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Efficient Pricing of Spread Options with Stochastic Rates and Stochastic Volatility

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Abstract: Spread options are notoriously difficult to price without the use of Monte Carlo simulation. Some strides have been made in recent years through the application of Fourier transform methods; however, to date, these methods have only been applied to specific underlying processes including two-factor geometric Brownian motion (gBm) and three-factor stochastic volatility models. In this paper, we derive the characteristic function for the two-asset Heston–Hull–White model with a full correlation matrix and apply the two-dimensional fast Fourier transform (FFT) method to price equity spread options. Our findings suggest that the FFT is up to 50 times faster than Monte Carlo and yields similar accuracy. Furthermore, stochastic interest rates can have a material impact on long-dated out-of-the-money spread options.

Keywords: spread option; two-asset Heston–Hull–White model; discounted characteristic function; fast Fourier transform; stochastic interest rates

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1. Introduction

Let S_1, S_2 denote the prices of two different equities, T the maturity date, K the strike price, and r the risk-free interest rate. Let $\mathbb{Q}$ be the risk-neutral measure associated with the bank account as numeraire, $B(t) = e^{\int_0^t r(s)ds}$. The value of a European equity spread call option at $t = 0$ is then given by:

$$V_{Spread}(0) = \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_0^T r(s)ds} \max\left(S_1(T) - S_2(T) - K, 0\right)\right]. \tag{1}$$

Equation (1) has no closed-form solution when the underlying assets are driven by stochastic interest rates and stochastic volatility. Monte Carlo simulation is one possible way of solving the problem, but it is too slow for practical use.

A breakthrough was made by Dempster and Hong (2002) where the authors extended the Carr and Madan (1999) Fourier transform method to two factors. The authors derived lower and upper bound expressions for the price of a European spread call option and showed how to compute these expressions by means of the two-dimensional FFT when the underlying asset processes are driven by gBm or stochastic volatility.

Hurd and Zhou (2010) proposed a method for pricing spread options based on a square integrable Fourier representation of the payoff function. The authors claim that the method can be applied to any model for which the characteristic function of the joint asset process is known in closed form. The authors considered gBm, three-factor stochastic volatility, and exponential Lévy models for the underlying asset processes.

Little has been said about the impact of stochastic interest rates on spread options. We do, however, know that stochastic interest rate risk dominates that of stochastic volatility for long-dated European call and put options (see Kammeyer and Kienitz 2012). Furthermore, there is empirical evidence that suggests changes in interest rates and changes in stock prices are negatively correlated in general (Alam and Uddin 2009). None of the models considered by Dempster and Hong (2002) or Hurd and Zhou (2010) account for stochastic interest rates or the correlation between interest rates and stock prices.

Grzelak and Oosterlee (2011) made a remarkable contribution to the literature, which derived approximations for the characteristic function of the Heston (1993) model with stochastic interest rates driven by a Hull and White (1990) process. This model is called the Heston–Hull–White model. Through these approximations, the model allows for a full matrix of correlations between stock, volatility, and interest rate processes. The authors showed that the model can be calibrated to European options efficiently via Fourier techniques.

In this paper, we extend the Heston–Hull–White model of Grzelak and Oosterlee (2011) to two assets and the FFT method of Hurd and Zhou (2010) to stochastic interest rates. We then compare the efficiency of the extended Hurd and Zhou (2010) method to a Monte Carlo simulation and assess the impact of stochastic interest rates on spread option prices.

The rest of this paper is organised as follows. The two-asset Heston–Hull–White model is introduced in Section 2. In Section 3, we discuss the result of Grzelak and Oosterlee (2011) that will be used to derive the discounted characteristic function. The discounted characteristic function for the two-asset Heston–Hull–White model is derived in Section 4. The extension of the Hurd and Zhou (2010) FFT algorithm to stochastic interest rates is detailed in Section 5. Our numerical results are shown in Section 6, and Section 7 concludes the paper.

2. The Two-Asset Heston–Hull–White Model

Combining the three-factor stochastic volatility model of Dempster and Hong (2002) and the Heston–Hull–White model of Grzelak and Oosterlee (2011) yields:

$$\begin{cases} dS_1(t) = (r(t) - \delta_1)S_1(t)dt + \sigma_1\sqrt{v(t)}S_1(t)dW_{x_1}(t) \\ dS_2(t) = (r(t) - \delta_2)S_2(t)dt + \sigma_2\sqrt{v(t)}S_2(t)dW_{x_2}(t) \\ dv(t) = \kappa\Big(\bar{v} - v(t)\Big)dt + \sigma_v\sqrt{v(t)}dW_v(t) \\ dr(t) = \lambda\Big(\theta(t) - r(t)\Big)dt + \eta dW_r(t), \end{cases} \tag{2}$$

where σ_1, σ_2 denote the volatility of S_1, S_2, respectively; δ_1, δ_2 are the dividend yields for S_1, S_2; κ and λ denote the mean reversion speed of the variance and short rate processes; σ_v and η denote the volatility of the variance and short rate processes, respectively, and $\bar{v}$ and $\theta(t)$ denote the mean level of the variance and short rate, respectively.

The underlying process are correlated as follows:

$$\begin{aligned} dW_{x_1}(t)dW_{x_2}(t) &= \rho_{x_1,x_2}dt, & dW_{x_1}(t)dW_v(t) &= \rho_{x_1,v}dt, \\ dW_{x_1}(t)dW_r(t) &= \rho_{x_1,r}dt, & dW_{x_2}(t)dW_v(t) &= \rho_{x_2,v}dt, \\ dW_{x_2}(t)dW_r(t) &= \rho_{x_2,r}dt, & dW_v(t)dW_r(t) &= \rho_{v,r}dt. \end{aligned}$$

Taking the log-transform $x_1(t) = \log S_1(t)$, $x_2(t) = \log S_2(t)$ and applying Itô's lemma, the system of stochastic differential equations (SDEs) in (2) can be rewritten as:

$$\begin{cases} dx_1(t) = \Big(r(t) - \delta_1 - \frac{1}{2}\sigma_1^2 v(t)\Big)dt + \sigma_1\sqrt{v(t)}dW_{x_1}(t) \\ dx_2(t) = \Big(r(t) - \delta_2 - \frac{1}{2}\sigma_2^2 v(t)\Big)dt + \sigma_2\sqrt{v(t)}dW_{x_2}(t) \\ dv(t) = \kappa\Big(\bar{v} - v(t)\Big)dt + \sigma_v\sqrt{v(t)}dW_v(t) \\ dr(t) = \lambda\Big(\theta(t) - r(t)\Big)dt + \eta dW_r(t). \end{cases} \tag{3}$$

The system of SDEs in (3) can be expressed as:

$$d\mathbf{X}(t) = \mu(\mathbf{X}(t))dt + \sigma(\mathbf{X}(t))d\mathbf{B}(t),$$

where

$$\mathbf{X}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ v(t) \\ r(t) \end{bmatrix},$$

$$\mu(\mathbf{X}(t)) = \begin{bmatrix} r(t) - \delta_1 - \frac{1}{2}\sigma_1^2 v(t) \\ r(t) - \delta_2 - \frac{1}{2}\sigma_2^2 v(t) \\ \kappa\Big(\bar{v} - v(t)\Big) \\ \lambda\Big(\theta(t) - r(t)\Big) \end{bmatrix},$$

$$\mathbf{B}(t) = \begin{bmatrix} B_{x_1}(t) \\ B_{x_2}(t) \\ B_v(t) \\ B_r(t) \end{bmatrix},$$

$$\begin{aligned} \mathbf{\Sigma}(\mathbf{X}(t)) &= \sigma(\mathbf{X}(t))\sigma(\mathbf{X}(t))^T \\ &= \begin{bmatrix} \sigma_1^2 v(t) & \rho_{x_1,x_2}\sigma_1\sigma_2 v(t) & \rho_{x_1,v}\sigma_1\sigma_v v(t) & \rho_{x_1,r}\sigma_1\eta\sqrt{v(t)} \\ * & \sigma_2^2 v(t) & \rho_{x_2,v}\sigma_2\sigma_v v(t) & \rho_{x_2,r}\sigma_2\eta\sqrt{v(t)} \\ * & * & \sigma_v^2 v(t) & \rho_{v,r}\sigma_v\eta\sqrt{v(t)} \\ * & * & * & \eta^2 \end{bmatrix}, \end{aligned}$$

with $\mathbf{B}(t)$ a vector of independent Brownian motions and $\sigma(\mathbf{X}(t))$ the Cholesky decomposition of the symmetric covariance matrix $\mathbf{\Sigma}(\mathbf{X}(t))$.

The influential paper by Duffie et al. (2000) states that each element in the drift and covariance matrices must be a linear function of the state variables in $\mathbf{X}(t)$ in order for a model to be in affine form. If this is the case, then the discounted characteristic function for the state vector $\mathbf{X}(t)$ can be written as:

$$\begin{aligned} \phi(\mathbf{u}, \mathbf{X}(t), t, T) &= \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^T r(s)ds + i\mathbf{u}^T\mathbf{X}(T)} \mid \mathcal{F}(t)\right] \\ &= e^{A(\mathbf{u},\tau)+B(\mathbf{u},\tau)x_1(t)+C(\mathbf{u},\tau)x_2(t)+D(\mathbf{u},\tau)v(t)+E(\mathbf{u},\tau)r(t)}, \end{aligned} \tag{4}$$

where $\tau := T - t$, and $\mathbf{u} = [u_1, u_2, 0, 0]^T$.

From the covariance matrix $\mathbf{\Sigma}(\mathbf{X}(t))$, it is clear that there are elements that are nonlinear functions of the state variables, in particular $\sqrt{v(t)}$.

Grzelak and Oosterlee (2011) proposed replacing the term $\sqrt{v(t)}$ with $\mathbb{E}[\sqrt{v(t)}]$ so that the Heston–Hull–White model could be expressed in affine form as in Equation (4). Their result follows in the next section.

3. The Result of Grzelak and Oosterlee

In order to write the Heston–Hull–White model in affine form, Grzelak and Oosterlee (2011) proposed the following approximation for $\sqrt{v(t)}$:

Lemma 1. $\left(\text{Approximation for } \mathbb{E}\left[\sqrt{v(t)}\right]\right)$. *Given that $v(t)$ follows a Cox et al. (1985) process, $\mathbb{E}\left[\sqrt{v(t)}\right]$ can be approximated by:*

$$\mathbb{E}\left[\sqrt{v(t)}\right] \approx \sqrt{c(t)(\lambda(t)-1)+c(t)d+\frac{c(t)d}{2(d+\lambda(t))}} =: \Lambda(t),$$

where $c(t) = \frac{1}{4\kappa}\sigma_v^2(1-e^{-\kappa t})$, $d = \frac{4\kappa\bar{v}}{\sigma_v^2}$, and $\lambda(t) = \frac{4\kappa v(0)e^{-\kappa t}}{\sigma_v^2(1-e^{-\kappa t})}$.

See Grzelak and Oosterlee (2011) for details.

The authors mention that the approximation for $\mathbb{E}\left[\sqrt{v(t)}\right]$ is still nontrivial and may lead to challenges when deriving the characteristic function for the Heston–Hull–White model. Therefore, a further simplified approximation for $\mathbb{E}\left[\sqrt{v(t)}\right]$ was proposed as shown in Lemma 2 below:

Lemma 2. $\left(\text{Further approximation for } \mathbb{E}\left[\sqrt{v(t)}\right]\right)$. *$\mathbb{E}\left[\sqrt{v(t)}\right]$ can be further approximated by a function of the form:*

$$\mathbb{E}\left[\sqrt{v(t)}\right] \approx a + be^{-ct} =: \tilde{\Lambda}(t),$$

where $a = \sqrt{\bar{v} - \frac{\sigma_v^2}{8\kappa}}$, $b = \sqrt{v(0)} - a$, and $c = -\log(b^{-1}(\Lambda(1) - a))$.

See Grzelak and Oosterlee (2011) for details.

Note that $\tilde{\Lambda}(t)$ in Lemma 2 is undefined for $\bar{v} < \sigma_v^2/8\kappa$; hence, parameter constraints must be imposed in order to use this result.

In the next section, we derive the discounted characteristic function for the two-asset Heston–Hull–White model using the approximation of Grzelak and Oosterlee (2011).

4. The Two-Asset Heston–Hull–White Characteristic Function

Replacing $\sqrt{v(t)}$ with $\mathbb{E}\left[\sqrt{v(t)}\right]$ in the covariance matrix $\mathbf{\Sigma}(\mathbf{X}(t))$ yields the approximated covariance matrix for the two-asset Heston–Hull–White model:

$$\tilde{\mathbf{\Sigma}}(\mathbf{X}(t)) = \begin{bmatrix} \sigma_1^2 v(t) & \rho_{x_1,x_2}\sigma_1\sigma_2 v(t) & \rho_{x_1,v}\sigma_1\sigma_v v(t) & \rho_{x_1,r}\sigma_1\eta\mathbb{E}\left[\sqrt{v(t)}\right] \\ * & \sigma_2^2 v(t) & \rho_{x_2,v}\sigma_2\sigma_v v(t) & \rho_{x_2,r}\sigma_2\eta\mathbb{E}\left[\sqrt{v(t)}\right] \\ * & * & \sigma_v^2 v(t) & \rho_{v,r}\sigma_v\eta\mathbb{E}\left[\sqrt{v(t)}\right] \\ * & * & * & \eta^2 \end{bmatrix}.$$

For the derivation of the discounted characteristic function, we drop the function arguments for convenience. Hence, $x_1 := x_1(t)$, $x_2 := x_2(t)$ $r := r(t)$, $v := v(t)$, and $\phi := \phi(\mathbf{u}, \mathbf{X}(t), t, T)$. Furthermore, to simplify the calculations, we consider a constant term structure of interest rates $\theta(t) = \theta$. The method can be generalised to include a term structure of interest rates, see Grzelak (2011) for details.

Using the drift vector $\mu(\mathbf{X}(t))$ and the covariance matrix $\tilde{\mathbf{\Sigma}}(\mathbf{X}(t))$ and applying the multidimensional Itô lemma to $\phi(\mathbf{u}, \mathbf{X}(t), t, T)$ yields the partial differential equation (PDE):

$$\begin{aligned}0 =&\frac{\partial\phi}{\partial t}+(r-\delta_1-\frac{1}{2}\sigma_1^2 v)\frac{\partial\phi}{\partial x_1}+(r-\delta_2-\frac{1}{2}\sigma_2^2 v)\frac{\partial\phi}{\partial x_2}+\kappa(\bar{v}-v)\frac{\partial\phi}{\partial v}+\lambda(\theta-r)\frac{\partial\phi}{\partial r}\\&+\frac{1}{2}\sigma_1^2 v\frac{\partial^2\phi}{\partial x_1^2}+\frac{1}{2}\sigma_2^2 v\frac{\partial^2\phi}{\partial x_2^2}+\frac{1}{2}\sigma_v^2 v\frac{\partial^2\phi}{\partial v^2}+\frac{1}{2}\eta^2\frac{\partial^2\phi}{\partial r^2}+\rho_{x_1,x_2}\sigma_1\sigma_2 v\frac{\partial^2\phi}{\partial x_1\partial x_2}\\&+\rho_{x_1,v}\sigma_1\sigma_v v\frac{\partial^2\phi}{\partial x_1\partial v}+\rho_{x_1,r}\sigma_1\eta\mathbb{E}\left[\sqrt{v}\right]\frac{\partial^2\phi}{\partial x_1\partial r}+\rho_{x_2,v}\sigma_2\sigma_v v\frac{\partial^2\phi}{\partial x_2\partial v}\\&+\rho_{x_2,r}\sigma_2\eta\mathbb{E}\left[\sqrt{v}\right]\frac{\partial^2\phi}{\partial x_2\partial r}+\rho_{v,r}\sigma_v\eta\mathbb{E}\left[\sqrt{v}\right]\frac{\partial^2\phi}{\partial v\partial r}-r\phi,\end{aligned} \tag{5}$$

subject to the terminal condition $\phi(\mathbf{u},\mathbf{X}(T),T,T)=e^{i(u_1x_1(T)+u_2x_2(T))}$.

The PDE in (5) is in affine form as a consequence of the linearisation technique proposed by Grzelak and Oosterlee (2011). Therefore, its solution is of the form:

$$\phi(\mathbf{u},\mathbf{X}(t),t,T)=e^{A(\mathbf{u},t,T)+B(\mathbf{u},t,T)x_1(t)+C(\mathbf{u},t,T)x_2(t)+D(\mathbf{u},t,T)v(t)+E(\mathbf{u},t,T)r(t)}.$$

Calculating the partial derivatives in (5) with $A:=A(\mathbf{u},t,T)$, $B:=B(\mathbf{u},t,T)$, $C:=C(\mathbf{u},t,T)$, $D:=D(\mathbf{u},t,T)$, and $E:=E(\mathbf{u},t,T)$ yields:

$$\begin{aligned}&\frac{\partial\phi}{\partial t}=\phi\left(\frac{\partial A}{\partial t}+x_1\frac{\partial B}{\partial x_1}+x_2\frac{\partial C}{\partial t}+v\frac{\partial D}{\partial v}+r\frac{\partial E}{\partial r}\right),\\&\frac{\partial\phi}{\partial x_1}=B\phi,\quad\frac{\partial\phi}{\partial x_1^2}=B^2\phi,\quad\frac{\partial\phi}{\partial x_1\partial x_2}=BC\phi,\\&\frac{\partial\phi}{\partial x_2}=C\phi,\quad\frac{\partial\phi}{\partial x_2^2}=C^2\phi,\quad\frac{\partial\phi}{\partial x_1\partial v}=BD\phi,\quad\frac{\partial\phi}{\partial x_2\partial r}=CE\phi,\\&\frac{\partial\phi}{\partial v}=D\phi,\quad\frac{\partial\phi}{\partial v^2}=D^2\phi,\quad\frac{\partial\phi}{\partial x_1\partial r}=BE\phi,\quad\frac{\partial\phi}{\partial v\partial r}=DE\phi,\\&\frac{\partial\phi}{\partial r}=E\phi,\quad\frac{\partial\phi}{\partial r^2}=E^2\phi,\quad\frac{\partial\phi}{\partial x_2\partial v}=CD\phi.\end{aligned}$$

Substituting the partial derivatives into (5) gives the following PDE:

$$\begin{aligned}0 =&\frac{\partial A}{\partial t}+x_1\frac{\partial B}{\partial t}+x_2\frac{\partial C}{\partial t}+v\frac{\partial D}{\partial t}+r\frac{\partial E}{\partial t}+(r-\delta_1-\frac{1}{2}\sigma_1^2 v)B+(r-\delta_2-\frac{1}{2}\sigma_2^2 v)C\\&+\kappa(\bar{v}-v)D+\lambda(\theta-r)E+\frac{1}{2}\sigma_1^2 vB^2+\frac{1}{2}\sigma_2^2 vC^2+\frac{1}{2}\sigma_v^2 vD^2+\frac{1}{2}\eta^2E^2\\&+\rho_{x_1,x_2}\sigma_1\sigma_2 vBC+\rho_{x_1,v}\sigma_1\sigma_v vBD+\rho_{x_1,r}\sigma_1\eta\mathbb{E}\left[\sqrt{v}\right]BE\\&+\rho_{x_2,v}\sigma_2\sigma_v vCD+\rho_{x_2,r}\sigma_2\eta\mathbb{E}\left[\sqrt{v}\right]CE+\rho_{v,r}\sigma_v\eta\mathbb{E}\left[\sqrt{v}\right]DE-r.\end{aligned}$$

Collecting the terms for x_1, x_2, v, and r and performing a change of variable $\tau=T-t$, the following set of ordinary differential equations (ODEs) must be solved:

$$\begin{cases}\frac{\partial B}{\partial\tau} = & 0,\\ \frac{\partial C}{\partial\tau} = & 0,\\ \frac{\partial D}{\partial\tau} = & -\frac{1}{2}\sigma_1^2B-\frac{1}{2}\sigma_2^2C-\kappa D+\frac{1}{2}\sigma_1^2B^2+\frac{1}{2}\sigma_2^2C^2+\frac{1}{2}\sigma_v^2D^2+\rho_{x_1,x_2}\sigma_1\sigma_2BC\\ & +\rho_{x_1,v}\sigma_1\sigma_vBD+\rho_{x_2,v}\sigma_2\sigma_vCD,\\ \frac{\partial E}{\partial\tau} = & B+C-\lambda E-1,\\ \frac{\partial A}{\partial\tau} = & \kappa\bar{v}D+\lambda\theta E+\frac{1}{2}\eta^2E^2+\rho_{x_1,r}\sigma_1\eta\mathbb{E}[\sqrt{v}]BE+\rho_{x_2,r}\sigma_2\eta\mathbb{E}[\sqrt{v}]CE\\ & +\rho_{v,r}\sigma_v\eta\mathbb{E}[\sqrt{v}]DE-\delta_1B-\delta_2C,\end{cases}$$

with initial conditions $B(\mathbf{u},\tau) = iu_1$, $C(\mathbf{u},\tau) = iu_2$, $D(\mathbf{u},\tau) = 0$, $E(\mathbf{u},\tau) = 0$, and $A(\mathbf{u},\tau) = 0$.

The solutions to the ODEs are given by:

$$\begin{aligned}
B(\mathbf{u},\tau) &= iu_1,\\
C(\mathbf{u},\tau) &= iu_2,\\
D(\mathbf{u},\tau) &= \frac{-Q-D_1}{2R(1-Ge^{-D_1\tau})}(1-e^{-D_1\tau}),\\
E(\mathbf{u},\tau) &= (iu_1+iu_2-1)\lambda^{-1}(1-e^{-\lambda\tau}),\\
A(\mathbf{u},\tau) &= \kappa\bar{v}I_1(\mathbf{u},\tau)+\lambda\theta I_2(\mathbf{u},\tau)+\frac{1}{2}\eta^2 I_3(\mathbf{u},\tau)+\rho_{x_1,r}\sigma_1\eta I_4(\mathbf{u},\tau)\\
&\quad+\rho_{x_2,r}\sigma_2\eta I_5(\mathbf{u},\tau)+\rho_{v,r}\sigma_v\eta I_6(\mathbf{u},\tau)-\delta_1 iu_1\tau-\delta_2 iu_2\tau,
\end{aligned}$$

where

$$\begin{aligned}
D_1 &= \sqrt{Q^2-4PR},\\
G &= \frac{-Q-D_1}{-Q+D_1}\\
P &= -\frac{1}{2}\left[\sigma_1^2u_1^2+\sigma_2^2u_2^2+2\rho_{x_1,x_2}\sigma_1\sigma_2u_1u_2+i(\sigma_1^2u_1+\sigma_2^2u_2)\right]\\
Q &= \rho_{x_1,v}\sigma_1\sigma_v iu_1+\rho_{x_2,v}\sigma_2\sigma_v iu_2-\kappa\\
R &= \frac{1}{2}\sigma_v^2.
\end{aligned}$$

The solutions to the integrals $I_1(\mathbf{u},\tau)$, $I_2(\mathbf{u},\tau)$, $I_3(\mathbf{u},\tau)$, $I_4(\mathbf{u},\tau)$, $I_5(\mathbf{u},\tau)$, and $I_6(\mathbf{u},\tau)$ are given by:

$$\begin{aligned}
I_1(\mathbf{u},\tau) &= \frac{1}{2R}\left[(-Q-D_1)\tau-2\log\left(\frac{1-Ge^{-D_1\tau}}{1-G}\right)\right],\\
I_2(\mathbf{u},\tau) &= \frac{1}{\lambda}(iu_1+iu_2-1)(\tau+\frac{1}{\lambda}(e^{-\lambda\tau}-1)),\\
I_3(\mathbf{u},\tau) &= \frac{1}{2\lambda^3}(i+u_1+u_2)^2[3+e^{-2\lambda\tau}-4e^{-\lambda\tau}-2\lambda\tau],\\
I_4(\mathbf{u},\tau) &= -\frac{1}{\lambda}(iu_1+u_1u_2+u_1^2)\left[\frac{b}{c}(e^{-ct}-e^{-cT})+a\tau+\frac{a}{\lambda}(e^{-\lambda\tau}-1)\right.\\
&\quad\left.+\frac{b}{c-\lambda}e^{-cT}\left(1-e^{-\tau(\lambda-c)}\right)\right],\\
I_5(\mathbf{u},\tau) &= -\frac{1}{\lambda}(iu_2+u_1u_2+u_2^2)\left[\frac{b}{c}(e^{-ct}-e^{-cT})+a\tau+\frac{a}{\lambda}(e^{-\lambda\tau}-1)\right.\\
&\quad\left.+\frac{b}{c-\lambda}e^{-cT}\left(1-e^{-\tau(\lambda-c)}\right)\right],\\
I_6(\mathbf{u},\tau) &= \int_0^\tau \mathbb{E}\left[\sqrt{v(T-s)}\right]D(\mathbf{u},s)E(\mathbf{u},s),
\end{aligned}$$

where $\mathbb{E}\left[\sqrt{v(T-s)}\right] \approx a+b^{-c(T-s)}$, with a, b, and c defined in Lemma 2.

This concludes the derivation of the discounted characteristic function for the two-asset Heston–Hull–White model. In the next section, we extend the FFT method of Hurd and Zhou (2010) to cater for stochastic interest rates.

5. The Result of Hurd and Zhou Extended

Hurd and Zhou (2010) stated the following theorem for the square integrable Fourier representation of the basic spread option payoff function $F(x_1, x_2) = \max(e^{x_1} - e^{x_2} - 1, 0)$:

Theorem 1. *For any real numbers* $\boldsymbol{\epsilon} = (\epsilon_1, \epsilon_2)$ *with* $\epsilon_2 > 0$, $\epsilon_1 + \epsilon_2 < -1$, *and* $\mathbf{x} = (x_1, x_2)$,

$$F(\mathbf{x}) = \frac{1}{4\pi^2} \int\int_{\mathbb{R}^2 + i\epsilon} e^{i\mathbf{u}\mathbf{x}'} \hat{F}(u_1, u_2) du_1 du_2,$$

$$\hat{F}(u_1, u_2) = \frac{\Gamma(i(u_1 + u_2) - 1)\Gamma(-iu_2)}{\Gamma(iu_1 + 1)},$$

where $\Gamma(z)$ *is the complex gamma function defined for* $\Re(z) > 0$ *by the integral* $\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt$ *and* $\mathbf{u}\mathbf{x}' = u_1 x_1 + u_2 x_2$, *where* $\mathbf{x}'$ *is the unconjugated transpose of* $\mathbf{x}$.

See Hurd and Zhou (2010) for the proof.

Lemma 3 below is adapted from Hurd and Zhou (2010) to account for stochastic interest rates:

Lemma 3. *Let* $N(T) = e^{-\int_0^T r(s)ds}$ *and* $\mathbf{x}(t) = \left[\log S_1(t), \log S_2(t)\right]$. *For any* $t > 0$, *the increment* $\mathbf{x}(t) - \mathbf{x}(0)$ *is independent of* $\mathbf{x}(0)$, *which implies:*

$$\mathbb{E}_{\mathbb{Q}}\left[N(T) e^{i\mathbf{u}\mathbf{x}(T)'}\right] = e^{i\mathbf{u}\mathbf{x}(0)'} \phi(\mathbf{u}; T),$$

with $\phi(\mathbf{u}; T) := \mathbb{E}_{\mathbb{Q}}\left[N(T) e^{i\mathbf{u}(\mathbf{x}(T) - \mathbf{x}(0))'}\right]$.

See Hurd and Zhou (2010) for details.

Using Theorem 1 and Lemma 3, the price of a European spread call option with stochastic interest rates can be written as a two-dimensional Fourier transform in the variable $\mathbf{x}(0)$. The derivation is shown below. Consider the spread option payoff function:

$$V_{Spread}(0) = \mathbb{E}_{\mathbb{Q}}\left[N(T) \max\left(e^{x_1(T)} - e^{x_2(T)} - 1, 0\right)\right].$$

Changing from the risk-neutral measure $\mathbb{Q}$ to the T-Forward measure using the zero-coupon bond $P(0, T)$ as numeraire, we obtain:

$$\begin{aligned} V_{Spread}(0) &= P(0,T)\mathbb{E}_T\left[\frac{1}{4\pi^2} \int\int_{\mathbb{R}^2 + i\epsilon} e^{i\mathbf{u}\mathbf{x}(T)'} \hat{F}(u_1, u_2) du_1 du_2\right] \\ &= \frac{1}{4\pi^2} P(0,T) \int\int_{\mathbb{R}^2 + i\epsilon} \mathbb{E}_T\left[e^{i\mathbf{u}\mathbf{x}(T)'}\right] \hat{F}(u_1, u_2) du_1 du_2. \end{aligned}$$

Using the result $P(0,T)\mathbb{E}_T\left[e^{i\mathbf{u}\mathbf{x}(T)'}\right] = \mathbb{E}_{\mathbb{Q}}\left[N(T) e^{i\mathbf{u}\mathbf{x}(T)'}\right]$, the price of the spread option becomes:

$$\begin{aligned} V_{Spread}(0) &= \frac{1}{4\pi^2} P(0,T) \left\{ \int\int_{\mathbb{R}^2 + i\epsilon} e^{i\mathbf{u}\mathbf{x}(0)'} \frac{1}{P(0,T)} \mathbb{E}_{\mathbb{Q}}\left[N(T) e^{i\mathbf{u}(\mathbf{x}(T) - \mathbf{x}(0))'}\right] \right. \\ &\quad \left. \times \hat{F}(u_1, u_2) du_1 du_2 \right\} \\ &= \frac{1}{4\pi^2} \int\int_{\mathbb{R}^2 + i\epsilon} e^{i\mathbf{u}\mathbf{x}(0)'} \phi(\mathbf{u}; T) \hat{F}(u_1, u_2) du_1 du_2. \end{aligned} \quad (6)$$

Equation (6) is for the specific case where $K = 1$. Roberts (2018) shows that the result can be extended to $K > 0$ by scaling the two initial stock prices as follows:

$$V_{Spread}(S_1(0), S_2(0), K, T) = K \times V_{Spread}\left(\frac{S_1(0)}{K}, \frac{S_2(0)}{K}, 1, T\right). \tag{7}$$

The implementation of Equation (7) using the FFT technique is outlined in Alfeus and Schlögl (2018). Algorithm 1 below follows directly from their paper:

Algorithm 1 2D FFT Algorithm

Input : N, a power of two; $\bar{u}$, truncation width; ϵ, damping factor.

Define : $\mathbf{u}(\mathbf{k}) = (u_1(k_1), u_2(k_2))$ and $\mathbf{x}(\mathbf{l}) = (x_1(l_1), x_2(l_2))$.

Set $\mathbf{x}(0) = \left[\log\left(\frac{S_1(0)}{K}\right), \log\left(\frac{S_2(0)}{K}\right)\right] \in \mathbf{x}(\mathbf{l})$.

for $\mathbf{k}, \mathbf{l} \in \{1, 2, \ldots, N-1\}^2$ **do**

$$H(\mathbf{k}) = (-1)^{k_1+k_2}\phi(\mathbf{u}(\mathbf{k}) + i\mathbf{\epsilon})\hat{F}(\mathbf{u}(\mathbf{k}) + i\epsilon);$$
$$C(\mathbf{l}) = (-1)^{l_1+l_2}\left(\frac{\gamma N}{2\pi}\right)^2 e^{-\epsilon\mathbf{x}(\mathbf{l})'};$$

end for

$V = \Re(C \times \text{ifft2}(H))$.

$P \leftarrow K \times V$ using an interpolation technique.
return P.

In Algorithm 1, the double integral in Equation (6) is approximated by a double sum over the lattice:

$$\Gamma = \{\mathbf{u}(\mathbf{k}) = (u_1(k_1), u_2(k_2)) \mid k = (k_1, k_2) \in \{0, 1, \ldots, N-1\}^2\},\ u_i(k_i) = -\bar{u} + k_i\gamma,$$

where $\gamma = \frac{2\bar{u}}{N}$ is the lattice spacing, and $\bar{u} = \frac{N\gamma}{2}$.

Furthermore, $\mathbf{x}(0) = \left[\log\left(\frac{S_1(0)}{K}\right), \log\left(\frac{S_2(0)}{K}\right)\right]$ is chosen to lie on the reciprocal lattice:

$$\Gamma^* = \{\mathbf{x}(\mathbf{l}) = (x_1(l_1), x_2(l_2)) \mid l = (l_1, l_2) \in \{0, 1, \ldots, N-1\}^2\},\ x_i(l_i) = -\bar{x} + l_i\gamma^*,$$

where $\gamma^* = \frac{\pi}{\bar{u}}$ is the reciprocal lattice spacing, and $\bar{x} = \frac{N\gamma^*}{2}$.

This concludes the extension of the Hurd and Zhou (2010) FFT algorithm to stochastic interest rates. In the next section, we test the accuracy of the two-asset Heston–Hull–White model and the impact of the stochastic interest rates on spread option prices.

6. Numerical Results

This section is divided into three parts. First, we compare our implementation of the Hurd and Zhou (2010) FFT algorithm to the results shown in their paper. Secondly, we compare the convergence of the FFT to the Monte Carlo simulation. Lastly, we assess the impact of stochastic rates on the spread option prices.

All code was implemented in Python on an HP laptop *Intel(R) Core(TM) i5 - 1.60GHz with 16 GB memory*.

6.1. Implementation Testing

The two-asset Heston–Hull–White model reduces to the three-factor stochastic volatility model of Dempster and Hong (2002) when $\eta = 0$. Hurd and Zhou (2010) published spread option prices for various strikes based on the three-factor stochastic volatility model and their FFT algorithm. Table 1 below compares our implementation with the results published in Hurd and Zhou (2010).

Table 1. $S_1(0) = 100, S_2(0) = 96, \delta_1 = 0.05, \delta_2 = 0.05, v(0) = 0.04, r(0) = 0.1, \sigma_1 = 1.0, \sigma_2 = 0.5, \kappa = 1, \bar{v} = 0.04, \sigma_v = 0.05, \lambda = 1, \theta = 0.1, \eta = 0, \rho_{x_1,x_v} = 0.5, \rho_{x_1,v} = -0.5, \rho_{x_1,r} = 0, \rho_{x_2,v} = 0.25, \rho_{x_2,r} = 0, \rho_{v,r} = 0, N = 256, \bar{u} = 40, \epsilon_1 = -3, \epsilon_2 = 1, T = 1.$

Strike	Hurd and Zhou Price	Model Price	Absolute Difference	Relative Difference
2.0	7.548502	7.549344	0.000842	0.011155%
2.2	7.453536	7.454381	0.000845	0.011337%
2.4	7.359381	7.360137	0.000756	0.010273%
2.6	7.266037	7.266787	0.000749	0.010308%
2.8	7.173501	7.174295	0.000794	0.011069%
3	7.081775	7.082660	0.000885	0.012497%
3.2	6.990857	6.991678	0.000821	0.011744%
3.4	6.900745	6.901351	0.000606	0.008782%
3.6	6.811440	6.812176	0.000736	0.010805%
3.8	6.722939	6.723817	0.000878	0.013060%
4.0	6.635242	6.635881	0.000639	0.009630%

The results confirm that our implementation of the FFT algorithm was accurate.

Next, we test the convergence of the FFT and Monte Carlo simulation for spread options.

6.2. Convergence

Table 2 below shows the FFT price and execution time for a European spread call option with varying K and N.

Table 2. Convergence of FFT using the parameters in Table 1.

N	FFT Price $K = 2$	FFT Price $K = 3$	FFT Price $K = 4$	Time (seconds)
4	4.354906	6.532359	8.709812	0.012491
8	1.488913	2.233370	2.977827	0.013598
16	0.697647	1.046470	1.395293	0.041658
32	0.450374	0.675562	0.900749	0.059231
64	0.936496	1.404743	1.872991	0.206991
128	7.553730	7.087006	6.640188	0.787242
256	7.549344	7.082660	6.635881	3.233390
512	7.549344	7.082660	6.635881	12.481379
1024	7.549344	7.082660	6.635881	50.788263
2048	7.549344	7.082660	6.635881	203.205588
4096	7.549344	7.082660	6.635881	817.879709

The FFT algorithm convergesdto the solution in approximately 3.23 s with $N = 256$ steps.

Figure 1 below shows the convergence and execution time for the Monte Carlo simulation.

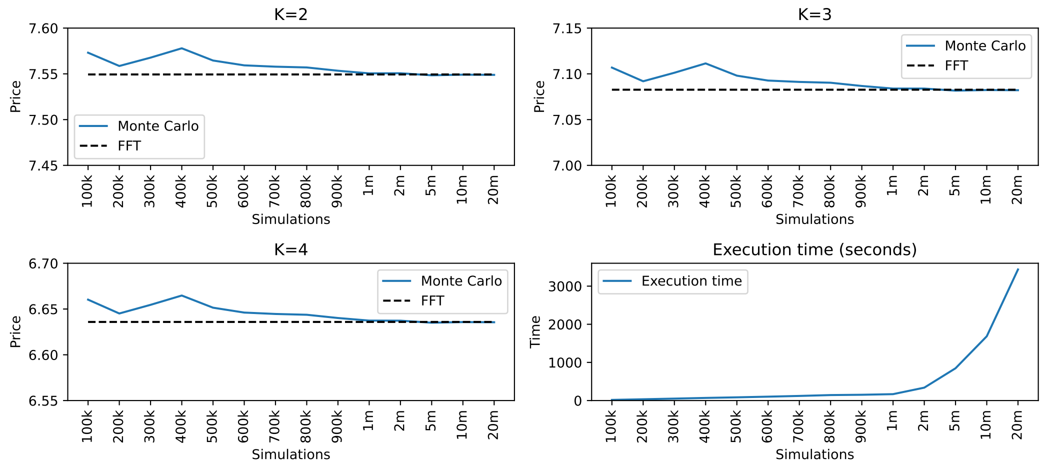

Figure 1. Monte Carlo convergence using the parameters in Table 1.

The Monte Carlo simulation converged to the FFT price with 1,000,000 simulations in approximately 168 s. The FFT significantly outperformed the Monte Carlo in terms of efficiency being up to 50 times faster.

We conclude this paper by assessing the impact of stochastic interest rates on the spread option prices.

6.3. Impact of Stochastic Interest Rates

We consider the following two cases to test the impact of stochastic interest rates: Case 1 where interest rates and equity prices are positively correlated; and Case 2 where interest rates and equity prices are negatively correlated.

Figure 2 below shows the results for Case 1 using the same parameters as in Table 1 except for η, $\rho_{x_1,r}$, and $\rho_{x_2,r}$.

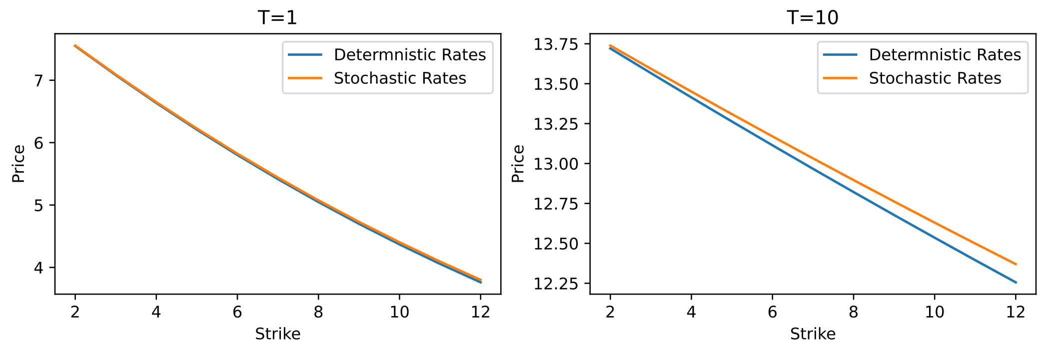

Figure 2. Impact of stochastic interest rates with $\eta = 0.05$, $\rho_{x_1,r} = 0.75$, and $\rho_{x_2,r} = 0.6$.

Similarly, Figure 3 below shows the results for Case 2.

For short-dated European spread call options, the impact of stochastic interest rates was insignificant. However, for long-dated European spread call options, the price difference between deterministic interest rates and stochastic interest rates widened for further out-the-money options. Moreover, when interest rates and stock prices were positively correlated, the spread option price under the stochastic interest rates was higher than under the deterministic interest rates. The opposite held when interest rates and stock prices were negatively correlated.

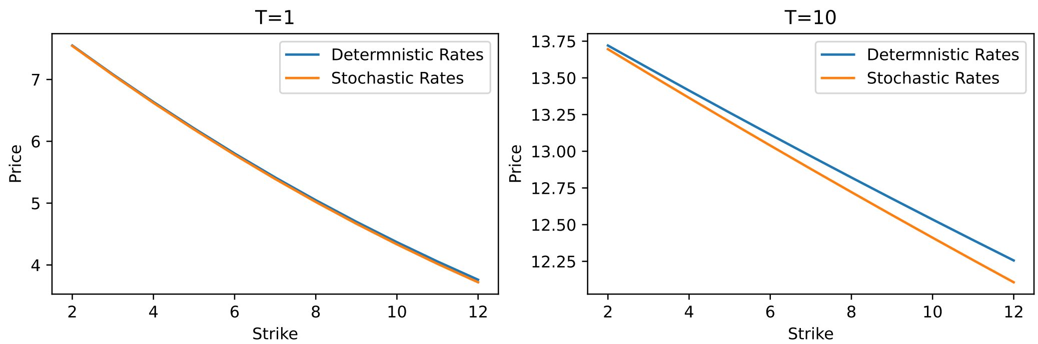

Figure 3. Impact of stochastic interest rates with $\eta = 0.05$, $\rho_{x_1,r} = -0.75$, and $\rho_{x_2,r} = -0.6$.

7. Conclusions

In this paper, we extended the Heston-Hull–White model of Grzelak and Oosterlee (2011) to two underlying assets and the FFT algorithm of Hurd and Zhou (2010) to account for stochastic interest rates.

Based on our implementation of the two-asset Heston–Hull–White model, we observed that the FFT algorithm was approximately 50 times faster than the Monte Carlo simulation. We also observed that the price of a long-dated European spread call option was sensitive to stochastic interest rates and equity-rate correlation. This difference became more significant for out-of-the-money options.

We hope that practitioners will find use in our extensions of the Heston–Hull–White model of Grzelak and Oosterlee (2011) and the FFT algorithm of Hurd and Zhou (2010), which will lead them to test the impact of stochastic interest rates on their spread option portfolios.

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References

Alam, Md. Mahmudul, and Gazi Uddin. 2009. Relationship between Interest Rate and Stock Price: Empirical Evidence from Developed and Developing Countries. *International Journal of Business and Management* 4: 43–51. [CrossRef]

Alfeus, Mesias, and Erik Schlögl. 2018. *On Numerical Methods for Spread Options*. Research Paper Series 388; Sydney: Quantitative Finance Research Centre, University of Technology.

Carr, Peter, and Dilip Madan. 1999. Option valuation using the fast Fourier transform. *Journal of Computational Finance* 2: 61–73. [CrossRef]

Cox, John C., Jonathan E. Ingersoll Jr, and Stephen A. Ross. 1985. A Theory of the Term Structure of Interest Rates. *Econometrica* 53: 61–73. [CrossRef]

Dempster, Michael Alan Howarth, and SS George Hong. 2002. Spread option valuation and the fast Fourier transform. In *Mathematical Finance-Bachelier Congress 2000*. Berlin/Heidelberg: Springer, pp. 203–220.

Duffie, Darrell, Jun Pan, and Kenneth Singleton. 2000. Transform Analysis and Asset Pricing for Affine Jump-Diffusions. *Econometrica* 68: 1343–76. [CrossRef]

Grzelak, Lech Aleksander. 2011. Equity and Foreign Exchange Hybrid Models for Pricing Long-Maturity Financial Derivatives. Ph.D. thesis, Delft University of Technology, Delft, The Netherlands.

Grzelak, Lech A., and Cornelis W. Oosterlee. 2011. On the Heston Model with Stochastic Interest Rates. *SIAM Journal on Financial Mathematics* 2: 255–86. [CrossRef]

Heston, Steven L. 1993. A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options. *The Review of Financial Studies* 6: 327–43. [CrossRef]

Hull, John, and Alan White. 1990. Pricing Interest-Rate-Derivative Securities. *The Review of Financial Studies* 3: 573–92. [CrossRef]

Hurd, Thomas R., and Zhuowei Zhou. 2010. A Fourier transform method for spread option pricing. *SIAM Journal on Financial Mathematics* 1: 142–57. [CrossRef]

Kammeyer, Holger, and Joerg Kienitz. 2012. The Heston-Hull-White Model Part I: Finance and Analytics. *Wilmott Magazine* 2012: 46–53. [CrossRef]

Roberts, Jessica Ellen. 2018. Fourier Pricing of Two-Asset Options: A Comparison of Methods. Master's thesis, University of Cape Town, Cape Town, South Africa. Available online: http://hdl.handle.net/11427/28126 (accessed on 25 February 2022).