The most interesting thing to take note of is this: CF's containing $\sim$-literals[5] in only one parameter, i.e $x \sim a$ and $x \not\sim a$ but not $x \sim b$ or $x \not\sim b$, define up to isomorphism the same graph as the given CF written

[5]These are literals using the only the symbols $\sim$ and $\not\sim$.

without $x \neq b$. This is obvious if the CF does not contain $x \neq b$ to begin with. It is however, harder to believe if the given CF contains $x \neq b$.

Note that if we want the CF to define a graph it can contain at most one of $\varphi_2$ and $\varphi_4$. We therefore only have to prove the claim for the CF's

1. $x \sim a \wedge x \neq b$,
2. $x \not\sim a \wedge x \neq b$,
3. $x = a \wedge x \sim a \wedge x \neq b$,
4. $x = a \wedge x \not\sim a \wedge x \neq b$,
5. $x \neq a \wedge x \sim a \wedge x \neq b$,
6. and $x \neq a \wedge x \not\sim a \wedge x \neq b$.

*Claim.* The graphs defined by $x \sim a \wedge x \neq b$ and $x \sim a$ are isomorphic.

*Proof.* Let $G$ be the graph defined by $x \sim a \wedge x \neq b$, so $G$ has vertex set {Neighbours of $a$} which might possibly include $b$. $G$ is, with or without $b$, isomorphic to $\boldsymbol{R}$ and hence isomorphic to the graph defined by $x \sim a$. □

*Claim.* The graphs defined by $x \not\sim a \wedge x \neq b$ and $x \not\sim a$ are isomorphic.

*Proof.* Let $G$ be the graph defined by $x \not\sim a \wedge x \neq b$. Then $G$ has vertex set $\{$Non-neighbours of $a\} \cup \{a\}$, where the {Non-neighbours of $a$} part without $a$ might possibly include $b$. {Non-neighbours of $a$} without $a$ is, with or without $b$, isomorphic to $\boldsymbol{R}$. Therefore $G$ consists of a copy of $\boldsymbol{R}$ and an isolated vertex $a$. But this is exactly (up to isomorphism) the graph defined by $x \not\sim a$. □

*Claim.* The graphs defined by $x = a \wedge x \sim a \wedge x \neq b$ and $x = a \wedge x \sim a$ are isomorphic.

*Proof.* Note that both $x = a \wedge x \sim a \wedge x \neq b$ and $x = a \wedge x \sim a$ are $\boldsymbol{R}$-equivalent to just $x = a$. It is clear in this case that both CF's thus define the trivial graph, giving the required result. □

*Claim.* The graphs defined by $x = a \wedge x \not\sim a \wedge x \neq b$ and $x = a \wedge x \not\sim a \wedge x \neq b$ are isomorphic.

*Proof.* As in the proof above, both these formulas are $\boldsymbol{R}$-equivalent to just $x = a$, implying the required result. □

*Claim.* The graphs defined by $x \neq a \wedge x \sim a \wedge x \neq b$ and $x \neq a \wedge x \sim a$ are isomorphic.

*Proof.* Let $G$ be the graph defined by $x \neq a \wedge x \sim a \wedge x \neq b$, then $G$ has vertex set {Neighbours of $a$} which might possibly include $b$. $G$ is, with or without $b$, isomorphic to $\boldsymbol{R}$ and hence isomorphic to the graph defined by $x \neq a \wedge x \sim a$. □

*Claim.* The graphs defined by $x \neq a \wedge x \not\sim a \wedge x \neq b$ and $x \neq a \wedge x \not\sim a$ are isomorphic.

*Proof.* Let $G$ be the graph defined by $x \neq a \wedge x \not\sim a \wedge x \neq b$. Then $G$ has vertex set {Non-neighbours of $a$} excluding $a$. This is, with or without $b$, isomorphic to $\boldsymbol{R}$, and hence isomorphic to the graph defined by $x \neq a \wedge x \not\sim a$. □

It should be clear that we can "delete" literals of the form $x \neq b$ from a CF, if the $\sim$-literals in the CF are only in terms of $a$, without changing (up to isomorphism) the graph defined. The question now becomes, can we do the same if such a CF is part of a DNF? Well, not exactly.

**Example 4.4.7.** Let $G$ be the graph defined by $x \sim b \vee (x \sim a \wedge x \neq b)$, and $H$ be the graph defined by $x \sim b \vee x \sim a$. Suppose that $a \sim b$. Then $H$ is countably infinite and not isomorphic to $\boldsymbol{R}$ since the sets $U = \emptyset$ and $V = \{a, b\}$ does not have a witness.

But $G$ is isomorphic to $\boldsymbol{R}$. To see this, note that $G$ has vertex set $\{\text{Neighbours of } b\} \cup (\{\text{Neighbours of } a\} \setminus \{b\})$. Now $G \setminus a$ will satisfy EP, by Proposition 2.1.1, so the only sets $U$ and $V$ that might possibly not have a witness are sets containing the vertex $a$. Let $U = \{a, u_1, \ldots, u_m\}$ and $V = \{v_1, \ldots, v_n\}$ be finite disjoint subsets of $G$. In this case they will also be finite disjoint subsets of $\boldsymbol{R}$ and hence has a witness $w_1$, with $w_1 \sim a$ and $w_1 \sim U$ and $w_1 \not\sim V$. So $w_1 \in G$ and witnesses EP for $U$ and $V$ in $G$. Now if $U = \{u_1, \ldots, u_m\}$ and $V = \{a, v_1, \ldots, v_n\}$, then $U' := U \cup \{b\}$ and $V$ are finite disjoint subsets of $\boldsymbol{R}$, and so has a witness to EP, say $w_2$, such that $w_2 \sim b$, $w_2 \sim U$ and $w_2 \not\sim V$. Hence $w_2 \in G$ and witnesses EP for $U$ and $V$ in $G$. This shows that $G$ has EP and is therefore isomorphic to $\boldsymbol{R}$. So $G$ and $H$ are not isomorphic. In this case, we see that we cannot "delete" the literal $x \neq b$ and get away with it.

We can, however, define up to isomorphism, the graph $G$ with just $x \sim a$. So, even though we cannot just "delete" the literal $x \neq b$, we can still define, up to isomorphism, the same graph with a DNF not containing $x \neq b$.

For the case of $n$ parameters we will prove in detail, that any DNF consisting of $L(\sim)$-literals defines up to isomorphism the same graph as a DNF without any literals in the symbol $\neq$. The fact that we can do without literals in the symbol $\neq$ is enough for now.

Our conclusion for the case of two parameters is then that, if the graph defined consists of a copy of $\boldsymbol{R}$ and extra vertex $a$, then, the way in which $a$ is connected to the copy of $\boldsymbol{R}$ depends only on whether or not the literals $x \sim a$ and $x \not\sim a$ are present in the DNF. This means we can effectively define, in the case of two parameters, all the graphs consisting of a copy of $\boldsymbol{R}$ and one extra vertex, by taking the literal $x = a$ (to add the extra vertex) and also every possible $\sim$-literal in the parameter $a$ to be CF's in the DNF. Similarly, we can define all the graphs consisting of a copy of $\boldsymbol{R}$ and two extra vertices by adding $x = a \vee x = b$ via disjunction to the DNF (to add the two extra vertices), and also every possible (unique up to $R$-equivalence) CF consisting of $\sim$-literals in possibly both the parameters $a$ and $b$ in the DNF.

We can now summarize all the above conclusions and discussion in the following proposition.

**Proposition 4.4.8.** *The graphs definable in $\boldsymbol{R}$ using two parameters are (and are defined for example by)*

1. *the trivial graph ($x = a$),*
2. *$\boldsymbol{R}$ itself ($x \sim a$)[6],*
3. *the countably infinite graph $G_{a\sim}$ consisting of a copy of $\boldsymbol{R}$ and one vertex connected to every vertex in the copy of $\boldsymbol{R}$ ($x = a \vee x \sim a$),*
4. *the countably infinite graph $G_{a\not\sim}$ consisting of a copy of $\boldsymbol{R}$ and one isolated vertex ($x = a \vee x \not\sim a$),*
5. *all possible graphs on two vertices ($x = a \vee x = b$),*
6. *the countably infinite graph $G_{\sim a}$ consisting of a copy of $\boldsymbol{R}$ and two extra vertices, such that one of the two extra vertices is connected to every vertex in the copy of $\boldsymbol{R}$ and $a \not\sim b$ ($x = a \vee x = b \vee x \sim a$),*
7. *the countably infinite graph $G_{\not\sim a}$ consisting of a copy of $\boldsymbol{R}$ and two extra vertices, such that one of the two extra vertices is not connected to any vertex in the copy of $\boldsymbol{R}$ and $a \sim b$ ($x = a \vee x = b \vee x \not\sim a$),*
8. *the countably infinite graph $G_{a\sim, b\sim}$ consisting of a copy of $\boldsymbol{R}$ where both $a$ and $b$ are connected to every vertex in the copy of $\boldsymbol{R}$ and either $a \sim b$ or $a \not\sim b$*

[6]The formula $x = x$ is perfectly acceptable to define $\boldsymbol{R}$ and might even be viewed as a "better" formula to do the job than we did here. We do however wish to highlight defining graphs with parameters, and so in this sense the formula $x \sim a$ does a "better" job than $x = x$.

$(x = a \vee x = b \vee (x \sim a \wedge x \sim b))$,

9. *the countably infinite graph $G_{a\sim,b\not\sim}$ consisting of a copy of **R** where $a$ is connected to every vertex in the copy of **R** and $b$ is connected to none of the vertices in copy of **R**, and either $a \sim b$ or $a \not\sim b$*
$(x = a \vee x = b \vee (x \sim a \wedge x \not\sim b))$,
10. *the countably infinite graph $G_{a\not\sim,b\not\sim}$ consisting of a copy of copy of **R** where neither of $a$ and $b$ are connected to any vertex in the copy of **R**, and either $a \sim b$ or $a \not\sim b$*
$(x = a \vee x = b \vee (x \not\sim a \wedge x \not\sim b))$,
11. *the countably infinite graph $G_{\sim a,b}$ consisting of a copy of **R** where every vertex in the copy of **R** is connected to at least one of $a$ and $b$, and either $a \sim b$ or $a \not\sim b$*
$(x = a \vee x = b \vee (x \sim a \vee x \sim b))$,
12. *the countably infinite graph $G_{\sim b \to \sim a}$ consisting of a copy of **R** where every vertex in the copy of **R** that is connected to $b$ will also be connected to $a$, and either $a \sim b$ or $a \not\sim b$*
$(x = a \vee x = b \vee (x \sim a \vee x \not\sim b))$,
13. *the countably infinite graph $G_{\not\sim a,b}$ consisting of a copy of **R** where every vertex in the copy of **R** is connected to at most one of $a$ and $b$, and either $a \sim b$ or $a \not\sim b$*
$(x = a \vee x = b \vee (x \not\sim a \vee x \not\sim b))$,
14. *the countably infinite graph $G_{\sim b \leftrightarrow \sim a}$ consisting of a copy of **R** where every vertex in the copy of **R** is connected to $b$ iff it is connected to $a$, and either $a \sim b$ or $a \not\sim b$*
$(x = a \vee x = b \vee (x \sim a \wedge x \sim b) \vee (x \not\sim a \wedge x \not\sim b))$,
15. *and the countably infinite graph $G_{\sim a,\sim b}$ consisting of a copy of **R** where every vertex in the copy of **R** is connected to either $a$ or $b$, but not both, and either $a \sim b$ or $a \not\sim b$*
$(x = a \vee x = b \vee (x \sim a \wedge x \not\sim b) \vee (x \not\sim a \wedge x \sim b))$.

Just to recap, what this proposition is saying, is that the graphs definable in **R** with two parameters are

1. all finite graphs of order $\leq 2$,
2. **R** itself,
3. and graphs consisting of a copy of **R** and a finite graph on either one or two vertices, with the vertices of the finite graph connected to the copy of **R** in a specific way.

Note that the "specific ways" includes all possible ways expressible by a first order formula in which the vertices can be connected to the copy of **R**.

**Remark 4.4.9.** To distinguish, for example, between the two graphs $G_{a\sim,b\sim}$ where $a \sim b$ and $a \not\sim b$, we may denote the two graphs as $G^{a\sim b}_{a\sim,b\sim}$ and $G^{a\not\sim b}_{a\sim,b\sim}$ respectively. We can use a similar notation for the other graphs of the same type, i.e. that are definable with the same $L(\sim)$-formula.

This gives us an idea for what will happen in the general case of $n$ parameters. We can see from the case of two parameters that there are two types of conjunctive forms in the disjunctive normal form that influence the graph defined. That is, literals with $=$ and conjunctions with either $\sim$, $\not\sim$ or a combination of the two.

We can write this formally as follows.

**Definition 4.4.10 (Moment).** *Let $\bar{a}$ be an $n$-tuple of vertices of **R**.*

*The formula $\varphi^{=}(x, a) : x = a$ is the* =-moment *of $a$.*

*The formula $\varphi^{\sim}(x, \bar{a}) : x \sim a_1 \wedge \cdots \wedge x \sim a_n$ is the* c-moment *("c" for **c**onnected) of $\bar{a}$.*

*The formula $\varphi^{\not\sim}(x, \bar{a}) : x \not\sim a_1 \wedge \cdots \wedge x \not\sim a_n$ is the* n-moment *("n" for **n**ot connected) of $\bar{a}$.*[7]

*Let $\bar{b}$ be a $j$-tuple and $\bar{c}$ be a $k$-tuple of vertices of **R** where $j + k = n$, with no vertices in common.*

[7] There are two n's in use at the moment, one for $n$ parameters, and one for an n-moment. Note that the one, the "$n$", denotes a number, specifically an integer. The other, the "n", denotes an alphabet letter. There should hopefully be no confusion between the two.

*The formula* $\varphi^*(x, \bar{b}, \bar{c}) : x \sim b_1 \wedge \cdots \wedge x \sim b_j \wedge x \not\sim c_1 \wedge \cdots \wedge x \not\sim c_k$ *is the* cn-moment *of* $\bar{b}$ *and* $\bar{c}$.

**Lemma 4.4.11.** *Let* $\varphi(x, \bar{a})$ *be a CF of* $L(\sim)$*-literals in at most* $n$ *parameters, such that* $\varphi(x, \bar{a})$ *contains* $x \neq a_1$*,* $\ldots$*,* $x \neq a_k$*, relabelling the parameters if necessary, but no* $\sim$*-literal in any of* $a_1, \ldots, a_k$*. Then the graph defined by* $\varphi(x, \bar{a})$ *in* $\boldsymbol{R}$ *is isomorphic to one defined by a moment.*

*Proof.* Let $\varphi(x, \bar{a})$ be as stated above and let $G$ be the graph defined by $\varphi(x, \bar{a})$. Note that if $\varphi(x, \bar{a})$ contains $x = a_j$ for some $j$, then either $\varphi(x, \bar{a}) \equiv_{\boldsymbol{R}} \perp$ or $\varphi(x, \bar{a}) \equiv_{\boldsymbol{R}} x = a_j$ and we have nothing to prove. We suppose therefore the $\varphi(x, \bar{a})$ does not contain $x = a_j$ for any $j$. Also, if $\varphi(x, \bar{a})$ contains only the literals $x \neq a_1, \ldots,$ $x \neq a_k$, then it defines $\boldsymbol{R}$ which is isomorphic to the graph defined by any c-moment and we are done.

So suppose that $\varphi(x, \bar{a})$ contains at least some literals of the form $x \sim a_i$ and/or $x \not\sim a_i$. We may write $\varphi(x, \bar{a})$ as $x \neq a_1 \wedge \cdots \wedge x \neq a_k \wedge x \sim a_{k+1} \wedge \cdots \wedge x \sim a_l \wedge x \not\sim a_{l+1} \wedge \cdots \wedge x \not\sim a_m$ where $m \leq n$. Let $G$ be the graph defined by $\varphi(x, \bar{a})$. Then $G$ has vertex set

$$A := \{x : x \neq a_1 \wedge \cdots \wedge x \neq a_k \wedge x \sim a_{k+1} \wedge \cdots \wedge x \sim a_l \wedge x \not\sim a_{l+1} \wedge \ldots x \not\sim a_m\}.$$

Let $A' = A \setminus \{a_{l+1}, \ldots, a_m\}$. Then $A'$ is isomorphic to $\boldsymbol{R}$, by Proposition 2.1.1 and Proposition 2.1.4. Hence, $G$ consists of a copy of $\boldsymbol{R}$ and extra vertices $\{a_{l+1}, \ldots, a_m\}$ such that the vertices in the copy of $\boldsymbol{R}$ are not connected to any of the vertices in $\{a_{l+1}, \ldots, a_m\}$.

Let $H$ be the graph defined by the n-moment $x \not\sim a_{l+1} \wedge \cdots \wedge x \not\sim a_m$. Then $H$ consists of a copy of $\boldsymbol{R}$ and extra vertices $\{a_{l+1}, \ldots, a_m\}$ such that the vertices in the copy of $\boldsymbol{R}$ are not connected to any of the vertices in $\{a_{l+1}, \ldots, a_m\}$.

Clearly $G \cong H$, and $H$ is defined by a moment, which is the required result. □

**Lemma 4.4.12.** *Let* $\varphi(x, \bar{a})$ *be a CF of* $L(\sim)$*-literals in at most* $n$ *parameters, such that* $\varphi(x, \bar{a})$ *contains* $x \neq a_1$*,* $\ldots$*,* $x \neq a_k$ *and* $x \not\sim a_1$*,* $\ldots$*,* $x \not\sim a_k$*, relabelling the parameters if necessary. Then the graph defined by* $\varphi(x, \bar{a})$ *in* $\boldsymbol{R}$ *is isomorphic to one defined by a moment.*

*Proof.* Let $\varphi(x, \bar{a})$ be as stated above and let $G$ be the graph defined by $\varphi(x, \bar{a})$. Note that if $\varphi(x, \bar{a})$ contains $x = a_j$ for some $a_j$, then either $\varphi(x, \bar{a}) \equiv_{\boldsymbol{R}} \perp$ or $\varphi(x, \bar{a}) \equiv_{\boldsymbol{R}} x = a_j$ and we have nothing to prove. Suppose therefore that $\varphi(x, \bar{a})$ does not contain $x = a_j$ for any $j$. Also, $\varphi(x, \bar{a})$ cannot contain $x \sim a_j$ for any $j \leq k$, otherwise $\varphi(x, \bar{a}) \equiv \perp$, and again there is nothing to prove. If $\varphi(x, \bar{a})$ contains only the literals $x \neq a_1, \ldots,$ $x \neq a_k$, and $x \not\sim a_1, \ldots, x \not\sim a_k$ then it defines $\boldsymbol{R}$ which is isomorphic to the graph defined by any c-moment and we are done.

Suppose therefore that $\varphi(x, \bar{a})$ contains at least some literals of the form $x \sim a_i$ and/or $x \not\sim a_i$ for $i > k$, so that we are not in the case above. Then we can write $\varphi(x, \bar{a})$ as

$$\bigwedge_{i=1}^{k}(x \neq a_i \wedge x \not\sim a_i) \wedge \bigwedge_{i=k+1}^{l}(x \sim a_i) \wedge \bigwedge_{i=l+1}^{m}(x \not\sim a_i).$$

So $G$ has vertex set

$$A := \left\{x : \bigwedge_{i=1}^{k}(x \neq a_i \wedge x \not\sim a_i) \wedge \bigwedge_{i=k+1}^{l}(x \sim a_i) \wedge \bigwedge_{i=l+1}^{m}(x \not\sim a_i)\right\}.$$

Let $A' = A \setminus \{a_{l+1}, \ldots, a_m\}$. Then $A'$ is isomorphic to $\boldsymbol{R}$, by Proposition 2.1.1. Hence, $G$ consists of a copy of $\boldsymbol{R}$ and extra vertices $\{a_{l+1}, \ldots, a_m\}$ such that the vertices in the copy of $\boldsymbol{R}$ are not connected to any of the vertices in $\{a_{l+1}, \ldots, a_m\}$.

Let $H$ be the graph defined by the n-moment $x \not\sim a_{l+1} \wedge \cdots \wedge x \not\sim a_m$. Then $H$ consists of a copy of $\boldsymbol{R}$ and extra vertices $\{a_{l+1}, \ldots, a_m\}$ such that the vertices in the copy of $\boldsymbol{R}$ are not connected to any of the vertices in $\{a_{l+1}, \ldots, a_m\}$.

Clearly $G \cong H$, and $H$ is defined by a moment, which is the required result. $\square$

**Lemma 4.4.13.** *Let $\varphi(x, \bar{a})$ be a CF of $L(\sim)$-literals in at most $n$ parameters, such that $\varphi(x, \bar{a})$ contains $x \neq a_1$, $\ldots$, $x \neq a_k$ and $x \not\sim a_1$, $\ldots$, $x \not\sim a_j$ with $j < k$, relabelling the parameters if necessary. Then the graph defined by $\varphi(x, \bar{a})$ in $\boldsymbol{R}$ is isomorphic to one defined by a moment.*

*Proof.* Let $\varphi(x, \bar{a})$ be as stated above and let $G$ be the graph defined by $\varphi(x, \bar{a})$. Note that if $\varphi(x, \bar{a})$ contains $x = a_j$ for some $a_j$, then either $\varphi(x, \bar{a}) \equiv_{\boldsymbol{R}} \perp$ or $\varphi(x, \bar{a}) \equiv_{\boldsymbol{R}} x = a_j$ and we have nothing to prove. Suppose therefore that $\varphi(x, \bar{a})$ does not contain $x = a_j$ for any $j$. Also, $\varphi(x, \bar{a})$ cannot contain $x \sim a_i$ for any $i \leq j$, otherwise $\varphi(x, \bar{a}) \equiv \perp$, and again there is nothing to prove. If $\varphi(x, \bar{a})$ contains only the literals $x \neq a_1$, $\ldots$, $x \neq a_k$, and $x \not\sim a_1$, $\ldots$, $x \not\sim a_j$ then it defines $\boldsymbol{R}$ (by Proposition 2.1.1 and Proposition 2.1.4) which is isomorphic to the graph defined by any c-moment and we are done.

Suppose therefore that $\varphi(x, \bar{a})$ contains at least some literals of the form $x \sim a_i$ and/or $x \not\sim a_i$ for $i > k$, so that we are not in the case above. Then we can write $\varphi(x, \bar{a})$ as

$$\bigwedge_{i=1}^{j}(x \neq a_i \wedge x \not\sim a_i) \wedge \bigwedge_{i=j+1}^{k}(x \neq a_i) \wedge \bigwedge_{i=k+1}^{l}(x \sim a_i) \wedge \bigwedge_{i=l+1}^{m}(x \not\sim a_i).$$

So $G$ has vertex set

$$A := \left\{x : \bigwedge_{i=1}^{j}(x \neq a_i \wedge x \not\sim a_i) \wedge \bigwedge_{i=j+1}^{k}(x \neq a_i) \wedge \bigwedge_{i=k+1}^{l}(x \sim a_i) \wedge \bigwedge_{i=l+1}^{m}(x \not\sim a_i)\right\}.$$

Let $A' = A \setminus \{a_{l+1}, \ldots, a_m\}$. Then $A'$ is isomorphic to $\boldsymbol{R}$, by Proposition 2.1.1 and Proposition 2.1.4. Hence, $G$ consists of a copy of $\boldsymbol{R}$ and extra vertices $\{a_{l+1}, \ldots, a_m\}$ such that the vertices in the copy of $\boldsymbol{R}$ are not connected to any of the vertices in $\{a_{l+1}, \ldots, a_m\}$.

Let $H$ be the graph defined by the n-moment $x \not\sim a_{l+1} \wedge \cdots \wedge x \not\sim a_m$. Then $H$ consists of a copy of $\boldsymbol{R}$ and extra vertices $\{a_{l+1}, \ldots, a_m\}$ such that the vertices in the copy of $\boldsymbol{R}$ are not connected to any of the vertices in $\{a_{l+1}, \ldots, a_m\}$.

Clearly $G \cong H$, and $H$ is defined by a moment, which is the required result. $\square$

**Proposition 4.4.14.** *A CF consisting of $L(\sim)$-literals in at most $n$ parameters is either $\boldsymbol{R}$ equivalent to $\perp$, or it is a moment itself, or it defines (up to isomorphism) the same graph as a moment.*

*Proof.* Let $\varphi(x, \bar{a})$ be as stated above and let $G$ be the graph defined by $\varphi(x, \bar{a})$. Note that if $\varphi(x, \bar{a})$ contains $x = a_j$ for some $a_j$, then either $\varphi(x, \bar{a}) \equiv_{\boldsymbol{R}} \perp$, or $\varphi(x, \bar{a}) \equiv_{\boldsymbol{R}} x = a_j$, in which case $\varphi(x, \bar{a})$ defines up to isomorphism the same graph as an $=$-moment.

We can now suppose that $\varphi(x, \bar{a})$ does not contain $x = a_j$ for any $a_j$. If $\varphi(x, \bar{a})$ contains only literals of the form $x \sim a_j$ and/or $x \not\sim a_j$ then it is either a c-, n- or cn-moment.

So suppose therefore that $\varphi(x, \bar{a})$ contains at least some literals of the form $x \neq a_j$. Then, we can write $\varphi(x, \bar{a})$ as

$$\bigwedge_{i \in I} x \neq a_i \wedge \bigwedge_{j \in J} x \sim a_j \wedge \bigwedge_{k \in K} x \not\sim a_k.$$

Note that if $J \cap K \neq \emptyset$ then $\varphi(x, \bar{a}) \equiv_{\boldsymbol{R}} \perp$. Suppose therefore that $J \cap K = \emptyset$.

Next, if $I \cap J \neq \emptyset$ then $\varphi(x, \bar{a})$ is $\boldsymbol{R}$-equivalent to

$$\bigwedge_{i \in I \setminus J} x \neq a_i \wedge \bigwedge_{j \in J} x \sim a_j \wedge \bigwedge_{k \in K} x \not\sim a_k$$

so we can, without loss of generality, suppose that $I \cap J = \emptyset$.

Now, if $I \cap K = \emptyset$, then the result follows from Lemma 4.4.11. Suppose, however, that $I \cap K \neq \emptyset$. In case $I \subseteq K$, then the result follows from Lemma 4.4.12. If $K \subseteq I$ then $\varphi(x, \bar{a})$ defines $\boldsymbol{R}$, by Proposition 2.1.1 and Proposition 2.1.4, and we can define up to isomorphism the same graph using any c-moment. Finally, it might be the case that $I$ and $K$ are not subsets of each other, but only have some elements in common, the result then follows from Lemma 4.4.13. □

**Lemma 4.4.15.** *Let $\varphi(x, \bar{a})$ and $\psi(x, \bar{a})$ both be CF's consisting of $L(\sim)$-literals in at most $n$ parameters. Then the graph defined by $\varphi(x, \bar{a}) \vee \psi(x, \bar{a})$ in $\boldsymbol{R}$ is isomorphic to one defined by a disjunction of moments.*

*Proof.* First note that, from Proposition 4.4.14, that $\varphi$ and $\psi$ is either $\boldsymbol{R}$-equivalent to $\perp$, or is a moment, or defines up to isomorphism the same graph as a moment. If both $\varphi$ and $\psi$ are $\boldsymbol{R}$-equivalent to $\perp$ then $\varphi(x, \bar{a}) \vee \psi(x, \bar{a})$ does not define a subgraph of $\boldsymbol{R}$ and we have nothing to prove. In only one of $\varphi$ and $\psi$, say $\varphi$ is $\boldsymbol{R}$-equivalent to $\perp$, then $\varphi \vee \psi \equiv \psi$ and the result follows directly from Proposition 4.4.14. So we suppose that neither of $\varphi$ and $\psi$ are $\boldsymbol{R}$-equivalent to $\perp$.

In case both $\varphi(x, \bar{a})$ and $\psi(x, \bar{a})$ are either =-, c-, n-, or cn-moments themselves we have the required result. So we suppose that at least one of $\varphi(x, \bar{a})$ and $\psi(x, \bar{a})$ is not a moment. Without loss of generality let $\psi(x, \bar{a})$ be the non-moment, then it must contain at least one literal of the form $x \neq a_i$.

Note that if $\psi(x, \bar{a}) := \bigwedge_i x \neq a_i$ then $\varphi(x, \bar{a}) \vee \psi(x, \bar{a})$ will just define a copy of $\boldsymbol{R}$ which is isomorphic to the graph defined by any c-moment, which is the required result. We suppose therefore that

$$\psi(x, \bar{a}) := \bigwedge_{i=1}^{j} (x \neq a_i \wedge x \not\sim a_i) \wedge \bigwedge_{i=j+1}^{k} (x \neq a_i) \wedge \bigwedge_{i=k+1}^{l} (x \sim a_i) \wedge \bigwedge_{i=l+1}^{m} (x \not\sim a_i)$$

where $I$, $J$, $K$ and $L$ are mutually disjoint. $J$ and $K$ can, without loss of generality be mutually disjoint, since $\boldsymbol{R} \models x \sim a_k \rightarrow x \neq a_k$.

<u>If $\varphi(x, \bar{a})$ is an =-moment:</u> Suppose that $\varphi(x, \bar{a}) : x = a_s$. Let

$$A = \{a_s\}$$

and

$$B = \left\{ x : \bigwedge_{i=1}^{j} (x \neq a_i \wedge x \not\sim a_i) \wedge \bigwedge_{i=j+1}^{k} (x \neq a_i) \wedge \bigwedge_{i=k+1}^{l} (x \sim a_i) \wedge \bigwedge_{i=l+1}^{m} (x \not\sim a_i) \right\}.$$

Then the graph defined by $\varphi(x, \bar{a}) \vee \psi(x, \bar{a})$ has vertex set $A \cup B$. Note that $B$ might possibly contain $a_s$. If $B$ does contain $a_s$ then $s \in \{l+1, \ldots, m\}$. Arguing as in Lemma 4.4.13 we can show that $B' := B \setminus \{a_{l+1}, \ldots, a_m\}$ is isomorphic to $\boldsymbol{R}$ and that the graph defined by $\varphi(x, \bar{a}) \vee \psi(x, \bar{a})$ is isomorphic to the one defined by the n-moment $\bigwedge_{i=l+1}^{m} (x \not\sim a_i)$. Suppose that $B$ does not contain $a_s$. Then it is either the case that $s \in \{1, \ldots, j\}$, or $s \in \{j+1, \ldots, k\}$, or $s \in \{k+1, \ldots, l\}$,or $s \notin \{1, \ldots, m\}$.

- In case $s \in \{1, \ldots, j\}$ then, $B'$ is isomorphic to $\boldsymbol{R}$, and hence the graph defined by $\varphi(x, \bar{a}) \vee \psi(x, \bar{a})$ consists of a copy of $\boldsymbol{R}$ and extra vertices $\{a_s, a_{l+1}, \ldots, a_m\}$ such that no vertex in the copy of $\boldsymbol{R}$ is connected to any of the vertices in $\{a_s, a_{l+1}, \ldots, a_m\}$. This is isomorphic to the graph defined by the n-moment $\bigwedge_{i=l+1}^{m} (x \not\sim a_i)$.

- In case $s \in \{j+1, \ldots, k\}$, we know that $B'$ is isomorphic to $\boldsymbol{R}$. Now $B' \cup \{a_s\}$ will also satisfy EP. To see this, let $U$ and $V$ be finite disjoint subsets of $B' \cup \{a_s\}$, then the only sets possibly without a witness are sets containing $a_s$. Let $a_s \in U$. Then $U' = U \cup \{a_{k+1}, \ldots, a_l\}$ and $V' = V \cup \{a_1, \ldots, a_j, a_{l+1}, \ldots, a_m\}$ will be finite disjoint subsets of $\boldsymbol{R}$ and hence have a witness, say $w$ such that $w \sim U'$ and $w \not\sim V'$. This implies $w \in B'$ and $w \sim U$, in particular $w \sim a_s$, and $w \not\sim V$, i.e $w$ is the needed witness in $B' \cup \{a_s\}$. A similar argument can be used if $a_s \in V$, hence $B' \cup \{a_s\}$ is isomorphic to $\boldsymbol{R}$. Therefore the graph defined by $\varphi(x, \bar{a}) \vee \psi(x, \bar{a})$ consists of a copy of $\boldsymbol{R}$ and extra vertices $\{a_{l+1}, \ldots, a_m\}$ such that no vertex in the copy of $\boldsymbol{R}$ is connected to any of the vertices in $\{a_{l+1}, \ldots, a_m\}$. This is isomorphic to the graph defined by the n-moment $\bigwedge_{i=l+1}^m (x \not\sim a_i)$.
- In case $s \in \{k+1, \ldots, l\}$ then $B'$ is isomorphic to $\boldsymbol{R}$, and hence the graph defined by $\varphi(x, \bar{a}) \vee \psi(x, \bar{a})$ consists of a copy of $\boldsymbol{R}$ and extra vertices $\{a_s, a_{l+1}, \ldots, a_m\}$ such that each vertex in the copy of $\boldsymbol{R}$ is connected to $a_s$ and not connected to any of the vertices in $\{a_{l+1}, \ldots, a_m\}$. This is isomorphic to the graph defined by the disjunction of moments $x = a_s \vee ((x \sim a_s) \wedge \bigwedge_{i=l+1}^m (x \not\sim a_i))$.
- In case $s \notin \{1, \ldots, m\}$ the set $B'' := B' \cup \{a_s\}$ is countably infinite, and we argue that it has EP using augmentation. Let $U$ and $V$ be disjoint finite subsets of $B''$. Then $U' = U \cup \{a_{k+1}, \ldots, a_l\}$ and $V' = V \cup \{a_{l+1}, \ldots, a_m\}$ are disjoint finite subsets of $\boldsymbol{R}$ and will therefore have a witness, $w$ with $w \sim U'$ and $w \not\sim V'$. This implies that $w \in B''$ and $w \sim U$ and $w \not\sim V$, and hence $w$ is the needed witness. So the graph defined by $\varphi(x, \bar{a}) \vee \psi(x, \bar{a})$ consists of a copy of $\boldsymbol{R}$ and extra vertices $\{a_{l+1}, \ldots, a_m\}$ such that no vertex in the copy of $\boldsymbol{R}$ is connected to any of the vertices in $\{a_{l+1}, \ldots, a_m\}$. This is isomorphic to the graph defined by the n-moment $\bigwedge_{i=l+1}^m (x \not\sim a_i)$.

In each case the graph defined by $\varphi(x, \bar{a}) \vee \psi(x, \bar{a})$ is isomorphic to a disjunction of moments, which is the required result.

If $\varphi(x, \bar{a})$ is a c-moment: Suppose that $\varphi(x, \bar{a}) : \bigwedge_{s \in S} (x \sim a_s)$. Let

$$A = \left\{ x : \bigwedge_{s \in S} (x \sim a_s) \right\}$$

and

$$B = \left\{ x : \bigwedge_{i=1}^{j} (x \neq a_i \wedge x \not\sim a_i) \wedge \bigwedge_{i=j+1}^{k} (x \neq a_i) \wedge \bigwedge_{i=k+1}^{l} (x \sim a_i) \wedge \bigwedge_{i=l+1}^{m} (x \not\sim a_i) \right\}.$$

Then the graph $G$ defined by $\varphi(x, \bar{a}) \vee \psi(x, \bar{a})$ has vertex set $A \cup B$.

Note that the indices $s$ and $i$ might possibly overlap, but this will not make a difference in the argument. Let $B' := B \setminus \{a_{l+1}, \ldots, a_m\}$ and let $U$ and $V$ be finite disjoint subsets of $A \cup B'$. Then $U' := U \cup \{a_s : s \in S\}$ and $V$ are finite disjoint subsets of $\boldsymbol{R}$ and has a witness $w$ such that $w \sim U'$ and $w \not\sim V$. This implies that $w \in A \cup B'$ and $w \sim U$ and $w \not\sim V$, so that $w$ is a witness to EP for $U$ and $V$ in $A \cup B'$. So $G$ consist of a copy of $\boldsymbol{R}$ and extra vertices $\{a_{l+1}, \ldots, a_m\}$ with the vertices of $\boldsymbol{R}$ connected to the vertices in $\{a_{l+1}, \ldots, a_m\}$ depending only on the appearance of the parameters $a_{l+1}, \ldots, a_m$ in $\varphi(x, \bar{a})$ and the $\bigwedge_{i=l+1}^m (x \not\sim a_i)$ part of $\psi(x, \bar{a})$. Therefore $G$ is isomorphic to the graph defined by the disjunction of moments

$$\bigwedge_{s \in S} (x \sim a_s) \vee \bigwedge_{i=l+1}^{m} (x \not\sim a_i).$$

If $\varphi(x, \bar{a})$ is an n-moment: Suppose that $\varphi(x, \bar{a}) : \bigwedge_{s \in S} (x \not\sim a_s)$. Let

$$A = \left\{ x : \bigwedge_{s \in S} (x \not\sim a_s) \right\}$$

and
$$B = \left\{ x : \bigwedge_{i=1}^{j} (x \neq a_i \wedge x \not\sim a_i) \wedge \bigwedge_{i=j+1}^{k} (x \neq a_i) \wedge \bigwedge_{i=k+1}^{l} (x \sim a_i) \wedge \bigwedge_{i=l+1}^{m} (x \not\sim a_i) \right\}.$$
Then the graph $G$ defined by $\varphi(x, \bar{a}) \vee \psi(x, \bar{a})$ has vertex set $A \cup B$.

Again the indices $s$ and $i$ might possibly overlap, but this will not make a difference in the argument. Let $A' := A \setminus \{a_s : s \in S\}$ and $B' := B \setminus \{a_{l+1}, \ldots, a_m\}$. Consider for a moment the graph on only the vertices $A' \cup B'$ and let $U$ and $V$ be finite disjoint subsets of $A' \cup B'$. Then $U$ and $V' := V \cup \{a_s : s \in S\}$ are finite disjoint subsets of $\boldsymbol{R}$ and has a witness $w$ such that $w \sim U$ and $w \not\sim V'$. This implies that $w \in A' \cup B'$ and $w \sim U$ and $w \not\sim V$, so that $w$ is a witness to EP for $U$ and $V$ in $A' \cup B'$. So $G$ consist of a copy of $\boldsymbol{R}$ and extra vertices $\{a_s : s \in S\} \cup \{a_{l+1}, \ldots, a_m\}$ with the vertices of $\boldsymbol{R}$ connected to the vertices in $\{a_s : s \in S\} \cup \{a_{l+1}, \ldots, a_m\}$ depending only on the appearance of the parameter $a_s$ for $s \in S$, and $a_{l+1}, \ldots, a_m$ in $\varphi(x, \bar{a})$ and the $\bigwedge_{i=1}^{j}(x \not\sim a_i) \wedge \bigwedge_{i=k+1}^{l}(x \sim a_i) \wedge \bigwedge_{i=l+1}^{m}(x \not\sim a_i)$ part of $\psi(x, \bar{a})$. Therefore $G$ is isomorphic to the graph defined by the disjunction of moments
$$\bigwedge_{s \in S} (x \not\sim a_s) \vee (\bigwedge_{i=1}^{j} (x \not\sim a_i) \wedge \bigwedge_{i=k+1}^{l} (x \sim a_i) \wedge \bigwedge_{i=l+1}^{m} (x \not\sim a_i)).$$

<u>If $\varphi(x, \bar{a})$ is a cn-moment:</u> Suppose that $\varphi(x, \bar{a}) : \bigwedge_{s \in S}(x \sim a_s) \wedge \bigwedge_{t \in T}(x \not\sim a_t)$. Let
$$A = \left\{ x : \bigwedge_{s \in S} (x \sim a_S) \wedge \bigwedge_{t \in T} (x \not\sim a_t) \right\}$$
and
$$B = \left\{ x : \bigwedge_{i=1}^{j} (x \neq a_i \wedge x \not\sim a_i) \wedge \bigwedge_{i=j+1}^{k} (x \neq a_i) \wedge \bigwedge_{i=k+1}^{l} (x \sim a_i) \wedge \bigwedge_{i=l+1}^{m} (x \not\sim a_i) \right\}.$$
Then the graph $G$ defined by $\varphi(x, \bar{a}) \vee \psi(x, \bar{a})$ has vertex set $A \cup B$.

The indices $s$ and $t$ might possibly overlap with the index $i$, but as before this will not make a difference in the argument. Let $A' := A \setminus \{a_t : t \in T\}$ and $B' := B \setminus \{a_{l+1}, \ldots, a_m\}$. Consider for a moment the graph on $A' \cup B'$ and let $U$ and $V$ be finite disjoint subsets of $A' \cup B'$. Then $U' := U \cup \{a_s : s \in S\}$ and $V' := V \cup \{a_t : t \in T\}$ are finite disjoint subsets of $\boldsymbol{R}$, and have a witness $w$ such that $w \sim U'$ and $w \not\sim V'$. This implies that $w \in A' \cup B'$ and $w \sim U$ and $w \not\sim V$, so that $w$ is a witness to EP for $U$ and $V$ in $A' \cup B'$. So $G$ consist of a copy of $\boldsymbol{R}$ and extra vertices $C := \{a_t : t \in T\} \cup \{a_{l+1}, \ldots, a + m\}$, with the vertices of $\boldsymbol{R}$ connected to the vertices in $C$ depending only on the appearance of the parameter from $C$ in $\varphi(x, \bar{a})$ and the $\bigwedge_{i=1}^{j}(x \not\sim a_i) \wedge \bigwedge_{i=k+1}^{l}(x \sim a_i) \wedge \bigwedge_{i=l+1}^{m}(x \not\sim a_i)$ part of $\psi(x, \bar{a})$. Therefore $G$ is isomorphic to the graph defined by the disjunction of moments
$$\bigwedge_{t \in T} (x \not\sim a_t) \vee \bigwedge_{i=1}^{j} (x \not\sim a_i) \wedge \bigwedge_{i=j+1}^{k} (x \neq a_i) \wedge \bigwedge_{i=k+1}^{l} (x \sim a_i) \wedge \bigwedge_{i=l+1}^{m} (x \not\sim a_i).$$

There is of course also the possibility of both $\varphi(x, \bar{a})$ and $\psi(x, \bar{a})$ being non-moments. In this case, let
$$\varphi(x, \bar{a}) := \bigwedge_{h \in H} (x \neq a_h \wedge x \not\sim a_h) \wedge \bigwedge_{i \in I} (x \neq a_i) \wedge \bigwedge_{j \in J} (x \sim a_j) \wedge \bigwedge_{k \in K}^{m} (x \not\sim a_k)$$

with $H$, $I$, $J$ and $K$ mutually disjoint, and let

$$\psi(x,\bar{a}) := \bigwedge_{l\in L}(x \neq a_l \wedge x \nsim a_l) \wedge \bigwedge_{m\in M}(x \neq a_m) \wedge \bigwedge_{s\in S}(x \sim a_s) \wedge \bigwedge_{t\in T}(x \nsim a_t)$$

with $L$, $M$, $S$ and $T$ mutually disjoint. There might be an overlap of indices, but we know from the cases above that this is not a problem. Let

$$A = \left\{x : \bigwedge_{h\in H}(x \neq a_h \wedge x \nsim a_h) \wedge \bigwedge_{i\in I}(x \neq a_i) \wedge \bigwedge_{j\in J}(x \sim a_j) \wedge \bigwedge_{k\in K}^{m}(x \nsim a_k)\right\}$$

and

$$B = \left\{x : \bigwedge_{l\in L}(x \neq a_l \wedge x \nsim a_l) \wedge \bigwedge_{m\in M}(x \neq a_m) \wedge \bigwedge_{s\in S}(x \sim a_s) \wedge \bigwedge_{t\in T}(x \nsim a_t)\right\}.$$

Let $G$ be the graph defined by $\varphi(x,\bar{a}) \vee \psi(x,\bar{a})$. Then $G$ has vertex set $A \cup B$.

Let $A' := A \setminus \{a_k : k \in K\}$ and $B' := B \setminus \{a_t : t \in T\}$. We show that $A' \cup B'$ satisfies EP. Let $U$ and $V$ be finite disjoint subsets of $A' \cup B'$. Then

$$U' := U \cup \{a_j : j \in J\}$$

and

$$V' := V \cup \{a_h : h \in H\} \cup \{a_k : k \in K\}$$

are finite disjoint subsets of $\boldsymbol{R}$. So we will be able to find a witness $w$ such that $w \sim U'$ and $w \nsim V'$. This $w$ will be in $A' \cup B'$, and $w \sim U$ and $w \nsim V$, so that $w$ is a witness to EP for $U$ and $V$ in $A' \cup B'$. This implies that $G$ consists of a copy of $\boldsymbol{R}$ and extra vertices $C := \{a_k : k \in K\} \cup \{a_t : t \in T\}$. Whether or not the vertices in the copy of $\boldsymbol{R}$ are connected to the vertices in $C$ depend only on the inclusion of these parameters in the

$$\bigwedge_{h\in H}(x \nsim a_h) \wedge \bigwedge_{j\in J}(x \sim a_j) \wedge \bigwedge_{k\in K}^{m}(x \nsim a_k)$$

part of $\varphi(x,\bar{a})$ and the

$$\bigwedge_{l\in L}(x \nsim a_l) \wedge \bigwedge_{s\in S}(x \sim a_s) \wedge \bigwedge_{t\in T}(x \nsim a_t)$$

part of $\psi(x,\bar{a})$. Hence $G$ is isomorphic to the graph defined by the disjunction of moments

$$\left(\bigwedge_{h\in H}(x \nsim a_h) \wedge \bigwedge_{j\in J}(x \sim a_j) \wedge \bigwedge_{k\in K}^{m}(x \nsim a_k)\right) \vee \left(\bigwedge_{l\in L}(x \nsim a_l) \wedge \bigwedge_{s\in S}(x \sim a_s) \wedge \bigwedge_{t\in T}(x \nsim a_t)\right).$$

□

**Remarks 4.4.16.** 1. Supposing in the above proof that

$$\psi(x,\bar{a}) := \bigwedge_{i=1}^{k}(x \neq a_i) \wedge \bigwedge_{i=k+1}^{l}(x \sim a_i) \wedge \bigwedge_{i=l+1}^{m}(x \nsim a_i)$$

or

$$\psi(x,\bar{a}) := \bigwedge_{i=1}^{k}(x \neq a_i \wedge x \nsim a_i) \wedge \bigwedge_{i=k+1}^{l}(x \sim a_i) \wedge \bigwedge_{i=l+1}^{m}(x \nsim a_i)$$

rather than

$$\psi(x,\bar{a}) := \bigwedge_{i=1}^{j}(x \neq a_i \wedge x \not\sim a_i) \wedge \bigwedge_{i=j+1}^{k}(x \neq a_i) \wedge \bigwedge_{i=k+1}^{l}(x \sim a_i) \wedge \bigwedge_{i=l+1}^{m}(x \not\sim a_i)$$

makes no difference in the steps or outcomes of the arguments, as should be clear from the two proofs of Lemma 4.4.11, Lemma 4.4.12, and Lemma 4.4.13.

2. Notice that the disjunction of moments, which define a graph $H$ that is isomorphic to the graph $G$ defined by $\varphi(x,\bar{a}) \vee \psi(x,\bar{a})$, can be written using only parameters that were present in $\varphi(x,\bar{a}) \vee \psi(x,\bar{a})$. So in case the graph $G$ defined by $\varphi(x,\bar{a}) \vee \psi(x,\bar{a})$ is finite, or consists of a copy of $\boldsymbol{R}$ and some finite number of extra vertices, then the finite part of $G$ is in fact exactly the same graph as the finite part of $H$ (not just isomorphic to) defined by a disjunction of $=$-moments.

**Proposition 4.4.17.** *Every DNF consisting of $L(\sim)$-literals in at most $n$ parameters defines, up to isomorphism, the same graph as a disjunction of moments.*

*Proof.* Let $\varphi(x,\bar{a})$ be a DNF as described above.

Suppose that $\varphi(x,\bar{a})$ does not contain a literal of the form $x \neq a_j$ then each constituent CF will only contain literals of the form $x = a_j$, $x \sim a_j$ and $x \not\sim a_j$, and hence is a moment. Therefore $\varphi(x,\bar{a})$ defines up to isomorphism the same graph as a disjunction of moments.

Next suppose that $\varphi(x,\bar{a})$ contains literals of the form $x \neq a_j$. If $\varphi(x,\bar{a})$ contains $x \neq a_j$ as a CF for some $j$, then $\varphi(x,\bar{a})$ defines a graph isomorphic to $\boldsymbol{R}$ and so defines the same graph, up to isomorphism, as any c-moment.

Suppose therefore that if $\varphi(x,\bar{a})$ contains literals of the form $x \neq a_j$ then it is part of a CF that contains other literals also. $\varphi(x,\bar{a})$ might possibly contain more than one CF with literals of the form $x \neq a_j$. Let's write $\varphi(x,\bar{a})$ as $\psi_1 \vee \cdots \vee \psi_m \vee \bigvee_{i\in I}\varphi_i$, where each $\psi_j$ is a CF containing literals of the form $x \neq a_j$ and each $\varphi_i$ is a CF with no such literals. In this case each $\varphi_i$ is a moment and so $\bigvee_i \varphi_i$ is a disjunction of moments.

Let $H$ be the graph defined by $\psi_m \vee \varphi_k$ for some $k \in I$ and $F$ be the graph defined by $\bigvee_{i\in I\setminus\{k\}}\varphi_i$. From Lemma 4.4.15 we have that $H$ is isomorphic to a graph $H'$ defined by a disjunction of moments, say $\bigvee_{j\in J}\varphi_j$. Simply considering the graph on $V(H') \cup V(F)$ it is easy to argue that this graph need not be isomorphic to the one defined by $\psi_m \vee \bigvee_{i\in I}\varphi_i$, i.e. the one on $V(H) \cup V(F)$, since we also have to consider how the vertices of $H$ might be connected to the vertices of $F$. But, from Remark 4.4.16 (2), the finite parts (if there are any) of $H$ and $H'$ are the same graphs, that is, on the exact same parameters, since $\bigvee_{j\in J}\varphi_j$ can be written using only the parameters present in $\psi_m \vee \varphi_k$. In this case the vertices in the finite part of $H'$ will be connected to the vertices in $F$ in exactly the same way as the vertices in the finite part of $H$. The only problem might then be the countably infinite parts (if there are any) of the graphs under consideration. We know however, from the proofs of Lemma 4.4.11, Lemma 4.4.12, Lemma 4.4.13 and Lemma 4.4.15, that these countably infinite parts are just copies of $\boldsymbol{R}$, and so we can establish, using an augmentation argument and the fact that $\boldsymbol{R}$ has EP (in the same way as in the proofs of the mentioned lemmas), the necessary connections between $H'$ and $F$ such that the graph on $V(H') \cup V(F)$ is isomorphic to the one on $V(H) \cup V(F)$.

This implies that $\psi_m \vee \bigvee_{i\in I}\varphi_i$ defines up to isomorphism the same graph in $\boldsymbol{R}$ as a disjunction of moments, say $\bigvee k \in K\varphi_k$.

We can repeat this argument to show that $\psi_{m-1} \vee \bigvee k \in K\varphi_k$ defines up to isomorphism the same graph in $\boldsymbol{R}$ as a disjunction of moments, say $\bigvee l \in L\varphi_l$, and again to show that $\psi_{m-2} \vee \bigvee l \in L\varphi_l$ defines up to isomorphism the same graph in $\boldsymbol{R}$ as a disjunction of moments. Repeating the argument $m-3$ more times, i.e. until $\psi_1 \vee \mu$, where $\mu$ is a disjunction of moments, shows that $\varphi(x,\bar{a})$ defines, up to isomorphism, the same graph as a disjunction of moments, which is the required result. □

We can now state and prove the main result of this section.

**Theorem 4.4.18.** *The graphs definable in $\boldsymbol{R}$ using $n$ parameters are*

1. *all finite graphs of order $\leq n$,*
2. *$\boldsymbol{R}$ itself,*
3. *and for each $k \leq n$, a graph consisting of a copy of $\boldsymbol{R}$ and a finite graph on say $\{a_1, \ldots, a_k\}$, with the vertices of $\boldsymbol{R}$ connected to the vertices of $\{a_1, \ldots, a_k\}$ depending only on use of the literals $x \sim a_j$ or $x \not\sim a_j$ for $a_j \in \{a_1, \ldots, a_k\}$, in the $L(\sim)$-formula which defines the graph.*

*Proof.*

1. From Theorem 2.1.6 every finite graph is an induced subgraph of $\boldsymbol{R}$. Let $G$ be such a graph and let each vertices in $G$ be a parameter in an $=$-moment. Then we can define $G$ in $\boldsymbol{R}$ with $\bigvee_{g \in G} x = g$.
2. The formula $x \sim a$ will be enough to define $\boldsymbol{R}$.
3. As discussed in the beginning of this chapter, every quantifier free formula is logically equivalent to a boolean combination of literals and hence to a disjunctive normal form of such a formula. Since Th$\boldsymbol{R}$ has quantifier elimination, i.e. every $L(\sim)$-formula is Th$\boldsymbol{R}$-equivalent to a quantifier free one, all possible graphs definable in $\boldsymbol{R}$ will be defined by formulas of the form $\bigvee_i \bigwedge_j \varphi_{ij}$, for $i$ and $j$ finite, with each $\varphi_{ij}$ a literal. Proposition 4.4.17 implies that each such DNF defines in $\boldsymbol{R}$ a graph isomorphic to one defined by a disjunction of moments. This means that each graph definable in $\boldsymbol{R}$ is definable by a disjunction of moments.

   Now, each graph definable in $\boldsymbol{R}$, consisting of a copy of $\boldsymbol{R}$ and a finite graph on say $\{a_1, \ldots, a_k\}$, defined using a disjunction of moments, and hence, the vertices of $\boldsymbol{R}$ will be connected to the vertices of $\{a_1, \ldots, a_k\}$ depending only on the use of $a_j \in \{a_1, \ldots, a_k\}$ as a parameter in this disjunction of moments, which is the required result.

□

**Remark 4.4.19.** Note that we could just as well have used $x = x$ to define $\boldsymbol{R}$ in the proof above. We used $x \sim a$ instead, to emphasize that all the graphs definable in $\boldsymbol{R}$, are definable using moments.

## 4.5 Age is not just a number: Fraïssé's Theorem and the Rado graph

We have seen many wonderful properties of our beast on our journey so far and now the time has come for us to look at its age.

**Definition 4.5.1** (**Generated substructure**, see [Rot00] pg. 76)**.** *Let $\mathcal{M}$ be an $L$-structure and $X \subseteq M$. The structure $\mathcal{M}_X$, called the* substructure generated *by $X$ in $\mathcal{M}$, is the substructure of $\mathcal{M}$ with universe $\cap\{N : X \subseteq N, \mathcal{N} \subseteq \mathcal{M}\}$.*

*In case $\mathcal{M} = \mathcal{M}_X$ we say that $\mathcal{M}$ is* generated *by $X$.*

When a structure is generated by a finite set $X$, we will say that it is *finitely generated*.

**Definition 4.5.2** (**Age**, see [Hod93] pg. 324)**.** *Let $\mathcal{M}$ be an $L$-structure. The class $\boldsymbol{K}$ consisting of all finitely generated structures embeddable in $\mathcal{M}$ is called the* age *of $\mathcal{M}$.*

Let $\boldsymbol{G}$ be the class of all finite graphs. Now for something much easier than carbon dating, it is clear from Theorem 2.1.6 that all finite graphs are embeddable in $\boldsymbol{R}$. This means that $\boldsymbol{G}$ is the age of $\boldsymbol{R}$. Let's see if we can get more information on the age of our beast.

Let $\boldsymbol{K}$ be the age of some $L$-structure $\mathcal{M}$. $\boldsymbol{K}$ can have any of the following properties.

**Property 4.5.3** (**The Hereditary Property**, HP for short, see [Hod93] pg. 324)**.** *Let $\mathcal{N}_0 \in \boldsymbol{K}$ and $\mathcal{N}_1$ be a finitely generated substructure of $\mathcal{N}_0$, then there is an $\mathcal{N} \in \boldsymbol{K}$ such that $\mathcal{N} \cong \mathcal{N}_1$.*

**Property 4.5.4** (**The Joint Embedding Property**, JEP for short, see [Hod93] pg. 324)**.** *For $\mathcal{N}_0, \mathcal{N}_1 \in \boldsymbol{K}$ there is an $\mathcal{N} \in \boldsymbol{K}$ such that $\mathcal{N}_0$ and $\mathcal{N}_1$ are embeddable in $\mathcal{N}$.*

**Property 4.5.5** (**The Amalgamation Property**, AP for short, see [Hod93] pg. 325)**.** *Let $\mathcal{N}_0, \mathcal{N}_1, \mathcal{N}_2 \in \boldsymbol{K}$ with $e : \mathcal{N}_0 \hookrightarrow \mathcal{N}_1$ and $f : \mathcal{N}_0 \hookrightarrow \mathcal{N}_2$. Then there is an $\mathcal{N} \in \boldsymbol{K}$ with $g : \mathcal{N}_1 \hookrightarrow \mathcal{N}$ and $h : \mathcal{N}_2 \hookrightarrow \mathcal{N}$ and $g \circ e = h \circ f$.*

It is not hard to show that $\boldsymbol{G}$ has HP and JEP, but what about AP? We have previously defined what it means for a graph to be ultrahomogeneous. We give a more general definition in terms of structures.

**Definition 4.5.6** (**Ultrahomogeneous**, see [Rot00] pg. 135)**.** *Consider an $L$-structure $\mathcal{M}$. If every isomorphism between finitely generated substructures of $\mathcal{M}$ can be extended to an automorphism of $\mathcal{M}$, then we say that $\mathcal{M}$ is* ultrahomogeneous.

**Definition 4.5.7** (**Weakly homogeneous**, see [Hod93] pg. 326)**.** *We say that an $L$-structure $\mathcal{M}$ is* weakly homogeneous *if for any finitely generated substructures $\mathcal{N}_0$ and $\mathcal{N}_1$ of $\mathcal{M}$ with $\mathcal{N}_0 \subseteq \mathcal{N}_1$ and $f : \mathcal{N}_0 \hookrightarrow \mathcal{M}$, there is an embedding $g : \mathcal{N}_1 \hookrightarrow \mathcal{M}$ which extends $f$.*

In case a structure is ultrahomogeneous, it is easy to see that it will also be weakly homogeneous.

The next result will help us to see whether $\boldsymbol{G}$ has AP.

**Theorem 4.5.8** (see [Hod93], Theorem 7.1.7, pg. 329)**.** *Let $L$ be a countable language and $\boldsymbol{K}$ be the age of a finite or countable ultrahomogeneous $L$-structure, $\mathcal{M}$. Then $\boldsymbol{K}$*

1. *is non-empty,*
2. *has at most countably many isomorphism classes and*
3. *has HP, JEP and AP.*

*Proof.* Let $\boldsymbol{K}$ be the age of a finite or countable ultrahomogeneous $L$-structure, $\mathcal{M}$.

$\boldsymbol{K}$ is non-empty: This follows from the fact that $\boldsymbol{K}$ will contain all finitely generated substructures of $\mathcal{M}$.

$\boldsymbol{K}$ has at most countably many isomorphism classes: Each structure in $\boldsymbol{K}$ will be isomorphic to a structure generated by a finite subset of $M$. Since there are at most countably many finite subsets of a countable set, there will be at most countably many isomorphism classes.

$\boldsymbol{K}$ has HP: Let $\mathcal{N}_0 \in \boldsymbol{K}$ with $f : \mathcal{N}_0 \hookrightarrow \mathcal{M}$. Now for any $\mathcal{N}_1$, which is a finitely generated substructure of $\mathcal{N}_0$, restricting $f$ to $\mathcal{N}_1$ gives and embedding of $\mathcal{N}_1$ into $\mathcal{M}$. This means that $\mathcal{N}_1 \in \boldsymbol{K}$, and of course $\mathcal{N}_1 \cong \mathcal{N}_1$, which is the required result.

$\boldsymbol{K}$ has JEP: Let $\mathcal{N}_0, \mathcal{N}_1 \in \boldsymbol{K}$. This means that $\mathcal{N}_0 \cong \mathcal{M}' \subseteq \mathcal{M}$ and $\mathcal{N}_1 \cong \mathcal{M}'' \subseteq \mathcal{M}$. Both $\mathcal{M}'$ and $\mathcal{M}''$ will be finitely generated substructures of $\mathcal{M}$, say by $X$ and $Y$. Then the structure $\mathcal{N}$ generated by $Z = X \cup Y$ will be in $\boldsymbol{K}$ with both $\mathcal{N}_0$ and $\mathcal{N}_1$ embeddable in $\mathcal{N}$.

$\boldsymbol{K}$ has AP: Let $\mathcal{N}_0, \mathcal{N}_1, \mathcal{N}_2 \in \boldsymbol{K}$ with $e : \mathcal{N}_0 \hookrightarrow \mathcal{N}_1$ and $f : \mathcal{N}_0 \hookrightarrow \mathcal{N}_2$. Since $\mathcal{N}_0$ is embeddable in $\mathcal{M}$ we can find an isomorphism $k_0 : \mathcal{N}_0 \cong \mathcal{N}_0'$ where $\mathcal{N}_0'$ is a substructure of $\mathcal{M}$. Note that $k_0 \circ e^{-1}$ will give an embedding of $e[\mathcal{N}_0]$ into $\mathcal{M}$. But $\mathcal{M}$ is ultrahomogeneous and hence weakly homogeneous, so there is a $g$ which extends $k_0 \circ e^{-1}$ to an embedding of $\mathcal{N}_1$ into $\mathcal{M}$. In a similar way we can find an $h$ which extends $k_0 \circ f^{-1}$ to an embedding of $\mathcal{N}_2$ into $\mathcal{M}$. Let $\mathcal{N}$ be the structure generated by $g[\mathcal{N}_1] \cup h[\mathcal{N}_2]$. Then $g : \mathcal{N}_1 \hookrightarrow \mathcal{N}$ and $h : \mathcal{N}_2 \hookrightarrow \mathcal{N}$. Also

$$(g \circ e)([\mathcal{N}_0]) = (k_0 \circ e^{-1} \circ e)([\mathcal{N}_0]) = k_0[\mathcal{N}_0] = (k_0 \circ f^{-1} \circ f)([\mathcal{N}_0]) = (h \circ f)([\mathcal{N}_0])$$

which is saying that $g \circ e = h \circ f$, as needed. Hence $\boldsymbol{K}$ has AP. $\square$

**Theorem 4.5.9.** $\boldsymbol{G}$ *has HP, JEP and AP.*

*Proof.* From Theorem 4.5.8, Proposition 3.1.2 ($\boldsymbol{R}$ is ultrahomogeneous) and the fact that $\boldsymbol{G}$ is the age of $\boldsymbol{R}$, we have that $\boldsymbol{G}$ has HP, JEP and AP. □

We can take this further.

**Theorem 4.5.10** (**Fraïssé's Theorem**, see [Hod93] pg. 326)**.** *Consider a countable language $L$ and let $\boldsymbol{K}$ be a non-empty finite or countable set of finitely generated $L$-structures and suppose that $\boldsymbol{K}$ has HP, JEP and AP. Then there is an $L$-structure $\mathcal{M}$ such that*

1. $|\mathcal{M}| \leq \omega$,
2. $\boldsymbol{K}$ *is the age of* $\mathcal{M}$ *and*
3. $\mathcal{M}$ *is ultrahomogeneous.*

$\mathcal{M}$ *is unique up to isomorphism and we call* $\mathcal{M}$ *the* Fraïssé limit *of* $\boldsymbol{K}$*.*

The proof of Fraïssé's Theorem consists of a uniqueness proof and an existence proof. We will not go into the details here, or anywhere along the journey. Those who are interested to see these proofs are referred to [Hod93].

**Theorem 4.5.11.** $\boldsymbol{R}$ *is the Fraïssé limit of the class of all finite graphs.*

*Proof.* We know from Theorem 4.5.9 that $\boldsymbol{G}$ has all the necessary properties, so from Theorem 4.5.10 there is a countable ultrahomogeneous graph $\mathcal{G}$ such that $\boldsymbol{G}$ is the age of $\mathcal{G}$. But this $\mathcal{G}$ is unique up to isomorphism, hence $\boldsymbol{R} \cong \mathcal{G}$, i.e. $\boldsymbol{R}$ is the Fraïssé limit of $\boldsymbol{G}$. □

## 4.6 Glowing splendor: The resplendence of the Rado graph

> There must be a beginning of any great matter, but the continuing unto the end until it be thoroughly finished yields the true glory.
>
> Sir Francis Drake

As we look upon the beast which we have pursued on this journey, we are captured by its brilliance. One might describe it as resplendent, without knowing how accurate the description is.

**Definition 4.6.1** (**Resplendent**, see [Kos11] pg. 813)**.** *Let $\mathcal{M}$ be an $L$-structure. We say that $\mathcal{M}$ is* resplendent *if for each sentence $\varphi(\bar{a}, R)$ with a new (not already in $L$) relation symbol $R$ and $\bar{a}$ from $\mathcal{M}$, if $\{\varphi(\bar{a}, R)\}$ is consistent with $Th(\mathcal{M}, \bar{a})$, then $\varphi(\bar{a}, R)$ is satisfied in some expansion of $\mathcal{M}$, i.e. $\varphi(\bar{a}, R)$ is true in $(\mathcal{M}, R_M)$ for some relation $R_M$ on $M$.*

This is the same as saying that if any elementary extension of $\mathcal{M}$ has a relation with a first-order property, then $\mathcal{M}$ also has such a relation. This is the definition of a resplendent structure in [BS76].

Before we see that $\boldsymbol{R}$ is, in every sense of the word, resplendent, we list some properties of resplendent structures in the next theorem, which enables us to see why resplendent structures are resplendent.

**Theorem 4.6.2** (see [BS76] pg. 534 and [Kos11] pg. 813)**.** *1. Every $L$-structure $\mathcal{M}$ has a resplendent elementary extension, say $\mathcal{M}'$, of the same cardinality.*

*2. Let $\mathcal{M}$ be a resplendent $L$-structure and $(\mathcal{M}, \bar{a}) \equiv (\mathcal{M}, \bar{b})$ with $n$-tuples $\bar{a}$ and $\bar{b}$ from $M$. Then there is an automorphism, $f$ of $\mathcal{M}$ such that $f(a_i) = b_i$ for all $i = 1, \ldots, n$. This is the same as saying that $\mathcal{M}$ is ultrahomogeneous.*

*3. Every infinite resplendent $L$-structure has nontrivial automorphisms.*

*4. Every countably infinite resplendent $L$-structure has continuum many nontrivial automorphisms.*

*5. Every infinite resplendent $L$-structure is isomorphic to one of its own proper elementary substructures.*

*6. Every infinite definable subset of a resplendent $L$-structure $\mathcal{M}$ has the same power as $\mathcal{M}$.*

*7. For every countably infinite resplendent $L$-structure $\mathcal{M}$, the expansion $(\mathcal{M}, R_M)$, with $R_M$ as in Definition 4.6.1, is also resplendent. This is known as* chronic *resplendence.*

This list is not exhaustive of all the properties or applications in which resplendent structures are involved, but we hope that it has shed some light on why resplendency is such a marvellous quality for a structure to have.

It is mentioned in [Kos11] (see pg. 813) that "any finite structure is resplendent". The proof of is fact is routine, and we consider it before looking at the resplendence of $\boldsymbol{R}$.

**Proposition 4.6.3.** *Every finite $L$-structure is resplendent.*

*Proof.* Let $\mathcal{M}$ be any finite $L$-structure, but suppose on the contrary that $\mathcal{M}$ is not resplendent. Then there is an $\mathcal{N} \succcurlyeq \mathcal{M}$ and an $L(R)$-formula $\exists R\varphi(\bar{x}, R)$ such that $\mathcal{N} \models \exists R\varphi(\bar{a}, R)$ but $\mathcal{M} \not\models \exists R\varphi(\bar{a}, R)$, with $\bar{a}$ from $\mathcal{M}$.

But $\mathcal{M} \preccurlyeq \mathcal{N}$ implies that $\mathcal{M} \equiv \mathcal{N}$ and since $\mathcal{M}$ is finite, this is equivalent to $\mathcal{M} \cong \mathcal{N}$, which is in contradiction with the statement that $\mathcal{N} \models \exists R\varphi(\bar{a}, R)$ but $\mathcal{M} \not\models \exists R\varphi(\bar{a}, R)$.

Hence, $\mathcal{M}$ must be resplendent. □

Let us return now to the topic of $\boldsymbol{R}$ and its resplendence.

**Theorem 4.6.4.** *$\boldsymbol{R}$ is resplendent.*

We will go about showing that $\boldsymbol{R}$ is resplendent in two ways.

**Definition 4.6.5** (**Recursive set**, see [BBJ02])**.** *A* recursive set *of $L$-formulas, $\Phi$, is a set for which there is an algorithm[8] to determine whether a given $L$-formula belongs to $\Phi$.*

**Definition 4.6.6** (**Recursively saturated**, see [BS76] pg. 531)**.** *We say that an $L$-structure $\mathcal{M}$ is* recursively saturated *if, for every recursive set $\Phi(x, \bar{y})$ of $L$-formulas, with $\bar{y}$ an $n$-tuple (possibly empty), we have that*

$$\mathcal{M} \models \forall \bar{y}\left( \bigwedge_{\Phi_0 \subseteq \Phi; \Phi_0 \ \mathit{finite}} \exists x \bigwedge \Phi_0(x, \bar{y}) \rightarrow \exists x \Phi(x, \bar{y}) \right).$$

Before we continue, we mention the following result that will help us make a connection between the concepts of saturated and recursively saturated.

**Proposition 4.6.7.** *The following are equivalent.*

*1. $\Phi$ is a type of $\mathcal{M}$.*

*2. $\mathcal{M} \models \exists \bar{x} \bigwedge \Phi_0$ for every finite subset $\Phi_0 \subseteq \Phi$.*

[8] Refer to Appendix B for more on algorithms.

*Proof.* Note firstly that a type of $\mathcal{M}$ is nothing more that a set of $L(M)$-formulas which is consistent with $\mathrm{Th}(\mathcal{M}, M)$. For an $n$-tuple $\bar{c}$ of new constant symbols, we can write this more clearly as $\mathrm{Th}(\mathcal{M}, M) \cup \Phi(\bar{c})$ is consistent.

It now follows, from the Compactness theorem (Theorem 4.1.9), that $\Phi$ is a type of $\mathcal{M}$ iff every finite subset of $\Phi$ is a type of $\mathcal{M}$. This is the same as saying, for every finite subset $\Phi_0 \subseteq \Phi$, that $\bigwedge \Phi_0$ is satisfied in some elementary extension of $\mathcal{M}$ and hence also in $\mathcal{M}$. This gives the required result. $\square$

Notice that the "$\bigwedge_{\Phi_0 \subseteq \Phi; \Phi_0 \text{ finite}} \exists x \bigwedge \Phi_0(x, \bar{y})$" part of Definition 4.6.6 together with Proposition 4.6.7 tells us that $\Phi$ is a 1-type of $\mathcal{M}$. If $\mathcal{M}$ is saturated then it would realize $\Phi$ so we get the "$\exists x \Phi(x, \bar{y})$" part of Definition 4.6.6. This means that any saturated structure will be recursively saturated.

Here is the result needed for our first method.

**Lemma 4.6.8** (see [BS76] pg. 534)**.** *Let $\mathcal{M}$ be any countably infinite $L$-structure. If $\mathcal{M}$ is recursively saturated, then $\mathcal{M}$ is resplendent.*

*Proof 1 of Theorem 4.6.4.* We know that $\boldsymbol{R}$ is countably infinite and so, by Lemma 4.6.8, if we can show that $\boldsymbol{R}$ is recursively saturated, then we will have that $\boldsymbol{R}$ is resplendent.

So let $\Phi(x, \bar{y})$ be any recursive set of $L$-formulas, with $\bar{y}$ an $n$-tuple of variables. Suppose that

$$\boldsymbol{R} \models \forall \bar{y} \big( \exists x \big( \bigwedge \Phi_0(x, \bar{y}) \big) \big)$$

for every finite subset $\Phi_0 \subseteq \Phi$.

We can restate this as $\boldsymbol{R} \models \exists x \big( \bigwedge \Phi_0(x, \bar{a}) \big)$ for all $n$-tuples $\bar{a}$ from $\boldsymbol{R}$. Using Proposition 4.6.7, this gives us that $\Phi$ is a type of $\boldsymbol{R}$.

We have seen, as stated in Proposition 4.3.8, that $\boldsymbol{R}$ is saturated, meaning that it realises all its $n$-types over finitely many parameters. So $\boldsymbol{R} \models \exists x \Phi(x, \bar{a})$ for all $n$-tuples $\bar{a}$ form $\boldsymbol{R}$. Now we have

$$\boldsymbol{R} \models \forall \bar{y} (\exists x \Phi(x, \bar{y}).$$

In conclusion, $\boldsymbol{R}$ is recursively saturated, and hence resplendent. $\square$

Now let's do the other proof.

*Proof 2 of Theorem 4.6.4.* Firstly, by Theorem 4.6.2 part 1, we are able to find a resplendent elementary extension $\mathcal{M}$ of $\boldsymbol{R}$ such that $\mathcal{M}$ is countably infinite. But now, using the $\aleph_0$-categoricity of $\mathrm{Th}\boldsymbol{R}$ we have that $\boldsymbol{R} \cong \mathcal{M}$. Therefore we have that $\boldsymbol{R}$ is resplendent. $\square$

We can extend this argument to give a more general result.

**Proposition 4.6.9** (see [Kos11] pg. 813)**.** *Any model of a $\kappa$-categorical theory of power $\kappa$, is resplendent.*

*Proof.* Let $T$ be a $\kappa$-categorical $L$-theory and $\mathcal{M}$ a model of $T$ with $|\mathcal{M}| = \kappa$.

Using Theorem 4.6.2 part 1, we can find a resplendent $\mathcal{N} \succcurlyeq \mathcal{M}$ with $|\mathcal{N}| = \kappa$. Since $T$ is $\kappa$-categorical, we have that $\mathcal{N} \cong \mathcal{M}$, giving the resplendency of $\mathcal{M}$. $\square$

# Chapter 5

# The 0-1 Forest

> How often have I said to you that when you have eliminated the impossible, whatever remains, however improbable, must be the truth?
>
> SHERLOCK HOLMES
> *The Sign of the Four*

When working in probability theory, a 0-1 law states that under specific conditions, certain events have either a probability of 0 or a probability of 1 of occurring. We are going to use $\boldsymbol{R}$ to obtain a 0-1 law for graphs, i.e. that an $L(\sim)$-sentence has either probability 0 or probability 1. We will also use this 0-1 law to prove some other nice results about $\boldsymbol{R}$.

## 5.1 Law-abiding citizens: A 0-1 law for graphs

Before deriving a 0-1 law for graphs, we need to define what we mean by the probability of a sentence, and give conditions under which they will have either probability 0 or 1. Let's tie up these loose ends.

**Definition 5.1.1** (**Probability of a sentence**, see [Abr18] pg. 5)**.** *Consider an $L$-sentence $\varphi$ and let $P_n(\varphi)$ be the fraction of all $L$-structures of size $n$ that model $\varphi$. Then the limit $P(\varphi) := \lim_{n\to\infty} P_n(\varphi)$, if it exists, is the* (labeled asymptotic) probability *of $\varphi$.*

*If $P(\varphi) = 1$ we say that $\varphi$ is* almost surely true *and in case $P(\varphi) = 0$ we say that $\varphi$ is* almost surely false*.*

Intuitively one would imagine that if a sentence is not almost surely true, then it must be almost surely false. Similarly, if two sentences are almost surely true, it feels natural to say that their conjunction must also be almost surely true. The next lemma shows that this is in deed the case.

**Lemma 5.1.2** (see [Abr18] pg. 6)**.** *1. $P(\varphi) = 1$ iff $P(\neg\varphi) = 0$.*

*2. $P(\varphi \wedge \psi) = 1$ iff $P(\varphi) = 1$ and $P(\psi) = 1$.*

*Proof.* 1. Note that if an $L$-structure models $\varphi$ then it cannot be a model of $\neg\varphi$ and vise versa. This means that $P_n(\neg\varphi) = 1 - P_n(\varphi)$. We thus have

$$\begin{aligned} 1 - P(\varphi) &= 1 - \lim_{n\to\infty} P_n(\varphi) \\ &= \lim_{n\to\infty} (1 - P_n(\varphi)) \\ &= \lim_{n\to\infty} P_n(\neg\varphi) \\ &= P(\neg\varphi) \end{aligned}$$

giving us $P(\neg\varphi) = 0$ in case $P(\varphi) = 1$.

The reverse implication is obtained in the same way.

2. Suppose that $P(\varphi \wedge \psi) = \lim_{n\to\infty} P_n(\varphi \wedge \psi) = 1$. For any $L$-structure to model $\varphi \wedge \psi$ it needs to model both $\varphi$ and $\psi$. This implies that $P_n(\varphi) > P_n(\varphi \wedge \psi)$ since a model of $\varphi$ need not be a model of $\psi$. For the same reason $P_n(\psi) > P_n(\varphi \wedge \psi)$. It now follows that both $P(\varphi) = \lim_{n\to\infty} P_n(\varphi) = 1$ and $P(\psi) = \lim_{n\to\infty} P_n(\psi) = 1$. For the other way, suppose that $P(\varphi) = 1$ and $P(\psi) = 1$. From the inclusion-exclusion principle we have that $P_n(\varphi) + P_n(\psi) - P_n(\varphi \wedge \psi) \leq 1$. This gives us $P_n(\varphi) + P_n(\psi) - 1 \leq P_n(\varphi \wedge \psi) \leq 1$. Finally we take the limit as $n$ goes to $\infty$ to obtain $1 \leq P(\varphi \wedge \psi) \leq 1$, which means that $P(\varphi \wedge \psi) = 1$.

□

Restricting our attention to graphs, we need only to consider $L(\sim)$-sentences. So for any $L(\sim)$-sentence, $\varphi$, $P_n(\varphi)$ is the number of graphs on $n$ vertices that model $\varphi$ divided by the total number of graphs on $n$ vertices.

**Theorem 5.1.3** (**0-1 Law for Graphs**, see [Fag76] pg. 52)**.** *Every $L(\sim)$-sentence is either almost surely true or almost surely false.*

As mentioned, we need the help of $\boldsymbol{R}$ or at least the essence of $\boldsymbol{R}$ to prove this law. Recall from Chapter 4 the extension axioms $\varphi_{m,n}$ which were used to describe EP.

$$\varphi_{m,n} := (\forall x_1, \ldots, x_m, y_1, \ldots, y_n)\left(\left(\bigwedge_{\forall i,\forall j} x_i \neq y_j\right) \rightarrow (\exists z)\left(\left(\bigwedge_{\forall i} z \sim x_i\right) \wedge \left(\bigwedge_{\forall j} z \neq y_j \wedge z \not\sim y_j\right)\right)\right)$$

**Lemma 5.1.4** (see [Hed04], Lemma 5.35, pg. 218)**.** *Each $\varphi_{m,n}$ is almost surely true.*

*Proof.* Fix $m, n \in \mathbb{N}_0$ and let $k = N + n + m$ for some $N \in \mathbb{N}$.

Let $\mathcal{G}$ be a graph on $k$ vertices and let $U = \{x_1, \ldots, x_m\}$ and $V = \{y_1, \ldots, y_n\}$ be sets of vertices of $\mathcal{G}$ with $U \cap V = \emptyset$. If $\mathcal{G} \models \varphi_{m,n}$ then there is a $z \in G \setminus (U \cup V)$ such that $z \sim U$ and $z \not\sim V$. We will calculate the probability that this is not the case.

For any of the $N$ vertices left to choose from $G$, the probability of being connected to each vertex in $U$ is $\frac{1}{2^m}$ and the probability of not being connected to any vertex in $V$ is $\frac{1}{2^n}$. Hence, the probability of a given $z$ witnessing $\varphi_{m,n}$ for $U$ and $V$ is $\frac{1}{2^{m+n}}$. The probability of a given $z$ not witnessing this sentence is therefore $1 - \frac{1}{2^{m+n}} := \delta$. This means that the chance of none of the $N$ vertices being the needed $z$ is $\delta^N$.

This needs to be done for all possible choices of sets $U$ and $V$. There are $\binom{k}{m+n}$ ways to pick vertices for $U$ and $V$ and $\binom{m+n}{m}$ ways to choose $m$ of these vertices for $U$ and the remaining $n$ vertices for $V$. This means that there are $\binom{k}{m+n}\binom{m+n}{m}$ possible choices of sets $U$ and $V$.

For $\mathcal{G} \models \neg\varphi_{m,n}$ we need to find only once choice of $U$ and $V$ for which there is no $z$. So

$$\begin{aligned} P_k(\neg\varphi_{m,n}) &= \binom{k}{m+n}\binom{m+n}{m}\delta^N \\ &= \frac{k!}{N!m!n!}\delta^N \end{aligned}$$

Note that

$$\begin{aligned}
\frac{k!}{N!} &= \frac{(N+m+n)!}{N!} \\
&= \frac{(N+m+n)(N+m+n-1)\cdots(N+1)(N)(N-1)\cdots(2)(1)}{(N)(N-1)\cdots(2)(1)} \\
&= (N+m+n)(N+m+n-1)\cdots(N+1) \\
&\leq (N+m+n)^{m+n} \\
&= k^{m+n}
\end{aligned}$$

and hence

$$\begin{aligned}
P_k(\neg\varphi_{m,n}) &\leq \frac{k^{m+n}}{m!n!}\delta^N \\
&\leq k^{m+n}\delta^{k-(m+n)} \\
&= \frac{k^{m+n}\delta^k}{\delta^{m+n}}.
\end{aligned}$$

Now $\lim\limits_{k\to\infty} k^{m+n}\delta^k = 0$ because the polynomial $k^{m+n}$ increases much slower than the exponential $\delta^k$ decays.[1] This implies that $P(\neg\varphi_{m,n}) = \lim\limits_{k\to\infty} P_k(\neg\varphi_{m,n}) = 0$, which, from Lemma 5.1.2, means that $P(\varphi_{m,n}) = 1$, i.e. each $\varphi_{m,n}$ is almost surely true. □

*Proof of Theorem 5.1.3.* Recall from Chapter 4 the axioms $(\forall x, y)((x \sim y) \to (x \neq y))$ and $(\exists x, y)(x \neq y)$ and $(\forall x, y)((x \sim y) \to (y \sim x))$, which are used to define the class of graphs. It is easy to see that of each of these sentences is almost surely true and from Lemma 5.1.4 each $\varphi_{m,n}$ is almost surely true. This means that each sentence in the axiomatization, say $\Sigma_{\boldsymbol{R}}$, of Th$\boldsymbol{R}$ is almost surely true.

Now each $\varphi \in$ Th$\boldsymbol{R}$ will be a consequence of $\Sigma_{\boldsymbol{R}}$, meaning that every graph which models $\Sigma_{\boldsymbol{R}}$ will also model $\varphi$. So $P_n(\varphi) \geq P_n(\psi)$ where $\psi \in \Sigma_{\boldsymbol{R}}$. This means that $P(\varphi) = \lim_{n\to\infty} P_n(\varphi) \geq \lim_{n\to\infty} P_n(\psi) = 1$. So $P(\varphi) = 1$ for each $\varphi \in$ Th$\boldsymbol{R}$.

Th$\boldsymbol{R}$ is complete, by Theorem 4.2.7, that is for each $L(\sim)$-sentence $\varphi$, either $\varphi \in$ Th$\boldsymbol{R}$ or $\neg\varphi \in$ Th$\boldsymbol{R}$. This implies that either $P(\varphi) = 1$ or $P(\neg\varphi) = 1$, so every $L(\sim)$-sentence is either almost surely true, or almost surely false. □

**Remark 5.1.5.** We have mentioned before that Th$\boldsymbol{R}$ is the set of all $L(\sim)$-sentences that hold in $\boldsymbol{R}$. We can now also say that Th$\boldsymbol{R}$ is the set of almost surely true $L(\sim)$-sentences.

It is quite surprising to be able to relate probability theory to graph theory in this way. It should be less surprising that the 0-1 Law for Graphs can be generalized to a 0-1 Law for Logic. We know that the language of graphs has only one relation symbol to worry about, and so we are tempted to wonder what will happen if we have to deal with a finite amount of relation symbols.

**Theorem 5.1.6** (**0-1 Law for Logic**, see [Abr18], Theorem 3.2, pg. 9)**.** *Let $L^*$ be a language with finitely many relation symbols. Then every $L^*$-sentence is either almost surely true or almost surely false.*

Obviously, relational $L$-structures need not be the same as graphs, and so to prove this new 0-1 law we would need to augment some of the necessary results. We are, being hunters of a great beast, only truly interested in the details of things which concern or are related to $\boldsymbol{R}$ is some direct way. So instead of stating and proving in full detail these results, we will just discuss the important principles and steps needed to prove Theorem 5.1.6. [Abr18] gives all the detail, for those who are interested.

[1] One can formally use L'Hospital's rule to prove that the limit is 0.

Firstly, we will need a property which does the same work as EP, but for finitely many relation symbols. This property will ensure the existence of some element $x$ in the structure, which relates to all the elements in a finite subset, precisely in the way that we want it to.

The next point will be that there is in fact a countably infinite $L$-structure, satisfying this property and that any two countably infinite $L$-structures with this property are isomorphic. This will ensure the completeness of the theory of structures with this property. To axiomatize this theory we will have to use relevant extension axioms[2]. These sentences will play the same role in the proof of Theorem 5.1.6 as the $\varphi_{m,n}$'s played in the proof of Theorem 5.1.3.

Each of these extension axioms will be almost surely true (just like before), and one can then argue in a similar fashion to what we did, using completeness of the theory, that each $L^*$-sentence is either almost surely true or almost surely false.

Theorem 5.1.3 puts certain restrictions on what can be said with $L(\sim)$-sentences.

**Example 5.1.7.** There is no $L(\sim)$-sentence which says that a graph has an even number of vertices. This is because the limit of any such sentence does not converge, with $P_n(\varphi)$ being 1 for all even $n$ and 0 for all odd $n$. For the same reason, there is no $L(\sim)$-sentence which says that a graph has an odd number of vertices.

Theorem 5.1.3 also tells us something about large finite graphs.

**Example 5.1.8.** In Proposition 2.2.1 we showed that $\boldsymbol{R}$ has a walk of length 2 between any two vertices. We can also say that $\boldsymbol{R}$ has diameter 2. This can be expressed with the $L(\sim)$-sentence

$$\delta_2 := \forall x, y(x \neq y \to \exists z(z \sim x \wedge z \sim y)).$$

So $\boldsymbol{R} \models \delta_2$ and $\delta_2 \in \mathrm{Th}\boldsymbol{R}$. It now follows that $\delta_2$ is almost surely true, so for a large $n$, a finite graph of order $n$ will likely have diameter 2.

## 5.2 Playing games: Finite graphs with $r$-extension

Wait...I know you...

GUARD
*The Elder Scrolls V: Skyrim*

Apart from putting restrictions on what we can say with $L(\sim)$-sentences, the 0-1 law for graphs helps us to determine if we can find finite graphs with certain properties.

Consider the extension axioms, or the $\varphi_{m,n}$'s and let $r = m + n$. We fix the notation $\epsilon_r = \varphi_{m,n}$ and we will say that a graph $G$ has *$r$-extension* if $G \models \epsilon_r$. It should be clear that $\boldsymbol{R}$ has $r$-extension for each $r$. The more interesting case will be a finite graph with $r$-extension. In this section we show that finite graphs with $r$-extension are $(r+1)$-equivalent to $\boldsymbol{R}$. We give explicit examples of graphs with 1- and 2-extension, and then determine some lower bounds for the order of finite graphs with $r$-extension. Most of the results follow easily enough from the relevant definitions.

**Proposition 5.2.1** (see [BH79], Theorem 1, pg. 228)**.** *For each $r$ there is a finite graph $G$ such that $G \models \epsilon_r$, i.e. that has $r$-extension.*

[2] These will be different to the extension axioms that we used.

*Proof.* Suppose on the contrary that there does not exist a finite graph with $r$-extension. Then for each $k \in \mathbb{N}$ no graph on $k$ vertices will satisfy $\epsilon_r = \varphi_{m,n}$. This implies that $P_k(\epsilon_r) = 0$ for each $k \in \mathbb{N}$ and hence that $P(\epsilon_r) = 0$, but we know that this is not the case from Lemma 5.1.4. □

It should be clear that if a graph has $r$-extension it will also have $n$-extension for each $n \leq r$. We explore this concept in a bit more depth.

**Definition 5.2.2** (**Quantifier rank**, see [Doe96] pg. 27)**.** *The* quantifier rank *of an $L$-formula $\varphi$, denoted $qr(\varphi)$, is recursively computed as follows.*

1. *An atomic formula has quantifier rank equal to 0.*
2. $qr(\neg\varphi) = qr(\varphi)$
3. $qr(\varphi \wedge \psi) = max(qr(\varphi), qr(\psi))$
4. $qr(\exists x\varphi) = qr(\varphi) + 1$

Intuitively, the quantifier rank of a formula is the largest number of nested quantifiers in the formula. For example, the $L(\sim)$-formula $\exists x(\forall y(y = x \sim y) \wedge \exists z(x \sim z))$ has quantifier rank 2.

**Definition 5.2.3** (**$n$-equivalence**, see [Doe96] pg. 28)**.** *$L$-structures $\mathcal{M}$ and $\mathcal{N}$ are* $n$-equivalent *if they satisfy the same $L$-sentences of quantifier rank $\leq n$. In this case we write $\mathcal{M} \equiv_n \mathcal{N}$.*

The easiest way to show that two structures are $n$-equivalent is through using *Ehrenfeucht-Fraïssé games.* We define such a game on graphs specifically, but these games can be played on arbitrary $L$-structures as in [Doe96].

**Definition 5.2.4** (**Ehrenfeucht-Fraïssé game**, see [Doe96] pg. 22)**.** *Let $G$ and $H$ be two graphs and $n \in \mathbb{N}$. The* Ehrenfeucht-Fraïssé game of length $n$ *on $G$ and $H$ consist of two players, called* **Di** *and* **Sy***, taking alternate turns.* ***Di*** *plays the first move, and both players are allowed $n$ moves. In each round of the game* ***Di*** *can choose a vertex from either $G$ or $H$, which* ***Sy*** *counters by choosing a vertex form the other graph. That is, if* ***Di*** *played a vertex from $G$ (resp. $H$) then* ***Sy*** *has to play a vertex of $H$ (resp. $G$).*

*Each game can only have one winner. Each game establishes $n$ pairs, which we can view as a mapping $f := \{(g_1, h_1), \ldots, (g_n, h_n)\}$ from $G$ to $H$. If $f$ is an isomorphism between the induced subgraphs on $\{g_1, \ldots, g_n\}$ and $\{h_1, \ldots, h_n\}$, of $G$ and $H$ respectively, then* ***Sy*** *has won. Otherwise* ***Di*** *has won.*

*In case a player is not able to play, due to a lack of vertices, then that player loses.*

[Doe96] describes the idea behind the game beautifully.

> **Di** sees *differences* all around; each of her moves is accompanied by some exclamation "hey, **Sy**, look: here I've found an extraordinary element in this [structure] you can't find the equal of in the other one!". On the other hand, to **Sy** every two [structures] appear to be *similar* and every move of **Di** is countered with some "oh yeah? then what about *this* one!"

We illustrate such a game in the following example.

**Example 5.2.5.** Consider the following two graphs.

(a) A graph $G$ (b) A graph $H$

**Di** and **Sy** are going to play a game of length 3 on these two graphs. **Di** starts by choosing the center vertex of $G$, and **Sy** responds by picking the center vertex of $H$. Now **Di** decides to play the top left vertex of $H$ and **Sy** counters with the bottom right vertex of $G$. Finally **Di** plays the top left vertex of $G$ to which **Sy** plays the top right vertex of $H$. The resulting induced subgraphs are drawn below.

(a) Induced subgraph of $G$ (b) Induced subgraph of $H$

Clearly the two induced subgraphs are isomorphic and hence **Sy** wins the game.

When considering Ehrenfeucht-Fraïssé games, the specific elements chosen aren't really that important. What we care about is who wins the game. In other words, we really want to know who has a winning strategy, **Di** or **Sy**? It can be shown that exactly one of the two players has a winning strategy for the game. We are rooting for **Sy** to win, and we write **Sy**$(G, H, n)$ if **Sy** has a winning strategy for the game on $G$ and $H$ of length $n$. The reason we like it when **Sy** wins is the following.

**Theorem 5.2.6** (see [Doe96], Theorem 3.18, pg. 28)**.** *Two graphs $G$ and $H$ are $n$-equivalent iff **Sy**$(G, H, n)$.*[3]

Let's bring this section back to $r$-extension.

**Theorem 5.2.7.** *Let $G$ be a finite graph with $r$-extension, then $G \equiv_{r+1} \boldsymbol{R}$.*

*Proof.* From Theorem 5.2.6 it is enough to show that **Sy** has a winning strategy for the game on $G$ and $\boldsymbol{R}$ of length $r+1$. Note that a non-empty graph satisfying $\epsilon_r$ will have at least $r+1$ vertices, so there is no danger of either **Di** or **Sy** running out of vertices to choose from.

We describe **Sy**'s responses for the first three rounds of the game, before giving a general winning strategy for **Sy**.

For the first round of the game **Di** can play a vertex from either $G$ or $\boldsymbol{R}$. Suppose that **Di** plays $g_1$ from $G$ as first move, then **Sy** can respond by playing any vertex of $\boldsymbol{R}$, say $r_1$. If on the other hand **Di** plays $r_1$ form $\boldsymbol{R}$ first, then **Sy** can respond by playing any vertex $g_1$ from $G$. Either way we have chosen $\{g_1\}$ from $G$ and $\{r_1\}$ from $\boldsymbol{R}$, and the induced subgraphs on $\{g_1\}$ and $\{r_1\}$ are isomorphic.

For the second round of the game **Di** can again choose from either $G$ or $\boldsymbol{R}$. Suppose that **Di** plays $g_2$ from $G$. In case $g_1 \sim g_2$ then **Sy** can counter with any of the infinitely many neighbours of $r_1$ (Proposition 2.1.2) and if $g_1 \not\sim g_2$ then **Sy** can counter with any of the infinitely many non-neighbours of $r_1$ (Proposition 2.1.3), call this vertex $r_2$. Suppose instead that **Di** plays $r_2$ from $\boldsymbol{R}$. If $r_1 \sim r_2$ then **Sy** has to choose $g_2$ from $G$ such that $g_1 \sim g_2$. Since $G$ has $r$-extension then $G$ also has 1-extension, which implies the existence of such a $g_2 \in G$. In case $r_1 \not\sim r_2$, then the $r$-extension, and hence 1-extension of $G$ again implies the existence of a $g_2 \in G$ such that $g_1 \not\sim g_2$. In either case **Sy** can respond to **Di**'s move by using the $r$-extension of $G$ to pick a vertex $g_2$. So the induced subgraphs of $G$ and $\boldsymbol{R}$ on $\{g_1, g_2\}$ and $\{r_1, r_2\}$ are isomorphic.

Suppose for the third round of the game that **Di** plays $g_3$ from $G$. We consider the following cases:

- $g_3 \sim g_1$ and $g_3 \sim g_2$: Let $U = \{r_1, r_2\}$ and $V = \emptyset$, then **Sy** can counter with any of the infinitely many witnesses to EP for $U$ and $V$ in $\boldsymbol{R}$ (Proposition 2.1.1).
- $g_3 \sim g_1$ and $g_3 \not\sim g_2$: Let $U = \{r_1\}$ and $V = \{r_2\}$, then **Sy** can counter with the infinitely many witnesses to EP for $U$ and $V$ in $\boldsymbol{R}$.

[3] Like Ehrenfeucht-Fraïssé games, this can be generalised to arbitrary $L$-structures.

- $g_3 \not\sim g_1$ and $g_3 \sim g_2$: Let $U = \{r_2\}$ and $V = \{r_1\}$. So **Sy** can counter with the infinitely many witnesses to EP for $U$ and $V$ in $\boldsymbol{R}$.
- $g_3 \not\sim g_1$ and $g_3 \not\sim g_2$: Let $U = \emptyset$ and $V = \{r_1, r_2\}$. Again **Sy** can counter using of the infinitely many witnesses to EP for $U$ and $V$ in $\boldsymbol{R}$.

**Di** might also start the third round by playing $r_3$ from $\boldsymbol{R}$.

- $r_3 \sim r_1$ and $r_3 \sim r_2$: Let $U = \{g_1, g_2\}$ and $V = \emptyset$, then since $G$ has $r$- and hence 2-extension there exists a vertex, say $g_3 \in G$, with which **Sy** can respond such that $g_3 \sim g_1$ and $g_3 \sim g_2$.
- $r_3 \sim r_1$ and $r_3 \not\sim r_2$: Let $U = \{g_1\}$ and $V = \{g_2\}$, then since $G$ has $r$- and hence 2-extension there exists a vertex, say $g_3 \in G$, with which **Sy** can respond such that $g_3 \sim g_1$ and $g_3 \not\sim g_2$.
- $r_3 \not\sim r_1$ and $r_3 \sim r_2$: Let $U = \{g_2\}$ and $V = \{g_1\}$, then since $G$ has $r$- and hence 2-extension there exists a vertex, say $g_3 \in G$, with which **Sy** can respond such that $g_3 \not\sim g_1$ and $g_3 \sim g_2$.
- $r_3 \not\sim r_1$ and $r_3 \not\sim r_2$: Let $U = \emptyset$ and $V = \{g_1, g_2\}$, then since $G$ has $r$- and hence 2-extension there exists a vertex, say $g_3 \in G$, with which **Sy** can respond such that $g_3 \not\sim g_1$ and $g_3 \not\sim g_2$.

After this round of the game the induced subgraphs of $G$ and $\boldsymbol{R}$ on $\{g_1, g_2, g_3\}$ and $\{r_1, r_2, r_3\}$ are isomorphic. The game will continue like this. We can put this into a winning strategy as follows.

If **Di** plays as first move $g_1 \in G$ (resp. $r_1 \in R$), then **Sy** can counter with any vertex of $\boldsymbol{R}$ (resp. $G$).

Suppose that $\{g_1, \ldots, g_n\}$ and $\{r_1, \ldots, r_n\}$ for $1 \leq n < r+1$ have already been chosen such that the induced subgraphs are isomorphic.

Now at step $n+1$ if **Di** plays $g_{n+1}$ from $G$ then **Sy** constructs sets $U = \{r_i : g_{n+1} \sim g_i \text{ for } 1 \leq i \leq n\}$ and $V = \{r_i : g_{n+1} \not\sim g_i \text{ for } 1 \leq i \leq n\}$. Using Proposition 2.1.1 **Sy** can play any of the infinitely many witnesses to EP of $U$ and $V$ in $\boldsymbol{R}$. Labelling the chosen witness $r_{n+1}$ results in the sets $\{g_1, \ldots, g_{n+1}\}$ and $\{r_1, \ldots, r_{n+1}\}$ such that $g_i \sim g_j$ iff $r_i \sim r_j$ for each $i, j \in \{1, \ldots, n+1\}$. Thus, the induced subgraphs of $G$ and $\boldsymbol{R}$ on $\{g_1, \ldots, g_{n+1}\}$ and $\{r_1, \ldots, r_{n+1}\}$ will be isomorphic.

If, at step $n+1$, **Di** plays $r_{n+1}$ from $\boldsymbol{R}$ then **Sy** constructs sets $U = \{g_i : r_{n+1} \sim r_i \text{ for } 1 \leq i \leq n\}$ and $V = \{g_i : r_{n+1} \not\sim r_i \text{ for } 1 \leq i \leq n\}$. Note that $|U| + |V|$ can be at most $n < r+1$. Since $G$ has $r$-extension, and hence also $n$-extension, there exists a vertex in $G$ which **Sy** can play as $g_{n+1}$ to give sets $\{g_1, \ldots, g_{n+1}\}$ and $\{r_1, \ldots, r_{n+1}\}$ such that $g_i \sim g_j$ iff $r_i \sim r_j$ for each $i, j \in \{1, \ldots, n+1\}$. So, the induced subgraphs of $G$ and $\boldsymbol{R}$ on $\{g_1, \ldots, g_{n+1}\}$ and $\{r_1, \ldots, r_{n+1}\}$ will be isomorphic.

This shows that **Sy** has a winning strategy for the game of length $r+1$ on $G$ with $r$-extension and $\boldsymbol{R}$, i.e. $G \equiv_{r+1} \boldsymbol{R}$. □

**Definition 5.2.8** (**Complement of a graph**, see [Die00] pg. 4)**.** *The* complement *of a graph $G$, denoted $\overline{G}$, is a graph on the same vertices as $G$ such that $g \sim h$ in $\overline{G}$ iff $g \not\sim h$ in $G$.*

**Proposition 5.2.9** (see [BEH81], pg. 438)**.** *A graph $G$ has $r$-extension iff its complement has $r$-extension.*

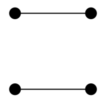
*Proof.* Suppose that $G$ has $r$-extension, but that $\overline{G}$ does not. Then there exist disjoint sets $U = \{u_1, \ldots, u_m\}$ and $V = \{v_1, \ldots, v_n\}$ of vertices of $\overline{G}$, with $m + n \leq r$, such that there does not exist $g \in \overline{G}$ with $g \sim U$ and $g \not\sim V$. Note that $U$ and $V$ are also disjoint subsets of vertices of $G$, and since $G$ has $r$-extension, there exists a vertex $h \in G$ such that $h \sim V$ and $h \not\sim U$. This vertex $h$ will also be in $\overline{G}$, but with $h \sim U$ and $h \not\sim V$, a contradiction to our previous claim. This implies that $\overline{G}$ must also have $r$-extension.

Supposing that $\overline{G}$ has $r$-extension, but that $G$ does not, gives the same contradiction.

Hence $G \models \epsilon_r$ iff $\overline{G} \models \epsilon_r$. □

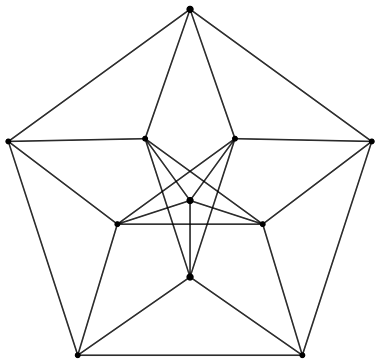
We can now ask ourselves, how do these finite graphs with $r$-extension look?

**Example 5.2.10.** The following graph has 1-extension.



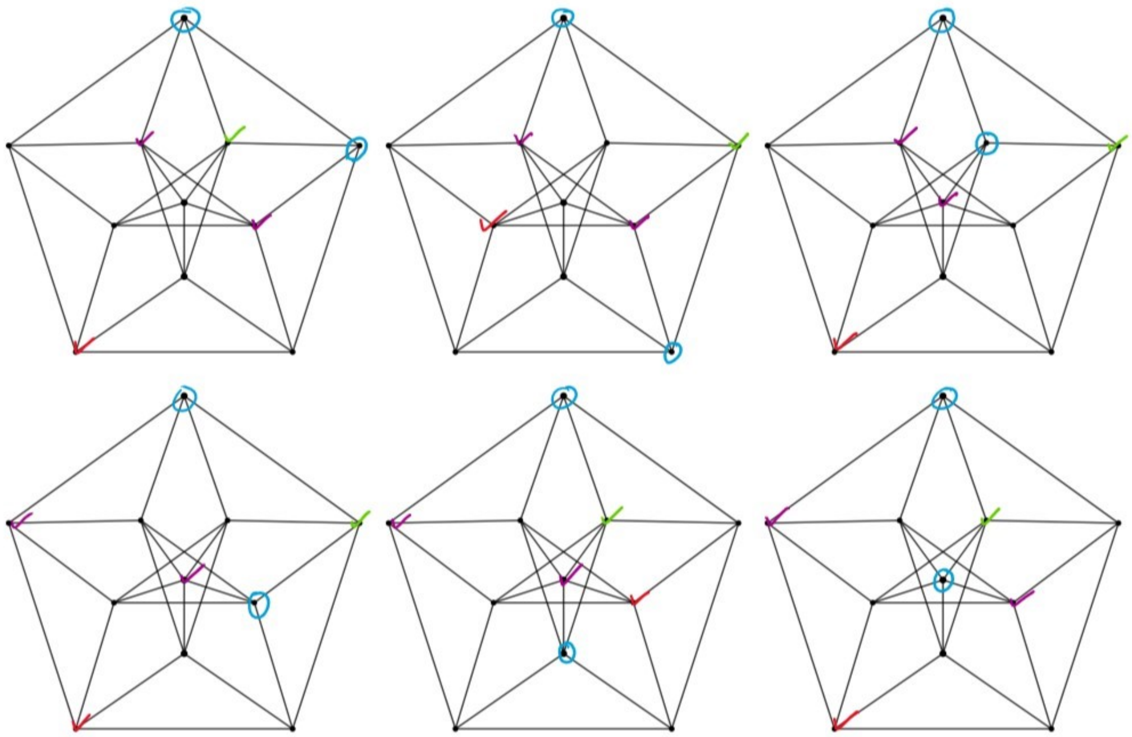
This means that for every vertex of the graph we should be able to find a vertex connected to it and one not connected to it. This is checked easily enough.

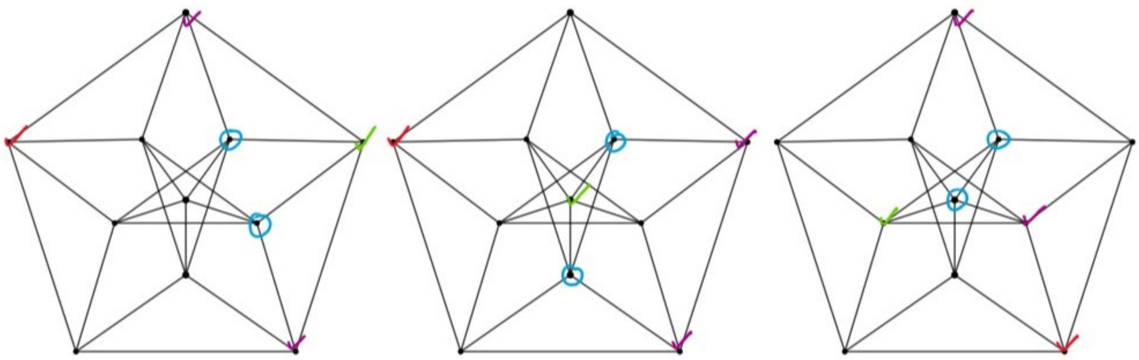
**Example 5.2.11.** The following graph has 2-extension.



So for any two vertices we should be able to find a vertex connected to both, one not connected to either, and one connected to the one vertex but not the other. Note that we do not literally need to check this for each pair of vertices. Due to the symmetry of the situation we only need to check some cases.

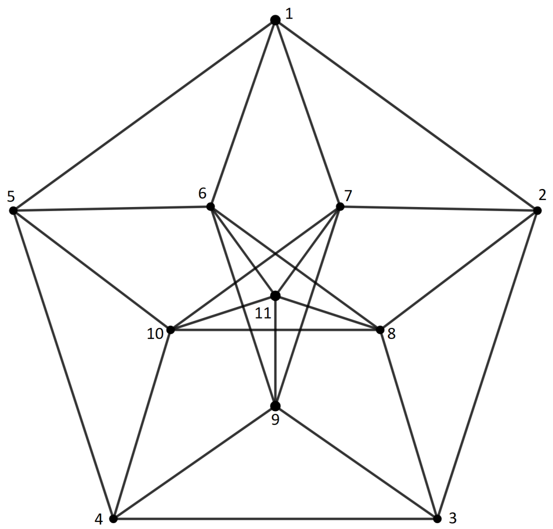
We will circle the chosen pair of vertices in blue, and mark with green a vertex connected to both, with red a vertex not connected to either, and with purple one connected to only one of the vertices.





We can also represent this graph with its *adjacency matrix*. This is just a square matrix with a row and column for each vertex, and entry $a_{i,j} = 1$ if vertices $i$ and $j$ are connected, and is 0 otherwise. Note that the diagonal of an adjacency matrix will consist only of 0's, since no vertex is connected to itself.

**Example 5.2.12.** We enumerate the vertices of the graph.




This gives the adjacency incidence matrix.

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

To see that 2-extension is satisfied, we need to check for every two rows, $i$ and $j$, that

1. $(a_{i,k}, a_{j,k}) = (0,0)$ for some $k \in \{1,\dots,11\} \setminus \{i,j\}$
   (this means that there is a vertex not connected to either of $i$ or $j$),
2. $(a_{i,l}, a_{j,l}) = (0,1)$ for some $l \in \{1,\dots,11\} \setminus \{i,j,k\}$
   (this means that there is a vertex connected to $j$ but not $i$),
3. $(a_{i,m}, a_{j,m}) = (1,0)$ for some $m \in \{1,\dots,11\} \setminus \{i,j,k,l\}$
   (this means that there is a vertex connected to $i$ but not $j$), and
4. $(a_{i,n}, a_{j,n}) = (1,1)$ for some $n \in \{1,\dots,11\} \setminus \{i,j,k,l,m\}$
   (this means that there is a vertex connected to both $i$ and $j$).

This can be checked easily with the relevant program found in Appendix D.

Let's see how this will work for the more general case of $r$-extension.

Consider the set of vectors $S := \{\bar{x} : \bar{x} \in \{0,1\}^r\}$. If we want to check that an $m \times m$ adjacency matrix has $r$-extension, then we have to check for every $r$ rows, $i_1$, $i_2$, $\dots$, $i_r$, that

1. $(a_{i_1,j}, a_{i_2,j}, \dots, a_{i_r,j}) = \bar{x}_1$ where $\bar{x}_1 \in S$ for some $j \in \{1,\dots,m\}$,
2. $(a_{i_1,k}, a_{i_2,k}, \dots, a_{i_r,k}) = \bar{x}_2$ where $\bar{x}_2 \in S \setminus \{\bar{x}_1\}$ for some $k \in \{1,\dots,m\} \setminus \{j\}$,
3. $(a_{i_1,l}, a_{i_2,l}, \dots, a_{i_r,l}) = \bar{x}_3$ where $\bar{x}_3 \in S \setminus \{\bar{x}_1, \bar{x}_2\}$ for some $l \in \{1,\dots,m\} \setminus \{j,k\}$,

$\vdots$

$2^r$. $(a_{i_1,n}, a_{i_2,n}, \dots, a_{i_r,n}) = \bar{x}_{2^r}$ where $\bar{x}_{2^r} \in S \setminus \{\bar{x}_1, \bar{x}_2, \dots, \bar{x}_{2^r-1}\}$
for some $n \in \{1,\dots,m\} \setminus \{$All previously used column indices$\}$.

Notice that in this process we exhaust $S$, i.e. we need to find all possible vectors in $\{0,1\}^r$ as column vectors for the matix's graph to satisfy $r$-extension.

Note that the graphs in Example 5.2.10 and Example 5.2.11 are not the only finite graphs with 1- and 2-extension respectively. Instead of trying to find all such graphs, we aim rather to discover some of the basic properties such graphs might possess.

Let's start with the possible orders of a graph with $r$-extension. It is clear from the work we did in Section 5.1 that the larger the graph is, the more likely it is to have $r$-extension. The interesting question is then, how small can these graphs be?

**Proposition 5.2.13.** *A graph with $r$-extension has at least $r+1$ vertices.*

*Proof.* We know from Proposition 5.2.1 that there is a nonempty graph $G$ with $r$-extension. Then $G$ has $k$-extension for each $k \in \{1,\dots,r\}$. Let $g_1$ be a vertex from $G$.

Since $G$ has 1-extension there exists a vertex $g_2 \in G \setminus \{g_1\}$ which is either connected to, or not connected to $g_1$, so $G$ has at least 2 vertices. Whether or not there is an edge is not important for this proof; we are only concerned with the number of vertices in $G$. Now, since $G$ has 2-extension, for sets $U$ and $V$ obtained from only the vertices $g_1$ and $g_2$, there exists a vertex $g_3 \in G \setminus \{g_1, g_2\}$ connected to the two vertices $g_1$ and $g_2$ in some way, so $G$ has at least 3 vertices. Next, considering sets $U$ and $V$ obtained from only the vertices $g_1$, $g_2$ and $g_3$, together with the fact that $G$ has 3-extension gives a vertex $g_4 \in G \setminus \{g_1, g_2, g_3\}$, hence $G$ has at least 4 vertices.

Continuing, in this way, we use the $k$-extension of $G$, for $k \leq r$, to show that there are at least $k+1$ vertices in $G$. The $r$-extension of $G$, i.e. where $k = r$, shows that $G$ has at least $r+1$ vertices. □

Alternatively, we can write for a graph $G$, if $G \models \epsilon_r$ then $|G| \geq r+1$.

So a graph with 1-extension has at least 2 vertices. But having only two vertices is surely not enough. If a graph has only two vertices labeled 1 and 2, these two vertices can either be connected or not. In case they are connected we won't be able to find a vertex not connected to 1. If they are not connected we won't be able to find a vertex connected to 1. So this graph will not satisfy 1-extension. We can make the bound more accurate by considering the ways in which the vertices should be connected to each other.

For a graph with 1-extension, for every vertex there is a vertex connected to it and also another vertex not connected to it. In this case there should be at least 3 vertices. We can write this as $1 + \binom{1}{0} + \binom{1}{1}$: 1 for the chosen vertex, $\binom{1}{0}$ for the number of ways in which a vertex can be not connected to the chosen vertex, and $\binom{1}{1}$ for the number of ways in which a vertex can be connected to the chosen vertex.

For a graph with 2-extension, for every two vertices there should be a vertex connected to both, a vertex connected to one vertex but not the other, and a vertex not connected to either. This means there should be at least 6 vertices. We can write this as $2 + \binom{2}{0} + \binom{2}{1} + \binom{2}{2}$, again counting the number of ways in which a vertex can be connected to none, one, or both of the vertices.

We can now give the following bound.

**Proposition 5.2.14.** *A graph with $r$-extension has at least $r + 2^r$ vertices.*

*Proof.* Let $G$ be a graph with $r$-extension. Then there are at least $r$ vertices from which we can make sets $U$ and $V$ for which $\epsilon_r$ will be satisfied.

We focus on the number of ways in which a vertex can be connected to exactly $k$ vertices, for $k \in \{0, \ldots, r\}$. This gives the following expression:

$$\binom{r}{0} + \binom{r}{1} + \cdots + \binom{r}{r}.$$

This counts one witness for each of the possible sets $U$ and $V$ made up of the $r$ vertices.

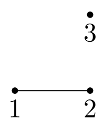
So $G$ has at least $r + \binom{r}{0} + \binom{r}{1} + \cdots + \binom{r}{r}$ vertices. We use the binomial expansion to simplify this expression.

Binomial Formula (see [Pin10] pg. 179)

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

If we let both $a$ and $b$ be 1 in the binomial formula, we get $2^n = (1+1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} 1^k = \sum_{k=0}^{n} \binom{n}{k}$. Hence, $G$ has at least $r + 2^r$ vertices. $\square$

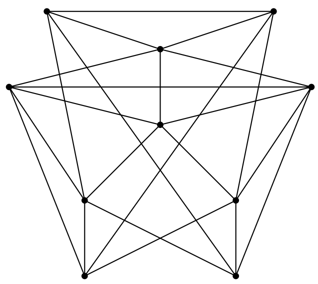
This bound does a lot better than just $r + 1$, but it is not perfect. Consider the following graph.

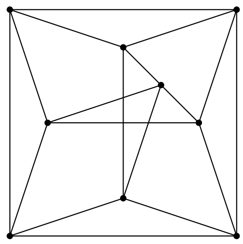



For vertex 3 there is no node connected to it, so the graph does not have 1-extension. Connecting it to either 1 or 2 does not fix the problem, since then either 1 or 2 will be connected to every other vertex, so we won't find a vertex not connected to it. This implies that we need a fourth point.

If we consider graphs with 2-extension then, according to our bound, the graph needs to have at least 6 vertices. The graph from Example 5.2.11 has 11 vertices. Our bound suggests that we can do with less, and in fact we can.

**Example 5.2.15.** The following two graphs, with 10 and 9 vertices respectively, each have 2-extension.[4]





We can, however, not do with less than 9 vertices. This can be tested with the sequential-binary-fill-program from Appendix D. We can say more about the order of finite graphs with $r$-extension.

**Proposition 5.2.16.** *If there is a graph of order $n$ that satisfies $r$-extension, then there is also a graph of order $> n$ that satisfies $r$-extension.*

*Proof.* Let $G$ be a graph such that $G \models \epsilon_r$ and $|G| = n$. Suppose, on the contrary, that $G$ is the largest graph with $r$-extension. That is, for each $H$ with $|H| > n$, $H \not\models \epsilon_r$. This implies that $P_k(\epsilon_r) = 0$ for each $k > n$. But then $\lim_{k\to\infty} P_k(\epsilon_r) = 0$, which is not the case. Therefore $G$ cannot be the largest graph with $r$-extension, i.e. there is a graph of order $> n$ with $r$-extension. $\square$

Apart from the order of graphs with $r$-extension, we might also look at how the vertices of such a graph are connected.

**Proposition 5.2.17** (see [BH79], Corollary 13, pg. 231)**.** *In every graph with $r$-extension, where $r \geq 2$, there is a walk of length $\leq 2$ between any two vertices, i.e. the graph has diameter 2.*

*Proof.* Let $r = 2$ and suppose that a graph $G$ has $r$-extension. Then, for any two vertices $u$ and $v$ of $G$ we will be able to find a third vertex (there are enough vertices to do this by Proposition 5.2.14) such that the third vertex is connected to both $u$ and $v$. This yields a walk of length 2 between $u$ and $v$. If the chosen vertices were already connected, then we have a walk of length 1.

Now let $r > 2$ and suppose that a graph $G$ has $r$-extension. Then certainly $G$ has 2-extension, and we use the same argument as above to find a walk of length 1 or 2.

This proves the claim. $\square$

In [BR05] it is mentioned that the existence or non-existence of finite triangle free graphs (i.e. graphs of which $K_3$ is not an induced subgraph) with $r$-extention is uncertain. The following result will clear up this uncertainty.

**Proposition 5.2.18** (see [BH79], Corollary 14, pg. 232)**.** *Every graph with $r$-extension contains a complete graph on $r + 1$ vertices. In fact, a graph with $r$-extension contains a complete graph on $k$ vertices for each $k \in \{2, \dots, r + 1\}$.*

[4]This can be checked by setting up an adjacency matrix and using the relevant program from Appendix D. The graph of order 10 was originally discovered using the random-binary-fill-program. The graph of order 9 was discovered after staring, for hours on end, at the graph of order 10.

*Proof.* Let $G$ be a graph with $r$-extension. Then $G$ will also satisfy $k$-extension for each $1 \leq k < r$. Let $g_0$ be a vertex from $G$.

Now since $G$ has 1-extension there is a $g_1 \in G$ such that $g_0 \sim g_1$, giving a complete graph on 2 vertices. Next, since $G$ has 2-extension there is a $g_2 \in G$ such that $g_2 \sim g_0$ and $g_2 \sim g_1$. So we have a complete graph on 3 vertices. Using the 3-extension of $G$ we can find a vertex $g_3 \in G$ connected to each of $g_0$, $g_1$ and $g_2$, resulting in a complete graph on 4 vertices. We can continue using the $i$-extension of $G$ to construct a complete graph on $i+1$ vertices. The final step would be to use the $r$-extension of $G$, giving a complete graph on $r+1$ vertices as required. □

So, in particular, if $G$ is a finite graph with $r$-extension, and $r \geq 2$, then it has $K_3$ as an induced subgraph and hence $G$ cannot be triangle free. In this case non-existence is clear. It is, however possible for a triangle free graph to satisfy 1-extension, as in Example 5.2.10, which makes existence clear.

Note that the vertex $g_0$ in the proof above was arbitrarily chosen. We can therefore give the following result.

**Proposition 5.2.19.** *Let $G$ be a graph with $r$-extension. Then, for $k \in \{2, \dots, r+1\}$, every vertex of $G$ is also a vertex of an induced subgraph $H$ of $G$, where $|H| = k$ and $H$ is complete.*

At this stage one might be wondering if there is a way to construct a finite graph with $r$-extension. There are, but these constructions are either very large or the details are more complicated than we care to go into. We will just outline the ideas of the constructions here. Readers who wish to see them in detail are referred to the relevant articles.

A graph is said to be *strongly regular* if it is *regular*, i.e. all vertices have the same number of neighbours, and there exist $m$ and $n$ such that each connected pair of vertices have $m$ mutual neighbours, and each disconnected pair of vertices have $n$ mutual neighbours. Cameron and Stark [CS02] use probabilistic methods to construct strongly regular graphs that have $r$-extension. The smallest graph, as constructed in [CS02], with 1-extension has 4 624 vertices, and with 2-extension has 1 065 024 vertices.

For a prime $p$ congruent to 1 modulo 4, take the elements of the field of order $p$, $\mathbb{F}_p$, to be the vertices of a graph $G_p$, and let two vertices be adjacent when their difference is a quadratic residue modulo $p$. The graph $G_p$ is called a *Paley graph* of order $p$. In [BEH81] it is shown that Paley graphs satisfy the following version of the $r$-extension property for sufficiently large primes.

> A graph satisfies Axiom $n$ if, for any sequence of $2n$ of its [vertices], there is another [vertex] adjacent to the first $n$ and not any of the last $n$.

Here, a sufficiently large prime is a prime $p$ with $p > n^2 2^{4n}$. The fact that (large enough) Paley graphs have $r$-extension, in the sense that we have defined it, comes from the proof of Theorem 3 in [BT81]. According to [BEH81], the smallest Paley graph guaranteed by their results to satisfy 1-extension then has 17 vertices; to satisfy 2-extension has 1 033 vertices.

[BR05] uses combinatorial methods to construct graphs that satisfy $r$-extension, but these are larger and more complicated than the Paley graphs.

We know from the examples above that there are much smaller graphs that satisfy the extension axioms and ideally we would want to give a better construction of a finite graph with $r$-extension than those mentioned above. Better in the sense that the construction is simpler, and the graph obtained is smaller. As it turns out this is not such a simple undertaking as we had hoped.

Having thought of and come up with no feasible solutions, at least not any that work to construct a graph with 3-extension or higher, we leave those who joined in the adventure with the constructions above and the few interesting properties of finite graphs with $r$-extension.

## 5.3 An unexpected model: The Rado graph as a non-standard model

> My mama always said, "Life is like a box of chocolates. You never know what you're gonna get."
>
> FORREST GUMP

In this section we will talk about the first-order theory of finite graphs, and one might be thinking, "why would we want to do this? Isn't $\boldsymbol{R}$ infinite?" Well, yes, $\boldsymbol{R}$ is infinite, but that is not the point. We have already seen some of the connections between finite graphs and $\boldsymbol{R}$, like the finite graphs being induced subgraphs of $\boldsymbol{R}$ or that $\boldsymbol{R}$ is the Fraïssé limit of the class of all finite graphs. It must be worthwhile then to talk about the theory of finite graphs.

If you are feeling faint you might wish to sit down for the next part. $\boldsymbol{R}$ is a model of the first-order theory of finite graphs. Now you must be thinking that I am out of it. Didn't we just say that $\boldsymbol{R}$ is infinite? And now I want you to believe that it is a model of something which describes finite structures! All right, so maybe the claim was stated in such a way as to vex you a bit, but let me put this right by explaining the matter.

**Theorem 5.3.1.** *$\boldsymbol{R}$ is a non-standard model of the first-order theory of the class of finite graphs.*

*Proof.* Consider the class of all finite graphs $\boldsymbol{G}$ and let $T$ be the theory of $\boldsymbol{G}$.

We know, for any $L(\sim)$-sentence, in particular for any $\varphi \in T$, that being true in $\boldsymbol{R}$ is equivalent to being almost surely true.

Now any $\varphi \in T$ is true in all finite graphs and so we will have that $P_n(\varphi) = 1$ for all $n$. Hence $P(\varphi) = 1$, i.e. $\varphi$ is almost surely true. This then is equivalent, as mentioned above, to $\boldsymbol{R} \models \varphi$.

In conclusion $\boldsymbol{R} \models \varphi$ for all $\varphi \in T$, i.e. $R \models T$ as required. □

Do you remember where this adventure began? Has the beast turned into a magnificent creature, one that you can admire for all its brilliant characteristics? Is it really possible that we got such beauty by just flipping a coin?

# Directions for future work

> Don't adventures ever have an end? I suppose not. Someone else always has to carry on the story.
>
> BILBO BAGGINS
> *The Lord of the Rings: The Fellowship of the Ring*

Our adventure has uncovered many interesting secrets of $\boldsymbol{R}$, but there are still many truths to be discovered.

We classified the definable subgraphs of $\boldsymbol{R}$ in this dissertation in which EP played a big role. Another problem in this direction would be to determine the number of non-isomorphic graphs definable in the Rado graph, with an arbitrary finite amount of parameters. Also, we did not consider definability in countably infinite elementary extensions of $\boldsymbol{R}$, nor definability in finite graphs with $r$-extension. These might also be interesting problems to consider.

Speaking of finite graphs with $r$-extension, our bound for the minimum amount of vertices a graph needs to have $r$-extension still needs a bit of work, as is clear from the examples. It has also been a tremendous struggle to find a construction better than those mentioned in Section 5.2 for producting graphs with $r$-extension. We were able to, given a graph with 1-extension, construct from it a graph with 2-extension, that does much better than the constructions from [CS02] and [BEH81]. In fact we found a few such constructions, but none of them worked to construct, from a graph with 2-extension, a graph with 3-extension; accordingly, we did not present those constructions here. Finding better constructions, in the sense that the construction is simpler and the graph produced is smaller, than those from [BEH81], [BR05], [BT81] and [CS02] is thus still a very open problem.

With this we greet all who joined in our exploration of the Rado graph, and bid them safe travels on their journey ahead.

# Appendix A

# Model theory background

## A.1 Structure

**Definition A.1.1** (**Signature**, see [Rot00] pg. 3)**.** *A* signature $\sigma = (C, F, R, \sigma')$ *is a quadruple consisting of*

1. *a (possibly empty) set $C$ of* constant symbols*,*
2. *a (possibly empty) set $F$ of* function symbols*,*
3. *a (possibly empty) set $R$ of* relation symbols *and*
4. *a* signature function $\sigma'$ *which assigns to each function and relation symbol an arity.*

*It is assumed that $C$, $F$ and $R$ are pairwise disjoint.*

Constant, function and relation symbols are called *non-logical symbols* and we will identify the signature with these. For this reason the cardinality of the signature, i.e. $|\sigma|$, is equal to $|C \cup F \cup R|$.

In the case where $C = \emptyset$ we say that the signature is *without constants*. Similarly $\sigma$ is without functions or without relations if $F = \emptyset$ or $R = \emptyset$ respectively. A signature with both $F = \emptyset$ and $C = \emptyset$ is called *purely relational*.

**Definition A.1.2** (**$\sigma$-structure**, see [Rot00] pg. 4)**.** *Let $\sigma$ be a signature.*
*A* $\sigma$-structure $\mathcal{M} = (M, C^{\mathcal{M}}, F^{\mathcal{M}}, R^{\mathcal{M}})$ *is a quadruple consisting of*

1. *an arbitrary set $M$, called the* universe *of $\mathcal{M}$,*
2. *$C^{\mathcal{M}} = (c^{\mathcal{M}} : c \in C)$ where $c^{\mathcal{M}} \in M$ for all $c \in C$,*
3. *$F^{\mathcal{M}} = (f^{\mathcal{M}} : f \in F)$ where $f^{\mathcal{M}}$ is a $\sigma'(f)$-ary function from $M$ to $M$ for each $f \in F$ and*
4. *$R^{\mathcal{M}} = (r^{\mathcal{M}} : r \in R)$ where $r^{\mathcal{M}}$ is a $\sigma'(r)$-ary relation on $M$ for each $r \in R$.*

*$c^{\mathcal{M}}$ is called the* interpretation *of the constant symbol $c$ in $\mathcal{M}$. Similarly $f^{\mathcal{M}}$ (resp. $r^{\mathcal{M}}$) is called the interpretation of the function (resp. relation) symbol $f$ (resp. $r$) in $\mathcal{M}$.*

The cardinality of $\mathcal{M}$ is just $|M|$.

**Definition A.1.3** (**Substructure**, see [Rot00] pg. 8)**.** *Let $\sigma$ be a signature, $\mathcal{M}$ be a $\sigma$-structure and $N \subseteq M$. If 1. $c^{\mathcal{M}} \in N$ for all $c \in C$ and 2. $f^{\mathcal{M}}(\bar{a}) \in N$ for all $f \in F$ and all $\sigma'(f)$-tuples $\bar{a}$ from $N$ then we get a $\sigma$-structure $\mathcal{N}$ by setting*

a. *$c^{\mathcal{N}} = c^{\mathcal{M}}$ for all $c \in C$,*
b. *$f^{\mathcal{N}}(\bar{a}) = f^{\mathcal{M}}(\bar{a})$ for all $f \in F$ and all $\sigma'(f)$-tuples $\bar{a}$ from $N$ and*
c. *$r^{\mathcal{N}}(\bar{a})$ whenever $r^{\mathcal{M}}(\bar{a})$ for all $r \in R$ and all $\sigma'(r)$-tuples $\bar{a}$ from $N$.*

*$\mathcal{N}$ is the restriction of $\mathcal{M}$ onto $N$. We call $\mathcal{N}$ a* substructure *of $\mathcal{M}$ and write $\mathcal{N} \subseteq \mathcal{M}$.*

We may also call $\mathcal{M}$ a *superstructure* or *extension* of $\mathcal{N}$ and write $\mathcal{M} \supseteq \mathcal{N}$.

**Remark A.1.4.** A constant symbol $c$ can be viewed as a 0-place function with constant value $c$. We can therefore reformulate conditions 1 and 2 of Definition A.1.3 by saying that $N$ is closed in $\mathcal{M}$ under functions from $\sigma$.

## A.2 Language

> Words - so innocent and powerless as they are, as standing in a dictionary, how potent for good and evil they become in the hands of one who knows how to combine them.
>
> NATHANIEL HAWTHORNE

We fix the signature $\sigma = (C, F, R, \sigma')$.

**Definition A.2.1** (**Alphabet**, see [Rot00] pg. 11)**.** *A* $\sigma$-alphabet *consists of the following.*

1. Logical symbols *which are*
   (a) $\neg$ *for negation (not),*
   (b) $\wedge$ *for conjunction (and),*
   (c) *the existential quantifier* $\exists$ *(there exists) and*
   (d) $=$.
2. *Countably many variables, denoted* $x, y, z$ *or* $x_0, x_1, \ldots$*, etc.*
3. *constant, function and relation symbols from* $\sigma$*, called* non-logical *symbols,*
4. *and parentheses, i.e. ( and ).*

The list of logical symbols might seem a bit scant, but it will become clear later that these are in fact adequate.

**Definition A.2.2** (**Term**, see [Rot00] pg. 12)**.** *A* $\sigma$-term *is defined recursively as follows:*

1. *all variables and constant symbols are* $\sigma$*-terms.*
2. *For* $f \in F$ *with* $\sigma'(f) = n$ *and* $\sigma$*-terms* $t_1, \ldots, t_n$*,* $f(t_1, \ldots, t_n)$ *is a* $\sigma$*-term.*
3. *Nothing else is a* $\sigma$*-term, i.e. all* $\sigma$*-terms can be obtained from 1 and 2 in finitely many steps.*

Terms will be interpreted as elements of structures, but we will see this in more detail later.

**Definition A.2.3** (**Formula**, see [Rot00] pg. 13)**.** *A* $\sigma$-formula *is defined recursively as follows:*

1. *for* $\sigma$*-terms* $t_1$ *and* $t_2$*,* $t_1 = t_2$ *is a* $\sigma$*-formula.*
2. *For* $r \in R$ *with* $\sigma'(r) = n$ *and* $\sigma$*-terms* $t_1, \ldots, t_n$*,* $r(t_1, \ldots, t_n)$ *is a* $\sigma$*-formula.*
3. *If* $\varphi$ *and* $\psi$ *are* $\sigma$*-formulas and* $x$ *is a variable, then*
   (a) $\neg\varphi$,
   (b) $\varphi \wedge \psi$ *and*
   (c) $\exists x \varphi$

   *are* $\sigma$*-formulas.*
4. *Nothing else is a* $\sigma$*-formula, i.e. all* $\sigma$*-formulas can be obtained from 1, 2 and 3 in finitely many steps.*

Formulas will be interpreted as statements about elements of structures.

Formulas in the form of 1 or 2 from Definition A.2.3 are known as *atomic* formulas and for a given signature we will denote the class containing all such formulas by **at**. Atomic formulas and negations of atomic formulas are also called *literals*.

**Definition A.2.4** (**Language**, see [Rot00] pg. 13)**.** *The* language $L = L(\sigma)$ *is the set containing all* $\sigma$*-formulas.*

Take note that languages can only differ in their non-logical symbols. This correspondence between a language and its signature allows us to write "$L$-terms" or "$L$-formulas".

We will write $t_1 \neq t_2$ instead of $\neg t_1 = t_2$.

We mentioned before that formulas will say something about the elements of a structure, and variables are the placeholders for elements in a formula. We therefore look at different occurrences of variables in formulas.

Consider the formula $\exists x\varphi$. Here $\varphi$ is the *scope* of the quantifier. The occurrence of $x$ after the quantifier $\exists$ as well as any occurrence of $x$ in the scope $\varphi$ is called a *bound* occurrence of $x$. An occurrence of a variable is called *free* if it is not bound by a quantifier. This means that in the formula $\exists x(x = y \vee \exists y(x \neq y))$ all occurrences of $x$ are bound and the first occurrence of $y$ is free, while the last two are bound (see [Rot00] pg. 16).

Formulas with no free variables are called *sentences*.

We discuss an example of substitution before giving the definition. Consider the language $L$ with constant symbol 0 and function symbol $+$. If we substitute $x$ by $y + y$ in the $L$-term $x + z$ we get $(y + y) + z$. If we substitute $x$ by $y + y$ in $\exists y(x + y = 0)$ then $y$ from the term $y + y$ will fall under the scope of $\exists y$. We call this a *collision of variables*. We can avoid this by renaming the variable $y$ to $z$, giving the formula $\exists z(x + z = 0)$, and then substituting $y + y$ for $x$ to obtain $\exists z((y + y) + z = 0)$.

**Definition A.2.5** (**Substitution**, see [Rot00] pg. 18)**.** *Let $x_1, \dots, x_n$ be pairwise distinct variables and $t_1, \dots, t_n$ be arbitrary $L$-terms.*

*Let $t$ be an $L$-term. Then $t_{x_1 \dots x_n}(t_1, \dots, t_n)$ denotes the result of* substituting *$t_i$ for each occurrence of $x_i$ in $t$ for all $i \in \{1, \dots, n\}$.*

*Let $\varphi$ be an $L$-formula. Then $\varphi_{x_1 \dots x_n}(t_1, \dots, t_n)$ denotes the result of* simultaneously substituting *$t_i$ for each free occurrence of $x_i$ in $t$ for all $i \in \{1, \dots, n\}$, renaming variables, if necessary, to avoid collision of variables.*

One might ask whether $t_{x_1 \dots x_n}(t_1, \dots, t_n)$ is an $L$-term or not, and indeed it is an $L$-term. This can be shown via induction on complexity, that is the number of function symbols occurring in $t_{x_1 \dots x_n}(t_1, \dots, t_n)$. In a similar way, one can do induction on the number of $\neg$, $\wedge$ and $\exists$ symbols in $\varphi_{x_1 \dots x_n}(t_1, \dots, t_n)$ to show that it is an $L$-formula.

## A.3 Semantics

We finally come to the connection between structures and languages. We remind ourselves that we have fixed a signature $\sigma = (C, F, R, \sigma')$ and are working with the corresponding language $L$.

Consider an $L$-structure $\mathcal{M}$. We choose, for every $a \in M$, a new (in the sense that it is not already in the alphabet of $L$) constant symbol, denoted $\underline{a}$, and we let $\underline{M} = \{\underline{a} : a \in M\}$. Adding these to our language as constant symbols, we *expand* our language, and denote this expansion by $L(\underline{M})$. Now $\mathcal{M}^* := (\mathcal{M}, \underline{M})$ denotes the $L(\underline{M})$-structure in which the constant symbol $\underline{a}$ is interpreted as $a$ for all $a \in M$. We make this more precise.

**Definition A.3.1** (**Interpretation**, see [Rot00] pg. 23)**.** *Let $\mathcal{M}$ be an $L$-structure and $t$ be an $L(\underline{M})$-term without any variables. We define the* value *or* interpretation *of $t$ in $\mathcal{M}^*$, denoted by $t^{\mathcal{M}^*}$, as follows.*

1. *If $t$ is a constant $c \in C$ then $t^{\mathcal{M}^*} = c^{\mathcal{M}}$.*
2. *If $t$ is a constant $\underline{a}$ for $a \in M$ then $t^{\mathcal{M}^*} = a$.*
3. *If $t$ is of the form $f(t_1, \dots, t_n)$, where $f \in F$ with $\sigma'(f) = n$ and all the $t_i$'s are $L(\underline{M})$-terms without any variables, then $t^{\mathcal{M}^*} = f^{\mathcal{M}}(t_1^{\mathcal{M}^*}, \dots, t_n^{\mathcal{M}^*})$.*

**Definition A.3.2** (**Satisfaction**, see [Rot00] pg. 23)**.** *Let $\mathcal{M}$ be an $L$-structure and $\varphi$ be an $L(\underline{M})$-sentence. We define that $\varphi$ is* true *in $\mathcal{M}^*$, denoted $\mathcal{M}^* \models \varphi$, as follows.*

1. *If $\varphi$ is $t_1 = t_2$ then $\mathcal{M}^* \models \varphi$ iff $t_1^{\mathcal{M}^*} = t_2^{\mathcal{M}^*}$.*
2. *If $\varphi$ is $r(t_1, \ldots, t_n)$, where $r$ is a relation symbol, then $\mathcal{M}^* \models \varphi$ iff $r^{\mathcal{M}}(t_1^{\mathcal{M}^*}, \ldots, t_n^{\mathcal{M}^*})$.*
3. *If $\varphi = \neg\psi$ then $\mathcal{M}^* \models \varphi$ iff $\mathcal{M}^* \not\models \psi$, i.e.* not *$\mathcal{M}^* \models \psi$.*
4. *If $\varphi$ is $\psi_1 \wedge \psi_2$ then $\mathcal{M}^* \models \varphi$ iff $\mathcal{M}^* \models \psi_1$ and $\mathcal{M}^* \models \psi_2$.*
5. *If $\varphi$ is $\exists x\psi$ then $\mathcal{M}^* \models \varphi$ iff there is $a \in M$ such that $\mathcal{M}^* \models \psi_x(\underline{a})$.*

*In case $\mathcal{M}^* \models \varphi$ we say that $\varphi$ is* true *or* holds *in $\mathcal{M}^*$. We may also say that $\mathcal{M}^*$* satisfies *$\varphi$.*

We extend the meaning of value and satisfaction to $L$-terms and -formulas in general. Let $\bar{x}$ be an $n$-tuple of variables, $\mathcal{M}$ an $L$-structure and $\bar{a}$ a $n$-tuple from $M$. We write $t(\bar{x})$ to denote that the variables of $t$ are amongst those in $\bar{x}$. Similarly we write $\varphi(\bar{x})$ to denote that the free variables of $\varphi$ are contained in $\bar{x}$.

The *value* of the $L$-term $t(\bar{x})$ at $\bar{a}$ in $\mathcal{M}$, denoted $t^{\mathcal{M}}(\bar{a})$, is $t_{\bar{x}}(\underline{\bar{a}})^{\mathcal{M}^*}$. The $L$-formula $\varphi(\bar{x})$ is *satisfied* by $\bar{a}$ in $\mathcal{M}$ if $\mathcal{M}^* \models \varphi_{\bar{x}}(\underline{\bar{a}})$ and we write $\mathcal{M} \models \varphi(\bar{a})$. This notation is extended to sets of formulas, so for $\Psi(\bar{x}) = \{\varphi_i(\bar{x}) : i \in I\}$, if we write $\mathcal{M} \models \Psi(\bar{a})$ this means that $\mathcal{M} \models \varphi_i(\bar{a})$ for each $i \in I$. We say that an $L$-formula $\varphi$ is *valid* in $\mathcal{M}$ if every tuple (of the correct length) in $\mathcal{M}$ satisfies $\varphi$. In this case we write $\mathcal{M} \models \varphi$.

**Definition A.3.3** (**Model**, see [Rot00] pg. 28)**.** *Let $\Sigma$ be a set of $L$-sentences and $\mathcal{M}$ be a non-empty $L$-structure. If $\mathcal{M} \models \Sigma$ we say that $\mathcal{M}$ is a* model *of $\Sigma$.*

Now back to the symbols that one might expect to be included in the language. We are missing $\top$ (verum) and $\perp$ (falsum). We also want $\vee$ for disjunction (or), $\rightarrow$ for implication (implies), $\leftrightarrow$ for equivalence (if and only if or iff) and the universal quantifier $\forall$ (for all). Thanks to logical equivalence, we can leave these out. Here is a reminder.

**Definition A.3.4** (**Logical equivalence**, see [Rot00] pg. 29)**.** *The $L$-sentences $\varphi$ and $\psi$ are said to be* logically equivalent, *denoted $\varphi \equiv \psi$, when for each $L$-structure $\mathcal{M}$, $\mathcal{M} \models \varphi$ iff $\mathcal{M} \models \psi$.*

$$\perp \equiv \exists x(x \neq x)$$

$$\top \equiv \neg\perp$$

$$\varphi \vee \psi \equiv \neg(\neg\varphi \wedge \neg\psi)$$

$$\varphi \rightarrow \psi \equiv \neg\varphi \vee \psi$$

$$\varphi \leftrightarrow \psi \equiv (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$$

$$\forall x\varphi \equiv \neg\exists x\neg\varphi$$

## A.4 Mappings

Mappings that preserve certain features of mathematical structures are useful when we want to compare different structures.

**Definition A.4.1** (**Homomorphism**, see [Rot00] pg. 5)**.** *Let $\mathcal{M}$ and $\mathcal{N}$ be $\sigma$-structures. $h : M \rightarrow N$ is a* homomorphism *if*

1. *$h(c^{\mathcal{M}}) = c^{\mathcal{N}}$ for all constant symbols $c \in C$,*
2. *$h(f^{\mathcal{M}}(\bar{a})) = f^{\mathcal{N}}(h[\bar{a}])$ for all function symbols $f \in F$ and all tuples $\bar{a} \in M^{\sigma'(f)}$ and*
3. *if $r^{\mathcal{M}}(\bar{a})$ then $r^{\mathcal{N}}(h[\bar{a}])$ for all relation symbols $r \in R$ and all tuples $\bar{a} \in M^{\sigma'(r)}$.*

*In this case we write $h : \mathcal{M} \rightarrow \mathcal{N}$.*

**Definition A.4.2** (**Monomorphism**, see [Rot00] pg. 5)**.** *A* monomorphsim *from $\mathcal{M}$ to $\mathcal{N}$ is an injective homomorphism $h : \mathcal{M} \to \mathcal{N}$ such that for all relation symbols $r \in R$ and all tuples $\bar{b} \in h[M]^{\sigma'(r)}$ with $r^{\mathcal{N}}(\bar{b})$, there is a tuple $\bar{a} \in M^{\sigma'(r)}$ such that $r^{\mathcal{M}}(\bar{a})$ and $h[\bar{a}] = \bar{b}$. We write $h : \mathcal{M} \hookrightarrow \mathcal{N}$.*

**Definition A.4.3** (**Isomorphism**, see [Rot00] pg. 5)**.** *A surjective monomorphism $h : \mathcal{M} \hookrightarrow \mathcal{N}$ is called an* isomorphism *and we write $h : \mathcal{M} \cong \mathcal{N}$.*

**Remarks A.4.4** (see [Rot00] pg. 6)**.**

1. A monomorphism between $\mathcal{M}$ and $\mathcal{N}$ may also be called an *isomorphic embedding* of $\mathcal{M}$ into $\mathcal{N}$.
2. An *endomorphism* is a homomorphism from a structure to itself.
3. An *automorphism* is an isomorphism from a structure to itself.

## A.5 Chains

**Definition A.5.1** (**Chain**, see [Rot00] pg. 151)**.** *Let $\alpha$ be an ordinal. A sequence of $L$-structures, $(\mathcal{M}_i : i < \alpha)$ is called a* chain *if $\mathcal{M}_i \subseteq \mathcal{M}_j$ for each $i < j < \alpha$.*

**Definition A.5.2** (**Continuous chain**, see [Rot00] pg. 151)**.** *If $\bigcup_{i<\delta} \mathcal{M}_i = \mathcal{M}_\delta$ for each limit ordinal $\delta < \alpha$ then the chain $(\mathcal{M}_i : i < \alpha)$ is said to be* continuous*.*

**Definition A.5.3** (**Union of chain**, see [Rot00] pg. 151)**.** *The* union *of the chain $(\mathcal{M}_i : i < \alpha)$, denoted $\bigcup_{i<\alpha} \mathcal{M}_i$, is the canonical $L$-structure on the set $\bigcup_{i<\alpha} M_i$.*

Note that $\mathcal{M}_j \subseteq \bigcup_{i<\alpha} \mathcal{M}_i$ for all $j < \alpha$ since $\mathcal{M}_i \subseteq \mathcal{M}_j$ for each $i < j < \alpha$.

**Definition A.5.4** (**Elementary chain**, see [Rot00] pg. 151)**.** *The chain $(\mathcal{M}_i : i < \alpha)$ is* elementary *if it is continuous and $\mathcal{M}_i \preccurlyeq \mathcal{M}_{i+1}$ for all $i < \alpha$.*

**Theorem A.5.5** (**On elementary chains**, see [Rot00] pg. 151)**.** *For any elementary chain $(\mathcal{M}_i : i < \alpha)$ we have that $\mathcal{M}_j \preccurlyeq \bigcup_{i<\alpha} \mathcal{M}_i$ for all $j < \alpha$.*

# Appendix B

# Turing machines

Turing machines are largely associated with computability and recursion theory. In this dissertation they are only used for the topic of decidability. We give an informal description first.

Imagine an infinite tape marked into squares, with all of the squares blank except for a finite number of squares which have a stroke. Now, place a cart with no bottom over some square of the tape. Inside the cart is a small man who will be carrying out computations on this tape.

The small man is able to read one square on the tape, erase or write a stroke in this square and then move one square left or right. He also has a finite list of instructions, telling him what to do when he is in a given state. The instructions can be any one of the following.

1. Erase whatever is in the square.
2. Write a stroke in the square.
3. Move one square left.
4. Move one square right.
5. Stop the computation.

The tape, cart and man make up a *Turing machine* (as described in [BBJ02]). Formally a Turing machine can be viewed as a transition function.

**Definition B.1** (see [End97] pg. 531)**.** *A Turing machine is a function $f$ such that, for some $n \in \mathbb{N}$,*

$$\mathit{domain}(f) \subseteq \{0, 1, \ldots, n\} \times \{0, 1\} \text{ and}$$

$$\mathit{range}(f) \subseteq \{0, 1\} \times \{L, R\} \times \{0, 1, \ldots, n\}.$$

*Pairs in the domain of $f$ have the form $(q, a)$ where*

1. *$q$ denotes the present "state" or "instruction" that the machine is in,*
2. *and $a$ is either a $0$ or a $1$, indicating whether the present square contains a stroke.*

*Triples in the range of $f$ have the form $(b, M, r)$ where*

1. *$b$ is either $0$ or $1$, indicating whether the new symbol in the present square should be a stroke or not,*
2. *$M$ is either $L$ or $R$, indicating whether the machine should move a square to the left or right of the present square once its symbol has been overridden,*
3. *and $r$ indicates the new state of the machine.*

**Example B.2** (see [End97] pg. 531)**.** If $f(3, 1) = (0, L, 2)$ this means that, whenever the machine (man in the cart) comes to instruction number 3 (on his finite list of instructions) while reading a square with a stroke, he has to erase the stroke, creating a blank square (represented by the 0), move the cart one square left, and then continue to instruction number 2.

The sense in which we use Turing machines is in the form of an algorithm. That is, we use it to answer the yes-no-question: "Is this sentence in the theory?" (Section 4.1). In more general terms a class of yes-no-problems is solvable if there is a fixed algorithm (or equivalently a Turing machine) such that, given a problem from the class as input, the computation stops at some point and gives the output "yes" or "no". The *Church-Turing Thesis* (see [BA05] pg. 2) states that a process is an *algorithm* iff it can be carried out by a Turing machine. A detailed description and explanation of this can be found in Section 1.5 of [Man03]. For the purpose of this dissertation, it is enough to know that a Turing machine can be used to determine whether or not a given $L$-sentence is in a given decidable $L$-theory.

# Appendix C

# Orderings

## C.1 Partial, linear, and dense orders

**Definition C.1** (**Partial order**, see [Rot00] pg. 59)**.** *A binary relation $<$ defines a* partial order *on a set $X$ if it is*

1. *irreflexive, i.e. $x \nless x$ for all $x \in X$*
2. *and transitive, i.e. if $x < y$ and $y < z$ then $x < z$ for all $x, y, z \in X$.*

We use $T_<$ to denote the first-order theory of all partial orderings.

**Definition C.2** (**Linear order**, see [Rot00] pg. 59)**.** *A binary relation $<$ defines a* linear order *(or total order) on a set $X$ if*

1. *it is a partial order*
2. *and for any $x, y \in X$ it is either the case that $x < y$ or $y = x$ or $y < x$.*

We use LO to denote the first-order theory of all linear orderings.

**Definition C.3** (**Dense linear order**, see [Rot00] pg. 59)**.** *A binary relation $<$ defines a* dense linear order *on a set $X$ if*

1. *it is a linear order*
2. *and for any $x, y \in X$ with $x < y$ there exists a $z \in X$ such that $x < z < y$.*

We use DLO to denote the first-order theory of all dense linear orderings.

We might add the sentence $\forall x \exists y \exists z (y < x \wedge x < z)$, saying that there are no left or right endpoints, to DLO. This gives the theory, $\mathrm{DLO}_{--}$, of all dense linear orderings without endpoints.

## C.2 The fellowship of Rational and Rado

See? He's her lobster.

PHOEBE
*Friends*

We have come across lots of properties of $\boldsymbol{R}$ and grouped them together in our guide to the Rado graph. But it all seems vaguely familiar. Haven't we seen these properties coming together before?

In this final chapter we discuss the resemblance between our beast, $\boldsymbol{R}$ and $(\mathbb{Q}, <)$, the rationals viewed as a linear order. We will give the results in parallel. We are not going to prove any of the results in this chapter, readers are referred to the relevant texts for proofs.

| | |
|---|---|
| $(\mathbb{Q},<)$ is countably infinite | $\boldsymbol{R}$ is countably infinite |
| $(\mathbb{Q},<)$ is dense and without endpoints | $\boldsymbol{R}$ satisfies EP |
| Any two countably infinite dense linear orderings without endpoints are isomorphic. (Cantor's Theorem, see [Rot00] Theorem 7.3.1 pg. 89) | Any two countably infinite graphs with EP are isomorphic. (Theorem 1.1.3) |
| Any countably infinite dense linear order without endpoints is isomorphic to $(\mathbb{Q},<)$. | Any countably infinite graph with EP is isomorphic to $\boldsymbol{R}$. |
| The ordering obtained by deleting an element (or finitely many elements) from $(\mathbb{Q},<)$ will be isomorphic to $(\mathbb{Q},<)$. | The graph obtained by deleting a vertex (or finitely many vertices) of $\boldsymbol{R}$ is isomorphic to $\boldsymbol{R}$. (Proposition 2.1.4) |
| Every countably infinite and finite linear order is isomorphic to a suborder of $(\mathbb{Q},<)$. ([Fre14] pg. 330) | Every countably infinite and finite graph is isomorphic to a induced subgraph of $\boldsymbol{R}$. (Theorem 2.1.6) |
| $(\mathbb{Q},<)$ is ultrahomogeneous. (See [Cam15]) | $\boldsymbol{R}$ is ultrahomogeneous. (Proposition 3.1.2) |
| $\mathrm{DLO}_{--}$ is $\aleph_0$-categorical. (See [Rot00] pg. 123) | $\mathrm{Th}\boldsymbol{R}$ is $\aleph_0$-categorical. (Proposition 4.2.5) |
| $\mathrm{DLO}_{--}$ is complete. (See [Rot00] pg. 123) | $\mathrm{Th}\boldsymbol{R}$ is complete. (Theorem 4.2.7) |
| $\mathrm{DLO}_{--}$ admits quantifier elimination. (See [Rot00] pg. 137) | $\mathrm{Th}\boldsymbol{R}$ admits quantifier elimination. (Theorem 4.2.16) |
| $(\mathbb{Q},<)$ is saturated. (See [Rot00] pg. 186) | $\boldsymbol{R}$ is saturated. (Proposition 4.3.8) |
| The class of all finite linear orderings is the age of $(\mathbb{Q},<)$. (See [Hod93] pg. 325) | The class of all finite graphs is the age of $\boldsymbol{R}$. |
| The class of all finite linear orderings has HP, JEP and AP. (See [Hod93] pg. 325) | The class of all finite graphs has HP, JEP and AP. (Theorem 4.5.8) |
| $(\mathbb{Q},<)$ is the Fraïssé limit of the class of all finite linear orderings. (See [Hod93] pg. 324) | $\boldsymbol{R}$ is the Fraïssé limit of the class of all finite graphs. (Theorem 4.5.11) |

# Appendix D

# Python code used for $r$-extension

Python was a great help in finding graphs with 2-extension. It also helped a great deal in testing whether certain methods of constructing graphs with $r$-extension would work or not. The code used for different parts are displayed and explained below.

## D.1 Testing a given graph for $r$-extension

The first step is to set up the hard-coded matrix you want to test. We use the matrix for the graph in Example 5.2.15 in the example below.

```
import numpy as np

test_matrix=np.array([
[0, 1, 1, 1, 0, 0, 1, 0, 0],
[1, 0, 1, 0, 1, 0, 0, 1, 0],
[1, 1, 0, 0, 0, 1, 0, 0, 1],
[1, 0, 0, 0, 1, 1, 1, 0, 0],
[0, 1, 0, 1, 0, 1, 0, 1, 0],
[0, 0, 1, 1, 1, 0, 0, 0, 1],
[1, 0, 0, 1, 0, 0, 0, 1, 1],
[0, 1, 0, 0, 1, 0, 1, 0, 1],
[0, 0, 1, 0, 0, 1, 1, 1, 0]
])
```

Next we have to set up the parameters of our program. We tell it the size of the matrix, 9 in this case. We also have to tell it what to look for. In this case specifically we want to test for 2-extension, so for each two rows of the matrix, there should be column vectors containing $(0, 0)$, $(0, 1)$, $(1, 0)$ and $(1, 1)$. We also set up a flag variable, `Eureka`, of which the value will change if the program does not find all the needed column vectors.

```
matrix_size=9

look_for=[
[0,0],
[0,1],
[1,0],
[1,1]
]

Eureka='Yes'
```

The program will loop through the rows and columns of the matrix to search for the column vectors. We initialize indices as follows.

```
row_index_1=0
col_index=0
```

We can now start with the loops. For two distinct rows $i$ and $j$ we only have to compare one of the rows to the other. So the first loop will run through all the rows of the matrix, and the second loop will run through all rows $i > j$ if the first loop is currently at row $j$. Next we loop through the columns to search for the column vectors.

```
while row_index_1<matrix_size :
    row_index_2=row_index_1+1

    while row_index_2<matrix_size :
        #This creates an empty array, we fill it in the next step.
        look_for_temp = [None] * len(look_for)

        for i in range(0, len(look_for)):
            #This fills the empty array with the same values as look_for.
            #If the column vector is found, it gets deleted from this new array.
            look_for_temp[i] = look_for[i];

        while col_index<matrix_size :
            #This ensures that the column checked is different from both rows.
            if col_index==row_index_1 or col_index==row_index_2 :
                col_index+=1

            else :
                #This creates the column vector for the two rows.
                temp_array=[
                test_matrix[row_index_1,col_index],
                test_matrix[row_index_2,col_index]
                ]
                if temp_array in look_for_temp :
                    #This deletes the column vector found.
                    look_for_temp.remove(temp_array)

                #This moves to the next column
                col_index+=1

        #This indicates if all the necessary column vectors were found.
        if len(look_for_temp)!=0 :
            print('Row',row_index_1+1,'and',row_index_2+1,'unsuccessful.')
            #This shows the missing column vector(s).
            print(look_for_temp)
            #Failure to find all the column vectors changes the flag variable...
            Eureka='No'
            #...and breaks out of the inner loop.
            break
        else :
            print('Row',row_index_1+1,'and',row_index_2+1,'works!')

        #This resets the column index.
        col_index=0
        row_index_2+=1

```

```
    if Eureka=='No' :
        break

    row_index_1+=1

#This indicates that the column vectors were found for all pairs of rows.
#In other words, the graph of the matrix has 2-extension.
if Eureka=='Yes' :
    print('Eureka!')

else :
    print('Matrix, you shall not pass!')
```

Note that this program tests only for 2-extension. If we want it to work for 3-extension we have to look for more column vectors, namely:

```
look_for=[
[0,0,0],
[0,0,1],
[0,1,0],
[1,0,0],
[0,1,1],
[1,0,1],
[1,1,0],
[1,1,1]
]
```

and then just add another loop on the outside of the current loops, running through the rows. Of course, some details in the inner part of the loops will have to change accordingly, but the idea of the program stays the same and can be followed for setting up these details.

## D.2 Random-binary-fill

Setting up and hard-coding possible matrices is not the most ideal way to find matrices of graphs with $r$-extension. The following program fills matrices of a given size with random binary numbers and checks if the matrix's graph will satisfy 2-extension. This is also not the ideal solution, but much better than trying to find matrices that work by hand.

```
#This imports the needed packages.
import numpy as np
import random

#Here we choose the size of the matrix to be filled.
matrix_size=10

#This generates a matrix of the given size filled with zeros.
matrix=np.zeros((matrix_size,matrix_size))

#We only need to fill the matrix above the diagonal.
#These entries will be reflected about the diagonal to create a symmetric matrix.
#Here we calculate the length of the binary string needed to fill the top half of the matrix.
length=(matrix_size**2-matrix_size)//2

#Next we work out the decimal value of the largest binary number,
#i.e. the longest sting of 1's,that can be filled in the top half of the matrix.
#We will use this number as an upper limit when generating random numbers.
```

```
power=0
max_num=0

while power<length:
    max_num+=2**power
    power_pos+=1

#Here we initialize the number of random numbers to be checked...
iteration=1
#...and the number of matrices found that works.
success=0
#We also choose how many randomly filled matrices we want to check.
check_up_to=100

while iteration<check_up_to+1 :
    #This generates a random number between 0 and max_num.
    num=random.randint(0,max_num)
    #This converts the random number to its equivalent in binary.
    bin_num=format(num,'b')
    #The binary number is then converted to a string, which we use to fill matrix.
    x=str(bin_num)
    bin_string=x.zfill(length)

    #Here we initialize loops to fill the top half of the matrix.
    row_pos=0
    col_pos=row_pos+1
    bin_pos=0

    while row_pos<matrix_size :
        while col_pos<matrix_size and bin_pos<length:
            #This fills the top half of the matrix with the binary string.
            matrix[row_pos][col_pos]=bin_string[bin_pos]
            #This reflects the entries about the diagonal.
            matrix[col_pos][row_pos]=bin_string[bin_pos]
            bin_pos+=1
            col_pos+=1

        row_pos+=1
        col_pos=row_pos+1

    #Here we start the process of checking if the matrix's graph has 2-extension.
    #This is the same as the code from the previous section.
    look_for=[[0,0],
              [0,1],
              [1,0],
              [1,1]]

    Eureka='Yes'

    row_index_1=0
    col_index=0

    while row_index_1<matrix_size :
        row_index_2=row_index_1+1

        while row_index_2<matrix_size :
```

```
            look_for_temp = [None] * len(look_for)

            for i in range(0, len(look_for)):
                look_for_temp[i] = look_for[i];

            while col_index<matrix_size :

                if col_index==row_index_1 or col_index==row_index_2 :
                    col_index+=1

                else :
                    temp_array=[
                    matrix[row_index_1,col_index],
                    matrix[row_index_2,col_index]
                    ]
                    if temp_array in look_for_temp :
                        look_for_temp.remove(temp_array)

                    col_index+=1

            if len(look_for_temp)!=0 :
                Eureka='No'
                break

            col_index=0
            row_index_2+=1

        if Eureka=='No' :
            break

        row_index_1+=1

    #Here we show the matrix if its graph has 2-extension.
    if Eureka=='Yes' :
        print('This matrix works!')
        print(matrix)
        success+=1

    iteration+=1

print(success, 'iterations successful.')
print(iteration-1-success,'iterations unsuccessful.')
```

## D.3 Sequential-binary-fill

We can do better than filling the matrix with random binary numbers and then checking to see if the matrix satisfies the necessary conditions. We only need to fill the half of the matrix, above the diagonal, and then reflect these entries to the half below the diagonal. This is because the diagonal entries are all 0 (since $\sim$ is irreflexive) and the adjacency matrix is symmetrical. So we might consider all possible binary strings of the correct length to fill the top half of the matrix with, and then check all of these matrices for 2-extension. This program is mostly the same as for random-binary-fill, except for a few minor differences.

```
#This imports the needed package.
import numpy as np
```

```

#Here we choose the size of the matrix to be filled.
matrix_size=6

#This generates a matrix of the given size filled with zeros.
matrix=np.zeros((matrix_size,matrix_size))

#Here we calculate the length of the string needed to fill half of the matrix.
length=(matrix_size**2-matrix_size)//2

#Here we set the number we start filling the matrix with.
num=0
success=0

power=0
max_num=0

while power<length :
    max_num+=2**power
    power+=1


while num<= max_num:
    bin_num=format(num,'b')
    x=str(bin_num)
    bin_string=x.zfill(length)

    row_pos=0
    col_pos=row_pos+1
    bin_pos=0

    while row_pos<matrix_size :
        while col_pos<matrix_size and bin_pos<length:
            matrix[row_pos][col_pos]=bin_string[bin_pos]
            matrix[col_pos][row_pos]=bin_string[bin_pos]
            bin_pos+=1
            col_pos+=1

        row_pos+=1
        col_pos=row_pos+1

    look_for=[[0,0],
              [0,1],
              [1,0],
              [1,1]]

    row_index_1=0
    col_index=0

    Eureka='Yes'

    while row_index_1<matrix_size :
        row_index_2=row_index_1+1

        while row_index_2<matrix_size :
            look_for_temp = [None] * len(look_for)
```

```python

            for i in range(0, len(look_for)):
                look_for_temp[i] = look_for[i];

            while col_index<matrix_size :

                if col_index==row_index_1 or col_index==row_index_2 :
                    col_index+=1

                else :
                    temp_array=[
                    matrix[row_index_1,col_index],
                    matrix[row_index_2,col_index]
                    ]
                    if temp_array in look_for_temp :
                        look_for_temp.remove(temp_array)

                    col_index+=1

            if len(look_for_temp)!=0 :
                Eureka='No'
                break

            col_index=0
            row_index_2+=1

        if Eureka=='No' :
            break

        row_index_1+=1

    if Eureka=='Yes' :
        print('This matrix works!')
        print(matrix)
        success+=1

    num+=1

print(success, 'matrices successful.')
print(num-success,'matrices unsuccessful.')
```

This program was used to show that there are no matrices of sizes 6, 7 and 8 with 2-extension.

# Index

# References

[Abr18] Anand Abraham. 0-1 laws in logic: An overview. 2018. [Online] Available at: http://math.uchicago.edu/~may/REU2018/REUPapers/Abraham.pdf.

[BA05] Amir M Ben-Amram. The Church-Turing thesis and its look-alikes. *ACM SIGACT News*, 36(3):113–114, 2005.

[BBJ02] George S Boolos, John P Burgess, and Richard C Jeffrey. *Computability and logic*. Cambridge University Press, 2002.

[BEH81] Andreas Blass, Geoffrey Exoo, and Frank Harary. Paley graphs satisfy all first-order adjacency axioms. *Journal of Graph Theory*, 5(4):435–439, 1981.

[BH79] Andreas Blass and Frank Harary. Properties of almost all graphs and complexes. *Journal of Graph Theory*, 3(3):225–240, 1979.

[BR05] Andreas Blass and Benjamin Rossman. Explicit graphs with extension properties. *Bull. EATCS*, 86:166–175, 2005.

[Bru10] Richard A Brualdi. *Introductory Combinatorics*. Prentice-Hall, 5th edition, 2010.

[BS76] Jon Barwise and John Schlipf. An introduction to recursively saturated and resplendent models. *The Journal of Symbolic Logic*, 41(2):531–536, 1976.

[BT81] Béla Bollobás and Andrew Thomason. Graphs which contain all small graphs. *European Journal of Combinatorics*, 2(1):13–15, 1981.

[Cam05] Peter J. Cameron. The random graph has the strong small index property. *Discrete Mathematics*, 291(1):41–43, 2005.

[Cam13] Peter J. Cameron. The random graph. In Ronald L. Graham, Jaroslav Nešetřil, and Steve Butler, editors, *The Mathematics of Paul Erdős II*, pages 353–378. Springer, New York, 2013.

[Cam15] Peter J. Cameron. Four precious jewels. In *Australasian Conference ON Combinatorial Mathematics and Combinatorial Computing*, 2015.

[CS02] Peter J Cameron and Dudley Stark. A prolific construction of strongly regular graphs with the $n$-ec property. *The electronic journal of combinatorics*, pages R31–R31, 2002.

[Die00] Reinhard Diestel. *Graph Theory*. Springer, 2000.

[Doe96] Kees Doets. *Basic model theory*. CSLI Publications, 1996.

[Dud09] Underwood Dudley. *Guide to Elementary Number Theory*, volume 41 of *Dolcaini Mathematical Expositions*. The Mathematical Association of America, 2009.

[EFT94] Heinz-Dieter Ebbinghaus, Jörg Flum, and Wolfgang Thomas. *Mathematical logic*, volume 1910. Springer, 1994.

[EHP75] Paul Erdos, András Hajnal, and Lajos Pósa. Strong embeddings of graphs into colored graphs. *Infinite and finite sets*, 1:585–595, 1975.

[End97] Herbert B. Enderton. *Handbook of Mathematical Logic.* Norht Holland Publishing Company, 1997.

[ER63] P Erdős and A Rényi. Asymmetric graphs. *Acta Mathematica Academiae Scientiarum Hungarica*, 14(3):295–315, 1963.

[Fag76] Ronald Fagin. Probabilities on finite models. *The Journal of Symbolic Logic*, 41(1):50–58, 1976.

[Fre14] Ronald C Freiwald. *An introduction to set theory and topology.* Washington University in St. Louis, 2014.

[Gai64] Haim Gaifman. Concerning measures in first order calculi. *Israel Journal of Mathematics*, 2:1–18, 1964.

[Hed04] Shawn Hedman. *A First Course in Logic: An introduction to model theory, proof theory, computability, and complexity.* Oxford University Press, 2004.

[Hen19] Nik Henderson. What is the rado graph? 2019. [Online]
Available at:https://math.osu.edu/sites/math.osu.edu/files/What_is_2019_Rado_Graph.pdf.

[Hod93] Wilfrid Hodges. *Model theory.* Cambridge University Press, 1993.

[IW19] Pierre Ille and Robert Woodrow. Fractal graphs. *Journal of Graph Theory*, 91(1):53–72, 2019.

[Kos11] Roman Kossak. What is a resplendent structure? *Notices of the AMS*, 58(6), 2011.

[Man03] Zohar Manna. *Mathematical theory of computation.* Dover Publications, Inc., 2003.

[Pin10] Charles C Pinter. *A book of abstract algebra.* Courier Corporation, 2010.

[PS96] Maurice Pouzet and Norbert Sauer. Edge partitions of the rado graph. *Combinatorica*, 16(4):505–520, 1996.

[Rad64] R. Rado. Universal graphs and universal functions. *Acta Arithmetica*, 9(4):331–340, 1964.

[Rot00] Phillip Rothmaler. *Introduction to Model Theory.* Taylor & Francis Group, 2000.

[Sel49] Atle Selberg. An Elementary Proof of Dirichlet's Theorem About Primes in an Arithmetic Progression. *Annals of Mathematics*, 50(2):297–304, 1949.

[Wil09] Robert Wilson. *The finite simple groups*, volume 251. Springer Science & Business Media, 2009.