# Avian-human influenza epidemic model with diffusion, nonlocal delay and spatial homogeneous environment.

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**Abstract**

In this paper, an avian-human influenza epidemic model with diffusion, nonlocal delay and spatial homogeneous environment is investigated. This model describes the transmission of avian influenza among poultry, humans and environment. The behavior of positive solutions to a reaction-diffusion system with homogeneous Neumann boundary conditions is investigated. By mean of linearization method and spectral analysis the local asymptotical stability is established. The global asymptotical stability for the poultry sub-system is studied by spectral analysis and by using a Lyapunov functional. For the full system, the global stability of the disease-free equilibrium is studied using the comparison Theorem for parabolic equations. Our result shows that the disease-free equilibrium is globally asymptotically stable, whenever the contact rate for the susceptible poultry is small. This suggests that the best policy to prevent the occurrence of an epidemic is not only to exterminate the asymptomatic poultry but also to reduce the contact rate between susceptible humans and the poultry environment. Numerical simulations are presented to illustrate the main results.

*Keywords:* Reaction-diffusion systems, Avian influenza, SI-SEIS-C model, Stability.

## 1. Introduction

The avian influenza is caused by viruses adapted to birds and it normally affects wild birds and poultry. The wild birds are natural reservoir for all the sub-types of influenza A viruses. Influenza viruses are widespread and due to their high mutation rate many subtypes exist. Furthermore, H5N1, H7N4, H7N7, H7N9, H9N2 and other avian influenza viruses with pathogenicity have great potential threat to human. Poultry farms are an important reservoir of avian influenza A virus (H7N9), which plays a critical role in the genesis of influenza pandemic [1]. Avian influenza virus (AIV) transmission to humans is largely facilitated by contact with animals and excretion of contaminated droplets or aerosols [2] and to a lesser extent through transport of (dead) birds or contaminated objects (vehicles, humans, or fomites), water, food and contact with infected wildfowl or insects [3]. Historically, the avian influenza splits into two classes: the "High Pathogenic Avian Influenza (HPAI)" and the "Low Pathogenic Avian Influenza (LPAI)". The HPAI can cause a series of systemic infections that can lead to high mortality. The LPAI causes mild or no symptoms.

Recently in [4], the authors proposed the following mathematical model to study the impact of

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environmental transmission on avian influenza infection:

$$\begin{cases} \dfrac{dX}{dt} = (1-q)A - \beta_v X \dfrac{Y}{1+\alpha Y} - \beta_e X \dfrac{C}{C+\kappa} - dX, \\ \dfrac{dY}{dt} = qA + \beta_v X \dfrac{Y}{1+\alpha Y} + \beta_e X \dfrac{C}{C+\kappa} - dY, \\ \dfrac{dS}{dt} = B + aE + \gamma I - \tau_v \dfrac{S}{N} Y - \tau_e \dfrac{S}{N} C - \delta S, \\ \dfrac{dE}{dt} = \tau_v \dfrac{S}{N} Y + \tau_e \dfrac{S}{N} C - (a+\delta+\epsilon)E, \\ \dfrac{dI}{dt} = \epsilon E - (\gamma+\rho+\delta)I, \\ \dfrac{dC}{dt} = \phi_2 Y - \xi C. \end{cases} \tag{1.1}$$

In (1.1), the first two equations and the last one describe the interactions between the birds and their biotope. Thus, the poultry population is divided into two classes: susceptible poultry $X$ and asymptomatic poultry contaminated with avian influenza viruses $Y$. The concentration of avian influenza viruses in the poultry living environment (biotope) is $C$. The remaining three equations form an SEIS model for humans, which describes the dynamics of human population divided in three mutually exclusive classes: susceptible humans $S$, latent humans $E$ and infected humans $I$.

It must be pointed out that System (1.1) neglects any spatial structure of disease spreading and is definitely not very realistic for moving individuals such as poultry and humans. For example, in our case, poultry on the farm can move from one point to another to feed or drink water and humans can migrate in large numbers from one area to another for supplies during the sales period (of poultry or eggs). During the rearing period, that is the time lag during which there is neither sale of poultry nor production of eggs, humans cannot be in the same location, so a rearing period will result in a delay. But whatever the reason for introducing a delay into any population model in which the individuals are moving, the corresponding term in the model must be nonlocal in space as well as in time. Thus it would be realistic to incorporate delay effects in the interaction terms. Furthermore, As the distribution of the individuals is in different spatial locations, the standard method of including the spatial effects consists in the introduction of diffusion terms. This lead is an extended version of the SI-SEIS-C avian-human epidemic model (1.1) in the form of a delayed reaction diffusion system of equations given below.

Therefore in this study, we propose a mathematical model for the transmission dynamics of AIV among poultry-human that incorporates both mobility of the poultry/human and spatial environmental homogeneity.

The outline of the remainder of the paper is as follows. In Section 2 we build an avian-human influenza epidemic model that incorporates diffusion, nonlocal delay and spatial homogeneous environment, and give the model's basic properties. Section 3 deals with the theoretical analysis of the continuous poultry model, while Section 4 presents an asymptotic analysis of the full model and numerical simulations are given in Section 5. Finally, we conclude the paper in Section 6 and provide some discussions that highlight few relevant perspectives.

## 2. Modelling framework and uniform bound

### *2.1. Modelling framework*

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain representing an industrial city in which humans live. We assume that poultry farms are built in human sparsely populated areas and that each farmer has already bought his poultry and will not do so until the end of the sale for broilers or until the end of egg laying for laying hens. Denote by $X(x,t), Y(x,t), S(x,t), E(x,t), I(x,t)$ the number of susceptible poultry, asymptomatic poultry, susceptible humans, latent humans and infected humans respectively at time $t$ and location $x$. $C(x,t)$ is the concentration of virus at time $t$ and location $x$.

*2.1.1. Poultry population dynamics*

We assume that a total number $A$ of poultry replenishes the farm per unit time due to importation and the proportion $(1-q)A$ is susceptible, while the remaining proportion $qA$ is asymptomatic. Susceptible and asymptomatic poultry die at rate $dX$ and $dY$, respectively. Upon direct transmission among poultry, susceptible poultry moves to asymptomatic class following a saturation type incidence at rate $\beta_v XY/(1+\alpha Y)$, such that $\beta_v Y$ measures the infection force of the infective poultry, the parameter $\alpha$ stands for the inhibitory effort, and $1/(1+\alpha Y)$ describes the saturation due to the protection measures of the poultry farmers or the crowding of infected poultry when the number of infective poultry increases [5]. Upon indirect transmission, $\beta_e XC/(C+\kappa)$ corresponds to the incidence rate between environmental contaminated food particles and susceptible poultry. In the latter, $\beta_e$ is the transmission coefficient such that $\beta_e \gg \beta_v$; $1/(C+\kappa)$ represents saturation due to the cleaning of farms when the concentration of excretion becomes larger; $\kappa$ is the concentration of avian viruses attached to aerosol particles in the farm, sufficient to guarantee 50% chance of catching the infection. In the farm, poultry move from point to other to feed or drink water. To model this displacement, we use diffusion Fick's law. Thus, the dynamics of poultry population is given by the following system:

$$\begin{cases} \dfrac{\partial X}{\partial t} - D_1 \Delta X = (1-q)A - \beta_v X \dfrac{Y}{1+\alpha Y} - \beta_e X \dfrac{C}{C+\kappa} - dX, \\ \dfrac{\partial Y}{\partial t} - D_2 \Delta Y = qA + \beta_v X \dfrac{Y}{1+\alpha Y} + \beta_e X \dfrac{C}{C+\kappa} - dY. \end{cases} \tag{2.1}$$

*2.1.2. Human population dynamics*

New born or immigrated humans are recruited susceptible at rate $B$ and die naturally at rate $\delta$. Since there are some medicines to fight against avian influenza A virus, the latent and the infected humans recover respectively at rate $a$ and $\gamma$. The transmission of avian influenza A from poultry to humans occurs at rate $\tau_v$, and $\tau_e$ is the transmission coefficient from the pathogenic or contaminated environment to humans. For the motivations on the choice of the different incidence functions in (2.2), we refer the reader to our previous paper [4] for details. The morbidity of the latent human is $\epsilon$ and the disease-related death rate is $\rho$, with $(\rho \gg \delta)$.

During the sales period (of poultry or eggs), humans migrate in large numbers from densely populated areas to these sparsely populated areas for supplies. This migration is similarly described by Fick's law of diffusion.

During the rearing period, that is the time lag during which there is neither sale of poultry nor production of eggs, humans cannot be in the same location in the industrial city. To model this phenomenon, we use a "nonlocal" delay: an average weight in space arises when the account is taken of the fact that humans have been at different points in space in previous times. Thus, for ecological reasons, it is necessary to incorporate a time delay into some equations of the model. In addition, it should be noted that the human population at all times will have some contribution in animal husbandry as in the sale or harvest of eggs. This contribution is modeled by a function $k(t)$ called the delay kernel and satisfies:

$$k(t) \geq 0, \quad \forall t \geq 0, \quad tk(t) \in L^1((0,+\infty), \mathbb{R}) \text{ and } \int_0^{+\infty} k(t)dt = 1.$$

Similarly a function $G$, defined as the spatial averaging kernel, informs that this delay is given and enjoys the following equalities:

$$\int_\Omega G(x,y,t)dx = \int_\Omega G(x,y,t)dy = 1.$$

For example, $G(x,y,t)$ is the Green's function of the operator $\frac{\partial}{\partial t} - D_3\Delta$ subject to homogeneous Neumann boundary condition, and $k(t) = \frac{1}{\tau}e^{-t/\tau}$ with a constant $\tau$ representing the delay.

We assume that humans at a typical time $s$ (with $s < t$) made a contribution so that the sale of poultry or the harvesting of eggs can take place at time $t$. To quantify this contribution, we first multiply the density at time $s$ by the function $k(t-s)$, because they have contributed at time $t-s$. Knowing that humans located at the point $x$ at time $t$ could have been anywhere in the industrial area at the previous instant $s$, we will now need to multiply this density by a function in space $G(x,y,t-s)$. Thus, the dynamics of human population is given by the following system:

$$\begin{cases} \dfrac{\partial S}{\partial t} - D_3\Delta S = B + aE + \gamma I - \delta S - \dfrac{S}{N}\displaystyle\int_\Omega \int_{-\infty}^{t} G(x,y,t-s)k(t-s)(\tau_v Y + \tau_e C)(y,s)dsdy, \\ \dfrac{\partial E}{\partial t} - D_4\Delta E = \dfrac{S}{N}\displaystyle\int_\Omega \int_{-\infty}^{t} G(x,y,t-s)k(t-s)(\tau_v Y + \tau_e C)(y,s)dsdy - (a+\delta+\epsilon)E, \\ \dfrac{\partial I}{\partial t} - D_5\Delta I = \epsilon E - (\gamma+\rho+\delta)I. \end{cases} \tag{2.2}$$

The term

$$\int_\Omega \int_{-\infty}^{t} G(x,y,t-s)k(t-s)(\tau_v Y + \tau_e C)(y,s)dsdy,$$

with

$$G(x,y,t-s)k(t-s) \geq 0, \;\; x,y \in \Omega, \quad t > 0,$$

accounts for the infection of individuals to their present position at time $t$, caused by the asymptomatic poultry and the infected aerosol from all possible positions at all previous times [6, 7, 8].

### *2.1.3. Virus concentration dynamics*

Since an emission rate for pathogens is defined as an amount released per unit of time, it depends on source type (pigs, poultry, industrial, humans, etc.), source characteristics (e.g., stable construction or animal activity), excretion route (e.g., exhaled air or feces), pathogen species or strain, particle size, etc. For a full quantitative risk assessment, quantified emission rates are required. Hence, the contribution by humans and poultry in the contamination of the poultry farm is respectively $\phi_1 I$ and $\phi_2 Y$; and the degradation or decontamination rate of viruses (inactivation) due to the temperature or humidity is $\xi$. It is worth stressing on the fact that the contribution of humans to the contamination of the environment can be neglected because of the precautions (disinfection, wearing of protective equipments) taken by poultry producers to prevent visitors from spreading the viruses in their farms. So we assume that only infected poultry can contaminate their living environment through feces and sneezing. If in addition we neglect the diffusion of avian influenza viruses in the living environment of the poultry, then the dynamics of their concentration is modeled by the following equation:

$$\frac{\partial C}{\partial t} = \phi_2 Y - \xi C. \tag{2.3}$$

So, in the above described framework, the full model governing the dynamics of avian-human influenza is the following partially degenerated reaction-diffusion system:

$$\begin{cases} \dfrac{\partial X}{\partial t} - D_1\Delta X = (1-q)A - \beta_v X\dfrac{Y}{1+\alpha Y} - \beta_e X\dfrac{C}{C+\kappa} - dX, \\ \dfrac{\partial Y}{\partial t} - D_2\Delta Y = qA + \beta_v X\dfrac{Y}{1+\alpha Y} + \beta_e X\dfrac{C}{C+\kappa} - dY, \\ \dfrac{\partial S}{\partial t} - D_3\Delta S = B + aE + \gamma I - \delta S - \dfrac{S}{N}\displaystyle\int_\Omega \int_{-\infty}^{t} G(x,y,t-s)k(t-s)(\tau_v Y + \tau_e C)(y,s)dsdy, \\ \dfrac{\partial E}{\partial t} - D_4\Delta E = \dfrac{S}{N}\displaystyle\int_\Omega \int_{-\infty}^{t} G(x,y,t-s)k(t-s)(\tau_v Y + \tau_e C)(y,s)dsdy - (a+\delta+\epsilon)E, \\ \dfrac{\partial I}{\partial t} - D_5\Delta I = \epsilon E - (\gamma+\rho+\delta)I, \\ \dfrac{\partial C}{\partial t} = \phi_2 Y - \xi C, \end{cases} \tag{2.4}$$

for $t > 0$, $x \in \Omega$. We emphasize that, the reaction part of system (2.4) corresponds the model we have proposed and studied in [4]. Therefore, the system (2.4) is its substantial extension and its analytical analysis calls for different mathematical techniques and approaches, as one will notice shortly. The parameters of the model (2.4), their biological significance and unit are gathered in Table 1.

Table 1: Biological significance of the parameters of PDE-model (2.4)–(2.6).

| Parameters | Biological significance | Units |
|---|---|---|
| $q$ | Proportion of asymptomatic imported poultry | no unit |
| $a$ | Recovery rate of the latent humans | $week^{-1}$ |
| $A$ | Numbers of imported poultry | ind/week |
| $\gamma$ | Recovery rate of the infected humans | $week^{-1}$ |
| $\beta_v$ | Direct contact rate in poultry host | $(\text{ind.week})^{-1}$ |
| $\rho$ | Disease-related death rate | $week^{-1}$ |
| $\beta_e$ | Indirect contact rate in poultry host | $week^{-1}$ |
| $D_1$ | Diffusion coefficient for susceptible poultry | no unit |
| $d$ | Natural death rate of poultry | $week^{-1}$ |
| $D_2$ | Diffusion coefficient for infected poultry | no unit |
| $\alpha$ | Parameter of the inhibitory effort | $ind^{-1}$ |
| $D_3$ | Diffusion coefficient for susceptible humans | no unit |
| $B$ | Recruitment rate for humans | ind/week |
| $D_4$ | Diffusion coefficient for latent humans | no unit |
| $\tau_v$ | Transmission rate of AIV from poultry to human | $week^{-1}$ |
| $\epsilon$ | Morbidity of the latent humans | $week^{-1}$ |
| $\delta$ | Natural death rate of humans | $week^{-1}$ |
| $\kappa$ | Half saturation rate ($eID_{50}$) | $g.m^3$ |
| $\xi$ | Degradation rate of virus | $week^{-1}$ |
| $\tau_e$ | Transmission rate of AIV from environment to human | ind $/(g.m^3.\text{week})$ |
| $\phi_2$ | Emission rate of poultry | $g.m^3/(\text{ind.week})$ |
| $D_5$ | Diffusion coefficient for infected humans | no unit |
| $\tau$ | Delay parameter | no unit |

We assume that during an epidemic, the borders between cities are closed. Thus, the sale and consumption (of hens or eggs) will only take place in the industrial area, that is, humans and poultry are banned to leave their industrial zone. So we use the homogeneous Neumann boundary conditions

$$\frac{\partial X}{\partial \eta} = \frac{\partial Y}{\partial \eta} = \frac{\partial S}{\partial \eta} = \frac{\partial E}{\partial \eta} = \frac{\partial I}{\partial \eta} = 0, \quad t > 0, \quad x \in \partial\Omega, \tag{2.5}$$

and initial conditions

$$\begin{cases} X(x,0) = \varphi_1(x),\ S(x,0) = \varphi_2(x),\ E(x,0) = \varphi_3(x),\ I(x,0) = \varphi_4(x), \\ Y(x,\theta) = \varphi_5(x,\theta),\ C(x,\theta) = \varphi_6(x,\theta),\ \ (x,\theta) \in \overline{\Omega} \times (-\infty, 0). \end{cases} \tag{2.6}$$

Here $\eta$ is the outward unit normal vector on the boundary and $\Delta$ is the usual Laplace operator. The positive constants $D_1$ and $D_2$ are the diffusion coefficients for poultry; $D_3$, $D_4$ and $D_5$ are the diffusion coefficients for humans. The initial function $\varphi_i$ for $i \in \{1 \cdots 6\}$ is nonnegative, Hölder continuous and satisfies $\frac{\partial \varphi_i}{\partial \eta} = 0$ on the boundary.

### 2.2. ***Uniform bound***

In this section, we provide an in-depth study of the dynamics of the initial boundary value problem (IBVP) (2.4)-(2.6) which yields various outcomes. Precisely, we prove the existence, uniqueness, positivity and boundedness of the solution for the IBVP (2.4)-(2.6). This is done by combining the variational method and semigroup techniques to some useful functional analysis arguments.

### 2.2.1. *Local existence and uniqueness for the IBVP*

We rewrite (2.4) in the following compact form:

$$\begin{cases} \dfrac{\partial u}{\partial t} + A_p u = f(u_1, u_2, \cdots, u_6) \quad \text{in } \ \Omega \times (0, +\infty), \\ \dfrac{\partial u_i}{\partial \eta} = 0 \quad \text{on } \ \partial\Omega \times (0, +\infty), \ \forall i \in \{1,2,3,4,5\} \\ u_i = \varphi_i \quad \text{in } \ \Omega \times (-\infty, 0], \ \forall i \in \{1,2,3,4,5,6\}, \end{cases} \tag{2.7}$$

where $u = (u_1, u_2, u_3, u_4, u_5, u_6)^t = (X, Y, S, E, I, C)^t$,
$A_p = diag\{-D_1\Delta + d, -D_2\Delta + d, -D_3\Delta + \delta, -D_4\Delta + (a + \delta + \epsilon), -D_5\Delta + (\gamma + \rho + \delta), \xi\}$
and $f = (f_1, f_2, f_3, f_4, f_5, f_6)^t$ with

$$\begin{aligned} f_1 &= (1-q)A - \beta_v X \frac{Y}{1+\alpha Y} - \beta_e X \frac{C}{C+\kappa}, \\ f_2 &= qA + \beta_v X \frac{Y}{1+\alpha Y} + \beta_e X \frac{C}{C+\kappa}, \\ f_3 &= B + aE + \gamma I - \frac{S}{N} \int_\Omega \int_{-\infty}^t G(x,y,t-s)k(t-s)(\tau_v Y + \tau_e C)(y,s)dsdy, \\ f_4 &= \frac{S}{N} \int_\Omega \int_{-\infty}^t G(x,y,t-s)k(t-s)(\tau_v Y + \tau_e C)(y,s)dsdy, \\ f_5 &= \epsilon E, \\ f_6 &= \phi_2 Y. \end{aligned}$$

The following Lemma is instrumental for Proposition 2.2 below.

**Lemma 2.1.** *[9] Let $K(x,y,t) = G(x,y,t)k(t), \quad x, y \in \Omega \subset \mathbb{R}^3$, where $k(t) \geq 0$ and $G(x,y,t)$ is the solution to*

$$\frac{\partial G}{\partial t} = D_2 \nabla^2 G, \quad \frac{\partial G}{\partial \eta} = 0 \ \text{ on } \ \partial\Omega, \quad G(x,y,0) = \delta(x-y). \tag{2.8}$$

*Then*

$$\left\| \int_\Omega \int_{-\infty}^t K(x,y,t-s)u(y,s)dsdy \right\|_2 \leq \int_{-\infty}^t k(t-s)\|u(\cdot,s)\|_2 ds$$

*for any function $u(x,t)$ such that $\partial u/\partial\eta = 0$ on $\partial\Omega$.*

The local existence result for the PDE system (2.7) can be established under the following condition on $f$.

**Proposition 2.2.** *Let $T > 0$. If $f : \mathbb{C}\left((-\infty; T]; \mathbb{C}(\overline{\Omega}; \mathbb{R}^6)\right) \to L^2(\Omega; \mathbb{R}^6)$, then $f$ is uniformly Lipschitz continuous on every bounded subset of $\mathbb{C}\left((-\infty; T]; \mathbb{C}(\overline{\Omega}; \mathbb{R}^6)\right)$.*

**Proof.** Set $u, v \in \mathbb{C}\left((-\infty; T]; \mathbb{C}(\overline{\Omega}; \mathbb{R}^6)\right)$ such that $u = (X_1, Y_1, S_1, E_1, I_1, C_1), \ \ v = (X_2, Y_2, S_2, E_2, I_2, C_2)$ and

$$\begin{aligned} &\|u_i\|_{\mathbb{C}(\overline{\Omega};\mathbb{R})} \leq T_m, \ \ \forall i \in \{1,2\}\,, \ \|u_i\|_{\mathbb{C}(\overline{\Omega};\mathbb{R})} \leq U_m, \ \ \forall i \in \{3,4,5\} \text{ and } \|u_6\|_{\mathbb{C}(\overline{\Omega};\mathbb{R})} \leq V_m, \\ &\|v_i\|_{\mathbb{C}(\overline{\Omega};\mathbb{R})} \leq T_m, \ \ \forall i \in \{1,2\}\,, \ \|v_i\|_{\mathbb{C}(\overline{\Omega};\mathbb{R})} \leq U_m, \ \ \forall i \in \{3,4,5\} \text{ and } \|v_6\|_{\mathbb{C}(\overline{\Omega};\mathbb{R})} \leq V_m. \end{aligned}$$

Recall that

$$\|u\|_{\mathbb{C}(\overline{\Omega};\mathbb{R}^6)} = \sum_{j=1}^6 \left\|u_j\right\|_{\mathbb{C}(\overline{\Omega};\mathbb{R})}, \ \left\|f(u) - f(v)\right\|_2 = \left\{ \sum_{j=1}^6 \left\|f_j(u) - f_j(v)\right\|^2_{L^2(\Omega;\mathbb{R})} \right\}^{\frac{1}{2}}. \tag{2.9}$$

Then

$$\begin{aligned}
\left\| f_1(u) - f_1(v) \right\|_{L^2(\Omega;\mathbb{R})} &\leq L_1^1 \left\| X_1(\cdot,s) - X_2(\cdot,s) \right\|_{\mathbb{C}(\overline{\Omega};\mathbb{R})} + L_2^1 \left\| Y_1(\cdot,s) - Y_2(\cdot,s) \right\|_{\mathbb{C}(\overline{\Omega};\mathbb{R})} \\
&\quad + L_3^1 \left\| C_1(\cdot,s) - C_2(\cdot,s) \right\|_{\mathbb{C}(\overline{\Omega};\mathbb{R})}, \\
&\leq L_1 \sup_{s\leq T} \left\| X_1(\cdot,s) - X_2(\cdot,s) \right\|_{\mathbb{C}(\overline{\Omega};\mathbb{R})} \\
&\quad + L_1 \sup_{s\leq T} \left\| Y_1(\cdot,s) - Y_2(\cdot,s) \right\|_{\mathbb{C}(\overline{\Omega};\mathbb{R})} \\
&\quad + L_1 \sup_{s\leq T} \left\| C_1(\cdot,s) - C_2(\cdot,s) \right\|_{\mathbb{C}(\overline{\Omega};\mathbb{R})},
\end{aligned}$$

where the non-vanishing $L_j^1$ for all $j \in \{1,2,3,4,5,6\}$ are

$$L_1^1 = \beta_v T_m + \beta_e \kappa V_m + \alpha \beta_v T_m^2 + \beta_e V_m^2 \ , \ L_2^1 = \beta_v T_m \ , \ L_3^1 = \beta_e \kappa T_m$$

and

$$L_1 = \max\left\{L_1^1, L_2^1, L_3^1\right\}.$$

Similarly, there exist $L_2, L_3, L_5, L_6 > 0$ such that:

$$\begin{aligned}
\left\| f_2(u) - f_2(v) \right\|_{L^2(\Omega;\mathbb{R})} &\leq L_1^2 \left\| X_1 - X_2 \right\|_{\mathbb{C}(\overline{\Omega};\mathbb{R})} + L_2^2 \left\| Y_1 - Y_2 \right\|_{\mathbb{C}(\overline{\Omega};\mathbb{R})} \\
&\quad + L_3^2 \left\| C_1 - C_2 \right\|_{\mathbb{C}(\overline{\Omega};\mathbb{R})}, \\
&\leq L_2 \sup_{s\leq T} \left\| X_1(\cdot,s) - X_2(\cdot,s) \right\|_{\mathbb{C}(\overline{\Omega};\mathbb{R})} \\
&\quad + L_2 \sup_{s\leq T} \left\| Y_1(\cdot,s) - Y_2(\cdot,s) \right\|_{\mathbb{C}(\overline{\Omega};\mathbb{R})} \\
&\quad + L_2 \sup_{s\leq T} \left\| C_1(\cdot,s) - C_2(\cdot,s) \right\|_{\mathbb{C}(\overline{\Omega};\mathbb{R})},
\end{aligned}$$

$$\begin{aligned}
\left\| f_5(u) - f_5(v) \right\|_{L^2(\Omega;\mathbb{R})} &\leq \epsilon \left\| E_1 - E_2 \right\|_{\mathbb{C}(\overline{\Omega};\mathbb{R})}, \\
&= L_5 \sup_{s\leq T} \left\| E_1(\cdot,s) - E_2(\cdot,s) \right\|_{\mathbb{C}(\overline{\Omega};\mathbb{R})},
\end{aligned}$$

$$\begin{aligned}
\left\| f_6(u) - f_6(v) \right\|_{L^p(\Omega;\mathbb{R})} &\leq \phi_2 \left\| Y_1 - Y_2 \right\|_{\mathbb{C}(\overline{\Omega};\mathbb{R})}, \\
&= L_6 \sup_{s\leq T} \left\| Y_1(\cdot,s) - Y_2(\cdot,s) \right\|_{\mathbb{C}(\overline{\Omega};\mathbb{R})},
\end{aligned}$$

$$\begin{aligned}
\left\| f_3(u) - f_3(v) \right\|_{L^2(\Omega;\mathbb{R})} &\leq L_3 \sup \left\| Y_1(\cdot,s) - Y_2(\cdot,s) \right\|_{\mathbb{C}(\overline{\Omega};\mathbb{R})} \\
&\quad + L_3 \sup \left\| C_1(\cdot,s) - C_2(\cdot,s) \right\|_{\mathbb{C}(\overline{\Omega};\mathbb{R})} \\
&\quad + L_3 \sup \left\| E_1(\cdot,s) - E_2(\cdot,s) \right\|_{\mathbb{C}(\overline{\Omega};\mathbb{R})} \\
&\quad + L_3 \sup \left\| I_1(\cdot,s) - I_2(\cdot,s) \right\|_{\mathbb{C}(\overline{\Omega};\mathbb{R})} \\
&\quad + L_3 \sup \left\| S_1(\cdot,s) - S_2(\cdot,s) \right\|_{\mathbb{C}(\overline{\Omega};\mathbb{R})}, \ \forall s \leq T.
\end{aligned}$$

Here

$$L_2 = \max\left\{L_1^2, L_2^2, L_3^2\right\}, \ L_3 = \max\left\{L_1^3, L_2^3, L_3^3, L_4^3, L_5^3\right\}, \ L_5 = \epsilon, \ L_6 = \phi_2,$$

$$L_1^2 = \beta_v T_m + \beta_e \kappa V_m + \alpha \beta_v T_m^2 + \beta_e V_m^2 \ , \ L_2^2 = \beta_v T_m, \ L_3^2 = \beta_e \kappa T_m, \ L_1^3 = 3\tau_v U_m^2, \ L_2^3 = 3\tau_e U_m^2,$$

$$L_3^3 = 4U_m(\tau_v T_m + \tau_e V_m), \ L_4^3 = U_m(\tau_v T_m + \tau_e V_m) + a, \ L_5^3 = U_m(\tau_v T_m + \tau_e V_m) + \gamma.$$

In the same manner, there exists $L_4 > 0$ such that:

$$\begin{aligned}
\left\| f_4(u) - f_4(v) \right\|_{L^2(\Omega;\mathbb{R})} &\leq L_4 \sup \left\| Y_1(\cdot,s) - Y_2(\cdot,s) \right\|_{\mathbb{C}(\overline{\Omega};\mathbb{R})} \\
&\quad + L_4 \sup \left\| C_1(\cdot,s) - C_2(\cdot,s) \right\|_{\mathbb{C}(\overline{\Omega};\mathbb{R})} \\
&\quad + L_4 \sup \left\| E_{1(\cdot,s)} - E_2(\cdot,s) \right\|_{\mathbb{C}(\overline{\Omega};\mathbb{R})} \\
&\quad + L_4 \sup \left\| I_1(\cdot,s) - I_2(\cdot,s) \right\|_{\mathbb{C}(\overline{\Omega};\mathbb{R})} \\
&\quad + L_4 \sup \left\| S_1(\cdot,s) - S_2(\cdot,s) \right\|_{\mathbb{C}(\overline{\Omega};\mathbb{R})}, \ \forall s \leq T.
\end{aligned}$$

Here,

$$L_4 = \max\left\{L_1^4, L_2^4, L_3^4, L_4^4, L_5^4\right\}, \ L_1^4 = 3\tau_v U_m^2, \ L_2^4 = 3\tau_e U_m^2, \ L_3^4 = 2U_m(\tau_v T_m + \tau_e V_m),$$

$$L_4^4 = U_m(\tau_v T_m + \tau_e V_m).$$

Finally, setting $K = \max\{L_1, L_2, L_3, L_4, L_5, L_6\}$, it follows that

$$\begin{aligned} \left\|f(u) - f(v)\right\|_2 &= \left\{\sum_{j=1}^{6} \left\|f_j(u) - f_j(v)\right\|^2_{L^2(\Omega;\mathbb{R})}\right\}^{\frac{1}{2}}, \\ &\leq K \sum_{j=1}^{6} \sup_{s\leq T} \left\|u_j(\cdot, s) - v_j(\cdot, s)\right\|_{\mathbb{C}(\overline{\Omega};\mathbb{R})}, \\ &= K \sup_{s\leq T} \|u(\cdot, s) - v(\cdot, s)\|_{\mathbb{C}(\overline{\Omega};\mathbb{R}^6)}. \end{aligned}$$

■

$A_p$ is a closed linear operator in $L^2(\Omega; \mathbb{R}^6)$, whose domain is given by

$$D(A_p) = \left\{ u = (u_1, u_2, u_3, u_4, u_5, u_6)^t \in W^{2,2}(\Omega; \mathbb{R}^6),\ \frac{\partial u_i}{\partial \eta} = 0 \ \text{ on } \ \partial\Omega\ \forall i \in \{1, 2, 3, 4, 5\} \right\}.$$

From [10], it is well known that $-A_p$ generates an analytic semi-group of bounded linear operators

$$G(t) = \left\{\exp(-tA_p)\right\}_{t\geq 0} \ \text{ on } L^2(\Omega; \mathbb{R}^6).$$

For each $0 < \alpha < 1$, we introduce the fractional power space $D(A_p^\alpha)$ equipped with the graph norm of $A_p^\alpha = -\Delta + \alpha I$

$$\|u\|_{2,\alpha} = \|u\|_2 + \left\|A_p^\alpha u\right\|_2 \quad \text{for } \ u \in D(A_p^\alpha).$$

We rewrite (2.4) in the following abstract form:

$$\begin{cases} \dfrac{du(t)}{dt} + A_p u(t) = f(u_t), \quad 0 < t < \infty \\ u(t) = \varphi(t), \quad -\infty < t \leq 0, \end{cases} \tag{2.10}$$

where $u = (u_1, u_2, \cdots, u_6,)^t$ and $u_t(\theta) = u(t + \theta)$ for $\theta \in (-\infty, 0]$.

**Lemma 2.3.** *([10]) $D(A_p^\alpha) \hookrightarrow \mathbb{C}^\mu(\overline{\Omega}; \mathbb{R}^6)$, if $\alpha > 3/4$ and $0 \leq \mu < 2\alpha - \frac{3}{2}$.*

here $\hookrightarrow$ means that the inclusion is continuous. Hence, for $3/4 < \alpha < 1$, there exists a positive number $c_\alpha$ satisfying

$$\|u\|_{\mathbb{C}(\overline{\Omega};\mathbb{R}^6)} + \sum_{i=1}^{n} \left\|\frac{\partial u}{\partial x_i}\right\|_{\mathbb{C}(\overline{\Omega};\mathbb{R}^6)} \leq c_\alpha \|u\|_{2,\alpha}, \quad \forall u \in D(A_p^\alpha). \tag{2.11}$$

**Proposition 2.4.** *[11] Assume that the initial function $\varphi$ satisfies $\varphi = (\varphi_1, \varphi_2, \cdots, \varphi_6)^t \in \mathbb{C}^\sigma\left((-\infty, T]; \mathbb{C}(\overline{\Omega}; \mathbb{R}^6)\right)$, with $0 < \sigma < 1$. Then,*

$$\sup_{t\leq 0} \left\|\varphi(t)\right\|_{\mathbb{C}(\overline{\Omega};\mathbb{R}^6)} + \sup_{t,s\leq 0\ ,\ t\neq s} \frac{\left\|\varphi(t) - \varphi(s)\right\|_{\mathbb{C}(\overline{\Omega};\mathbb{R}^6)}}{|t - s|^\sigma} < \infty. \tag{2.12}$$

**Corollary 2.5.** *([10]) Let G be the analytic semigroup generated by $-A_p$: The following properties hold for the semigroup G and the fractional power space $D(A_p^\alpha)$ :*

*(1) $G(t) : L^2(\Omega) \longrightarrow D(A_p^\alpha)\ \forall t > 0$,*
*(2) $\left\|A_p^\alpha G(t)u\right\|_2 \leq M_\alpha t^{-\alpha} e^{\nu t} \|u\|_2,\ \forall t > 0,\ \alpha \geq 0$ and $u \in L^2(\Omega; \mathbb{R}^6)$,*
*(3) $\|(G(t) - I)u\|_2 \leq \frac{1}{\alpha} M_{1-\alpha} t^\alpha \left\|A_p^\alpha u\right\|_2 \quad \forall t > 0 \quad , \ 0 < \alpha \leq 1$ and $u \in L^2(\Omega; \mathbb{R}^6)$,*
*(4) $G(t)A_p^\alpha u = A_p^\alpha G(t)u,\ \forall\ t > 0,\ u \in D(A_p^\alpha)$.*

*Here $M_\alpha$ and $\nu$ are some positive numbers.*

**Theorem 2.6.** *Assume Proposition 2.2 and $3/4 < \alpha < 1$ hold true. Then, for each $\varphi$ satisfying (2.12) and $\varphi(0) \in D(A_p^\alpha)$, there exists a positive number $T$ such that (2.10) has a unique strong solution $u$ on $(-\infty, T]$ satisfying $u \in \mathbb{C}\left([0,T]; D(A_p^\alpha)\right)$.*

**Proof.** It is easy to see that

$$u(t) = G(t)\varphi(0) + \int_0^t G(t-s)f(u_s)ds, \tag{2.13}$$

for $t \geq 0$ is a mild solution of (2.10).

Let $r$ denote a sufficiently large number satisfying $r > \left\|\varphi(0)\right\|_{2,\alpha}$ and $Q$ the complete metric space

$$Q = \left\{u \in \mathbb{C}\left([0,T]; D(A_p^\alpha)\right);\;\; u(0) = \varphi(0) \;\; and \;\; \sup_{0 \leq s \leq T} \left\|u(s) - \varphi(0)\right\|_{2,\alpha} \leq r\right\}.$$

For $u \in Q$, define $P(u) : [0,T] \to \mathbb{C}(\overline{\Omega}; \mathbb{R}^6)$ by

$$P(u)(t) = G(t)\varphi(0) + \int_0^t G(t-s)f(u_s)ds \quad for \;\; 0 \leq t \leq T.$$

We show that $P$ maps $Q$ into itself, and is a strict contraction.

By virtue of Proposition 2.2, Corollary 2.5 and (2.11), we have:

$$\begin{aligned}\left\|P(u)(t) - \varphi(0)\right\|_{2,\alpha} \;\; \leq \;\; & \frac{1}{\alpha} M_{1-\alpha} t^\alpha \left\|\varphi(0)\right\|_{2,\alpha} \\ & + e^{\nu t}\left(\frac{M_0 c_\alpha r + M_0\left\|f(u_0)\right\|_2}{\nu} + \frac{M_\alpha c_\alpha r + M_\alpha \left\|f(u_0)\right\|_2}{1-\alpha} t^{1-\alpha}\right).\end{aligned}$$

Thus, for $0 < t < T_1 < T$ such that

$$\frac{1}{\alpha} M_{1-\alpha} T_1^\alpha \left\|\varphi(0)\right\|_{2,\alpha} + e^{\nu T_1}\left(\frac{M_0 c_\alpha r + M_0\left\|f(u_0)\right\|_2}{\nu} + \frac{M_\alpha c_\alpha r + M_\alpha \left\|f(u_0)\right\|_2}{1-\alpha} T_1^{1-\alpha}\right) \leq r,$$

we conclude that $P$ maps $Q$ into itself.

Similarly, we obtain

$$\|P(u)(t) - P(v)(t)\|_{2,\alpha} \;\; \leq \;\; K c_\alpha e^{\nu t}\left\{\frac{M_0}{\nu} + \frac{M_\alpha}{1-\alpha} t^{1-\alpha}\right\} \sup_{0 \leq s \leq t} \|u(s) - v(s)\|_{2,\alpha},$$

for all $u, v \in Q$. It follows that $\|P(u)(t) - P(v)(t)\|_{2,\alpha} \leq \frac{1}{2} \sup_{0 \leq s \leq T_2} \|u(s) - v(s)\|_{2,\alpha}$ for $0 < t < T_2 < T$ such that

$$K c_\alpha e^{\nu T_2}\left\{\frac{M_0}{\nu} + \frac{M_\alpha}{1-\alpha} T_2^{1-\alpha}\right\} \leq \frac{1}{2}.$$

Therefore, $P$ is a strict contraction mapping $Q$ into itself if $T = \min\{T_1; T_2\}$ is sufficiently small. Hence, applying the fixed point Theorem shows that (2.13) has a unique solution $u \in \mathbb{C}\left((-\infty, T]; \mathbb{C}(\overline{\Omega}; \mathbb{R}^6)\right) \cap \mathbb{C}\left([0,T]; D(A_p^\alpha)\right)$.

We prove that this solution $u$ actually satisfies (2.10). It is well known (see [10]) that, if $f(u_t) : (0,T] \to L^2(\Omega; \mathbb{R}^6)$ is Hölder continuous, the function $u$ given by (2.13) is a strong solution of (2.10). Therefore, in view of Proposition 2.2 and Equation (2.12), it suffices to show the Hölder continuity of $u : [0,T] \to \mathbb{C}(\overline{\Omega}; \mathbb{R}^6)$. For this purpose, we employ the method used in [12].

Let $t, t+h \in [0,T]$ with $h > 0$. From (2.13) we have

$$\begin{aligned} u(t+h) - u(t) &= G(t+h)\varphi(0) + \int_0^{t+h} G(t+h-s) f(u_s) ds \\ & \quad -G(t)\varphi(0) + \int_0^t G(t-s) f(u_s) ds, \\ &= G(t)[G(h) - I]u_0 + \int_t^{t+h} G(t+h-s) f(u_s) ds \\ & \quad + \int_0^t G(t-s) f(u_s)[G(h)-I] ds, \\ &= I_1 + I_2 + I_3. \end{aligned}$$

For any $0 \le \beta < \alpha$, each $A_p^\beta$ will be estimated separately. we have,

$$A_p^\beta I_1 = \int_t^{t+h} A_p^\beta \tfrac{d}{ds} \exp(-sA_p)\varphi(0) ds = -\int_t^{t+h} A_p^\beta \exp(-sA_p) A_p \varphi(0) ds = -\int_t^{t+h} A_p^{1+\beta-\alpha} \exp(-sA_p) A_p^\alpha \varphi(0) ds.$$

It follows from Corollary 2.5 that if $0 < \delta < 1-\beta$ with $0 < \delta \le 1$, then:

$$\left\| A_p^\beta I_1 \right\|_2 \le M_{1+\beta-\alpha} \left\| A_p^\alpha \varphi(0) \right\|_2 e^{\nu T} \left( (t+h)^{\alpha-\beta} - t^{\alpha-\beta} \right) \le C_1 h^{\alpha-\beta}, \tag{2.14a}$$

$$\left\| A_p^\beta I_2 \right\|_2 \le M_\beta \left( K M_\alpha c_\alpha r + \left\| f(u_0) \right\|_2 \right) \int_t^{t+h} (t+h-s)^{-\beta} e^{\nu(t+h-s)} ds \le C_2 h^{1-\beta}, \tag{2.14b}$$

$$\left\| A_p^\beta I_3 \right\|_2 \le M_{\beta+\delta} \frac{1}{\delta} M_{1-\delta} h^\delta \left( K M_\alpha c_\alpha r + \left\| f(u_0) \right\|_2 \right) \int_0^t (t-s)^{-(\beta+\delta)} e^{\nu(t-s)} ds \le C_3 h^\delta. \tag{2.14c}$$

These estimates (2.14a)–(2.14c) yield the Hölder continuity of $A_p^\beta u : [0,T] \to L^2(\Omega; \mathbb{R}^6)$, with exponent $\alpha - \beta$ for any $0 \le \beta < \alpha$. This fact together with Lemma 2.3 imply that $u \in \mathbb{C}^{\alpha-\beta}\left([0,T]; \mathbb{C}(\overline{\Omega}; \mathbb{R}^6)\right)$ for $3/4 < \beta < \alpha$. Thus the proof is complete. ■

#### 2.2.2. *Positivity of solutions for the IBVP*

We rewrite the IBVP (2.4)-(2.6) in the form:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \overline{D}\Delta u + g(u)u = f(u) & \text{in } \Omega \times (0,T), \\ \dfrac{\partial u}{\partial \eta} = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,\theta) = u_{\theta i} & \text{in } \overline{\Omega} \times (-\infty, 0], \end{cases} \tag{2.15}$$

where $u = (u_1, u_2, u_3, u_4, u_5, u_6)^t = (X, Y, S, E, I, C)^t$,
$g(u) = \text{diag}(g_1, g_2, g_3, g_4, g_5, g_6)$, $f(u) = (f_1, f_2, f_3, f_4, f_5, f_6)^t, \overline{D} = \text{diag}(D_1, D_2, \cdots, D_5, 0)$,
with $g_1 = \beta_v \dfrac{u_2}{1+\alpha u_2} + \beta_e \dfrac{u_6}{u_6 + \kappa} + d$ , $g_2 = -\beta_v \dfrac{u_1}{1+\alpha u_2} + d$,
$g_3 = \delta + \dfrac{1}{u_3 + u_4 + u_5} \int_\Omega \int_{-\infty}^t G(x,y,t-s) k(t-s)(\tau_v u_2 + \tau_e u_6)(y,s) ds dy$,
$g_4 = (a + \delta + \epsilon)$, $g_5 = (\gamma + \rho + \delta)$, $g_6 = \xi$, $f_1 = (1-q)A$,
$f_2 = qA + \beta_e \dfrac{u_1 u_6}{\kappa + u_6}$ , $f_5 = \epsilon u_4$ , $f_3 = B + a u_4 + \gamma u_5$ , $f_6 = \phi_2 u_2$,
$f_4 = \dfrac{u_3}{u_3 + u_4 + u_5} \int_\Omega \int_{-\infty}^t G(x,y,t-s) k(t-s)(\tau_v u_2 + \tau_e u_6)(y,s) ds dy$.

Note that $D_i > 0$, for $i = \{1, 2, .., 5\}$. Denote $\mathcal{H} = L^2(\Omega)$ and $\mathcal{V} = H^1(\Omega)$. Following [13], define the Hilbert space

$$W(0,T,\mathcal{V},\mathcal{V}') = \left\{ u \in L^2((0,T),\mathcal{V}) \ / \ \frac{\partial u}{\partial t} \in L^2((0,T),\mathcal{V}') \right\}$$

equipped with the norm

$$\|u\|^2_{W(0,T,\mathcal{V},\mathcal{V}')} = \|u\|^2_{L^2((0,T),\mathcal{V})} + \left\|\frac{\partial u}{\partial t}\right\|^2_{L^2((0,T),\mathcal{V}')},$$

and the following hypotheses for initial conditions:

$$u_{\theta 1}, u_{\theta 2}, u_{\theta 6} \in L^\infty(\Omega) \ , \ u_{\theta i} \in \mathcal{H} \text{ for } i \in \{3,4,5\} \ , \ u_{\theta i} \geq 0 \text{ for } i \in \{1, ..., 6\}. \tag{2.16}$$

Moreover, define

$$a(u,v) = \sum_{j=1}^{n} \int_\Omega \frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_j}dx. \tag{2.17}$$

The variational parabolic problem associated to the triple $(\mathcal{H}, \mathcal{V}, a(t,\cdot,\cdot))$ , is

$$\begin{cases} \dfrac{d}{dt}(u(t),v)_{\mathcal{H}} + \overline{D}a(u(t),v) + (g(u_t)u(t),v)_{\mathcal{H}} = (f(u_t),v) \qquad \forall v \in \mathcal{V}. \\ u(\theta) = u_{\theta i}, \end{cases} \tag{2.18}$$

Given $f(u_t) \in L^2((0,T),\mathcal{V}')$ and $u_{\theta i} \in \mathcal{H}$, there exists $u \in W(0,T,\mathcal{V},\mathcal{V}')$ such that (2.18) holds, since this problem is equivalent to (2.10).

**Proposition 2.7.** *[13] For $u_0 \in \mathcal{H}$ and $f \in L^2((0,T),\mathcal{V}')$, Problem (2.18) which consists in finding $u \in W(0,T,\mathcal{V},\mathcal{V}')$ such that*

$$\frac{du}{dt} + A_p u = f, \quad \text{with} \quad u(0) = u_0, \tag{2.19}$$

*admits a unique solution given by*

$$u(t) = G(t)u_0 + \int_0^t G(t-s)f(u_s)ds. \tag{2.20}$$

We first present a positivity lemma, which can be found in any standard textbook on PDE.

**Lemma 2.8.** *[14] Let $u_i \in \mathbb{C}(\overline{\Omega} \times [0,T]) \cap \mathbb{C}^{2,1}(\Omega \times (0,T))$ be such that*

$$\begin{cases} \dfrac{\partial u_i}{\partial t} - D\Delta u_i + c_i u_i \geq 0 \qquad \text{in } \Omega \times (0,T], \\ \dfrac{\partial u_i}{\partial \eta} \geq 0 \qquad \text{on } \partial\Omega \times (0,T], \\ u_i(x,0) = u_i^0(x) \geq 0 \qquad x \in \overline{\Omega}, \end{cases} \tag{2.21}$$

*and $c_i \equiv c_i(x,t)$ is a bounded function in $\overline{\Omega} \times [0,T]$, D>0. Then $u_i(x,t) \geq 0$ in $\overline{\Omega} \times [0,T]$. Moreover $u_i(x,t) > 0$ in $\Omega \times (0,T]$ unless it is identically zero.*

As a consequence of Lemma 2.8, we have the following positivity result.

**Lemma 2.9.** *Any solution of (2.4)–(2.6) with a non negative initial function is positive.*

**Proof.** Here, one approaches the solution of (2.15) by a sequence of solutions $(u_i^n)$ of linear equations. For $n = 0$, $u_i^0$ denotes the solution of

$$\begin{cases} \dfrac{\partial u_i^0}{\partial t} - \overline{D}_i \Delta u_i^0 = 0 \qquad \text{in } \Omega \times (0,T), \\ \dfrac{\partial u_i^0}{\partial \eta} = 0 \qquad \text{on } \partial\Omega \times (0,T], \\ u_i^0(\theta) = u_{\theta i} \qquad \text{in } \overline{\Omega} \times (-\infty, 0]. \end{cases} \tag{2.22}$$

This equation admits a strong solution and $u_i^0 \geq 0$. By induction, $u_i^n$ denotes the solution of

$$\begin{cases} \dfrac{\partial u_i^n}{\partial t} - \overline{D}_i \Delta u_i^n + g_i(u^{n-1}) u_i^n = f_i(u^{n-1}) & \text{in } \Omega \times (0,T), \\ \dfrac{\partial u_i^n}{\partial \eta} = 0 \quad \text{on } \partial\Omega \times (0,T], \\ u_i^n(\theta) = u_{\theta i} \quad \text{in } \overline{\Omega} \times (-\infty, 0]. \end{cases} \tag{2.23}$$

Suppose that there exists a unique nonnegative solution $u^{n-1}$. Assuming by induction that $u_i^j \geq 0$ for $0 \leq j \leq n-1$, we have

$$0 \leq \beta_v \frac{u_2^{n-1}}{1+\alpha u_2^{n-1}} \leq \beta_v \text{ and } 0 \leq \beta_e \frac{u_6^{n-1}}{u_6^{n-1}+\kappa} \leq \beta_e,$$

which implies that

$$d \leq g_1(u^{n-1}) \leq \beta_v + \beta_e + d. \tag{2.24}$$

Note that $f_i(u^{n-1}) \geq 0$ for all $i$. Since $g_4, g_5$ and $g_6$ are constants , we have $g_i(u^{n-1}) \in L^\infty(\Omega \times (0,T))$ for $i \in \{1,4,5,6\}$. It remains to show that $g_i(u^{n-1}) \in L^\infty(\Omega \times (0,T))$ for $i \in \{2,3\}$.

For this, we need to prove that $u_i^n \in L^\infty((0,T); L^\infty(\Omega))$, for $i \in \{1,2,6\}$.

- Case of $u_i^0$

  Let $k \in \mathbb{N}^*$. We multiply the first equality in (2.22) by $(u_i^0)^{2k-1}$, integrate over $\Omega$ and use Green formula, to get

$$\frac{1}{2k}\frac{d}{dt}\int_\Omega (u_i^0)^{2k} dx + D_i(2k-1)\int_\Omega (u_i^0)^{2k-2} |\nabla u_i^0|^2 dx - D_i \int_{\partial\Omega} \frac{\partial u_i^0}{\partial \eta} u_i^0 d\eta = 0. \tag{2.25}$$

  Then we have

$$\frac{1}{2k}\frac{d}{dt}\int_\Omega (u_i^0)^{2k} dx \leq 0. \tag{2.26}$$

  By integrating over $(\theta, t)$, we obtain

$$\|u_i^0(t)\|_{L^{2k}(\Omega)} \leq \|u_i^0(\theta)\|_{L^{2k}(\Omega)}. \tag{2.27}$$

  When $k$ tends to $\infty$, we obtain,

$$\|u_i^0(t)\|_{L^\infty(\Omega)} \leq \|u_{\theta i}\|_{L^\infty(\Omega)}. \tag{2.28}$$

  This implies that $u_i^0 \in L^\infty((0,T); L^\infty(\Omega))$.

- Case of $u_i^n$ with $n \in \mathbb{N}^*$

  By induction, we suppose that $u_i^0, u_i^1, \cdots, u_i^{n-1} \in L^\infty((0,T); L^\infty(\Omega))$.

  For $i \in \{1,6\}$ we multiply the first equality in (2.23) by $(u_i^n)^{2k-1}$, integrate over $\Omega$ and use Green formula, to have

$$\begin{aligned} \frac{1}{2k}\frac{d}{dt}\int_\Omega (u_i^n)^{2k} dx \quad + \quad & D_i(2k-1)\int_\Omega (u_i^n)^{2k-2}|\nabla u_i^n|^2 dx \\ & + \int_\Omega g_i(u^{n-1})(u_i^n)^{2k} dx \\ & = \int_\Omega f_i(u^{n-1})(u_i^n)^{2k-1} dx. \end{aligned}$$

  Then we have

$$\frac{1}{2k}\frac{d}{dt}\int_\Omega (u_i^n)^{2k} dx \leq 0. \tag{2.29}$$

  By integrating over $(\theta, t)$, we obtain

$$\|u_i^n(t)\|_{L^{2k}(\Omega)} \leq \|u_i^0(\theta)\|_{L^{2k}(\Omega)}. \tag{2.30}$$

When $k$ tends to $\infty$, we get,

$$\|u_i^n(t)\|_{L^\infty(\Omega)} \le \|u_{\theta i}\|_{L^\infty(\Omega)}. \tag{2.31}$$

This implies that $u_i^n \in L^\infty((0,T);L^\infty(\Omega))$.

**Remark 2.10.** *Since the function $g_2(u^{n-1})$ is undervalued, we make the change $w_2^n = e^{-\lambda t}u_2^n$, to obtain:*

$$\frac{\partial w_2^n}{\partial t} - D_2\Delta w_2^n + (\lambda + g_2(e^{\lambda t}w^{n-1})w_2^n = f_i(e^{\lambda t}w^{n-1})e^{-\lambda t}. \tag{2.32}$$

*We can choose $\lambda \ge 0$ such that*

$$\lambda + g_2(e^{\lambda t}w^{n-1}) \ge 0.$$

*Doing the same manipulation as before, we obtain*

$$\|w_i^n(t)\|_{L^\infty(\Omega)} \le \|w_{\theta i}\|_{L^\infty(\Omega)} \le \|u_{\theta i}\|_{L^\infty(\Omega)}. \tag{2.33}$$

*As a result, we obtain that $w_2^n \in L^\infty((0,T);L^\infty(\Omega))$ and since $u_2^n = e^{\lambda t}w_2^n$, we have $u_2^n \in L^\infty((0,T);L^\infty(\Omega))$.*

As $u_i^n \in L^\infty((0,T);L^\infty(\Omega))$ for $i \in \{1,2,6\}$ and $\forall n \in \mathbb{N}$ we have

$$d - \beta_v T_m \le g_2(u^{n-1}) \le d \ \ and \ \ \delta \le g_3(u^{n-1}) \le \tau_v V_m + \tau_e U_m + \delta, \tag{2.34}$$

since $\int_\Omega \int_{-\infty}^t G(x,y,t-s)k(t-s)dsdy = 1$.

*Conclusion 1.* It then follows that $g_i(u^{n-1}) \in L^\infty(\Omega \times (0,T))$ for all $i$. Thus, by Lemma 2.8, $u_i^n \ge 0$.

Let us show that the sequence is bounded. From (2.18), we have

$$\frac{\partial}{\partial t}\left(u_i^n, v\right)_{\mathcal{H}} + D_i a\left(u_i^n, v\right) + \left(g_i(u^{n-1})u_i^n, v\right)_{\mathcal{H}} = \left\langle f_i(u^{n-1}), v\right\rangle \qquad \forall v \in \mathcal{V}. \tag{2.35}$$

Since

$$\frac{\partial}{\partial t}\left(u_i^n, v\right)_{\mathcal{H}} = \left\langle \frac{\partial u_i^n}{\partial t}, v\right\rangle, \tag{2.36}$$

by density and choosing $v = u_i^n$, we have

$$\left\langle \frac{\partial u_i^n}{\partial t}, u_i^n\right\rangle = \frac{1}{2}\frac{d}{dt}\left(u_i^n(t), u_i^n(t)\right)_{\mathcal{H}} = \frac{1}{2}\frac{d}{dt}\|u_i^n(t)\|_{\mathcal{H}}^2. \tag{2.37}$$

Hence,

$$\frac{1}{2}\frac{d}{dt}\|u_i^n\|_{\mathcal{H}}^2 + D_i a\left(u_i^n, u_i^n\right) + \left(g_i(u^{n-1})u_i^n, u_i^n\right)_{\mathcal{H}} = \left\langle f_i(u^{n-1}), u_i^n\right\rangle. \tag{2.38}$$

For $i \in \{1,3,4,5,6\}$, the form $D_i a$ is $\mathbf{V}$-coercive that is, there exists $\alpha > 0$ such that $D_i a(u,u) \ge \alpha\|u\|_{\mathcal{V}}^2$ for all v in $\mathcal{V}$. Moreover $g_i$ are bounded, that is there exists $l_1, l_2 > 0$ such that $l_1 \le g_i(u) \le l_2$, for all $u \ge 0$. Therefore,

$$\frac{1}{2}\frac{d}{dt}\|u_i^n\|_{\mathcal{H}}^2 + \alpha\|u_i^n\|_{\mathcal{V}}^2 + l_1\|u_i^n\|_{\mathcal{H}}^2 \le \|f_i(u^{n-1})\|_{\mathcal{V}'}\|u_i^n\|_{\mathcal{V}}. \tag{2.39}$$

Then by the Young inequality, we have

$$\frac{1}{2}\frac{d}{dt}\|u_i^n\|_{\mathcal{H}}^2 + \alpha\|u_i^n\|_{\mathcal{V}}^2 + l_1\|u_i^n\|_{\mathcal{H}}^2 \le \frac{1}{2\epsilon_1}\|f_i(u^{n-1})\|_{\mathcal{V}'}^2 + \frac{\epsilon_1}{2}\|u_i^n\|_{\mathcal{V}}^2. \tag{2.40}$$

We take $\epsilon_1$ small enough such that $\alpha - (\epsilon_1/2) = \epsilon_2$.

Hence

$$\frac{1}{2}\frac{d}{dt}\|u_i^n\|_{\mathcal{H}}^2 + \epsilon_2\|u_i^n\|_{\mathcal{V}}^2 + l_1\|u_i^n\|_{\mathcal{H}}^2 \leq \frac{1}{2\epsilon_1}\|f_i(u^{n-1})\|_{\mathcal{V}'}^2. \tag{2.41}$$

Therefore by integration, one has

$$\tfrac{1}{2}\|u_i^n(t)\|_{\mathcal{H}}^2 + \epsilon_2\int_\theta^t \|u_i^n(s)\|_{\mathcal{V}}^2 ds + l_1\int_\theta^t \|u_i^n(s)\|_{\mathcal{H}}^2 ds \quad \leq \quad \frac{1}{2\epsilon_1}\int_\theta^t \|f_i(u^{n-1})\|_{\mathcal{V}'}^2 ds + \tfrac{1}{2}\|u_i^n(\theta)\|_{\mathcal{H}}^2. \tag{2.42}$$

**Remark 2.11.** *For i=2, we make the following change of variable $w_2^n = e^{-\lambda t}u_2^n$ where we can take $\lambda = \beta_1 + \beta_2$. Taking into account the fact that $g_2$ is bounded and that the form $D_i a$ is $\mathcal{H}$-coercive, we have the same result as* (2.42).

As $f_1(u) = (1-q)A$, we deduce that $(u_1^n)$ remains bounded in $\mathbb{C}^0([0;T],\mathcal{H})$ and $L^2((0;T),\mathcal{V})$. As $f_2(u) = qA + \beta_e\dfrac{u_1u_6}{\kappa + u_6}$, we get $f_2(u^{n-1}) \leq qA + \beta_e u_1^{n-1}$, which remains bounded in $L^2((0;T),\mathcal{V})$. Therefore, $u_2^n$ has the same property as $u_1^n$. The same result holds for $u_6^n$, because $f_6(u^{n-1}) = \phi_2 u_2^{n-1}$.
We have $f_4(u) = \dfrac{u_3}{u_3+u_4+u_5}\int_\Omega\int_{-\infty}^t G(x,y,t-s)k(t-s)(\tau_v u_2 + \tau_e u_6)(y,s)dsdy$. Therefore, $f_4(u^{n-1}) \leq \int_\Omega\int_{-\infty}^t G(x,y,t-s)k(t-s)(\tau_v u_2^{n-1} + \tau_e u_6^{n-1})(y,s)dsdy$, which remains bounded in $L^2((0;T),\mathcal{V})$. A similar result holds for $u_5^n$, because $f_5(u^{n-1}) = \epsilon u_4^{n-1}$. Since $f_3 = B + au_4 + \gamma u_5$, we have the same conclusion for $u_3^n$.

Now, we deduce that for the positive bounded sequence $\left(u_i^n\right)_{n\geq 0}$ one can extract subsequence $\left(u_i^m\right)_{m\geq 0}$ which converges uniformly for almost all $t$ by some compact operator in $\mathbb{C}^0([0;T],\mathcal{H})$ to $u_i$. Applying Proposition 2.7, for all $n$ it holds that

$$u_i^n(t) = \int_0^t G_i(t-s)q_i^n(s)ds + G_i(t)u_{\theta i}, \tag{2.43}$$

where $G_i(t)$ is the semigroup generated by the unbounded operator $-\overline{D}_i A_p$. Let us denote

$$q_i^n(s) = -g_i(u^{n-1}(s))u_i^n(s) + f_i(u^{n-1}(s)). \tag{2.44}$$

We deduce that $q_i^n \in L^2((0;T),\mathcal{V})$.

Moreover, the sequence $\left(u_i^n\right)_{n\geq 0}$ is bounded in $\mathbb{C}^0([0;T],\mathcal{H})$, which implies that the sequence $\left(q_i^n\right)_{n\geq 0}$ is bounded in $\mathbb{C}^0([0;T],\mathcal{H})$ for all $i$.

Then, we can conclude by showing that operator $\mathcal{G}_i$ which maps $\mathbb{C}^0([0;T],\mathcal{H})$ into $\mathbb{C}^0([0;T],\mathcal{H})$ and given by

$$\mathcal{G}_i^n(f) = \int_0^t G_i(t-s)f(s)ds, \tag{2.45}$$

is compact.

Considering the triple $\left(L^2(\Omega), H^1(\Omega), a\right)$, the unbounded variational operator $A_p$ associated to $a$ is a positive symmetric operator with compact resolvent. It admits a sequence $(\lambda_k)_{k\geq 0}$ of positive eigenvalues with $\lim_{k\to+\infty}\lambda_k = \infty$ and a Hilbert basis $(e_k)_{k\geq 0}$ of $\mathcal{H}$ consisting of eigenvectors of $A_p$. Since $(G(t))_{t>0}$ is the semigroup generated by $-A_p$, then for all $u_0 \in \mathcal{H}$,

$$G_i(t)u_0 = \sum_{k=0}^{+\infty} e^{-tD_i\lambda_k}(u_0, e_k)e_k, \tag{2.46}$$

which proves that the operator is compact for all $t > 0$, because

$$\lim_{k\to+\infty} e^{-tD_i\lambda_k} = 0. \tag{2.47}$$

Setting

$$G_N(t)u = \sum_{k=0}^{N} e^{-tD\lambda_k}(u, e_k)e_k, \tag{2.48}$$

one sees that $G_N(t)$ is an operator with finite rank which converges to $G(t)$. The following Theorem is relevant in the sequel.

**Theorem 2.12.** *[13] Let $t \to G(t)$ be an application from $[0, +\infty[$ into $\mathcal{L}(\mathcal{H})$. One assumes that there exists a sequence of operators $(G_N(t))_{N\geq 0}$ of $\mathcal{H}$ with the following properties:*

*(1) : for all $N$ and all $t > 0$, $G_N(t)$ is of finite rank and independent of $t$,*

*(2) : $t \to G_N(t)$, is continuous from $[0, +\infty)$ into $\mathcal{L}(\mathcal{H})$ for all $N$,*

*(3) : for $N \to \infty$, $G_N(t)$ converges to $G(t)$ in $L^1(]0, T[, \mathcal{L}(\mathcal{H}))$ for all $T > 0$.*

*Then the operator $\mathcal{G}$ is compact from $\mathbb{C}^0([0; T], \mathcal{H})$ into $\mathbb{C}^0([0; T], \mathcal{H})$ for all $T > 0$.*

From Theorem 2.12 since $\mathcal{G}_i$ is compact for all $i$, we have

$$u_i^n(t) = G_i(t)u_i^0 + \mathcal{G}_i(q_i^n)(t). \tag{2.49}$$

Then $(u_i^n)_n \geq 0$ belong to a relatively compact set of $\mathbb{C}^0([0; T], \mathcal{H})$. Therefore from $(u_i^n)_{n\geq 0}$ we can extract a subsequence $(u_i^m)_{m\geq 0}$ which converges uniformly to $u_i \in \mathbb{C}^0([0; T], \mathcal{H})$ for each $i$.

*Conclusion 2.*

$$u_i^m \longrightarrow u_i \text{ in } \mathbb{C}^0([0; T], \mathcal{H}). \tag{2.50}$$

Thus, combining *Conclusion 1.* and *Conclusion 2.* yield $u_i \geq 0$ and $u_i(\theta) = u_{\theta i}$. ■

### 2.2.3. ***Boundedness of the solutions for IBVP***

**Lemma 2.13.** *Let $u(x, t)$ satisfy*

$$\begin{cases} \dfrac{\partial u}{\partial t} - D\Delta u = f(u, x, t), & \text{in } \Omega \times (0, \infty), \\ u\dfrac{\partial u}{\partial \eta} \leq 0, & \text{on } \partial\Omega \times (0, \infty), \\ u(x, 0) = u^0(x), & x \in \overline{\Omega}. \end{cases} \tag{2.51}$$

*where $D > 0$ and $\left\| f(u, x, t) \right\| \leq K\|u\|$. If there exists $p$ with $1 \leq p < \infty$ such that $\|u(x, t)\|_{L^p(\Omega)}$ is uniformly bounded for $t \geq 0$, then $\|u(x, t)\|_{L^q(\Omega)}$ is uniformly bounded for $t \geq 0$, where $q = p \times 2^N, N = 1, 2, \ldots$. In particular $\|u(x, t)\|_{L^\infty(\Omega)}$ is uniformly bounded for $t \geq 0$.*

The following result shows that the solution of (2.4)-(2.6) is uniformly bounded, and global in time.

**Theorem 2.14.** *Let $(X, Y, S, E, I, C) \in \left[\mathbb{C}(\overline{\Omega} \times [0, T)) \cap \mathbb{C}^{2,1}(\Omega \times (0, T))\right]^6$ be the solution of problem (2.4)-(2.6) with non-negative non-trivial initial value. Then $T = \infty$ and there exist $M_2$, $M_3$ and $M_4$ such that:*

$$0 < X + Y \leq M_2,\ 0 < S + E + I \leq M_3 \text{ and } 0 \leq C \leq M_4\ ,\ (x, t) \in \Omega \times (0, \infty).$$

**Proof.** Clearly, we have

$$\frac{\partial(X+Y)}{\partial t} - \Delta(D_1X + D_2Y) = A - d(X+Y), \tag{2.52a}$$

$$\frac{\partial(S+E+I)}{\partial t} - \Delta(D_3S + D_4E + D_5I) = B - \delta(S+E+I) - \rho I \leq B - \delta(S+E+I). \tag{2.52b}$$

Integrating (2.52a) and (2.52b) over $\Omega$ yields

$$\frac{d}{dt}\int_\Omega (X+Y)dx = A|\Omega| - d\int_\Omega (X+Y)dx, \tag{2.53a}$$

$$\frac{d}{dt}\int_\Omega (S+E+I)dx \leq B|\Omega| - \delta\int_\Omega (S+E+I)dx. \tag{2.53b}$$

Applying Gronwall inequality yields

$$\begin{aligned} \|X+Y\|_{L^1(\Omega)} &= \frac{A|\Omega|}{d}(1-e^{-dt}) + \sup_{\theta\leq 0}\|\varphi_1(\cdot)+\varphi_5(\cdot,\theta)\|_{L^1(\Omega)}e^{-dt}, \\ &\leq \max\left\{\sup_{\theta\leq 0}\|\varphi_1(\cdot)+\varphi_5(\cdot,\theta)\|_{L^1(\Omega)}, \frac{A|\Omega|}{d}\right\}, \end{aligned} \tag{2.54a}$$

$$\begin{aligned} \|S+E+I\|_{L^1(\Omega)} &\leq \frac{B|\Omega|}{\delta} + \left(\|\varphi_2(x)+\varphi_3(x)+\varphi_4(x)\|_{L^1(\Omega)} - \frac{B|\Omega|}{\delta}\right)e^{-\delta t}, \\ &\leq \max\left\{\|\varphi_2(x)+\varphi_3(x)+\varphi_4(x)\|_{L^1(\Omega)}, \frac{B|\Omega|}{\delta}\right\}. \end{aligned} \tag{2.54b}$$

According to Lemma 2.13, we obtain the uniform bounds of $X, Y, S, E$ and $I$.

Knowing from (2.54a) that $Y$ is bounded, we have

$$\frac{\partial C}{\partial t} = \phi_2 Y - \xi C \Rightarrow \frac{\partial C}{\partial t} \leq \frac{A\phi_2|\Omega|}{d} - \xi C.$$

By the comparison principle

$$C(x,t) \leq \frac{A\phi_2|\Omega|}{d\xi} + \left(\sup_{\theta\leq 0}\varphi_6(\cdot,\theta) - \frac{A\phi_2|\Omega|}{d\xi}\right)e^{-\xi t} \leq \max\left\{\sup_{\theta\leq 0}\varphi_6(\cdot,\theta), \frac{A\phi_2|\Omega|}{d\xi}\right\}. \tag{2.55}$$

The proof is completed. ■

Moreover, from the above results, we conclude that the solution of IBVP (2.4)–(2.6) enters and stays in the region.

$$\Sigma = \left\{(X,Y,S,E,I,C) \in (\Omega\times\mathbb{R}_+)^6 : 0 < X+Y \leq M_2\ ,\ 0 < S+E+I \leq M_3\ ,\ 0 \leq C \leq M_4\right\},$$

where

$$M_2 = \max\left\{\sup_{\theta\leq 0}\|\varphi_1(\cdot)+\varphi_5(\cdot,\theta)\|_{L^\infty(\Omega)}, \frac{A|\Omega|}{d}\right\},$$

$$M_3 = \max\left\{\left\|\sum_{k=2}^{4}\varphi_k(x)\right\|_{L^\infty(\Omega)}, \frac{B|\Omega|}{\delta}\right\},$$

$$M_4 = \max\left\{\sup_{\theta\leq 0}\varphi_6(\cdot,\theta), \frac{A\phi_2|\Omega|}{d\xi}\right\}.$$

Hence the region $\Sigma$ of biological interest, is positively-invariant under the flow induced by IBVP (2.4)–(2.6).

## 3. Asymptotic analysis of the poultry system (when q=0)

We start by studying the poultry sub-system as it decouples from the human sub-system. It is given by:

$$\begin{cases} \dfrac{\partial X}{\partial t} - D_1\Delta X = A - \beta_v X\dfrac{Y}{1+\alpha Y} - \beta_e X\dfrac{C}{C+\kappa} - dX, \\ \dfrac{\partial Y}{\partial t} - D_2\Delta Y = \beta_v X\dfrac{Y}{1+\alpha Y} + \beta_e X\dfrac{C}{C+\kappa} - dY, \\ \dfrac{\partial C}{\partial t} = \phi_2 Y - \xi C, \\ \dfrac{\partial X}{\partial \eta} = \dfrac{\partial Y}{\partial \eta} = 0, \\ X(x,0) = \varphi_1(x),\ Y(x,\theta) = \varphi_5(x,\theta),\ C(x,\theta) = \varphi_6(x,\theta). \end{cases} \tag{3.1}$$

Since the disease starts in poultry population, the basic reproduction number of the full model (2.4) can be computed by using the poultry sub-system (3.1). By letting the densities of the diseased compartments $Y$ and $C$ be zero, we get $P^0 = \left(\dfrac{A}{d}, 0, 0\right)$ as the disease-free equilibrium of (3.1).

Let $\mathbf{X} := \mathbb{C}(\overline{\Omega}, \mathbb{R}^3)$ be the Banach space, with the usual supremum form $\|.\|_{\mathbf{X}}$. Define $\mathbf{X}^+ = \mathbb{C}(\overline{\Omega}, \mathbb{R}^3_+)$. Then $(\mathbf{X}, \mathbf{X}^+)$ is a strongly ordered space. Assume that $T_1(t), T_2(t), T_3(t) : \mathbb{C}(\overline{\Omega}, \mathbb{R}) \to \mathbb{C}(\overline{\Omega}, \mathbb{R})$ are the $C_0$ semigroups associated with $D_1\Delta - d, D_2\Delta - d$ and $0\times\Delta - \xi$ subject to the Neumann boundary condition, respectively. It follows that for any $\varphi \in \mathbb{C}(\overline{\Omega}, \mathbb{R}), t \geq 0$, one has

$$\begin{aligned} T_1(t)\varphi(x) &= e^{-dt}\int_\Omega \Gamma_1(x,y,t)\varphi(y)dy, \\ T_2(t)\varphi(x) &= e^{-dt}\int_\Omega \Gamma_2(x,y,t)\varphi(y)dy, \\ T_3(t)\varphi(x) &= e^{-\xi t}\varphi(x), \end{aligned}$$

where $\Gamma_1$ and $\Gamma_2$ are the Green functions associated with $D_1\Delta - d, D_2\Delta - d$ subject to the Neumann boundary condition, respectively. It follows from [15, Section 7.1 and Corollary 7.2.3] that $T_i(t) : \mathbb{C}(\overline{\Omega}, \mathbb{R}) \to \mathbb{C}(\overline{\Omega}, \mathbb{R}) \quad (i = 1, 2,\ t > 0)$ is compact and strongly positive. Linearizing (3.1) at the disease-free equilibrium $P^0$, we obtain:

$$\begin{cases} \dfrac{\partial \omega_1}{\partial t} = -\dfrac{\beta_v A}{d}\omega_2 - \dfrac{\beta_e A}{d\kappa}\omega_6 - d\omega_1 + D_1\Delta\omega_1, \\ \dfrac{\partial \omega_2}{\partial t} = \left(\frac{\beta_v A}{d} - d\right)\omega_2 + \dfrac{\beta_e A}{d\kappa}\omega_6 + D_2\Delta\omega_2, \\ \dfrac{\partial \omega_6}{\partial t} = \phi_2\omega_2 - \xi\omega_6, \end{cases} \tag{3.2}$$

subject to the boundary conditions

$$\frac{\partial \omega_1}{\partial \eta} = \frac{\partial \omega_2}{\partial \eta} = 0, \quad \forall x \in \partial\Omega, t > 0,$$

and initial conditions

$$\omega_1 = \varphi_1(x,0),\ \omega_2 = \varphi_5(x,\theta) \text{ and } \omega_6 = \varphi_6(x,\theta), \quad \forall (x,\theta) \in \overline{\Omega}\times(-\infty, 0).$$

We can observe that the equations for $\omega_2$ and $\omega_6$, corresponding to the infectious compartments, are decoupled from $\omega_1$. These two equations form the following cooperative system,

$$\begin{cases} \dfrac{\partial \omega_2}{\partial t} = \left(\frac{\beta_v A}{d} - d\right)\omega_2 + \dfrac{\beta_e A}{d\kappa}\omega_6 + D_2\Delta\omega_2, \\ \dfrac{\partial \omega_6}{\partial t} = \phi_2\omega_2 - \xi\omega_6, \end{cases} \tag{3.3}$$

supplemented by initial conditions and the boundary condition $\dfrac{\partial \omega_2}{\partial \eta} = 0,\ \forall x \in \partial\Omega, t > 0$. For every initial value $\varphi = (\varphi_1; \varphi_2) \in \mathbf{X}$; the solution semiflows $\Pi_t : \mathbf{X} \to \mathbf{X}$ associated with the linear system (3.3) is defined by

$$\Pi_t(\varphi) = (\omega_2(., t, \varphi), \omega_6(., t, \varphi)).$$

$\Pi_t$ is obviously a positive $C_0$-semigroup on $\mathbb{C}(\overline{\Omega}, \mathbb{R}^3)$ generated by

$$\mathcal{B} = \begin{pmatrix} D_2\Delta - d & 0 \\ \phi_2 & -\xi \end{pmatrix}.$$

Setting $\omega_2(x,t) = e^{\lambda_0 t}\varphi_1(x), \omega_6(x,t) = e^{\lambda_0 t}\varphi_2(x)$, with $\varphi = (\varphi_1, \varphi_2) \in \mathbf{X} \times \mathbf{X}$ and substituting them into the equations for $\omega_2$ and $\omega_6$, we obtain the following eigenvalue problem

$$\begin{cases} \lambda_0\varphi_1(x) = \left(\dfrac{\beta_v A}{d} - d\right)\varphi_1(x) + \dfrac{\beta_e A}{d\kappa}\varphi_2(x) + D_2\Delta\varphi_1(x), \\ \lambda_0\varphi_2(x) = \phi_2\varphi_1(x) - \xi\varphi_2(x), \\ \dfrac{\partial\varphi_1(x)}{\partial\eta} = 0,\ \forall x \in \partial\Omega, t > 0. \end{cases} \tag{3.4}$$

The result below about the existence of the principal eigenvalue of (3.4) follows from [16, Lemma 2.7].

**Lemma 3.1.** *[16]. Suppose $s(\mathcal{B})$ is the spectral bound of $\mathcal{B}$. Since all the parameters are constant, then $\lambda_{\frac{A}{d}} = s(\mathcal{B})$ is the principal eigenvalue of the eigenvalue problem (3.4) which has a strongly positive eigenfunction.*

This means that $\lambda_{\frac{A}{d}}$ is a real eigenvalue with algebraic multiplicity one, and $\mathcal{R}_e(\lambda) < \lambda_{\frac{A}{d}}$ for any other eigenvalue $\lambda$ of (3.4). Furthermore, $\lambda_{\frac{A}{d}}$ has a corresponding eigenvector $\varphi_0(x) = (\varphi_{01}, \varphi_{02})$ satisfying $\varphi_0(x) \gg 0$, and any other nonnegative eigenvector of (3.4) is a positive multiple of $\varphi_0(x)$.

In the paper by Wang and Zhao [17], the concept of the basic reproduction number is extended to reaction-diffusion epidemic systems with Neumann boundary conditions. Based on the theory of principle eigenvalues, they defined the basic reproduction number $\mathcal{R}_0$ for a reaction-diffusion epidemic model as the spectral radius of the "next generator" operator $\mathbb{L}$ defined by

$$\mathbb{L}(\varphi(x)) = \int_0^\infty F(x)T(t)\varphi dt = F(x)\int_0^\infty T(t)\varphi dt. \tag{3.5}$$

Consequently, they showed that if $\mathcal{B} = \nabla \cdot (d_I\nabla) - V_T$ then

$$\int_0^\infty T(t)\varphi dt = -\mathcal{B}^{-1}\varphi, \tag{3.6}$$

and the next generation operator is

$$\mathbb{L} = -F\mathcal{B}^{-1}. \tag{3.7}$$

In (3.6) and (3.7), $F$ is the matrix characterizing the generation of secondary infectious cases/agents, and $V_T$ is the matrix of transition rates between compartments. Both are analogues to the next-generation matrices associated with the corresponding ODE system (i.e. without diffusion terms). $T(t) = (T_2(t); T_3(t))$ is the solution semigroup for the linearized reaction-diffusion system; it denotes the distribution of the initial infection, and $d_I = \mathrm{diag}[D_2, 0]$ is the diffusion matrix.

Following [17], the basic reproduction number of PDE system (2.4)–(2.6) is defined by

$$\mathcal{R}_0 = \rho(\mathbb{L}), \tag{3.8}$$

where

$$F = \begin{bmatrix} \frac{\beta_v A}{d} & \frac{\beta_e A}{\kappa d} \\ 0 & 0 \end{bmatrix}, \quad V_T = \begin{bmatrix} d & 0 \\ -\phi_2 & \xi \end{bmatrix},$$

and

$$\mathcal{B} = \begin{pmatrix} D_2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) - d & 0 \\ \phi_2 & -\xi \end{pmatrix}.$$

Since all parameters are spatially homogeneous, we can actually find an explicit formula for the basic reproduction number $\mathcal{R}_0$. Indeed, applying [17, Theorem 3.4], we obtain the following result.

**Theorem 3.2.** *Suppose that $D_2$ is a positive constant. Then one has*

$$\mathcal{R}_0 = \frac{\beta_v A}{d^2} + \frac{\beta_e A \phi_2}{\kappa \xi d^2}. \tag{3.9}$$

### 3.1. *Existence of equilibrium points*

In this section, we investigate the existence of constant endemic equilibria of PDE poultry system (3.1). For this purpose, let $P^* = (X^*, Y^*, C^*)$ be an endemic steady state of system (3.1), then it is straightforward that

$$\begin{cases} \beta_v(\frac{A}{d} - Y^*)\frac{Y^*}{1+\alpha Y^*} + \beta_e(\frac{A}{d} - Y^*)\frac{C^*}{C^* + \kappa} - dY^* = 0, \\ \phi_2 Y^* - \xi C^* = 0, \\ X^* + Y^* = \frac{A}{d}. \end{cases} \tag{3.10}$$

System (3.10) yields

$$X^* = \frac{A}{d} - Y^*, \quad C^* = \frac{\phi_2}{\xi} Y^*, \tag{3.11}$$

and $Y^*$ is a positive root of the following quadratic polynomial:

$$Q(Y^*) = \alpha_4 Y^{*2} + \alpha_5 Y^* + \alpha_6, \tag{3.12}$$

whose coefficients are given by

$$\alpha_4 = -\frac{\beta_v \phi_2}{\xi} - \frac{\beta_e \alpha \phi_2}{\xi} - \frac{d\alpha\phi_2}{\xi}, \tag{3.13a}$$

$$\alpha_5 = -\kappa\beta_v - \frac{\beta_e\phi_2}{\xi} - \left(d\alpha\kappa + \frac{d\phi_2}{\xi}\right)(1 - \mathcal{R}_0) - \frac{\alpha\kappa\beta_v A}{d} - \frac{\beta_e A \phi_2^2}{\kappa d \xi^2}, \tag{3.13b}$$

$$\alpha_6 = \kappa d(\mathcal{R}_0 - 1). \tag{3.13c}$$

Investigating the signs of $\alpha_4, \alpha_5$ and $\alpha_6$ lead to the following straightforward result.

**Proposition 3.3.** *The model (3.1) has:*

1. *a unique positive endemic equilibrium whenever $\mathcal{R}_0 > 1$,*
2. *no positive endemic equilibrium whenever $\mathcal{R}_0 \leq 1$.*

3.2. ***Local stability of the equilibrium points***

As in references [18], let $0 = \mu_0 < \mu_i < \mu_{i+1}, i = 1, 2, \cdots$ denote the eigenvalues of $-\Delta$ on $\Omega$ with homogeneous Neumann boundary condition, $E(\mu_i)$ the space of eigenfunctions corresponding to $\mu_i$ and $\left\{\Phi_{ij} : j = 1, 2, \cdots, \dim E(\mu_i)\right\}$ an orthonormal basis of $E(\mu_i)$. Then $\mathbb{X} = [C(\overline{\Omega})]^3$ can be decomposed as

$$\mathbb{X} = \bigoplus_{i=1}^{\infty} \mathbb{X}_i \quad , \quad \mathbb{X}_i = \bigoplus_{i=1}^{\dim E(\mu_i)} \mathbb{X}_{ij}, \text{ where } \mathbb{X}_{ij} = \{c\Phi_{ij} : c \in \mathbb{R}^3\}.$$

**Theorem 3.4.** *The disease-free equilibrium $P^0$ of the poultry system (3.1) is locally asymptotically stable whenever $\mathcal{R}_0 < 1$, but unstable when $\mathcal{R}_0 > 1$.*

**Proof.** The linearization of system (3.1) at $P^0$ gives

$$\frac{\partial Z(x,t)}{\partial t} = \overline{D}\Delta Z(x,t) + \mathcal{A}Z(x,t), \tag{3.14}$$

where $\overline{D} = \text{diag}(D_1, D_2, 0)$ and

$$\mathcal{A} = \begin{pmatrix} -d & -\beta_v \dfrac{A}{d} & -\beta_e \dfrac{A}{\kappa d} \\ 0 & \beta_v \dfrac{A}{d} - d & \beta_e \dfrac{A}{\kappa d} \\ 0 & \phi_2 & -\xi \end{pmatrix}.$$

For each $i \geq 1$, $\mathbb{X}_i$ is invariant under the operator $\mathcal{L}$ and $\lambda$ is an eigenvalue of $\mathcal{L}$ if and only if it is an eigenvalue of the matrix $-\mu_i\overline{D} + \mathcal{A}$ for $i \geq 1$; in which case, there is an eigenvector in $\mathbb{X}_i$.

The characteristic equation of $-\mu_i\overline{D} + \mathcal{A}$ at $P^0$ is

$$(-\mu_i D_1 - d - \lambda)\left\{\lambda^2 + \lambda(\mu_i D_2 + \xi + d - \frac{\beta_v A}{d}) + \mu_i D_2 \xi + d\xi - \frac{\beta_v A \xi}{d} - \frac{\beta_e A \phi_2}{\kappa d}\right\} = 0. \tag{3.15}$$

It is obvious that (3.15) has an eigenvalue

$$\lambda_1 = -\mu_i D_1 - d < 0,$$

and the other two eigenvalues $\lambda_2$ and $\lambda_3$ solve the following equation

$$\lambda^2 + \lambda(\mu_i D_2 + \xi + d - \frac{\beta_v A}{d}) + \mu_i D_2 \xi + d\xi - \frac{\beta_v A \xi}{d} - \frac{\beta_e A \phi_2}{\kappa d} = 0.$$

It is easy to see that

$$\begin{aligned} \lambda_2 + \lambda_3 &= -\mu_i D_2 \xi - \xi - d + \frac{\beta_v A}{d} = -\mu_i D_2 \xi - \xi - \frac{\beta_e A \phi_2}{\kappa d^2 \xi} + d(\mathcal{R}_0 - 1), \\ \lambda_2 \times \lambda_3 &= \mu_i D_2 \xi + d\xi - \frac{\beta_v A \xi}{d} - \frac{\beta_e A \phi_2}{\kappa d} = d\xi(1 - \mathcal{R}_0) + \mu_i D_2 \xi. \end{aligned}$$

Clearly, If $\mathcal{R}_0 < 1$, then $\lambda_2 \times \lambda_3 > 0$ and $\lambda_2 + \lambda_3 < 0$. Thus, $Re(\lambda_2) < 0$ and $Re(\lambda_3) < 0$. Hence, $P^0$ is locally asymptotically stable whenever $\mathcal{R}_0 < 1$.

On the other hand, if $\mathcal{R}_0 > 1$, at least one of the eigeinvalues has a positive real part, which implies that $P^0$ is unstable. In fact, set

$$h_1(\lambda) = \lambda^2 + \lambda\left(\mu_i D_2 + \xi + d - \frac{\beta_v A}{d}\right) + d\xi(1 - \mathcal{R}_0) + \mu_i D_2 \xi.$$

If $\mathcal{R}_0 > 1$, it is easy to show that for $\lambda$ real and $i = 0$ (in this case, $\mu_0 = 0$),

$$h_1(0) = d\xi(1 - \mathcal{R}_0) < 0 \quad \textit{and} \quad \lambda_2 \times \lambda_3 = h_1(0).$$

This completes the proof. ∎

**Theorem 3.5.** *The endemic equilibrium $P^*$ of the poultry system (3.1) is locally asymptotically stable whenever $\mathcal{R}_0 > 1$.*

**Proof.** Linearizing system (3.1) at $P^*$ gives

$$\frac{\partial Z(x,t)}{\partial t} = \overline{D}\Delta Z(x,t) + \mathcal{B}Z(x,t), \tag{3.16}$$

where $\overline{D} = \mathrm{diag}(D_1, D_2, 0)$ and

$$\mathcal{B} = \begin{pmatrix} -P^{**} - d & -Q^{**} & -R^{**} \\ P^{**} & Q^{**} - d & R^{**} \\ 0 & \phi_2 & -\xi \end{pmatrix},$$

where

$$P^{**} = \beta_v \frac{Y^*}{1+\alpha Y^*} + \beta_e \frac{C^*}{C^*+\kappa}, \; Q^{**} = \beta_v \frac{X^*}{(1+\alpha Y^*)^2}, \; R^{**} = \kappa\beta_e \frac{X^*}{(\kappa + C^*)^2}.$$

The characteristic equation of $-\mu_i\overline{D} + \mathcal{B}$ at $Z^*$ is

$$\lambda^3 + c_1\lambda^2 + c_2\lambda + c_3 = 0, \tag{3.17}$$

where

$$\begin{aligned}
c_1 &= \mu_i D_1 + \beta_v \frac{Y^*}{1+\alpha Y^*} + \beta_e \frac{C^*}{C^*+\kappa} + d + \mu_i D_2 - \beta_v \frac{X^*}{(1+\alpha Y^*)^2} + d + \xi \\
&= \beta_v \frac{Y^*}{1+\alpha Y^*} + \beta_e \frac{C^*}{C^*+\kappa} + \beta_v \frac{X^*}{1+\alpha Y^*}\left(1 - \frac{1}{1+\alpha Y^*}\right) + \beta_e \frac{X^* C^*}{Y^*(C^*+\kappa)} + \mu_i D_1 + d + \mu_i D_2 + \xi > 0,
\end{aligned}$$

$$\begin{aligned}
c_2 &= \xi\left(\mu_i D_1 + \beta_v \frac{Y^*}{1+\alpha Y^*} + \beta_e \frac{C^*}{C^*+\kappa} + d\right) + \xi\left(\mu_i D_2 - \beta_v \frac{X^*}{(1+\alpha Y^*)^2} + d\right) \\
&\quad + \left(\mu_i D_1 + \beta_v \frac{Y^*}{1+\alpha Y^*} + \beta_e \frac{C^*}{C^*+\kappa} + d\right)\left(\mu_i D_2 - \beta_v \frac{X^*}{(1+\alpha Y^*)^2} + d\right) \\
&\quad + \kappa\beta_e\phi_2 \frac{X^*}{(\kappa+C^*)^2} + \beta_v \frac{X^*}{(1+\alpha Y^*)^2}\left(\beta_v \frac{Y^*}{1+\alpha Y^*} + \beta_e \frac{C^*}{C^*+\kappa}\right), \\
&= \xi\left(\mu_i D_1 + \beta_v \frac{Y^*}{1+\alpha Y^*} + \beta_e \frac{C^*}{C^*+\kappa} + d\right) + \xi\left(\mu_i D_2 + \beta_v \frac{X^*}{1+\alpha Y^*}\left(1 - \frac{1}{1+\alpha Y^*}\right) + \beta_e \frac{X^* C^*}{Y^*(C^*+\kappa)}\right) \\
&\quad + \left(\mu_i D_1 + \beta_v \frac{Y^*}{1+\alpha Y^*} + \beta_e \frac{C^*}{C^*+\kappa} + d\right)\left(\mu_i D_2 + \beta_v \frac{X^*}{1+\alpha Y^*}\left(1 - \frac{1}{1+\alpha Y^*}\right) + \beta_e \frac{X^* C^*}{Y^*(C^*+\kappa)}\right) \\
&\quad + \kappa\beta_e\phi_2 \frac{X^*}{(\kappa+C^*)^2} + \beta_v \frac{X^*}{(1+\alpha Y^*)^2}\left(\beta_v \frac{Y^*}{1+\alpha Y^*} + \beta_e \frac{C^*}{C^*+\kappa}\right) > 0,
\end{aligned}$$

$$\begin{aligned}
c_3 &= \kappa\beta_e\phi_2 \frac{X^*}{(\kappa+C^*)^2}\left(\mu_i D_1 + \beta_v \frac{Y^*}{1+\alpha Y^*} + \beta_e \frac{C^*}{C^*+\kappa} + d\right) + \kappa\beta_e\phi_2 \frac{X^*}{(\kappa+C^*)^2} \\
&\quad + \beta_v\xi \frac{X^*}{(1+\alpha Y^*)^2}\left(\beta_v \frac{Y^*}{1+\alpha Y^*} + \beta_e \frac{C^*}{C^*+\kappa}\right) \\
&\quad + \xi\left(\mu_i D_1 + \beta_v \frac{Y^*}{1+\alpha Y^*} + \beta_e \frac{C^*}{C^*+\kappa} + d\right)\left(\mu_i D_2 + \beta_v \frac{X^*}{1+\alpha Y^*}\left(1 - \frac{1}{1+\alpha Y^*}\right) + \beta_e \frac{X^* C^*}{Y^*(C^*+\kappa)}\right) \\
&\quad + \kappa\beta_e\phi_2 \frac{X^*}{(\kappa+C^*)^2}\left(\beta_v \frac{Y^*}{1+\alpha Y^*} + \beta_e \frac{C^*}{C^*+\kappa}\right) > 0,
\end{aligned}$$

$$
\begin{aligned}
c_1c_2 - c_3 \;=\; & \xi\left(\mu_iD_1 + \beta_v\frac{Y^*}{1+\alpha Y^*} + \beta_e\frac{C^*}{C^*+\kappa} + d\right)^2 + \xi\left(\mu_iD_2 - \beta_v\frac{X^*}{(1+\alpha Y^*)^2} + d\right)^2 \\
& + \left(\mu_iD_1 + \beta_v\frac{Y^*}{1+\alpha Y^*} + \beta_e\frac{C^*}{C^*+\kappa} + d\right)^2\left(\mu_iD_2 - \beta_v\frac{X^*}{(1+\alpha Y^*)^2} + d\right) \\
& + \left(\mu_iD_1 + \beta_v\frac{Y^*}{1+\alpha Y^*} + \beta_e\frac{C^*}{C^*+\kappa} + d\right)\left(\mu_iD_2 - \beta_v\frac{X^*}{(1+\alpha Y^*)^2} + d\right)^2 \\
& + \xi^2\left(\mu_iD_1 + \beta_v\frac{Y^*}{1+\alpha Y^*} + \beta_e\frac{C^*}{C^*+\kappa} + d\right) + \xi^2\left(\mu_iD_2 - \beta_v\frac{X^*}{(1+\alpha Y^*)^2} + d\right) \\
& + \beta_v\frac{X^*}{(1+\alpha Y^*)^2}\left(\beta_v\frac{Y^*}{1+\alpha Y^*} + \beta_e\frac{C^*}{C^*+\kappa}\right)\left(\mu_iD_1 + \beta_v\frac{Y^*}{1+\alpha Y^*} + \beta_e\frac{C^*}{C^*+\kappa} + d\right) \\
& + 2\xi\left(\mu_iD_1 + \beta_v\frac{Y^*}{1+\alpha Y^*} + \beta_e\frac{C^*}{C^*+\kappa} + d\right)\left(\mu_iD_2 - \beta_v\frac{X^*}{(1+\alpha Y^*)^2} + d\right) \\
& + \kappa\beta_e\phi_2\frac{X^*}{(\kappa+C^*)^2}\left(\mu_iD_2 + \beta_v\frac{X^*}{1+\alpha Y^*}\left(1 - \frac{1}{1+\alpha Y^*}\right) + \beta_e\frac{X^*C^*}{Y^*(C^*+\kappa)}\right) \\
& + \beta_v\frac{X^*}{(1+\alpha Y^*)^2}\left(\beta_v\frac{Y^*}{1+\alpha Y^*} + \beta_e\frac{C^*}{C^*+\kappa}\right)\left(\mu_iD_2 - \beta_v\frac{X^*}{(1+\alpha Y^*)^2} + d\right) \\
& + \kappa\beta_e\xi\phi_2\frac{X^*}{(\kappa+C^*)^2} - \kappa\beta_e\phi_2\frac{X^*}{(\kappa+C^*)^2}\left(\beta_v\frac{Y^*}{1+\alpha Y^*} + \beta_e\frac{C^*}{C^*+\kappa}\right) > 0.
\end{aligned}
$$

Then, by using Routh-Hurwitz criterion, the endemic equilibrium $P^*$ of system (3.1) is locally asymptotically stable. This completes the proof. ■

### 3.3. *Global stability analysis of the equilibrium points*

Here, we establish the global stability of the equilibria for the continuous system (3.1). This is achieved by constructing suitable Lyapunov functions. We first introduce the function $\Phi(x) = x - 1 - \ln x$. Clearly, $\Phi(x) \geq 0$ for all $x > 0$ and the equality holds if and only if $x = 1$.

**Theorem 3.6.** *The disease-free equilibrium $P^0$ of the poultry system (3.1) is globally asymptotically stable (GAS) in $\Sigma$ if $\mathcal{R}_0 \leq 1$.*

**Proof.** Define the Lyapunov function

$$L(t) = \int_\Omega L_1(x,t)dx,$$

with

$$L_1(x,t) = X - X^0 - X^0\ln\left(\frac{X}{X^0}\right) + Y + \frac{\beta_eX^0}{\kappa\xi}C.$$

Using the fact that $A = dX^0$, the derivative of $L_1(x,t)$ in the direction of the vector field given by the right-hand side of system (3.1) is

$$
\begin{aligned}
\frac{\partial L_1(x,t)}{\partial t} \;=\; & \left[1 - \frac{X^0}{X}\right]\left[dX^0 - \beta_vX\frac{Y}{1+\alpha Y} - \beta_eX\frac{C}{C+\kappa} - dX + D_1\Delta X\right] \\
& + \left[\beta_vX\frac{Y}{1+\alpha Y} + \beta_eX\frac{C}{C+\kappa} - dY + D_2\Delta Y\right] + \frac{\beta_eX^0}{\kappa\xi}(\phi_2Y - \xi C), \\
=\; & -\frac{d}{X}(X-X^0)^2 + \beta_vX^0\frac{Y}{1+\alpha Y} + \beta_eX^0\frac{C}{C+\kappa} + \frac{\beta_eX^0}{\kappa\xi}\phi_2Y - dY - \frac{\beta_eX^0}{\kappa\xi}\xi C \\
& + D_1\Delta X + D_2\Delta Y - D_1X^0\frac{\Delta X}{X}.
\end{aligned}
$$

Direct calculations lead to

$$\frac{\partial L_1(x,t)}{\partial t} \;\leq\; -\frac{d}{X}(X-X^0)^2 + d\,(\mathcal{R}_0 - 1)\,Y + D_1\Delta X + D_2\Delta Y - D_1X^0\frac{\Delta X}{X}.$$

Since

$$\int_\Omega \Delta X dx = \int_\Omega \Delta Y dx = 0 \text{ and } \int_\Omega \frac{\Delta X}{X} dx = \int_\Omega \frac{|\nabla X|^2}{X^2} dx,$$

we have

$$\begin{aligned} \frac{dL(t)}{dt} &= \int_\Omega \frac{\partial L_1(x,t)}{\partial t} dx, \\ &\leq -d \int_\Omega \frac{1}{X}(X - X^0)^2 dx + d(\mathcal{R}_0 - 1) \int_\Omega Y(x,t) dx - D_1 X^0 \int_\Omega \frac{|\nabla X|^2}{X^2} dx. \end{aligned}$$

Consequently, $\frac{dL(t)}{dt} < 0$ if and only if $\mathcal{R}_0 < 1$. $\frac{dL(t)}{dt} = 0$, if and only if $\mathcal{R}_0 = 1$ and $X = X^0$, for all $t > 0$ and $x \in \Omega$. It is easy to see that the largest invariant subset included in the set $\left\{(X, Y, C) \in \Sigma / \frac{dL(t)}{dt} = 0\right\}$ is the singleton $\left\{P^0\right\}$. Thus, by the generalized LaSalle's Invariance Principle [19, Theorem 4.2] (see also [20]), the disease-free equilibrium $P^0$ is globally asymptotically stable in $\Sigma$. This completes the proof. ∎

**Theorem 3.7.** *The endemic equilibrium $P^*$ of the poultry system (3.1) is globally asymptotically stable (GAS) in the interior of $\Sigma$ if $\mathcal{R}_0 > 1$.*

**Proof.**

$$H(t) = \int_\Omega H_1(x,t) dx,$$

where the Volterra-type Lyapunov function $H_1$ is given by

$$H_1(x,t) = c_1\left(X - X^* - X^* \ln\left(\frac{X}{X^*}\right)\right) + c_2\left(Y - Y^* - Y^* \ln\left(\frac{Y}{Y^*}\right)\right) + c_3\left(C - C^* - C^* \ln\left(\frac{C}{C^*}\right)\right),$$

with $c_1, c_2$ and $c_3$ being three positive constants to be determined shortly. Denote

$$O_1 = X - X^* - X^* \ln\left(\frac{X}{X^*}\right),\ O_2 = Y - Y^* - Y^* \ln\left(\frac{Y}{Y^*}\right),$$

$$O_3 = C - C^* - C^* \ln\left(\frac{C}{C^*}\right),\ f(Y) = \frac{Y}{1 + \alpha Y} \text{ and } g(C) = \frac{C}{C + \kappa}.$$

We have

$$\begin{aligned} \frac{\partial O_1}{\partial t} &= \left(1 - \frac{X^*}{X}\right)[X^* f(Y^*) + X^* g(C^*) - d(X - X^*) - X f(Y) - X g(C) + D_1 \Delta X], \\ &= -d\frac{(X - X^*)^2}{X} + X^* f(Y^*)\left[1 - \frac{X^*}{X} - \frac{X f(Y)}{X^* f(Y^*)} + \frac{f(Y)}{f(Y^*)}\right] \\ &\quad + X^* g(C^*)\left[1 - \frac{X^*}{X} - \frac{X g(C)}{X^* g(C^*)} + \frac{g(C)}{g(C^*)}\right] + \left(1 - \frac{X^*}{X}\right) D_1 \Delta X, \\ &= -d\frac{(X - X^*)^2}{X} + \left(1 - \tfrac{X^*}{X}\right) D_1 \Delta X + a_{12} G_{12} + a_{13} G_{13}. \end{aligned}$$

$$\begin{aligned} \frac{\partial O_2}{\partial t} &= \left(1 - \frac{Y^*}{Y}\right)\left[X f(Y) + X g(C) - \frac{Y}{Y^*} X^* f(Y^*) - \frac{Y}{Y^*} X^* g(C^*)\right] + \left(1 - \frac{Y^*}{Y}\right) D_2 \Delta Y, \\ &= X^* f(Y^*)\left[\frac{X f(Y)}{X^* f(Y^*)} + 1 - \frac{Y}{Y^*} - \frac{X Y^* f(Y)}{X^* Y f(Y^*)}\right] \\ &\quad + X^* g(C^*)\left[\frac{X g(C)}{X^* g(C^*)} + 1 - \frac{C}{C^*} - \frac{X Y^* g(C)}{X^* Y g(C^*)}\right] + \left(1 - \frac{Y^*}{Y}\right) D_2 \Delta Y, \\ &= \left(1 - \frac{Y^*}{Y}\right) D_2 \Delta Y + a_{21} G_{21} + a_{23} G_{23}. \end{aligned}$$

$$\frac{\partial O_3}{\partial t} = \left(1-\frac{C^*}{C}\right)\left[\phi_2 Y - \phi_2 Y^* \frac{C}{C^*}\right] = \phi_2 Y^* \left[1 - \frac{C}{C^*} + \frac{Y}{Y^*} - \frac{YC^*}{Y^*C}\right] = a_{31}G_{31},$$

where $a_{12} = a_{21} = X^* f(Y^*), a_{13} = a_{23} = X^* g(C^*), a_{31} = \phi_2 Y^*$ and all other $a_{ij} = 0$, for all others $(i,j), 1 \le i,j \le 3$. The associated weighted digraph $(\mathcal{G}, \mathcal{A})$ has three vertices and three cycles. We consider the following two kind of cycles: cycles involving direct transmission and cycles involving indirect transmission. By [21, Theorem 3.5] there exists $c_i, 1 \le i \le 3$, such that $H_1 = \sum_{i=1}^{3} c_i O_i$ is a Lyapunov function for (3.1). Futhermore, following [21], $c_1 = c_2$ and $c_3 = \dfrac{X^* g(C^*)}{\phi_2 Y^*} c_1$. Thus,

$$H_1(x,t) = c_1 O_1 + c_1 O_2 + \frac{X^* g(C^*)}{\phi_2 Y^*} c_1 O_3.$$

We have

$$\begin{aligned}
\frac{\partial H_1(x,t)}{\partial t} &= c_1 \left[\frac{\partial O_1(x,t)}{\partial t} + \frac{\partial O_1(x,t)}{\partial t} + \frac{X^* g(C^*)}{\phi_2 Y^*}\frac{\partial O_3(x,t)}{\partial t}\right], \\
&= -dc_1 \frac{(X-X^*)^2}{X} + X^* f(Y^*) c_1 \left[1 - \frac{X^*}{X} - \frac{Xf(Y)}{X^* f(Y^*)} + \frac{f(Y)}{f(Y^*)}\right] \\
&\quad + X^* g(C^*) c_1 \left[1 - \frac{X^*}{X} - \frac{Xg(C)}{X^* g(C^*)} + \frac{g(C)}{g(C^*)}\right] \\
&\quad + X^* f(Y^*) c_1 \left[\frac{Xf(Y)}{X^* f(Y^*)} + 1 - \frac{Y}{Y^*} - \frac{XY^* f(Y)}{X^* Y f(Y^*)}\right] \\
&\quad + X^* g(C^*) c_1 \left[\frac{Xg(C)}{X^* g(C^*)} + 1 - \frac{Y}{Y^*} - \frac{XY^* g(C)}{X^* Y g(C^*)}\right] \\
&\quad + X^* g(C^*) c_1 \left[1 - \frac{C}{C^*} + \frac{Y}{Y^*} - \frac{YC^*}{Y^* C}\right] \\
&\quad + c_1 \left(1 - \frac{X^*}{X}\right) D_1 \Delta X + c_1 \left(1 - \frac{Y^*}{Y}\right) D_2 \Delta Y, \\
&= -dc_1 \frac{(X-X^*)^2}{X} + X^* f(Y^*) c_1 \left[2 - \frac{X^*}{X} - \frac{Y}{Y^*} - \frac{XY^* f(Y)}{X^* Y f(Y^*)} + \frac{f(Y)}{f(Y^*)}\right] \\
&\quad + X^* g(C^*) c_1 \left[2 - \frac{X^*}{X} - \frac{Y}{Y^*} - \frac{XY^* g(C)}{X^* Y g(C^*)} + \frac{g(C)}{g(C^*)}\right] \\
&\quad + X^* g(C^*) c_1 \left[1 - \frac{C}{C^*} + \frac{Y}{Y^*} - \frac{YC^*}{Y^* C}\right] \\
&\quad + c_1 \left(1 - \frac{X^*}{X}\right) D_1 \Delta X + c_1 \left(1 - \frac{Y^*}{Y}\right) D_2 \Delta Y, \\
&= -dc_1 \frac{(X-X^*)^2}{X} \\
&\quad - X^* f(Y^*) c_1 \left[\frac{X^*}{X} + \frac{Y}{Y^*} + \frac{XY^* f(Y)}{X^* Y f(Y^*)} - \frac{f(Y)}{f(Y^*)} - 2\right] \\
&\quad - X^* g(C^*) c_1 \left[\frac{X^*}{X} + \frac{Y}{Y^*} + \frac{XY^* g(C)}{X^* Y g(C^*)} - \frac{g(C)}{g(C^*)} - 3\right] \\
&\quad - X^* g(C^*) c_1 \left[\frac{C}{C^*} - \frac{Y}{Y^*} + \frac{YC^*}{Y^* C}\right] \\
&\quad + c_1 \left(1 - \frac{X^*}{X}\right) D_1 \Delta X + c_1 \left(1 - \frac{Y^*}{Y}\right) D_2 \Delta Y,
\end{aligned}$$

$$
\begin{aligned}
&= -dc_1\frac{(X-X^*)^2}{X} \\
&\quad -X^*f(Y^*)c_1\left[\phi\left(\frac{X^*}{X}\right)+\phi\left(\frac{Y}{Y^*}\right)+\phi\left(\frac{XY^*f(Y)}{X^*Yf(Y^*)}\right)-\phi\left(\frac{f(Y)}{f(Y^*)}\right)\right] \\
&\quad -X^*f(Y^*)c_1\left[\ln\left(\frac{X^*Y}{XY^*}\right)+\ln\left(\frac{XY^*f(Y)}{X^*Yf(Y^*)}\right)-\ln\left(\frac{f(Y)}{f(Y^*)}\right)\right] \\
&\quad -X^*g(C^*)c_1\left[\phi\left(\frac{X^*}{X}\right)+\phi\left(\frac{Y}{Y^*}\right)+\phi\left(\frac{XY^*g(C)}{X^*Yg(C^*)}\right)-\phi\left(\frac{g(C)}{g(C^*)}\right)-1\right] \\
&\quad -X^*g(C^*)c_1\left[\ln\left(\frac{X^*Y}{XY^*}\right)+\ln\left(\frac{XY^*g(C)}{X^*Yg(C^*)}\right)-\ln\left(\frac{g(C)}{g(C^*)}\right)\right] \\
&\quad -X^*g(C^*)c_1\left[\phi\left(\frac{C}{C^*}\right)-\phi\left(\frac{Y}{Y^*}\right)+\phi\left(\frac{YC^*}{Y^*C}\right)+1\right] \\
&\quad +c_1\left(1-\frac{X^*}{X}\right)D_1\Delta X+c_1\left(1-\frac{Y^*}{Y}\right)D_2\Delta Y, \\
&= -dc_1\frac{(X-X^*)^2}{X} \\
&\quad -X^*f(Y^*)c_1\left[\phi\left(\frac{X^*}{X}\right)+\phi\left(\frac{Y}{Y^*}\right)+\phi\left(\frac{XY^*f(Y)}{X^*Yf(Y^*)}\right)-\phi\left(\frac{f(Y)}{f(Y^*)}\right)\right] \\
&\quad -X^*g(C^*)c_1\left[\phi\left(\frac{X^*}{X}\right)+\phi\left(\frac{C}{C^*}\right)+\phi\left(\frac{YC^*}{Y^*C}\right)+\phi\left(\frac{XY^*g(C)}{X^*Yg(C^*)}\right)-\phi\left(\frac{g(C)}{g(C^*)}\right)\right] \\
&\quad +c_1\left(1-\frac{X^*}{X}\right)D_1\Delta X+c_1\left(1-\frac{Y^*}{Y}\right)D_2\Delta Y.
\end{aligned}
$$

Note that

$$
\begin{cases}
f(0)=g(0)=0\ ,\quad f(Y)>0\ ,\ g(C)>0\quad \forall Y>0\ ,\ C>0, \\
f'(Y)\ ,\ g'(C)>0\ \text{ and }\ f''(Y)\ ,\ g''(C)\le 0,
\end{cases}
$$

and

$$
\phi\left(\frac{f(Y)}{f(Y^*)}\right)-\phi\left(\frac{Y}{Y^*}\right)\le\left(\frac{f(Y)}{f(Y^*)}-\frac{Y}{Y^*}\right)\left(1-\frac{f(Y^*)}{f(Y)}\right)=-\frac{\alpha YY^*(Y-Y^*)^2}{Y^*f(Y)(f(Y^*))^2(1+\alpha Y)(1+\alpha Y^*)},
$$

$$
\phi\left(\frac{g(C)}{g(C^*)}\right)-\phi\left(\frac{C}{C^*}\right)\le\left(\frac{g(C)}{g(C^*)}-\frac{C}{C^*}\right)\left(1-\frac{g(C^*)}{g(C)}\right)=-\frac{\kappa CC^*(C-C^*)^2}{C^*g(C)(g(C^*))^2(\kappa+C)(\kappa+C^*)}.
$$

Finally,

$$
\begin{aligned}
\frac{dH(t)}{dt} &= \int_\Omega \frac{\partial H_1(x,t)}{\partial t}dx, \\
&\le -dc_1\int_\Omega\frac{(X-X^*)^2}{X}dx \\
&\quad -X^*f(Y^*)c_1\int_\Omega\left[\phi\left(\frac{X^*}{X}\right)+\phi\left(\frac{Y}{Y^*}\right)+\phi\left(\frac{XY^*f(Y)}{X^*Yf(Y^*)}\right)-\phi\left(\frac{f(Y)}{f(Y^*)}\right)\right]dx \\
&\quad -X^*g(C^*)c_1\int_\Omega\left[\phi\left(\frac{C}{C^*}\right)-\phi\left(\frac{g(C)}{g(C^*)}\right)\right]dx, \\
&\quad -X^*g(C^*)c_1\int_\Omega\left[\phi\left(\frac{X^*}{X}\right)+\phi\left(\frac{YC^*}{Y^*C}\right)+\phi\left(\frac{XY^*g(C)}{X^*Yg(C^*)}\right)\right]dx \\
&\quad -D_1X^*\int_\Omega\frac{|\nabla X|^2}{X^2}dx-D_2Y^*\int_\Omega\frac{|\nabla Y|^2}{Y^2}dx.
\end{aligned}
$$

Consequently, $\frac{dH(t)}{dt}<0$ and $\frac{dH(t)}{dt}=0$ if and only if $X=X^*, Y=Y^*$ and $C=C^*$, for all $t>0$ and $x\in\Omega$. Moreover, the largest invariant subset contained in $\left\{(X,Y,C)\in\Sigma/\frac{dH}{dt}(t)=0\right\}$ is the singleton $\{P^*\}$. It follows from the generalized LaSalle's Invariance Principle [19, Theorem 4.2 ] (see also [20]) that $P^*$ is globally asymptotically stable. ■

**Remark 3.8.** *When $q \neq 0$, the poultry system has only one endemic equilibruim, which is locally asymptotically stable.*

## 4. Asymptotic analysis of the full system (when q=0)

In the absence of infection, that is $Y = E = I = C = 0$, the model (2.4)-(2.6) has a disease-free equilibrum

$$Z^0 = \left(\frac{A}{d}, 0, \frac{B}{\delta}, 0, 0, 0\right).$$

### 4.1. *Existence of endemic equilibrium point*

Suppose that

$$\mathcal{R}_0 = \frac{\beta_e A \phi_2}{\kappa d^2 \xi} + \frac{\beta_v A}{d^2} > 1.$$

Then the full system (2.4)-(2.6) has the endemic equilibrium $Z^* = (X^*, Y^*, S^*, E^*, I^*, C^*)$, where $X^*, Y^*$ and $C^*$) are given by (3.11) and (3.12) and

$$S^* = N^* - E^* - I^*, \quad I^* = \frac{1}{\rho}\left(B - \delta N^*\right), \quad E^* = \frac{\gamma + \rho + \delta}{\rho \epsilon}\left(B - \delta N^*\right),$$

with $N^*$ being the positive root of the following quadratic equation:

$$\alpha_1 N^{*2} + \left(\alpha_3 Y^* - \alpha_1 \frac{B}{\delta}\right) N^* - \alpha_2 Y^* = 0, \tag{4.1}$$

where

$$\alpha_1 = \frac{(a + \delta + \epsilon)(\gamma + \rho + \delta)\delta}{\rho \epsilon}, \ \alpha_2 = \frac{B}{\rho}\left(\frac{(\gamma + \rho + \delta)}{\epsilon} + 1\right)\left(\tau_v + \tau_e \frac{\phi_2}{\xi}\right),$$

$$\alpha_3 = \left(\frac{(\gamma + \rho + \delta)\delta}{\rho \epsilon} + \frac{\delta}{\rho} + 1\right)\left(\tau_v + \tau_e \frac{\phi_2}{\xi}\right).$$

Thanks to the Descarte's rule of sign, $N^*$ is unique.

### 4.2. *Local stability of the equilibrium points*

The local stability of the equilibria $Z^0$ and $Z^*$ follows from linearization method of (2.4)-(2.6) and detailed spectral analysis of the corresponding characteristic equation.

**Theorem 4.1.** *If $\mathcal{R}_0 < 1$, the disease-free equilibrium $Z^0$ of the full system (2.4)-(2.6) is locally asymptotically stable, but unstable when $\mathcal{R}_0 \geq 1$.*

**Proof.** The linearization of system (2.4) at $Z^0$ is

$$\frac{\partial Z(x,t)}{\partial t} = \mathcal{L} Z(x,t) = \overline{D} \Delta Z(x,t) + C Z(x,t), \tag{4.2}$$

where $\overline{D} = \mathrm{diag}(D_1, D_2, D_3, D_4, D_5, 0)$ and

$$C = \begin{pmatrix} -d & -\beta_v \frac{A}{d} & 0 & 0 & 0 & -\beta_e \frac{A}{\kappa d} \\ 0 & \beta_v \frac{A}{d} - d & 0 & 0 & 0 & \beta_e \frac{A}{\kappa d} \\ 0 & -\tau_v & -\delta & a & \gamma & -\tau_e \\ 0 & \tau_v & 0 & -(a + \delta + \epsilon) & 0 & \tau_e \\ 0 & 0 & 0 & \epsilon & -(\gamma + \rho + \delta) & 0 \\ 0 & \phi_2 & 0 & 0 & 0 & -\xi \end{pmatrix}.$$

The characteristic equation of $-\mu_i\overline{D} + \mathcal{C}$ at $Z^0$ is

$$\begin{aligned}(-\mu_i D_1 - d - \lambda)(-\mu_i D_3 - \delta - \lambda)(-\mu_i D_4 - (a+\delta+\epsilon) - \lambda)(-\mu_i D_5 - (\gamma+\rho+\delta) - \lambda)& \\ \times\left\{\lambda^2 + \lambda(\mu_i D_2 + \xi + d - \frac{\beta_v A}{d}) + \mu_i D_2 \xi + d\xi - \frac{\beta_v A \xi}{d} - \frac{\beta_e A \phi_2}{\kappa d}\right\} = 0.& \end{aligned} \tag{4.3}$$

According to the local stability of $P^0$ for the poultry sub-system, all eigenvalues of (4.3) have negative real parts when $\mathcal{R}_0 < 1$. Hence, $Z^0$ is locally asymptotically stable. ■

**Theorem 4.2.** *If $\mathcal{R}_0 \geq 1$, the endemic equilibrium $Z^*$ of the full system (2.4)-(2.6) is locally asymptotically stable.*

**Proof.** Linearizing system (2.4) at $Z^*$ gives

$$\frac{\partial Z(x,t)}{\partial t} = \mathcal{L}Z(x,t) = \overline{D}\Delta Z(x,t) + \mathcal{D}Z(x,t), \tag{4.4}$$

where $\overline{D} = \mathrm{diag}(D_1, D_2, D_3, D_4, D_5, 0)$ and

$$\mathcal{D} = \begin{pmatrix} -P^{**} - d & -Q^{**} & 0 & 0 & 0 & -R^{**} \\ P^{**} & Q^{**} - d & 0 & 0 & 0 & R^{**} \\ 0 & -\dfrac{\tau_v S^*}{N^*} & -\dfrac{\tau_v Y^* - \tau_e C^*}{N^*} - \delta & a & \gamma & -\dfrac{\tau_e S^*}{N^*} \\ 0 & \dfrac{\tau_v S^*}{N^*} & \dfrac{\tau_v Y^* + \tau_e C^*}{N^*} & -(a+\delta+\epsilon) & 0 & \dfrac{\tau_e S^*}{N^*} \\ 0 & 0 & 0 & \epsilon & -(\gamma+\rho+\delta) & 0 \\ 0 & \phi_2 & 0 & 0 & 0 & -\xi \end{pmatrix}.$$

Here

$$P^{**} = \beta_v \frac{Y^*}{1+\alpha Y^*} + \beta_e \frac{C^*}{C^*+\kappa}, \quad Q^{**} = \beta_v \frac{X^*}{(1+\alpha Y^*)^2}, \quad R^{**} = \kappa\beta_e \frac{X^*}{(\kappa + C^*)^2}.$$

The characteristic equation of $-\mu_i\overline{D} + \mathcal{D}$ at $Z^*$ is

$$(\lambda^3 + c_1\lambda^2 + c_2\lambda + c_3)(\lambda^3 + \bar{c}_1\lambda^2 + \bar{c}_2\lambda + \bar{c}_3) = 0, \tag{4.5}$$

where

$$\begin{aligned}
\bar{c}_1 &= \mu_i D_3 + \tau_v \frac{Y^*}{N^*} + \tau_e \frac{C^*}{N^*} + \delta + \mu_i D_4 + \mu_i D_5 + a + \delta + \epsilon + \gamma + \rho + \delta > 0, \\
\bar{c}_2 &= \left(\mu_i D_3 + \tau_v \frac{Y^*}{N^*} + \tau_e \frac{C^*}{N^*} + \delta\right)(\mu_i D_5 + \gamma + \rho + \delta) \\
&\quad + \left(\mu_i D_3 + \tau_v \frac{Y^*}{N^*} + \tau_e \frac{C^*}{N^*} + \delta\right)(\mu_i D_4 + a + \delta + \epsilon) \\
&\quad + (\mu_i D_5 + \gamma + \rho + \delta)(\mu_i D_4 + a + \delta + \epsilon) - a\left(\tau_v \frac{Y^*}{N^*} + \tau_e \frac{C^*}{N^*}\right), \\
&= \left(\mu_i D_3 + \tau_v \frac{Y^*}{N^*} + \tau_e \frac{C^*}{N^*} + \delta\right)(\mu_i D_5 + \gamma + \rho + \delta) \\
&\quad + \left(\mu_i D_3 + \tau_v \frac{Y^*}{N^*} + \tau_e \frac{C^*}{N^*} + \delta\right)(\mu_i D_4 + \delta + \epsilon) \\
&\quad + (\mu_i D_5 + \gamma + \rho + \delta)(\mu_i D_4 + a + \delta + \epsilon) + a(\mu_i D_3 + \delta) > 0,
\end{aligned}$$

$$
\begin{aligned}
\bar{c}_3 \quad &= -a\left(\mu_i D_5+\gamma+\rho+\delta\right)\left(\tau_v\frac{Y^*}{N^*}+\tau_e\frac{C^*}{N^*}\right)+\epsilon\gamma\left(\tau_v\frac{Y^*}{N^*}+\tau_e\frac{C^*}{N^*}\right)\\
&\quad+\left(\mu_i D_3+\tau_v\frac{Y^*}{N^*}+\tau_e\frac{C^*}{N^*}+\delta\right)\left(\mu_i D_5+\gamma+\rho+\delta\right)\left(\mu_i D_4+a+\delta+\epsilon\right),\\
&= \epsilon\gamma\left(\tau_v\frac{Y^*}{N^*}+\tau_e\frac{C^*}{N^*}\right)\\
&\quad+\left(\mu_i D_3+\tau_v\frac{Y^*}{N^*}+\tau_e\frac{C^*}{N^*}+\delta\right)\left(\mu_i D_5+\gamma+\rho+\delta\right)\left(\mu_i D_4+\delta+\epsilon\right)\\
&\quad+a\left(\mu_i D_5+\gamma+\rho+\delta\right)\left(\mu_i D_3+\delta\right)>0,
\end{aligned}
$$

$$
\begin{aligned}
\bar{c}_1\bar{c}_2-\bar{c}_3 &= \left(\mu_i D_5+\gamma+\rho+\delta\right)\left(\mu_i D_3+\tau_v\frac{Y^*}{N^*}+\tau_e\frac{C^*}{N^*}+\delta\right)^2\\
&\quad+\left(\mu_i D_5+\gamma+\rho+\delta\right)\left(\mu_i D_4+a+\epsilon+\delta\right)^2\\
&\quad+2\left(\mu_i D_5+\gamma+\rho+\delta\right)\left(\mu_i D_4+a+\epsilon+\delta\right)\left(\mu_i D_3+\tau_v\frac{Y^*}{N^*}+\tau_e\frac{C^*}{N^*}+\delta\right)\\
&\quad-\epsilon\gamma\left(\tau_v\frac{Y^*}{N^*}+\tau_e\frac{C^*}{N^*}\right)\\
&\quad-\left(\mu_i D_3+\tau_v\frac{Y^*}{N^*}+\tau_e\frac{C^*}{N^*}+\delta+\mu_i D_4+a+\epsilon+\delta\right)\\
&\quad\times\left[a\left(\tau_v\frac{Y^*}{N^*}+\tau_e\frac{C^*}{N^*}\right)-\left(\mu_i D_4+a+\epsilon+\delta\right)\left(\mu_i D_3+\tau_v\frac{Y^*}{N^*}+\tau_e\frac{C^*}{N^*}+\delta\right)\right],\\
&= \left(\mu_i D_5+\gamma+\rho+\delta\right)\left(\mu_i D_3+\tau_v\frac{Y^*}{N^*}+\tau_e\frac{C^*}{N^*}+\delta\right)^2\\
&\quad+\left(\mu_i D_5+\gamma+\rho+\delta\right)\left(\mu_i D_4+a+\epsilon+\delta\right)^2\\
&\quad+2\left(\mu_i D_5+\gamma+\rho+\delta\right)\left(\mu_i D_4+a+\epsilon+\delta\right)\left(\mu_i D_3+\tau_v\frac{Y^*}{N^*}+\tau_e\frac{C^*}{N^*}+\delta\right)\\
&\quad-\epsilon\gamma\left(\tau_v\frac{Y^*}{N^*}+\tau_e\frac{C^*}{N^*}\right)\\
&\quad+\left(\mu_i D_3+\tau_v\frac{Y^*}{N^*}+\tau_e\frac{C^*}{N^*}+\delta+\mu_i D_4+a+\epsilon+\delta\right)\\
&\quad\times\left[a\left(\mu_i D_3+\delta\right)+\left(\mu_i D_4+\epsilon+\delta\right)\left(\mu_i D_3+\tau_v\frac{Y^*}{N^*}+\tau_e\frac{C^*}{N^*}+\delta\right)\right],
\end{aligned}
$$

$$
\begin{aligned}
&= \left(\mu_i D_5+\gamma+\rho+\delta\right)\left(\mu_i D_3+\tau_v\frac{Y^*}{N^*}+\tau_e\frac{C^*}{N^*}+\delta\right)^2\\
&\quad+\left(\mu_i D_5+\gamma+\rho+\delta\right)\left(\mu_i D_4+a+\epsilon+\delta\right)^2\\
&\quad+2\left(\mu_i D_5+\rho+\delta\right)\left(\mu_i D_4+a+\epsilon+\delta\right)\left(\mu_i D_3+\tau_v\frac{Y^*}{N^*}+\tau_e\frac{C^*}{N^{**}}+\delta\right)\\
&\quad+2\gamma\left(\mu_i D_4+a+\delta\right)\left(\mu_i D_3+\tau_v\frac{Y^*}{N^*}+\tau_e\frac{C^*}{N^*}+\delta\right)+2\gamma\epsilon\left(\mu_i D_3+\delta\right)\\
&\quad+\epsilon\gamma\left(\tau_v\frac{Y^*}{N^*}+\tau_e\frac{C^*}{N^*}\right)\\
&\quad+\left(\mu_i D_3+\tau_v\frac{Y^*}{N^*}+\tau_e\frac{C^*}{N^*}+\delta+\mu_i D_4+a+\epsilon+\delta\right)\\
&\quad\times\left[a\left(\mu_i D_3+\delta\right)+\left(\mu_i D_4+\epsilon+\delta\right)\left(\mu_i D_3+\tau_v\frac{Y^*}{N^*}+\tau_e\frac{C^*}{N^*}+\delta\right)\right]>0.
\end{aligned}
$$

Thanks to Routh-Hurwitz criterion, the endemic equilibrium $Z^*$ of the full model is locally asymptotically stable. ■

### 4.3. *Global stability analysis of the DFE*

To establish the global stability of the full system (2.4) − (2.6), we first give two lemmas about the global stability of the scalar equations.

**Lemma 4.3.** *Let $u \in \mathbb{C}(\overline{\Omega}\times[0,\infty))\cap\mathbb{C}^{2,1}(\Omega\times(0,\infty))$ be a nonnegative nontrivial solution of the scalar problem:*

$$\begin{cases} \dfrac{\partial u}{\partial t} - D\Delta u = f(x,t) + A_1 u(x,t) & \text{in } \Omega\times(0,\infty), \\ \dfrac{\partial u}{\partial \eta} = 0 & \text{on } \partial\Omega\times(0,\infty), \\ u(x,0)\geq 0 & x\in\overline{\Omega}, \end{cases} \tag{4.6}$$

*where $A_1 > 0$ and $f(x,t)$ is a nonnegative continuous function. Then $u$ tends to $A_2/A_1$ as $t$ tends to $\infty$ uniformly on $\overline{\Omega}$, whenever $f(x,t)$ tends to $A_2$ as $t$ tends to $\infty$ uniformly on $\overline{\Omega}$.*

The proof follows directly from the comparison principle for the parabolic equations. We omit it here.

**Lemma 4.4.** *[7] If $u(x,t)$ is a bounded function and $\lim_{t\to\infty}\|u(x,t)-A_1\|_\infty = 0$, then*

$$\int_\Omega\int_{-\infty}^t G(x,y,t-s)k(t-s)u(y,s)dsdy \to A_1 \ \text{ as } \ t\to\infty$$

*uniformly on $\overline{\Omega}$.*

Lemma 4.4, which is a consequence of Lemma 2.1, implies that the nonlocal integral term do not affect the long time behavior of the solution.

**Theorem 4.5.** *The disease-free equilibrium of the full system (2.4) is globally asymptotically stable (GAS) in $\Sigma$ if $\mathcal{R}_0\leq 1$.*

**Proof.** For $\mathcal{R}_0\leq 1$, it follows from the global stability of $P^0$ of the poultry system that $\lim_{t\to\infty}\left\|X(x,t)-\dfrac{A}{d}\right\|_\infty = 0$, $\lim_{t\to\infty}\|Y(x,t)-0\|_\infty = 0$ and $\lim_{t\to\infty}\|C(x,t)-0\|_\infty = 0$. Thus, by Lemma 4.4,

$$\frac{S}{N}\int_\Omega\int_{-\infty}^t G(x,y,t-s)k(t-s)(\tau_v Y+\tau_e C)(y,s)dsdy \to 0 \ \text{ as } \ t\to\infty,$$

uniformly on $\overline{\Omega}$. Therefore $\lim_{t\to\infty}\|E(x,t)-0\|_\infty = 0$, according to Lemma 4.3. Applying once more Lemma 4.3 gives $\lim_{t\to\infty}\|I(x,t)-0\|_\infty = 0$.

For the third equation of the full system (2.4)–(2.6), since

$$\lim_{t\to\infty}\|E(x,t)-0\|_\infty = 0, \lim_{t\to\infty}\|I(x,t)-0\|_\infty = 0,$$

and the fact that Lemma 4.3 applies again, we have $\lim_{t\to\infty}\left\|S(x,t)-\dfrac{B}{\delta}\right\|_\infty = 0$. Therefore, $Z^0$ is GAS for $\mathcal{R}_0\leq 1$. ■

**Remark 4.6.** *When $q\neq 0$, the full system has only one endemic equilibrium, which is locally asymptotically stable.*

## 5. Numerical simulations

In this section, we present some numerical simulations to illustrate the spread of avian influenza. For simplicity, we choose $\Omega = [0,\pi]$, $K(x,y,t) = G(x,y,t)k(t)$, where

$$k(t) = \frac{1}{\tau}e^{-t/\tau}\ ;\ G(x,y,t) = \frac{1}{\pi} + \frac{2}{\pi}\sum_{n=1}^{\infty} e^{-D_3 n^2 t}\cos(nx)\cos(ny).$$

To circumvent the difficulty caused by the nonlocal integral terms, we introduce the following new variables

$$U(x,t) = \int_0^{\pi}\int_{-\infty}^{t} G(x,y,t-s)k(t-s)Y(y,s)dsdy\ ,\ V(x,t) = \int_0^{\pi}\int_{-\infty}^{t} G(x,y,t-s)k(t-s)C(y,s)dsdy.$$

Then system (2.4) becomes:

$$\begin{cases} \dfrac{\partial X}{\partial t} - D_1\Delta X = (1-q)A - \beta_v X\dfrac{Y}{1+\alpha Y} - \beta_e X\dfrac{C}{C+\kappa} - dX, \\ \dfrac{\partial Y}{\partial t} - D_2\Delta Y = qA + \beta_v X\dfrac{Y}{1+\alpha Y} + \beta_e X\dfrac{C}{C+\kappa} - dY, \\ \dfrac{\partial S}{\partial t} - D_3\Delta S = B + aE + \gamma I - \delta S - \dfrac{S}{N}(\tau_v U + \tau_e V), \\ \dfrac{\partial E}{\partial t} - D_4\Delta E = \dfrac{S}{N}(\tau_v U + \tau_e V) - (a+\delta+\epsilon)E, \\ \dfrac{\partial I}{\partial t} - D_5\Delta I = \epsilon E - (\gamma+\rho+\delta)I, \\ \dfrac{\partial C}{\partial t} = \phi_2 Y - \xi C, \\ \dfrac{\partial U}{\partial t} - D_3\Delta U = \dfrac{1}{\tau}(Y-U), \\ \dfrac{\partial V}{\partial t} - D_3\Delta V = \dfrac{1}{\tau}(C-V). \end{cases}$$

Every variables of the previous system enjoys the homogenous Neumann boundary conditions. Additionally, we need the following initial conditions

$$U(x,0) = \int_0^{\pi}\int_{-\infty}^{0} G(x,y,-s)k(-s)Y(y,s)dsdy \text{ and } V(x,0) = \int_0^{\pi}\int_{-\infty}^{0} G(x,y,-s)k(-s)C(y,s)dsdy.$$

The parameters are fixed in the Table 2 below

Table 2: Numerical values of the parameters of PDE-model (2.4)–(2.6).

| Parameters | values | Source | Parameters | values | Source |
|---|---|---|---|---|---|
| $q$ | 0, 0.1 | [22] | $a$ | 1 | [23] |
| $A$ | 100 | [22] | $\gamma$ | 0.9 | [23] |
| $\beta_v$ | $1.7143 \cdot 10^{-6}$ | [23] | $\rho$ | 0.001 | [22] |
| $\beta_e$ | 0.002 week$^{-1}$ | Assumed | $D_1$ | 4 | Assumed |
| $d$ | 1/72 week$^{-1}$ | [24] | $D_2$ | 3 | Assumed |
| $\alpha$ | 0.001 ind$^{-1}$ | [23] | $D_3$ | 2 | Assumed |
| $B$ | 1.5 | [22] | $D_4$ | 1.5 | Assumed |
| $\tau_v$ | 0.6 | [22] | $\epsilon$ | 1 | [22] |
| $\delta$ | 0.00025641 | [24] | $\kappa$ | $10^6$ | [22] |
| $\xi$ | 35 | Assumed | $\tau_e$ | 0.1 | Assumed |
| $\phi_2$ | · | variable | $D_5$ | 1 | Assumed |
| $\tau$ | 3 | Assumed | | | |

### 5.1. *General dynamics*

Figure 1 illustrates Theorem 4.5, which states that the disease-free equilibrium $Z^0$ of the full system (2.4)–(2.6) is globally asymptotically stable. That is, aviain influenza ultimately disappears in the poultry, human population and in the environment irrespective of the initial conditions whenever $\mathcal{R}_0 < 1$. Thus, reducing the contact rates (poultry-to-poultry and poultry-to-environment) for susceptible poultry in order to keep ($\mathcal{R}_0 < 1$), is a good policy to control the spread of avian influenza virus.

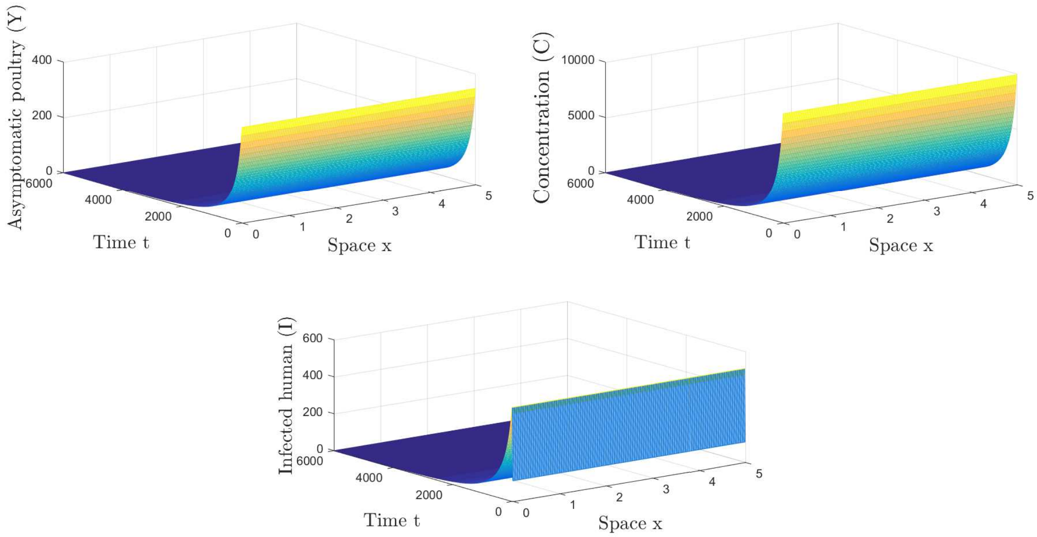


Figure 1: Simulations of IBVP (2.4)–(2.6) using various initial conditions when $q = 0$ and $\phi_2 = 10^3$ (so that $\mathcal{R}_0 = 0.9183 < 1$). All other parameter values are as in Table 2.

Figure 2 illustrates Theorem 4.2, which states that the endemic equilibrium $Z^*$ of the full system (2.4)–(2.6) is locally asymptotically stable. That is, avian influenza are still present in poultry, human population and in the environment irrespective of the initial conditions whenever ($\mathcal{R}_0 > 1$). So, reducing contact rates (poultry-to-human, environment-to-human) for susceptible humans seems to be a recommended measure to control the spread of avian influenza within the human population.

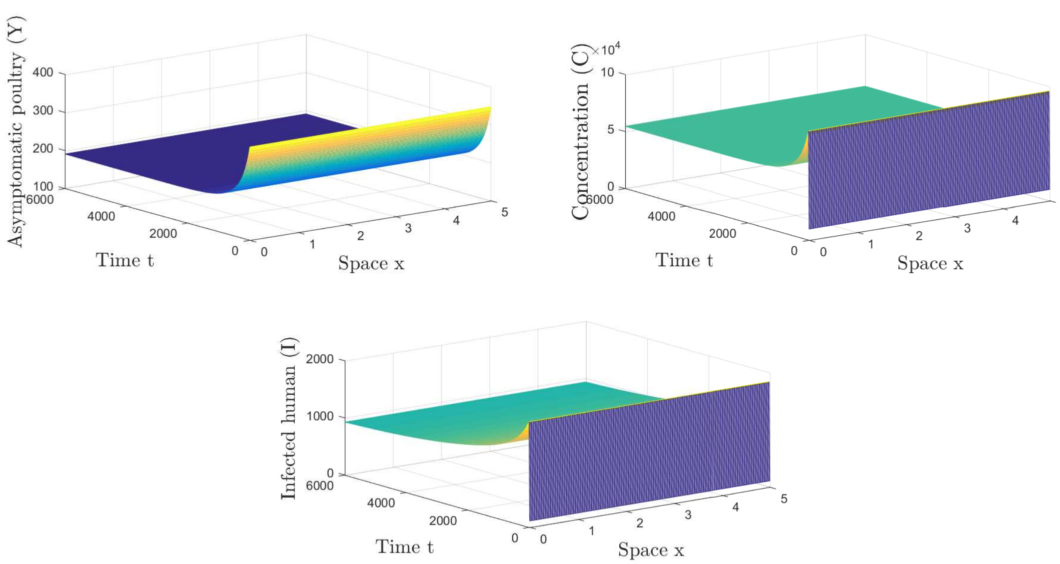


Figure 2: Simulations of IBVP (2.4)–(2.6) using various initial conditions when $q = 0$ and $\phi_2 = 10^4$ (so that $\mathcal{R}_0 = 1.1849 > 1$). All other parameter values are as in Table 2.

Figure 3 illustrates Remark 4.6, which states that the endemic equilibrium of the full system (2.4)–(2.6), when $q \neq 0$, is locally asymptotically stable. It not only shows that asymptomatic poultry and infected humans are still present in the industrial zone, but also that only 10% of infected imported poultry can multiply the number of asymptomatic poultry by 7 (that is from 200 to 1400 poultry), while keeping the number of infected humans at the same level.

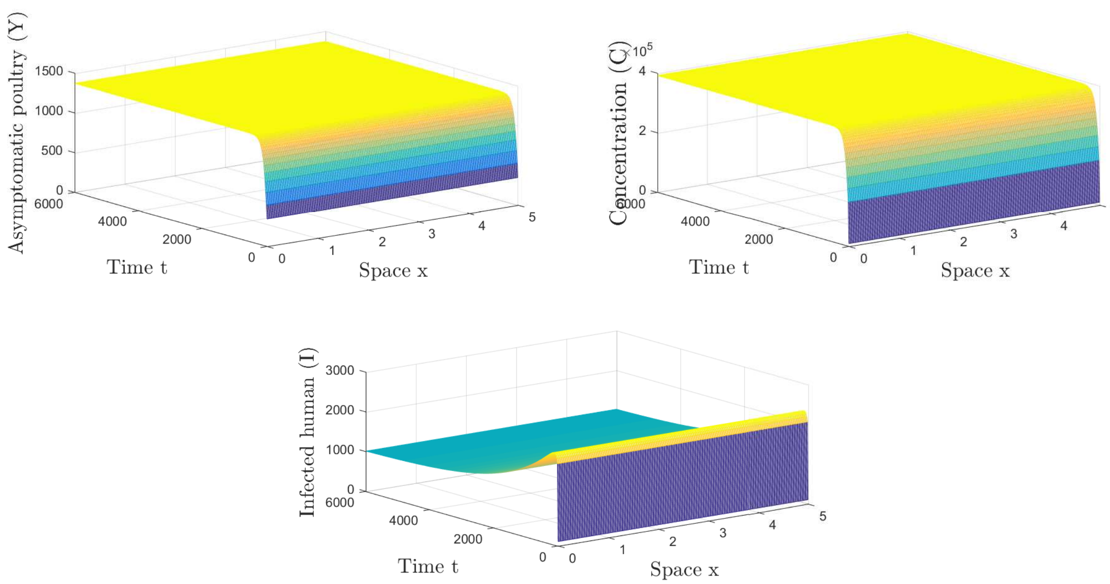


Figure 3: Simulations of IBVP (2.4)–(2.6) using various initial conditions when $\phi_2 = 10^4$ and $q = 0.1$. All other parameter values are as in Table 2.

### *5.2. Impact of some parameters on the model dynamics*

As we can see from Figure 4, the diffusion of poultry and humans has no impact on the transmission dynamics of avian influenza. This is because: Indirect transmission through the environment is the most devastating one during an avian influenza outbreak on the one hand (see [4]) and infected humans can't transmit the virus on the other hand.

Figure 5 illustrates the impact of the delay parameter $\tau$ on the transmission dynamics of avian influenza. We observed that for very large values of $\tau$, the number of infected humans decreases. Which is realistic because a significant delay by humans in feeding poultry can result in less contact between humans and poultry.

Figure 6 illustrates the impact of the transmission coefficient of the disease from the environment to humans. A significant impact on infected humans is observed when this parameter increases from 10% to 15%.

The effect of the transmission coefficient of the disease from the environment to the poultry is shown on Figure 7. We observe a significant impact on the three infected classes (i.e. human, poultry and virus concentration) when this parameter varies from 0.002 to 0.004.

We can conclude from Figures 6 and 7 that the environment has a significant impact on the dynamics of the model.

## 6. Conclusion and discussion.

The main objective of this work was to add more realism to the modelling and analysis of the transmission of AIV. It was achieved by taking the authors's previous work [4] to the next level in two main directions:

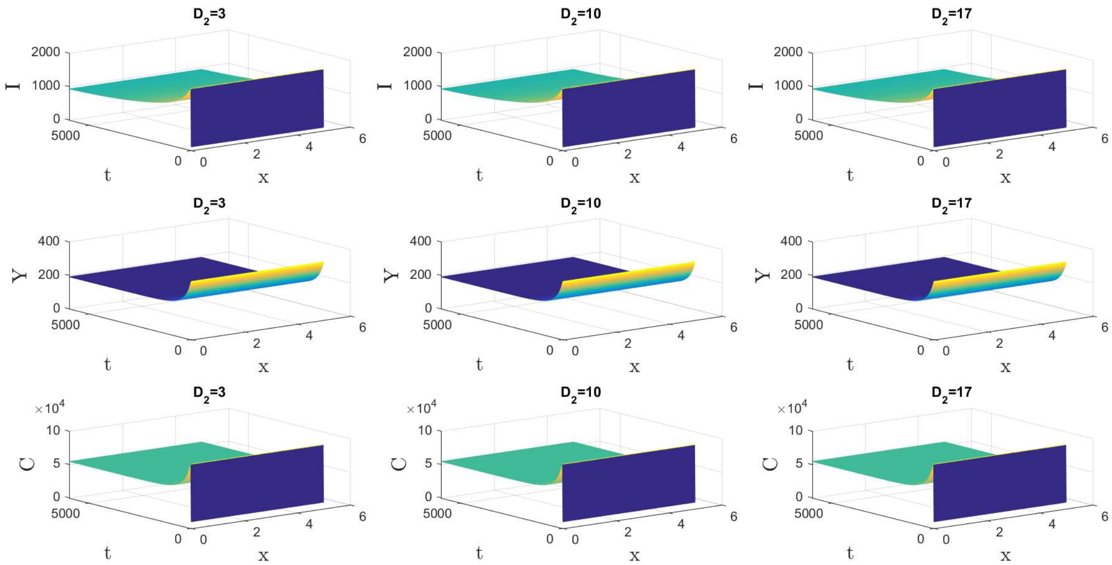


Figure 4: Simulations of IBVP (2.4)–(2.6) with various values of $D_2$ (so that $\mathcal{R}_0 = 1.1849 > 1$). All other parameter values are as in Table 2.

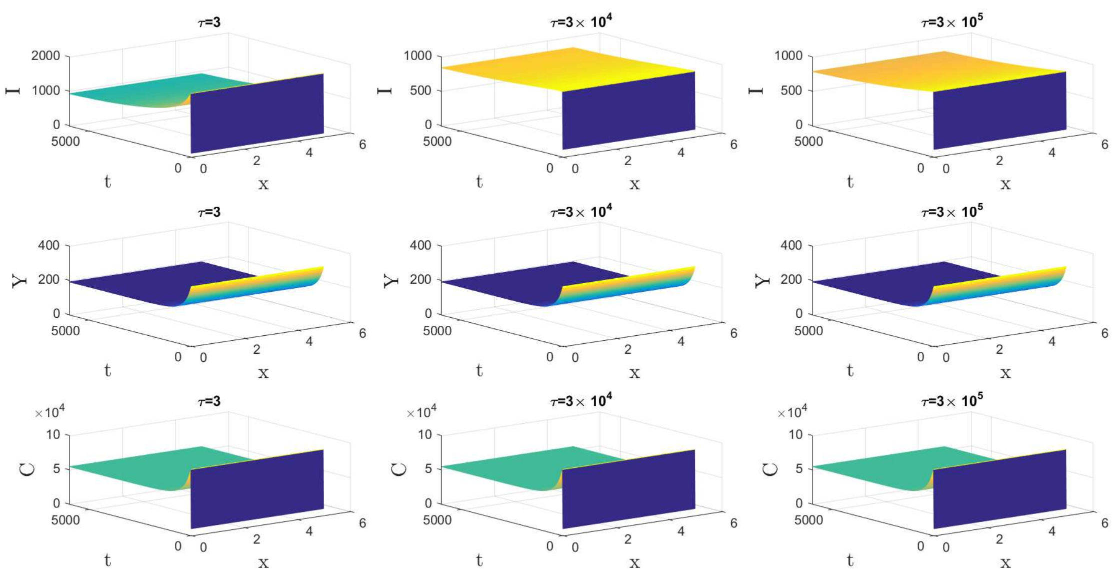


Figure 5: Simulations of IBVP (2.4)–(2.6) with various values of $\tau$ (so that $\mathcal{R}_0 = 1.1849 > 1$). All other parameter values are as in Table 2.

From the modelling perspective, the diffusion of poultry and humans were considered, as well as the delay in the trading of poultry and production of eggs (new poultry). The resulted more realistic model was a system of delayed reaction-diffusion equations.

From the theoretical perspective, we used the semigroup theory to deal with the well-posedness of the system. Moreover, the qualitative analysis of the model was insightfully performed and the main findings are as follows: An explicit formula for the reproduction number, given by the method in [17], allowed us to conclude whether the disease should persist or disappear in populations and in the environment. We obtained results on asymptotic behavior and numerical simulations were

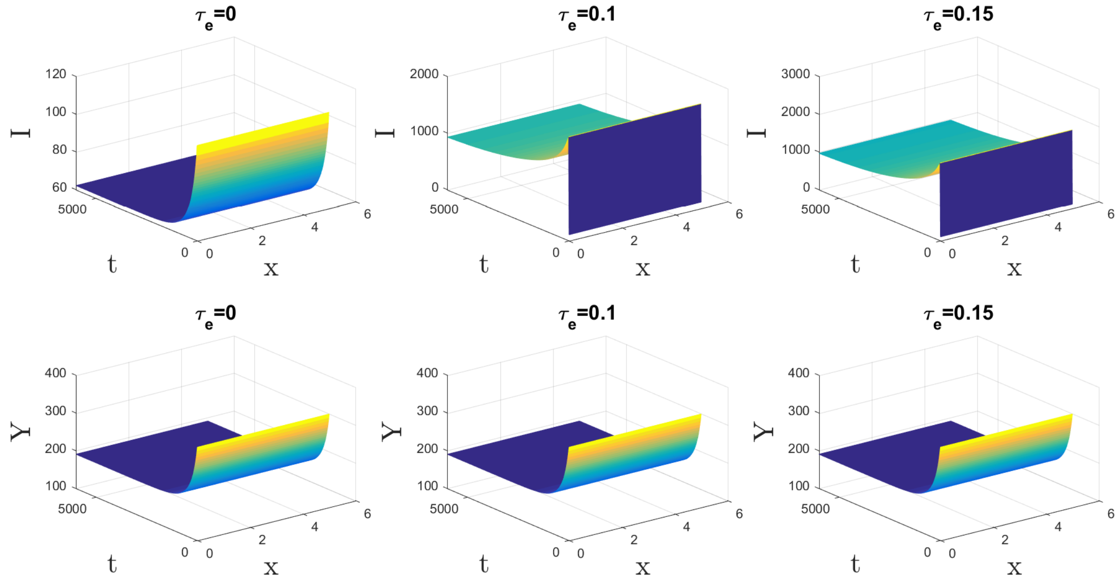


Figure 6: Simulations of IBVP (2.4)–(2.6) with various values of $\tau_e$. All other parameter values are as in Table 2.

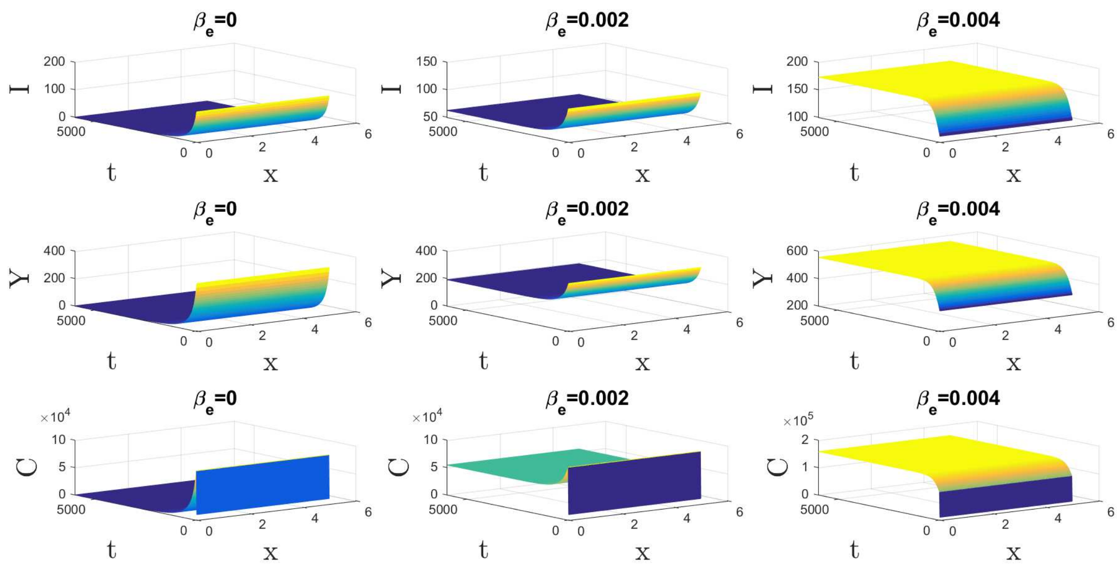


Figure 7: Simulations of IBVP (2.4)–(2.6) with various values of $\beta_e$. All other parameter values are as in Table 2.

presented to interpret the results. It is observed that if $\mathcal{R}_0 < 1$, the disease-free equilibrium $Z^0$ is globally asymptotically stable, implying that poultry, humans are safe and the environment is healthy if the contact rate for susceptible poultry is small. Our results also show that avian influenza spreads in the industrial zone when at least one of the two conditions is fulfilled: $\mathcal{R}_0 > 1$ or in the recruitment of poultry a proportion is asymptomatic.

From the computational aspect, we observed on the one hand that the importation of infected poultry can boost the endemic level of AIV in poultry and do not affect much the human population; on the other hand, in an epidemic situation, a significant delay can lead to a decrease in the number of infected humans. Moreover, we noticed that the environment has a significant impact on the dynamics of the model. It should be noted that viruses live in poultry excrements, which are small particles that can be transported by the effect of the wind and diffused into the atmosphere. In view of this, it is very realistic to extend this work by taking into account the transport and spread of the virus. Thus, we will obtain an advection-diffusion model whose main investigation will be the study of impact of virus transport and diffusion on the transmission dynamics of this disease.

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