

QUADRATIC WASSERSTEIN METRICS FOR VON NEUMANN ALGEBRAS VIA TRANSPORT PLANS

ROCCO DUVENHAGE

ABSTRACT. We show how one can obtain a class of quadratic Wasserstein metrics, that is to say, Wasserstein metrics of order 2, on the set of faithful normal states of a von Neumann algebra A , via transport plans, rather than through a dynamical approach. Two key points to make this work, are a suitable formulation of the cost of transport arising from Tomita-Takesaki theory and relative tensor products of bimodules (or correspondences in the sense of Connes). The triangle inequality, symmetry and $W_2(\mu, \mu) = 0$ all work quite generally, but to show that $W_2(\mu, \nu) = 0$ implies $\mu = \nu$, we need to assume that A is finitely generated.

1. INTRODUCTION

In recent years, several papers have studied Wasserstein metrics of order 2, or quadratic Wasserstein metrics, in a noncommutative context using a dynamical approach inspired by the dynamical formulation of Benamou and Brenier [5] in the classical case. These papers, often involving finite dimensional algebras, include [12, 13, 14, 37, 38, 15, 16, 49]. The papers [37, 49] investigate this approach for W^* -algebras, i.e., von Neumann algebras, motivated by, among other goals, finding a noncommutative analogue of Ricci curvature bounds. In turn, the papers [7, 8] made use of these developments to obtain logarithmic Sobolev inequalities for quantum Markov semigroups

Classically, however, there is also another approach, using transport plans (i.e., couplings of probability measures). This approach has also been followed in the literature on the noncommutative case, but it typically seems to lead to a Wasserstein “distance” which is not zero between a state and itself, while the triangle equality also has to be adapted. In other words, so far in the literature on the noncommutative transport plan approach, the Wasserstein distance on the state space of the algebra concerned, is not actually a metric. Quadratic Wasserstein distance functions obtained from transport plans have nevertheless already had applications in the noncommutative case, specifically in quantum physics, despite not being metrics. See in particular [33, 34, 35, 11, 24]. This motivates a further study of the transport plan

Date: 2022-7-28.

2020 Mathematics Subject Classification. Primary: 49Q22. Secondary: 46L30, 46L55.

formulation, in particular finding an approach where they do deliver metrics, which is exactly our aim in this paper.

In [2], difficulties in the noncommutative transport plan approach were pointed out. They obtained simple Wasserstein-like metrics for the states on the 2×2 matrices using noncommutative transport plans, but by deviating somewhat from the standard conceptual framework of optimal transport.

In the case of the space of tracial states on a C^* -algebra, [6] did obtain Wasserstein metrics (of any order) via a coupling approach, through a somewhat different formulation of couplings (transport plans) than in this paper. Furthermore, our setup involves bimodules for von Neumann algebras in a central way, which is again different from the approach in [6]. Nevertheless, there are some analogies between the two approaches, which will be pointed out where relevant.

We can also mention the well-known work of Connes [21] on noncommutative geometry, where a metric on the state space is obtained as a Wasserstein distance of order 1. However, this is not based on couplings, but rather extends the classical dual definition based on Lipschitz seminorms.

There has been at least one other approach to a quantum Wasserstein distance, also of order 1, namely [23], based on the notion of neighbouring states, where a metric was obtained.

In this paper we develop an approach to noncommutative quadratic Wasserstein distances using transport plans, where the distance functions on the space of faithful normal states of a von Neumann algebra are indeed metrics. Below we outline our approach, starting with brief remarks on the classical case.

We are interested in obtaining a non-commutative analogue of the classical case where the cost function is taken as the usual distance squared in \mathbb{R}^n , giving the following cost of transport from one probability measure to another on some closed subset X of \mathbb{R}^n :

$$(1) \quad I(\omega) = \int_{X \times X} \|x - y\|^2 d\omega(x, y)$$

where ω is a transport plan from the one probability measure μ to the other ν , i.e. ω is a coupling of the two measures. The goal is to find the minimum of $I(\omega)$, for a given pair (μ, ν) , the square root of which then defines the quadratic Wasserstein distance between μ and ν , giving a metric on an appropriate set of probability measures. This is a specific but important case of the optimal transport problem. A clear discussion of the original motivation for optimal transport, as well as its modern implications, can be found in the books by Villani [47, 48].

As we only consider the quadratic case in this paper, we henceforth simply refer to Wasserstein distances (should they not yet be shown to be metrics) and Wasserstein metrics.

Our noncommutative version will correspond to the case where X is a bounded set, ensuring that the coordinate functions $x \mapsto x_l$, in terms of $x = (x_1, \dots, x_n)$, are bounded. We replace these coordinate functions by elements k_1, \dots, k_n of a von Neumann algebra A . The k_l are then bounded operators, which is why this corresponds to bounded X . In this approach the closest analogy to the classical coordinate functions would be to assume that the k_l are self-adjoint, but for our development this is not actually needed, though we do ultimately need a weaker assumption, as will be pointed out below.

A direct translation of the classical cost function $c(x, y) := \|x - y\|^2$ above, is then

$$\sum_{l=1}^n |k_l \otimes 1 - 1 \otimes k_l|^2$$

where 1 denotes the unit of A and $|a|^2 := a^*a$ for a in any $*$ -algebra. However, noncommutativity of A makes this direct translation untenable.

Instead, for the cost of transport between two faithful normal states μ and ν on A , we, at least heuristically, propose to use

$$(2) \quad c = \sum_{l=1}^n |k_l \otimes 1 - 1 \otimes (S_\nu k_l^* S_\nu)|^2$$

where S_ν is the conjugate linear (and typically unbounded) linear operator from Tomita-Takesaki theory associated with ν . As this c appears to depend on one of the states, namely ν , it is immediately clear that this is quite different from the classical case, where no such dependence on the probability measures are present. However, even though c has been “contextualized” in this sense for the states involved, the elements k_l of A do not depend on the states. Furthermore, it is not clear that this can lead to a distance that is indeed symmetric, but the apparent asymmetry between the two states μ and ν in c can be circumvented using a condition involving the modular group. The first indication that we should use (2), was that it leads to zero distance between a state and itself (see Proposition 6.1).

The fact that S_ν is typically unbounded and $S_\nu k_l^* S_\nu$ not an element of A (in standard form) nor of its commutant A' , is a complication which we can sidestep by rewriting the cost for a noncommutative transport plan in a more convenient form which will give us a rigorous definition of the cost of transportation from μ to ν . We then do not refer to c above directly, but only to the cost of transport from one state to another in terms of the k_l . A detailed discussion of this appears in Section 3.

This approach turns out to fit perfectly with bimodules, or correspondences in the sense of Connes, and the relative tensor product of bimodules provides the ideal structure to prove the triangle inequality for the Wasserstein distance W_2 which we are going to define along the lines above.

An important point is the following: To complete the proof that our W_2 is a metric, we assume that A is finitely generated, in analogy to \mathbb{R}^n being finite dimensional. More precisely, that A is generated by k_1, \dots, k_n above, with $\{k_1^*, \dots, k_n^*\} = \{k_1, \dots, k_n\}$; note that $k_l^* = k_l$ is not required. This is needed to show that $W_2(\mu, \nu) = 0$ for faithful normal states on A , implies that $\mu = \nu$. However, the triangle inequality, symmetry and $W_2(\mu, \mu) = 0$ all work without this assumption.

We note that our use of generators parallels the approach of [6]. In particular, they also assume their C^* -algebra to be finitely generated to obtain Wasserstein metrics on the space of tracial states.

Transport plans and related cyclic representations are discussed in Section 2, setting up the basic framework and much notation for the rest of the paper. Our basic definitions related to the cost of transport and Wasserstein distances follow in Section 3. The subsequent three sections then prove the various properties of a metric in turn. After that, the last section gives an indication of how the setting we use may be expanded.

2. TRANSPORT PLANS AND CYCLIC REPRESENTATIONS

This section sets up the framework to be used in the rest of the paper, in particular the formulation of transport plans. Transport plans lead to cyclic representation which play a central role in this paper. These representations and the bimodule structure on their Hilbert spaces, are also discussed in this section. The notation introduced here will be used throughout the sequel.

Although our focus is eventually on the faithful normal states of one von Neumann algebra at a time, a number of aspects of our approach are clarified by starting off with three von Neumann algebra. This is ultimately related to the three elements of a metric space appearing in the formulation of the triangle inequality.

We consider three (necessarily σ -finite) von Neumann algebras A , B and C with faithful normal states μ , ν and ξ respectively. We assume that they are all in standard form, meaning that A is a von Neumann algebra on a Hilbert space G_μ with a cyclic and separating vector Λ_μ for A such that

$$\mu(a) = \langle \Lambda_\mu, a\Lambda_\mu \rangle$$

for all $a \in A$, which allows us to define a state μ' on the commutant A' by

$$\mu'(a') = \langle \Lambda_\mu, a'\Lambda_\mu \rangle$$

for all $a' \in A'$. The corresponding notation will be used for (B, ν) and (C, ξ) as well.

The starting point of our development is the definition of *transport plans*, also known as *couplings*.

Definition 2.1. A *transport plan* from μ to ν is a state ω on the algebraic tensor product $A \odot B'$ such that

$$\omega(a \otimes 1) = \mu(a) \quad \text{and} \quad \omega(1 \otimes b') = \nu'(b')$$

for all $a \in A$ and $b' \in B'$. Denote the set of all transport plans from μ to ν by $T(\mu, \nu)$.

A transport plan from ν to ξ will be denoted by ψ , and from μ to ξ by φ .

To obtain symmetry of a Wasserstein distance on the faithful normal states, we are going to use a restricted set of transport plans. To define them, we use a property which was called balance between dynamical systems in [28]:

Definition 2.2. Given *dynamical systems* $\mathbf{A} = (A, \alpha, \mu)$ and $\mathbf{B} = (B, \beta, \nu)$, where α and β are unital completely positive (u.c.p.) maps $\alpha : A \rightarrow A$ and $\beta : B \rightarrow B$ such that $\mu \circ \alpha = \mu$ and $\nu \circ \beta = \nu$, we say that \mathbf{A} and \mathbf{B} (in this order) are in *balance* with respect to a transport plan ω from μ to ν , written as

$$\mathbf{A}\omega\mathbf{B},$$

if

$$\omega(\alpha(a) \otimes b') = \omega(a \otimes \beta'(b'))$$

for all $a \in A$ and $b' \in B'$, where $\beta' : B' \rightarrow B'$ is the dual of β defined by

$$\langle \Lambda_\nu, b\beta'(b')\Lambda_\nu \rangle = \langle \Lambda_\nu, \beta(b)b'\Lambda_\nu \rangle$$

for all $b \in B$ and $b' \in B'$ (see [1] for the theory behind such duals and [28, Section 2] for a summary).

Our restricted set of transport plans are then defined as follows:

Definition 2.3. The set of *modular* transport plans from μ to ν is

$$T_\sigma(\mu, \nu) := \{\omega \in T(\mu, \nu) : (A, \sigma_t^\mu, \mu)\omega(B, \sigma_t^\nu, \nu) \text{ for all } t \in \mathbb{R}\}$$

where σ^μ and σ^ν are the modular groups associated with μ and ν respectively.

Note that $T_\sigma(\mu, \nu)$ is not empty, as it contains $\mu \odot \nu'$.

In the remainder of this section we consider general transport plans, but from the next section modular transport plans will become the focus.

Transport plans lead to cyclic representations and we need to keep track how the various cyclic representations relate to one another. Let

$(H_\omega, \pi_\omega, \Omega)$ be a cyclic representation of $(A \odot B', \omega)$, i.e. $\Omega \in H_\omega$ is cyclic for $A \odot B'$ and

$$\omega = \langle \Omega, \pi_\omega(\cdot)\Omega \rangle_\omega,$$

where for emphasis we write the inner product of H_ω as $\langle \cdot, \cdot \rangle_\omega$. This induces cyclic representations $(H_\mu^\omega, \pi_\mu^\omega, \Omega)$ and $(H_\nu^\omega, \pi_\nu^\omega, \Omega)$ of (A, μ) and (B', ν') respectively, by setting

$$H_\mu^\omega := \overline{\pi_\omega(A \otimes 1)\Omega} \quad \text{and} \quad \pi_\mu^\omega(a) := \pi_\omega(a \otimes 1)|_{H_\mu^\omega}$$

for all $a \in A$, and

$$H_\nu^\omega := \overline{\pi_\omega(1 \otimes B')\Omega} \quad \text{and} \quad \pi_\nu^\omega(b') := \pi_\omega(1 \otimes b')|_{H_\nu^\omega}$$

for all $b' \in B'$. A unitary equivalence

$$(3) \quad u_\nu : G_\nu \rightarrow H_\nu^\omega$$

from $(G_\nu, \text{id}_{B'}, \Lambda_\nu)$ to $(H_\nu^\omega, \pi_\nu^\omega, \Omega)$ is given by

$$u_\nu b' \Lambda_\nu := \pi_\nu^\omega(b')\Omega$$

for all $b' \in B'$. Then

$$\pi_\nu^\omega(b') = u_\nu b' u_\nu^*$$

for all $b' \in B'$. By setting

$$\pi_\nu^\omega(b) := u_\nu b u_\nu^*$$

for all $b \in B$, we also obtain a cyclic representation $(H_\nu^\omega, \pi_\nu^\omega, \Omega)$ of (B, ν) .

Denoting the modular conjugation for B associated to Λ_ν by J_ν , while the modular conjugation for B associated to Ω is denoted by J_ν^ω , one finds that

$$J_\nu^\omega u_\nu = u_\nu J_\nu$$

from which it follows that

$$\pi_\nu^\omega = j_\nu^\omega \circ \pi_\nu^\omega \circ j_\nu$$

where $j_\nu^\omega := J_\nu^\omega(\cdot)^* J_\nu^\omega$ and $j_\nu := J_\nu(\cdot)^* J_\nu$ on $\mathcal{B}(H_\nu^\omega)$ and $\mathcal{B}(G_\nu)$ respectively. Here $\mathcal{B}(G_\nu)$ is the von Neumann algebra of all bounded linear operators $G_\nu \rightarrow G_\nu$.

A key structure that emerges from the transport plan ω , is that H_ω is an A - B -bimodule via

$$axb := \pi_\omega(a \otimes j_\nu(b))x$$

for all $a \in A$, $b \in B$ and $x \in H_\omega$. The required normality conditions follow from [4, Theorem 3.3].

The forgoing discussion of course also holds for ψ and φ , using corresponding notation for the induced representations, and in particular denoting the cyclic vectors by

$$\Psi \text{ and } \Phi$$

respectively.

A central fact which will be of great importance to us, is that transport plans from μ to ν are in a one-to-one correspondence with u.c.p. maps $E : A \rightarrow B$ such that $\nu \circ E = \mu$. Let

$$\varpi_B : B \odot B' \rightarrow \mathcal{B}(G_\nu)$$

be the unital $*$ -homomorphism defined by extending $\varpi_B(b \otimes b') = bb'$ using the universal property of tensor products. Then set

$$\delta_\nu = \langle \Lambda_\nu, \varpi_B(\cdot) \Lambda_\nu \rangle.$$

Note that δ_ν is a transport plan from ν to itself. It now follows that there is a unique map

$$E_\omega : A \rightarrow B$$

such that

$$(4) \quad \omega(a \otimes b') = \delta_\nu(E_\omega(a) \otimes b')$$

for all $a \in A$ and $b' \in B'$. This map E_ω is linear, normal (i.e. σ -weakly continuous), u.c.p. (unital and completely positive) and satisfies

$$\nu \circ E_\omega = \mu.$$

Conversely, given a u.c.p. map $E : A \rightarrow B$ such that $\nu \circ E = \mu$, it defines a transport plan ω_E from μ to ν by

$$\omega_E(a \otimes b') = \delta_\nu(E(a) \otimes b')$$

which satisfies $E = E_{\omega_E}$. Technical details can be found in [28, Section 3]. This correspondence is well-known and widely used in various forms and degrees of generality, for example in finite dimensions in quantum information theory, where it is known as the Choi-Jamiołkowski duality (see [18, 25, 39] for the origins and [40] for an overview), and has also found application in the theory of noncommutative joinings [4]. In effect we can now view u.c.p. maps as transport plans, as was done in [24], however, we reserve the term “transport plan” for how it is used in Definition 2.1. In classical probability theory the correspondence between couplings and Markov operators was studied for couplings of a measure with itself in [10], and more generally in [41], though it seems not to be widely used in classical optimal transport.

By [28, Theorem 4.1] we can then express $\mathbf{A}\omega\mathbf{B}$ in Definition 2.2 as

$$E_\omega \circ \alpha = \beta \circ E_\omega$$

which is often a convenient way to check if a transport plan is modular.

As in [28, Section 5], this correspondence allows us to compose transport plans $\omega \in T(\mu, \nu)$ and $\psi \in T(\nu, \xi)$ to obtain a transport plan $\omega \circ \psi \in T(\mu, \xi)$ defined via

$$E_{\omega \circ \psi} = E_\psi \circ E_\omega.$$

In the remainder of this section we are particularly interested in this transport plan, and how it relates to the relative tensor product of the bimodules H_ω and H_ψ . So we set

$$\varphi = \omega \circ \psi$$

and consider the relative tensor product

$$H := H_\omega \otimes_\nu H_\psi.$$

An extended discussion of these products can be found in [46, Section IX.3], but also see [30] and the early work [44]. The original source is Connes' work on correspondences, which has not been published in full, but see [22, Appendix V.B] for a partial exposition.

The relative tensor product H is itself an A - C -bimodule, and contains the vector

$$\Phi := \Omega \otimes_\nu \Psi$$

which can indeed be taken as the cyclic vector in the cyclic representation $(H_\varphi, \pi_\varphi, \Phi)$ of $(A \odot C', \varphi)$. This is because the representation can be obtained from the bimodule H using the not necessarily cyclic representation π of $A \odot C'$ defined through

$$\pi(a \otimes c')z := azj_\xi(c')$$

for all $z \in H$, by restriction:

$$H_\varphi := \overline{\pi(A \odot C')\Phi} \quad \text{and} \quad \pi_\varphi(t) := \pi(t)|_{H_\varphi}$$

for all $t \in A \odot C'$. That this indeed provides a cyclic representation for φ , follows from [28, Corollary 5.7].

3. COST OF TRANSPORT AND THE DISTANCE W_2

Here we introduce the cost of transport as will be used in this paper, followed by the paper's central definition, namely that of Wasserstein distance W_2 .

An important formula related to cost of transport, expressed in terms of the setup and notation from the previous section and norm $\|\cdot\|_\omega$ of the Hilbert space H_ω , is the following:

$$(5) \quad \|\pi_\mu^\omega(a)\Omega - \pi_\nu^\omega(b)\Omega\|_\omega^2 = \mu(a^*a) + \nu(b^*b) - \nu(E_\omega(a)^*b) - \nu(b^*E_\omega(a))$$

for all $a \in A$ and $b \in B$. This is derived by a straightforward sequence of manipulations, in particular making use of the projection P_ν of H_ω onto H_ν^ω , in terms of which one has

$$\begin{aligned} \langle \pi_\mu^\omega(a)\Omega, \pi_\nu^\omega(b)\Omega \rangle_\omega &= \langle \pi_\omega(a \otimes 1)\Omega, P_\nu \pi_\nu^\omega(b)\Omega \rangle_\omega \\ &= \langle u_\nu^* P_\nu \pi_\omega(a \otimes 1) u_\nu \Lambda_\nu, b \Lambda_\nu \rangle \\ &= \langle E_\omega(a) \Lambda_\nu, b \Lambda_\nu \rangle \\ &= \nu(E_\omega(a)^* b) \end{aligned}$$

by [28, Proposition 3.1].

Now set

$$A = B = C.$$

By the theory of standard forms [3, 20, 36] (also see [9, Theorem 2.5.31]) we can consequently take $G = G_\mu = G_\nu = G_\xi$ and represent the faithful normal states μ, ν and ξ on A by three cyclic and separating vectors $\Lambda_\mu, \Lambda_\nu, \Lambda_\xi \in G$ for A . As usual in Tomita-Takesaki theory, we define a closed conjugate linear operator S_ν in G through

$$S_\nu a \Lambda_\nu = a^* \Lambda_\nu$$

for all $a \in A$. Following the idea (2) in the introduction, we use the form of the cyclic representation to formally manipulate the expression

$$(6) \quad \omega(|a \otimes 1 - 1 \otimes (S_\nu b^* S_\nu)|^2)$$

for a transport plan ω from μ to ν and $a, b \in A$ as if $S_\nu b^* S_\nu$ lies in A' (it is actually only affiliated with A'), to obtain

$$(7) \quad \|\pi_\mu^\omega(a)\Omega - \pi_\nu^\omega(b)\Omega\|_\omega^2.$$

Alternatively, one can obtain the right hand side of (5) from (6) by a different sequence of formal manipulations using (4). The expression (7) will be used as the basis for rigorously defining the cost of transport, as each term in the heuristic expression $\omega(c)$ for the cost of transport is of the form (6). Note that $\omega(c)$ is a noncommutative translation of the integral in (1).

Definition 3.1. Given $k_1, \dots, k_n \in A$ and writing $k = (k_1, \dots, k_n)$, the associated *transport cost function* I , which gives the cost of transport $I(\omega)$ from μ to ν for the transport plan $\omega \in T(\mu, \nu)$ and faithful normal states μ and ν on A , is defined to be

$$I(\omega) = \|\pi_\mu^\omega(k)\Omega - \pi_\nu^\omega(k)\Omega\|_{\oplus\omega}^2$$

where we have written

$$\pi_\mu^\omega(k)\Omega \equiv (\pi_\mu^\omega(k_1)\Omega, \dots, \pi_\mu^\omega(k_n)\Omega) \in \bigoplus_{l=1}^n H_\omega$$

and $\|\cdot\|_{\oplus\omega}$ denotes the norm on $\bigoplus_{l=1}^n H_\omega$. I.e.,

$$(8) \quad I(\omega) = \sum_{l=1}^n \|\pi_\mu^\omega(k_l)\Omega - \pi_\nu^\omega(k_l)\Omega\|_\omega^2.$$

This realizes the idea outlined in the introduction, and parallels the use of generators in [6]'s definition of Wasserstein metrics. It can be written in a second equivalent form in terms of the u.c.p. map E_ω corresponding to ω , using (5), which is more convenient for certain purposes:

$$(9) \quad I(\omega) = \sum_{l=1}^n [\mu(k_l^* k_l) + \nu(k_l^* k_l) - \nu(E_\omega(k_l)^* k_l) - \nu(k_l^* E_\omega(k_l))].$$

A more complete notation for I would be I_k , but no confusion will arise.

We are now in a position to define the distance function W_2 on the faithful normal states, which we aim to prove is a metric. By a *distance function* we simply mean a function $d : X \times X \rightarrow \mathbb{R}$ on some set X , which is potentially a metric on X , in order to distinguish it from a verified metric.

If we directly generalize the classical case, one would define the distance function as the infimum of the square root of the cost over all transport plans ω from μ to ν . However, in our noncommutative setup, this causes problems with symmetry of the distance function. We solve this problem by only allowing the modular transport plans $T_\sigma(\mu, \nu)$ from Definition 2.3. This is a natural restriction on the allowed transport plans which becomes trivial when μ and ν are traces. Similar restrictions have been used in a related context in the theory of noncommutative joinings, also involving couplings (not viewed as transport plans in that case). See in particular [4, 27]. Modular transport plans are exactly what we need to obtain symmetry in a natural way, as will be seen in the next section.

Thus we present the main definition of this paper:

Definition 3.2. Denote the set of faithful normal states on a σ -finite von Neumann algebra A by $\mathfrak{F}(A)$. Given $k_1, \dots, k_n \in A$, we define the associated *Wasserstein distance* W_2 on $\mathfrak{F}(A)$ by

$$W_2(\mu, \nu) := \inf_{\omega \in T_\sigma(\mu, \nu)} I(\omega)^{1/2}$$

for all $\mu, \nu \in \mathfrak{F}(A)$, in terms of Definition 3.1.

More completely W_2 can be called a Wasserstein distance of order 2, or a quadratic Wasserstein distance.

Having defined our candidate W_2 , we have to investigate if it is indeed a metric. This is done in the next three sections.

4. THE TRIANGLE INEQUALITY

In this section we intend to show that W_2 satisfies the triangle inequality. To do this we use the bimodule structure and relative tensor product discussed in Section 2. This is analogous to [6]'s use of free products with amalgamation to prove the triangle inequality in their approach.

The main technical point in this regard is the following lemma, which is a key (and we presume well-known) feature of the quotient construction of the relative tensor product, essentially telling us that the imbeddings of H_ν into H_ω and H_ψ respectively, are identified in the relative tensor product $H = H_\omega \otimes_\nu H_\psi$. This is precisely the reason that the relative tensor product is the natural setting to prove the triangle inequality. Please refer to [46, Section IX.3] for further background on

relative tensor products and detail on related operators used in the lemma's proof.

Lemma 4.1. *In terms of the setup and notation from Section 2 and the imbeddings*

$$\iota_\omega : H_\omega \rightarrow H : x \mapsto x \otimes_\nu \Psi$$

and

$$\iota_\psi : H_\psi \rightarrow H : y \mapsto \Omega \otimes_\nu y$$

one has

$$\iota_\omega(\pi_\nu^\omega(b)\Omega) = \iota_\psi(\pi_\nu^\psi(b)\Psi)$$

for all $b \in B$.

Proof. Given any $b \in B$, write $x = \pi_\nu^\omega(b)\Omega$ and $y = \pi_\nu^\psi(b)\Psi$. In setting up the relative tensor product, one considers an operator $L_\nu(x) : G_\nu \rightarrow H_\omega$ which in our setup can be written as $L_\nu(x) = \pi_\nu^\omega(b)u_\nu$ in terms of (3). From the bilinear form used in the quotient construction of the relative tensor product (see [46, Proposition IX.3.15 and Definition IX.3.16]) one finds

$$\begin{aligned} \|\iota_\omega(x) - \iota_\psi(y)\|^2 &= \langle \Psi, L_\nu(x)^* L_\nu(x) \Psi \rangle_\psi - \langle y, L_\nu(\Omega)^* L_\nu(x) \Psi \rangle_\psi \\ &\quad - \langle \Psi, L_\nu(x)^* L_\nu(\Omega) \Psi \rangle_\psi + \langle y, L_\nu(\Omega)^* L_\nu(\Omega) \Psi \rangle_\psi \\ &= 0, \end{aligned}$$

since one has $L_\nu(x)^* L_\nu(x) = u_\nu^* \pi_\nu^\omega(b^*b)u_\nu = b^*b$ etc., and where we are using the left B -module structure of H_ψ . (Here $\langle \cdot, \cdot \rangle_\psi$ is the inner product of H_ψ .) \square

The second fact we need is the following:

Lemma 4.2. *In terms of the setup and notation from Section 2, with $\varphi = \omega \circ \psi$, we have*

$$\iota_\omega(\pi_\mu^\omega(a)\Omega) = \pi_\mu^\varphi(a)\Phi$$

and

$$\iota_\psi(\pi_\xi^\psi(c)\Psi) = \pi_\xi^\varphi(c)\Phi$$

for all $a \in A$ and $c \in C$.

Proof. The first is easy to verify, while the second follows by first showing that

$$J_\xi^\varphi(\Omega \otimes_\nu y) = \Omega \otimes_\nu J_\xi^\psi y$$

for all $y \in H_\xi^\psi$. \square

The triangle inequality for W_2 given in Definition 3.2 can now be proven.

Proposition 4.3. *Consider any $k_1, \dots, k_n \in A$, and let W_2 be the associated Wasserstein distance on $\mathfrak{F}(A)$. Then*

$$W_2(\mu, \xi) \leq W_2(\mu, \nu) + W_2(\nu, \xi)$$

for all $\mu, \nu, \xi \in \mathfrak{F}(A)$.

Proof. For $\omega \in T_\sigma(\mu, \nu)$ and $\psi \in T_\sigma(\nu, \xi)$ in terms of Definition 2.3, we set $\varphi = \omega \circ \psi$. Note that $\varphi \in T_\sigma(\mu, \xi)$, since

$$E_\varphi \circ \sigma_t^\mu = E_\psi \circ E_\omega \circ \sigma_t^\mu = E_\psi \circ \sigma_t^\nu \circ E_\omega = \sigma_t^\xi \circ E_\psi \circ E_\omega = \sigma_t^\xi \circ E_\varphi,$$

hence $(A, \sigma^\mu, \mu) \varphi (A, \sigma^\xi, \xi)$ by [28, Theorem 4.1]. According to Definition 3.1, and applying ι_ω and ι_ψ componentwise to elements of direct sums, we have

$$\begin{aligned} I(\varphi)^{1/2} &= \left\| \pi_\mu^\varphi(k) \Phi - \pi_\xi^\varphi(k) \Phi \right\|_{\oplus \varphi} \\ &= \left\| \iota_\omega(\pi_\mu^\omega(k) \Omega) - \iota_\psi(\pi_\xi^\psi(k) \Psi) \right\|_{\oplus H} \\ &\leq \left\| \iota_\omega(\pi_\mu^\omega(k) \Omega) - \iota_\omega(\pi_\nu^\omega(k) \Omega) \right\|_{\oplus H} \\ &\quad + \left\| \iota_\omega(\pi_\nu^\omega(k) \Omega) - \iota_\psi(\pi_\nu^\psi(k) \Psi) \right\|_{\oplus H} \\ &\quad + \left\| \iota_\psi(\pi_\nu^\psi(k) \Psi) - \iota_\psi(\pi_\xi^\psi(k) \Psi) \right\|_{\oplus H} \\ &= \left\| \pi_\mu^\omega(k) \Omega - \pi_\nu^\omega(k) \Omega \right\|_{\oplus \omega} + \left\| \pi_\nu^\psi(k) \Psi - \pi_\xi^\psi(k) \Psi \right\|_{\oplus \psi} \\ &= I(\omega)^{1/2} + I(\psi)^{1/2} \end{aligned}$$

where we employed the triangle inequality in $(\bigoplus_{l=1}^n H, \|\cdot\|_{\oplus H})$, applied Lemmas 4.1 and 4.2, and used the fact that ι_ω and ι_ψ preserve the inner products. Now take the infimum on the left over all of $T_\sigma(\mu, \xi)$, which includes the compositions $\omega \circ \psi$ for all $\omega \in T_\sigma(\mu, \nu)$ and $\psi \in T_\sigma(\nu, \xi)$, followed in turn by the infima over all $\omega \in T_\sigma(\mu, \nu)$ and $\psi \in T_\sigma(\nu, \xi)$ on the right. \square

Note that the middle term in the application of triangle inequality in $\bigoplus_{l=1}^n H$ above is zero because we use the relative tensor product $H = H_\omega \otimes_\nu H_\psi$ containing H_φ . Exactly this point of the triangle inequality tends to run astray in other transport plan based approaches to the Wasserstein distance. See in particular [24], which in turn built on [33]; also see [35, Footnote 4].

The triangle inequality actually still works if one were to use all transport plans, rather than just modular transport plans, a point which we come back to in the final section.

5. SYMMETRY

We now proceed to prove that W_2 is symmetric. This is where the modular property of our allowed transport plans (Definition 2.3) becomes important.

For clarity, in this section we initially work with two von Neumann algebras A and B with faithful normal states μ and ν respectively, again using the notation and conventions from Section 2. After setting up some general facts in this setting, we return to a single von Neumann algebra when proving that W_2 is symmetric.

A suitable path to symmetry is provided by KMS-duals. Such duals and KMS-symmetry were studied and applied in [43, 42, 31, 32, 19, 29, 28]. As in Section 2 we define $j_\mu := J_\mu(\cdot)^* J_\mu$ on $\mathcal{B}(G_\mu)$ in terms of the modular conjugation J_μ for A associated to $\Lambda_\mu \in G_\mu$.

Definition 5.1. Given a u.c.p. map $E : A \rightarrow B$ such that $\nu \circ E = \mu$, we define its *KMS-dual* (w.r.t. μ and ν) as

$$E^\sigma := j_\mu \circ E' \circ j_\nu : B \rightarrow A$$

in terms of the dual $E' : B' \rightarrow A'$ of E defined by

$$\langle \Lambda_\mu, aE'(b')\Lambda_\mu \rangle = \langle \Lambda_\nu, E(a)b'\Lambda_\nu \rangle$$

for all $a \in A$ and $b' \in B'$. (Regarding E' , see [1], and [28, Section 2] for a summary).

The basic properties of the KMS-dual needed here are set out in the next lemma.

Lemma 5.2. *In Definition 5.1, E^σ is u.c.p. and $\mu \circ E^\sigma = \nu$. Furthermore, $(E^\sigma)^\sigma = E$. The map E has a Hilbert space representation as a contraction $K : G_\mu \rightarrow G_\nu$ defined through $Ka\Lambda_\mu = E(a)\Lambda_\nu$ for all $a \in A$, and E^σ is similarly represented by the contraction $J_\nu K^* J_\nu : G_\nu \rightarrow G_\mu$. Consequently, the following equivalence holds:*

$$\mu(aE^\sigma(b)) = \nu(E(a)b)$$

for all $a \in A$ and $b \in B$, if and only if

$$J_\nu K = K J_\mu.$$

Proof. Note that E' is u.c.p. and satisfies $\mu' \circ E' = \nu'$ according to [1, Proposition 3.1]. The corresponding properties for E^σ then follow easily, while $(E^\sigma)^\sigma = E$ results from $(E')' = E$. That K is well-defined, follows from $\nu \circ E = \mu$ (which implies that $\nu(E(a)^*E(a)) \leq \nu(E(a^*a)) = \mu(a^*a)$ by Kadison's inequality). This implies that E' is represented by K^* , since

$$\langle E'(b')\Lambda_\mu, a\Lambda_\mu \rangle = \langle K^*b'\Lambda_\nu, a\Lambda_\mu \rangle$$

by the definition of E' , which in turn means that E^σ is represented by $J_\nu K^* J_\nu$ by Definition 5.1. The mentioned equivalence now follows by routine manipulations. \square

This allows us to prove the following lemma in terms of the modular transport plans from Definition 2.3, which subsequently leads to symmetry of W_2 .

Lemma 5.3. *For faithful normal states μ and ν on von Neumann algebras A and B respectively, we have*

$$\|\pi_\mu^\omega(a)\Omega_\omega - \pi_\nu^\omega(b)\Omega_\omega\|_\omega = \|\pi_\nu^{\omega^\sigma}(b)\Omega_{\omega^\sigma} - \pi_\mu^{\omega^\sigma}(a)\Omega_{\omega^\sigma}\|_{\omega^\sigma}$$

for every $\omega \in T_\sigma(\mu, \nu)$, where $\omega^\sigma \in T_\sigma(\nu, \mu)$ is determined by $E_{\omega^\sigma} = E_\omega^\sigma$. For emphasis we have written Ω_ω for the cyclic vector appearing in a cyclic representation associated to ω , and similarly for ω^σ .

Proof. Since ω is modular, for the Hilbert space representation K_ω of E_ω we have $\Delta_\nu^{it} K_\omega = K_\omega \Delta_\mu^{it}$ by [28, Theorem 4.1], Δ_μ and Δ_ν being the modular operators associated to Λ_μ and Λ_ν respectively. Consequently $J_\nu K_\omega = K_\omega J_\mu$. From (5) and Lemma 5.2, it then follows that

$$\begin{aligned} \|\pi_\mu^\omega(a)\Omega_\omega - \pi_\nu^\omega(b)\Omega_\omega\|_\omega^2 &= \mu(a^*a) + \nu(b^*b) - \nu(E_\omega(a)^*b) - \nu(b^*E_\omega(a)) \\ &= \nu(b^*b) + \mu(a^*a) - \mu(E_\omega^\sigma(b)^*a) - \mu(a^*E_\omega^\sigma(b)) \\ &= \|\pi_\nu^{\omega^\sigma}(b)\Omega_{\omega^\sigma} - \pi_\mu^{\omega^\sigma}(a)\Omega_{\omega^\sigma}\|_{\omega^\sigma}^2 \end{aligned}$$

where $\omega^\sigma \in T(\nu, \mu)$ is determined by $E_{\omega^\sigma} = E_\omega^\sigma$ according to [28, Section 4]. Note that $\omega^\sigma \in T_\sigma(\nu, \mu)$ by [28, Theorem 4.1], since $(\sigma_t^\mu)^\sigma = \sigma_{-t}^\mu$, so $E_{\omega^\sigma} \circ \sigma_t^\nu = (\sigma_{-t}^\nu \circ E_\omega)^\sigma = (E_\omega \circ \sigma_{-t}^\mu)^\sigma = \sigma_t^\mu \circ E_{\omega^\sigma}$. \square

In particular we have symmetry of W_2 :

Proposition 5.4. *Consider any $k_1, \dots, k_n \in A$, and let W_2 be the associated Wasserstein distance on $\mathfrak{F}(A)$. Then*

$$W_2(\mu, \nu) = W_2(\nu, \mu)$$

for all $\mu, \nu \in \mathfrak{F}(A)$.

Proof. This follows from Definition 3.2 of W_2 and Lemma 5.3, since for each ω , every term in $I(\omega)$ (see (8)) is equal to the corresponding term in $I(\omega^\sigma)$, while $(\omega^\sigma)^\sigma = \omega$ because of $(E_\omega^\sigma)^\sigma = E_\omega$, giving a one-to-one correspondence between $T_\sigma(\mu, \nu)$ and $T_\sigma(\nu, \mu)$, which means we retain equality in the infima over $T_\sigma(\mu, \nu)$ and $T_\sigma(\nu, \mu)$ respectively on the two sides. \square

6. W_2 IS A METRIC

So far in our proof that W_2 is a metric, we have not assumed that A is finitely generated. This will shortly become necessary. However, we postpone making this assumption until it becomes essential.

One of the remaining properties of a metric is the following:

Proposition 6.1. *For W_2 in Definition 3.2, we have*

$$W_2(\mu, \mu) = 0$$

for all $\mu \in \mathfrak{F}(A)$.

Proof. This follows directly from (9) by setting $\omega = \delta_\mu \in T_\sigma(\mu, \mu)$, i.e. $E_\omega = \text{id}_A$, which is obviously a modular transport plan, since trivially $E_\omega \circ \sigma_t^\mu = \sigma_t^\mu \circ E_\omega$. \square

Aiming to achieve this simple result, was in fact the initial hint to approach the cost I heuristically through (2) and consequently rigorously by Definition 3.1.

All that remains, is to show that $W_2(\mu, \nu) = 0$ implies $\mu = \nu$. The first of two lemmas towards this, is the following:

Lemma 6.2. *Consider any $k_1, \dots, k_n \in A$, and let W_2 be the associated Wasserstein distance on $\mathfrak{F}(A)$. For any $\mu, \nu \in \mathfrak{F}(A)$, there then exists a modular transport plan $\omega \in T_\sigma(\mu, \nu)$ such that*

$$W_2(\mu, \nu) = I(\omega)^{1/2}.$$

Proof. It is a routine exercise to show that $T_\sigma(\mu, \nu)$ in Definition 2.3 is weakly* compact, since without loss we can view each element of $T_\sigma(\mu, \nu)$ as a state on the maximal C*-tensor product $A \otimes_{\max} B'$ (see for example [26, Proposition 4.1]). By the definition of W_2 there is a sequence $\omega_q \in T_\sigma(\mu, \nu)$ such that $I(\omega_q)^{1/2} \rightarrow W_2(\mu, \nu)$, which then must have a weak* cluster point $\omega \in T_\sigma(\mu, \nu)$. The lemma now follows by the following approximation:

Given $\varepsilon > 0$, there is a q_0 such that

$$|I(\omega_q) - W_2(\mu, \nu)^2| < \varepsilon$$

for all $q > q_0$. There exist $a'_1, \dots, a'_n \in A'$ such that

$$\|k_l \Lambda_\nu - a'_l \Lambda_\nu\| < \varepsilon$$

for $l = 1, \dots, n$. Furthermore, there is a $q > q_0$ such that

$$|\omega_q(k_l \otimes a'_l) - \omega(k_l \otimes a'_l)| < \varepsilon$$

for $l = 1, \dots, n$. Using (9) we then find

$$\begin{aligned}
|I(\omega) - I(\omega_q)| &\leq 2 \sum_{l=1}^n |\nu(k_l^* E_{\omega_q}(k_l)) - \nu(k_l^* E_{\omega}(k_l))| \\
&= 2 \sum_{l=1}^n |\langle k_l \Lambda_{\nu}, (E_{\omega_q}(k_l) - E_{\omega}(k_l)) \Lambda_{\nu} \rangle| \\
&\leq 2 \sum_{l=1}^n |\langle k_l \Lambda_{\nu} - a_l'^* \Lambda_{\nu}, (E_{\omega_q}(k_l) - E_{\omega}(k_l)) \Lambda_{\nu} \rangle| \\
&+ 2 \sum_{l=1}^n |\langle \Lambda_{\nu}, (E_{\omega_q}(k_l) - E_{\omega}(k_l)) a_l' \Lambda_{\nu} \rangle| \\
&\leq 4 \sum_{l=1}^n \|k_l \Lambda_{\nu} - a_l'^* \Lambda_{\nu}\| \|k_l\| \\
&+ 2 \sum_{l=1}^n |\omega_q(k_l \otimes a_l') - \omega(k_l \otimes a_l')| \\
&< 4\varepsilon \sum_{l=1}^n \|k_l\| + 2n\varepsilon.
\end{aligned}$$

Consequently,

$$|I(\omega) - W_2(\mu, \nu)^2| < 4\varepsilon \sum_{l=1}^n \|k_l\| + 2n\varepsilon + \varepsilon$$

for all $\varepsilon > 0$. □

This lemma is of some independent interest, since it says that the distance $W_2(\mu, \nu)$ is always reached by some modular transport plan. To prove that W_2 is a metric, it will be applied to the special case $W_2(\mu, \nu) = 0$.

The second lemma is where we see why A is eventually taken to be finitely generated, as is to be expected given the analogy to \mathbb{R}^n in the Introduction.

Lemma 6.3. *Consider any $k_1, \dots, k_n \in A$ such that $\{k_1^*, \dots, k_n^*\} = \{k_1, \dots, k_n\}$ and let I be the associated transport cost function. Let R be the von Neumann subalgebra of A generated by $\{k_1, \dots, k_n\}$. Let $\mu, \nu \in \mathfrak{F}(A)$. If $I(\omega) = 0$ for some transport plan $\omega \in T(\mu, \nu)$, then the restriction of E_{ω} to R is the identity map:*

$$E_{\omega}|_R = \text{id}_R$$

and consequently $\mu|_R = \nu|_R$.

Proof. Note that due to (5) and (9)

$$\begin{aligned} & \nu(|k_l - E_\omega(k_l)|^2) + \nu(E_\omega(|k_l|^2) - |E_\omega(k_l)|^2) \\ &= \mu(k_l^* k_l) + \nu(k_l^* k_l) - \nu(E_\omega(k_l)^* k_l) - \nu(k_l^* E_\omega(k_l)) \\ &= 0 \end{aligned}$$

where $E_\omega(|k_l|^2) - |E_\omega(k_l)|^2 \geq 0$ by Kadison's inequality, hence

$$\nu(|k_l - E_\omega(k_l)|^2) = 0 \quad \text{and} \quad \nu(E_\omega(|k_l|^2) - |E_\omega(k_l)|^2) = 0$$

for $l = 1, \dots, n$. Since ν is faithful, the former implies

$$(10) \quad E_\omega(k_l) = k_l$$

while the latter implies

$$E_\omega(k_l^* k_l) = E_\omega(k_l)^* E_\omega(k_l)$$

for $l = 1, \dots, n$.

Setting

$$A_\omega := \{a \in A : E_\omega(a^* a) = E_\omega(a)^* E_\omega(a)\},$$

it follows from [17, Theorem 3.1] that A_ω is a (norm closed) subalgebra of A and that

$$(11) \quad A_\omega = \{a \in A : E_\omega(ba) = E_\omega(b)E_\omega(a) \text{ for all } b \in A\}.$$

Since $\{k_1^*, \dots, k_n^*\} = \{k_1, \dots, k_n\}$, we have $k_1, \dots, k_n, k_1^*, \dots, k_n^* \in A_\omega$, so A_ω contains the $*$ -algebra R_0 generated by $\{1, k_1, \dots, k_n\}$. Moreover, since E_ω is positive and therefore preserves the involution, we see from (11) that $E_\omega|_{R_0}$ is a unital $*$ -homomorphism, thus $E_\omega|_{R_0} = \text{id}_{R_0}$ because of (10). As E_ω is normal, it follows that $E_\omega|_R = \text{id}_R$, which implies that $\mu(r) = \nu \circ E_\omega(r) = \nu(r)$ for all $r \in R$. \square

The following corollary of this lemma is what we need to complete the proof that W_2 is a metric:

Corollary 6.4. *If A is generated by $\{k_1, \dots, k_n\}$ in Lemma 6.3 and $I(\omega) = 0$ for some $\omega \in T(\mu, \nu)$, then $E_\omega = \text{id}_A$, $\omega = \delta_\mu$ and $\mu = \nu$.*

Proof. This follows from Lemma 6.3, since $R = A$, and $\omega = \delta_\nu \circ (E_\omega \odot \text{id}_{A'}) = \delta_\mu$. \square

In particular, this tells us that δ_μ is the unique transport plan from μ to itself attaining zero transport cost, but only the $\mu = \nu$ result is needed next.

Finally, we reach this paper's main result:

Theorem 6.5. *Let A be a σ -finite von Neumann algebra. Assume that A is generated by $k_1, \dots, k_n \in A$ such that $\{k_1^*, \dots, k_n^*\} = \{k_1, \dots, k_n\}$. Let W_2 be the Wasserstein distance on $\mathfrak{F}(A)$ associated to k_1, \dots, k_n in Definition 3.2. Then W_2 is a metric.*

Proof. By its definition, W_2 is real-valued and never negative. According to Propositions 4.3 and 5.4 we know that W_2 satisfies the triangle inequality and is symmetric. Proposition 6.1 tells us that $W_2(\mu, \mu) = 0$. If $W_2(\mu, \nu) = 0$, it follows from Lemma 6.2 followed by Corollary 6.4 that $\mu = \nu$. \square

If A is not generated by $\{k_1, \dots, k_n\}$ in Lemma 6.3, then $\mu \neq \nu$ is possible, as we now show, in which case W_2 is then only a pseudometric.

Suppose we have a von Neumann subalgebra F of A such that $\sigma_t^\mu(F) = F$ for all t and $R \subset F$. Then, by [45], there is a unique conditional expectation E from A onto F such that $\mu \circ E = \mu$. Assuming that

$$\mu|_F = \nu|_F,$$

this gives a transport plan $\omega = \delta_\nu \circ (E \odot \text{id}_{A'})$ from μ to ν . (This ω is an example of a relatively independent coupling, which comes up in the theory of noncommutative joinings; see [27, Section 3].) Then $I(\omega) = 0$ by (9), since $E_\omega = E$ restricted to F is the identity map. In particular $W_2(\mu, \nu) = 0$.

In this setup there are cases with $\mu \neq \nu$, for example: Consider $A = M \bar{\otimes} N$ for von Neumann algebras M and N , and set $\mu = \lambda \bar{\otimes} \zeta$ and $\nu = \lambda \bar{\otimes} \eta$ for faithful normal states λ on M , and ζ and η on N . Taking $F = M \otimes 1$ and $k_1, \dots, k_n \in F$, we satisfy all the requirements of the previous paragraph, but $\mu \neq \nu$ if $\zeta \neq \eta$.

7. EXPANDING THE SETTING

To conclude the paper, we briefly outline two ways of expanding the setting above, to obtain somewhat weaker results, which emphasize the role our assumptions play in proving that W_2 is a metric, and as an indication of possible further avenues to explore. This is followed by questions regarding further generalization.

First, we can allow all transport plans in Definition 3.2 of the distance. Given $k_1, \dots, k_n \in A$, we define an associated distance function d on $\mathfrak{F}(A)$ by

$$d(\mu, \nu) := \inf_{\omega \in T(\mu, \nu)} I(\omega)^{1/2}$$

in terms of Definition 3.1.

With minor modifications to the foregoing work, one then obtains the following variation on our main result:

Proposition 7.1. *Let A be a σ -finite von Neumann algebra. Assume that A is generated by $k_1, \dots, k_n \in A$ such that $\{k_1^*, \dots, k_n^*\} = \{k_1, \dots, k_n\}$. Let d be the distance function above associated to k_1, \dots, k_n . Then d is an asymmetric metric, that is to say, d satisfies all the requirements of a metric, except that it may not be symmetric.*

That d may not be symmetric, could be very natural, as we have a direction of transport involved, which now appears to be reflected in the distance function.

On the other hand, we have not shown that the modular property of transport plans is the weakest condition ensuring symmetry, nor have we explicitly shown d not to be symmetric. The modular condition is very natural, but the possibility of symmetry more generally is an open question.

The second variation is to drop the assumption that the states we work with are faithful. That is to say, we define a distance function on the set $\mathfrak{S}(M)$ of all normal states on a von Neumann algebra M . To do this, let p_ζ be the support projection of $\zeta \in \mathfrak{S}(M)$. Restrict to the von Neumann algebra $A = p_\zeta M p_\zeta$ and replace ζ by its restriction μ to A , which is a faithful normal state. Similarly for $\eta \in \mathfrak{S}(M)$ to obtain the von Neumann algebra B and faithful normal state ν .

For $k_1, \dots, k_n \in M$, we consider

$$k_l^\zeta := p_\zeta k_l p_\zeta \in A$$

and write $k^\zeta = (k_1^\zeta, \dots, k_n^\zeta)$. Similarly for η . Then apply the representation machinery from Section 2 to (A, μ) and (B, ν) , strictly speaking after setting up a cyclic representation for both, to define

$$\mathcal{I}(\omega)^{1/2} = \left\| \pi_\mu^\omega(k^\zeta)\Omega - \pi_\nu^\omega(k^\eta)\Omega \right\|_{\oplus\omega}$$

for $\omega \in T_\sigma(\mu, \nu)$, in place of Definition 3.1.

Now we define a distance function ρ on $\mathfrak{S}(M)$ associated to k_1, \dots, k_n , by

$$\rho(\zeta, \eta) = \inf_{\omega \in T_\sigma(\mu, \nu)} \mathcal{I}(\omega)^{1/2}$$

which leads to the next result, again by minor modifications to our previous work:

Proposition 7.2. *Let M be a von Neumann algebra and consider any $k_1, \dots, k_n \in M$. Let ρ be the distance function above associated to k_1, \dots, k_n . Then ρ is an pseudometric on $\mathfrak{S}(M)$, that is to say, ρ satisfies all the requirements of a metric, except that we may have $\rho(\zeta, \eta) = 0$ with $\zeta \neq \eta$.*

There are some natural further questions:

We focussed on a finite set $\{k_1, \dots, k_n\}$ of bounded operators to define cost, in analogy to the coordinate functions on a bounded closed set in \mathbb{R}^n . Can one expand on this and use appropriate infinite sets of k_l as well? Or can the k_l be unbounded, but affiliated to the von Neumann algebra in question?

Lastly, what would be the best way of adapting the approach of this paper to Wasserstein metrics of order other than 2?

REFERENCES

- [1] L. ACCARDI, C. CECCHINI, Conditional expectations in von Neumann algebras and a theorem of Takesaki, *J. Funct. Anal.* **45**(1982), 245–273.
- [2] J. AGREDO, F. FAGNOLA, On quantum versions of the classical Wasserstein distance, *Stochastics* **89**(2017), 910–922.
- [3] H. ARAKI, Some properties of modular conjugation operator of von Neumann algebras and a non-commutative Radon-Nikodym theorem with a chain rule, *Pacific J. Math.* **50**(1974), 309–354.
- [4] J.P. BANNON, J. CAMERON, K. MUKHERJEE, On noncommutative joinings, *Int. Math. Res. Not. IMRN* **2018**, 4734–4779.
- [5] J.-D. BENAMOU, Y. BRENIER, A computational fluid mechanics solution to the Monge-Kantorovich mass transfer problem, *Numer. Math.* **84**(2000), 375–393.
- [6] P. BIANE, D. VOICULESCU, A free probability analogue of the Wasserstein metric on the trace-state space, *Geom. Funct. Anal.* **11**(2001), 1125–1138.
- [7] M. BRANNAN, L. GAO, M. JUNGE, Complete Logarithmic Sobolev inequalities via Ricci curvature bounded below, *Adv. Math.* **394**(2022), 108129.
- [8] M. BRANNAN, L. GAO, M. JUNGE, Complete Logarithmic Sobolev inequalities via Ricci curvature bounded below II, arXiv:2008.12038.
- [9] O. BRATTELI, D.W. ROBINSON, *Operator algebras and quantum statistical mechanics 1*, second edition, Springer-Verlag, New York, 1987.
- [10] J. W. BROWN, Approximation theorems for Markov operators, *Pacific J. Math.* **16**(1966), 13–23.
- [11] E. CAGLIOTI, F. GOLSE, T. PAUL, Quantum optimal transport is cheaper, *J. Stat. Phys.* **181**(2020), 149–162.
- [12] E.A. CARLEN, J. MAAS, An analog of the 2-Wasserstein metric in non-commutative probability under which the fermionic Fokker-Planck equation is gradient flow for the entropy, *Comm. Math. Phys.* **331**(2014), 887–926.
- [13] E.A. CARLEN, J. MAAS, Gradient flow and entropy inequalities for quantum Markov semigroups with detailed balance, *J. Funct. Anal.* **273**(2017), 1810–1869.
- [14] E.A. CARLEN, J. MAAS, Non-commutative calculus, optimal transport and functional inequalities in dissipative quantum systems, *J. Stat. Phys.* **178**(2020), 319–378.
- [15] Y. CHEN, T.T. GEORGIU, A. TANNENBAUM, Matrix optimal mass transport: a quantum mechanical approach, *IEEE Trans. Automat. Control* **63**(2018), 2612–2619.
- [16] Y. CHEN, W. GANGBO, T.T. GEORGIU, A. TANNENBAUM, On the matrix Monge-Kantorovich problem, *European J. Appl. Math.* **31**(2020), 574–600.
- [17] M.-D. CHOI, A Schwarz inequality for positive linear maps on C^* -algebras, *Illinois J. Math.* **18**(1974), 565–574.
- [18] M.-D. CHOI, Completely positive linear maps on complex matrices, *Linear Algebra Appl.* **10**(1975), 285–290.
- [19] F. CIPRIANI, Dirichlet forms and Markovian semigroups on standard forms of von Neumann algebras, *J. Funct. Anal.* **147**(1997), 259–300.
- [20] A. CONNES, Caractérisation des espaces vectoriels ordonnés sous-jacents aux algèbres de von Neumann, *Ann. Inst. Fourier (Grenoble)* **24**(1974), 121–155.
- [21] A. CONNES, Compact metric spaces, Fredholm modules, and hyperfiniteness, *Ergodic Theory Dynam. Systems* **9**(1989), 207–220.
- [22] A. CONNES, *Noncommutative geometry*, Academic Press, San Diego, CA, 1994.

- [23] G. DE PALMA, M. MARVIAN, D. TREVISAN, S. LLOYD, The quantum Wasserstein distance of order 1, *IEEE Trans. Inform. Theory* **67**(2021), 6627–6643.
- [24] G. DE PALMA, D. TREVISAN, Quantum optimal transport with quantum channels, *Ann. Henri Poincaré* **22**(2021), 3199–3234.
- [25] J. DE PILLIS, Linear transformations which preserve hermitian and positive semidefinite operators, *Pacific J. Math.* **23**(1967), 129–137.
- [26] R. DUVENHAGE, Ergodicity and mixing of W^* -dynamical systems in terms of joinings, *Illinois J. Math.* **54**(2010), 543–566.
- [27] R. DUVENHAGE, Relatively independent joinings and subsystems of W^* -dynamical systems, *Studia Math.* **209**(2012), 21–41.
- [28] R. DUVENHAGE, M. SNYMAN, Balance between quantum Markov semigroups, *Ann. Henri Poincaré* **19**(2018), 1747–1786.
- [29] F. FAGNOLA, R. REBOLLEDO, Entropy production for quantum Markov semigroups, *Comm. Math. Phys.* **335**(2015), 547–570.
- [30] T. FALCONE, L^2 -von Neumann modules, their relative tensor products and the spatial derivative, *Illinois J. Math.* **44**(2000), 407–437.
- [31] S. GOLDSTEIN, J.M. LINDSAY, Beurling-Deny conditions for KMS-symmetric dynamical semigroups, *C. R. Acad. Sci. Paris Sér. I Math.* **317**(1993), 1053–1057.
- [32] S. GOLDSTEIN, J.M. LINDSAY, KMS-symmetric Markov semigroups, *Math. Z.* **219**(1995), 591–608.
- [33] F. GOLSE, C. MOUHOT, T. PAUL, On the mean field and classical limits of quantum mechanics, *Comm. Math. Phys.* **343**(2016), 165–205.
- [34] F. GOLSE, T. PAUL, The Schrödinger equation in the mean-field and semi-classical regime, *Arch. Ration. Mech. Anal.* **223**(2017), 57–94.
- [35] F. GOLSE, T. PAUL, Wave packets and the quadratic Monge-Kantorovich distance in quantum mechanics, *C. R. Math. Acad. Sci. Paris* **356**(2018), 177–197.
- [36] U. HAAGERUP, The standard form of von Neumann algebras, *Math. Scand.* **37**(1975), 271–283.
- [37] D.F. HORNSHAW, L^2 -Wasserstein distances of tracial W^* -algebras and their disintegration problem, arXiv:1806.01073.
- [38] D.F. HORNSHAW, Quantum optimal transport for AF- C^* -algebras, arXiv:1910.03312.
- [39] A. JAMIOŁKOWSKI, Linear transformations which preserve trace and positive semidefiniteness of operators, *Rep. Mathematical Phys.* **3**(1972), 275–278.
- [40] M. JIANG, S. LUO, S. FU, Channel-state duality, *Phys. Rev. A*, **87**(2013) 022310.
- [41] P. MIKUSIŃSKI, M. D. TAYLOR, Markov operators and n -copulas, *Ann. Polon. Math.* **96**(2009), 75–95.
- [42] M. OHYA, D. PETZ, *Quantum entropy and its use*, Texts and Monographs in Physics. Springer-Verlag, Berlin, 1993.
- [43] D. PETZ, A dual in von Neumann algebras with weights, *Quart. J. Math. Oxford Ser. (2)* **35**(1984), 475–483.
- [44] J.-L. SAUVAGEOT, Sur le produit tensoriel relatif d’espaces de Hilbert, *J. Operator Theory* **9**(1983), 237–252.
- [45] M. TAKESAKI, Conditional expectations in von Neumann algebras, *J. Funct. Anal.* **9**(1972), 306–321.
- [46] M. TAKESAKI, *Theory of operator algebras. II*, Encyclopaedia of Mathematical Sciences, 125. Operator Algebras and Non-commutative Geometry, 6. Springer-Verlag, Berlin, 2003.

- [47] C. VILLANI, *Topics in optimal transportation*, Graduate Studies in Mathematics, 58. American Mathematical Society, Providence, RI, 2003.
- [48] C. VILLANI, *Optimal transport. Old and new*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 338. Springer-Verlag, Berlin, 2009.
- [49] M. WIRTH, A noncommutative transport metric and symmetric quantum Markov semigroups as gradient flows of the entropy, arXiv:1808.05419.

DEPARTMENT OF PHYSICS, UNIVERSITY OF PRETORIA, PRETORIA 0002, SOUTH AFRICA

Email address: `rocco.duvenhage@up.ac.za`