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Doctoral Thesis

Applications of direct and inverse limits in analysis

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by

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Declaration

I, the undersigned, declare that the dissertation, which I hereby submit for the degree Doctor of Philosophy in Mathematical Sciences, is my own work and has not been previously submitted by me for a degree at this or any other tertiary institution.

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Signature:

Date:

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Summary

Title: Applications of direct and inverse limits in analysis
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In this dissertation, we use the categorical notions of direct and inverse limits to solve certain problems in analysis; in particular, in the field of vector lattices. Chapter 1 provides a general overview and motivation of the problems we will focus on. Specifically, these are a decomposition theorem for $\mathrm{C}(X)$ spaces that are order dual spaces, and the problem of existence of free objects in certain categories of locally convex structures. The connecting thread between these two disparate problems will be our extensive and fundamental use of direct and inverse limits in their solutions.

Chapter 2 deals with the first of these two problems. After settling some preliminaries, the first few sections of Chapter 2 develops the basic theory of direct and inverse limits in categories of vector lattices. This includes results on existence, permanence properties, as well as some examples. After this, we give some results on the duality between direct and inverse limits. In particular, we will show that the order (continuous) dual of a direct limit of vector lattices is an inverse limit of order (continuous) duals, and (under more strict conditions) the order (continuous) dual of an inverse limit of vector lattices is a direct limit of order (continuous) duals. The rest of Chapter 2 deals with applications of this duality theory in various contexts, among these will be our desired decomposition result for certain $\mathrm{C}(X)$ spaces, which is formulated in terms of an inverse limit.

Chapter 3 starts with some further preliminaries we need in order to define certain categories of algebraic structures, normed structures, and locally convex structures forming the setting of this chapter. After this, we cover some material from universal algebra to prove the existence of free objects in these algebraic categories. We use the existence of these algebraic free objects to expand upon the existing literature regarding certain 'free objects' in categories of normed structures. As we shall detail below, these are not bona fide free objects in our sense of the term. Inverse limits re-enter the picture at this point: We will prove a general categorical result involving inverse limits that allows us to use our results for categories of normed structures to obtain genuine free objects in categories of locally convex structures. The abstract material in Chapter 3 will be interspersed with some concrete examples chosen from two particular cases. We conclude Chapter 3 by giving concrete descriptions of two free objects in certain categories of locally convex structures whose existence was proven using our general abstract methods.

Contents

CHAPTER 1

General overview and preliminaries

1.1. General overview and motivation of problems

This study demonstrates how the notions of *direct* and *inverse* limits can be used to address some problems in analysis, particularly in the field of vector lattices. Direct and inverse limits are special cases of the more general notions of *colimits* and *limits of a diagram* respectively. Informally speaking, a *diagram*,[1] in a category **C**, is some collection of objects S_0 in **C** along with a (possibly empty) collection of morphisms S_1 connecting some (or none) the objects in S_0. If we denote by $\mathcal{S}$ the diagram consisting of the *data* S_0 and S_1, a limit or colimit of $\mathcal{S}$ can be thought of as an 'abstract method' of using the data in $\mathcal{S}$ to construct a new object X in **C** along with a collection of new morphisms M in **C** which connect X with the objects in S_0. The pair (X, M) forming the limit or colimit of the diagram $\mathcal{S}$ is not just any such pair, but is in some sense a '*universal pair*' for $\mathcal{S}$. Many of the standard methods of constructing new objects from old ones used in mathematics are particular examples of such limit and colimit constructions. Familiar examples of limits include terminal objects, products, kernels (more generally, equalisers), pullbacks, and the aforementioned inverse limits. Examples of colimits include initial objects, coproducts (more specfically, disjoint unions), pushouts, and direct limits. As implied by the names, limits and colimits are *categorically dual* concepts, meaning that every example of a limit (resp. colimit) in a category **C** is an example of a colimit (resp. limit) in the opposite category[2] $\mathbf{C}^{\mathbf{op}}$. Much information on these general constructions can be found in [**2**], [**12**], [**52**], and [**54**].

Direct and inverse limits[3] have become fairly standard tools in analysis: The direct or inverse limits of systems of topological vector spaces or locally convex spaces are well-known constructions, see for example [**22**], [**25**, Chapter IV, § 5], [**55**], and [**60**]. The notion of an inverse limit of measure spaces has also been studied, see [**19**, Chapter 5] and [**24**]. The notions of direct and inverse limits are used extensively in the book of Beattie and Butzmann on convergence spaces [**15**].

In the field of vector lattices, the notions of direct and inverse limits have received comparatively little attention. One source that stands out is the work of Filter in [**37**] where he studies the properties of direct limits of vector lattices. This has been supplemented recently by Ding and de Jeu in [**32**] where they study direct limits of normed vector lattices and Banach lattices. However, the literature for inverse limits of vector lattices is more sporadic: Two relevant sources we could find are

[1] Category theory books often describe a diagram with data from a category **C** as a *functor* $F : \mathbf{J} \to \mathbf{C}$ where **J** is an '*index category*', see [**12**, Definition 5.15].

[2] See [**52**, p. 16].

[3] Which are also called *inductive* and *projective* limits in the literature.

[**31**] and [**51**] which deal with the inverse limits of Banach lattices. Accordingly, the work in this study will address some of the gaps in the existing literature. Chapter 2 in particular will develop the basic theory of direct and inverse limits in categories of vector lattices. Results on (normed) vector lattices and (normed) vector lattice algebras are found in Chapter 3, although vector lattices are not the particular focus of that chapter.

With this brief overview of direct and inverse limits in the context of analysis in mind, we now give some context and motivation for the particular problems we seek to address.

1.1.1. Decomposition theorem for $\mathrm{C}(X)$. The first major question we seek to address was originally inspired by some connections between the theory of C*-algebras and the theory of vector lattices. Denote by $\mathrm{C}(X,\mathbb{C})$ the space of continuous complex-valued functions on a topological space X and denote by $\mathrm{C}_0(X,\mathbb{C})$ the space of continuous complex-valued functions on X that vanish at infinity. Of course, for K a compact Hausdorff topological space, the space $\mathrm{C}(K,\mathbb{C})$ is a unital commutative C*-algebra. In addition, it is well-known that every unital commutative C*-algebra A is isometrically $\ast$-isomorphic to a $\mathrm{C}(K,\mathbb{C})$ for some uniquely determined compact Hausdorff space K, see for instance [**25**, Chapter VII, Theorem 8.6; Chapter VIII, Theorem 2.1]. More generally, a (not necessarily unital) commutative C*-algebra A is isometrically $\ast$-isomorphic to a $\mathrm{C}_0(L,\mathbb{C})$ for some locally compact topological space L, see [**34**, Theorem 1.4.1].

Given these representations, it is clear that the self-adjoint part of commutative C*-algebra is a Banach lattice $\mathrm{C}_0(L) \coloneqq \mathrm{C}_0(L,\mathbb{R})$. In particular, by the Kakutani Representation Theorem [**56**, Theorem 2.1.3], the self-adjoint parts of unital commutative C*-algebras are precisely the Archimedean relatively uniformly complete vector lattices with a strong order unit.

These results relate to the following classic result of Dixmier and Grothendieck.

THEOREM 1.1.1. *Let K be a compact Hausdorff space. The following statements are equivalent.*

(i) K is hyper-Stonean[4]*.*

(ii) $\mathrm{C}(K)$ is isometrically isomorphic to a dual Banach space.

The forward implication is found in [**33**, p. 21] while the reverse implication is in [**43**, Théorème 2]. In addition to this, a theorem of Sakai in [**59**] tells us that a C*-algebra A is a W^*-algebra[5] precisely when A is a dual Banach space. With this in mind, the result in Theorem 1.1.1 gives us a characterisation of the unital commutative W^*-algebras among the unital commutative C*-algebras.

Given the importance of this characterisation and the clear connection between the theories of commutative C*-algebras and vector lattices, it seems natural to ask the following question: We call a vector lattice E an *order dual space* if there exists a

[4] See Section 2.1.3.

[5] See [**63**, Chapter III, § 3].

vector lattice F such that E and $F^\sim$ are lattice isomorphic. For which topological spaces X is the vector lattice $\mathrm{C}(X)$ an order dual space? This was answered by Xiong in [**69**]. Undefined terms and notation used in the following result are defined in Sections 2.1.1 and 2.1.2.

Theorem 1.1.2 ([**69**, Theorems 1 and 2]). *Let X be a realcompact space. Denote by S the union of supports of compactly supported normal measures on X. The following statements are equivalent.*

(i) X is extremally disconnected and $\upsilon S = X$.

(ii) $\mathrm{C}(X)$ is an order dual space.

Theorem 1.1.2 is a generalisation of Theorem 1.1.1 to the non-compact case since the compact Hausdorff spaces satisfying (i) in the above theorem are precisely the hyper-Stonean spaces. The recent book of Dales, Dashiell, Lau, and Strauss on spaces of continuous functions [**26**] contains a comprehensive list of characterisations for when a $\mathrm{C}(K)$ is a dual Banach space. Among these is a decomposition result for $\mathrm{C}(K)$.

Theorem 1.1.3 ([**26**, Theorem 6.4.1]). *Let K be a compact Hausdorff space. Consider the following statements.*

(i) K is hyper-Stonean.

(ii) $\mathrm{C}(K)$ is isometrically isomorphic to a dual Banach space.

(iii) Let $\mathcal{F}$ be a maximal singular family of normal probability measures on K, and for each $\mu \in \mathcal{F}$ let S_μ denote its support. Then

$$\mathrm{C}(K) \ni f \longmapsto \left(f|_{S_\mu}\right)_{\mu\in\mathcal{F}} \in \bigoplus_{\mu\in\mathcal{F}}^{\infty} \mathrm{C}(S_\mu)$$

is an isometric lattice isomorphism.

The statements (i) and (ii) are equivalent, and both (i) and (ii) imply (iii). If K is Stonean, then all three statements are equivalent.

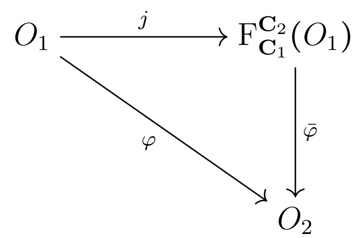
We can now state the question that we seek to address: In the same vein as [**69**], can we find a similar decomposition result for $\mathrm{C}(X)$ spaces that are order dual spaces?

Indeed, this is possible and this is precisely where the study of inverse limits in particular enters our research: We will show that for an extremally disconnected realcompact space X, the vector lattice $\mathrm{C}(X)$ is an order dual space precisely when $\mathrm{C}(X)$ is lattice isomorphic to the inverse limit of the carriers of its order continuous linear functionals. In its first form, the material in Sections 2.3 - 2.5 was developed with a view towards answering this question. Given the wider relevance discussed above, the material in Chapter 2 has taken on a more general form since and we have added a number of other applications along with the aforementioned decomposition theorem.

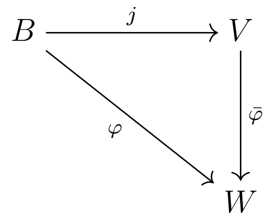
To conclude this section, we give a brief outline of the structure of Chapter 2. In Section 2.1, we record some preliminary definitions and results that are used in the rest of the chapter. This includes some topology, topological measure theory, and a brief section on measurable cardinals. In Section 2.2, we define the categories of vector lattices which from the setting of Chapter 2 and record a number of results on the properties of various linear operators between vector lattices. Sections 2.3 and 2.4 contain our treatment of direct and inverse limits of vector lattices. The structure of these two sections are the same: After stating the definitions of direct and inverse limits and noting that these constructions are essentially unique in some sense, we give some results on the existence of direct and inverse limits in our categories of vector lattices. In Section 2.4, we supplement the work of Filter by showing that certain properties of vector lattices have permanence under the construction of inverse limits, similar to what was done in [**37**] for direct limits. We conclude both sections with a few natural examples of objects that may be regarded as direct or inverse limits of vector lattices. While the examples are illustrative of the ideas involved, some of these examples will be instrumental for applications to follow. Section 2.5 contains a number of duality results: We will show that the order (continuous) dual of a direct limit of vector lattices (in some category of vector lattices) is an inverse limit of order (continuous) duals (in a related category of vector lattices). Similarly, but under more strict conditions, the order (continuous) dual of an inverse limit of vector lattices is a direct limit of order (continuous) duals. The results we have for order (continuous) duals of direct limits of vector lattices are fairly general while the results for order (continuous) duals of inverse limits require stronger assumptions. The impediment to more general results in the latter case will be identified. The rest of the chapter contains various applications of the preceding theory: Section 2.6 shows how the duality results from Section 2.5 can be applied to examples in Sections 2.3 and 2.4 to obtain order (continuous) duals of some function spaces. Section 2.7 contains results on the permanence of the vector lattice property of perfectness under the construction of direct and inverse limits along with a decomposition theorem for perfect vector lattices. Section 2.8 contains results on the permanence of the order dual space property under inverse limits. In Section 2.9, we record the decomposition result for $\mathrm{C}(X)$ spaces that are order dual spaces. In particular, we use the decomposition result in Section 2.7 to show that $\mathrm{C}(X)$ can be decomposed as the inverse limit of carriers of its order continuous functionals. Finally, in Section 2.10, we use the theory of direct and inverse limits of vector lattices to concretely characterise the class of Archimedean relatively uniformly complete vector lattices and their order duals.

1.1.2. Free objects. The notion of a free object is ubiquitous in the fields of category theory and algebra. In order to facilitate the exposition on free objects to follow, we state the general definition of a free object[6] upfront.

DEFINITION 1.1.4. Let $\mathbf{C}_1$ be a category and let $\mathbf{C}_2$ be a subcategory of $\mathbf{C}_1$. Fix $O_1 \in \mathbf{C}_1$ and consider an object $\mathrm{F}_{\mathbf{C}_1}^{\mathbf{C}_2}(O_1)$ in $\mathbf{C}_2$ and a morphism $j : O_1 \to \mathrm{F}_{\mathbf{C}_1}^{\mathbf{C}_2}(O_1)$ in $\mathbf{C}_1$. The pair $\left(\mathrm{F}_{\mathbf{C}_1}^{\mathbf{C}_2}(O_1), j\right)$ is called *a free object over* O_1 *of* $\mathbf{C}_2$ if it has the property that for every $O_2 \in \mathbf{C}_2$ and every morphism $\varphi : O_1 \to O_2$ in $\mathbf{C}_1$, there exists a unique morphism $\bar{\varphi} : \mathrm{F}_{\mathbf{C}_1}^{\mathbf{C}_2}(O_1) \to O_2$ in $\mathbf{C}_2$ such that the following diagram commutes in $\mathbf{C}_1$.



Indeed, one may consider any vector space as a free object over its basis: Let V be a vector space over a field $\mathbb{K}$ with basis B and denote by $j : B \to V$ the inclusion map. Then for any vector space W over $\mathbb{K}$ and any set map $\varphi : B \to W$ there exists a unique linear map $\bar{\varphi} : V \to W$ such that the following diagram commutes.



We say that the linear map $\bar{\varphi}$ is the unique *factoring morphism* through the pair (V, j), which is called a *free vector space over* B. Conversely, given any non-empty set S we can construct a vector space V_S and a map $j : S \to V_S$ satisfying the necessary universal property: Let V_S denote the collection of functions $f : S \to \mathbb{K}$ where the set $\{s \in S : f(s) \neq 0\}$ is finite. Define the addition and scalar multiplication operations on V_S by setting

$$(f+g)(s) := f(s) + g(s), \qquad (\alpha f)(s) := \alpha f(s) \qquad (f, g \in V_S, \quad \alpha \in \mathbb{K}).$$

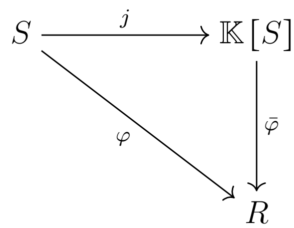
Further, for every $s \in S$ define $e_S \in V_S$ where

$$e_s(t) := \begin{cases} 1 \text{ if } t = s, \\ 0 \text{ if } t \neq s. \end{cases}$$

Define $j : S \to V_S$ where $j(s) := e_s$ for $s \in S$. It is clear that $\{e_s : s \in S\}$ is a basis for V_S and that for every vector space W and every set map $\varphi : S \to W$ there exists a unique linear map $\bar{\varphi} : V_S \to W$ such that $\bar{\varphi} \circ j = \varphi$. Thus the pair (V_S, j) may be called a *solution to a free object problem* between the category of sets (with functions as morphisms) and the category of vector spaces over $\mathbb{K}$ (with linear maps as morphisms) for the set S.

[6]See Section 1.2 for clarification of the categorical terminology used in the following definition.

Similarly, the set $\mathbb{K}[S]$ of polynomials with indeterminates $\{X_s : s \in S\}$ equipped with the standard operations along with the map $j : S \to \mathbb{K}[S]$ where $j(s) := X_s$ is a *free (unital) commutative associative algebra* over the set S. Thus for every (unital) commutative associative algebra R over $\mathbb{K}$ and every set map $\varphi : S \to R$ there exists a unique (unital) algebra homomorphism $\bar{\varphi} : \mathbb{K}[S] \to R$ such that the following diagram commutes.



In addition, if S is a singleton $\{X\}$, it is clear that $(\mathbb{K}[X], j)$ is a *free (unital) associative algebra* over the singleton $\{X\}$. That is to say, $(\mathbb{K}[X], j)$ satisfies the above universal property for both commutative and non-commutative (unital) algebras R.

It can also be shown that a *free vector lattice over a set* S exists: For every $s \in S$, define $d_s : \mathbb{R}^S \to \mathbb{R}$ where $d_s(f) := f(s)$ for $f \in \mathbb{R}^S$. Denote by $\mathrm{FVL}(S)$ the vector sublattice of $\mathbb{R}^{\mathbb{R}^S}$ generated by $\{d_s : s \in S\}$ and define $j : S \to \mathrm{FVL}(S)$ where $j(s) := d_s$. The pair $(\mathrm{FVL}(S), j)$ is then a free vector lattice over S. Results on free vector lattices are found in [**13**] and [**18**]. More examples of free objects in the context of algebra can be found in [**2**, Examples 8.23, p. 135]. In fact, using the language of universal algebra, one can show that for any *abstract algebraic structure* (i.e. a set equipped with operations satisfying certain identities) there exists a free object of this abstract algebraic type over any non-empty set (see Theorem 3.2.19).

In contrast with this rather complete picture of free objects we have in the context of algebra, the picture appears to be more sparse in the context of analysis. One positive result we have is the following.

PROPOSITION 1.1.5. *For every $n \in \mathbb{N}$, let $S_n := \{s_1, s_2, \ldots, s_n\}$ be a set. Consider the n-dimensional Banach space $(\mathbb{R}^n, \|\cdot\|_1)$ along with the map $j : S \to \mathbb{R}^n$ sending the i^{th} element of S_n to the i^{th} basis vector of $\mathbb{R}^n$. Then $((\mathbb{R}^n, \|\cdot\|_1), j)$ is a free Banach space over S_n.*

However, this result does not hold when we consider infinite sets.

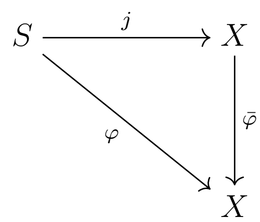
PROPOSITION 1.1.6. *Given any infinite set S, there is no free Banach space over S.*

PROOF. Suppose for the sake of a contradiction that there exists a free Banach space (X, j) over an infinite set S. Let $S_{\mathbb{N}} := \{s_1, s_2, \ldots\}$ be a denumerable subset of S. We show that the sequence $(j(s_n))$ is unbounded in X: Consider the Banach space $\mathbb{R}$ and the map $\psi : S \to \mathbb{R}$ where $\psi(s_n) := n$ for $s_n \in S_{\mathbb{N}}$ and $\psi(s) = 0$ for $s \in S \setminus S_{\mathbb{N}}$. There exists a unique bounded linear map $\bar{\psi} : X \to \mathbb{R}$ such that for all $n \in \mathbb{N}$,

$$n = |\psi(s_n)| = \left|\bar{\psi}(j(s_n))\right| \le \left\|\bar{\psi}\right\| \|j(s_n)\| .$$

If we assumed that there exists $N \in \mathbb{N}$ uniformly bounding the norms of the sequence $(j(s_n))$, this would contradict the Archimedean property of $\mathbb{N}$. Thus we may choose a sequence of indices (n_k) in $\mathbb{N}$ such that $j(s_{n_k}) \neq 0$ for all $k \in \mathbb{N}$ and $\|j(s_{n_{k+1}})\| \geq k\,\|j(s_{n_k})\|$ for all $k \in \mathbb{N}$. Denote by S_0 the subset of $S_{\mathbb{N}}$ indexed by the sequence (n_k).

Now, define the map $\varphi : S \to X$ where $\varphi(s_{n_k}) \coloneqq j\left(s_{n_{k+1}}\right)$ for $k \in \mathbb{N}$ and $\varphi(s) = 0$ for $s \in S \setminus S_0$. Then there exists a unique bounded linear map $\bar{\varphi} : X \to X$ such that the following diagram commutes.



Thus for every $k \in \mathbb{N}$, we have

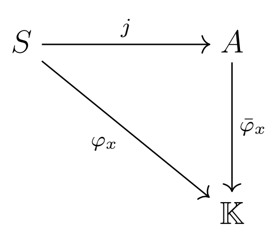
$$\|\bar{\varphi}\|\,\|j(s_{n_k})\| \geq \|\bar{\varphi}(j(s_{n_k}))\| = \|\varphi(s_{n_k})\| = \|j(s_{n_{k+1}})\| \geq k\,\|j(s_{n_k})\|$$

which implies that $\|\bar{\varphi}\| \geq k$ for all $k \in \mathbb{N}$, which is impossible. $\square$

The picture breaks down even further if consider Banach algebras. The following result is found in [**27**, Examples 1(7)]. We reproduce a short proof here since it is so striking.

PROPOSITION 1.1.7. *Given any non-empty set S, there is no free Banach algebra over S.*

PROOF. Suppose for the sake of a contradiction that there exists a free Banach algebra (A, j) over a non-empty set S. Fix a point $s_0 \in S$ and for every $x \in \mathbb{K} \setminus \{0\}$, define the map $\varphi_x : S \to \mathbb{K}$ where $\varphi_x(s_0) = x$ and $\varphi(s) = 0$ for $s \in S \setminus \{s_0\}$. Thus, for every $x \in \mathbb{K} \setminus \{0\}$, there exists a unique bounded algebra homomorphism $\bar{\varphi}_x : A \to \mathbb{K}$ such that the following diagram commutes.



Then for every $n \in \mathbb{N}$,

$$\begin{aligned} |x|^n &= \|[\varphi_x(s_0)]^n\| \\ &= \|[\bar{\varphi}_x(j(s_0))]^n\| \\ &= \|\bar{\varphi}_x\left([j(s_0)]^n\right)\| \\ &\leq \|\bar{\varphi}_x\|\,\|[j(s_0)]^n\| \\ &\leq \|\bar{\varphi}_x\|\,\|j(s_0)\|^n . \end{aligned}$$

Thus $(|x|/\|j(s_0)\|)^n \leq \|\bar{\varphi}_x\|$ for all $n \in \mathbb{N}$ which implies that $|x| \leq \|j(s_0)\|$ for all $x \in \mathbb{K} \setminus \{0\}$, which is impossible. □

It is readily seen that the proofs of Propositions 1.1.6 and 1.1.7 respectively show that there is no free normed space over an infinite set and no free normed algebra over any non-empty set. Indeed, if we consider Proposition 1.1.6, the result will also hold if we consider any subcategory **C** of the category of normed spaces with bounded linear maps as morphisms where $\mathbb{R} \in \mathbf{C}$. A similar statement is also true for Proposition 1.1.7.

It is important to note that the non-existence of free objects proven in Propositions 1.1.6 and 1.1.7 is done with respect to the definition of a free object given in Definition 1.1.4. In light of these results, we make some observations. In Proposition 1.1.6, we constructed a set map $\varphi : S \to \mathbb{R}$ that grows too quickly for there to exist a *bounded* linear factoring morphism $\bar{\varphi} : X \to \mathbb{R}$. In Proposition 1.1.7 on the other hand, a contradiction arises since there are essentially too many morphisms that we are required to factor through the fictional free Banach algebra. One way to negotiate these problems is to restrict ourselves to a smaller class of morphisms that are required to factor through a free object. For example, the contradictions arrived at in Propositions 1.1.6 and 1.1.7 will not occur if we only consider morphisms $\varphi : S \to Y$ such that $\|\varphi(s)\| \leq M$ for all $s \in S$ for some fixed $M > 0$. Results of this sort already exist in the literature: Let S be a non-empty, possibly infinite set and consider

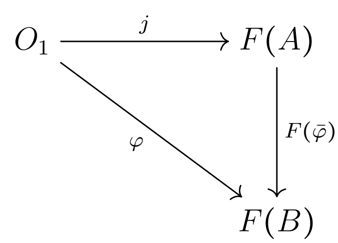
$$\ell^1(S) := \left\{ f \in \mathbb{R}^S : \sum_{s \in S} |f(s)| < \infty \right\}$$

along with the map $j : S \to \ell^1(S)$ where $j(s) := e_s$ and e_s denotes the indicator function of $\{s\}$ as defined above for the free vector space over a set. Then for every Banach space Y and every set map $\varphi : S \to Y$ with $\|\varphi(s)\| \leq 1$ for all $s \in S$ there exists a unique bounded linear map $\bar{\varphi} : \ell^1(S) \to Y$ such that $\bar{\varphi} \circ j = \varphi$. The pair $(\ell^1(S), j)$ is often called *the free Banach space over* S in the literature. This idea is generalised in [**41**] where the author introduces the notion of a 'normed set' (S, η) where S is a set equipped with a 'sizing function' $\eta : S \to [0, \infty)$ serving the role of a norm. It is then shown that *the free Banach space over a normed set* (S, η) is the ℓ^1-space weighted by the sizing function η. The rest of [**41**] deals with the (non-)existence of free normed structures over these normed sets. Further, it was shown in [**29**] that for any non-empty set S, there exists a Banach lattice $\mathrm{FBL}(S)$ and a map $j : S \to \mathrm{FBL}(S)$ with the property that for every Banach lattice B and every set map $\varphi : S \to B$ with $\|\varphi(s)\| \leq 1$ for all $s \in S$ there exists a unique bounded vector lattice homomorphism $\bar{\varphi} : \mathrm{FBL}(S) \to B$ such that $\bar{\varphi} \circ j = \varphi$. The pair $(\mathrm{FBL}(S), j)$ is called *the free Banach lattice over* S in [**29**]. Given the discrepancy between the universal properties of the pairs $(\ell^1(S), j)$ and $(\mathrm{FBL}(S), j)$ and Definition 1.1.4, we will introduce alternative terminology for these kind of objects, which we describe shortly.

In recent years, the notion of a *free Banach lattice* over various different structures has become a very active area of research. The *free Banach lattice over a Banach* *space* is defined in [**8**] and [**64**], and the *free Banach lattice over a lattice* is defined in [**10**]. Properties of these free Banach lattices are studied in [**7**], [**9**], [**11**], and [**46**].

It is encouraging to see that a lot more can be said about free objects in categories of normed structures once one has accepted the compromise of only considering bounded morphisms. At least, this seems like a compromise when considering the definition of a free object given in Definition 1.1.4. If we instead consider the more general categorical definition of a free object given in [**2**, Definition 8.22], it turns out that the pairs $(\ell^1(S), j)$ and $(\mathrm{FBL}(S), j)$ defined above are in fact free objects.

DEFINITION 1.1.8. Let $\mathbf{C}_1$ and $\mathbf{C}_2$ be categories with $F : \mathbf{C}_2 \to \mathbf{C}_1$ a faithful functor.[7] Fix $O_1 \in \mathbf{C}_1$ and consider an object A in $\mathbf{C}_2$ and a morphism $j : O_1 \to F(A)$ in $\mathbf{C}_1$. The pair (A, j) is called a *free object over* O_1 *of the functor* F if for every $B \in \mathbf{C}_2$ and every morphism $\varphi : O_1 \to F(B)$ in $\mathbf{C}_1$ there exists a unique morphism $\bar{\varphi} : A \to B$ such that the following diagram commutes in $\mathbf{C}_1$.

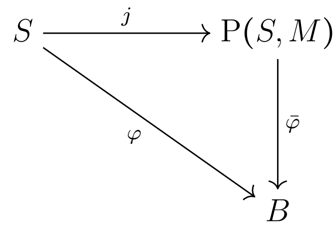


We note that Definition 1.1.4 is a special case of Definition 1.1.8 since Definition 1.1.4 implicitly makes use of the *inclusion functor* $I : \mathbf{C}_2 \to \mathbf{C}_1$ where $\mathbf{C}_2 \subseteq \mathbf{C}_1$. From the point of view of Definition 1.1.8, the pair $(\ell^1(S), j)$ is a free object if we consider the so-called '*unit ball functor*' $O : \mathbf{Ban} \to \mathbf{Set}$ which sends every Banach space X to the underlying set of its closed unit ball $\mathbf{B}_X$, see [**2**, Examples 8.23 (12)]. The same is true of the pair $(\mathrm{FBL}(S), j)$ from [**29**] if we consider the associated unit ball functor $O : \mathbf{BL} \to \mathbf{Set}$.

We will make no further use of the definition of a free object given in Definition 1.1.8 and only use the simpler version of the definition of a free object given in Definition 1.1.4. Instead, we will consider the free Banach space $(\ell^1(S), j)$ over a set S and the free Banach lattice over a set in [**29**] as particular examples of *pseudo-solutions* of free object problems. The term '*pseudo*' is used to denote that a pair (P, j) is not a free object in the full sense of Definition 1.1.4, but where some constraint has been imposed on the morphisms we require to factor through (P, j). One particular pseudo-solution we will find is the following: Let S be an arbitrary set and $M : S \to \mathbb{R}$ any non-negative function. We will construct a *positive unital Banach lattice algebra* $\mathrm{P}(S, M)$ (see Section 3.1.2) along with a map $j : S \to \mathrm{P}(S, M)$ such that for every positive unital Banach lattice algebra B and every map $\varphi : S \to B$ such that $\|\varphi(s)\| \le M(s)$ for all $s \in S$, there exists a unique bounded unital vector lattice algebra homomorphism $\bar{\varphi} : \mathrm{P}(S, M) \to B$ such that the following diagram

[7]Given categories $\mathbf{X}$ and $\mathbf{Y}$, a functor $F : \mathbf{X} \to \mathbf{Y}$ is *faithful* if for every $A, B \in \mathbf{X}$, the induced function $F_{A,B} : \mathrm{Hom}_{\mathbf{X}}(A, B) \to \mathrm{Hom}_{\mathbf{Y}}(F(A), F(B))$ mapping a morphism $f : A \to B$ to $F(f) : F(A) \to F(B)$ is injective.

commutes.



The existence of this pseudo-solution will be proven in detail in Theorem 3.3.6. These pseudo-solutions are similar to the idea of the free Banach space over a normed set in [**41**], although they will not be introduced in an attempt to reconcile the necessity of bounds in this context with the definition of a free object as was done in [**41**]. Instead, these pseudo-solutions will be put to a different use in Chapter 3, which we describe shortly.

In contrast to this approach of constraining the morphisms required to factor through a free object, another way to negotiate the problem of free objects in categories of normed structures is to *enlarge the category* in which a free object is required to reside from a category of *normed structures* to a category of *locally convex structures*. This may not immediately seem like useful compromise: If we borrow the notation in Definition 1.1.4 for a moment, in this larger category there would be more candidate objects and morphisms among which we can find a pair (F, j) satisfying the necessary universal property, however, there will also be more O_2 objects and thus more $\varphi : O_1 \to O_2$ morphisms that need to factor through (F, j). Nevertheless, this does turn out to be a viable strategy. We may call the pseudo-solutions approach and the enlargement of solution-category approach the two alternatives to the problem of free objects in categories of normed structures.

Indeed, it is not difficult to show that there exists a free (complete) locally convex space over an arbitrary set S: Fix any set S and consider the free vector space (V_S, j) over S. Take the class of morphisms $\varphi : S \to W$ where W is any (complete) locally convex space and consider the associated class of unique factoring linear maps $\bar{\varphi} : V_S \to W$. We endow V_S with the initial topology τ generated by these factoring morphisms $\bar{\varphi}$ ranging over all (complete) locally convex spaces W. The details in [**60**, p. 51] show that (V_S, τ) is a (Hausdorff) locally convex space and by the definition of the initial topology this makes $((V_S, \tau), j)$ into the free locally convex space over S and by taking a completion of (V_S, τ) we obtain the free complete locally convex space over S.

It seems reasonable to expect that the above approach of starting with the abstract existence of an algebraic free object as a foundation and then adding an initial topology should also work for the proof of existence of free objects in other categories of locally convex structures (see for example Table 4 in Section 3.1.3). However, the above proof and its modification to other categories would not yield much more information other than the existence of these free objects. In the sequel, we will construct free objects in certain categories of (complete) locally convex structures as *inverse limits of (complete) normed structures*. It is known in general that the complete locally convex spaces are precisely the inverse limits of Banach spaces [**60**, Chapter II, Theorem 5.4] and that the complete locally m-convex algebras (defined in Section 3.1.3) are precisely the inverse limits of Banach algebras [**14**, Theorem 4.5.3]. However, our approach will show *which* normed structures can be used in the construction of these free locally convex structures via inverse limits. These will turn out to be nothing else but these *pseudo-solutions* alluded to above. As a result, the inverse limit construction we detail in Chapter 3 will allow us to transmute one alternative to the problem of free objects in categories of normed structures (restriction to the bounded morphisms) into the other alternative (enlargement of the solution category).

Beyond their application to the construction of free objects in categories of locally convex structures, the significant number of pseudo-solutions we will obtain are also interesting in their own right since they expand the picture of the existing literature we mentioned above. In particular, the existence of pseudo-solutions in categories of Banach lattice algebras allows us to give some partial answers to both Problems 13 and 15 in [**67**] in Section 3.3.

We now give a brief overview of the structure of Chapter 3. In Section 3.1, we record notation, terminology, and some preliminary results used in the rest of the chapter. For the sake of keeping the chapter self-contained we will repeat the definition of a free object in Section 3.1.4 before we discuss why free objects are essentially unique, when they exist. Section 3.2 gives an account of results from universal algebra that are used to prove the existence of free objects in categories of algebraic structures. In Section 3.3, we define the notion of a pseudo-solution of a free object problem and give a uniform approach for the construction of various pseudo-solutions. In Section 3.4, we use the approach developed in Section 3.3 to concretely describe the pseudo-solution of free object problems between the categories **Ban** (Banach spaces with bounded linear maps as morphisms) and **Set** (sets with functions as morphisms) as well as the pseudo-solution of a free object problem between the categories $\mathbf{BA^1}$ (unital Banach algebras with bounded unital algebra homomorphisms as morphisms) and **Set**. In Section 3.5, we prove general categorical results which we will use as our tools to transmute our *inverse systems of pseudo-solutions* into free objects in categories of locally convex structures. In order to apply these tools, we need to show that certain categories of locally convex structures are in fact *categories of inverse limits* of normed structures; this is done in Section 3.6. Armed with the pseudo-solutions found in Section 3.3, the tools developed in Section 3.5, and a characterisation of certain categories of locally convex structures in Section 3.6, we will be able to prove the existence of free objects in categories of locally convex structures in Section 3.7. Lastly, using the concrete pseudo-solutions found in Section 3.4 and the general approach outlined in Section 3.7, we will give concrete descriptions of the free complete unital locally m-convex algebra over a point in Section 3.8 as well as the free (complete) locally convex space over a arbitrary set in Section 3.9.

This concludes our motivation of the problems we will consider in this study. The remaining sections in this chapter record some terminology and notation used in the rest of the document. Readers who are acquainted with both category theory and vector lattices may happily skip these sections.

1.2. Category theory

As outlined in the previous section, this study will make extensive use of some themes from category theory. However, we note upfront that the conceptual sophistication of the category theory we use will be fairly modest throughout. With a few exceptions in Chapter 3, notably Section 3.5, category theory will mostly be harnessed as a natural organisational tool for our work.

Given how extensive our use of the notions of categories and subcategories are, we record these definitions here for the sake of completeness.

DEFINITION 1.2.1. A *category* $\mathbf{C}$ consists of a pair of classes $(\mathrm{Obj}_{\mathbf{C}}, \mathrm{Hom}_{\mathbf{C}})$ where the elements in $\mathrm{Obj}_{\mathbf{C}}$ and $\mathrm{Hom}_{\mathbf{C}}$ are called $\mathbf{C}$-*objects* and $\mathbf{C}$-*morphisms* respectively. These classes satisfy the following statements.

(i) *Domains and co-domains:* For every $\mathbf{C}$-morphism f, there are two unique $\mathbf{C}$-objects called the *domain* and *co-domain* of f. We denote these as $\mathrm{d}(f)$ and $\mathrm{c}(f)$ respectively. If g is a $\mathbf{C}$-morphism with $A, B \in \mathrm{Obj}_{\mathbf{C}}$ such that $\mathrm{d}(g) = A$ and $\mathrm{c}(f) = B$, then we encode this information by using the *function notation* $g : A \to B$.

(ii) *Composition of morphisms:* For $\mathbf{C}$-morphisms $f : A \to B$ and $g : B \to C$, there exists a $\mathbf{C}$-morphism $h : A \to C$ which we call the *composition of* f *and* g and denote by $g \circ f := h$.

(iii) *Identity morphisms:* For each $\mathbf{C}$-object A, there exists an *identity morphism* $\mathbf{1}_A : A \to A$ in $\mathbf{C}$.

(iv) *Associativity of composition:* Let $f : A \to B$, $g : B \to C$, and $h : C \to D$ be $\mathbf{C}$-morphisms. The composition of morphisms is associative, i.e.

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

(v) *Composition with identity:* For each $\mathbf{C}$-morphism $f : A \to B$, we have that

$$f \circ \mathbf{1}_A = f = \mathbf{1}_B \circ f.$$

DEFINITION 1.2.2 (Subcategories). Consider a category $\mathbf{C} := (\mathrm{Obj}_{\mathbf{C}}, \mathrm{Hom}_{\mathbf{C}})$. Then a pair of subclasses $\mathrm{Obj}_{\mathbf{D}} \subseteq \mathrm{Obj}_{\mathbf{C}}$ and $\mathrm{Hom}_{\mathbf{D}} \subseteq \mathrm{Hom}_{\mathbf{C}}$ forms a *subcategory* $\mathbf{D} := (\mathrm{Obj}_{\mathbf{D}}, \mathrm{Hom}_{\mathbf{D}})$ of $\mathbf{C}$ if the following conditions are satisfied:

(i) For every $A \in \mathrm{Obj}_{\mathbf{D}}$, the identity morphism $\mathbf{1}_A$ is in $\mathrm{Hom}_{\mathbf{D}}$.

(ii) If $f : A \to B$ is in $\mathrm{Hom}_{\mathbf{D}}$, then A and B are in $\mathrm{Obj}_{\mathbf{D}}$.

(iii) If $f : A \to B$ and $g : B \to C$ are in $\mathrm{Hom}_{\mathbf{D}}$, then the composition $g \circ f : A \to C$ is also in $\mathrm{Hom}_{\mathbf{D}}$.

We denote the fact that $\mathbf{D}$ is a subcategory of $\mathbf{C}$ by writing $\mathbf{D} \subseteq \mathbf{C}$. Occasionally, we will call the category $\mathbf{C}$ the *larger* or *weaker* category when compared with $\mathbf{D}$.

The function notation $f : \mathrm{d}(f) \to \mathrm{c}(f)$ for morphisms used above is standard in category theory. This is due to its evident utility, despite the fact that objects and morphisms in arbitrary categories *need not* be sets or functions between sets.

However, all categories we consider in the sequel will be subcategories of **Set**, thus the morphisms we consider will indeed be functions between sets.

Even though a category consists of both objects and morphisms, we will occasionally commit an abuse of notation by using the shorthand $X \in \mathbf{C}$ when a structure X is an object in a category $\mathbf{C}$. For morphisms, we will always write that $f : A \to B$ is a morphism in $\mathbf{C}$ or say that $f : A \to B$ is a $\mathbf{C}$-morphism. Further, if $\mathbf{C}$ is a category and $X, Y \in \mathbf{C}$, we denote by $\mathrm{Hom}_{\mathbf{C}}(X, Y)$ the collection of all morphisms $f : X \to Y$ in $\mathbf{C}$. A morphism $f : A \to B$ in $\mathbf{C}$ is called a $\mathbf{C}$-*isomorphism* if there exists a morphism $g : B \to A$ such that $g \circ f = \mathbf{1}_A$ and $f \circ g = \mathbf{1}_B$. If $f : A \to B$ is an $\mathbf{C}$-isomorphism then the objects A and B are *isomorphic* in $\mathbf{C}$.

The following basic definition is, despite its simplicity, a fundamental example of a *universal property.*

Definition 1.2.3 (Initial objects and terminal objects). Let $\mathbf{C}$ be any category. An object $I \in \mathbf{C}$ is called an *initial object* if for any object $C \in \mathbf{C}$, there exists a unique morphism $f : I \to C$. Similarly, an object $T \in \mathbf{C}$ is called a *terminal object* if for any object $C \in \mathbf{C}$, there exists a unique morphism $g : C \to T$.

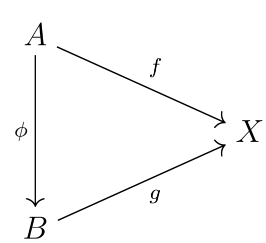
The following result states that initial and terminal objects are '*essentially unique*' with respect their universal property, when they exist. A proof is found in [**2**, Chapter II, Proposition 7.3]

Proposition 1.2.4. *Initial objects are unique up to a unique isomorphism and all objects isomorphic to an initial object are themselves initial objects. Dually, terminal objects are unique up to a unique isomorphism and all objects isomorphic to a terminal object are themselves terminal objects.*

This last result shows that we get a lot of information for free when a particular object is identified as an initial or terminal object in some category. In the relevant sections, we will note how the universal property of direct limits, inverse limits, and free objects make these categorical objects into initial or terminal objects in some *derived category.* We briefly motivate this concept of a derived category: If $\mathbf{C}$ is any category, then a *derived category* $\mathbf{C}'$ is any category where the objects and morphisms in $\mathbf{C}'$ are built up from objects and morphisms in $\mathbf{C}$. For example, fix an object X in $\mathbf{C}$ and construct the category $\mathbf{C}'$ in the following way:

(i) *Objects in* $\mathbf{C}'$: Pairs (A, f) where $A \in \mathbf{C}$ and $f : A \to X$ is a morphism in $\mathbf{C}$.

(ii) *Morphisms in* $\mathbf{C}'$: A morphism between objects (A, f) and (B, g) in $\mathbf{C}'$ is a morphism $\phi : A \to B$ such that the following diagram commutes.



It is easily verified from Definition 1.2.1 that $\mathbf{C}'$ is indeed a category.

These few definitions and pieces of terminology from category theory will be sufficient for the moment. We will introduce the central categorical concepts of *direct limits*, *inverse limits*, and *free objects* mentioned above in the chapters where these are relevant. This is done for the sake of readability and ease of reference.

1.3. Vector lattices

Since the notion of a vector lattice is used in both Chapter 2 and 3, we take the opportunity here to very briefly recall a few concepts and facts from the theory of vector lattices. For undeclared terms and notation we refer to the reader to any of the standard texts in the field, for instance [**4**], [**6**], [**53**], and [**70**].

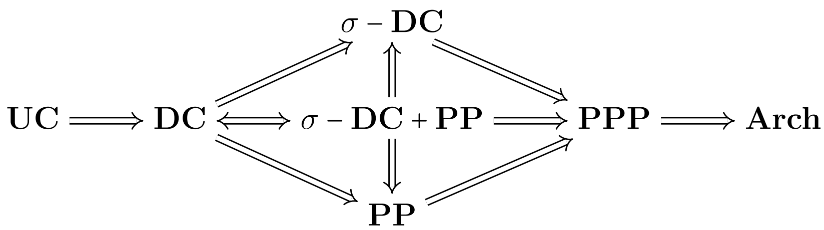
We will only consider real vector lattices in this document. Let E be a vector lattice. For $u \in E$, the elements $u^+ \coloneqq u \vee 0$, $u^- \coloneqq (-u) \vee 0$, and $|u| \coloneqq u \vee (-u)$ are called the *positive part*, *negative part*, and *absolute value* of u, respectively. Vectors $u, v \in E$ are *disjoint* if $|u| \wedge |v| = 0$. For subsets $A, B \subseteq E$, denote $A \vee B \coloneqq \{u \vee v \ : \ u \in A, \ v \in B\}$. The sets $A \wedge B$, A^+, A^- and $|A|$ are defined similarly. In particular, the set E^+ is called the *positive cone of* E. The set $A^d \coloneqq \{u \in E \ : \ |u| \wedge |v| = 0 \text{ for all } v \in A\}$ is called the *disjoint complement* of A. For $u, v \in E$, the collection $[u, v] \coloneqq \{x \in E : u \leq x \leq v\}$ is an *order interval* and subsets of order intervals are *order bounded*. We write $A \downarrow u$ if A is downwards directed and $\inf A = u$. Similarly, we write $B \uparrow u$ if B is upwards directed and $\sup B = u$.

A vector lattice E is *Archimedean* if $\frac{1}{n} u \downarrow 0$ for every $u \in E^+$. In both Chapters 2 and 3 we do *not* assume that vector lattices are Archimedean unless this is stated explicitly.

Given $u \in E^+$, a sequence (v_n) in E converges *u-uniformly* to $v \in E$ if for every $\epsilon > 0$ there exists $N_\epsilon \in \mathbb{N}$ such that if $n \geq N_\epsilon$ then $|v_n - v| \leq \epsilon u$. Further, the sequence (v_n) is *u-uniformly Cauchy* if for every $\epsilon > 0$ there exists $M_\epsilon \in \mathbb{N}$ such that if $n, m \geq M_\epsilon$ then $|v_n - v_m| \leq \epsilon u$. A vector lattice E is *relatively uniformly complete* if for every $u \in E^+$ we have that every u-uniformly Cauchy sequence has a u-uniform limit in E.

A subset $S \subseteq E$ is *solid in* E if for every $u \in S$ and $v \in E$, $|v| \leq |u|$ implies that $v \in S$. A linear subspace A of E is a *vector sublattice* if for every $u, v \in E$ we have that $u \vee v \in A$ and $u \wedge v \in A$. Solid linear subspaces are *order ideals*. Further, an order ideal B is a *band* if B has the following property: If $D \subseteq B$ such that $\sup D$ exists in E, then $\sup D \in B$. Given a subset $A \subseteq E$, the *order ideal generated by* A is the smallest order ideal in E containing A, which we denote as E_A. The *band generated by* A is defined similarly and is denoted by B_A. In particular, if $A = \{u\}$ for some $u \in E$, we write E_u and B_u to denote the ideal and band generated by $\{u\}$, respectively. These are respectively referred to as the *principal order ideal generated by* u and the *principal band generated by* u. If E is Archimedean and $A \subseteq E$, then $B_A = A^{dd}$ [**6**, Theorem 1.39].

A band B in E is a *projection band* if $B \oplus B^d = E$. A vector lattice E has the *projection property* if all of the bands in E are projection bands. If the previous statement only holds for principal bands, then E has the *principal projection property*. A vector lattice E is *Dedekind complete* if every non-empty subset of E which is bounded above (resp. bounded below) in E has a supremum (resp. infimum) in E. If the previous statement only holds for sequences (u_n) in E which are bounded above (resp. bounded below), then E is called *σ-Dedekind complete*. Further, a vector lattice E is *laterally complete* if every non-empty subset of pairwise disjoint vectors has a supremum [**6**, p. 106]. Lastly, a vector lattice E is *universally complete* if E is both Dedekind complete and laterally complete. Using the obvious abbreviations for the vector lattice properties defined thus far, we have the following implications for any vector lattice E, see [**53**, Theorems 25.1,42.8].



Let E and F be vector lattices and $T : E \to F$ be a linear operator. Recall that T is *positive* if $T[E^+] \subseteq F^+$, and *regular* if T is the difference of two positive operators. T is *order bounded* if T maps order bounded sets in E to order bounded sets in F. It is an important result that when F is Dedekind complete, T is order bounded precisely when T is regular [**71**, Theorem 20.2]. Further, T is *order continuous* if $\inf |T[A]| = 0$ whenever $A \downarrow 0$ in E. Every order continuous operator is necessarily order bounded [**6**, Theorem 1.54]. Denote by $E^\sim$ the collection of order bounded linear functionals $\phi : E \to \mathbb{R}$ and denote by $E^\sim_{\mathrm{n}}$ the collection of order continuous linear functionals. We refer to $E^\sim$ and $E^\sim_{\mathrm{n}}$ respectively as the *order dual* and the *order continuous dual* of E. For any vector lattice E, it is known that $E^\sim_{\mathrm{n}}$ is a band in $E^\sim$ [**71**, Theorem 22.2]. If $A \subseteq E$ and $B \subseteq E^\sim$ we define the *annihilator of A* and the *pre-annihilator of B* as the sets

$$A^\circ := \{\varphi \in E^\sim \ : \ \varphi(u) = 0, \ u \in A\}, \qquad {}^\circ B := \{u \in E \ : \ \varphi(u) = 0, \ \varphi \in B\}$$

respectively. For $\varphi \in E^\sim$, the *null ideal* (or *absolute kernel*) of φ is

$$\mathrm{N}_\varphi := \{u \in E \ : \ |\varphi|(|u|) = 0\}.$$

The *carrier* of φ is $\mathrm{C}_\varphi := \mathrm{N}_\varphi^d$. The null ideal N_φ of φ is an order ideal in E and its carrier C_φ is a band; if φ is order continuous then N_φ is also a band in E, see for instance [**70**, §90].

A linear operator $T : E \to F$ between vector lattices is a *lattice homomorphism* if it preserves suprema and infima of finite sets, and a *normal lattice homomorphism* if it preserves suprema and infima of arbitrary sets. Equivalently, T is a normal lattice homomorphism if and only if T is an order continuous lattice homomorphism, see [**53**, p. 103]. Further, T is a *lattice isomorphism* if it is bijective lattice homomorphism; equivalently, if it is bijective and bipositive (i.e. both T and T^{-1} are positive operators) [**71**, Theorem 19.3]. Lastly, we say that T is *interval preserving* if for all $u \in E^+$, we have $T[[0,u]] = [0, T(u)]$. An interval preserving map need not be a lattice homomorphism, nor is a (normal) lattice homomorphism in general interval preserving, see for instance [**6**, p. 95]. However, if T is bijective then T is a lattice homomorphism if and only if T is interval preserving, if and only if T is bipositive.

Lastly, let E be a vector lattice with B an ideal in $E^{\sim}$. For $u \in E$, define $\Psi_{B,u} \in B^{\sim}_{\mathrm{n}}$ where $\Psi_{B,u}(\varphi) \coloneqq \varphi(u)$ for $\varphi \in B$. The map $\sigma : E \to B^{\sim}_{\mathrm{n}}$ where $u \longmapsto \Psi_{B,u}$ is a lattice homomorphism and is injective precisely when ${}^{\circ}B = \{0\}$, see [**70**, p. 404-405]. Given that $B^{\sim}_{\mathrm{n}}$ is a band in $B^{\sim}$, we will occasionally also consider the map $\sigma : E \to B^{\sim}$. When ${}^{\circ}B = \{0\}$, we refer to the map $\sigma : E \to B^{\sim}_{\mathrm{n}}$ as a *canonical embedding*. We will most often consider the cases where $B = E^{\sim}$ or $B = E^{\sim}_{\mathrm{n}}$. In particular for $B = E^{\sim}_{\mathrm{n}}$, a vector lattice E is called *perfect* if $\sigma[E] = (E^{\sim}_{\mathrm{n}})^{\sim}_{\mathrm{n}}$.

This concludes our general overview and preliminaries. The next chapter starts with some additional brief preliminaries specific to Chapter 2.

CHAPTER 2

Direct and inverse limits of vector lattices

2.1. Preliminaries

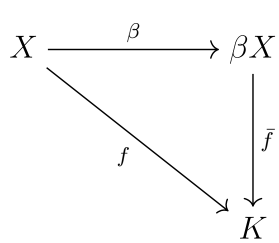
2.1.1. Topological preliminaries. Let X and Y be topological spaces and denote by $\mathrm{C}(X,Y)$ the collection of all continuous functions $f : X \to Y$. In particular, we write $\mathrm{C}(X) \coloneqq \mathrm{C}(X,\mathbb{R})$ where $\mathbb{R}$ is given the standard topology. We recall some topological definitions and results pertaining to the structure of $\mathrm{C}(X)$.

Denote by $\mathrm{C_b}(X)$ the order ideal of bounded continuous functions in $\mathrm{C}(X)$. Both $\mathrm{C}(X)$ and $\mathrm{C_b}(X)$ are relatively uniformly complete vector lattices [**53**, Theorem 43.1]. A topological space X is *Hausdorff* if for every $x, y \in X$ with $x \neq y$ there exists open sets U and V with $x \in U$ and $y \in V$ such that $U \cap V = \varnothing$. Further, X is *completely regular* if for every every closed set C in X and $x \notin C$, there exists $f \in \mathrm{C}(X)$ such that $f(x) = 1$ and $f|_C = 0$. Completely regular Hausdorff spaces are called *Tychonoff spaces*. The following result shows that there is no loss of generality in only considering Tychonoff spaces X when studying the vector lattice structure of a $\mathrm{C}(X)$ space.

Theorem 2.1.1 ([**39**, Theorem 3.9]). *For every topological space X there exists a Tychonoff space Y and a surjective continuous map $\tau : X \to Y$ such that the map $T : \mathrm{C}(Y) \to \mathrm{C}(X)$ where $f \longmapsto f \circ \tau$ is a vector lattice isomorphism.*

A *compactification* of a Tychonoff space X is any compact Hausdorff space, say cX, into which X can be homeomorphically embedded as a dense subset. The *Stone-Čech compactification*, which is in some sense the 'maximal' compactification of a Tychonoff space X (see [**36**, Sections 3.5, 3.6]), is characterised by the following result.

Theorem 2.1.2 (The Stone-Čech compactification). *Let X be a Tychonoff space. There exists a unique (up to a unique homeomorphism) compact Hausdorff space βX and a homeomorphic embedding $\beta : X \to \beta X$ with $\beta[X]$ dense in βX such that for every compact Hausdorff space K and every $f \in \mathrm{C}(X,K)$ there exists a unique $\bar{f} \in \mathrm{C}(\beta X, K)$ such that the following diagram commutes.*



We call the compact Hausdorff space βX *satisfying the above conditions the* Stone-Čech compactification *of* X.

Since the closure of the range of elements in $\mathrm{C_b}(X)$ is bounded, hence compact, in $\mathbb{R}$, we have the following result.

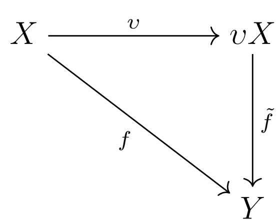
Corollary 2.1.3. *For every Tychonoff space* X, *the restriction map*

$$R : \mathrm{C}(\beta X) \to \mathrm{C_b}(X)$$

is a vector lattice isomorphism.

Recall that a subspace S of a topological space X is C-*embedded in* X if every continuous function on S admits a continuous extension to X. Similarly, the subspace S is C*-*embedded in* X if every bounded continuous function on S admits a continuous bounded extension to X. A Tychonoff space X is *realcompact* if X cannot be embedded into a Tychonoff space $\tilde{X}$ as a proper, dense C-embedded subspace [**36**, p. 214]. Realcompact spaces are characterised as the closed subspaces of $\mathbb{R}^m$ where m is some cardinal [**36**, Theorem 3.11.3]. As a result, all Euclidean spaces $\mathbb{R}^n$ are realcompact. Further, X is a compact topological space if and only if X is realcompact and pseudocompact [**36**, Theorem 3.11.1]. The *realcompactification* υX of a Tychonoff space X is constructed as the largest subspace of βX in which X is C-embedded [**66**, § 1.53, p. 30] and satisfies the following universal property.

Theorem 2.1.4 (Universal property of the realcompactification). *Let* X *be a Tychonoff space. There exists a unique (up to a unique homeomorphism) realcompact space* υX *and a homeomorphic embedding* $\upsilon : X \to \upsilon X$ *with* $\upsilon[X]$ *dense in* υX *such that for every realcompact space* Y *and every* $f \in \mathrm{C}(X, Y)$ *there exists a unique* $\tilde{f} \in \mathrm{C}(\upsilon X, Y)$ *such that the following diagram commutes.*



It is clear that a Tychonoff space X is realcompact if and only if $X = \upsilon X$ and it follows from the construction of the realcompactification that X is pseudocompact if and only if $\upsilon X = \beta X$. Further, since $\mathbb{R}$ is realcompact, we have the following result.

Corollary 2.1.5. *Let* X *be a Tychonoff space. The spaces of continuous functions* $\mathrm{C}(X)$ *and* $\mathrm{C}(\upsilon X)$ *are isomorphic as vector lattices.*

As a result, in view of Theorem 2.1.1, there is no loss of generality in only considering realcompact spaces X when studying the vector lattice structure of $\mathrm{C}(X)$ spaces. In fact, realcompact spaces are homeomorphic precisely when the associated spaces of continuous functions are isomorphic as vector lattices.

THEOREM 2.1.6 ([**42**, Theorems 9.1, 9.2]). *Realcompact topological spaces X and Y are homeomorphic if and only if the vector lattices* $\mathrm{C}(X)$ *and* $\mathrm{C}(Y)$ *are isomorphic.*

As discussed in Section 1.1.1, we are interested in the case when the vector lattice $\mathrm{C}(X)$ is an order dual space. Since order dual spaces are necessarily Dedekind complete, we need to consider what topological properties on X make $\mathrm{C}(X)$ into a Dedekind complete vector lattice.

A topological space X is *extremally disconnected* if the closure of every open set is open. An extremally disconnected compact Hausdorff space is called a *Stonean* space. This terminology is motivated by the fact that the Stone spaces of Dedekind complete Boolean algebras are precisely the extremally disconnected compact Hausdorff spaces [**44**, Chapter F-06]. It is well-known that the Stone-Čech compactification of a infinite discrete space is extremally disconnected. Extremally disconnected spaces are also characterised by the fact that the collection of regular open sets[1] (resp. regular closed sets) coincide with the collection of clopen sets [**28**, Definition 12.11, Exercise 12.F]. Recall that a topology is *semi-regular* if it has a base of regular open sets. All regular topological spaces are semi-regular [**68**, Exercise 14E, p. 98], and therefore every extremally disconnected Tychonoff space has a base of clopen sets.

Using [**26**, Proposition 1.5.9], [**26**, Theorem 2.3.3], and [**53**, Theorem 43.2] along with Corollary 2.1.5, we have the following characterisation of extremally disconnected spaces.

THEOREM 2.1.7. *Let X be a Tychonoff space. The following statements are equivalent.*

(i) *X is extremally disconnected.*
(ii) *υX is extremally disconnected.*
(iii) *βX is Stonean.*
(iv) $\mathrm{C}(X)$ *is Dedekind complete.*
(v) $\mathrm{C_b}(X)$ *is Dedekind complete.*
(vi) $\mathrm{C}(X)$ *has the projection property.*
(vii) $\mathrm{C_b}(X)$ *has the projection property.*

2.1.2. Measures on topological spaces. Since the usage of terminology related to measures on topological spaces is inconsistent across the literature, we declare our conventions explicitly. Let X be a Hausdorff topological space. For a function $u : X \to \mathbb{R}$ we denote by $\mathbf{Z}_u$ the *zero set* of u and by $\mathbf{Z}_u^c$ its *co-zero set*, that is, the complement of $\mathbf{Z}_u$. If $A \subseteq X$ then $\mathbf{1}_A$ denotes the indicator function of A.

Denote by $\mathfrak{B}_X$ the Borel σ-algebra generated by the open sets in X. A *(signed) Borel measure* on X is a real-valued and σ-additive function on $\mathfrak{B}_X$. We denote the space of all signed Borel measures on X by $\mathrm{M}_\sigma(X)$. This space is a Dedekind

[1] See [**28**, Definition 12.8]

complete vector lattice with respect to the standard pointwise operations and order [**71**, Theorem 27.3]. In particular, for $\mu, \nu \in \mathrm{M}_\sigma(X)$,

$$(\mu \vee \nu)(B) = \sup\{\mu(A) + \nu(B \setminus A) \ : \ A \subseteq B, \ \ A \in \mathfrak{B}_X\}, \qquad B \in \mathfrak{B}_X.$$

For any upward directed set $D \subseteq \mathrm{M}_\sigma(X)^+$ with $\sup D = \nu$ in $\mathrm{M}_\sigma(X)$,

$$\nu(B) = \sup\{\mu(B) \ : \ \mu \in D\}, \qquad B \in \mathfrak{B}_X.$$

Following Bogachev [**20**], we call a Borel measure μ on X a *Radon measure* if for every $B \in \mathfrak{B}_X$,

$$|\mu|(B) = \sup\{|\mu|(K) \ : \ K \subseteq B \text{ is compact}\}.$$

Equivalently, μ is Radon if for every $B \in \mathfrak{B}_X$ and every $\epsilon > 0$ there exists a compact set $K \subseteq B$ so that $|\mu|(B \setminus K) < \epsilon$. Observe that if μ is Radon, then also

$$|\mu|(B) = \inf\{|\mu|(U) \ : \ U \supseteq B \text{ is open}\}.$$

Denote the space of Radon measures on X by $\mathrm{M}(X)$. Recall that the *support* of a Borel measure μ on X is defined as

$$S_\mu := \{x \in X \ : \ |\mu|(U) > 0 \text{ for all } U \ni x \text{ open}\}.$$

One may verify the following equality for the support of a Borel measure μ on X.

$$S_\mu = \bigcap \{C \in \mathfrak{B}_X \ : \ C \text{ closed}, \ \mu(X \setminus C) = 0\}.$$

Thus the support of a Borel measure is always closed. A non-zero Borel measure μ may have empty support, and even if S_μ is non-empty, S_μ may have measure zero, see for instance [**20**, Vol. II, Example 7.1.3]. However, if μ is a nonzero Radon measure, then $S_\mu \neq \varnothing$ and $|\mu|(S_\mu) = |\mu|(X)$; in fact, for every $B \in \mathfrak{B}_X$, $|\mu|(B) = |\mu|(B \cap S_\mu)$. We list the following useful properties of the support of a measure; the proofs are straightforward and therefore omitted.

PROPOSITION 2.1.8. *Let μ and ν be Radon measures on X. The following statements are true.*

(i) *If $|\mu| \leq |\nu|$ then $S_\mu \subseteq S_\nu$.*

(ii) $S_{\mu+\nu} \subseteq S_{|\mu|+|\nu|}$

(iii) $S_{|\mu|+|\nu|} = S_\mu \cup S_\nu$.

A Radon measure μ is called *compactly supported* if S_μ is compact. We denote the space of all compactly supported Radon measures on X as $\mathrm{M}_c(X)$. Further, a Radon measure μ on X is called a *normal measure* if $|\mu|(L) = 0$ for all closed nowhere dense sets L in X. The space of all normal Radon measures on X is denoted $\mathrm{N}(X)$, and the space of compactly supported normal Radon measures by $\mathrm{N}_c(X)$.

THEOREM 2.1.9. *The following statements are true.*

(i) $\mathrm{M}(X)$ *is a band in* $\mathrm{M}_\sigma(X)$

(ii) $\mathrm{M}_c(X)$ *is an order ideal in* $\mathrm{M}(X)$.

(iii) $\mathrm{N}(X)$ *is a band in* $\mathrm{M}(X)$.

(iv) $\mathrm{N}_c(X)$ *is a band in* $\mathrm{M}_c(X)$.

PROOF. For the proof of (i), let $\mu, \nu \in \mathrm{M}(X)$. Consider a Borel set B and a real number $\epsilon > 0$. There exists a compact set $K \subseteq B$ so that $|\mu|(B \setminus K) < \epsilon/2$ and $|\nu|(B \setminus K) < \epsilon/2$. We have $|\mu + \nu|(B \setminus K) \leq |\mu|(B \setminus K) + |\nu|(B \setminus K) < \epsilon$. Therefore $\mu + \nu \in \mathrm{M}(X)$. A similar argument shows that $a\mu \in \mathrm{M}(X)$ for all $a \in \mathbb{R}$. A similar argument also shows that for all $\nu \in \mathrm{M}_\sigma(X)$ and $\mu \in \mathrm{M}(X)$, if $|\nu| \leq |\mu|$ then $\nu \in \mathrm{M}(X)$. By definition of a Radon measure, $|\mu| \in \mathrm{M}(X)$ whenever $\mu \in \mathrm{M}(X)$. Therefore $\mathrm{M}(X)$ is an order ideal in $\mathrm{M}_\sigma(X)$.

To see that $\mathrm{M}(X)$ is a band in $\mathrm{M}_\sigma(X)$, consider an upward directed subset D of $\mathrm{M}(X)^+$ so that $\sup D = \nu$ in $\mathrm{M}_\sigma(X)$. Fix a Borel set B and a real number $\epsilon > 0$. There exists $\mu \in D$ so that $\nu(B) - \epsilon/2 < \mu(B)$. But μ is a Radon measure, so there exists a compact subset K of B so that $\mu(K) > \mu(B) - \epsilon/2$. Therefore $\nu(K) \geq \mu(K) > \mu(B) - \epsilon/2 > \nu(B) - \epsilon$. Therefore $\nu \in \mathrm{M}(X)$ so that $\mathrm{M}(X)$ is a band in $\mathrm{M}_\sigma(X)$.

The statement in (ii) follows immediately from the definition of the support of a measure and Proposition 2.1.8. It is clear that $\mathrm{N}(X)$ is an order ideal in $\mathrm{M}(X)$, and that it is a band follows from the expression for suprema in $\mathrm{M}_\sigma(X)$. Hence (iii) is true. The fact that (iv) is true follows immediately from (iii). □

The results in this subsection and the previous subsection are supplemented by the results in Appendix A.1. We state the main results of Appendix A.1 here since they are used in the following two subsections.

THEOREM 2.1.10. *Let X be a realcompact space. There is a lattice isomorphism $\Psi : \mathrm{C}(X)^\sim \to \mathrm{M}_c(X)$ where $\phi \longmapsto \nu_\phi$ so that for every $\phi \in \mathrm{C}(X)^\sim$,*

$$\phi(f) = \int_{S_{\nu_\phi}} f \, d\nu_\phi, \qquad f \in \mathrm{C}(X).$$

THEOREM 2.1.11. *Let X be a realcompact space. Consider the lattice isomorphism $\Psi : \mathrm{C}(X)^\sim \to \mathrm{M}_c(X)$ where $\phi \longmapsto \nu_\phi$ defined in Theorem 2.1.10. Then $\Psi\left[\mathrm{C}(X)^\sim_{\mathrm{n}}\right] = \mathrm{N}_c(X)$.*

2.1.3. Hyper-Stonean spaces. Recall that a compact Hausdorff space K is *hyper-Stonean* if K is Stonean and the union of supports of normal measures on K is dense in K. The following result shows that the definition of a hyper-Stonean space can be expressed purely in the language of vector lattices.

PROPOSITION 2.1.12. *Let X be a realcompact space. The union of supports of measures in $\mathrm{N}_c(X)$ is dense in X if and only if $\mathrm{C}(X)^\sim_{\mathrm{n}}$ separates $\mathrm{C}(X)$.*

PROOF. First, denote by S the union of supports of elements in $\mathrm{N}_c(X)$ and assume that S is dense in X. Take $0 \neq f \in \mathrm{C}(X)$ and assume without loss of generality that the open set $U := \{x \in X : f(x) > 0\}$ is non-empty. Since S is dense, there exists $\mu \in \mathrm{N}_c(X)^+$ such that $U \cap S_\mu \neq \varnothing$. Take any $x \in U \cap S_\mu$, then there exists an open neighbourhood V of x and $\epsilon > 0$ such that $\mu(V) > 0$ and $f|_V > \epsilon$.

Let $\phi \in \mathrm{C}(X)_{\mathrm{n}}^{\sim}$ denote the order continuous functional identified with $\mu \in \mathrm{N}_c(X)^+$, which exists by Theorem 2.1.11. Then $\phi \geq 0$ and we have

$$\phi(f) = \int_{S_\mu} f \; d\mu = \int_X f \; d\mu \geq \int_V f \; d\mu > \epsilon \cdot \mu(V) > 0.$$

For the reverse implication, we consider the contrapositive statement. Assume that S not dense in X. Then there exists a non-empty open set U where $U \cap S_\mu = \varnothing$ for every $\mu \in \mathrm{N}_c(X)$. Thus $S_\mu \subseteq X \setminus U$ for all $\mu \in \mathrm{N}_c(X)$. Since X is Tychonoff, for every $x \in U$ there exists $g_x \in C(X)$ such that $g_x|_{X \setminus U} = 0$ and $g_x(x) = 1$. Let $\phi_\mu \in \mathrm{C}(X)_n^{\sim}$ denote the order continuous functional identified with $\mu \in \mathrm{N}_c(X)$, which exists by Theorem 2.1.11. Then for every $\mu \in \mathrm{N}_c(X)$ and every such $g_x \in \mathrm{C}(X)$, we have

$$\phi_\mu(g_x) = \int_{S_\mu} g_x \; d\mu = 0.$$

Thus $\mathrm{C}(X)_{\mathrm{n}}^{\sim}$ does not separate $\mathrm{C}(X)$. □

Corollary 2.1.13. *Let K be a compact Hausdorff space. The following statements are equivalent.*

(i) K is hyper-Stonean.

(ii) $\mathrm{C}(K)$ is Dedekind complete and $\mathrm{C}(K)_{\mathrm{n}}^{\sim}$ separates $\mathrm{C}(K)$.

2.1.4. Measurable cardinals. We will occasionally need to refer to non-measurable cardinals in some of our results. Let κ be a cardinal. A set function $\mu : \mathcal{P}(\kappa) \to [0,1]$ is called *κ-additive measure* if the following holds.

(i) $\mu(\kappa) = 1$.

(ii) For all $A \subsetneq \kappa$, and all collections $\{B_i \; : \; i \in A\}$ of pairwise disjoint subsets of κ, we have

$$\mu\left(\bigcup_{i \in A} B_i\right) = \sum_{i \in A} \mu(B_i).$$

Such a κ-additive measure μ is *non-trivial* if $\mu(\{i\}) = 0$ for all $i \in \kappa$, and μ is *two-valued* if $\mu(A) \in \{0,1\}$ for all $A \in \mathcal{P}(\kappa)$. An infinite cardinal κ is *measurable* if there exists a non-trivial two-valued κ-additive measure on κ, see [**35**, Chapter 6, Definition 1.6]. If no such measure exists, then κ is *non-measurable*. It is known that all measurable cardinals are strongly inaccessible and, in fact, the non-existence of a measurable cardinal is consistent with ZFC, see [**35**, Chapter 6].

The following result is a combination of [**39**, Theorem 12.2, p.163] and Corollary A.1.9.

Theorem 2.1.14. *Let X be a non-empty set equipped with the discrete topology. The following statements are equivalent.*

(i) $|X|$ is non-measurable.

(ii) The discrete topology on X is realcompact.

(iii) For every $\phi \in \mathrm{C}(X)^\sim$, *there exists* $\mu \in \mathrm{M}_c(X)$ *such that*

$$\phi(f) = \int_{S_\mu} f\, d\mu, \qquad f \in \mathrm{C}(X).$$

2.2. Operators on vector lattices

In this section, we record a number of results regarding various kinds of linear operators between vector lattices that we need in the rest of this chapter. Before we proceed with this, we define the categories that form the setting of this chapter. It is readily verified that these are indeed categories.

	Objects	Morphisms
VL	Vector lattices	Lattice homomorphisms
NVL	Vector lattices	Normal lattice homomorphisms
IVL	Vector lattices	Interval preserving lattice homomorphisms
NIVL	Vector lattices	Normal, interval preserving lattice homomorphisms

Table 1. Table of categories of vector lattices under consideration

We refer to these four categories as *categories of vector lattices*. Below we depict the subcategory relationships between these categories.

$$\mathbf{VL} \supseteq \mathbf{NVL} \supseteq \mathbf{NIVL}, \qquad \mathbf{VL} \supseteq \mathbf{IVL} \supseteq \mathbf{NIVL}$$

The results in this section will give us information on the morphisms in these categories of vector lattices. Our first result is simple, however, since we could not find a reference in the literature we include the proof.

Proposition 2.2.1. *Let* E *and* F *be vector lattices and* $T : E \to F$ *a positive operator. The following statements are true.*

(i) If T *is injective and interval preserving, then* T *is a lattice isomorphism onto an order ideal in* F, *hence a normal lattice homomorphism into* F.

(ii) If T *is a lattice homomorphism and* $T[E]$ *is an order ideal in* F, *then* T *is interval preserving.*

Proof of (i). Assume that T is injective and interval preserving. $T[E]$ is an order ideal in F by [**49**, Proposition 14.7]. Therefore, because T is injective, it suffices to show that T is a lattice homomorphism. To this end, consider $u, v \in E^+$. Then $0 \leq T(u) \wedge T(v) \leq T(u)$ and $0 \leq T(u) \wedge T(v) \leq T(v)$. Since T is interval preserving and injective there exists $w \in [0,u] \cap [0,v] = [0, u \wedge v]$ so that $T(w) = T(u) \wedge T(v)$. We have

$$T(w) \leq T(u \wedge v) \leq T(u) \text{ and } T(w) \leq T(u \wedge v) \leq T(v).$$

Hence $T(u)\wedge T(v) = T(w) \leq T(u\wedge v) \leq T(u)\wedge T(v)$ so that $T(u\wedge v) = T(u)\wedge T(v)$.

To see that T is a normal lattice homomorphism, let $A \downarrow 0$ in E. Then $T[A] \downarrow 0$ in $T[E]$ because T is a lattice isomorphism onto $T[E]$. But $T[E]$ is an order ideal in F, so $T[A] \downarrow 0$ in F. □

Proof of *(ii)*. Assume that T is a lattice homomorphism and $T[E]$ is an order ideal in F. Let $0 \leq u \in E$ and $0 \leq v \leq T(u)$. Because $T[E]$ is an order ideal in F there exists $w \in E$ so that $T(w) = v$. Let $w' = (w \vee 0)\wedge u$. Then $0 \leq w' \leq u$ and $T(w') = (v\vee 0)\wedge u = v$. □

We list some properties of band projections which will be used frequently in the sequel.

Proposition 2.2.2. *Let E be a vector lattice, A and B projection bands in E, P_A and P_B the band projections of E onto A and B, respectively, and I_E the identity operator on E. Assume that $A \subseteq B$. The following statements are true.*

(i) *P_A is an order continuous lattice homomorphism.*

(ii) *$P_A \leq I_E$.*

(iii) *$P_A P_B = P_B P_A = P_A$.*

(iv) *P_A is interval preserving.*

Proof. For (i), see [**53**, Theorem 24.6 and Exercise 24.11]. For (ii) and (iii), see [**6**, Theorem 1.44] and [**6**, Theorem 1.46] respectively. Lastly, (iv) follows from Proposition 2.2.1 (ii), since $P_A[E] = A$ is a band, hence an order ideal, in E. □

In the following theorem, we briefly recall some basic facts concerning the order adjoint of a positive operator $T : E \to F$ which we will make extensive use of in the sequel.

Theorem 2.2.3. *Let E and F be vector lattices and $T : E \to F$ a positive operator. Denote by $T^\sim : F^\sim \to E^\sim$ its order adjoint, $\varphi \mapsto \varphi\circ T$. The following statements are true.*

(i) *$T^\sim$ is positive and order continuous.*

(ii) *If T is order continuous then $T^\sim[F^\sim_{\mathrm{n}}] \subseteq E^\sim_{\mathrm{n}}$.*

(iii) *If T is interval preserving then $T^\sim$ is a lattice homomorphism.*

(iv) *If T is a lattice homomorphism then $T^\sim$ is interval preserving. The converse is true if ${}^\circ F^\sim = \{0\}$.*

Proof. For (i), the positivity of $T^\sim$ is easily verified and order continuity follows directly from the proof of [**70**, Theorem 83.4]. The statement in (ii) follows directly from the fact that composition of order continuous operators is order continuous. The statements in (iii) and (iv) are special cases of [**6**, Theorem 2.16]. We note that although [**6**] declares a blanket assumption at the start of the book that all vector lattices under consideration in [**6**] are Archimedean, the proof of [**6**, Theorem 2.16] does not make use of this assumption. □

PROPOSITION 2.2.4. *Let E and F be vector lattices and $T : E \to F$ a lattice homomorphism onto F. The following statements are true.*

(i) $T^{\sim}[F^{\sim}] = \ker(T)^{\circ}$.

(ii) *If T is order continuous then* $T^{\sim}[F^{\sim}_{\mathrm{n}}] = \ker(T)^{\circ} \cap E^{\sim}_{\mathrm{n}}$.

PROOF OF (*i*). Let $\varphi \in F^{\sim}$. If $u \in \ker(T)$ then $T^{\sim}(\varphi)(u) = \varphi(T(u)) = \varphi(0) = 0$. Hence $\varphi \in \ker(T)^{\circ}$. On the other hand, let $\psi \in \ker(T)^{\circ}$. Define $\varphi : F \to \mathbb{R}$ by setting $\varphi(v) = \psi(u)$ if $v = T(u)$. Then $\varphi \in F^{\sim}$ and $T^{\sim}(\varphi) = \psi$. □

PROOF OF (*ii*). It follows from (i) and Theorem 2.2.3 (ii) that $T^{\sim}[F^{\sim}_{\mathrm{n}}] \subseteq \ker(T)^{\circ} \cap E^{\sim}_{\mathrm{n}}$. We show that if $T^{\sim}(\varphi) \in E^{\sim}_{\mathrm{n}}$ for some $\varphi \in F^{\sim}$ then $\varphi \in F^{\sim}_{\mathrm{n}}$. From this and (i) it follows that $T^{\sim}[F^{\sim}_{\mathrm{n}}] = \ker(T)^{\circ} \cap E^{\sim}_{\mathrm{n}}$. We observe that it suffices to consider positive $\varphi \in F^{\sim}$. Indeed, T is a surjective lattice homomorphism and therefore also interval preserving. Hence by Theorem 2.2.3 (iii), $T^{\sim}$ is a lattice homomorphism.

Suppose that $0 \le \varphi \in F^{\sim}$ and that $T^{\sim}(\varphi) \in E^{\sim}_{\mathrm{n}}$. Let $A \downarrow 0$ in F. Define $B \coloneqq T^{-1}[A] \cap E_{+}$. Then B is downward directed and $T[B] = A$. In particular, $\varphi[A] = T^{\sim}(\varphi)[B]$. Let $C := \{w \in E \ : \ 0 \le w \le v \text{ for all } v \in B\}$. If $w \in C$ then $0 \le T(w) \le u$ for all $u \in A$ so that $T(w) = 0$. Hence $C \subseteq \ker(T)$. We have $B - C \downarrow 0$ in E. Since $T^{\sim}(\varphi)$ is order continuous, $T^{\sim}(\varphi)[B - C] \downarrow 0$. That is, for every $\epsilon > 0$ there exists $v \in B$ and $w \in C$ so that $\varphi(T(v)) = \varphi(T(v - w)) = T^{\sim}(\varphi)(v - w) < \epsilon$. Hence, for every $\epsilon > 0$ there exists $u \in A$ so that $\varphi(u) < \epsilon$. This shows that $\varphi[A] \downarrow 0$ so that $\varphi \in F^{\sim}_{\mathrm{n}}$ as required. □

2.2.1. Products of vector lattices. Let I be a non-empty set and $\{E_\alpha\}_{\alpha \in I}$ a collection of vector lattices. The product $\prod_{\alpha \in I} E_\alpha$ is a vector lattice when equipped with the standard coordinate-wise operations. In the sequel, if the index set of a product is clear form the context, we will omit it and write $\prod E_\alpha$. For $\beta \in I$, let $\pi_\beta : \prod E_\alpha \to E_\beta$ be the coordinate projection onto E_β and $\iota_\beta : E_\beta \to \prod E_\alpha$ the right inverse of π_β where

$$\pi_\alpha(\iota_\beta(u)) = \begin{cases} u & \text{if} \quad \alpha = \beta \\ 0 & \text{if} \quad \alpha \neq \beta. \end{cases}$$

We denote by $\bigoplus E_\alpha$ the order ideal in $\prod E_\alpha$ consisting of $u \in \prod E_\alpha$ for which $\pi_\alpha(u) \neq 0$ for only finitely many $\alpha \in I$. The following properties of $\prod E_\alpha$ and $\bigoplus E_\alpha$ are used frequently in the sequel and so we record them here for ease of reference.

PROPOSITION 2.2.5. *Let I be a non-empty set and $\{E_\alpha\}_{\alpha \in I}$ a collection of vector lattices with $U \subseteq \prod E_\alpha$ and $u = (u_\alpha)_{\alpha \in I} \in \prod E_\alpha$. The following statements are true.*

(i) $\inf \pi_\alpha[U] = u_\alpha$ *in E_α for all $\alpha \in I$ if and only if* $\inf U = u$ *in* $\prod E_\alpha$.

(ii) $\sup \pi_\alpha[U] = u_\alpha$ *in E_α for all $\alpha \in I$ if and only if* $\sup U = u$ *in* $\prod E_\alpha$.

PROOF. We only prove (i) since the proof of (ii) follows similarly. For the forward implication, consider $v \in U$. Then for every $\alpha \in I$, we have $\pi_\alpha(v) \geq \inf \pi_\alpha[U] = u_\alpha$ in E_α. Thus u is a lower bound of U in $\prod E_\alpha$. Let l be any lower bound for U in $\prod E_\alpha$, then $\pi_\alpha(l)$ is a lower bound for $\pi_\alpha[U]$ in E_α which implies that $u \geq l$. Thus $\inf U = u$ in $\prod E_\alpha$.

For the reverse implication, fix $\alpha \in I$. Since u is a lower bound for U, we have that u_α is a lower bound for $\pi_\alpha[U]$ in E_α. Consider any lower bound m of $\pi_\alpha[U]$ in E_α and define $\tilde{m} \in \prod E_\alpha$ where $\pi_\alpha(\tilde{m}) = m$ and $\pi_\beta(\tilde{m}) = u_\beta$ for $\beta \in I \setminus \{\alpha\}$. Then $\tilde{m}$ is a lower bound for U in $\prod E_\alpha$, which implies that $u_\alpha \geq m$. □

THEOREM 2.2.6. *Let I be a non-empty set with E_α a vector lattice for every $\alpha \in I$. The following statements are true.*

(i) The coordinate projections π_β and their right inverses ι_β are normal, interval preserving lattice homomorphisms.

(ii) $\prod E_\alpha$ is Archimedean if and only if E_α is Archimedean for every $\alpha \in I$.

(iii) $\prod E_\alpha$ is Dedekind complete if and only if E_α is Dedekind complete for every $\alpha \in I$.

(iv) If I has non-measurable cardinal, then the order dual of $\prod E_\alpha$ is $\bigoplus E_\alpha^\sim$.

(v) The order continuous dual of $\prod E_\alpha$ is $\bigoplus_{\alpha\in I} (E_\alpha)_{\mathrm{n}}^\sim$.

(vi) The order dual of $\bigoplus E_\alpha$ is $\prod E_\alpha^\sim$.

(vii) The order continuous dual of $\bigoplus E_\alpha$ is $\prod (E_\alpha)_{\mathrm{n}}^\sim$.

PROOF OF (*i*). Fix $\beta \in I$. The coordinate-wise operations defined on $\prod E_\alpha$ make it clear that π_β is a lattice homomorphism. Fix $x \in E^+$. By positivity, it follows that $\pi_\beta[0,x] \subseteq [0,\pi_\beta(x)]$. For $y \in [0,\pi_\beta(x)]$, we have $\iota_\beta(y) \in [0,x]$ and $y = \pi_\beta(\iota_\beta(y)) \in \pi_\beta[0,x]$. Thus π_β is interval preserving. It follows from Proposition 2.2.5 (i) that coordinate projections are order continuous and thus normal lattice homomorphisms. Similarly, it is easy to verify that ι_β is both a lattice homomorphism and interval preserving and the order continuity of ι_β also follows from Proposition 2.2.5 (i). □

PROOF OF (*ii*). This is easily verified from the definition of an Archimedean vector lattice. □

PROOF OF (*iii*). First, assume that E_α is Dedekind complete for every $\alpha \in I$. Take $D \subseteq \prod E_\alpha$ with $d \in \prod E_\alpha$ an upper bound for D. By the positivity of the coordinate projections, the element $\pi_\alpha(d)$ is an upper bound for $\pi_\alpha[D]$ in E_α and by Proposition 2.2.5 (ii) it follows that $\sup D = (\sup \pi_\alpha[D])_{\alpha\in I}$.

Next, assume that $\prod E_\alpha$ is Dedekind complete and fix $\beta \in I$. Consider a collection $G \subseteq E_\beta$ with $g \in E_\beta$ an upper bound for G. Then $\iota_\beta(g)$ is an upper bound for

$\iota_\beta[G]$ in $\prod E_\alpha$ and thus $\sup \iota_\beta[G]$ exists in $\prod E_\alpha$ and since π_β is a normal lattice homomorphism we conclude that $\sup G = \pi_\beta(\sup \iota_\beta[G])$ exists. □

Proof of (*iv*). Assume that I has non-measurable cardinal. By (i) of this theorem and Theorem 2.2.3 (iii) and (iv), $\iota_\beta^\sim : \left(\prod E_\alpha\right)^\sim \to E_\beta^\sim$ is an interval preserving normal lattice homomorphism for every $\beta \in I$. Because each $\varphi \in \left(\prod E_\alpha\right)^\sim$ is linear and order bounded, the set $I_\varphi := \{\beta \in I \;:\; \iota_\beta^\sim(\varphi) \neq 0\}$ is finite. Define $S : \left(\prod E_\alpha\right)^\sim \to \bigoplus E_\alpha^\sim$ by setting

$$S(\varphi) := (\iota_\alpha^\sim(\varphi))_{\alpha \in I}, \qquad \varphi \in \left(\prod E_\alpha\right)^\sim.$$

Then S is a lattice homomorphism. It remains to verify that S is bijective.

We show that S is injective. Let $0 \neq \varphi \in \left(\prod E_\alpha\right)^\sim$. Fix $0 \leq u \in \prod E_\alpha$ so that $\varphi(u) \neq 0$. For $f \in \mathbb{R}^I$ let $fu \in \prod E_\alpha$ be defined by $\pi_\alpha(fu) = f(\alpha)\pi_\alpha(u)$, $\alpha \in I$. Define $\hat{\varphi} : \mathbb{R}^I \to \mathbb{R}$ by setting

$$\hat{\varphi}(f) := \varphi(fu), \qquad f \in \mathbb{R}^I.$$

Then $\hat{\varphi}$ is a non-zero order bounded linear functional on $\mathbb{R}^I$. Since I has non-measurable cardinal, by Theorem 2.1.14, I equipped with the discrete topology is realcompact and there exists a non-zero finitely supported countably additive measure $\mu : \mathcal{P}(I) \to \mathbb{R}$ such that

$$\hat{\varphi}(f) = \int_I f\, d\mu = \sum_{\alpha \in I} f(\alpha)\mu(\alpha), \qquad f \in \mathbb{R}^I.$$

Let α be in the support of μ, and let g be the indicator function of $\{\alpha\}$. Then $0 \neq \mu(\alpha) = \hat{\varphi}(g) = \varphi(gu) = \iota_\alpha^\sim(\varphi)(u)$. Therefore $S\varphi \neq 0$ so that S is injective.

To see that S is surjective, observe that for every $\beta \in I$, $\pi_\beta^\sim : E_\beta^\sim \to \left(\prod E_\alpha\right)^\sim$ is an interval preserving normal lattice homomorphism by (i) of this theorem and Theorem 2.2.3 (iii) and (iv). Define $T : \bigoplus E_\alpha^\sim \to \left(\prod E_\alpha\right)^\sim$ by setting

$$T(\psi) := \sum \pi_\alpha^\sim(\psi_\alpha), \qquad \psi = (\psi_\alpha) \in \bigoplus E_\alpha^\sim.$$

Then T is a positive operator. We claim that $S \circ T$ is the identity on $\bigoplus E_\alpha^\sim$. Indeed, for any $\psi \in \bigoplus E_\alpha^\sim$ we have

$$S \circ T(\psi) = \sum_{\alpha \in I} (\iota_\beta^\sim(\pi_\alpha^\sim(\psi_\alpha)))_{\beta \in I} = \sum_{\alpha \in I} (\psi_\alpha \circ \pi_\alpha \circ \iota_\beta)_{\beta \in I}.$$

By definition of the ι_β it follows that $S \circ T(\psi) = \psi$ which verifies our claim. Therefore S is a lattice isomorphism. □

Proof of (*v*). We point out that, unlike the proof of (iv), the following proof is independent of the cardinality of I. Define $S : \left(\prod E_\alpha\right)^\sim \to \bigoplus E_\alpha^\sim$ as in the proof of (iv). By Theorem 2.2.3 (ii), S maps $\left(\prod E_\alpha\right)_{\mathrm{n}}^\sim$ into $\bigoplus (E_\alpha)_{\mathrm{n}}^\sim$. A similar argument to that given in proof of (iv) shows that S is a surjective lattice homomorphism. Hence it remains to show that S is injective.

Denote by $\mathcal{F}(I)$ the collection of finite subsets of I. Let $0 \leq \varphi \in \left(\prod E_\alpha\right)^\sim_{\mathrm{n}}$ and suppose that $S(\varphi) = 0$. Then $\iota^\sim_\beta(\varphi) = 0$ for every $\beta \in I$. But for any $0 \leq u \in \prod E_\alpha$,

$$u = \sup\left\{\sum_{\alpha\in F} \iota_\alpha(u) \ : \ F \in \mathcal{F}(I)\right\}.$$

Therefore by the order continuity of φ,

$$\varphi(u) = \sup\left\{\sum_{\alpha\in F} \iota^\sim_\alpha(\varphi)(u) \ : \ F \in \mathcal{F}(I)\right\} = 0$$

for all $0 \leq u \in \prod E_\alpha$; hence $\varphi = 0$. Because S is a lattice homomorphism it follows that, for all $\varphi \in \left(\prod E_\alpha\right)^\sim_{\mathrm{n}}$, if $S(\varphi) = 0$ then $\varphi = 0$; that is, S is injective. □

Proof of (*vi*). Define $S : (\bigoplus E_\alpha)^\sim \to \prod E^\sim_\alpha$ by setting

$$S(\varphi) := (\iota^\sim_\alpha(\varphi))_{\alpha\in I}.$$

By (i) of this theorem and Theorem 2.2.3 (iii), it follows that the maps $\iota^\sim_\alpha : (\bigoplus E_\alpha)^\sim \to E^\sim_\alpha$ are lattice homomorphisms and thus S is a lattice homomorphism. It is then enough to verify that S is bijective. To show that S is injective, fix $\varphi \in (\bigoplus E_\alpha)^\sim$ such that $S\varphi = 0$. Take $u \in \bigoplus E_\alpha$ where $u = \sum_{i=1}^n \iota_{\alpha_i}(u_{\alpha_i})$ for $\{\alpha_1, \ldots, \alpha_n\} \in \mathcal{F}(I)$. Then

$$\varphi(u) = \varphi\left(\sum_{i=1}^n \iota_{\alpha_i}(u_{\alpha_i})\right) = \sum_{i=1}^n \iota^\star_{\alpha_i}(\varphi)(u_{\alpha_i}) = 0.$$

Thus $\varphi = 0$. For $\psi = (\psi_\alpha)_{\alpha\in I} \in \prod E^\sim_\alpha$, define $\psi_0 \in (\bigoplus E_\alpha)^\sim$ where

$$\psi_0(u) := \sum_{i=1}^n \psi_{\alpha_i}(u_{\alpha_i}), \qquad u = \sum_{i=1}^n \iota_{\alpha_i}(u_{\alpha_i}) \in \bigoplus E_\alpha.$$

Define $T : \prod E^\sim_\alpha \to (\bigoplus E_\alpha)^\sim$ by setting $T(\psi) = \psi_0$ for $\psi = (\psi_\alpha)_{\alpha\in I} \in \prod E^\sim_\alpha$. It is clear that $S \circ T(\psi) = \psi$ and thus S is surjective. □

Proof of (*vii*). The proof of (vi) is easily modified to prove (vii). □

2.3. Direct limits

First, we introduce the notions of *direct systems* and *direct limits*. After discussing some basic properties of direct systems and direct limits, we will consider the existence of direct limits in our categories of vector lattices and list some properties of vector lattices that have permanence under the construction of a direct limit. Additional results are found in [**37**]. Lastly, we give a number of examples of direct limits which we make use of later.

Definition 2.3.1. Let **C** be any category and $(I, \preccurlyeq)$ an upwards directed set. Consider a family of objects $\{E_\alpha\}_{\alpha\in I}$ in **C**. For all $\alpha \preccurlyeq \beta$ in I, let $e_{\alpha,\beta} : E_\alpha \to E_\beta$ be a morphism in $\mathbf{C}$. The pair $\mathcal{D} \coloneqq ((E_\alpha)_{\alpha \in I}, (e_{\alpha,\beta})_{\alpha \preccurlyeq \beta})$ is called a *direct system* in $\mathbf{C}$ if, for all $\alpha \preccurlyeq \beta \preccurlyeq \gamma$ in I, the diagram

$$\begin{array}{ccc} E_\alpha & \xrightarrow{e_{\alpha,\gamma}} & E_\gamma \\ & {\scriptstyle e_{\alpha,\beta}} \searrow \quad \nearrow {\scriptstyle e_{\beta,\gamma}} & \\ & E_\beta & \end{array}$$

commutes in $\mathbf{C}$. We refer to the maps $e_{\alpha,\beta}$ for $\alpha, \beta \in I$ as the *linking maps of* $\mathcal{D}$.

Let E be an object in $\mathbf{C}$ and $e_\alpha : E_\alpha \to E$ a morphism in $\mathbf{C}$ for every $\alpha \in I$. The pair $(E, (e_\alpha)_{\alpha \in I})$ is called a *compatible system over* $\mathcal{D}$ if for all $\alpha \preccurlyeq \beta$ in I the diagram

$$\begin{array}{ccc} E_\alpha & \xrightarrow{e_\alpha} & E \\ & {\scriptstyle e_{\alpha,\beta}} \searrow \quad \nearrow {\scriptstyle e_\beta} & \\ & E_\beta & \end{array}$$

commutes in $\mathbf{C}$.

A *direct limit of* $\mathcal{D}$ *in* $\mathbf{C}$ is a compatible system $(E, (e_\alpha)_{\alpha \in I})$ over $\mathcal{D}$ satisfying the universal property that for any compatible system $(E', (e'_\alpha)_{\alpha \in I})$ over $\mathcal{D}$ there exists a unique morphism $r : E \to E'$ so that for all $\alpha \in I$ the diagram

$$\begin{array}{ccc} E & \xrightarrow{r} & E' \\ & {\scriptstyle e_\alpha} \nwarrow \quad \nearrow {\scriptstyle e'_\alpha} & \\ & E_\alpha & \end{array}$$

commutes in $\mathbf{C}$. Where convenient, we denote by $\varinjlim \mathcal{D}$ the direct limit of $\mathcal{D}$ in $\mathbf{C}$.

In order to show that direct limits of a direct system $\mathcal{D}$ are essentially unique when they exist, we define the *derived category of compatible systems over* $\mathcal{D}$: Denote by $\mathbf{C}(\mathcal{D})$ the category whose objects are the compatible systems over $\mathcal{D}$ and where a morphism between compatible systems $(E, (e_\alpha)_{\alpha \in I})$ and $(E', (e'_\alpha)_{\alpha \in I})$ is a morphism $f : E \to E'$ in $\mathbf{C}$ such that for all $\alpha \in I$ the diagram

$$\begin{array}{ccc} E & \xrightarrow{f} & E' \\ & {\scriptstyle e_\alpha} \nwarrow \quad \nearrow {\scriptstyle e'_\alpha} & \\ & E_\alpha & \end{array}$$

commutes in $\mathbf{C}$. The universal property of a direct limit in Definition 2.3.1 makes it clear that a compatible system $\mathcal{S} \coloneqq (E, (e_\alpha)_{\alpha \in I})$ over $\mathcal{D}$ is a direct limit of $\mathcal{D}$ in $\mathbf{C}$ precisely when $\mathcal{S}$ is the initial object in $\mathbf{C}(\mathcal{D})$. As a result, any two compatible systems satisfying the universal property of a direct limit are connected by a unique isomorphism in $\mathbf{C}$ that makes the above diagram commute. In the sequel, we will therefore refer to *the* direct limit of a direct system in a fixed category, when it exists.

The following simple categorical result will be indispensable in the sequel.

PROPOSITION 2.3.2. *Let* $\mathbf{C}$ *be any category and* $(I,\preccurlyeq)$ *an upwards directed set. Consider direct systems* $\mathcal{D} := ((E_\alpha)_{\alpha\in I}, (e_{\alpha,\beta})_{\alpha\preccurlyeq\beta})$ *and* $\mathcal{D}' := \big((E'_\alpha)_{\alpha\in I}, (e'_{\alpha,\beta})_{\alpha\preccurlyeq\beta}\big)$ *in* $\mathbf{C}$ *with direct limits* $\mathcal{S} := (E, (e_\alpha)_{\alpha\in I})$ *and* $\mathcal{S}' := (E', (e'_\alpha)_{\alpha\in I})$ *in* $\mathbf{C}$ *respectively. For every* $\alpha\in I$*, let* $f_\alpha : E_\alpha \to E'_\alpha$ *be a morphism in* $\mathbf{C}$ *so that the diagram*

$$\begin{array}{ccc} E_\alpha & \xrightarrow{\ f_\alpha\ } & E'_\alpha \\ {\scriptstyle e_{\alpha,\beta}}\downarrow & & \downarrow{\scriptstyle e'_{\alpha,\beta}} \\ E_\beta & \xrightarrow[\ f_\beta\ ]{} & E'_\beta \end{array} \tag{2.3.1}$$

commutes in $\mathbf{C}$ *for all* $\alpha\preccurlyeq\beta$ *in* I*. The following statements are true.*

(i) *There exists a unique lattice homomorphism* $f : E \to E'$ *so that the diagram*

$$\begin{array}{ccc} E_\alpha & \xrightarrow{\ f_\alpha\ } & E'_\alpha \\ {\scriptstyle e_{\alpha}}\downarrow & & \downarrow{\scriptstyle e'_{\alpha}} \\ E & \xrightarrow[\ f\ ]{} & E' \end{array} \tag{2.3.2}$$

commutes in $\mathbf{C}$ *for every* $\alpha\in I$*.*

(ii) *If* f_α *is an isomorphism in* $\mathbf{C}$ *for every* $\alpha\in I$*, then so is* f*.*

PROOF. For (i), we note that the pair $(E', (e'_\alpha\circ f_\alpha)_{\alpha\in I})$ is a compatible system over $\mathcal{D}$: Fix $\alpha\preccurlyeq\beta$ in I, then by (2.3.1) and the fact that $\mathcal{S}'$ is a compatible system over $\mathcal{D}'$, we have

$$\big(e'_\beta\circ f_\beta\big)\circ e_{\alpha,\beta} = e'_\beta\circ(f_\beta\circ e_{\alpha,\beta}) = e'_\beta\circ\big(e'_{\alpha,\beta}\circ f_\alpha\big) = e'_\alpha\circ f_\alpha.$$

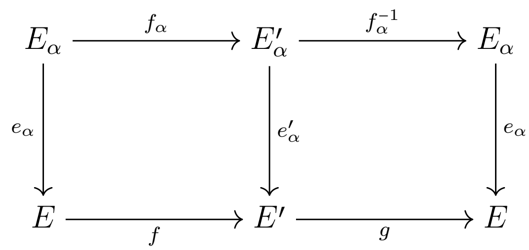
By the universal property of the direct limit, there exists a unique morphism $f : E\to E'$ in $\mathbf{C}$ such that the following diagram commutes in $\mathbf{C}$ for all $\alpha\in I$.

$$\begin{array}{ccccc} & & E_\alpha & & \\ & {\scriptstyle e_\alpha}\swarrow & & \searrow{\scriptstyle e'_\alpha\circ f_\alpha} & \\ E & & \xrightarrow[\ f\ ]{} & & E' \end{array} \tag{2.3.3}$$

Decomposing the morphism on the right side of the above triangle gives us exactly the square in (2.3.2). For (ii), assume that f_α is an isomorphism for every $\alpha\in I$. We start by imitating the first step in (i): It is easy to verify that the pair $(E, (e_\alpha\circ f_\alpha^{-1})_{\alpha\in I})$ is a compatible system over $\mathcal{D}'$. Thus there exists a unique morphism $g : E'\to E$ in $\mathbf{C}$ such that the following diagram commutes in $\mathbf{C}$.

$$\begin{array}{ccccc} E' & & \xrightarrow{\ g\ } & & E \\ & {\scriptstyle e'_\alpha}\nwarrow & & \nearrow{\scriptstyle e_\alpha\circ f_\alpha^{-1}} & \\ & & E'_\alpha & & \end{array} \tag{2.3.4}$$

Putting (2.3.3) and (2.3.4) together gives us the following diagram for every $\alpha \in I$.



Thus for every $\alpha \in I$, we have

$$(g \circ f) \circ e_\alpha = g \circ (f \circ e_\alpha) = g \circ (e'_\alpha \circ f_\alpha) = (g \circ e'_\alpha) \circ f_\alpha = \left(e_\alpha \circ f_\alpha^{-1}\right) \circ f_\alpha = e_\alpha.$$

However, it is also clear that $\mathbf{1}_E \circ e_\alpha = e_\alpha$ holds in $\mathbf{C}$ for all $\alpha \in I$. By the universal property of the direct limit we conclude that $g \circ f = \mathbf{1}_E$. A similar argument will show that $f \circ g = \mathbf{1}_{E'}$ and thus the morphism $f : E \to E'$ making diagram (2.3.2) commute is indeed an isomorphism. □

2.3.1. Existence and permanence properties of direct limits. In [**37**], Filter *defines* the direct limit of a direct system $\mathcal{D} \coloneqq ((E_\alpha)_{\alpha \in I}, (e_{\alpha,\beta})_{\alpha \preccurlyeq \beta})$ in **VL** to be the set-theoretic direct limit of $\mathcal{D}$ (see [**21**, Chapter III, §7.5]) equipped with suitable vector space and order structures. It is not difficult to see that this construction is exactly the direct limit in the sense of Definition 2.3.1 since it satisfies the necessary universal property. We briefly recall the details.

Let $\mathcal{D} \coloneqq ((E_\alpha)_{\alpha \in I}, (e_{\alpha,\beta})_{\alpha \preccurlyeq \beta})$ be a direct system in **VL**. For u in the disjoint union $\biguplus E_\alpha$ of the collection $\{E_\alpha\}_{\alpha \in I}$, denote by $\alpha(u)$ that element of I so that $u \in E_{\alpha(u)}$. Define an equivalence relation on $\biguplus E_\alpha$ by setting $u \sim v$ if and only if there exists $\beta \succcurlyeq \alpha(u), \alpha(v)$ in I so that $e_{\alpha(u),\beta}(u) = e_{\alpha(v),\beta}(v)$. Let $E \coloneqq \biguplus E_\alpha / \sim$ and denote the equivalence class generated by $u \in \biguplus E_\alpha$ by $\dot{u}$.

Let $\dot{u}, \dot{v} \in E$. We set $\dot{u} \leq \dot{v}$ if and only if there exists $\beta \succcurlyeq \alpha(u), \alpha(v)$ in I so that $e_{\alpha(u),\beta}(u) \leq e_{\alpha(v),\beta}(v)$. Further, for $a, b \in \mathbb{R}$ define

$$a\dot{u} + b\dot{v} \coloneqq \dot{\overbrace{a e_{\alpha(u),\beta}(u) + b e_{\alpha(v),\beta}(v)}},$$

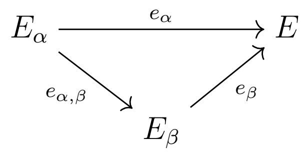
where $\beta \succcurlyeq \alpha(u), \alpha(v)$ in I is arbitrary. With addition, scalar multiplication and the partial order defined in this way, E is a vector lattice. The lattice operations are given by

$$\dot{u} \wedge \dot{v} = \dot{\overbrace{e_{\alpha(u),\beta}(u) \wedge e_{\alpha(v),\beta}(v)}}$$

and

$$\dot{u} \vee \dot{v} = \dot{\overbrace{e_{\alpha(u),\beta}(u) \vee e_{\alpha(v),\beta}(v)}},$$

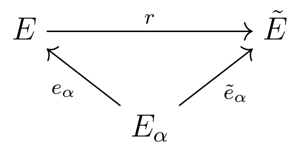
with $\beta \succcurlyeq \alpha(u), \alpha(v)$ in I arbitrary. The reader may verify that the lattice operations will be well-defined when the linking maps in $\mathcal{D}$ are lattice homomorphisms. For each $\alpha \in I$ define $e_\alpha : E_\alpha \to E$ by setting $e_\alpha(u) := \dot{u}$ for $u \in E_\alpha$. Each e_α is a lattice homomorphism and the diagram



commutes in **VL** for all $\alpha \preccurlyeq \beta$ in I so that $\mathcal{S} := (E, (e_\alpha)_{\alpha \in I})$ is a compatible system of $\mathcal{D}$ in **VL**. Further, if $\tilde{\mathcal{S}} = (\tilde{E}, (\tilde{e}_\alpha)_{\alpha \in I}$ is another compatible system over $\mathcal{D}$ in **VL** then

$$r : E \ni \dot{u} \longmapsto \tilde{e}_{\alpha(u)}(u) \in \tilde{E}$$

is the unique lattice homomorphism so that the diagram



commutes for every $\alpha \in I$. Hence $\mathcal{S}$ is indeed the direct limit of $\mathcal{D}$ in **VL**. We give two further existence results for direct limits of direct systems in other categories of vector lattices.

THEOREM 2.3.3. *Let $\mathcal{D} := ((E_\alpha)_{\alpha \in I}, (e_{\alpha,\beta})_{\alpha \preccurlyeq \beta})$ be a direct system in* **IVL**, *and let $\mathcal{S} := (E, (e_\alpha)_{\alpha \in I})$ be the direct limit of $\mathcal{D}$ in* **VL**. *Then $\mathcal{S}$ is the direct limit of $\mathcal{D}$ in* **IVL**.

PROOF. We show that each e_α is interval preserving. To this end, fix $\alpha \in I$ and $0 < u \in E_\alpha$. Suppose that $\dot{0} \le \dot{v} \le e_\alpha(u) = \dot{u}$. Then there exists a $\beta \succcurlyeq \alpha, \alpha(v)$ in I so that $0 \le e_{\alpha(v),\beta}(v) \le e_{\alpha,\beta}(u)$. But $e_{\alpha,\beta}$ is interval preserving, so there exists $0 \le w \le u$ in E_α so that $e_{\alpha,\beta}(w) = e_{\alpha(v),\beta}(v)$. Therefore $e_\alpha(w) = \dot{w} = \dot{v}$. Hence e_α is interval preserving.

Let $\tilde{\mathcal{S}} := (\tilde{E}, (\tilde{e}_\alpha)_{\alpha \in I})$ be a compatible system over $\mathcal{D}$ in **IVL**, thus also in **VL**. We show that the unique lattice homomorphism $r : E \to \tilde{E}$ is interval preserving. Consider $\dot{u} \in E^+$. Let $0 \le v \le r(\dot{u})$ in $\tilde{E}$, that is, $0 \le v \le \tilde{e}_{\alpha(u)}(u)$. But $\tilde{e}_{\alpha(u)}$ is interval preserving so there exists $0 \le w \le u$ in $E_{\alpha(u)}$ so that $v = \tilde{e}_{\alpha(u)}(w)$. Thus $\dot{0} \le \dot{w} \le \dot{u}$ and $r(\dot{w}) = v$ in E. Therefore r is interval preserving. □

THEOREM 2.3.4. *Let $\mathcal{D} := ((E_\alpha)_{\alpha \in I}, (e_{\alpha,\beta})_{\alpha \preccurlyeq \beta})$ be a direct system in* **NIVL**, *and let $\mathcal{S} := (E, (e_\alpha)_{\alpha \in I})$ be the direct limit of $\mathcal{D}$ in* **VL**. *Assume that $e_{\alpha,\beta}$ is injective for all $\alpha \preccurlyeq \beta$ in I. Then $\mathcal{S}$ is the direct limit of $\mathcal{D}$ in* **NIVL**.

PROOF. We start by proving that $e_\alpha : E_\alpha \to E$ is injective for every $\alpha \in I$: Fix $\alpha \in I$ and $u \in E_\alpha$ so that $e_\alpha(u) = \dot{0}$ in E. Then there exists $\beta \succcurlyeq \alpha$ in I so that $e_{\alpha,\beta}(u) = 0$. But $e_{\alpha,\beta}$ is injective, so $u = 0$. Hence e_α is injective.

By Theorem 2.3.3, $e_\alpha : E_\alpha \to E$ is an injective interval preserving lattice homomorphism for every $\alpha \in I$. It follows from Proposition 2.2.1 (i) that e_α is a **NIVL**-morphism for every $\alpha \in I$.

Let $\tilde{S} \coloneqq (\tilde{E}, (\tilde{e}_\alpha)_{\alpha \in I})$ be a compatible system over $\mathcal{D}$ in **NIVL**. By Theorem 2.3.3, the canonical map $r : E \to \tilde{E}$ is an interval preserving lattice homomorphism. We claim that r is a normal lattice homomorphism. To this end, let $A \downarrow \dot{0}$ in E. Without loss of generality we may suppose that A is bounded from above in E, say by $\dot{u}_0$. There exists $\alpha \in I$ and $u_0 \in E_\alpha$ so that $\dot{u}_0 = e_\alpha(u_0)$. Because e_α is injective and interval preserving, there exists for every $\dot{u} \in A$ a unique $u \in [0, u_0] \subseteq E_\alpha$ so that $e_\alpha(u) = \dot{u}$. In particular, $e_\alpha^{-1}[A] \subseteq [0, u_0]$. We claim that $\inf e_\alpha^{-1}[A] = 0$ in E_α. Let $0 \leq v \in E_\alpha$ be a lower bound for $e_\alpha^{-1}[A]$. Then $e_\alpha(v) \geq 0$ is a lower bound for A in E, hence $e_\alpha(v) = 0$. But e_α is injective, so $v = 0$. This verifies our claim. By definition, $r[A] = \tilde{e}_\alpha[e_\alpha^{-1}[A]]$. Because $\tilde{e}_\alpha$ is a normal lattice homomorphism it follows that $\inf r[A] = 0$ in $\tilde{E}$. □

The following list of vector lattice properties that have permanence under the construction of direct limits is taken from [**37**].

THEOREM 2.3.5. *Let $\mathcal{D} \coloneqq ((E_\alpha)_{\alpha \in I}, (e_{\alpha,\beta})_{\alpha \preccurlyeq \beta})$ be a direct system in a category* **C** *of vector lattices. Assume that $e_{\alpha,\beta}$ is injective for all $\alpha \preccurlyeq \beta$ in I. Let $\mathcal{S} \coloneqq (E, (e_\alpha)_{\alpha \in I})$ be the direct limit of $\mathcal{D}$ in* **VL**. *Then the following statements are true.*

(i) *E is Archimedean if and only if E_α is Archimedean for all $\alpha \in I$.*

(ii) *If* **C** *is* **IVL** *then E is order separable if and only if E_α is order separable for every $\alpha \in I$.*

(iii) *If* **C** *is* **IVL** *then E has the (principal) projection property if and only if E_α has the (principal) projection property for every $\alpha \in I$.*

(iv) *If* **C** *is* **IVL** *then E is (σ-)Dedekind complete if and only if E_α is (σ-)Dedekind complete for every $\alpha \in I$.*

(v) *If* **C** *is* **IVL** *then E is relatively uniformly complete if and only if E_α is relatively uniformly complete for every $\alpha \in I$.*

Before we proceed to discuss examples of direct limits, we make some clarifying remarks about the structure of the direct limit of vector lattices.

REMARK 2.3.6. Let $\mathcal{D} \coloneqq ((E_\alpha)_{\alpha \in I}, (e_{\alpha,\beta})_{\alpha \preccurlyeq \beta})$ be a direct system in **VL** and let $\mathcal{S} \coloneqq (E, (e_\alpha)_{\alpha \in I})$ be the direct limit of $\mathcal{D}$ in **VL**.

(i) Unless clarity demands it, we henceforth cease to explicitly express elements of E as equivalence classes; that is, we write $u \in E$ instead of $\dot{u} \in E$.

(ii) For every $u \in E$ there exists at least one $\alpha \in I$ and $u_\alpha \in E_\alpha$ so that $u = e_\alpha(u_\alpha)$. If $u = e_\beta(u_\beta)$ for some other $\beta \in I$ and $u_\beta \in E_\beta$, then there exists $\gamma \succcurlyeq \alpha, \beta$ in I so that $e_{\alpha,\gamma}(u_\alpha) = e_{\beta,\gamma}(u_\beta)$, and hence

$$e_\gamma(e_{\alpha,\gamma}(u_\alpha)) = u = e_\gamma(e_{\beta,\gamma}(u_\beta)).$$

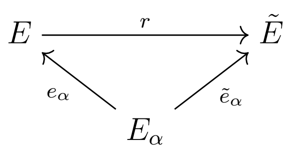
(iii) It is proven in Theorem 2.3.4 that if $e_{\alpha,\beta}$ is injective for all $\alpha \preccurlyeq \beta$ in I then e_α is injective for all $\alpha \in I$. In this case we identify E_α with the sublattice $e_\alpha[E_\alpha]$ of E.

(iv) An element $u \in E$ is positive if and only if there exists $\alpha \preccurlyeq \beta$ in I and $u_\alpha \in E_\alpha$ so that $e_\alpha(u_\alpha) = u$ and $e_{\alpha,\beta}(u_\alpha) \geq 0$ in E_β. Combining this observation with (ii) we see that $u \geq 0$ if and only if there exists $\alpha \in I$ and $0 \leq u_\alpha \in E_\alpha$ so that $u = e_\alpha(u_\alpha)$.

2.3.2. Examples of direct limits. In [**37**], a number of examples are presented of naturally occurring vector lattices which can be expressed as direct limits of vector lattices. We provide further examples which will be used in the sequel.

Example 2.3.7. Let E be a vector lattice. Let $\{E_\alpha\}_{\alpha \in I}$ be a collection of order ideals in E where $E_\alpha \subseteq E_\beta$ if and only if $\alpha \preccurlyeq \beta$. Assume that $\bigcup E_\alpha = E$. For all $\alpha \preccurlyeq \beta$ in I, let $e_{\alpha,\beta} : E_\alpha \to E_\beta$ and $e_\alpha : E_\alpha \to E$ be the inclusion mappings. Then $\mathcal{D} := ((E_\alpha)_{\alpha \in I}, (e_{\alpha,\beta})_{\alpha \preccurlyeq \beta})$ is a direct system in **NIVL** and $\mathcal{S} := (E, (e_\alpha)_{\alpha \in I})$ is the direct limit of $\mathcal{D}$ in **NIVL**.

Proof. It is clear that $\mathcal{D}$ is a direct system in **NIVL** and that $\mathcal{S}$ is a compatible system over $\mathcal{D}$ in **NIVL**. Let $\tilde{\mathcal{S}} := (\tilde{E}, (\tilde{e}_\alpha)_{\alpha \in I})$ be any compatible system over $\mathcal{D}$ in **NIVL**. We show that there exists a unique **NIVL**-morphism $r : E \to \tilde{E}$ so that for all $\alpha \in I$ the diagram



commutes. If $u \in E$ and $\alpha, \beta \in I$ are such that $u \in E_\alpha, E_\beta$, then $\tilde{e}_\alpha(u) = \tilde{e}_\beta(u)$. Indeed, for any $\gamma \succcurlyeq \alpha, \beta$ in I

$$\tilde{e}_\gamma(u) = \tilde{e}_\gamma(e_{\alpha,\gamma}(u)) = \tilde{e}_\alpha(u)$$

and

$$\tilde{e}_\gamma(u) = \tilde{e}_\gamma(e_{\beta,\gamma}(u)) = \tilde{e}_\beta(u)$$

Therefore the map $r : E \to \tilde{E}$ given by

$$r(u) = \tilde{e}_\alpha(u), \qquad u \in E_\alpha$$

is well-defined. It is clear that this map makes the above diagram commute. Further, if $u, v \in E$ then there exists $\alpha \in I$ so that $u, v \in E_\alpha$. Then for all $a, b \in \mathbb{R}$ we have $au + bv, u \vee v \in E_\alpha$ so that

$$r(au + bv) = \tilde{e}_\alpha(au + bv) = a\tilde{e}_\alpha(u) + b\tilde{e}_\alpha(v) = a\,r(u) + b\,r(v)$$

and

$$r(u \vee v) = \tilde{e}_\alpha(u \vee v) = \tilde{e}_\alpha(u) \vee \tilde{e}_\alpha(v) = r(u) \vee r(v).$$

Hence r is a lattice homomorphism. A similar argument shows that r is interval preserving. To see that r is a normal lattice homomorphism, let $A \downarrow 0$ in E. Without loss of generality, assume that there exists $0 < u_0 \in E$ so that $u \leq u_0$ for all $u \in A$. Then $A \subseteq E_\alpha$ for some $\alpha \in I$ so that $r[A] = \tilde{e}_\alpha[A]$. Hence, because $\tilde{e}_\alpha$ is a normal lattice homomorphism, $\inf r[A] = 0$. Therefore r is a **NIVL**-morphism.

It remains to show that r is the unique **NIVL**-morphism making the diagram above commute. Suppose that $\tilde{r}$ is another such morphism. Let $u \in E$. There exists $\alpha \in I$

so that $u \in E_\alpha$. We have $\tilde{r}(u) = \tilde{r}(e_\alpha(u)) = \tilde{e}_\alpha(u) = r(u)$, which completes the proof. □

The remaining examples in this section may readily been seen to be special cases of Example 2.3.7. Therefore we omit the proofs.

EXAMPLE 2.3.8. Let E be a vector lattice. For every $0 < u \in E$ let E_u be the order ideal generated by u in E. For all $0 < u \leq v$, let $e_{u,v} : E_u \to E_v$ and $e_u : E_u \to E$ be the inclusion mappings. Let I be an upward directed subset of $E^+ \setminus \{0\}$ so that $E = \bigcup E_u$. Then $\mathcal{D} := ((E_u)_{u\in I}, (e_{u,v})_{u\leq v})$ is a direct system in **NIVL** and $\mathcal{S} := (E, (e_u)_{u\in I})$ is the direct limit of $\mathcal{D}$ in **NIVL**.

EXAMPLE 2.3.9. Let (X, Σ, μ) be a complete σ-finite measure space. Let $\Xi := (X_n)$ be an increasing sequence (w.r.t. inclusion) of measurable sets with positive measure so that $X = \bigcup X_n$. For $n \leq m$ in $\mathbb{N}$ let $e_{n,m} : \mathrm{L}^p(X_n) \to \mathrm{L}^p(X_m)$ be defined (a.e.) by setting

$$e_{n,m}(u)(t) := \begin{cases} u(t) & \text{if} \quad t \in X_n \\ 0 & \text{if} \quad t \in X_m \setminus X_n \end{cases}$$

for each $u \in \mathrm{L}^p(X_n)$. Further, define

$$\mathrm{L}^p_{\Xi-c}(X) := \{u \in \mathrm{L}^p(X) \ : \ u = 0 \text{ a.e. on } X \setminus X_n \text{ for some } n \in \mathbb{N}\}.$$

For $n \in \mathbb{N}$ let $e_n : \mathrm{L}^p(X_n) \to \mathrm{L}^p_{\Xi-c}(X)$ be given by

$$e_n(u)(t) := \begin{cases} u(t) & \text{if} \quad t \in X_n \\ 0 & \text{if} \quad t \in X \setminus X_n \end{cases}$$

The following statements are true.

(i) $\mathcal{D}^p_{\Xi-c} := ((\mathrm{L}^p(X_n))_{n\in\mathbb{N}}, (e_{n,m})_{n\leq m})$ is a direct system in **NIVL**.

(ii) $\mathcal{S}^p_{\Xi-c} := (\mathrm{L}^p_{\Xi-c}(X), (e_n)_{n\in\mathbb{N}})$ is the direct limit of $\mathcal{D}^p_{\Xi-c}$ in **NIVL**.

EXAMPLE 2.3.10. Let X be a locally compact Hausdorff space. Let $\Gamma := (X_\alpha)_{\alpha\in I}$ be an upward directed (w.r.t. inclusion) collection of non-empty open precompact subsets of X so that $\bigcup X_\alpha = X$. For each $\alpha \in I$, let $\mathrm{M}(\bar{X}_\alpha)$ be the space of Radon measures on $\bar{X}_\alpha$ and $\mathrm{M}_c(X)$ the space of compactly supported Radon measures on X. For all $\alpha \preccurlyeq \beta$ in I, let $e_{\alpha,\beta} : \mathrm{M}(\bar{X}_\alpha) \to \mathrm{M}(\bar{X}_\beta)$ be defined by setting

$$e_{\alpha,\beta}(\mu)(B) = \mu(B \cap \bar{X}_\alpha) \text{ for all } \mu \in \mathrm{M}(\bar{X}_\alpha) \text{ and } B \in \mathfrak{B}_{\bar{X}_\beta}.$$

Likewise, for $\alpha \in I$, define $e_\alpha : \mathrm{M}(\bar{X}_\alpha) \to \mathrm{M}_c(X)$ by setting

$$e_\alpha(\mu)(B) = \mu(B \cap \bar{X}_\alpha) \text{ for all } \mu \in \mathrm{M}(X_\alpha) \text{ and } B \in \mathfrak{B}_X.$$

The following statements are true.

(i) $\mathcal{D}_\Gamma := \big((\mathrm{M}(\bar{X}_\alpha)_{\alpha\in I}, (e_{\alpha,\beta})_{\alpha\preccurlyeq\beta}\big)$ is a direct system in **NIVL** and $e_{\alpha,\beta}$ is injective for all $\alpha \preccurlyeq \beta$ in I.

(ii) $\mathcal{S}_\Gamma := (\mathrm{M}_c(X), (e_\alpha)_{\alpha\in I})$ is the direct limit of $\mathcal{D}_\Gamma$ in **NIVL**.

Example 2.3.11. Let X be a locally compact Hausdorff space. Let $\Gamma \coloneqq (X_\alpha)_{\alpha\in I}$ be an upward directed (w.r.t. inclusion) collection of open precompact subsets of X so that $\bigcup X_\alpha = X$. For each $\alpha \in I$, let $\mathrm{N}(\bar{X}_\alpha)$ be the space of normal Radon measures on $\bar{X}_\alpha$ and $\mathrm{N}_c(X)$ the space of compactly supported normal Radon measures on X. For all $\alpha \preccurlyeq \beta$ in I, let $e_{\alpha,\beta} : \mathrm{N}(\bar{X}_\alpha) \to \mathrm{N}(\bar{X}_\beta)$ be defined by setting

$$e_{\alpha,\beta}(\mu)(B) = \mu(B \cap \bar{X}_\alpha) \text{ for all } \mu \in \mathrm{N}(\bar{X}_\alpha) \text{ and } B \in \mathfrak{B}_{\bar{X}_\beta}.$$

Likewise, for $\alpha \in I$, define $e_\alpha : \mathrm{N}(\bar{X}_\alpha) \to \mathrm{N}_c(X)$ by setting

$$e_\alpha(\mu)(B) = \mu(B \cap \bar{X}_\alpha) \text{ for all } \mu \in \mathrm{N}(X_\alpha) \text{ and } B \in \mathfrak{B}_X.$$

The following statements are true.

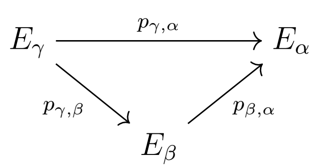
(i) $\mathcal{E}_\Gamma \coloneqq \big((\mathrm{N}(\bar{X}_\alpha)_{\alpha\in I}, (e_{\alpha,\beta})_{\alpha\preccurlyeq\beta}\big)$ is a direct system in **NIVL** and $e_{\alpha,\beta}$ is injective for all $\alpha \preccurlyeq \beta$ in I.

(ii) $\mathcal{T}_\Gamma \coloneqq (\mathrm{N}_c(X), (e_\alpha)_{\alpha\in I})$ is the direct limit of $\mathcal{J}_\Gamma$ in **NIVL**.

2.4. Inverse limits

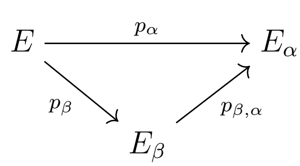
Next, we introduce the concepts of *inverse systems* and *inverse limits*. We investigate analogous questions of existence and permanence for inverse limits in our categories of vector lattices below.

Definition 2.4.1. Let $\mathbf{C}$ be any category and $(I, \preccurlyeq)$ an upwards directed set. Consider a family of objects $\{E_\alpha\}_{\alpha\in I}$ in $\mathbf{C}$. For all $\beta \succcurlyeq \alpha$ in I, let $p_{\beta,\alpha} : E_\beta \to E_\alpha$ be a morphism in $\mathbf{C}$. The pair $\mathcal{I} \coloneqq ((E_\alpha)_{\alpha\in I}, (p_{\beta,\alpha})_{\beta\succcurlyeq\alpha})$ is called an *inverse system* in $\mathbf{C}$ if, for all $\alpha \preccurlyeq \beta \preccurlyeq \gamma$ in I, the diagram



commutes in $\mathbf{C}$. We will refer to the maps $p_{\beta,\alpha}$ for $\alpha, \beta \in I$ as the *linking maps of* $\mathcal{I}$.

Let E be an object in $\mathbf{C}$ and $p_\alpha : E \to E_\alpha$ a morphism in $\mathbf{C}$ for every $\alpha \in I$. The pair $(E, (p_\alpha)_{\alpha\in I})$ is called a *compatible system over* $\mathcal{I}$ if for all $\alpha \preccurlyeq \beta$ in I the diagram



commutes in $\mathbf{C}$.

An inverse limit of $\mathcal{I}$ in $\mathbf{C}$ is a compatible system $(E, (p_\alpha)_{\alpha\in I})$ over $\mathcal{I}$ satisfying the universal property that for any compatible system $(E', (p'_\alpha)_{\alpha\in I})$ over $\mathcal{I}$ there exists a unique morphism $s : E' \to E$ so that for all $\alpha \in I$ the diagram

$$\begin{array}{ccc} E' & \xrightarrow{\quad s \quad} & E \\ & {\scriptstyle p'_\alpha} \searrow \quad \swarrow {\scriptstyle p_\alpha} & \\ & E_\alpha & \end{array}$$

commutes in $\mathbf{C}$. Where convenient, we denote by $\varprojlim \mathcal{I}$ the inverse limit of $\mathcal{I}$ in $\mathbf{C}$.

As was done for direct limits, we define the *derived category of compatible systems over* $\mathcal{I}$: Denote by $\mathbf{C}(\mathcal{I})$ the category whose objects are the compatible systems over $\mathcal{I}$ and where a morphism between compatible systems $(E, (p_\alpha)_{\alpha \in I})$ and $(E', (p'_\alpha)_{\alpha \in I})$ is a morphism $f : E \to E'$ in $\mathbf{C}$ such that for all $\alpha \in I$ the diagram

$$\begin{array}{ccc} E & \xrightarrow{\quad f \quad} & E' \\ & {\scriptstyle p_\alpha} \searrow \quad \swarrow {\scriptstyle p'_\alpha} & \\ & E_\alpha & \end{array}$$

commutes in $\mathbf{C}$. The universal property of an inverse limit in Definition 2.4.1 makes it clear that a compatible system $\mathcal{S} \coloneqq (E, (p_\alpha)_{\alpha \in I})$ over $\mathcal{I}$ is an inverse limit of $\mathcal{I}$ in $\mathbf{C}$ precisely when $\mathcal{S}$ is the terminal object in $\mathbf{C}(\mathcal{I})$. Thus, as with direct limits, we consider the inverse limit of an inverse system in a category to be essentially unique and therefore we may refer to *the* inverse limit of an inverse system, when it exists.

The following result is the analogue of Proposition 2.3.2 for inverse systems and inverse limits. We omit the proof since the same approach may be used.

Proposition 2.4.2. *Let* $\mathbf{C}$ *be any category and* $(I, \preccurlyeq)$ *an upwards directed set. Consider inverse systems* $\mathcal{I} \coloneqq ((E_\alpha)_{\alpha \in I}, (p_{\beta,\alpha})_{\alpha \preccurlyeq \beta})$ *and* $\mathcal{I}' \coloneqq \left((E'_\alpha)_{\alpha \in I}, (p'_{\beta,\alpha})_{\alpha \preccurlyeq \beta}\right)$ *in* $\mathbf{C}$ *with inverse limits* $\mathcal{S} \coloneqq (E, (p_\alpha)_{\alpha \in I})$ *and* $\mathcal{S}' \coloneqq (E', (p'_\alpha)_{\alpha \in I})$ *in* $\mathbf{C}$ *respectively. For every* $\alpha \in I$*, let* $f_\alpha : E_\alpha \to E'_\alpha$ *be a morphism in* $\mathbf{C}$ *so that the diagram*

$$\begin{array}{ccc} E_\beta & \xrightarrow{\quad f_\beta \quad} & E'_\beta \\ {\scriptstyle p_{\beta,\alpha}} \downarrow & & \downarrow {\scriptstyle p'_{\beta,\alpha}} \\ E_\alpha & \xrightarrow[\quad f_\alpha \quad]{} & E'_\alpha \end{array} \tag{2.4.1}$$

commutes for all $\alpha \preccurlyeq \beta$ *in* I*. The following statements are true.*

(i) There exists a unique lattice homomorphism $f : E \to E'$ *so that the diagram*

$$\begin{array}{ccc} E & \xrightarrow{\quad f \quad} & E' \\ {\scriptstyle p_\alpha} \downarrow & & \downarrow {\scriptstyle p'_\alpha} \\ E_\alpha & \xrightarrow[\quad T_\alpha \quad]{} & E'_\alpha \end{array} \tag{2.4.2}$$

commutes for every $\alpha \in I$.

(ii) *If* f_α *is an isomorphism in* $\mathbf{C}$ *for every* $\alpha \in I$, *then so is* f.

2.4.1. Existence of inverse limits. Our first task is to establish the existence of inverse limits in our categories of vector lattices. As one might expect, the analogue of Filter's approach can be used where one starts with the set-theoretic inverse limit of an inverse system of vector lattices (see [**21**, Chapter III, §7.1]) after which one adds the 'obvious' vector space and order structures.

THEOREM 2.4.3. *Let* $\mathcal{I} := ((E_\alpha)_{\alpha\in I}, (p_{\beta,\alpha})_{\beta\succcurlyeq\alpha})$ *be an inverse system in* $\mathbf{VL}$. *Define the set*

$$E := \left\{u \in \prod E_\alpha \ : \ \pi_\alpha(u) = p_{\beta,\alpha}(\pi_\beta(u)) \text{ for all } \alpha \preccurlyeq \beta \text{ in } I\right\}.$$

For every $\alpha \in I$ *define* $p_\alpha := \pi_\alpha|_E$. *The following statements are true.*

(i) E *is a vector sublattice of* $\prod E_\alpha$.

(ii) *The pair* $\mathcal{S} := (E, (p_\alpha)_{\alpha\in I})$ *is the inverse limit of* $\mathcal{I}$ *in* $\mathbf{VL}$.

PROOF OF (i). We verify that E is closed under the lattice operations on $\prod E_\alpha$; that it is a linear subspace follows by a similar argument, as the reader may readily verify. Consider u and v in E. Then $\pi_\alpha(u \vee v) = \pi_\alpha(u) \vee \pi_\alpha(v)$ for all $\alpha \in I$. Fix any $\alpha, \beta \in I$ so that $\beta \succcurlyeq \alpha$. Then

$$p_{\beta,\alpha}(\pi_\beta(u \vee v)) = p_{\beta,\alpha}(\pi_\beta(u)) \vee p_{\beta,\alpha}(\pi_\beta(u)) = \pi_\alpha(u) \vee \pi_\alpha(v) = \pi_\alpha(u \vee v).$$

Therefore $u \vee v \in E$. One can show similarly that $u \wedge v \in E$. ☐

PROOF OF (ii). From the definitions of E and the p_α it is clear that $\mathcal{S}$ is a compatible system over $\mathcal{I}$ in $\mathbf{VL}$. Let $\tilde{\mathcal{S}} := (\tilde{E}, (\tilde{p}_\alpha)_{\alpha\in I})$ be any compatible system over $\mathcal{I}$ in $\mathbf{VL}$. Define $s : \tilde{E} \to E$ by setting $s(u) := (\tilde{p}_\alpha(u))_{\alpha\in I}$. Let $\beta \succcurlyeq \alpha$ in I. Because $\tilde{\mathcal{S}}$ is a compatible system

$$p_{\beta,\alpha}(\tilde{p}_\beta(u)) = \tilde{p}_\alpha(u), \qquad u \in \tilde{E}.$$

Therefore $s(u) \in E$ for all $u \in \tilde{E}$. Because each $\tilde{p}_\alpha$ is a lattice homomorphism, so is s. By the definitions of s and the p_α, respectively, it follows that $p_\alpha \circ s = \tilde{p}_\alpha$ for every $\alpha \in I$. We show that s is the unique lattice homomorphism with this property. To this end, let $\tilde{s} : \tilde{E} \to E$ be a lattice homomorphism so that $p_\alpha \circ \tilde{s} = \tilde{p}_\alpha$ for every $\alpha \in I$. Fix $u \in \tilde{E}$. Then for every $\alpha \in I$,

$$\pi_\alpha(\tilde{s}(u)) = p_\alpha(\tilde{s}(u)) = \tilde{p}_\alpha(u) = \pi_\alpha(s(u)).$$

Hence $s = \tilde{s}$ and therefore $(E, (p_\alpha)_{\alpha\in I})$ is the inverse limit of $\mathcal{I}$ in $\mathbf{VL}$. ☐

THEOREM 2.4.4. *Let* $\mathcal{I} := ((E_\alpha)_{\alpha\in I}, (p_{\beta,\alpha})_{\beta\succcurlyeq\alpha})$ *be an inverse system in* $\mathbf{NVL}$ *and* $\mathcal{S} := (E, (p_\alpha)_{\alpha\in I})$ *its inverse limit in* $\mathbf{VL}$. *The following statements are true.*

(i) *Let* $A \subseteq E$ *and assume that* $\inf A = u$ *or* $\sup A = u$ *in* $\prod_{\alpha\in I} E_\alpha$, *then* $u \in E$.

(ii) *If* E_α *is Dedekind complete for every* $\alpha \in I$, *then* $\mathcal{S}$ *is the inverse limit of* $\mathcal{I}$ *in* $\mathbf{NVL}$.

Proof of (i). It is sufficient for us to consider infima of downward directed subsets of E. Let $A \subseteq E$ and assume that $A \downarrow u$ in $\prod E_\alpha$. Since the coordinate projections are normal homomorphisms (Theorem 2.2.6 (i)), for every $\alpha \in I$ we have $p_\alpha[A] = \pi_\alpha[A] \downarrow \pi_\alpha(u)$ in E_α. Since the linking maps in $\mathcal{I}$ are normal homomorphisms, for $\beta \geqslant \alpha$ in I, we have

$$\pi_\alpha(u) = \inf p_\alpha[A] = \inf p_{\beta,\alpha}[p_\beta[A]] = p_{\beta,\alpha}(\inf p_\beta[A]) = p_{\beta,\alpha}(\pi_\beta(u)).$$

Therefore $u \in E$. □

Proof of (ii). First, we prove that the p_α are normal homomorphisms: Fix $\alpha \in I$ and let $A \downarrow 0$ in E. Since E_α is Dedekind complete for every $\alpha \in I$, by Theorem 2.2.6 (iii), the product $\prod E_\alpha$ is also Dedekind complete. Therefore $A \downarrow u$ in $\prod E_\alpha$ for some $u \in \prod E_\alpha$. By (i), we have $u \in E$ which implies $A \downarrow u$ in E. Since $A \downarrow 0$ in E, we conclude that $A \downarrow 0$ in $\prod E_\alpha$. By the normality of the coordinate projections π_α, it follows that

$$\inf p_\alpha[A] = \inf \pi_\alpha[A] = \pi_\alpha[\inf A] = 0.$$

Hence the pair $\mathcal{S}$ is a compatible system over $\mathcal{I}$ in **NVL**. It remains to verify that $\mathcal{S}$ satisfies the universal property of the inverse limit in **NVL**: Let $\tilde{\mathcal{S}} \coloneqq (\tilde{E}, (\tilde{p}_\alpha)_{\alpha\in I})$ be a compatible system in **NVL**. Following the proof of Theorem 2.4.3, we need only show that $s : \tilde{E} \to E$ where $s(v) \coloneqq (\tilde{p}_\alpha(v))_{\alpha\in I}$ is a **NVL**-morphism: Let $A \downarrow 0$ in $\tilde{E}$, then since each $\tilde{p}_\alpha$ is a normal homomorphism, we have $p_\alpha[s[A]] = \tilde{p}_\alpha[A] \downarrow 0$ in E_α for each $\alpha \in I$. By Proposition 2.2.5, it follows that $s[A] \downarrow 0$ in $\prod E_\alpha$, and by (i) above $s[A] \downarrow 0$ in E. Therefore s is a **NVL**-morphism. □

2.4.2. Permanence properties. In this section, we establish some permanence properties for inverse limits along the same vein as those for direct limits given in Theorem 2.3.5. These follow easily from the construction of inverse limits given in Theorem 2.4.3 and the properties of products of vector lattices given in Theorem 2.2.6

Theorem 2.4.5. *Let* $\mathcal{I} \coloneqq ((E_\alpha)_{\alpha\in I}, (p_{\beta,\alpha})_{\beta\geqslant\alpha})$ *an inverse system in* **VL** *and* $\mathcal{S} \coloneqq (E, (p_\alpha)_{\alpha\in I})$ *its inverse limit in* **VL**. *The following statements are true.*

- *(i) If* E_α *is Archimedean for every* $\alpha \in I$, *then so is* E.
- *(ii) If* E_α *is Archimedean and relatively uniformly complete for every* $\alpha \in I$, *then* E *is relatively uniformly complete.*

Proof. We note that (i) follows immediately from Theorems 2.2.6 (ii) and the construction of an inverse limit in **VL**.

For (ii), assume that E_α is relatively uniformly complete for every $\alpha \in I$. We show that every relatively uniformly Cauchy sequence in E is relatively uniformly convergent: Because E is Archimedean by (i), it follows from [**53**, Theorem 39.4] that it suffices to consider increasing sequences. Let (u_n) be an increasing, relatively uniformly Cauchy sequence in E. Then for every $\alpha \in I$, $(p_\alpha(u_n))$ is an increasing sequence in E_α. According to [**53**, Theorem 59.3], $(p_\alpha(u_n))$ is relatively uniformly Cauchy in E_α. Because each E_α is relatively uniformly complete, there exists $u_\alpha \in E_\alpha$

so that $(p_\alpha(u_n))$ converges relatively uniformly to u_α. In fact, because $(p_\alpha(u_n))$ is increasing, $u_\alpha = \sup\{p_\alpha(u_n) : n \in \mathbb{N}\}$. Therefore $u := (u_\alpha)_{\alpha\in I} = \sup\{u_n : n \in \mathbb{N}\}$ in $\prod E_\alpha$ by Proposition 2.2.5 (ii). By Theorem 2.4.4 (i), $u \in E$ so that $u = \sup\{u_n : n \in \mathbb{N}\}$ in E. Therefore (u_n) converges relatively uniformly to u by [**53**, Lemma 39.2]. We conclude that E is relatively uniformly complete. □

THEOREM 2.4.6. *Let $\mathcal{I} := ((E_\alpha)_{\alpha\in I}, (p_{\beta,\alpha})_{\beta\succcurlyeq\alpha})$ be an inverse system in* **NVL** *and $\mathcal{S} := (E, (p_\alpha)_{\alpha\in I})$ its inverse limit in* **VL**. *The following statements are true.*

(i) If E_α is σ-Dedekind complete for every $\alpha \in I$ then so is E.

(ii) If E_α is Dedekind complete for every $\alpha \in I$ then so is E.

(iii) If E_α is laterally complete for every $\alpha \in I$ then so is E.

(iv) If E_α is universally complete for every $\alpha \in I$ then so is E.

PROOF. We prove (ii). The statements in (i) and (iii) follow by almost identical arguments, and (iv) follows immediately from (ii) and (iii).

Let $D \subseteq E$ be an upwards directed set bounded above by $u \in E$. For every $\alpha \in I$, the set $p_\alpha[D]$ is bounded above in E_α by $\pi_\alpha(u)$. Since E_α is Dedekind complete for every $\alpha \in I$, $v_\alpha := \sup p_\alpha[D]$ exists in E_α for all $\alpha \in I$ and by Proposition 2.2.5 (ii), we have that $\sup D = (v_\alpha)_{\alpha\in I}$ in $\prod E_\alpha$. By Theorem 2.4.4 (i), we have $v \in E$ and since E forms a sublattice of $\prod E_\alpha$ it follows that $v = \sup D$ in E. □

2.4.3. Examples of inverse limits. In this section, we present a number of examples of inverse systems and inverse limits in our categories of vector lattices. These will be used in conjunction with the examples in Section 2.3.2 in the sequel. Our first example is related to Example 2.3.9.

EXAMPLE 2.4.7. Let (X, Σ, μ) be a complete σ-finite measure space. Let $\Xi := (X_n)$ be an increasing sequence (w.r.t. inclusion) of measurable sets with positive measure so that $X = \bigcup X_n$. For $1 \le p \le \infty$ let $\mathrm{L}^p_{\Xi-\ell oc}(X)$ denote the set of (equivalence classes of) measurable functions $u : X \to \mathbb{R}$ so that $u\mathbf{1}_{X_n} \in \mathrm{L}^p(X)$ for every $n \in \mathbb{N}$. For $m \ge n$ in $\mathbb{N}$ let $r_{m,n} : \mathrm{L}^p(X_m) \to \mathrm{L}^p(X_n)$ and $r_n : \mathrm{L}^p_{\Xi-\ell oc}(X) \to \mathrm{L}^p(X_n)$ be the restriction maps. The following statements are true.

(i) $\mathcal{I}^p_{\Xi-\ell oc} := ((\mathrm{L}^p(X_n))_{n\in\mathbb{N}}, (r_{m,n})_{m\ge n})$ is an inverse system in **NVL**.

(ii) $\mathcal{S}^p_{\Xi-\ell oc} := (\mathrm{L}^p_{\Xi-\ell oc}(X), (r_n)_{n\in\mathbb{N}})$ is a compatible system over $\mathcal{I}^p_{\Xi-\ell oc}$ in **NVL**.

(iii) $\mathcal{S}^p_{\Xi-\ell oc}$ is the inverse limit of $\mathcal{I}^p_{\Xi-\ell oc}$ in **NVL**.

PROOF. The validity of (i) and (ii) is clear. We prove (iii).

By Theorem 2.4.4, since $\mathrm{L}^p(X_n)$ is Dedekind complete for every $n \in \mathbb{N}$, the inverse limit $(F, (p_n)_{n\in\mathbb{N}})$ of $\mathcal{I}^p_{\Xi-\ell oc}$ exists in **NVL**. Since $\mathcal{S}^p_{\Xi-\ell oc}$ is a compatible system over $\mathcal{I}^p_{\Xi-\ell oc}$ in **NVL**, there exists a unique normal lattice homomorphism $s : \mathrm{L}^p_{\Xi-\ell oc}(X) \to$

F so that the diagram

$$\begin{array}{ccc} \mathrm{L}^p_{\Xi-\ell oc}(X) & \xrightarrow{\quad s \quad} & F \\ & {\scriptstyle r_n}\searrow \quad \swarrow{\scriptstyle p_n} & \\ & \mathrm{L}^p(X_n) & \end{array}$$

commutes for every $n \in \mathbb{N}$. We show that s is bijective. To see that s is injective, suppose that $s(u) = 0$ for some $u \in \mathrm{L}^p_{\Xi-\ell oc}(X)$. Then $r_n(u) = 0$ for every $n \in \mathbb{N}$; that is, the restriction of u to each set X_n is 0. Since $\bigcup X_n = X$ it follows that $u = 0$. To see that s is surjective, consider $u \in F$. If $m \geq n$ then $p_n(u) = r_{m,n}(p_m(u))$; that is, $p_n(u) = p_m(u)$ a.e. on X_n. Therefore $v : X \to \mathbb{R}$ given by

$$v(x) \coloneqq p_n(u)(x), \qquad x \in X_n$$

is a.e. well-defined on $X = \bigcup X_n$. For $n \in \mathbb{N}$, v restricted to X_n is $p_n(u) \in \mathrm{L}^p(X_n)$. Therefore $v \in \mathrm{L}^p_{\Xi-\ell oc}(X)$. Furthermore, $p_n(s(v)) = r_n(v) = p_n(u)$ for all $n \in \mathbb{N}$ so that $s(v) = u$. We conclude that s is a lattice isomorphism. □

Our second example is a companion result for Examples 2.3.10 and 2.3.11.

Example 2.4.8. Let X be a topological space and $\mathcal{O} \coloneqq \{O_\alpha \ : \ \alpha \in I\}$ collection of non-empty open subsets of X which is upward directed with respect to inclusion; that is, $\alpha \preccurlyeq \beta$ if and only if $O_\alpha \subseteq O_\beta$. Assume that $\bigcup O_\alpha$ is dense and C-embedded in X. For $\beta \succcurlyeq \alpha$, denote by $r_{\beta,\alpha} : \mathrm{C}(\bar{O}_\beta) \to \mathrm{C}(\bar{O}_\alpha)$ and $r_\alpha : \mathrm{C}(X) \to \mathrm{C}(\bar{O}_\alpha)$ the restriction maps. The following statements are true.

(i) $\mathcal{I}_{\mathcal{O}} \coloneqq ((\mathrm{C}(\bar{O}_\alpha))_{\alpha\in I}, (r_{\beta,\alpha})_{\beta\succcurlyeq\alpha})$ is an inverse system in **VL**.

(ii) $\mathcal{S}_{\mathcal{O}} \coloneqq (\mathrm{C}(X), (r_\alpha)_{\alpha\in I})$ is a compatible system over $\mathcal{I}_{\mathcal{O}}$ in **VL**.

(iii) $\mathcal{S}_{\mathcal{O}}$ is the inverse limit of $\mathcal{I}_{\mathcal{O}}$ in **VL**.

(iv) If X is a Tychonoff space and O_α is precompact for every $\alpha \in I$ then $\mathcal{I}_{\mathcal{O}}$ is an inverse system in **NIVL**, and $\mathcal{S}_{\mathcal{O}}$ is a compatible system of $\mathcal{I}_{\mathcal{O}}$ in **NIVL**.

Proof. The validity of (i), (ii), and (iii) follows from arguments similar to those used in the proof of Example 2.4.7. We therefore omit the proofs of these statements. We only note that for (iii), we use the fact that every $u \in \mathrm{C}(\bigcup O_\alpha)$ has a unique continuous and real-valued extension to X; that is, restriction from X to $\bigcup O_\alpha$ defines a lattice isomorphism from $\mathrm{C}(\bigcup O_\alpha)$ onto $\mathrm{C}(X)$.

To verify (iv), it is sufficient to show that the r_α and $r_{\alpha,\beta}$ maps are order continuous and interval preserving. The fact that these maps are order continuous follows from [**48**, Theorem 3.4]. That they are interval preserving will follow from the fact that every compact subset of a Tychonoff space is C^*-embedded: We show that the r_α maps are interval preserving, the proof for $r_{\alpha,\beta}$ being identical. Consider an $\alpha \in I$, $u \in \mathrm{C}(X)^+$ and $v \in \mathrm{C}(\bar{O}_\alpha)$ so that $0 \leq v \leq r_\alpha(u)$. Because $\bar{O}_\alpha$ is C^*-embedded in X there exists a continuous function $v' \in \mathrm{C}(X)$ so that $r_\alpha(v') = v$. Let $w \coloneqq (0 \vee v') \wedge u$. Then $0 \leq w \leq u$ and, because r_α is a lattice homomorphism, $r_\alpha(w) = v$. Therefore $[0, r_\alpha(u)] \subseteq r_\alpha[[0,u]]$. The reverse inclusion follows since r_α is positive. □

Our next example is of a more general nature. It is an essential ingredient in our solution of the decomposition problem for $\mathrm{C}(X)$ mentioned in Section 1.1.1.

EXAMPLE 2.4.9. Let E be an Archimedean vector lattice. Denote by $\mathbf{B}_E$ the Boolean algebra of projection bands in E where the ordering on $\mathbf{B}_E$ is inclusion. Let M be a non-trivial ideal in $\mathbf{B}_E$. That is, $\mathrm{M} \subset \mathbf{B}_E$ is downward closed, upward directed and does not consist of the trivial band $\{0\}$ only. For notational convenience we express M as indexed by a directed set I, $\mathrm{M} = \{B_\alpha \ : \ \alpha \in I\}$, so that $\alpha \preccurlyeq \beta$ if and only if $B_\alpha \subseteq B_\beta$.

For $B_\alpha \subseteq B_\beta$ in M, denote by P_α the band projection of E onto B_α and by $P_{\beta,\alpha}$ the band projection of B_β onto B_α; that is, $P_{\beta,\alpha} = P_\alpha|_{B_\beta}$. The following statements are true.

(i) $\mathcal{I}_\mathrm{M} := (\mathrm{M}, (P_{\beta,\alpha})_{\beta\succcurlyeq\alpha})$ is an inverse system in **NIVL** and $(E, (P_\alpha)_{\alpha\in I})$ is a compatible system over $\mathcal{I}_\mathrm{M}$ in **NIVL**.

(ii) The inverse limit $\varprojlim \mathcal{I}_\mathrm{M} := (F, (p_\alpha)_{\alpha\in I})$ exists in **VL**. If E is Dedekind complete then $(F, (p_\alpha)_{\alpha\in I})$ is the inverse limit of $\mathcal{I}_\mathrm{M}$ in **NVL**.

(iii) The map $P_\mathrm{M} : E \ni u \mapsto (P_\alpha(u))_{\alpha\in I} \in F$ is the unique lattice homomorphism so that the diagram

$$\begin{array}{ccc} E & \xrightarrow{P_\mathrm{M}} & F \\ & {\scriptstyle P_\alpha}\searrow \quad \swarrow{\scriptstyle p_\alpha} & \\ & B_\alpha & \end{array}$$

commutes in **VL** for every $\alpha \in I$. Furthermore, $P_\mathrm{M}[E]$ is an order dense sublattice of F. If E is Dedekind complete then $P_\mathrm{M}[E]$ is an order ideal in F.

(iv) P_M is injective if and only if $\{P_\alpha : \alpha \in I\}$ separates the points of E. In this case, P_M is a lattice isomorphism onto an order dense sublattice of F.

PROOF. The statement in (i) follows immediately from Proposition 2.2.2 as band projections are both interval preserving and order continuous. The statement in (ii) follows immediately from (i) and Theorems 2.4.3 and 2.4.4. The fact that (iv) is true is a direct consequence of the definition of P_M.

It remains to prove (iii). Since P_α is a lattice homomorphism for every $\alpha \in I$, P_M is a lattice homomorphism into $\prod B_\alpha$. If $u \in E$ and $\alpha \preccurlyeq \beta$ then $P_{\beta,\alpha}(P_\alpha(u)) = P_\alpha(u)$ by Proposition 2.2.2 (iii). Hence $P_\mathrm{M}[E]$ is a sublattice of F. It follows from the construction of F as a sublattice of $\prod B_\alpha$ given in Theorem 2.4.3 that $p_\alpha \circ P_\mathrm{M} = P_\alpha$ for all $\alpha \in I$.

Let $0 < u = (u_\alpha) \in F$. There exists $\alpha_0 \in I$ so that $u_{\alpha_0} > 0$ in $B_{\alpha_0} \subseteq E$. Then $0 < P_\mathrm{M}(u_{\alpha_0}) \le u$ in F. Hence $P_\mathrm{M}[E]$ is order dense in F.

Assume that E is Dedekind complete. We show that $P_\mathrm{M}[E]$ is an order ideal in F: Consider $v \in E^+$ and $u = (u_\alpha) \in F^+$ so that $0 \le u \le P_\mathrm{M}(v)$. Then $u_\alpha \le P_\alpha(v) \le v$ for all $\alpha \in I$. Let $w = \sup\{u_\alpha \ : \ \alpha \in I\}$ in E. We claim that $P_\mathrm{M}(w) = u$. Because $u_\alpha \le w$

for all $\alpha \in I$, $u_\alpha = P_\alpha(u_\alpha) \leq P_\alpha(w)$. Therefore $u \leq P_{\mathrm{M}}(w)$. For the reverse inequality we note that for all $\beta \in I$,

$$P_\beta(w) = \sup\{P_\beta u_\alpha \ : \ \alpha \in I\}.$$

We claim that $P_\beta(u_\alpha) \leq u_\beta$ for all $\alpha, \beta \in I$. It follows from this claim that $P_\beta(w) \leq u_\beta$ so that $P_{\mathrm{M}}(w) \leq u$. Thus we need only verify that, indeed, $P_\beta(u_\alpha) \leq u_\beta$ for all $\alpha, \beta \in I$. To this end, fix $\alpha, \beta \in I$. Let $\gamma \in I$ be a mutual upper bound for α and β. Because $u = (u_\alpha) \in F$, $\tilde{\mathcal{S}}$ is compatible with $\mathcal{I}_{\mathrm{M}}$ and $u_\gamma, u_\alpha \in E$ we have

$$P_\beta(u_\alpha) = P_\beta(P_{\gamma,\alpha}(u_\gamma)) \leq P_\beta(u_\gamma) = P_{\gamma,\beta}(P_\gamma(u_\gamma)) = P_{\gamma,\beta}(u_\gamma) = u_\beta.$$

This completes the proof. □

REMARK 2.4.10. Consider the setting of Example 2.4.9 where $\varprojlim \mathcal{I}_{\mathrm{M}} \coloneqq (F, (p_\alpha)_{\alpha\in I})$ is the inverse limit of $\mathcal{I}_{\mathrm{M}}$ in **VL**. Assume that $\{P_\alpha \ : \ \alpha \in I\}$ separates the points of E. It may happen that $P_{\mathrm{M}} : E \to F$ is surjective, but this is not always the case. If this is the case, then $(E, (P_\alpha)_{\alpha\in I})$ is the inverse limit of $\mathcal{I}_{\mathrm{M}}$ in **NIVL**. A sufficient, but not necessary, condition for P_{M} to be surjective is that $E \in \mathrm{M}$. Consider the following examples:

(i) Consider the vector lattice $\mathbb{R}^{\mathbb{N}}$. For $G \subseteq \mathbb{N}$ let

$$B_G \coloneqq \{u \in \mathbb{R}^{\mathbb{N}} \ : \ \mathrm{supp}(u) \subseteq G\}.$$

Then $\mathrm{M} \coloneqq \{B_G \ : \ \varnothing \neq G \subseteq \mathbb{N} \text{ finite}\}$ is an order ideal in the Boolean algebra of projection bands $\mathbf{B}_{\mathbb{R}^{\mathbb{N}}}$ and $\{P_G : \varnothing \neq G \subseteq \mathbb{N} \text{ finite}\}$ separates the points of $\mathbb{R}^{\mathbb{N}}$. It is easy to see that P_{M} is surjective and $\mathbb{R}^{\mathbb{N}} \notin \mathrm{M}$.

(ii) Consider the vector lattice ℓ^1. As in (i), for $G \subseteq \mathbb{N}$ define

$$B_G \coloneqq \{u \in \ell^1 \ : \ \mathrm{supp}(u) \subseteq G\}$$

Then $\mathrm{M} \coloneqq \{B_G \ : \ \varnothing \neq G \subseteq \mathbb{N} \text{ finite}\}$ is an order ideal in $\mathbf{B}_{\ell^1}$ and $\{P_G : \varnothing \neq G \subseteq \mathbb{N} \text{ finite}\}$ separates the points of ℓ^1. However, $F = \mathbb{R}^{\mathbb{N}}$ and so P_{M} is not surjective.

Based on the observations in Example 2.4.9 and Remark 2.4.10 we ask the following question: Given a Dedekind complete vector lattice E, does there exist a proper ideal M in $\mathbf{B}_E$ so that the map $P_{\mathrm{M}} : E \to F$ defined in Example 2.4.9 (iii) is an isomorphism onto F? We do not pursue this question any further here, except to note the following example.

EXAMPLE 2.4.11. Let X be an extremally disconnected Tychonoff space. Let $\mathcal{O} \coloneqq \{O_\alpha : \alpha \in I\}$ be a proper, non-trivial ideal in the Boolean algebra $\mathbf{R}_X$ of clopen subsets of X. Assume that $\bigcup O_\alpha$ is dense and C-embedded in X. Then $\mathrm{M} \coloneqq \{\mathrm{C}(O_\alpha) \ : \ \alpha \in I\}$ is a proper, non-trivial ideal in $\mathbf{B}_{\mathrm{C}(X)}$ and $P_{\mathrm{M}} : \mathrm{C}(X) \to F$ defined in Example 2.4.9 (iii) is a lattice isomorphism onto F.

PROOF. The Boolean algebras $\mathbf{R}_X$ and $\mathbf{B}_{\mathrm{C}(X)}$ are isomorphic. In particular, the isomorphism is given by

$$\mathbf{R}_X \ni O \longmapsto B_O = \{u \in \mathrm{C}(X) \ : \ \mathrm{supp}(u) \subseteq O\},$$

see [**28**, Theorem 12.9]. Therefore M is a proper, nontrivial ideal in $\mathbf{B}_{\mathrm{C}(X)}$. Moreover, for $O \in \mathbf{R}_X$ the band projection onto B_O is given by restriction to O. Finally, we note that for $O \in \mathbf{R}_X$ the band B_O may be identified with $\mathrm{C}(O)$. It follows from Example 2.4.8 that $F = \mathrm{C}(X)$, i.e. $P_{\mathrm{M}} : \mathrm{C}(X) \to F$ is a lattice isomorphism onto F. □

2.5. Dual spaces and the duality between direct and inverse limits

The results presented in this section form the technical heart of this chapter. Roughly speaking, we will show, under fairly general assumptions, that the order (continuous) dual of a direct limit is an inverse limit. On the other hand, more restrictive conditions are needed to show that the order (continuous) dual of an inverse limit is a direct limit. These results form the basis of the applications to follow after this section.

2.5.1. Duals of direct limits.

Definition 2.5.1. Let $\mathcal{D} := ((E_\alpha)_{\alpha\in I}, (e_{\alpha,\beta})_{\alpha\preccurlyeq\beta})$ be a direct system in **IVL**. The *dual system* of $\mathcal{D}$ is the pair $\mathcal{D}^\sim \coloneqq \big((E_\alpha^\sim)_{\alpha\in I}, (e_{\alpha,\beta}^\sim)_{\alpha\preccurlyeq\beta}\big)$.

If $\mathcal{D}$ is a direct system in **NIVL**, define the *order continuous dual system* of $\mathcal{D}$ as the pair $\mathcal{D}_{\mathrm{n}}^\sim \coloneqq \big(((E_\alpha)_{\mathrm{n}}^\sim)_{\alpha\in I}, (e_{\alpha,\beta}^\sim)_{\alpha\preccurlyeq\beta}\big)$ with $e_{\alpha,\beta}^\sim : (E_\beta)_{\mathrm{n}}^\sim \to (E_\alpha)_{\mathrm{n}}^\sim$.

Proposition 2.5.2. *Let* $\mathcal{D} := ((E_\alpha)_{\alpha\in I}, (e_{\alpha,\beta})_{\alpha\preccurlyeq\beta})$ *be a direct system in* **VL**. *The following statements are true.*

(i) If $\mathcal{D}$ is a direct system in **IVL**, *then the dual system $\mathcal{D}^\sim$ is an inverse system in* **NIVL**.

(ii) If $\mathcal{D}$ is a direct system in **NIVL**, *then the order continuous dual system $\mathcal{D}_{\mathrm{n}}^\sim$ is an inverse system in* **NIVL**.

Proof. We present the proof of (i). The validity of (ii) follows by a similar argument, so we omit the proof.

Assume that $\mathcal{D}$ is a direct system in **IVL**. Then the maps $e_{\alpha,\beta} : E_\alpha \to E_\beta$ are interval preserving lattice homomorphisms for all $\alpha \preccurlyeq \beta$. By Theorem 2.2.3, the adjoint maps $e_{\alpha,\beta}^\sim : E_\beta^\sim \to E_\alpha^\sim$ are all normal interval preserving lattice homomorphisms. Fix $\alpha, \beta, \gamma \in I$ such that $\alpha \preccurlyeq \beta \preccurlyeq \gamma$. Since $\mathcal{D}$ is a direct system in **NIVL**, we have $e_{\alpha,\gamma} = e_{\beta,\gamma} \circ e_{\alpha,\beta}$ so that $e_{\alpha,\gamma}^\sim = e_{\alpha,\beta}^\sim \circ e_{\beta,\gamma}^\sim$. Thus the dual system $\mathcal{D}^\sim = \big((E_\alpha^\sim)_{\alpha\in I}, (e_{\alpha,\beta}^\sim)_{\alpha\preccurlyeq\beta}\big)$ is an inverse system in **NIVL**. □

Proposition 2.5.3. *Let* $\mathcal{D} := ((E_\alpha)_{\alpha\in I}, (e_{\alpha,\beta}))$ *be a direct system in* **IVL** *and* $\mathcal{S} := (E, (e_\alpha)_{\alpha\in I})$ *a compatible system over $\mathcal{D}$ in* **IVL**. *The following statements are true.*

(i) $\mathcal{S}^\sim := (E^\sim, (e_\alpha^\sim)_{\alpha\in I})$ *is a compatible system over the inverse system $\mathcal{D}^\sim$ in* **NIVL**.

(ii) If $\mathcal{D}$ is a direct system in **NIVL**, *then* $\mathcal{S}_{\mathrm{n}}^\sim := (E_{\mathrm{n}}^\sim, (e_\alpha^\sim)_{\alpha\in I})$ *is a compatible system over the inverse system $\mathcal{D}_{\mathrm{n}}^\sim$ in* **NIVL**.

PROOF. Again, we only prove (i) as the proof of (ii) is similar. By Theorem 2.2.3, $e_\alpha^\sim : E^\sim \to E_\alpha^\sim$ is a normal interval preserving lattice homomorphism for every $\alpha \in I$. Furthermore, if $\alpha \preccurlyeq \beta$ then $e_\alpha = e_\beta \circ e_{\alpha,\beta}$ so that $e_\alpha^\sim = e_{\alpha,\beta}^\sim \circ e_\beta^\sim$. Therefore $\mathcal{S}^\sim$ is a compatible system of $\mathcal{D}^\sim$ in **NIVL**. □

The main results of this section are the following.

THEOREM 2.5.4. *Let $\mathcal{D} := ((E_\alpha)_{\alpha\in I}, (e_{\alpha,\beta})_{\alpha\preccurlyeq\beta})$ be a direct system in* **IVL**, *and let $\mathcal{S} := (E, (e_\alpha)_{\alpha\in I})$ be the direct limit of $\mathcal{D}$ in* **IVL**. *The following statements are true.*

(i) $\varprojlim \mathcal{D}^\sim := (F, (p_\alpha)_{\alpha\in I})$ exists in **NVL**.

(ii) $\left(\varinjlim \mathcal{D}\right)^\sim \cong \varprojlim \mathcal{D}^\sim$ in **NVL**. *That is, there exists a lattice isomorphism $T : E^\sim \to F$ such that the following diagram commutes for all $\alpha \in I$.*

$$\begin{array}{ccc} E^\sim & \xrightarrow{\quad T \quad} & F \\ & {\scriptstyle e_\alpha^\sim}\searrow \quad \swarrow{\scriptstyle p_\alpha} & \\ & E_\alpha^\sim & \end{array} \tag{2.5.1}$$

PROOF. The fact that (i) is true follows from Proposition 2.5.2 (i) and Theorem 2.4.4 (ii) because $E_\alpha^\sim$ is Dedekind complete for every $\alpha \in I$.

We prove (ii): By Proposition 2.5.3 (i), $\mathcal{S}^\sim := (E^\sim, (e_\alpha^\sim)_{\alpha\in I})$ is a compatible system over $\mathcal{D}^\sim$ in **NIVL**, hence also in **NVL**. Therefore there exists a unique normal lattice homomorphism $T : E^\sim \to F$ so that the diagram (2.5.1) commutes. We show that T is bijective.

To see that T is injective, let $\psi \in E^\sim$ and suppose that $T(\psi) = 0$. Consider any $u \in E$. There exists $\alpha \in I$ and $u_\alpha \in E_\alpha$ so that $u = e_\alpha(u_\alpha)$, see Remark 2.3.6. Then $\psi(u) = \psi(e_\alpha(u_\alpha)) = e_\alpha^\sim(\psi)(u_\alpha) = p_\alpha(T(\psi))(u) = 0$. This holds for all $u \in E$ so that $\psi = 0$. Therefore T is injective.

It remains to show that T maps $E^\sim$ onto F. To this end, consider $(\varphi_\alpha)_{\alpha\in I} \in F^+$. We construct a functional $0 \leq \varphi \in E^\sim$ so that $T(\varphi) = (\varphi_\alpha)_{\alpha\in I}$.

Let $u \in E$. Consider any $\alpha, \beta \in I$, $u_\alpha \in E_\alpha$ and $u_\beta \in E_\beta$ so that $e_\alpha(u_\alpha) = u = e_\beta(u_\beta)$, see Remark 2.3.6. We claim that $\varphi_\alpha(u_\alpha) = \varphi_\beta(u_\beta)$. Indeed, there exists $\gamma \succcurlyeq \alpha, \beta$ in I so that $e_{\alpha,\gamma}(u_\alpha) = e_{\beta,\gamma}(u_\beta)$. Furthermore, $e_\gamma(e_{\alpha,\gamma}(u_\alpha)) = u = e_\gamma(e_{\beta,\gamma}(u_\beta))$. Because $(\varphi_\alpha)_{\alpha\in I} \in F$ we have $\varphi_\alpha = e_{\alpha,\gamma}^\sim(\varphi_\gamma)$ and $\varphi_\beta = e_{\beta,\gamma}^\sim(\varphi_\gamma)$; that is,

$$\varphi_\alpha(u_\alpha) = \varphi_\gamma(e_{\alpha,\gamma}(u_\alpha)) = \varphi_\gamma(e_{\beta,\gamma}(u_\beta)) = \varphi_\beta(u_\beta).$$

Thus our claim is verified.

For $u \in E$ define $\varphi(u) := \varphi_\alpha(u_\alpha)$ if $u = e_\alpha(u_\alpha)$. By our above claim, φ is a well-defined map from E into $\mathbb{R}$. We show that φ is linear. Consider $u, v \in E$ and $a, b \in \mathbb{R}$. Let $u = e_\alpha(u_\alpha)$ and $v = e_\beta(v_\beta)$ where $\alpha, \beta \in I$, $u_\alpha \in E_\alpha$ and $v_\beta \in E_\beta$. There exists $\gamma \succcurlyeq \alpha, \beta$ in I so that

$$au + bv = e_\gamma(ae_{\alpha,\gamma}(u_\alpha) + be_{\beta,\gamma}(v_\beta)).$$

Then

$$\varphi(au+bv) = \varphi_\gamma(ae_{\alpha,\gamma}(u_\alpha)+be_{\beta,\gamma}(v_\beta)) = a\varphi_\gamma(e_{\alpha,\gamma}(u_\alpha)) + b\varphi_\gamma(e_{\beta,\gamma}(v_\beta)).$$

But $e_\gamma(e_{\alpha,\gamma}(u_\alpha)) = e_\alpha(u_\alpha) = u$ and $e_\gamma(e_{\beta,\gamma}(v_\beta)) = e_\beta(v_\beta) = v$. Hence $\varphi_\gamma(e_{\alpha,\gamma}(u_\alpha)) = \varphi(u)$ and $\varphi_\gamma(e_{\beta,\gamma}(v_\beta)) = \varphi(v)$. Therefore $\varphi(au+bv) = a\varphi(u)+b\varphi(v)$.

We show that φ is positive. If $0 \le u \in E$ then there exist $\alpha \in I$ and $0 \le u_\alpha \in E_\alpha$ so that $u = e_\alpha(u_\alpha)$, see Remark 2.3.6. Then $\varphi(u) = \varphi_\alpha(e_\alpha(u_\alpha)) \ge 0$, the final inequality following from the fact that $(\varphi_\alpha)_{\alpha\in I} \in F^+$.

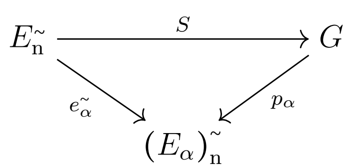
It follows from the definition of φ and the commutativity of the diagram (2.5.1) that $p_\alpha(T(\varphi)) = e_\alpha^\sim(\varphi) = \varphi_\alpha$ for every $\alpha \in I$. Hence $T(\varphi) = (\varphi_\alpha)_{\alpha\in I}$ so that T is surjective. □

Theorem 2.5.5. *Let $\mathcal{D} := ((E_\alpha)_{\alpha\in I}, (e_{\alpha,\beta})_{\alpha\preccurlyeq\beta})$ be a direct system in* **NIVL**, *and let $\mathcal{S} := (E, (e_\alpha)_{\alpha\in I})$ be the direct limit of $\mathcal{D}$ in* **IVL**. *The following statements are true.*

(i) $\varprojlim \mathcal{D}_{\mathrm{n}}^\sim := (G, (p_\alpha)_{\alpha\in I})$ *exists in* **NVL**.

(iii) If $e_{\alpha,\beta}$ is injective for all $\alpha \preccurlyeq \beta$ in I, then $\left(\varinjlim \mathcal{D}\right)_{\mathrm{n}}^\sim \cong \varprojlim \mathcal{D}_{\mathrm{n}}^\sim$ *in* **NVL**. *That is, there exists a lattice isomorphism $S : E_{\mathrm{n}}^\sim \to G$ such that the following diagram commutes for all $\alpha \in I$.*

(2.5.2)



Proof. The proof proceeds in a similar fashion to that of Theorem 2.5.4. The fact that (i) is true follows from Proposition 2.5.2 and Theorem 2.4.4.

For the proof of (ii), assume that $e_{\alpha,\beta}$ is injective for all $\alpha \preccurlyeq \beta$ in I. By Proposition 2.5.3, $\mathcal{S}_{\mathrm{n}}^\sim$ is a compatible system over $\mathcal{D}_{\mathrm{n}}^\sim$ in **NIVL**, hence in **NVL**. Therefore there exists a unique normal lattice homomorphism $S : E_{\mathrm{n}}^\sim \to G$ so that the diagram (2.5.2) commutes.

It follows by exactly the same reasoning as employed in the proof of Theorem 2.5.4 that S is injective. It remains to verify that S maps $E_{\mathrm{n}}^\sim$ onto G. Let $(\varphi_\alpha)_{\alpha\in I} \in G^+$. As in the proof of Theorem 2.5.4 we define a positive functional $\varphi \in E^\sim$ by setting, for each $u \in E$,

$$\varphi(u) := \varphi_\alpha(u_\alpha) \text{ if } u = e_\alpha(u_\alpha).$$

We claim that φ is order continuous. To see that this is so, let $A \downarrow 0$ in E. Without loss of generality, we may assume that A is bounded above by some $0 \le w \in E$. By Remark 2.3.6 (ii), there exists an $\alpha \in I$ and a $0 \le w_\alpha \in E_\alpha$ so that $e_\alpha(w_\alpha) = w$, and, by Remark 2.3.6 (iii), e_α is injective for all $\alpha \in I$. Because e_α is also interval preserving, there exists for every $u \in A$ a unique $0 \le u_\alpha \le w_\alpha$ in E_α so that $e_\alpha(u_\alpha) = u$. Let $A_\alpha := \{u_\alpha \ : \ u \in A\}$. Then $A_\alpha \downarrow 0$ in E_α. Indeed, let $0 \le v \in E_\alpha$ be a lower bound for A_α. Then $0 \le e_\alpha(v) \le e_\alpha(u_\alpha) = u$ for all $u \in A$. Because $A \downarrow 0$ in E it follows that $e_\alpha(v) = 0$, hence $v = 0$. By definition of φ and the order continuity of φ_α we now have $\varphi[A] = \varphi_\alpha[A_\alpha] \downarrow 0$. Hence $\varphi \in E_{\mathrm{n}}^\sim$.

By definition of φ and the commutativity of the diagram (2.5.2), it follows that $S(\varphi) = (\varphi_\alpha)_{\alpha\in I}$. Therefore S is surjective. □

REMARK 2.5.6. Let $\mathcal{D} := ((E_\alpha)_{\alpha\in I}, (e_{\alpha,\beta})_{\alpha\preccurlyeq\beta})$ be a direct system in **IVL**, and let $\mathcal{S} := (E, (e_\alpha)_{\alpha\in I})$ be the direct limit of $\mathcal{D}$ in **IVL**. In general, the fact that ${}^\circ E^\sim = \{0\}$ does not follow from ${}^\circ(E_\alpha)^\sim = \{0\}$ for all $\alpha \in I$, even if all the E_α are non-trivial and the e_α injective. Indeed, it is well known that $\mathrm{L}^0[0,1]$, the space of Lebesgue measurable functions on the unit interval $[0,1]$, has trivial order dual, see for instance [**70**, Example 85.1]. However, by Example 2.3.8, $\mathrm{L}^0[0,1]$ can be expressed as the direct limit of its principal order ideals, each of which has a separating order dual.

In view of the above remark, the following proposition is of interest.

PROPOSITION 2.5.7. *Let* $\mathcal{D} := ((E_\alpha)_{\alpha\in I}, (e_{\alpha,\beta})_{\alpha\preccurlyeq\beta})$ *be a direct system in* **IVL**, *and let* $\mathcal{S} := (E, (e_\alpha)_{\alpha\in I})$ *be the direct limit of* $\mathcal{D}$ *in* **IVL**. *Assume that for every* $\alpha \in I$, e_α *is injective and* $e_\alpha[E_\alpha]$ *is a projection band in* E. *The following statements are true.*

(i) *If* ${}^\circ(E_\alpha)^\sim = \{0\}$ *for every* $\alpha \in I$ *then* ${}^\circ E^\sim = \{0\}$.

(ii) *If* ${}^\circ(E_\alpha)^\sim_{\mathrm{n}} = \{0\}$ *for every* $\alpha \in I$ *then* ${}^\circ E^\sim_{\mathrm{n}} = \{0\}$.

PROOF. For (i), assume that ${}^\circ(E_\alpha)^\sim = \{0\}$ for every $\alpha \in I$. Let $u \in E$ be non-zero. Then there exists $\alpha \in I$ and a non-zero $u_\alpha \in E_\alpha$ so that $e_\alpha(u_\alpha) = u$, see Remark 2.3.6. By assumption, there exists $\varphi_\alpha \in E_\alpha{}^\sim$ so that $\varphi_\alpha(u_\alpha) \neq 0$. Denote by $P_\alpha : E \to e_\alpha[E_\alpha]$ the projection onto $e_\alpha[E_\alpha]$. We note that e_α is an isomorphism onto $e_\alpha[E_\alpha]$. Let $\varphi := (e_\alpha^{-1} \circ P_\alpha)^\sim(\varphi_\alpha)$. Then $\varphi \in E^\sim$ and $\varphi(u) = \varphi_\alpha(e_\alpha^{-1}(P_\alpha(u))) = \varphi_\alpha(u_\alpha) \neq 0$. Hence ${}^\circ E^\sim = \{0\}$.

The proof of (ii) is identical to the above with the additional note that for all $\alpha \in I$, e_α and e_α^{-1} are order continuous by Proposition 2.2.1 (i). □

2.5.2. Duals of inverse limits. We now study analogous results for duals of inverse limits. In the special case of inverse systems indexed by $\mathbb{N}$, we are able to prove similar results to those of Theorems 2.5.4 and 2.5.5. Beyond this, we will identify the main obstacle to more general results for inverse systems over arbitrary index sets: Positive (order continuous) functionals defined on a proper sublattice of a vector lattice E do not necessarily extend to E.

DEFINITION 2.5.8. Let $\mathcal{I} := ((E_\alpha)_{\alpha\in I}, (p_{\beta,\alpha})_{\beta\succcurlyeq\alpha})$ be an inverse system in **IVL**. The *dual system* of $\mathcal{I}$ is the pair $\mathcal{I}^\sim := \left((E_\alpha^\sim)_{\alpha\in I}, (p_{\beta,\alpha}^\sim)_{\beta\succcurlyeq\alpha}\right)$.

If $\mathcal{I}$ is an inverse system in **NVL**, define the *order continuous dual system* of $\mathcal{I}$ as the pair $\mathcal{I}^\sim_{\mathrm{n}} := \left(((E_\alpha)^\sim_{\mathrm{n}})_{\alpha\in I}, (p_{\beta,\alpha}^\sim)_{\beta\succcurlyeq\alpha}\right)$ with $p_{\beta,\alpha}^\sim : (E_\alpha)^\sim_{\mathrm{n}} \to (E_\beta)^\sim_{\mathrm{n}}$.

The following preliminary results, analogous to Propositions 2.5.2 and 2.5.3, are proven in the same way as the corresponding results for direct limits. As such, we omit the proofs.

PROPOSITION 2.5.9. *Let* $\mathcal{I} := ((E_\alpha)_{\alpha\in I}, (p_{\beta,\alpha})_{\beta\succcurlyeq\alpha})$ *be an inverse system in* **VL**. *The following statements are true.*

(i) *If $\mathcal{I}$ is an inverse system in* **IVL**, *then the dual system $\mathcal{I}^\sim$ is a direct system in* **NIVL**.

(ii) *If $\mathcal{I}$ is an inverse system in* **NIVL**, *then the order continuous dual system $\mathcal{I}^\sim_{\mathrm{n}}$ is a direct system in* **NIVL**.

PROPOSITION 2.5.10. *Let $\mathcal{I} \coloneqq ((E_\alpha)_{\alpha\in I}, (p_{\beta,\alpha})_{\beta\succcurlyeq\alpha})$ be an inverse system in* **IVL** *and $\mathcal{S} \coloneqq (E, (p_\alpha)_{\alpha\in I})$ a compatible system of $\mathcal{I}$ in* **IVL**. *The following statements are true.*

(i) *$\mathcal{S}^\sim \coloneqq (E^\sim, (p^\sim_\alpha)_{\alpha\in I})$ is a compatible system for the direct system $\mathcal{I}^\sim$ in* **NIVL**.

(ii) *If $\mathcal{I}$ is an inverse system in* **NIVL**, *then $\mathcal{S}^\sim_{\mathrm{n}} \coloneqq (E^\sim_{\mathrm{n}}, (p^\sim_\alpha)_{\alpha\in I})$ is a compatible system for the direct system $\mathcal{I}^\sim_{\mathrm{n}}$ in* **NIVL**.

LEMMA 2.5.11. *Let $\mathcal{I} \coloneqq ((E_n)_{n\in\mathbb{N}}, (p_{m,n})_{m\geq n})$ be an inverse system in* **IVL** *and let $\mathcal{S} \coloneqq (E, (p_n)_{n\in\mathbb{N}})$ be the inverse limit of $\mathcal{I}$ in* **VL**. *Assume that $p_{m,n}$ is a surjection for all $m \geq n$ in $\mathbb{N}$. Then p_n is surjective and interval preserving for every $n \in \mathbb{N}$.*

PROOF. Fix $n_0 \in \mathbb{N}$. Consider any $u_{n_0} \in E_{n_0}$. For $n < n_0$ let $u_n = p_{n_0,n}(u_{n_0})$. Because p_{n_0+1,n_0} is a surjection, there exists $u_{n_0+1} \in E_{n_0+1}$ so that $p_{n_0+1,n_0}(u_{n_0+1}) = u_{n_0}$. Inductively, for each $n > n_0$ there exists $u_n \in E_n$ so that $p_{n,n-1}(u_n) = u_{n-1}$.

We show that $(u_n) \in E$. Let $n < m$ in $\mathbb{N}$. By the definition of an inverse system, it follows that $p_{m,n} = p_{n+1,n} \circ p_{n+2,n+1} \circ \cdots \circ p_{m-1,m-2} \circ p_{m,m-1}$. It thus follows that $p_{m,n}(u_m) = u_n$ so that $(u_n) \in E$. We have $p_{n_0}((u_n)) = u_{n_0}$ so that p_{n_0} is a surjection. It follows from Proposition 2.2.1 that p_{n_0} is interval preserving. ☐

THEOREM 2.5.12. *Let $\mathcal{I} \coloneqq ((E_n)_{n\in\mathbb{N}}, (p_{m,n})_{m\geq n})$ be an inverse system in* **IVL**, *and let $\mathcal{S} \coloneqq (E, (p_n)_{n\in\mathbb{N}})$ be the inverse limit of $\mathcal{I}$ in* **VL**. *Assume that $p_{m,n}$ is a surjection for all $m \geq n$ in $\mathbb{N}$. Then the following statements are true.*

(i) *$\varinjlim \mathcal{I}^\sim \coloneqq (F, (e_n)_{n\in\mathbb{N}})$ exists in* **NIVL**.

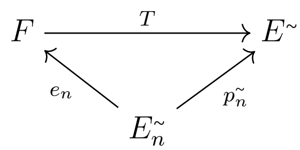
(ii) *$\left(\varprojlim \mathcal{I}\right)^\sim \cong \varinjlim \mathcal{I}^\sim$ in* **NIVL**. *That is, there exists a lattice isomorphism $T : F \to E^\sim$ such that the following diagram commutes for all $n \in \mathbb{N}$.*

$$\begin{array}{ccc} F & \xrightarrow{\quad T \quad} & E^\sim \\ & {\scriptstyle e_n}\nwarrow \quad \nearrow{\scriptstyle p_n^\sim} & \\ & E_n^\sim & \end{array}$$

PROOF. By Proposition 2.5.9, $\mathcal{I}^\sim$ is a direct system in **NIVL**. Because the $p_{m,n}$ are surjections their adjoints are injective. Thus by Theorem 2.3.4, $\varinjlim \mathcal{I}^\sim$ exists in **NIVL**.

We proceed to prove (ii). Because the $p^\sim_{m,n} : (E_n)^\sim \to (E_m)^\sim$ are injective, so are the $e_n : (E_n)^\sim \to F$, see Remark 2.3.6. By Lemma 2.5.11, each $p_n : E \to E_n$ is surjective and interval preserving. This implies that $\mathcal{S}$ is a compatible system over $\mathcal{I}$ in **IVL** and that $p^\sim_n : (E_n)^\sim \to E^\sim$ is an injection for every n in $\mathbb{N}$.

By Proposition 2.5.10, $\mathcal{S}^{\sim} = (E^{\sim}, (p_n^{\sim})_{n\in\mathbb{N}})$ is a compatible system over $\mathcal{I}^{\sim}$ in **NIVL**. Therefore there exists a unique interval preserving normal lattice homomorphism $T : F \to E^{\sim}$ so that the diagram



commutes for all $n \in \mathbb{N}$. We show that T is a lattice isomorphism.

Our first goal is to establish that T is injective. Consider $\varphi \in F$ so that $T(\varphi) = 0$. There exists an $n \in \mathbb{N}$ and a unique $\varphi_n \in E_n^{\sim}$ so that $e_n(\varphi_n) = \varphi$. Then $p_n^{\sim}(\varphi_n) = T(e_n(\varphi_n)) = T(\varphi) = 0$. But $p_n^{\sim}$ is injective so that $\varphi_n = 0$, hence $\varphi = e_n(\varphi_n) = 0$.

It remains to show that T maps F onto $E^{\sim}$. This will follow from the equality

$$E^{\sim} = \bigcup p_n^{\sim}[(E_n)^{\sim}],$$

which we now establish: Suppose that $E^{\sim} \neq \bigcup p_n^{\sim}[E_n^{\sim}]$ and take $0 \leq \psi \in E^{\sim} \setminus \bigcup p_n^{\sim}[E_n^{\sim}]$. By Proposition 2.2.4 (i), $p_n^{\sim}[E_n^{\sim}] = \ker(p_n)^{\circ}$ for every $n \in \mathbb{N}$ so that $\psi \notin \ker(p_n)^{\circ}$ for $n \in \mathbb{N}$. Hence, for every $n \in \mathbb{N}$, there exists $0 \leq u^{(n)} \in \ker(p_n)$ so that $\psi(u^{(n)}) = 1$. We claim that there exists $w \in E$ so that $w \geq u^{(1)} + \cdots + u^{(n)}$ for all $n \in \mathbb{N}$. This claim leads to $\psi(w) \geq \psi(u^{(1)} + \cdots + u^{(n)}) = n$ for every $n \in \mathbb{N}$ which is impossible, contradicting the supposition that $E^{\sim} \neq \bigcup p_n^{\sim}[E_n^{\sim}]$.

It remains to prove the claim: Write $u^{(n)} = (u_m^{(n)}) \in E \subseteq \prod E_m$. Fix $m \in \mathbb{N}$. If $n > m$ then $u_m^{(n)} = p_{n,m}(p_n(u^{(n)})) = 0$ because $u^{(n)} \in \ker(p_n)$. Let $w_m := u_m^{(1)} + \cdots + u_m^{(m)}$ and $w := (w_m)$. Then $w \geq u^{(1)} + \cdots + u^{(n)}$ for every $n \in \mathbb{N}$ because $u_m^{(n)} \geq 0$ for all $m, n \in \mathbb{N}$. To see that $w \in E$ consider $m_1 \geq m_0$ in $\mathbb{N}$. Then

$$p_{m_1,m_0}(w_{m_1}) = p_{m_1,m_0}(u_{m_1}^{(1)}) + \cdots + p_{m_1,m_0}(u_{m_1}^{(m_1)}).$$

But $u^{(n)} = (u_m^{(n)}) \in E$ for all $n \in \mathbb{N}$, so

$$p_{m_1,m_0}(w_{m_1}) = u_{m_0}^{(1)} + \cdots + u_{m_0}^{(m_1)}.$$

Finally, because $u_m^{(n)} = 0$ for all $n > m$ in $\mathbb{N}$ we have

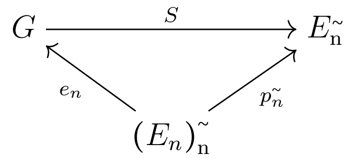
$$p_{m_1,m_0}(w_{m_1}) = u_{m_0}^{(1)} + \cdots + u_{m_0}^{(m_0)} = w_{m_0}.$$

Hence $w \in E$, which verifies our claim. This completes the proof. □

THEOREM 2.5.13. *Let* $\mathcal{I} := ((E_n)_{n\in\mathbb{N}}, (p_{m,n})_{m\geq n})$ *be an inverse system in* **NIVL**, *and let* $\mathcal{S} := (E, (p_n)_{n\in\mathbb{N}})$ *be the inverse limit of* $\mathcal{I}$ *in* **VL**. *Assume that* $p_{m,n}$ *is a surjection for all* $m \geq n$ *in* $\mathbb{N}$. *The following statements are true.*

(i) $\varinjlim \mathcal{I}_{\mathrm{n}}^{\sim} := (G, (e_n)_{n\in\mathbb{N}})$ *exists in* **NIVL**.

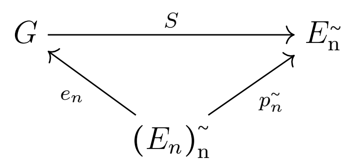
(ii) $\left(\varprojlim \mathcal{I}\right)_{\mathrm{n}}^{\sim} \cong \varinjlim \mathcal{I}_{\mathrm{n}}^{\sim}$ *in* **NIVL**. *That is, there exists a lattice isomorphism* $S : G \to E_{\mathrm{n}}^{\sim}$ *such that the following diagram commutes for all* $n \in \mathbb{N}$.



Proof. The existence of $\varinjlim \mathcal{I}^{\sim}_{\mathrm{n}}$ in **NIVL** follows by the same reasoning as given in Theorem 2.5.12.

For (ii), as in the proof of Theorem 2.5.12, we see that $e_n : (E_n)^{\sim}_{\mathrm{n}} \to G$ and $p^{\sim}_n : (E_n)^{\sim}_{\mathrm{n}} \to E^{\sim}_{\mathrm{n}}$ are injective interval preserving maps for all $n \in \mathbb{N}$. In addition, $\mathcal{S}$ is a compatible system over $\mathcal{I}$ in **IVL**.

By Proposition 2.5.10, $\mathcal{S}^{\sim}_{\mathrm{n}} = (E^{\sim}_{\mathrm{n}}, (p^{\sim}_n)_{n\in\mathbb{N}})$ is a compatible system over $\mathcal{I}^{\sim}_{\mathrm{n}}$ in **NIVL**. Therefore there exists a unique interval preserving normal lattice homomorphism $S : G \to E^{\sim}_{\mathrm{n}}$ so that the diagram

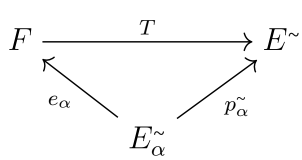


commutes for all $n \in \mathbb{N}$. The reader may verify that exactly the same argument as used in the proof of Theorem 2.5.12 shows that S is a lattice isomorphism, this time making use of Proposition 2.2.4 (ii). □

We observe that the proofs of Theorems 2.5.12 and 2.5.13 cannot be generalised to systems over an arbitrary directed set I. Indeed, the assumption that the inverse system $\mathcal{I}$ is indexed by $\mathbb{N}$ is used in essential ways to show that the mappings T and S in Theorems 2.5.12 and 2.5.13, respectively, are both injective and surjective: The injectivity of S and T follows from the surjectivity of the maps p_n, which in turn follows from Lemma 2.5.11 where the total ordering of $\mathbb{N}$ is used explicitly. We are not aware of any conditions on a general inverse system $\mathcal{I}$ in **VL**, indexed by an arbitrary directed set, which implies that the projections from inverse limit to the components are necessarily surjective. Furthermore, the method of proof for surjectivity of S and T cannot be generalised to systems over arbitrary directed sets. As we show next, this issue is related to the extension of positive linear functionals.

Theorem 2.5.14. *Let $\mathcal{I} := ((E_\alpha)_{\alpha\in I}, (p_{\beta,\alpha})_{\alpha\preccurlyeq\beta})$ be an inverse system in* **IVL** *and $\mathcal{S} := (E, (p_\alpha)_{\alpha\in I})$ its inverse limit in* **VL**. *Assume that $p_{\beta,\alpha}$ and p_α are surjections for all $\beta \succcurlyeq \alpha$ in I. Then the following statements are true.*

- *(i) $\varinjlim \mathcal{I}^{\sim} := (F, (e_\alpha)_{\alpha\in I})$ exists in* **NIVL**.
- *(ii) There exists an injective normal interval preserving lattice homomorphism $T : F \to E^{\sim}$ so that the diagram*

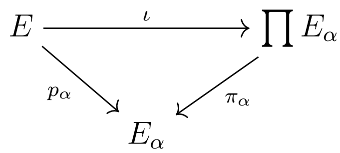


commutes for every $\alpha \in I$.

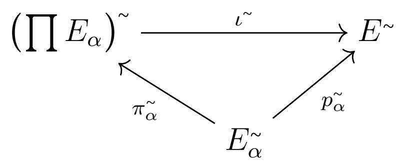
(iii) If T *is a bijection, hence a lattice isomorphism, then every positive linear functional on* E *has a positive linear extension to* $\prod E_\alpha$. *The converse is true if* I *has non-measurable cardinal.*

Proof. The fact that (i) and (ii) are true follow as in the proof of Theorem 2.5.12.

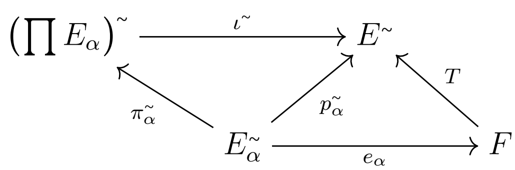
We verify (iii): Let $\iota : E \to \prod E_\alpha$ be the inclusion map. The diagram



commutes in **VL** for every $\alpha \in I$, and therefore the diagram



also commutes in **VL** for each $\alpha \in I$. Hence, for each $\alpha \in I$, the diagram



commutes in **VL**. Assume that T is a lattice isomorphism, and therefore a surjection. Let $\varphi \in E^\sim$. There exists a $\psi \in F$ so that $T(\psi) = \varphi$. By Remark 2.3.6, there exist $\alpha \in I$ and $\psi_\alpha \in E_\alpha^\sim$ so that $e_\alpha(\psi_\alpha) = \psi$. Then

$$\iota^\sim(\pi_\alpha^\sim(\psi_\alpha)) = p_\alpha^\sim(\psi_\alpha) = T(e_\alpha(\psi_\alpha)) = \varphi.$$

Therefore $\iota^\sim$ is a surjection; that is, every $\varphi \in E^\sim$ has an order bounded linear extension to $\prod E_\alpha$.

Next, assume that I has non-measurable cardinal and every order bounded linear functional on E extends to an order bounded linear functional on $\prod E_\alpha$. Then $\iota^\sim : \left(\prod E_\alpha\right)^\sim \to E^\sim$ is a surjection. Fix $\varphi \in E^\sim$. By assumption, there exists $\psi \in \left(\prod E_\alpha\right)^\sim$ so that $\varphi = \iota^\sim(\psi)$. By Theorem 2.2.6 (iv), there exists $\alpha_1, \ldots, \alpha_n \in I$ and $\psi_1 \in E_{\alpha_1}^\sim, \ldots, \psi_n \in E_{\alpha_n}^\sim$ so that $\psi = \pi_{\alpha_1}^\sim(\psi_{\alpha_1}) + \ldots + \pi_{\alpha_n}^\sim(\psi_{\alpha_n})$. Then

$$\varphi = \iota^\sim\left(\sum_{i=1}^n \pi_{\alpha_i}^\sim(\psi_i)\right) = \sum_{i=1}^n \iota^\sim(\pi_{\alpha_i}^\sim(\psi_i)) = \sum_{i=1}^n p_{\alpha_i}^\sim(\psi_i) = \sum_{i=1}^n T(e_{\alpha_i}(\psi_i)) = T\left(\sum_{i=1}^n e_{\alpha_i}(\psi_i)\right).$$

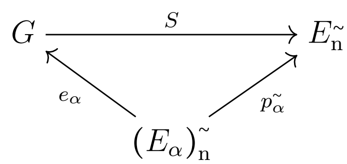
Therefore T is surjective, and hence a lattice isomorphism. □

A similar result holds for the order continuous dual of an inverse limit. We omit the proof of the next theorem, which is virtually identical to that of Theorem 2.5.14. Note, however, that unlike in Theorem 2.5.14, we make no assumption on the cardinality of I.

THEOREM 2.5.15. *Let* $\mathcal{I} \coloneqq ((E_\alpha)_{\alpha\in I}, (p_{\beta,\alpha})_{\beta\succcurlyeq\alpha})$ *be an inverse system in* **NIVL** *and* $\mathcal{S} \coloneqq (E, (p_\alpha)_{\alpha\in I})$ *its inverse limit in* **VL**. *Assume that* $p_{\beta,\alpha}$ *and* p_α *are surjections for all* $\beta \succcurlyeq \alpha$ *in* I. *Then the following statements are true.*

(i) $\varinjlim (\mathcal{I}^\sim_{\mathrm{n}}) \coloneqq (G, (e_\alpha)_{\alpha\in I})$ *exists in* **NIVL**.

(ii) *There exists an injective and interval preserving normal lattice homomorphism* $S : G \to E^\sim_{\mathrm{n}}$ *so that the diagram*

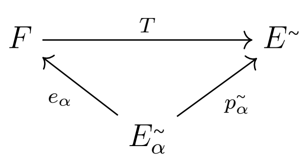


commutes for every $\alpha \in I$.

(iii) S *is a lattice isomorphism if and only if every order continuous linear functional on* E *has an order continuous linear extension to* $\prod E_\alpha$.

The following two results are consequences of Theorems 2.5.14 and 2.5.15, respectively.

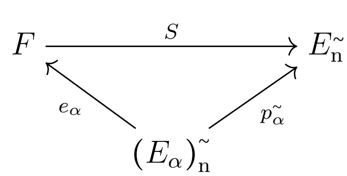
COROLLARY 2.5.16. *Let* $\mathcal{I} := ((E_\alpha)_{\alpha\in I}, (p_{\beta,\alpha})_{\alpha\preccurlyeq\beta})$ *be an inverse system in* **IVL**, $\varprojlim \mathcal{I} := (E, (p_\alpha)_{\alpha\in I})$ *its inverse limit in* **VL**, *and* $\varinjlim \mathcal{I}^\sim := (F, (e_\alpha)_{\alpha\in I})$ *the direct limit of* $\mathcal{I}^\sim$ *in* **NIVL**. *Assume that* $p_{\beta,\alpha}$ *and* p_α *are surjections for all* $\beta \succcurlyeq \alpha$ *in* I. *If* E *is majorising in* $\prod E_\alpha$ *then* $\left(\varprojlim \mathcal{I}\right)^\sim \cong \varinjlim \mathcal{I}^\sim$ *in* **NIVL**. *That is, there exists a lattice isomorphism* $T : F \to E^\sim$ *such that the diagram*



commutes for all $\alpha \in I$.

PROOF. This follows immediately from [**6**, Theorem 1.32] and Theorem 2.5.14. □

COROLLARY 2.5.17. *Let* $\mathcal{I} := ((E_\alpha)_{\alpha\in I}, (p_{\beta,\alpha})_{\alpha\preccurlyeq\beta})$ *be an inverse system in* **NIVL**, $\varprojlim \mathcal{I} := (E, (p_\alpha)_{\alpha\in I})$ *its inverse limit in* **VL**, *and* $\varinjlim \mathcal{I}^\sim_{\mathrm{n}} := (F, (e_\alpha)_{\alpha\in I})$ *the direct limit of* $\mathcal{I}^\sim_{\mathrm{n}}$ *in* **NIVL**. *Assume that* $p_{\beta,\alpha}$ *and* p_α *are surjections for all* $\beta \succcurlyeq \alpha$ *in* I. *If* E *is majorising and order dense in* $\prod E_\alpha$ *then* $\left(\varprojlim \mathcal{I}\right)^\sim_{\mathrm{n}} \cong \varinjlim \mathcal{I}^\sim_{\mathrm{n}}$ *in* **NIVL**. *That is, there exists a lattice isomorphism* $S : F \to E^\sim_{\mathrm{n}}$ *such that the diagram*



commutes for all $\alpha \in I$.

PROOF. This follows immediately from [**6**, Theorem 1.65] and Theorem 2.5.15. □

In contrast with direct limits, the inverse limit construction always preserves the property of having a separating order (continuous) dual.

PROPOSITION 2.5.18. *Let $\mathcal{I} \coloneqq ((E_\alpha)_{\alpha\in I}, (p_{\beta,\alpha})_{\beta\succcurlyeq\alpha})$ be an inverse system in* **VL** *and $\mathcal{S} \coloneqq (E, (p_\alpha)_{\alpha\in I})$ its inverse limit in* **VL**. *Then the following statements are true.*

(i) If ${}^\circ(E_\alpha)^\sim = \{0\}$ for every $\alpha \in I$ then ${}^\circ E^\sim = \{0\}$.

(ii) If ${}^\circ(E_\alpha)^\sim_{\mathrm{n}} = \{0\}$ and p_α is order continuous for every $\alpha \in I$ then ${}^\circ E^\sim_{\mathrm{n}} = \{0\}$.

PROOF. The proofs of (i) and (ii) are identical. Hence we omit the proof of (ii).

Assume that ${}^\circ(E_\alpha)^\sim = \{0\}$ for every $\alpha \in I$. Let $u \in E$ be non-zero. Then there exists $\alpha \in I$ so that $p_\alpha(u) \neq 0$. Since ${}^\circ(E_\alpha)^\sim = \{0\}$, there exists $\varphi \in (E_\alpha)^\sim$ so that $\varphi(p_\alpha(u)) \neq 0$; that is, $p_\alpha^\sim(\varphi)(u) \neq 0$. Hence ${}^\circ E^\sim = \{0\}$. □

In the last few sections of this chapter, we put the duality theory developed in Section 2.5 to use. First, we use the duality theory to easily obtain the order (continuous) duals of some function spaces. After this, we investigate the permanence of the vector lattice property of perfectness under both direct and inverse limits and also give a decomposition result for perfect vector lattices. We will also show that order dual spaces have permanence under the construction of inverse limits. Our decomposition result for perfect vector lattices will then be used in the particular case of $\mathrm{C}(X)$ spaces to solve the decomposition problem mentioned in Section 1.1.1. Lastly, we use the Kakutani representation theorem for unital AM-spaces [**56**, Theorem 2.1.3] along with the result in Example 2.3.8 and our duality theory to characterise to give a characterisation of Archimedean relatively uniformly complete vector lattices and their order duals using direct and inverse limits.

2.6. Duals of function spaces

First, we apply the duality theory to the examples in Sections 2.3.2 and 2.4.3 to obtain characterisations of the order and order continuous duals of some function spaces. All of these results follow immediately from the corresponding examples and the appropriate duality result.

THEOREM 2.6.1. *Let (X,Σ,μ) be a complete σ-finite measure space. Let $\Xi \coloneqq (X_n)$ be an increasing sequence (w.r.t. inclusion) of measurable sets with positive measure so that $X = \bigcup X_n$. Let $1 \leq p < \infty$ and $1 \leq q \leq \infty$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$. For $n \in \mathbb{N}$ let e_n and r_n be as in Examples 2.3.9 and 2.4.7, respectively.*

For every $n \in \mathbb{N}$, let $T_n : \mathrm{L}^q(X_n) \to \mathrm{L}^p(X_n)^\sim$ be the usual (isometric) lattice isomorphism,

$$T_n(u)(v) = \int_{X_n} uv\, d\mu, \qquad u \in \mathrm{L}^q(X_n),\ v \in \mathrm{L}^p(X_n).$$

There exists a unique lattice isomorphism $T : \mathrm{L}^q_{\Xi-\ell oc}(X) \to \mathrm{L}^p_{\Xi-c}(X)^\sim$ *so that the diagram*

$$\begin{array}{ccc} \mathrm{L}^q_{\Xi-\ell oc}(X) & \xrightarrow{\;T\;} & \mathrm{L}^p_{\Xi-c}(X)^\sim \\ \Big\downarrow{\scriptstyle r_n} & & \Big\downarrow{\scriptstyle e_n^\sim} \\ \mathrm{L}^q(X_n) & \xrightarrow[\;T_n\;]{} & \mathrm{L}^p(X_n)^\sim \end{array}$$

commutes for every $n \in \mathbb{N}$.

Proof. The result follows immediately from Examples 2.3.9 and 2.4.7, Theorem 2.5.4, and Proposition 2.4.2. □

Theorem 2.6.2. *Let* (X, Σ, μ) *be a complete* σ*-finite measure space. Let* $\Xi \coloneqq (X_n)$ *be an increasing sequence (w.r.t. inclusion) of measurable sets with positive measure so that* $X = \bigcup X_n$. *Let* $1 \le p \le \infty$ *and* $1 \le q \le \infty$ *satisfy* $\frac{1}{p} + \frac{1}{q} = 1$. *For* $n \in \mathbb{N}$ *let* e_n *and* r_n *be as in Examples 2.3.9 and 2.4.7, respectively.*

For every $n \in \mathbb{N}$, *let* $S_n : \mathrm{L}^q(X_n) \to \mathrm{L}^p(X_n)^\sim_{\mathrm{n}}$ *be the usual (isometric) lattice isomorphism,*

$$S_n(u)(v) = \int_{X_n} uv \, d\mu, \qquad u \in \mathrm{L}^q(X_n), \; v \in \mathrm{L}^p(X_n).$$

There exists a unique lattice isomorphism $S : \mathrm{L}^q_{\Xi-\ell oc}(X) \to \mathrm{L}^p_{\Xi-c}(X)^\sim_{\mathrm{n}}$ *so that the diagram*

$$\begin{array}{ccc} \mathrm{L}^q_{\Xi-\ell oc}(X) & \xrightarrow{\;S\;} & \mathrm{L}^p_{\Xi-c}(X)^\sim_{\mathrm{n}} \\ \Big\downarrow{\scriptstyle r_n} & & \Big\downarrow{\scriptstyle e_n^\sim} \\ \mathrm{L}^q(X_n) & \xrightarrow[\;S_n\;]{} & \mathrm{L}^p(X_n)^\sim_{\mathrm{n}} \end{array}$$

commutes for every $n \in \mathbb{N}$.

Proof. We observe that the mappings $e_{n,m}$ in Example 2.3.9 are injective for all $n \le m$ in $\mathbb{N}$. Therefore the result follows immediately from Examples 2.3.9 and 2.4.7, Theorem 2.5.5, and Proposition 2.4.2. □

Theorem 2.6.3. *Let* (X, Σ, μ) *be a complete* σ*-finite measure space. Let* $\Xi \coloneqq (X_n)$ *be an increasing sequence (w.r.t. inclusion) of measurable sets with positive measure so that* $X = \bigcup X_n$. *Let* $1 \le p < \infty$ *and* $1 \le q \le \infty$ *satisfy* $\frac{1}{p} + \frac{1}{q} = 1$. *For* $n \in \mathbb{N}$ *let* e_n *and* r_n *be as in Examples 2.3.9 and 2.4.7, respectively.*

For every $n \in \mathbb{N}$, *let* $T_n : \mathrm{L}^q(X_n) \to \mathrm{L}^p(X_n)^\sim$ *be the usual (isometric) lattice isomorphism,*

$$T_n(u)(v) = \int_{X_n} uv \, d\mu, \qquad u \in \mathrm{L}^q(X_n), \; v \in \mathrm{L}^p(X_n).$$

There exists a unique lattice isomorphism $R : \mathrm{L}^q(X)_{\Xi-c} \to \mathrm{L}^p_{\Xi-\ell oc}(X)^\sim$ *so that the diagram*

$$\begin{array}{ccc} \mathrm{L}^q(X_n) & \xrightarrow{\;T_n\;} & \mathrm{L}^p(X_n)^\sim \\ \Big\downarrow{\scriptstyle e_n} & & \Big\downarrow{\scriptstyle r_n^\sim} \\ \mathrm{L}^q_{\Xi-c}(X) & \xrightarrow[\;R\;]{} & \mathrm{L}^p_{\Xi-\ell oc}(X)^\sim \end{array}$$

commutes for every $n \in \mathbb{N}$.

Proof. We note that the mappings $p_{m,n}$ in Example 2.4.7 are surjective for all $m \geq n$ in $\mathbb{N}$. Therefore the result follows immediately from Examples 2.3.9 and 2.4.7, Theorem 2.5.12, and Proposition 2.3.2. □

Theorem 2.6.4. *Let* (X, Σ, μ) *be a complete σ-finite measure space. Let* $\Xi := (X_n)$ *be an increasing sequence (w.r.t. inclusion) of measurable sets with positive measure so that* $X = \bigcup X_n$. *Let* $1 \leq p \leq \infty$ *and* $1 \leq q \leq \infty$ *satisfy* $\frac{1}{p} + \frac{1}{q} = 1$. *For* $n \in \mathbb{N}$ *let* e_n *and* r_n *be as in Examples 2.3.9 and 2.4.7, respectively.*

For every $n \in \mathbb{N}$, *let* $S_n : \mathrm{L}^q(X_n) \to \mathrm{L}^p(X_n)^\sim_{\mathrm{n}}$ *be the usual (isometric) lattice isomorphism,*

$$S_n(u)(v) = \int_{X_n} uv \, d\mu, \quad u \in \mathrm{L}^q(X_n), \; v \in \mathrm{L}^p(X_n).$$

There exists a unique lattice isomorphism $Q : \mathrm{L}^p_{\Xi-c}(X) \to \mathrm{L}^q_{\Xi-\ell oc}(X)^\sim_{\mathrm{n}}$ *so that the diagram*

$$\begin{array}{ccc} \mathrm{L}^q(X_n) & \xrightarrow{\;S_n\;} & \mathrm{L}^p(X_n)^\sim_{\mathrm{n}} \\ \Big\downarrow{\scriptstyle e_n} & & \Big\downarrow{\scriptstyle r_n^\sim} \\ \mathrm{L}^q_{\Xi-c}(X) & \xrightarrow[\;Q\;]{} & \mathrm{L}^p_{\Xi-\ell oc}(X)^\sim_{\mathrm{n}} \end{array}$$

commutes for every $n \in \mathbb{N}$.

Proof. Because the mappings $p_{m,n}$ in Example 2.4.7 are surjective for all $m \geq n$ in $\mathbb{N}$, the result follows immediately from Examples 2.3.9 and 2.4.7, Theorem 2.5.13, and Proposition 2.3.2. □

The following result is a special case of the Riesz Representation Theorem [**25**, Chapter III, Theorem 5.7].

Theorem 2.6.5. *Let* X *be a locally compact and σ-compact Hausdorff space. Let* $\Gamma := (X_n)$ *be an increasing sequence (w.r.t. inclusion) of open precompact sets in* X *so that* $X = \bigcup X_n$. *For* $n \in \mathbb{N}$ *let* e_n *and* r_n *be as in Examples 2.3.10 and 2.4.8, respectively.*

For every $n \in \mathbb{N}$, let $T_n : \mathrm{M}(\bar{X}_n) \to \mathrm{C}(\bar{X}_n)^{\sim}$ denote the usual (isometric) lattice isomorphism,

$$T_n(\mu)(u) = \int_{X_n} u \, d\mu, \quad \mu \in \mathrm{M}(\bar{X}_n), \; u \in \mathrm{C}(\bar{X}_n).$$

There exists a unique lattice isomorphism $T : \mathrm{M}_c(X) \to \mathrm{C}(X)^{\sim}$ so that the diagram

$$\begin{CD} \mathrm{M}(\bar{X}_n) @>{T_n}>> \mathrm{C}(\bar{X}_n)^{\sim} \\ @V{e_n}VV @VV{r_n^{\sim}}V \\ \mathrm{M}_c(X) @>>{T}> \mathrm{C}(X)^{\sim} \end{CD}$$

commutes for every $n \in \mathbb{N}$.

Proof. The result follows immediately from Examples 2.3.10 and 2.4.8, Theorem 2.5.12, and Proposition 2.3.2. ☐

Theorem 2.6.6. *Let X be a locally compact and σ-compact Hausdorff space. Let $\Gamma := (X_n)$ be an increasing sequence (with respect to inclusion) of open precompact sets in X so that $X = \bigcup X_n$. For $n \in \mathbb{N}$ let e_n and r_n be as in Examples 2.3.11 and 2.4.8, respectively.*

For every $n \in \mathbb{N}$, let $S_n : \mathrm{N}(\bar{X}_n) \to \mathrm{C}(\bar{X}_n)^{\sim}_{\mathrm{n}}$ denote the (isometric) lattice isomorphism,

$$S_n(\mu)(u) = \int_{X_n} u \, d\mu, \quad \mu \in \mathrm{N}(\bar{X}_n), \; u \in \mathrm{C}(\bar{X}_n).$$

There exists a unique lattice isomorphism $S : \mathrm{N}_c(X) \to \mathrm{C}(X)^{\sim}_{\mathrm{n}}$ so that the diagram

$$\begin{CD} \mathrm{N}(\bar{X}_n) @>{S_n}>> \mathrm{C}(\bar{X}_n)^{\sim}_{\mathrm{n}} \\ @V{e_n}VV @VV{r_n^{\sim}}V \\ \mathrm{N}_c(X) @>>{S}> \mathrm{C}(X)^{\sim}_{\mathrm{n}} \end{CD}$$

commutes for every $n \in \mathbb{N}$.

Proof. The result follows immediately from Examples 2.3.11 and 2.4.8, Theorem 2.5.13, and Proposition 2.3.2. ☐

2.7. Perfect spaces

Recall that a vector lattice E is *perfect* if the canonical embedding $E \ni u \longmapsto \Psi_u \in (E^{\sim}_{\mathrm{n}})^{\sim}_{\mathrm{n}}$ is a lattice isomorphism [**70**, p. 409]. We say that a vector lattice E is *is an order continuous dual*, or has an *order continuous predual* if there exists a vector lattice F so that E and $F^{\sim}_{\mathrm{n}}$ are isomorphic vector lattices. From the definition it is clear that every perfect vector lattice has an order continuous dual. On the other hand, see [**70**, Theorem 110.3], $F^{\sim}_{\mathrm{n}}$ is perfect for any vector lattice F. Therefore, E is perfect if and only if E has an order continuous predual.

LEMMA 2.7.1. *Let E be a vector lattice and $0 \leq \varphi, \psi \in E_{\mathrm{n}}^{\sim}$. The following statements are true.*

(i) There exist functionals $0 \leq \varphi_1, \psi_1 \in E_{\mathrm{n}}^{\sim}$ so that $\varphi_1 \wedge \psi_1 = 0$, $\varphi_1 \leq \varphi$, $\psi_1 \leq \psi$ and $\varphi \vee \psi = \varphi_1 \vee \psi_1$.

(ii) If E has the principal projection property and φ is strictly positive, then for all $u \in E$, if $\eta(u) = 0$ for all functionals $0 \leq \eta \leq \varphi$ then $u = 0$.

PROOF. The statement in (i) follows from [**56**, Lemma 1.28 (ii) & Exercise 1.2.E1].

We prove the contrapositive of (ii). Let $u \neq 0$ in E. Without loss of generality assume that $u^+ \neq 0$. Denote by B the band generated by u^+ in E. Define $\eta \coloneqq \varphi \circ P_B$. Then η is order continuous, $0 \leq \eta \leq \varphi$ and $\eta(u) = \varphi(u^+) \neq 0$. ☐

Using the duality results obtained in Section 2.5, we can now show that the vector lattice property of perfectness also has permanence under the inverse limit construction.

THEOREM 2.7.2. *Let $\mathcal{I} \coloneqq ((E_\alpha)_{\alpha\in I}, (p_{\beta,\alpha})_{\beta\succcurlyeq\alpha})$ be an inverse system in* **NIVL**, *and let $\mathcal{S} \coloneqq (E, (p_\alpha)_{\alpha\in I})$ be its inverse limit in* **VL**. *Assume that $p_{\beta,\alpha}$ is surjective for all $\beta \succcurlyeq \alpha$ in I. If E_α is perfect for every $\alpha \in I$ then so is E.*

PROOF. By Proposition 2.5.9, the pair $\mathcal{I}_{\mathrm{n}}^{\sim} \coloneqq \big(((E_\alpha)_{\mathrm{n}}^{\sim})_{\alpha\in I}, (p_{\beta,\alpha}^{\sim})_{\alpha\preccurlyeq\beta}\big)$ is a direct system in **NIVL**. Because every $p_{\beta,\alpha}$ is surjective, each $p_{\beta,\alpha}^{\sim}$ is injective. Hence, by Theorem 2.3.4, the direct limit of $\mathcal{I}_{\mathrm{n}}^{\sim}$ exists in **NIVL**. Let $\mathcal{S} \coloneqq (F, (e_\alpha)_{\alpha\in I})$ be the direct limit of $\mathcal{I}_{\mathrm{n}}^{\sim}$ in **NIVL**.

By Proposition 2.5.2, the pair $\mathcal{I}_{\mathrm{nn}}^{\sim\sim} \coloneqq \big(((E_\alpha)_{\mathrm{nn}}^{\sim\sim})_{\alpha\in I}, (p_{\beta,\alpha}^{\sim\sim})_{\alpha\preccurlyeq\beta}\big)$ is an inverse system in **NIVL**, and $\mathcal{S}_{\mathrm{n}}^{\sim} \coloneqq (F_{\mathrm{n}}^{\sim}, (e_\alpha^{\sim})_{\alpha\in I})$ is the inverse limit of $\mathcal{I}_{\mathrm{nn}}^{\sim\sim}$ in **NVL** by Theorem 2.5.5. For every $\alpha \in I$, let $\sigma_\alpha : E_\alpha \to (E_\alpha)_{\mathrm{nn}}^{\sim\sim}$ denote the canonical lattice isomorphism. We observe that the diagram

$$\begin{array}{ccc} E_\beta & \xrightarrow{\sigma_\beta} & (E_\beta)_{\mathrm{nn}}^{\sim\sim} \\ {\scriptstyle p_{\beta,\alpha}}\downarrow & & \downarrow{\scriptstyle p_{\beta,\alpha}^{\sim\sim}} \\ E_\alpha & \xrightarrow[\sigma_\alpha]{} & (E_\alpha)_{\mathrm{nn}}^{\sim\sim} \end{array}$$

commutes for all $\beta \succcurlyeq \alpha$ in I. By Proposition 2.4.2, there exists a unique lattice isomorphism $\Sigma : E \to F_{\mathrm{n}}^{\sim}$ so that the diagram

$$\begin{array}{ccc} E & \xrightarrow{\Sigma} & F_{\mathrm{n}}^{\sim} \\ {\scriptstyle p_\alpha}\downarrow & & \downarrow{\scriptstyle e_\alpha^{\sim}} \\ E_\alpha & \xrightarrow[\sigma_\alpha]{} & (E_\alpha)_{\mathrm{nn}}^{\sim\sim} \end{array}$$

commutes for every $\alpha \in I$. Since $F_{\mathrm{n}}^{\sim}$ is perfect, we conclude that E is also perfect. ☐

Our next result is a decomposition theorem for perfect vector lattices (i.e. vector lattices with an order continuous dual). The result follows as an application of Example 2.4.9 and the duality results in Section 2.5.

Theorem 2.7.3. *Let E be a Dedekind complete vector lattice. Recalling the terminology and notation introduced in Example 2.4.9, denote by $\mathrm{M_n} \subseteq \mathbf{B}_E$ the collection of carriers for positive, order continuous functionals on E; that is,*

$$\mathrm{M_n} \coloneqq \{\mathrm{C}_\varphi \ : \ 0 \leq \varphi \in E_\mathrm{n}^\sim\}.$$

For $\mathrm{C}_\varphi \subseteq \mathrm{C}_\psi$ in $\mathrm{M_n}$, denote by P_φ the band projection of E onto C_φ and by $P_{\psi,\varphi}$ the band projection of C_ψ onto C_φ. Consider the inverse system $\mathcal{I}_{\mathrm{M_n}} \coloneqq \left(\mathrm{M_n}, (P_{\psi,\varphi})_{\psi\geq\varphi}\right)$ with $\varprojlim \mathcal{I}_{\mathrm{M_n}} \coloneqq \left(F, (p_\varphi)_{0\leq\varphi\in E_\mathrm{n}^\sim}\right)$ in $\mathbf{VL}$. The following statements are true.

(i) *$\mathrm{M_n}$ is an ideal in $\mathbf{B}_E$.*

(ii) *$\mathrm{M_n}$ is a non-trivial ideal in $\mathbf{B}_E$ if and only if E admits a non-zero order continuous functional.*

(iii) *$\mathrm{M_n}$ is a proper ideal in $\mathbf{B}_E$ if and only if E admits no strictly positive order continuous functional.*

(iv) *The map $P_{\mathrm{M_n}} : E \ni u \mapsto (P_\varphi(u))_{0\leq\varphi\in E_\mathrm{n}^\sim} \in F$ is injective if and only if ${}^\circ E_\mathrm{n}^\sim = \{0\}$.*

(v) *If E is perfect then $P_{\mathrm{M_n}}$ is a lattice isomorphism.*

Proof of (*i*). For $0 \leq \psi, \varphi \in E_\mathrm{n}^\sim$, we have $\mathrm{C}_\psi, \mathrm{C}_\varphi \subseteq \mathrm{C}_{\varphi\vee\psi} \in \mathrm{M_n}$ and therefore $\mathrm{M_n}$ is upwards directed.

Take $B \in \mathbf{B}_E$ and $0 \leq \varphi \in E_\mathrm{n}^\sim$ such that $B \subseteq \mathrm{C}_\varphi$. Define $\psi \coloneqq \varphi \circ P_B$. Then $\psi \geq 0$ and by the order continuity of band projections, we have $\psi \in E_\mathrm{n}^\sim$. We show that $\mathrm{N}_\psi = B^d$: For $u \in B^d$, we have $P_B(|u|) = 0$ so that $\psi(|u|) = \varphi\left(P_B(|u|)\right) = \varphi(0) = 0$. Therefore $B^d \subseteq \mathrm{N}_\psi$. For the reverse inclusion, let $v \in \mathrm{N}_\psi$. Then $\varphi\left(P_B(|v|)\right) = 0$ so that $P_B(|v|) \in \mathrm{N}_\varphi \subseteq B^d$. Hence $P_B(|v|) = 0$ so that $v \in B^d$. We conclude that $B = \mathrm{C}_\psi$. Therefore $B \in \mathrm{M_n}$ so that $\mathrm{M_n}$ is downward closed, hence an ideal in $\mathbf{B}_E$. □

Proof of (*ii*). This is clear. □

Proof of (*iii*). A functional $0 \leq \varphi \in E_\mathrm{n}^\sim$ is strictly positive if and only if $\mathrm{N}_\varphi = \{0\}$, if and only if $\mathrm{C}_\varphi = E$; hence the result follows. □

Proof of (*iv*). According to Example 2.4.9 (iii), $P_{\mathrm{M_n}}$ is injective if and only if $\{P_\varphi \ : \ 0 \leq \varphi \in E_\mathrm{n}^\sim\}$ separates the points of E. It therefore suffices to prove that ${}^\circ E_\mathrm{n}^\sim = \{0\}$ if and only if $\{P_\varphi : 0 \leq \varphi \in E_\mathrm{n}^\sim\}$ separates the points of E.

Assume that ${}^\circ E_\mathrm{n}^\sim = \{0\}$. Fix $u \in E$ with $u \neq 0$. Then there exists $\varphi \in E_\mathrm{n}^\sim$ such that $\varphi(u) \neq 0$. Therefore $0 < |\varphi(u)| \leq |\varphi|(|u|)$. Hence $u \notin \mathrm{N}_{|\varphi|}$ and thus $P_{|\varphi|}(u) \neq 0$.

Conversely, assume that $\{P_\varphi : 0 \leq \varphi \in E_\mathrm{n}^\sim\}$ separates the points of E. We first prove the statement for positive elements: Let $0 < v \in E^+$. There exists $0 \leq \varphi \in E_\mathrm{n}^\sim$ such that $P_\varphi(v) > 0$. Since every positive functional is strictly positive on its carrier, it follows that $\varphi(v) \geq \varphi(P_\varphi(v)) > 0$. Now, consider any non-zero $w \in E$. There exists $0 \leq \varphi \in E_{\mathrm{n}}^{\sim}$ such that $\varphi(w^+) \neq 0$. Let B denote the band generated by w^+ in E and define the functional $\psi := \varphi \circ P_B$. Then $0 \leq \psi \in E_{\mathrm{n}}^{\sim}$ and $\psi(w) = \varphi(w^+) \neq 0$. ☐

Proof of (v). It follows from Example 2.4.9 (ii) that $P_{\mathrm{M_n}}$ is a lattice homomorphism. Since E is perfect, we have ${}^\circ E_{\mathrm{n}}^{\sim} = \{0\}$ [**70**, Theorem 110.1] and so by (iv), $P_{\mathrm{M_n}}$ is injective. We show that $P_{\mathrm{M_n}}$ is surjective.

Let $0 \leq u = (u_\varphi) \in F$. Define the map $\Upsilon : (E_{\mathrm{n}}^{\sim})^+ \to \mathbb{R}$ by setting $\Upsilon(\varphi) := \varphi(u_\varphi)$ for $\varphi \in (E_{\mathrm{n}}^{\sim})^+$. We claim that Υ is additive. Let $0 \leq \varphi, \psi \in E_{\mathrm{n}}^{\sim}$. Then

$$\begin{aligned}\Upsilon(\varphi+\psi) &= (\varphi+\psi)(u_{\varphi+\psi}) \\ &= \varphi(u_{\varphi+\psi}) + \psi(f_{\varphi+\psi}) \\ &= \varphi \circ P_\varphi(u_{\varphi+\psi}) + \psi \circ P_\psi(u_{\varphi+\psi}).\end{aligned}$$

Because $(u_\varphi) \in F$ we have $P_\varphi(u_{\varphi+\psi}) = P_{\varphi+\psi,\varphi}(u_{\varphi+\psi}) = u_\varphi$ and $P_\psi(u_{\varphi+\psi}) = P_{\varphi+\psi,\psi}(u_{\varphi+\psi}) = u_\psi$. Hence

$$\Upsilon(\varphi+\psi) = \varphi(u_\varphi) + \psi(u_\psi) = \Upsilon(\varphi) + \Upsilon(\psi).$$

By [**3**, Theorem 1.10] Υ extends to a positive linear functional on $E_{\mathrm{n}}^{\sim}$, which we denote by Υ as well.

We claim that Υ is order continuous. To see this, consider any $D \downarrow 0$ in $E_{\mathrm{n}}^{\sim}$. Fix $\epsilon > 0$ and $\varphi \in D$. By [**6**, Theorem 1.18] there exists $\psi_0 \leq \varphi$ in D so that $0 \leq \psi(u_\varphi) < \epsilon$ for all $\psi \leq \psi_0$ in D. Consider $\psi \leq \psi_0$. Since $u \in F$ we have $u_\psi = P_{\varphi,\psi}(u_\varphi) \leq u_\varphi$ so that $0 \leq \psi(u_\psi) \leq \psi(u_\varphi) < \epsilon$. That is, $0 \leq \Upsilon(\psi) < \epsilon$ for all $\psi \leq \psi_0$. Therefore $\Upsilon[D] \downarrow 0$ in $\mathbb{R}$ so that Υ is order continuous, as claimed.

Since E is perfect, there exists $v \in E^+$ so that $\Upsilon(\varphi) = \varphi(v)$ for all $\varphi \in E_{\mathrm{n}}^{\sim}$. We claim that $P_{\mathrm{M_n}}(v) = u$; that is, $P_\varphi(v) = u_\varphi$ for every $0 \leq \varphi \in E_{\mathrm{n}}^{\sim}$. For every $0 \leq \varphi \in E_{\mathrm{n}}^{\sim}$ we have $\varphi(u_\varphi) = \Upsilon(\varphi) = \varphi(v) = \varphi(P_\varphi(v))$. Let $0 \leq \eta \leq \varphi$ in $E_{\mathrm{n}}^{\sim}$. Then

$$\eta(u_\varphi) = \eta(P_\eta u_\varphi) = \eta(u_\eta) = \Upsilon(\eta) = \eta(v),$$

and,

$$\eta(P_\varphi(v)) = \eta(P_\eta P_\varphi(v)) = \eta(P_\eta(v)) = \eta(v).$$

Thus $\eta(u_\varphi - P_\varphi(v)) = 0$. By Lemma 2.7.1 (ii), applied on C_φ, we conclude that $P_\varphi(v) = u_\varphi$. This verifies our claim. Therefore $P_{\mathrm{M_n}}$ maps E^+ onto F^+ which shows that $P_{\mathrm{M_n}}$ is surjective. ☐

Remark 2.7.4. We observe that the converse of Theorem 2.7.3 (v) is false. Indeed, $(\mathrm{c}_0)_{\mathrm{nn}}^{\sim\sim} = \ell^\infty$ so that c_0 is not perfect. However, there exists a strictly positive functional $\varphi \in (\mathrm{c}_0)_{\mathrm{n}}^{\sim} = \ell^1$. Therefore $\mathrm{c}_0 = \mathrm{C}_\varphi \in \mathrm{M_n}$ so that $P_{\mathrm{M_n}}$ maps c_0 lattice isomorphically onto F, see Remark 2.4.10.

Corollary 2.7.5. *Let E be a Dedekind complete vector lattice. Let $\mathrm{M}_p \subseteq \mathbf{B}_E$ consist of the carriers of all positive, order continuous functionals on E which are perfect; that is,*

$$\mathrm{M}_p := \{\mathrm{C}_\varphi \ : \ 0 \leq \varphi \in E_{\mathrm{n}}^{\sim} \text{ and } \mathrm{C}_\varphi \text{ is perfect}\}.$$

The following statements are true.

(i) *M_p is an ideal in $\mathbf{B}_E$.*

(ii) P_{M_p} *is a lattice isomorphism if and only if* E *is perfect.*

Proof of (i). It follows from Theorem 2.7.3 (i) and the fact that bands in a perfect vector lattices are themselves perfect [**70**, Theorem 110.3] that M_p is downwards closed in $\mathbf{B}_E$. To see that M_p is upwards directed, fix $C_\varphi, C_\psi \in \mathrm{M}_p$. By Lemma 2.7.1 (i) there exist functionals $0 \leq \varphi_1 \leq \varphi$ and $0 \leq \psi_1 \leq \psi$ in $E^\sim_{\mathrm{n}}$ such that $\varphi_1 \wedge \psi_1 = 0$ and $\varphi_1 \vee \psi_1 = \varphi \vee \psi$. Because $0 \leq \varphi_1 \leq \varphi$ and $0 \leq \psi_1 \leq \psi$ it follows that $\mathrm{C}_{\varphi_1} \subseteq \mathrm{C}_\varphi$ and $\mathrm{C}_{\psi_1} \subseteq \mathrm{C}_\psi$. Therefore C_{φ_1} and C_{ψ_1} are perfect. By [**70**, Theorem 90.7], we have

$$\mathrm{C}_{\varphi_1 \vee \psi_1} = (\mathrm{C}_{\varphi_1} + \mathrm{C}_{\psi_1})^{dd} = \mathrm{C}_{\varphi_1} + \mathrm{C}_{\psi_1}.$$

By [**70**, Theorem 90.6], since $\varphi_1 \wedge \psi_1 = 0$, we have $\mathrm{C}_{\varphi_1} \perp \mathrm{C}_{\psi_1}$. Thus $\mathrm{C}_{\varphi_1} \cap \mathrm{C}_{\psi_1} = \{0\}$ which implies $\mathrm{C}_{\varphi_1 \vee \psi_1} = \mathrm{C}_{\varphi_1} \oplus \mathrm{C}_{\psi_1}$. Hence it follows from Theorem 2.2.6 (v) and (vii) that $(\mathrm{C}_{\varphi_1 \vee \psi_1})^{\sim\sim}_{\mathrm{nn}} \cong \mathrm{C}_{\varphi_1 \vee \psi_1}$. That is, $\mathrm{C}_{\varphi \vee \psi} = \mathrm{C}_{\varphi_1 \vee \psi_1}$ is perfect. Since $\mathrm{C}_\varphi, \mathrm{C}_\psi \subseteq \mathrm{C}_{\varphi \vee \psi}$ it follows that M_p is upward directed, hence an ideal in $\mathbf{B}_E$. ☐

Proof of (ii). If E is perfect then $\mathrm{M}_p = \mathrm{M}_{\mathrm{n}}$, and so the result follows from Theorem 2.7.3 (v). Conversely, if P_{M_p} is an isomorphism then Theorem 2.7.2 implies that E is perfect. ☐

Given the duality results obtained in Section 2.5, one would expect there to be at least some analogue of Theorem 2.7.2 for direct limits. Due to the inherent limitations of duality results for inverse limits observed in Section 2.5.2, these results will be less general than the preceding results in this section.

Theorem 2.7.6. *Let* $\mathcal{D} \coloneqq ((E_n)_{n\in\mathbb{N}}, (e_{n,m})_{n\leq m})$ *be a direct system in* **NIVL**, *and let* $\mathcal{S} \coloneqq (E, (e_n)_{n\in\mathbb{N}})$ *be the direct limit of* $\mathcal{D}$ *in* **IVL**. *Assume that* $e^\sim_{n,m}$ *is surjective for all* $n \leq m$ *in* $\mathbb{N}$. *If* E_n *is perfect for every* $n \in \mathbb{N}$ *then so is* E.

Proof. By Proposition 2.5.2, the pair $\mathcal{D}^\sim_{\mathrm{n}} \coloneqq \left(((E_n)^\sim_{\mathrm{n}})_{n\in\mathbb{N}}, (e^\sim_{n,m})_{n\leq m}\right)$ is an inverse system in **NIVL**, and by Theorem 2.4.4, the inverse limit $\mathcal{S}_0 \coloneqq (F, (p_n)_{n\in\mathbb{N}})$ of $\mathcal{D}^\sim_{\mathrm{n}}$ exists in **NVL**.

By Proposition 2.5.9, the pair $\mathcal{D}^{\sim\sim}_{\mathrm{nn}} \coloneqq \left(((E_n)^{\sim\sim}_{\mathrm{nn}})_{n\in\mathbb{N}}, (e^{\sim\sim}_{n,m})_{n\leq m}\right)$ is a direct system in **NIVL**. Since we assumed that the $e^\sim_{n,m}$ are surjective, it follows by Theorem 2.5.13 that $(\mathcal{S}_0)^\sim_{\mathrm{n}}$ is the direct limit of $\mathcal{D}^{\sim\sim}_{\mathrm{nn}}$ in **NIVL**. For every $n \in \mathbb{N}$, let $\sigma_n : E_n \to (E_n)^{\sim\sim}_{\mathrm{nn}}$ denote the canonical lattice isomorphism. The diagram

$$\begin{array}{ccc} E_n & \xrightarrow{\ \sigma_n\ } & (E_n)^{\sim\sim}_{\mathrm{nn}} \\ {\scriptstyle e_{n,m}}\big\downarrow & & \big\downarrow{\scriptstyle e^{\sim\sim}_{n,m}} \\ E_m & \xrightarrow[\ \sigma_m\ ]{} & (E_m)^{\sim\sim}_{\mathrm{nn}} \end{array}$$

commutes in **VL** for all $n \leq m$ in $\mathbb{N}$. By Proposition 2.3.2, there exists a unique lattice isomorphism $\Sigma : E \to F^{\sim}_{\mathrm{n}}$ so that the diagram

$$\begin{array}{ccc} E_n & \xrightarrow{\sigma_n} & (E_n)^{\sim\sim}_{\mathrm{nn}} \\ \Big\downarrow{\scriptstyle e_n} & & \Big\downarrow{\scriptstyle e^{\sim\sim}_{n,m}} \\ E & \xrightarrow[\Sigma]{} & F^{\sim}_{\mathrm{n}} \end{array}$$

commutes in **VL** for every $n \in \mathbb{N}$. Since $F^{\sim}_{\mathrm{n}}$ is perfect, we conclude that E is also perfect. □

COROLLARY 2.7.7. *Let $\mathcal{D} := ((E_n)_{n\in\mathbb{N}}, (e_{n,m})_{n\leq m})$ be a direct system in* **NIVL**, *and let $\mathcal{S} := (E, (e_n)_{n\in\mathbb{N}})$ be the direct limit of $\mathcal{D}$ in* **IVL**. *Assume that $e_{n,m}$ is injective and $e_{n,m}[E_n]$ is a band in E_m for all $n \leq m$ in $\mathbb{N}$. If E_n is perfect for every $n \in \mathbb{N}$ then so is E.*

PROOF. The result will follow directly from Theorem 2.7.6 if we can show that $e^{\sim}_{n,m}$ is surjective for all $n \leq m$ in $\mathbb{N}$. We observe that each E_n is Dedekind complete and thus has the projection property. Fix $n \leq m$ in $\mathbb{N}$. Let $P_{m,n} : E_m \to e_{n,m}[E_n]$ be the band projection onto $e_{n,m}[E_n]$. The diagram

$$\begin{array}{ccccc} E_n & & \xrightarrow{\quad e_{n,m} \quad} & & E_m \\ & {\scriptstyle e_{n,m}} \searrow & & \swarrow {\scriptstyle P_{m,n}} & \\ & & e_{n,m}[E_n] & & \end{array}$$

commutes in **NIVL**. Therefore

$$\begin{array}{ccccc} (E_m)^{\sim}_{\mathrm{n}} & & \xrightarrow{\quad e^{\sim}_{n,m} \quad} & & (E_n)^{\sim}_{\mathrm{n}} \\ & {\scriptstyle P^{\sim}_{m,n}} \nwarrow & & \nearrow {\scriptstyle e^{\sim}_{n,m}} & \\ & & (e_n[E_n])^{\sim}_{\mathrm{n}} & & \end{array}$$

commutes as well in **NIVL**. Since $e_{m,n} : E_n \to e_{n,m}[E_n]$ is a lattice isomorphism, the adjoint $e^{\sim}_{n,m} : (e_{n,m}[E_n])^{\sim}_{\mathrm{n}} \to (E_n)^{\sim}_{\mathrm{n}}$ is also a lattice isomorphism. Therefore it follows from the last diagram that $e^{\sim}_{n,m} : (E_m)^{\sim}_{\mathrm{n}} \to (E_n)^{\sim}_{\mathrm{n}}$ is indeed a surjection. □

COROLLARY 2.7.8. *Let E be an Archimedean vector lattice. Assume that there exists an increasing sequence (φ_n) of positive order continuous functionals on E such that (φ_n) separates E and, for every $n \in \mathbb{N}$, C_{φ_n} is perfect. Then E is perfect.*

PROOF. Since (φ_n) is increasing, we have that $\mathrm{C}_{\varphi_n} \subseteq \mathrm{C}_{\varphi_{n+1}}$ for every $n \in \mathbb{N}$. Further, since (φ_n) separates E we also have that $\bigcup \mathrm{C}_{\varphi_n} = E$.

For all $n \leq m$ denote by $e_{n,m} : \mathrm{C}_{\varphi_n} \to \mathrm{C}_{\varphi_m}$ and $e_n : \mathrm{C}_{\varphi_n} \to E$ the inclusion maps. By Example 2.3.7, the pair $\mathcal{D} := ((\mathrm{C}_{\varphi_n})_{n\in\mathbb{N}}, (e_{n,m})_{n\leq m})$ is a direct system in **NIVL**, and $\mathcal{S} := (E, (e_n)_{n\in\mathbb{N}})$ is the direct limit of $\mathcal{D}$ in **NIVL**. By Corollary 2.7.7, E is perfect. □

2.8. Order dual spaces

In this section, we study the permanence of order dual spaces under the inverse limit construction. As mentioned in Section 1.1.1, a reasonable definition for a vector lattice E to be an order dual space is to simply require that there exists a vector lattice F such that E and $F^\sim$ are lattice isomorphic. Hence, by [**70**, Theorem 110.2], all order dual spaces are perfect vector lattices.

However, this definition appears to be problematic for our purposes for the following reason: For perfect vector lattices it was clear how we can approach questions of permanence under direct and inverse limits since a vector lattice is perfect when the canonical embedding $\sigma_E : E \to (E^\sim_{\mathrm{n}})^\sim_{\mathrm{n}}$ is a lattice isomorphism. It is then easily seen that canonical embedding σ_E interacts nicely with linking maps of direct and inverse systems in the square diagrams in Propositions 2.3.2 and 2.4.2. However, our 'reasonable' definition of an order dual space only gives us *some* lattice isomorphism $T : E \to F^\sim$ and it is not immediately obvious that this isomorphism T can also be used in the same way with Propositions 2.3.2 and 2.4.2. Fortunately, it can be shown that this reasonable definition is equivalent to a more specific statement for which it is clear that a similar approach as that for perfect vector lattices via Propositions 2.3.2 and 2.4.2 can be used.

First, we make the following simple but very important observation.

PROPOSITION 2.8.1. *Let F be a vector lattice. There exists a vector lattice G with separating order dual $G^\sim$ such that $F^\sim$ and $G^\sim$ are lattice isomorphic.*

PROOF. Note that since $F^\sim$ is an order ideal in itself, the pre-annihilator ${}^\circ(F^\sim)$ is an order ideal in F, see [**71**, Theorem 30.2]. Thus the quotient $G \coloneqq F/{}^\circ(F^\sim)$ is a vector lattice and the quotient map $Q : F \to G$ is a surjective lattice homomorphism. Since $Q[F] = G$ is an order ideal in G, it follow by Proposition 2.2.1 (ii) that Q is interval preserving. Therefore, by Theorem 2.2.3 (iii) and (iv), the order adjoint $Q^\sim : G^\sim \to F^\sim$ is an injective interval preserving lattice homomorphism. Since Q is a surjective lattice homomorphism, it follows by Proposition 2.2.4 (i) that $Q^\sim[G^\sim] = (\ker Q)^\circ = ({}^\circ(F^\sim))^\circ = F^\sim$. Thus $Q^\sim$ is a lattice isomorphism.

The definition of the vector lattice G guarantees that $G^\sim$ separates G: Take $u \in G$ with $v \in F$ such that $Q(v) = u$ and assume that $\psi(u) = 0$ for all $\psi \in G^\sim$. Then for all $\psi \in G^\sim$,

$$Q^\sim(\psi)(v) = \psi(Q(v)) = \psi(u) = 0.$$

Since $Q^\sim$ is a lattice isomorphism it follows that $\varphi(v) = 0$ for all $\varphi \in F^\sim$. Hence, $v \in {}^\circ(F^\sim) = \ker Q$ which implies $u = 0$. □

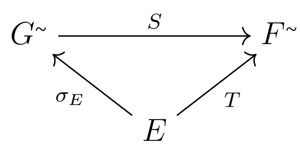
PROPOSITION 2.8.2 ([**4**, Theorem 3.11]). *Let E be a vector lattice with B an order ideal in $E^\sim$. The canonical map $\sigma_E : E \to B^\sim_{\mathrm{n}}$ is a vector lattice embedding if and only if B separates the points of E.*

PROPOSITION 2.8.3. *Let E be a vector lattice. The following statements are equivalent.*

(i) There exists a vector lattice F such that E and $F^\sim$ are lattice isomorphic.

(ii) There exists a vector sublattice G of $E_{\mathrm{n}}^{\sim}$ such that $G^{\sim}$ separates G and the canonical embedding $\sigma_E : E \to G^{\sim}$ is a lattice isomorphism.

PROOF. It is clear that (ii) is a special case of (i). We prove that (i) implies (ii): By Proposition 2.8.1, we may assume without loss of generality that $F^{\sim}$ separates F. Thus the canonical map $\sigma_F : F \to (F^{\sim})_{\mathrm{n}}^{\sim}$ is a vector lattice embedding by Proposition 2.8.2. Let $T : E \to F^{\sim}$ be a lattice isomorphism. Then $T^{\sim} : (F^{\sim})_{\mathrm{n}}^{\sim} \to E_{\mathrm{n}}^{\sim}$. Denote $G := T^{\sim} \circ \sigma_F[F]$ which is a vector sublattice of $E_{\mathrm{n}}^{\sim}$. Then $T^{\sim} \circ \sigma_F : F \to G$ and $S : G^{\sim} \to F^{\sim}$ where $S := (T^{\sim} \circ \sigma_F)^{\sim}$ are lattice isomorphisms. Since $F^{\sim}$ separates F, it follows that $G^{\sim}$ separates G. Define the canonical map $\sigma_E : E \to G^{\sim}$ where $(\sigma_E(f))(\varphi) := \varphi(f)$ for $f \in E$ and $\varphi \in G \subseteq E_{\mathrm{n}}^{\sim}$. We claim that the following diagram commutes.



Take $x \in E$ and $u \in F$, then

$$\begin{aligned}(S \circ \sigma_E(x))(u) &= ((T^{\sim} \circ \sigma_F)^{\sim} \circ \sigma_E(x))(u) = (\sigma_E(x))(T^{\sim} \circ \sigma_F(u)) \\ &= (T^{\sim} \circ \sigma_F(u))(x) = (\sigma_F(u))(Tx) = (Tx)(u).\end{aligned}$$

Therefore $S \circ \sigma_E = T$, which implies that $\sigma_E = S^{-1} \circ T$ is a lattice isomorphism. $\square$

In light of this result, we take the statement in (ii) above as the definition of an order dual space.

DEFINITION 2.8.4. A vector lattice E is an *order dual space* if there exists a vector sublattice F of $E_{\mathrm{n}}^{\sim}$ such that $F^{\sim}$ separates F and the canonical map $\sigma_E : E \to F^{\sim}$ is a lattice isomorphism. The sublattice F of $E_{\mathrm{n}}^{\sim}$ is called an *order predual* of E.

In the particular case where $E_{\mathrm{n}}^{\sim}$ is an order predual of E, we call E an *immaculate vector lattice.*

A similar result to that of Proposition 2.8.3 can be proven in the particular case of immaculate vector lattices.

PROPOSITION 2.8.5. *Let E be a vector lattice. The following statements are equivalent.*

(i) There exists a lattice isomorphism $T : E \to (E_{\mathrm{n}}^{\sim})^{\sim}$.

(ii) E is an immaculate vector lattice.

PROOF. As before, the statement in (ii) is just a special case of (i). For (i) implies (ii), we note that since E is a perfect vector lattice, the canonical map $\sigma_E : E \to (E_{\mathrm{n}}^{\sim})^{\sim}$ is a vector lattice embedding with $\sigma_E[E] = (E_{\mathrm{n}}^{\sim})_{\mathrm{n}}^{\sim}$. However, since T is a lattice isomorphism and E is perfect it follows that there is a lattice isomorphism $S : (E_{\mathrm{n}}^{\sim})_{\mathrm{n}}^{\sim} \to (E_{\mathrm{n}}^{\sim})^{\sim}$. Since $(E_{\mathrm{n}}^{\sim})_{\mathrm{n}}^{\sim}$ is a band in $(E_{\mathrm{n}}^{\sim})^{\sim}$, we conclude that $(E_{\mathrm{n}}^{\sim})_{\mathrm{n}}^{\sim} = (E_{\mathrm{n}}^{\sim})^{\sim}$ and thus $\sigma_E : E \to (E_{\mathrm{n}}^{\sim})^{\sim}$ is a lattice isomorphism. Lastly, since $E_{\mathrm{n}}^{\sim}$ is a perfect vector lattice it follows by [**70**, Theorem 110.1] that $(E_{\mathrm{n}}^{\sim})^{\sim}$ separates $E_{\mathrm{n}}^{\sim}$ and thus (ii) follows. $\square$

The specific formulations of Definition 2.8.4 and Proposition 2.8.1 allows us to show that the order dual space property has, under fairly general conditions, permanence under the inverse limit construction.

THEOREM 2.8.6. *Let $\mathcal{I} := ((E_\alpha)_{\alpha\in I}, (p_{\beta,\alpha})_{\beta\geqslant\alpha})$ be an inverse system in* **NIVL***, and let $\mathcal{S} := (E, (p_\alpha)_{\alpha\in I})$ be the inverse limit of $\mathcal{I}$ in* **VL***. Assume that E_α is an order dual space for every $\alpha \in I$ with F_α the order predual of E_α. Further, assume that $p^\sim_{\beta,\alpha}[F_\alpha]$ is an order ideal in F_β for every $\alpha, \beta \in I$ with $\beta \geqslant \alpha$. Then E is an order dual space.*

PROOF. By Proposition 2.5.9, the pair $\mathcal{I}^\sim_{\mathrm{n}} := \left(((E_\alpha)^\sim_{\mathrm{n}})_{\alpha\in I}, (p^\sim_{\beta,\alpha})_{\alpha\leqslant\beta}\right)$ is a direct system in **NIVL**. For every $\alpha, \beta \in I$ with $\beta \geqslant \alpha$, define $e_{\alpha,\beta} : F_\alpha \to F_\beta$ where $e_{\alpha,\beta} := p^\sim_{\beta,\alpha}\big|_{F_\alpha}$ and denote $\mathcal{R} := ((F_\alpha)_{\alpha\in I}, (e_{\alpha,\beta})_{\alpha\leqslant\beta})$. Since the linking maps in $\mathcal{I}^\sim_{\mathrm{n}}$ are lattice homomorphisms it follows that the $e_{\alpha,\beta}$ maps are lattice homomorphisms and it follows precisely by the assumption that $e_{\alpha,\beta}[F_\alpha]$ is an order ideal in F_β that the $e_{\alpha,\beta}$ maps are also interval preserving (Proposition 2.2.1 (ii)). Thus $\mathcal{R}$ is a direct system in **IVL** and by Theorem 2.3.3 the direct limit $(F, (e_\alpha)_{\alpha\in I})$ of $\mathcal{R}$ exists in **IVL**.

By Proposition 2.5.2, the pair $\mathcal{R}^\sim := \left((F^\sim_\alpha)_{\alpha\in I}, (e^\sim_{\alpha,\beta})_{\alpha\leqslant\beta}\right)$ is an inverse system in **NIVL** and by Theorem 2.5.4, $(F^\sim, (e^\sim_\alpha)_{\alpha\in I})$ is the inverse limit of $\mathcal{R}^\sim$ in **NVL**. We observe that the diagram

$$\begin{array}{ccc} E_\beta & \xrightarrow{\ \sigma_\beta\ } & F^\sim_\beta \\ {\scriptstyle p_{\beta,\alpha}}\big\downarrow & & \big\downarrow{\scriptstyle e^\sim_{\alpha,\beta}} \\ E_\alpha & \xrightarrow[\ \sigma_\alpha\ ]{} & F^\sim_\alpha \end{array}$$

commutes in **VL** for all $\beta \geqslant \alpha$ in I. By Proposition 2.4.2, there exists a unique lattice isomorphism $T : E \to F^\sim$ so that the diagram

$$\begin{array}{ccc} E & \xrightarrow{\ T\ } & F^\sim \\ {\scriptstyle p_\alpha}\big\downarrow & & \big\downarrow{\scriptstyle e^\sim_\alpha} \\ E_\alpha & \xrightarrow[\ \sigma_\alpha\ ]{} & F^\sim_\alpha \end{array}$$

commutes for every $\alpha \in I$. By Proposition 2.8.3, it follows that E is an order dual space. □

We note that the assumptions in the previous theorem cannot be weakened in an obvious way and still deliver the same outcome: We need to assume that $p^\sim_{\beta,\alpha}[F_\alpha] \subseteq F_\beta$ in order to form the direct system of preduals $\mathcal{R}$. Further, the assumption that $p^\sim_{\beta,\alpha}[F_\alpha]$ is an order ideal in F_β is used to guarantee that $\mathcal{R}$ is a direct system in **IVL**. This step cannot be dispensed with since the duality result in Proposition 2.5.2 requires the linking maps in $\mathcal{R}$ to be interval preserving in order to ensure that $\mathcal{R}^\sim$

is at least an inverse system in **VL**. For similar reasons, the duality result in Proposition 2.5.9 requires $\mathcal{I}$ to be an inverse system in **NIVL**.

The analogous result to Theorem 2.8.6 for immaculate vector lattices follows as a corollary of Theorem 2.8.6 since the linking maps $p^{\sim}_{\beta,\alpha} : (E_\alpha)^{\sim}_{\mathrm{n}} \to (E_\beta)^{\sim}_{\mathrm{n}}$ are **NIVL**-morphisms and so by [**49**, Proposition 14.7], $p^{\sim}_{\beta,\alpha}\left[(E_\alpha)^{\sim}_{\mathrm{n}}\right]$ is an order ideal in $(E_\beta)^{\sim}_{\mathrm{n}}$.

COROLLARY 2.8.7. *Let $\mathcal{I} := ((E_\alpha)_{\alpha\in I}, (p_{\beta,\alpha})_{\beta\succcurlyeq\alpha})$ be an inverse system in* **NIVL**, *and let $\mathcal{S} := (E, (p_\alpha)_{\alpha\in I})$ be the inverse limit of $\mathcal{I}$ in* **VL**. *Assume that E_α is an immaculate vector lattice for every $\alpha \in I$. Then E is an immaculate vector lattice.*

We show that more can be said in Theorem 2.8.6 if we consider Banach lattices where the underlying vector lattice is an order dual space.

THEOREM 2.8.8. *Let E be a Banach lattice that is an order dual space. Then there exists a Banach lattice G such that E and $G^\star$ are isomorphic*[2] *as Banach lattices.*

PROOF. Let F be a vector lattice such that $F^\sim$ and E are lattice isomorphic. By Proposition 2.8.1, we may assume without loss of generality that $F^\sim$ separates F. By Proposition 2.8.2, the canonical map $\sigma_F : F \to (F^\sim)^\sim = (F^\sim)^\star$ is a vector lattice embedding. Define $F_0 := \sigma_F[F]$, then F_0 is a sublattice of $(F^\sim)^\star$ and thus F_0 is a normed vector lattice. We claim that $F_0^\star = F_0^\sim$: In general, we know that $F_0^\star$ is an order ideal in $F_0^\sim$.

For the reverse inclusion, take $\psi \in F_0^\sim$. Consider a sequence $(\sigma_F(u_n))$ in $F_0 \subseteq (F^\sim)^\star$ such that $(\sigma_F(u_n))$ converges to zero in the norm topology. Since the weak-$\star$ topology on $(F^\sim)^\star$ is weaker than the norm topology on $(F^\sim)^\star$, it follows that

$$(\sigma_F(u_n))(\varphi) = \varphi(u_n) \longrightarrow 0$$

in $\mathbb{R}$ for every $\varphi \in F^\sim$. In particular, consider $\varphi_0 := \psi \circ \sigma_F \in F^\sim$, then

$$\psi(\sigma_F(u_n)) = \varphi_0(u_n) = (\sigma_F(u_n))(\varphi_0) \longrightarrow 0$$

in $\mathbb{R}$. Hence $\psi \in F_0^\star$, which implies that $F_0^\star = F_0^\sim$.

Denote by G the completion of F_0; then G is a Banach lattice and we know that $F_0^\star$ and $G^\star$ are isomorphic as Banach lattices. Since E and $F_0^\sim$ are lattice isomorphic it follows that there is a lattice isomorphism $T : E \to G^\star$. Since positive operators between Banach lattices are automatically norm bounded [**71**, Theorem 18.4], we conclude that T is norm bounded and T^{-1} is also automatically norm bounded by the Bounded Inverse Theorem [**57**, Theorem 14.5.1]. □

We call a Banach lattice E an *isomorphic dual Banach lattice* if there exists a Banach lattice G such that E and $G^\star$ are isomorphic as Banach lattices. The next result is a direct consequence of Theorems 2.8.6 and 2.8.8.

COROLLARY 2.8.9. *Let E be a Banach lattice and $\mathcal{I} := ((E_\alpha)_{\alpha\in I}, (p_{\beta,\alpha})_{\beta\succcurlyeq\alpha})$ an inverse system in* **NIVL** *such that $(E, (p_\alpha)_{\alpha\in I})$ is the inverse limit of $\mathcal{I}$ in* **VL**. *That is to say, the underlying vector lattice of the Banach lattice E is obtained as the inverse*

[2]A bi-continuous lattice isomorphism between Banach lattices.

limit of $\mathcal{I}$ in **VL**. *If E_α is an order dual space for every $\alpha \in I$, then E is an isomorphic dual Banach lattice.*

To end this section, we make some comments on the properties of being an immaculate vector lattice, an order dual space, and a perfect vector lattice. It is clear that the following implications hold in general.

$$\text{Immaculate} \implies \text{Order dual space} \implies \text{Perfect}$$

However, there are examples which demonstrate that the reverse implications do not hold in general. Therefore these three notions are indeed distinct.

EXAMPLE 2.8.10 (An order dual space which is not immaculate). Since $\ell^1 \cong (c_0)^\sim$, it follows that ℓ^1 is an order dual space. By the Riesz Representation Theorem [**61**, Theorem 18.4.1],

$$\left(\ell^1{}^{\sim}_{\mathrm{n}}\right)^\sim \cong (\ell^\infty)^\sim \cong \mathrm{M}(\beta\mathbb{N}).$$

Since ℓ^1 is an atomic Banach lattice [**56**, p. 113] but $\mathrm{M}(\beta\mathbb{N})$ is not, we conclude that ℓ^1 is not an immaculate vector lattice.

EXAMPLE 2.8.11 (A perfect vector lattice that is not an order dual space). The Banach lattice $\mathrm{L}^1[0,1]$ is perfect [**3**, Theorem 9.22, Theorem 9.34]. Suppose for a contradiction that $\mathrm{L}^1[0,1]$ is an order dual space. By Theorem 2.8.8, there exists a Banach lattice G such that $\mathrm{L}^1[0,1]$ and $G^\star$ are isomorphic as Banach lattices. Since $\mathrm{L}^1[0,1]$ is separable, so is $G^\star$. Every separable dual Banach space has the Radon-Nikodým property [**30**, Appendix D3]. Therefore $G^\star$, hence also $\mathrm{L}^1[0,1]$, has the Radon-Nikodým property. But $\mathrm{L}^1[0,1]$ does not possess the Radon-Nikodým property [**58**, Example 5.15]. Therefore $\mathrm{L}^1[0,1]$ is not an order dual space.

2.9. Decomposition theorem for $\mathrm{C}(X)$

This section deals with decomposition theorems for spaces $\mathrm{C}(X)$ which are order dual spaces: We will show that a naive generalisation of the decomposition theorem for $\mathrm{C}(K)$ in Theorems 1.1.3 to the non-compact case fails and present an alternative approach using the theory of inverse limits of vector lattices we have developed. In order to facilitate the discussion to follow, we recall some basic facts concerning the structure of the carriers of positive functionals on $\mathrm{C}(X)$. Throughout this section X will denote a realcompact space.

Let $0 \leq \varphi \in \mathrm{C}(X)^\sim$. According to Theorem A.1.7 there exists a measure $\mu_\varphi \in \mathrm{M}_c(X)^+$ so that

$$\varphi(u) = \int u \, d\mu_\varphi, \quad u \in \mathrm{C}(X).$$

Denote by S_φ the support of the measure μ_φ. The null ideal of φ is given by

$$\mathrm{N}_\varphi = \{u \in \mathrm{C}(X) \ : \ u(x) = 0 \text{ for all } x \in S_\varphi\}.$$

Indeed, the inclusion $\{u \in \mathrm{C}(X) \ : \ u(x) = 0 \text{ for all } x \in S_\varphi\} \subseteq \mathrm{N}_\varphi$ is clear. For the reverse inclusion, consider $u \in \mathrm{C}(X)$ so that $u(x_0) \neq 0$ for some $x_0 \in S_\varphi$. Then there exists a neighbourhood V of x_0 and $\epsilon > 0$ so that $|u|(x) > \epsilon$ for all $x \in V$. Because $x_0 \in S_\varphi$, we have $\mu_\varphi(V) > 0$. Therefore

$$\varphi(|u|) \geq \int_V |u| \, d\mu_\varphi \geq \epsilon\mu_\varphi(V) > 0$$

so that $u \notin \mathrm{N}_\varphi$. It therefore follows that

$$\mathrm{C}_\varphi = \{u \in \mathrm{C}(X) \ : \ u(x) = 0 \text{ for all } x \in X \setminus S_\varphi\}.$$

The band C_φ is a projection band if and only if S_φ is open, hence compact and open, see [**48**, Theorem 6.3]. In this case we identify C_φ with $\mathrm{C}(S_\varphi)$ and the band projection $P_\varphi : \mathrm{C}(X) \to \mathrm{C}_\varphi$ is given by restriction of $u \in \mathrm{C}(X)$ to S_φ.

PROPOSITION 2.9.1. *Let X be extremally disconnected. Then C_φ is perfect for every $0 \neq \varphi \in \mathrm{C}(X)^\sim_\mathrm{n}$.*

PROOF. Fix $0 \neq \varphi \in \mathrm{C}(X)^\sim_\mathrm{n}$. By Theorem 2.1.7, since X is extremally disconnected, then $\mathrm{C}(X)$, and hence also C_φ, is Dedekind complete. Furthermore, $|\varphi|$ is strictly positive and order continuous on C_φ. It follows by Corollary 2.1.13 that S_φ is hyper-Stonean and thus by Theorem 1.1.2, we know that $\mathrm{C}_\varphi = \mathrm{C}(S_\varphi)$ is an order dual space, hence perfect. □

The work of Xiong in [**69**] characterises those $\mathrm{C}(X)$ spaces which are order dual spaces. A slight rearrangement of the material in [**69**] allows us to expand this equivalence.

THEOREM 2.9.2. *Let X be a realcompact space. Denote by S the union of the supports of all compactly supported normal measures on X. The following statements are equivalent.*

- *(i)* $\mathrm{C}(X)$ *is an immaculate vector lattice.*
- *(ii)* $\mathrm{C}(X)$ *is an order dual space.*
- *(iii)* $\mathrm{C}(X)$ *is perfect.*
- *(iv)* X *is extremally disconnected and $\upsilon S = X$; that is,* $\mathrm{C}(X)$ *is Dedekind complete and*

$$\mathrm{C}(X) \ni f \longmapsto f|_S \in \mathrm{C}(S)$$

is a lattice isomorphism.

PROOF. The implications (i) implies (ii) implies (iii) holds for general vector lattices. It is proven in [**69**, Theorem 1] that (iv) implies (i) and the argument in the proof of [**69**, Theorem 2] shows that (iii) implies (iv). □

A naive attempt to generalise the decomposition theorem in Theorem 1.1.3 (iii) is to replace the ℓ^∞-direct sum of carriers of a maximal singular family $\mathcal{F}$ in $\mathrm{C}(K)^\sim_\mathrm{n}$ with simply the Cartesian product of the carriers in $\mathcal{F}$. In next two results, we show that this is approach is not correct.

PROPOSITION 2.9.3. *Let X be an extremally disconnected realcompact space, and let $\mathcal{F}$ be a maximal (w.r.t. inclusion) singular family of positive order continuous linear functionals on* $\mathrm{C}(X)$. *Consider the following statements.*

(i) The map

$$\mathrm{C}(X) \ni f \longmapsto (P_\varphi f) \in \prod_{\varphi \in \mathcal{F}} \mathrm{C}_\varphi$$

is a lattice isomorphism.

(ii) $\mathrm{C}(X)$ *is an immaculate vector lattice.*

(iii) $\mathrm{C}(X)$ *is an order dual space.*

(iv) $\mathrm{C}(X)$ *is perfect.*

Then (i) implies (ii), (iii), and (iv) while (ii), (iii), and (iv) are equivalent.

PROOF. The equivalence of (ii), (iii), and (iv) is given in Theorem 2.9.2. Assume that (i) is true. By Theorem 2.2.6 (v) and (vii), $\mathrm{C}(X)^{\sim\sim}_{\mathrm{nn}}$ is isomorphic to $\prod (\mathrm{C}_\varphi)^{\sim\sim}_{\mathrm{nn}}$. But each C_φ is perfect so that $\prod (\mathrm{C}_\varphi)^{\sim\sim}_{\mathrm{nn}}$ is isomorphic to $\prod \mathrm{C}_\varphi$, hence $\mathrm{C}(X)$ is isomorphic to $\mathrm{C}(X)^{\sim\sim}_{\mathrm{nn}}$. □

EXAMPLE 2.9.4. As is well known, $\mathrm{C}(\beta\mathbb{N}) = \ell^\infty = (\ell^1)^\sim$, hence $\mathrm{C}(\beta\mathbb{N})$ is an order dual space. For every $x \in \mathbb{N}$, denote by $\delta_x : \mathrm{C}(\beta\mathbb{N}) \to \mathbb{R}$ the point mass centred at x. Then $\mathcal{F} = \{\delta_x \ : \ x \in \mathbb{N}\}$ is a maximal singular family in $\mathrm{C}(\beta\mathbb{N})^\sim_{\mathrm{n}} \cong \ell^1$. Since $\mathrm{C}_{\delta_x} = \mathbb{R}$ for every $x \in \mathbb{N}$, it follows that $\prod \mathrm{C}_{\delta_x} = \mathbb{R}^{\mathbb{N}}$. Since $\prod \mathrm{C}_{\delta_x}$ does not have a strong order unit while $\mathrm{C}(\beta\mathbb{N})$ does have a strong order unit, we conclude that

$$\mathrm{C}(\beta\mathbb{N}) \ni u \longmapsto (P_{\delta_x} u) \in \prod \mathrm{C}_{\delta_x}$$

is not an isomorphism.

The final result of this section resolves the decomposition problem for $\mathrm{C}(X)$ which are order dual spaces. We briefly recall the notation in Theorem 2.7.3: Let $\mathrm{M}_{\mathrm{n}} \subseteq \mathbf{B}_E$ denote the collection of carriers for positive, order continuous functionals on E. Consider the inverse system $\mathcal{I}_{\mathrm{M}_{\mathrm{n}}} := \left(\mathrm{M}_{\mathrm{n}}, (P_{\psi,\varphi})_{\psi \geq \varphi}\right)$ with $\varprojlim \mathcal{I}_{\mathrm{M}_{\mathrm{n}}} := \left(F, (p_\varphi)_{0 \leq \varphi \in E^\sim_{\mathrm{n}}}\right)$ in **VL**.

THEOREM 2.9.5. *Let X be an extremally disconnected realcompact space. Denote by S the union of the supports of all order continuous functionals on* $\mathrm{C}(X)$. *The following statements are equivalent.*

(i) $\mathrm{C}(X)$ *is an immaculate vector lattice.*

(ii) $\mathrm{C}(X)$ *is an order dual space.*

(iii) $\mathrm{C}(X)$ *is perfect.*

(iv) $\upsilon S = X$.

(v) $P_{\mathrm{M}_{\mathrm{n}}} : \mathrm{C}(X) \to F$ *is a lattice isomorphism.*

PROOF. By Theorem 2.9.2, it suffices to show that (iii) and (v) are equivalent. By Proposition 2.9.1, the carriers C_φ are perfect for every $0 \le \varphi \in C(X)_n^\sim$. The desired equivalence then follows immediately from Corollary 2.7.5. ☐

2.10. Structure theorems for Archimedean relatively uniformly complete vector lattices

Let E be an Archimedean vector lattice. In Example 2.3.8, it is shown that the principal order ideals of E form a direct system in **NIVL** and that E can be expressed as the direct limit of this system. In this section, we exploit this result and the duality results in Section 2.5 to obtain structure theorems for Archimedean relatively uniformly complete vector lattices and their order duals.

A frequently used technique in the theory of vector lattices is to reduce a problem to one confined to a fixed principal order ideal E_u of a space E. Once this is achieved, the problem becomes equivalent to one in a $C(K)$ space for a compact Hausdorff topological space K via the Kakutani Representation Theorem, see [**47**] or [**56**, Theorem 2.1.3]. For instance, this technique is used in [**56**, Theorem 3.8.6] to study tensor products of Banach lattices. The following result is essentially a formalisation of this method in the language of direct limits.

THEOREM 2.10.1. *Let E be an Archimedean, relatively uniformly complete vector lattice. For all $0 < u \le v$ there exists compact Hausdorff spaces K_u and K_v and injective, interval preserving normal lattice homomorphisms $e_{u,v} : C(K_u) \to C(K_v)$ and $e_u : C(K_u) \to E$ so that the following is true.*

- *(i) E_u is lattice isomorphic to $C(K_u)$ for every $0 < u \in E$.*
- *(ii) $\mathcal{D}_E := ((C(K_u))_{0<u\in E}, (e_{u,v})_{u\le v})$ is a direct system in* **NIVL** *with injective linking maps.*
- *(iii) $\mathcal{S}_E := (E, (e_u)_{0<u\in E})$ is the direct limit of $\mathcal{D}_E$ in* **NIVL**.
- *(iv) E is Dedekind complete if and only if K_u is Stonean for every $0 < u \in E$.*
- *(v) If E is perfect, then K_u is hyper-Stonean for every $0 < u \in E$.*

PROOF. By [**56**, Proposition 1.2.13], we know that every principal order ideal in E is a unital AM-space. Therefore the statements in (i), (ii), and (iii) follow immediately from Example 2.3.8 and Kakutani's Representation Theorem for unital AM-spaces. The proof of (iv) follows immediately from Theorem 2.3.5 and [**56**, Proposition 2.1.4].

For the proof of (v), assume that E is perfect. Then, in particular, E is Dedekind complete and has a separating order continuous dual [**70**, Theorem 110.1]. Therefore the same is true for each E_u. By (i), $C(K_u)$ is Dedekind complete and has a separating order continuous dual, i.e. K_u is hyper-Stonean. ☐

A converse to the statement in (ii) in the previous result follows directly from the permanence results for direct limits of vector lattices in [**37**].

COROLLARY 2.10.2. *Let I be an index set and K_α a compact Hausdorff topological space for every $\alpha \in I$. If $\mathcal{D} := ((\mathrm{C}(K_\alpha))_{\alpha\in I}, (e_{\alpha,\beta})_{\alpha\preccurlyeq\beta})$ is a direct system in* **IVL** *with injective linking maps, then the direct limit $(E, (e_\alpha)_{\alpha\in I})$ exists in* **IVL** *and E is an Archimedean, relatively uniformly complete vector lattice.*

PROOF. The result follows directly by the existence result for direct limits in Theorem 2.3.3 and the permanence results in Theorem 2.3.5 (i) and (v) along with the fact that every $\mathrm{C}(K)$ space is Archimedean and relatively uniformly complete [**53**, Theorem 43.1]. ☐

Thus Archimedean relatively uniformly complete vector lattices are characterised as precisely those vector lattices obtained in the direct limit of a direct system of $\mathrm{C}(K)$ spaces for compact Hausdorff K with injective linking maps. In the last few results of this section and this chapter, we examine the order duals of Archimedean relatively uniformly complete vector lattices.

COROLLARY 2.10.3. *Let E be a Archimedean relatively uniformly complete vector lattice. There exists an inverse system $\mathcal{I} := ((\mathrm{M}(K_\alpha))_{\alpha\in I}, (p_{\beta,\alpha})_{\beta\succcurlyeq\alpha})$ in* **NIVL**, *with each K_α a compact Hausdorff space, and normal lattice homomorphisms $p_\alpha : E^\sim \to \mathrm{M}(K_\alpha)$, so that $\mathcal{S} := (E^\sim, (p_\alpha)_{\alpha\in I})$ is the inverse limit of $\mathcal{I}$ in* **NVL**.

PROOF. The result follows immediately from Theorems 2.10.1 and 2.5.4, and the Riesz Representation Theorem [**25**, Chapter III, Theorem 5.7]. ☐

In order to obtain a converse of Corollary 2.10.3, we require a more detailed description of the interval preserving normal lattice homomorphisms $e_{u,v} : \mathrm{C}(K_u) \to \mathrm{C}(K_v)$ in Theorem 2.10.1. Let X and Y be topological spaces and $p : X \to Y$ a continuous function. Recall that p is *almost open* if for every non-empty open subset U of X, $\mathrm{int}\left(\overline{p[U]}\right) \neq \varnothing$. It is clear that all open maps are almost open and thus every homeomorphism is almost open.

PROPOSITION 2.10.4. *Let K and L be compact Hausdorff spaces and $T : \mathrm{C}(K) \to \mathrm{C}(L)$ a positive linear map. T is a lattice homomorphism if and only if there exist a unique $0 < w \in \mathrm{C}(L)$ and a unique continuous function $p : \mathbf{Z}_w^c \to K$ so that*

$$T(u)(x) = \begin{cases} w(x)u(p(x)) & \text{if } x \in \mathbf{Z}_w^c \\ 0 & \text{if } x \in \mathbf{Z}_w \end{cases} \qquad (2.10.1)$$

for all $u \in \mathrm{C}(K)$. In particular, $w = T(\mathbf{1}_K)$.

Assume that T is a lattice homomorphism. Then the following statements are true.

(i) T is order continuous if and only if p is almost open.

(ii) T is injective if and only if $p[\mathbf{Z}_w^c]$ is dense in K.

(iii) T is interval preserving if and only if $p[\mathbf{Z}_w^c]$ is C^*-embedded in K and p is a homeomorphism onto $p[\mathbf{Z}_w^c]$.

PROOF. The statement in (i) is well known, see for instance [**1**, Theorem 4.25]. Now suppose that T is a lattice homomorphism. The statement (i) follows from [**65**, Theorem 4.4], or, from [**17**, Theorem 7.1 (iii)].

We prove (ii). Assume that $p[\mathbf{Z}_w^c]$ is dense in K. Let $u \in \mathrm{C}(K)$ satisfy $T(u) = 0$. Then $w(x)u(p(x)) = 0$ for all $x \in \mathbf{Z}_w^c$. Hence $u(z) = 0$ for all $z \in p[\mathbf{Z}_w^c]$. Since $p[\mathbf{Z}_w^c]$ is dense in K it follows that $u = 0$. Thus T is injective. Conversely, suppose that $p[\mathbf{Z}_w^c]$ is not dense in K. Then there exists $0 < u \in \mathrm{C}(K)$ so that $u(z) = 0$ for all $z \in p[\mathbf{Z}_w^c]$; that is, $u(p(x)) = 0$ for all $x \in \mathbf{Z}_w^c$. Hence $T(u)(x) = w(x)u(p(x)) = 0$ for all $x \in \mathbf{Z}_w^c$. By definition $T(u)(x) = 0$ for all $x \in \mathbf{Z}_w$ so that $T(u) = 0$. Therefore T is not injective. Thus (ii) is proved.

Lastly we verify (iii). Suppose that T is interval preserving. We first show that $p[\mathbf{Z}_w^c]$ is C*-embedded in K. Consider $0 \leq f \in \mathrm{C_b}(p[\mathbf{Z}_w^c])$. We must show that there exists a function $g \in \mathrm{C}(K)$ so that $g(z) = f(z)$ for all $z \in p[\mathbf{Z}_w^c]$. We may assume that $f \leq \mathbf{1}_{p[\mathbf{Z}_w^c]}$. Define $v : L \to \mathbb{R}$ by setting

$$v(x) \coloneqq \begin{cases} w(x)f(p(x)) & \text{if} \quad x \in \mathbf{Z}_w^c \\ 0 & \text{if} \quad x \in \mathbf{Z}_u \end{cases}$$

for every $x \in K$. It is clear that v is continuous on $\mathbf{Z}_w^c$ and on the interior of $\mathbf{Z}_w$. For all other point $x \in K$, continuity of v follows from the inequality $0 \leq v \leq w$. From this last inequality and the fact that T is interval preserving it follows that there exists $0 \leq g \leq \mathbf{1}_K$ so that $Tg = v$. If $x \in p[\mathbf{Z}_w^c]$ then $w(x)f(p(x)) = v(x) = Tg(x) = w(x)g(p(x))$ so that $f(p(x)) = g(p(x))$; that is, $g(z) = f(z)$ for all $z \in p[\mathbf{Z}_w^c]$.

Next we show that p is a homeomorphism onto $p[\mathbf{Z}_w^c]$. First we show that p is injective. Consider distinct $x_0, x_1 \in \mathbf{Z}_w^c$ and suppose that $p(x_0) = p(x_1)$. There exists $v \in \mathrm{C}(L)$ with $0 < v \leq w$ such that $v(x_0) = 0$ and $v(x_1) > 0$. Because T is interval preserving there exists $0 < u \leq \mathbf{1}_K$ in $\mathrm{C}(K)$ so that $T(u) = v$. Then $u(p(x_0)) = 0$ and $u(p(x_1)) > 0$, contradicting the assumption that $p(x_0) = p(x_1)$. Therefore p is injective.

It remains to verify that p^{-1} is continuous. Let (x_i) be a net in $\mathbf{Z}_w^c$ and $x \in \mathbf{Z}_w^c$ so that $(p(x_i))$ converges to $p(x)$ in K. Suppose that (x_i) does not converge to x. Passing to a subnet of (x_i) if necessary, we obtain a neighbourhood V of x so that $x_i \notin V$ for all i. There exists a function $0 < v \leq w$ in $\mathrm{C}(L)$ so that $v(x) > 0$ and $v(x_i) = 0$ for all i. Because T is interval preserving there exists a function $u \in \mathrm{C}(K)$ so that $T(u) = v$. In particular, $w(x)u(p(x)) = v(x) > 0$ so that $u(p(x)) > 0$, but $w(x_i)u(p(x_i)) = v(x_i) = 0$ so that $u(p(x_i)) = 0$ for all i. Therefore $(u(p(x_i)))$ does not converge to $u(x)$, contradicting the continuity of u. Hence (x_i) converges to x so that p^{-1} is continuous.

Conversely, suppose that p is a homeomorphism onto $p[\mathbf{Z}_w^c]$, and that $p[\mathbf{Z}_w^c]$ is C*-embedded in K. Let $0 < u \in \mathrm{C}(K)$ and $0 \leq v \leq T(u)$ in $\mathrm{C}(L)$. Define $f : p[\mathbf{Z}_w^c] \to \mathbb{R}$ by setting

$$f(z) \coloneqq \frac{1}{w(p^{-1}(z))} v(p^{-1}(z)), \quad z \in p[\mathbf{Z}_w^c].$$

Because $p^{-1} : p[\mathbf{Z}_w^c] \to \mathbf{Z}_w^c$ is continuous, f is continuous. Furthermore, $0 \leq f(z) \leq u(z)$ for all $z \in p[\mathbf{Z}_w^c]$. Therefore f is a bounded continuous function on $p[\mathbf{Z}_w^c]$. By assumption, there exists a continuous function $g : K \to \mathbb{R}$ so that $g(z) = f(z)$ for all $z \in p[\mathbf{Z}_w^c]$. Since $0 \leq f \leq u$ on $p[\mathbf{Z}_w^c]$, the function g may to chosen so that $0 \leq g \leq u$. For $x \in \mathbf{Z}_w^c$ we have

$$T(g)(x) = w(x)g(p(x)) = w(x)f(p(x)) = \frac{w(x)v(x)}{w(x)} = v(x),$$

and for $x \in \mathbf{Z}_w$ we have $v(x) = 0 = T(g)(x)$. Therefore $Tg = v$ so that T is interval preserving. □

THEOREM 2.10.5. *Let E be a vector lattice. The following statements are equivalent.*

(i) *$E \cong F^\sim$ where F is some Archimedean relatively uniformly complete vector lattice.*

(ii) *There exists an inverse system $\mathcal{I} \coloneqq ((\mathrm{M}(K_\alpha))_{\alpha\in I}, (p_{\beta,\alpha})_{\beta\succcurlyeq\alpha})$ in* **NIVL**, *with each K_α a compact Hausdorff space, such that the following holds.*

- *(a)* *For each $\beta \succcurlyeq \alpha$ in I there exist a function $w \in \mathrm{C}(K_\beta)^+$ and homeomorphism $t : \mathbf{Z}_w^c \to t[\mathbf{Z}_w^c] \subseteq K_\alpha$ onto a dense $\mathrm{C}^\star$-embedded subspace of K_α so that for every $\mu \in \mathrm{M}(K_\beta)$,*

$$p_{\beta,\alpha}(\mu)(A) = \int_{t^{-1}[A]} w\, d\mu, \quad A \in \mathfrak{B}_{K_\alpha}.$$

- *(b)* *For every $\alpha \in I$ there exists a normal lattice homomorphism $p_\alpha : E \to \mathrm{M}(K_\alpha)$ such that $\varprojlim \mathcal{I} = (E, (p_\alpha)_{\alpha\in I})$.*

PROOF OF (i) IMPLIES (ii). By Theorem 2.10.1, there exists a direct system $\mathcal{D} \coloneqq ((\mathrm{C}(K_\alpha))_{\alpha\in I}, (e_{\alpha,\beta})_{\alpha\preccurlyeq\beta})$ in **NIVL**, with each K_α a compact Hausdorff space, and interval preserving normal lattice homomorphisms $e_\alpha : \mathrm{C}(K_\alpha) \to F$ so that $\mathcal{S} \coloneqq (F, (e_\alpha)_{\alpha\in I})$ is the direct limit of $\mathcal{D}$ in **NIVL**. By Theorem 2.5.4 and the Riesz Representation Theorem [**61**, Theorem 18.4.1], $\mathcal{S}^\sim \coloneqq (E, (e_\alpha^\sim)_{\alpha\in I})$ is the inverse limit of the inverse system $\mathcal{D}^\sim \coloneqq \left(\mathrm{M}(K_\alpha), (e_{\alpha,\beta}^\sim)_{\alpha\preccurlyeq\beta}\right)$ in **NVL**. Thus the claim in (b) holds.

Fix $\beta \succcurlyeq \alpha$ in I. We show that $e_{\alpha,\beta}^\sim$ is of the form given in (a). By Proposition 2.10.4, there exists $w \in \mathrm{C}(K_\beta)^+$ and a homeomorphism $t : \mathbf{Z}_w^c \to t[\mathbf{Z}_w^c] \subseteq K_\alpha$ onto a dense $\mathrm{C}^\star$-embedded subspace of K_α so that

$$e_{\alpha,\beta}(u)(x) = \begin{cases} w(x)u(t(x)) & \text{if } \ x \in \mathbf{Z}_w^c \\ 0 & \text{if } \ x \in \mathbf{Z}_w \end{cases}$$

for all $u \in \mathrm{C}(K_\alpha)$. Let $T : \mathrm{C}(K_\alpha) \to \mathrm{C_b}(\mathbf{Z}_w^c)$ and $M_w : \mathrm{C_b}(\mathbf{Z}_w^c) \to \mathrm{C}(K_\beta)$ be given by $T(u) = u \circ t$ and $M_w(v) = wv$ for all $u \in \mathrm{C}(K_\alpha)$ and $v \in \mathrm{C_b}(\mathbf{Z}_w^c)$, with wv defined as identically zero outside $\mathbf{Z}_w^c$. Then T and M_w are positive operators and $e_{\alpha,\beta} = M_w \circ T$; hence $e_{\alpha,\beta}^\sim = T^\sim \circ M_w^\sim$. It follows from [**20**, Theorems 3.6.1 & 9.1.1] that $T^\sim(\mu)(A) = \mu(t^{-1}[A])$ for every $\mu \in \mathrm{M}(\mathbf{Z}_w^c)$ and $A \in \mathfrak{B}_{K_\alpha}$. The Riesz Representation Theorem shows that, for each $\nu \in \mathrm{M}(K_\beta)$ and every Borel set B in $\mathbf{Z}_w^c$,

$$M_w^\sim(\nu)(B) = \int_B w\, d\nu.$$

Hence for $\mu \in \mathrm{M}(K_\beta)$ and $A \in \mathfrak{B}_{K_\alpha}$,

$$e^{\sim}_{\alpha,\beta}(\mu)(A) = \int_{t^{-1}[A]} w\, d\mu$$

as claimed. □

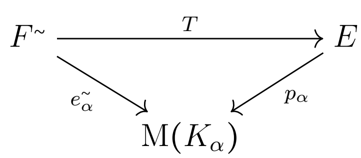
Proof of (ii) implies (i). Fix $\beta \succcurlyeq \alpha$ in I and consider the function $w \in \mathrm{C}(K_\beta)^+$ and the homeomorphism $t : \mathbf{Z}_w^c \to t[\mathbf{Z}_w^c] \subseteq K_\alpha$ given in (b). Define the map $e_{\alpha,\beta} : \mathrm{C}(K_\alpha) \to \mathrm{C}(K_\beta)$ as

$$e_{\alpha,\beta}(u)(x) = \begin{cases} w(x)u(t(x)) & \text{if} \quad x \in \mathbf{Z}_w^c \\ 0 & \text{if} \quad x \in \mathbf{Z}_w \end{cases}$$

We show that $\mathcal{D} \coloneqq ((\mathrm{C}(K_\alpha))_{\alpha\in I}, (e_{\alpha,\beta})_{\alpha\prec\beta})$ is a direct system in **NIVL**.

It follows by Proposition 2.10.4 that each $e_{\alpha,\beta}$ is an injective interval preserving normal lattice homomorphism. It remains to show that $e_{\alpha,\gamma} = e_{\beta,\gamma} \circ e_{\alpha,\beta}$ for all $\alpha \preccurlyeq \beta \preccurlyeq \gamma$ in I. An argument similar to that in the proof that (i) implies (ii) shows that $e^{\sim}_{\alpha,\beta} = p_{\beta,\alpha}$ for all $\alpha \preccurlyeq \beta$; hence $e^{\sim\sim}_{\alpha,\beta} = p^{\sim}_{\beta,\alpha}$. By Proposition 2.5.9, $\mathcal{I}^{\sim} \coloneqq \big((\mathrm{M}(K_\alpha)^{\sim}_{\alpha})_{\alpha\in I}, (p^{\sim}_{\beta,\alpha})_{\beta\succcurlyeq\alpha}\big)$ is a direct system in **NIVL** and therefore $e^{\sim\sim}_{\alpha,\gamma} = e^{\sim\sim}_{\beta,\gamma} \circ e^{\sim\sim}_{\alpha,\beta}$ for all $\alpha \preccurlyeq \beta \preccurlyeq \gamma$ in I. Since $\mathrm{C}(K_\alpha)$ has a separating order dual for every $\alpha \in I$, it follows that $e_{\alpha,\gamma} = e_{\beta,\gamma} \circ e_{\alpha,\beta}$. Hence $\mathcal{D}$ is a direct system in **NIVL**.

Since each $e_{\alpha,\beta}$ is injective, $\varinjlim \mathcal{D} \coloneqq (F, (e_\alpha)_{\alpha\in I})$ exists in **NIVL** by Theorem 2.3.4. Since $\mathrm{C}(K_\alpha)$ is relatively uniformly complete for each $\alpha \in I$ it follows from Theorem 2.3.5 (v) that F is also relatively uniformly complete. Because $e^{\sim}_{\alpha,\beta} = p_{\beta,\alpha}$ for all $\alpha \preccurlyeq \beta$ in I, $\mathcal{D}^{\sim} = \mathcal{I}$. Therefore, by Theorem 2.5.4, there exists a lattice isomorphism $T : F^{\sim} \to E$ such that the diagram



commutes for all $\alpha \in I$. This completes the proof. □

CHAPTER 3

Free objects

We now move on to our second major problem regarding the existence of free objects in some categories we have not yet defined. We start with the definitions and terminology we need to define these categories.

3.1. Preliminaries

3.1.1. Algebraic structures. Denote by $\mathbb{K}$ either $\mathbb{R}$ or $\mathbb{C}$. We refer to associative algebras (over $\mathbb{K}$) as *algebras*. Algebras possessing a multiplicative identity are called *unital*. Algebra homomorphisms between unital algebras need not preserve multiplicative identities unless explicitly indicated.

We make use of the same notation and conventions for vector lattices that was introduced in Chapter 1, Section 1.3. We do not assume that vector lattices are Archimedean unless stated explicitly. A *vector lattice algebra* A (also called a *Riesz algebra* in the literature) is an algebra over $\mathbb{R}$ equipped with a partial order $\leq$ that makes A into a vector lattice with the additional property that the positive cone A^+ is closed under multiplication. Further, A is called a *unital vector lattice algebra* if the underlying algebra possesses a multiplicative identity 1_A, and A is a *positive unital vector lattice algebra* if $1_A \in A^+$. The vector lattice algebra homomorphisms are the maps that are both lattice and algebra homomorphisms, also not necessarily unital unless indicated.

This gives us the following categories:

	Objects	Morphisms
Set	Sets	Total functions
VS	Vector spaces	Linear maps
VL	Vector lattices	Vector lattice homomorphisms
Alg	Algebras	Algebra homomorphisms
$\mathbf{Alg^1}$	Unital algebras	Unital algebra homomorphisms
VLA	Vector lattice algebras	Vector lattice algebra homomorphisms
$\mathbf{VLA^1}$	Unital vector lattice algebras	Unital vector lattice algebra homomorphisms
$\mathbf{VLA^{1+}}$	Positive unital vector lattice algebras	Unital vector lattice algebra homomorphisms

Table 1. Table of algebraic categories under consideration

$$
\begin{array}{ccccccccc}
\mathbf{Set} & \supseteq & \mathbf{VS} & \supseteq & \mathbf{VL} & & & & \\
 & & \rotatebox{-45}{$\supseteq$} & & \rotatebox{-90}{$\supseteq$} & & & & \\
 & & \mathbf{Alg} & \supseteq & \mathbf{VLA} & \supseteq & \mathbf{VLA^1} & \supseteq & \mathbf{VLA^{1+}} \\
 & & & & \rotatebox{-90}{$\supseteq$} & \rotatebox{-135}{$\supseteq$} & & & \\
 & & & & \mathbf{Alg^1} & & & &
\end{array}
$$

FIGURE 1. Figure depicting subcategory relationships between the various algebraic categories of interest.

We will refer to the categories in Table 1, excluding **Set**, as *algebraic categories*. A *subspace* in each of the above contexts is a subset of that algebraic object that is closed with respect to all the different operations defined on that space.

Recall that a subset S of a vector lattice E is *solid in* E if for $x \in S$ and $y \in E$, the condition that $|y| \leq |x|$ implies $y \in S$. A linear solid subspace of a vector lattice is an *order ideal*.

Given an algebra R, a linear subspace I is a *left (resp. right) algebra ideal* if for all $x \in R$ and $a \in I$ we have $xa \in I$ (resp. $ax \in I$). If I is both a left and right algebra ideal then I is called a *two-sided algebra ideal*. It is clear that every algebra ideal (left, right, or two-sided) is closed under multiplication and is thus an algebra subspace (i.e. a subalgebra). A linear subspace of a vector lattice algebra that is both an order ideal and a two-sided algebra ideal is called a *bi-ideal*.

3.1.2. Normed structures. A vector lattice E equipped with a norm $\|\cdot\| : E \to \mathbb{R}^+$ is called a *normed vector lattice* if the underlying vector space is a normed space and if the norm satisfies the following property: For $x, y \in E$ such that $|x| \leq |y|$ we have $\|x\| \leq \|y\|$. Norms that satisfy this property are called *Riesz norms* and Riesz seminorms are defined in the same way. We make the following observations for Riesz (semi)norms: For $x \in E$, since $|x| = ||x||$, it follows that $\|x\| = \||x|\|$.

An algebra or unital algebra R equipped with a norm $\|\cdot\| : R \to \mathbb{R}^+$ is a *normed algebra* if the norm is *submultiplicative*: For $x, y \in R$ we have $\|xy\| \leq \|x\| \, \|y\|$. One defines a submultiplicative seminorm in the same way. A vector lattice algebra (non-unital, unital, or positive unital) equipped with a submultiplicative Riesz norm is a *normed vector lattice algebra*. The multiplicative identity in a normed vector lattice algebra is not assumed to have norm 1 when it exists.

Complete normed vector lattices, normed algebras, and normed vector lattice algebras are known as *Banach lattices*, *Banach algebras*, and *Banach lattice algebras* respectively. These definitions give us the following two collections of categories.

	Objects	Morphisms
NS	Normed spaces	Bounded linear maps
NVL	Normed vector lattices	Bounded vector lattice homomorphisms
NA	Normed algebras	Bounded algebra homomorphisms
$\mathbf{NA^1}$	Unital normed algebras	Bounded unital algebra homomorphisms
NVLA	Normed vector lattice algebras	Bounded V.L.A. homomorphisms
$\mathbf{NVLA^1}$	Unital normed vector lattice algebras	Bounded unital V.L.A. homomorphisms
$\mathbf{NVLA^{1+}}$	Positive unital normed vector lattice algebras	Bounded unital V.L.A. homomorphisms

Table 2. Table of categories of normed structures under consideration.

	Objects	Morphisms
Ban	Banach spaces	Bounded linear maps
BL	Banach lattices	Bounded vector lattice homomorphisms
BA	Banach algebras	Bounded algebra homomorphisms
$\mathbf{BA^1}$	Unital Banach algebras	Bounded unital algebra homomorphisms
BLA	Banach lattice algebras	Bounded V.L.A. homomorphisms
$\mathbf{BLA^1}$	Unital Banach lattice algebras	Bounded unital V.L.A. homomorphisms
$\mathbf{BLA^{1+}}$	Positive unital Banach lattice algebras	Bounded unital V.L.A. homomorphisms

Table 3. Table of categories of complete normed structures under consideration.

For a category $\mathbf{C}$ of normed structures from Table 2 or 3 above, we denote by $\mathbf{C_1}$ the subcategory whose morphisms are contractive. For a category of normed structures $\mathbf{Y}$ in Table 2 or 3, denote by $\mathcal{A}(\mathbf{Y})$ the smallest category $\mathbf{X}$ in Table 1 such that $\mathbf{X} \supseteq \mathbf{Y}$. For example, if we consider the category **BLA**, the categories $\mathbf{X}$ such that $\mathbf{X} \supseteq \mathbf{Y}$ form the collection $\{\mathbf{Set}, \mathbf{VS}, \mathbf{Alg}, \mathbf{VL}, \mathbf{VLA}\}$. Consulting Figure 1, we see that **VLA** is the smallest category among the above with the necessary property, so we write $\mathcal{A}(\mathbf{BLA}) = \mathbf{VLA}$. Thus $\mathcal{A}(\cdot)$ associates with each category of normed structures from Table 2 and 3 the 'canonical' algebraic supercategory.

Let $\mathbf{Y}$ be a category of normed structures from Table 2 and denote by $\hat{\mathbf{Y}}$ the associated category of complete normed structures in Table 3. For $Y \in \mathbf{Y}$, a *completion* of Y is an object $\hat{Y} \in \hat{\mathbf{Y}}$ for which there exists an isometric $\mathbf{Y}$-isomorphism $T : Y \to \hat{Y}$ where $T[Y]$ is a dense subspace of $\hat{Y}$. For objects $N \in \mathbf{Y}$ and $B \in \hat{\mathbf{Y}}$ and a morphism $f : N \to B$ in $\mathbf{Y}$, one may verify that there exists a unique morphism $\hat{f} : \hat{N} \to B$ in $\hat{\mathbf{Y}}$ extending f. This means that, for example, if N is a normed space then the completion $\hat{N}$ along with the inclusion $j : N \to \hat{N}$ is the *free Banach space over the normed space* N.

3.1.3. Locally convex structures. All topological vector spaces are assumed to be Hausdorff. A topological vector space is a *locally convex space* if the origin has a neighbourhood basis of convex sets. An algebra (non-unital or unital) equipped with a linear topology is a *locally multiplicatively-convex algebra* (or a *locally m-convex algebra* for short) if the origin has a neighbourhood basis of convex sets that are closed under multiplication,[1] see [**55**, Definition 1.3, p. 5]. Further, a vector

[1] *Locally m-convex algebras* are a particular sort of *locally convex algebra*. Some care must be taken when dealing with definitions of *topological algebras* and *locally convex algebras* since these definitions vary across the literature: A *topological algebra* is defined in [**55**, Definition 1.1, p. 4] and [**38**, Definition 1.6] as an algebra equipped with a linear

lattice equipped with a linear topology is a *locally convex-solid vector lattice* if the origin has a neighbourhood basis of convex solid sets. Lastly, a vector lattice algebra (non-unital, unital, or positive unital) equipped with a linear topology is a *locally multiplicatively-convex-solid vector lattice algebra* (or *locally m-convex-solid vector lattice algebra* for shorter) if the origin has a neighbourhood basis of convex solid sets that are closed under multiplication. This gives us another batch of categories.

	OBJECTS	MORPHISMS
LCS	Locally convex spaces	Continuous linear maps
LC-SVL	Locally convex-solid vector lattices	Cts. vector lattice homomorphisms
LM-CA	Locally m-convex algebras	Cts. algebra homomorphisms
$\mathbf{LM\text{-}CA^1}$	Unital locally m-convex algebras	Cts. unital algebra homomorphisms
LM-C-SVLA	Locally m-convex-solid vector lattice algebras	Cts. V.L.A. homomorphisms
$\mathbf{LM\text{-}C\text{-}SVLA^1}$	Unital locally m-convex-solid vector lattice algebras	Cts. unital V.L.A. homomorphisms
$\mathbf{LM\text{-}C\text{-}SVLA^{1+}}$	Positive unital locally m-convex-solid vector lattice algebras	Cts. unital V.L.A. homomorphisms

TABLE 4. Table of categories of locally convex structures under consideration.

The names of the above categories of locally convex structures are unfortunately not very elegant, but at least they are not misleading.

For a category $\mathbf{Y}$ of normed structures from Table 2, we denote by $\mathbf{LCY}$ the associated category of locally convex structures in Table 4 such that $\mathcal{A}(\mathbf{LCY}) = \mathcal{A}(\mathbf{Y})$ (with the notation $\mathcal{A}(\cdot)$ defined for Table 4 as above for Table 2 and 3).

A locally convex structure in Table 4 is *complete* if every Cauchy net converges. The details of the construction of a completion of a locally convex space are recorded in [**50**, p. 208] and completions of other locally convex structures in Table 4 follow by modification. We add the prefix **Com** to the categories of locally convex structures above to denote the subcategory consisting of complete objects. Thus we denote (rather clumsily, but at least descriptively and not misleadingly) by $\mathbf{ComLM\text{-}C\text{-}SVLA^{1+}}$ the category of complete positive unital locally multiplicatively-convex-solid vector lattice algebras with continuous unital vector lattice algebra homomorphisms as morphisms. Similarly, for a category of complete normed structures $\hat{\mathbf{Y}}$ in Table 3, denote by $\mathbf{ComLC\hat{Y}}$ the associated category of complete locally convex structures such that $\mathcal{A}\left(\mathbf{ComLC\hat{Y}}\right) = \mathcal{A}\left(\hat{\mathbf{Y}}\right)$.

Let $\mathbf{C}$ and $\mathbf{D}$ be categories with $\mathbf{C} \subseteq \mathbf{D}$. Recall that $\mathbf{C}$ is a *full subcategory* of $\mathbf{D}$ when all morphisms in $\mathbf{D}$ between objects in $\mathbf{C}$ are included in $\mathbf{C}$. Thus for any set $S \subseteq \mathrm{Obj}_{\mathbf{D}}$ there exists a unique full subcategory $\mathbf{C}_S$ of $\mathbf{D}$ where $\mathrm{Obj}_{\mathbf{C}_S} = S$. Thus for any category $\mathbf{Y}$ from Table 2, it is clear that $\hat{\mathbf{Y}}$ is a full subcategory of $\mathbf{Y}$ and $\mathbf{ComLC\hat{Y}}$ is a full subcategory of $\mathbf{LCY}$.

topology making multiplication *separately* continuous whereas an algebra equipped with a linear topology making multiplication *jointly* continuous is called a *topological algebra with continuous multiplication* in [**55**, Definition 1.1, p. 4] and [**38**, Example 1.8]. On the other hand, these formulations of topological algebras with separate continuity and joint continuity are called *weak topological algebras* and just *topological algebras* respectively in [**14**, p. 84]. Taking the above difference in terminology into account, the authors of [**14**], [**38**], and [**55**] define a *locally convex algebra* as a topological algebra where the topology makes the underlying vector space into a locally convex space (see [**55**, Definition 1.1, p. 4], [**38**, Definition 1.7], and [**14**, Definition 4.4.1]). **We adopt the terminology of** [**55**] **and** [**38**] **for these definitions.** In any case, it is easy to see that a *locally m-convex algebra* is then in fact also a *locally convex algebra with continuous multiplication* in our terminology.

Although the above definitions for the locally convex structures in Table 4 should be considered the 'natural' definitions, in practice, it is often preferable to work with *seminorm* characterisations for locally convex structures. We record these seminorm characterisations for the locally convex structures in Table 4 in the next result.

PROPOSITION 3.1.1. *The following statements are true.*

(i) A topological vector space X is a locally convex space if and only if there exists a separating family of seminorms generating the topology on X.

(ii) A topological algebra R (non-unital or unital) is a locally m-convex algebra if and only if there exists a separating family of submultiplicative seminorms generating the topology on R.

(iii) A topological vector lattice[2] E is a locally convex-solid vector lattice if and only if there exists a separating family of Riesz seminorms generating the topology on E.

(iv) A topological vector lattice algebra[3] A (non-unital, unital, or positive unital) is a locally m-convex-solid vector lattice algebra if and only if there exists a separating family of submultiplicative Riesz seminorms generating the topology on A.

PROOF. The proofs of (i), (ii), and (iii) are found in [**25**, Chapter IV, Proposition 1.15], [**55**, Theorem 3.1, p. 18] or [**14**, Lemma 4.4.2], and [**5**, Theorem 2.25] respectively. We prove (iv): First, assume that A locally m-convex-solid vector lattice algebra and let V be any convex solid neighbourhood of the origin that is closed under multiplication. Since every solid set is balanced and every neighbourhood of the origin is absorbing, it follows by [**25**, Chapter IV, Proposition 1.14] that the Minkowski functional $\rho_V : A \to \mathbb{R}$ is a seminorm and that $V = \{x \in A : \rho_V(x) < 1\}$. It follows by [**5**, p. 59] that ρ_V is a Riesz seminorm since V is solid and it follows by [**38**, Proposition 1.5] that ρ_V is submultiplicative since V is closed under multiplication. Thus there exists a separating family of submultiplicative Riesz seminorms generating the topology on A. Conversely, let A be a topological vector lattice algebra whose topology is generated by a separating family of submultiplicative Riesz seminorms P. Let N be any neighbourhood of the origin, then there exists $\rho \in P$ and $\epsilon \in (0,1]$ such that

$$U_\epsilon := \{x \in A : \rho(x) < \epsilon\} \subseteq N.$$

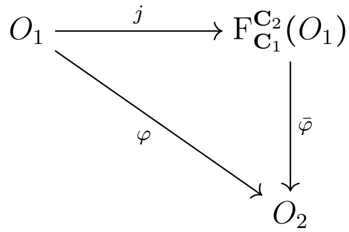
The properties of ρ make it easy to verify that U_ϵ is convex, solid, and closed under multiplication. □

3.1.4. Free objects. For the sake of recording all the important definitions used in the rest of this chapter all in one place, we repeat the definition of a free object before discussing some of its properties.

[2]A *topological vector lattice* is a vector lattice equipped with a linear topology that is locally solid, i.e. the origin has a neighbourhood basis of solid sets, see [**60**, Chapter V, Section 7].

[3]A *topological vector lattice algebra* is a vector lattice algebra equipped with a linear topology that makes the underlying algebra into a topological algebra and the underlying vector lattice into a topological vector lattice.

DEFINITION 3.1.2. Let $\mathbf{C}_1$ and $\mathbf{C}_2$ be categories with $\mathbf{C}_1 \supseteq \mathbf{C}_2$. Fix $O_1 \in \mathbf{C}_1$ and consider an object $\mathrm{F}_{\mathbf{C}_1}^{\mathbf{C}_2}(O_1)$ in $\mathbf{C}_2$ and a morphism $j : O_1 \to \mathrm{F}_{\mathbf{C}_1}^{\mathbf{C}_2}(O_1)$ in $\mathbf{C}_1$. The pair $\left(\mathrm{F}_{\mathbf{C}_1}^{\mathbf{C}_2}(O_1), j\right)$ is called *a free object over* O_1 *of* $\mathbf{C}_2$ if it has the property that for every $O_2 \in \mathbf{C}_2$ and every morphism $\varphi : O_1 \to O_2$ in $\mathbf{C}_1$, there exists a unique morphism $\bar{\varphi} : \mathrm{F}_{\mathbf{C}_1}^{\mathbf{C}_2}(O_1) \to O_2$ in $\mathbf{C}_2$ such that the following diagram commutes in $\mathbf{C}_1$.

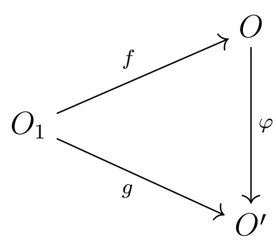


If the above holds, the pair $\left(\mathrm{F}_{\mathbf{C}_1}^{\mathbf{C}_2}(O_1), j\right)$, or by abuse of notation just the object $\mathrm{F}_{\mathbf{C}_1}^{\mathbf{C}_2}(O_1)$, is also called a *solution for the free object problem over* O_1 *between the categories* $\mathbf{C}_1$ *and* $\mathbf{C}_2$.

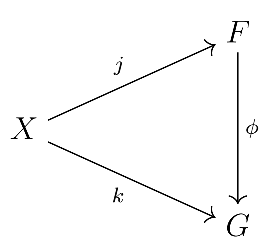
As was done for direct and inverse limits in Chapter 2, Sections 2.3 and 2.4, a slight change in perspective will allow us to easily conclude that free objects (F, j) are essentially unique in a particular sense: Consider categories $\mathbf{C}_1$ and $\mathbf{C}_2$ with $\mathbf{C}_1 \supseteq \mathbf{C}_2$ and fix $O_1 \in \mathbf{C}_1$. We construct the derived category $\mathbf{C}_{F[\mathbf{C}_1,\mathbf{C}_2]}$ where:

(i) Objects in $\mathbf{C}_{F[\mathbf{C}_1,\mathbf{C}_2]}$ are pairs (O_2, f) where O_2 is an object in $\mathbf{C}_2$ and $f : O_1 \to O_2$ is a morphism in $\mathbf{C}_1$.

(ii) A morphism between (O, f) and (O', g) in $\mathbf{C}_{F[\mathbf{C}_1,\mathbf{C}_2]}$ is a morphism $\varphi : O \to O'$ in $\mathbf{C}_2$ such that the following diagram commutes in $\mathbf{C}_1$.



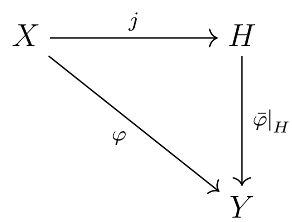
Thus, it is clear that an object (F, j) in $\mathbf{C}_{F[\mathbf{C}_1,\mathbf{C}_2]}$ is a free object over O_1 of $\mathbf{C}_2$ in the sense of Definition 3.1.2 if and only if the object (F, j) is the initial object in the derived category $\mathbf{C}_{F[\mathbf{C}_1,\mathbf{C}_2]}$. As a result (F, j) is unique up to a unique isomorphism in $\mathbf{C}_{F[\mathbf{C}_1,\mathbf{C}_2]}$, meaning that for any other candidate free object (G, k) there exists a unique isomorphism $\phi : F \to G$ in $\mathbf{C}_2$ such that the following diagram commutes in $\mathbf{C}_1$.



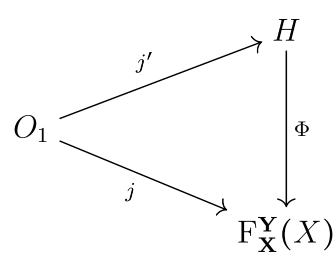
Beyond establishing this essential uniqueness, the universal property of the free object also gives us information on the structure of free objects in categories of normed structures.

PROPOSITION 3.1.3. *Let* $\mathbf{Y}$ *be any category of normed structures from Table 2 or 3 with* $\mathbf{X}$ *any category from Table 1, 2, or 3 such that* $\mathbf{X} \supseteq \mathbf{Y}$*. Fix* $X \in \mathbf{X}$ *and assume that the free object* $(\mathrm{F}_{\mathbf{X}}^{\mathbf{Y}}(X), j)$ *exists. Consider the associated category* $\mathcal{A}(\mathbf{Y})$ *in Table 1 and denote by* G *the object in* $\mathcal{A}(\mathbf{Y})$ *generated by the subset* $j[X]$ *in* $\mathrm{F}_{\mathbf{X}}^{\mathbf{Y}}(X)$*. Then* G *is dense in* $\mathrm{F}_{\mathbf{X}}^{\mathbf{Y}}(X)$*.*

PROOF. Let H be the closure of G in $\mathrm{F}_{\mathbf{X}}^{\mathbf{Y}}(X)$, then H is an object in $\mathbf{Y}$. Define the morphism $j' : X \to H$ where $j'(x) := j(x)$. We claim that the pair (H, j') is a free object over X in $\mathbf{Y}$: Let Y be an object in $\mathbf{Y}$ and $\varphi : X \to Y$ a morphism in $\mathbf{X}$. Then there exists a unique morphism $\bar{\varphi} : \mathrm{F}_{\mathbf{X}}^{\mathbf{Y}}(X) \to Y$ such that $\bar{\varphi} \circ j = \varphi$. The restriction $\bar{\varphi}|_H : H \to Y$ is also a morphism in $\mathbf{Y}$ and satisfies the diagram



Let $\bar{\psi} : H \to Y$ be any other morphism in $\mathbf{Y}$ such that $\bar{\psi} \circ j' = \varphi$. Thus $\bar{\varphi}|_H$ and $\bar{\psi}$ coincide on G and since the morphisms in $\mathbf{Y}$ are continuous, it follows that $\bar{\varphi}|_H = \bar{\psi}$. Thus (H, j') is a free object over X in $\mathbf{Y}$ and so there exists an isomorphism $\Phi : H \to \mathrm{F}_{\mathbf{X}}^{\mathbf{Y}}(X)$ in $\mathbf{Y}$ such that the following diagram commutes in $\mathbf{X}$.



We conclude that G is dense in $\mathrm{F}_{\mathbf{X}}^{\mathbf{Y}}(X)$. $\square$

With these preliminaries in place, the real work of this chapter can begin in the next section. We start by reviewing existence results for certain algebraic free objects. These will be used in the sequel as the foundation for the construction of other free objects.

3.2. Existence of algebraic free objects

The main goal of this section is to summarise all the major results in [**27**] which show that we can construct free objects $\left(\mathrm{F}_{\mathbf{X}}^{\mathbf{X}'}(X), j\right)$ for categories $\mathbf{X}$ and $\mathbf{X}'$ from Table 1 with $\mathbf{X} \supseteq \mathbf{X}'$. This will be done using methods from universal algebra where the principle object of study is *abstract algebras*. Although abstract algebras are usually just called ‘algebras’ in the universal algebra literature, we will have to make use the term ‘abstract algebras’ to distinguish this general notion from the particular example of a vector space equipped with an associative multiplication, which are usually called 'algebras' in analysis.

Along with the development of the theory we need to achieve the stated goal for this section, we will also explain how the algebraic categories in Table 1 may be viewed as categories of abstract algebras of particular *types*.

First, we define the notion of an *abstract algebra*. Let A be a set and $n \in \mathbb{N}$. A function $f : A^n \to A$ is called an *operation on A of rank n*. We write $\operatorname{rank}(f) = n$. In particular, operations of rank one and two are most often called *unary* and *binary* operations respectively.

We fix the notation $\mathbb{N}_0 \coloneqq \{0, 1, 2, \ldots\}$.

Definition 3.2.1 (Abstract algebras). Let $\mathcal{F}$ be a non-empty and possibly infinite set and let $\rho : \mathcal{F} \to \mathbb{N}_0$ be any function. The pair $(\mathcal{F}, \rho)$ is called a *type*. Let A be a non-empty set and $(\mathcal{F}, \rho)$ a type. Suppose that for every $f \in \mathcal{F}$, the following is given:

(I) If $\rho(f) = 0$, f^A is an element in A, which is called a *constant*.

(II) If $\rho(f) \geq 1$, $f^A : A^{\rho(f)} \to A$ is an *operation of rank* $\rho(f)$ on A.

Define the collection $\mathcal{F}^A := \{f^A : f \in \mathcal{F}\}$. The pair $(A, \mathcal{F}^A)$ is called an *abstract algebra of type* $(\mathcal{F}, \rho)$.

The set A is called the *universe* of the abstract algebra $(A, \mathcal{F}^A)$ and the set $\mathcal{F}^A$ is called the collection of *basic operations on A*. We view the constants as operations on A, which can also be thought of as operations of rank zero [**16**, p. 3].

As an example, let $(M, \cdot, e)$ be a monoid, i.e. a non-empty set equipped with an associative binary operation $\cdot : M \times M \to M$ with identity element e. Let $\mathcal{F} \coloneqq \{f_0, f_1\}$ be a set and $\rho : \mathcal{F} \to \mathbb{N}_0$ a function where $\rho(f_0) = 0$ and $\rho(f_1) = 2$. The pair $(\mathcal{F}, \rho)$ is then the type associated with all monoids. In particular, define $f_0^M := e$ and $f_1^M : M \times M \to M$ where $f_1^M(x, y) := x \cdot y$. The pair $(M, \{f_0^M, f_1^M\})$ is then the abstract algebra of type $(\mathcal{F}, \rho)$ associated with the monoid $(M, \cdot, e)$. However, a monoid is not just a non-empty set equipped with an arbitrary pair of rank zero and rank two operations, but also has to satisfy certain *identities*: For $x, y, z \in M$,

$$x \cdot e = x, \qquad e \cdot x = x,$$
$$x \cdot (y \cdot z) = (x \cdot y) \cdot z.$$

The formal definition of what it means for an abstract algebra to *satisfy an identity* will be given below (Definition 3.2.14) as we continue to develop the theory.

Definition 3.2.2 (Abstract algebra homomorphisms). Let $(A, \mathcal{F}^A)$ and $(B, \mathcal{F}^B)$ be abstract algebras of type $(\mathcal{F}, \rho)$ with $h : A \to B$ a function. We call h an *abstract algebra homomorphism of type* $(\mathcal{F}, \rho)$ if

(i) $h(f^A) = f^B$ for all constants in $f^A \in \mathcal{F}^A$,

(ii) $h\left(f^A(a_1, \ldots, a_{\rho(f)})\right) = f^B\left(h(a_1), \ldots, h(a_{\rho(f)})\right)$ for all operations $f^A \in \mathcal{F}^A$ with $\rho(f) \geq 1$.

With the above example of a monoid viewed as an abstract algebra in mind, it is not difficult to see that we can also construct a type associated with all vector spaces over $\mathbb{K}$: Informally speaking, this will consist of a constant 0 (zero vector), a binary map $\oplus$ (vector addition), a unary map $\ominus$ (additive inverse), and for every $\lambda \in \mathbb{K}$ a unary map m_λ (scalar multiplication by λ). One can similarly also associate a type with all (unital) algebras. Leaving aside the matter of what it means for an abstract algebra to satisfy an identity for the moment, it seems rather natural that the categories **VS**, **Alg**, and $\mathbf{Alg^1}$ can be envisaged as categories of abstract algebras, each of these categories with its own associated type $(\mathcal{F}, \rho)$. The morphisms in these categories can then also naturally be viewed as abstract algebra homomorphisms.

Indeed, in general, any type $(\mathcal{F}, \rho)$ will encode a particular category with abstract algebras of this type as objects and the associated abstract algebra homomorphisms as morphisms.

Definition 3.2.3. Let $(\mathcal{F}, \rho)$ be a type. The class of all abstract algebras of type $(\mathcal{F}, \rho)$ and the class of all abstract algebra homomorphisms of type $(\mathcal{F}, \rho)$ forms a subcategory of **Set**, which we denote as $\mathbf{AbsAlg}_{(\mathcal{F},\rho)}$.

Although it needs to be made precise, we have already made a tentative natural connection between the categories **VS**, **Alg**, and $\mathbf{Alg^1}$ and categories of abstract algebras associated with an appropriate type $(\mathcal{F}, \rho)$. However, this connection is not as immediately obvious for subcategories of **VL** from Table 1 since the objects in these categories are not only axiomatised by equalities (i.e. identities), but also *inequalities*, as well as the assumption of the existence of suprema and infima of two-element sets. Interestingly, it turns out that vector lattices and vector lattice algebras can in fact be represented as abstract algebras in the sense of Definition 3.2.1. We follow the exposition in [**27**, Section 4] for this, which in turn traces back to [**16**, Definition 1.7, Exercise 2.4.1].

Definition 3.2.4 ([**27**, Definition 4.1]). Let S be a non-empty set.

(i) Let $\leq$ be a partial order on S, then $(S, \leq)$ is called a *partially ordered lattice* if, for all $x, y \in S$, the supremum $x \vee y$ and the infimum $x \wedge y$ exist in S.

(ii) Suppose S has binary operations $\owedge$ and $\ovee$. Then the triple $(S, \owedge, \ovee)$ is called an *algebraic lattice* if, for all $x, y, z \in S$,

$$\begin{aligned} x \owedge (y \owedge z) &= (x \owedge y) \owedge z, & x \ovee (y \ovee z) &= (x \ovee y) \ovee z, \\ x \owedge x &= x, & x \ovee x &= x, \\ x \owedge y &= y \owedge x, & x \ovee y &= y \ovee x, \\ x \owedge (x \ovee y) &= x, & x \ovee (x \owedge y) &= x. \end{aligned}$$

We note that the operations $\owedge$ and $\ovee$ need not satisfy distributive properties.

LEMMA 3.2.5 ([**27**, Lemma 4.2]). *Let S be a non-empty set.*

(i) Let $\leq$ be a partial order on S making $(S,\leq)$ into a partially ordered lattice. For $x,y\in S$, define binary operations $\owedge$ and $\ovee$ where

$$x \owedge y := x \wedge y$$

and

$$x \ovee y := x \vee y.$$

Then the triple $(S,\ovee,\owedge)$ is an algebraic lattice.

(ii) Given binary operations $\owedge$ and $\ovee$ on S such that $(S,\owedge,\ovee)$ forms an algebraic lattice. Then define a relation $\leq$ on S where $x\leq y$ if and only if

$$x \owedge y = x.$$

Then $(S,\leq)$ is a partially ordered lattice. Moreover, for $x,y\in S$ the supremum and infimum of the set $\{x,y\}$ with respect to the partial order $\leq$ exists and we have

$$x \wedge y = x \owedge y$$

and

$$x \vee y = x \ovee y.$$

The notion of a partially ordered lattice was described in Definition 3.2.4 as a triple $(S,\owedge,\ovee)$ consisting of a set and two binary operations. This can be made more precise using the language of abstract algebras by encoding the type $(\mathcal{F},\rho)$ of all partially ordered lattices, as was done for monoids above.

In the next result, we will describe a positive unital vector lattice algebra as a set equipped with a collection of operations satisfying a long list of identities. We stick to this more informal description over the formalism of Definition 3.2.1 for the sake of readability. It is in any case routine (but tedious) to encode positive unital vector lattice algebras as abstract algebras of a certain type.

We will see in the results below, however, that the formalism of abstract algebras is useful for the formulation of elegant results and that this formalism is not just a rather esoteric way of describing the well-liked and well-understood notion of 'a set equipped with operations and identities' that is seen throughout mathematics.

LEMMA 3.2.6 ([**27**, Lemma 4.4 and p. 124]). *Let A be a set equipped with (not necessarily different) constants 0 and 1, a binary map $\oplus$, a unary map $\ominus$, a unary map m_λ for every $\lambda\in\mathbb{R}$, a binary map $\odot$, and binary maps $\owedge$ and $\ovee$. Assume that the following identities are satisfied:*

(1) $(x\oplus y)\oplus z = x\oplus(y\oplus z)$ for all $x,y,z\in A$;

(2) $x\oplus 0 = x$ for all $x\in A$;

(3) $x\oplus(\ominus x) = 0$ for all $x\in A$;

(4) $x\oplus y = y\oplus x$ for all $x,y\in A$;

(5) $m_\lambda(x \oplus y) = m_\lambda(x) \oplus m_\lambda(y)$ *for all* $\lambda \in \mathbb{R}$ *and* $x, y \in A$;

(6) $m_{\lambda+\mu}(x) = m_\lambda(x) \oplus m_\mu(x)$ *for all* $\lambda, \mu \in \mathbb{R}$ *and* $x \in A$;

(7) $m_{\lambda\mu}(x)) = m_\lambda(m_\mu(x))$ *for all* $\lambda, \mu \in \mathbb{R}$ *and* $x \in A$;

(8) $m_1(x) = x$ *for all* $x \in A$;

(9) $(x \odot y) \odot z = x \odot (y \odot z)$ *for all* $x, y, z \in A$;

(10) $x \odot (y \oplus z) = (x \odot y) \oplus (x \odot z)$ *for all* $x, y, z \in A$;

(11) $(x \oplus y) \odot z = (x \odot z) \oplus (y \odot z)$ *for all* $x, y, z \in A$;

(12) $m_\lambda(x \odot y) = m_\lambda(x) \odot y = x \odot m_\lambda(y)$ *for all* $\lambda \in \mathbb{R}$ *and* $x, y \in A$;

(13) $1 \odot x = x \odot 1 = x$ *for all* $x \in A$;

(14) $x \owedge (y \owedge z) = (x \owedge y) \owedge z$ *and* $x \ovee (y \ovee z) = (x \ovee y) \ovee z$ *for all* $x, y, z \in A$;

(15) $x \owedge x = x$ *and* $x \ovee x = x$ *for all* $x \in A$;

(16) $x \owedge y = y \owedge x$ *and* $x \ovee y = y \ovee x$ *for all* $x, y \in A$;

(17) $x \owedge (x \ovee y) = x$ *and* $x \ovee (x \owedge y) = x$ *for all* $x, y \in A$;

(18) $x \oplus (y \owedge z) = (x \oplus y) \owedge (x \oplus z)$ *for all* $x, y, z \in A$;

(19) $m_\lambda(0 \owedge x) = 0 \owedge (m_\lambda(x))$ *for all* $\lambda \in \mathbb{R}^+$ *and* $x \in A$;

(20) $0 \owedge ((x \owedge (\ominus x)) \odot (y \owedge (\ominus y))) = 0$ *for all* $x, y \in A$;

(21) $0 \owedge 1 = 0$.

Define

(a) $x + y := x \oplus y$ *for* $x, y \in A$;

(b) $\lambda x := m_\lambda(x)$ *for* $\lambda \in \mathbb{R}$ *and* $x \in A$;

(c) $xy := x \odot y$ *for* $x, y \in A$.

Equipping the set A *with the operations in (a)-(c) makes* A *into a unital algebra over* $\mathbb{R}$ *with zero vector* 0 *and multiplicative identity* 1. *Further, define a relation* $\leq$ *on* A *where* $x \leq y$ *if and only if* $x \owedge y = x$. *Then* $(A, \leq)$ *forms a partially ordered lattice where for* $x, y \in A$, *we have* $x \wedge y = x \owedge y$ *and* $x \vee y = x \ovee y$. *Finally, the partial order* $\leq$ *together with the operations defined in (a)-(c) make* A *into a positive unital vector lattice algebra with zero vector* 0 *and multiplicative identity* 1.

The operations (a)-(c) along with the constants 0 and 1 give us an informal description of the type associated with the category $\mathbf{VLA^{1+}}$. By making appropriate omissions to this list, we can obtain an informal description of the type associated with any other algebraic category $\mathbf{X}$ in Table 1. Further, appropriate omissions from the above list of 21 identities will be used in the sequel to describe the algebraic categories in Table 1 in the language of abstract algebras.

We now return to the further development of the basic universal algebra theory that we need. We have defined the notion of an abstract algebra as well as the notion of a structure-preserving map between abstract algebras, the next concept one would want to consider is that of a *subobject*. This will be given in two different flavours, one where the subobject is considered as an entity within a larger whole, and one where the subobject is considered as a separate entity in its own right. It will often be more convenient to make use of one formulation over the other.

DEFINITION 3.2.7 (Subuniverses and abstract subalgebras). Let $(A, \mathcal{F}^A)$ be an abstract algebra of type $(\mathcal{F}, \rho)$. A subset $B \subseteq A$ is called a *subuniverse* of the abstract algebra $(A, \mathcal{F}^A)$ if B contains all the constants in $\mathcal{F}^A$ and if for every $n \in \mathbb{N}$ and every $f^A \in \mathcal{F}^A$ with $\operatorname{rank}(f^A) = n$ we have that $b_1, \dots, b_n \in B$ implies that $f^A(b_1, \dots, b_n) \in B$.

Further, if $B \subseteq A$ is a subuniverse of $(A, \mathcal{F}^A)$, define

$$\mathcal{F}^B := \left\{ f^A \ : \ \rho(f) = 0 \right\} \cup \left\{ f^A\big|_{B^{\rho(f)}} \ : \ \rho(f) \geq 1 \right\}.$$

Then the pair $(B, \mathcal{F}^B)$ forms an abstract algebra of type $(\mathcal{F}, \rho)$ and is called an *abstract subalgebra of* $(A, \mathcal{F}^A)$.

Let $(A, \mathcal{F}^A)$ be an abstract algebra of type $(\mathcal{F}, \rho)$. The above definitions imply that every abstract subalgebra of $(A, \mathcal{F}^A)$ forms a non-empty subuniverse of $(A, \mathcal{F}^A)$ and every non-empty subuniverse of $(A, \mathcal{F}^A)$ gives rise to an abstract subalgebra. Another important concept is that of a subuniverse *generated* by a subset.

DEFINITION 3.2.8 (Generated subuniverse). Let $(A, \mathcal{F}^A)$ be an abstract algebra of type $(\mathcal{F}, \rho)$ with X a subset of A. The set of all subuniverses of the abstract algebra $(A, \mathcal{F}^A)$ is denoted by $\operatorname{Sub}(A)$. The *subuniverse generated by* X is defined as

$$\operatorname{Sg}^A(X) := \bigcap \{ U \in \operatorname{Sub}(A) \ : \ X \subseteq U \}.$$

In particular, if $A = \operatorname{Sg}^A(X)$, we say that the abstract algebra $(A, \mathcal{F}^A)$ is *generated* by the set X.

The idea of the subuniverse generated by a subset will play a very important role in the sequel.

PROPOSITION 3.2.9. *Let $(A, \mathcal{F}^A)$ be an abstract algebra of type $(\mathcal{F}, \rho)$ with $X \subseteq A$. The pair $\left(Sg^A(X), \mathcal{F}^{Sg^A(X)} \right)$ is the smallest abstract subalgebra of $(A, \mathcal{F}^A)$ that contains X.*

The following result gives a useful characterisation of the subuniverse generated by a set.

THEOREM 3.2.10 ([**16**, Theorem 1.14]). *Let $(A, \mathcal{F}^A)$ be an abstract algebra of type $(\mathcal{F}, \rho)$ with $X \subseteq A$. Define a sequence of subsets $(X_n)_{n \in \mathbb{N}_0}$ of A by recursion:*

$$X_0 := X \cup \left\{ f^A : \rho(f) = 0 \right\},$$
$$X_{n+1} := X_n \cup \left\{ f^A \left(a_1, \dots, a_{\rho(f)} \right) \ : \ a_1, \dots, a_{\rho(f)} \in X_n, f \in \mathcal{F} \text{ st. } \rho(f) \geq 1 \right\}.$$

Then $Sg^A(X) = \bigcup_{n \in \mathbb{N}_0} X_n$.

PROPOSITION 3.2.11. *Let $(A,\mathcal{F}^A)$ and $(B,\mathcal{F}^B)$ be abstract algebras of type $(\mathcal{F},\rho)$ with $h_1 : A \to B$ and $h_2 : A \to B$ abstract algebra homomorphisms. If $X \subseteq A$ is such that $h_1|_X = h_2|_X$, then $h_1|_{Sg^A(X)} = h_2|_{Sg^A(X)}$.*

PROOF. The result follows directly from Proposition 3.2.9 and the fact that the set of elements in A on which h_1 and h_2 coincide forms subuniverse of $(A,\mathcal{F}^A)$ which contains X. □

We can take the idea of generation given above one step further by defining the *abstract algebra* of a given type *generated over a set*. Given a non-empty set X, a *word* on X is defined as a finite string of elements from X. For example, if $a, b \in X$ then '*aaa*' and '*ababbb*' are words on X.

DEFINITION 3.2.12 (Abstract term algebras). Let $(\mathcal{F},\rho)$ be a type and let S be a non-empty set that is disjoint from $\mathcal{F}$. We define a sequence of sets $(T_n(S))_{n\in\mathbb{N}_0}$ consisting of words on $S \cup \mathcal{F}$ by recursion:

$$T_0(S) \coloneqq S \cup \{f : \rho(f) = 0\},$$
$$T_{n+1}(S) \coloneqq T_n(S) \cup \left\{ft_1 \dots t_{\rho(f)} \ : \ t_1,\dots,t_{\rho(f)} \in T_n(S), f \in \mathcal{F} \text{ st. } \rho(f) \geq 1\right\}.$$

Define $T_{(\mathcal{F},\rho)}(S) \coloneqq \bigcup_{n\in\mathbb{N}_0} T_n(S)$ which we call the *terms of type* $(\mathcal{F},\rho)$ *over* S.

For every $f \in \mathcal{F}$, define the following:

(i) If $\rho(f) = 0$, define $f^{T_{(\mathcal{F},\rho)}(S)} \coloneqq f$.

(ii) If $\rho(f) \geq 1$, define $f^{T_{(\mathcal{F},\rho)}(S)}\left(t_1,\dots,t_{\rho(f)}\right) \coloneqq ft_1 \dots t_{\rho(f)}$ for $t_1,\dots,t_{\rho(f)} \in T_{(\mathcal{F},\rho)}(S)$.

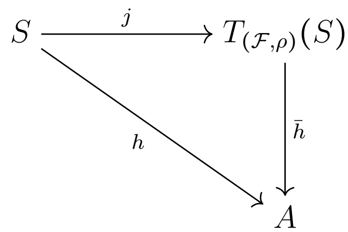
Define $\mathcal{F}^{T_{(\mathcal{F},\rho)}(S)} \coloneqq \left\{f^{T_{(\mathcal{F},\rho)}(S)} : f \in \mathcal{F}\right\}$, then the pair $\left(T_{(\mathcal{F},\rho)}(S), \mathcal{F}^{T_{(\mathcal{F},\rho)}(S)}\right)$ forms an abstract algebra of type $(\mathcal{F},\rho)$, which we call the *abstract term algebra* of type $(\mathcal{F},\rho)$ over the set S.

Let $j : S \to T_{(\mathcal{F},\rho)}(S)$ denote the inclusion map. It follows directly from Theorem 3.2.10 and Proposition 3.2.9 that $\left(T_{(\mathcal{F},\rho)}(S), \mathcal{F}^{T_{(\mathcal{F},\rho)}(S)}\right)$ is the smallest abstract subalgebra of itself that contains $j[S]$.

The idea behind the construction of the abstract term algebra of a given type over a set can be viewed as a more refined version of the construction of a *free monoid* over a set S (see [**12**, § 1.7]). The free monoid over a set S consists of words on S equipped with concatenation as an associative binary operation. Indeed, the abstract term algebra of type $(\mathcal{F},\rho)$ over a set S turns out to be nothing but the free abstract algebra of type $(\mathcal{F},\rho)$ over a set S.

THEOREM 3.2.13 ([**16**, Theorem 4.21]). *Let $(\mathcal{F},\rho)$ be a type and let S be a non-empty set disjoint from $\mathcal{F}$. For every abstract algebra $(A,\mathcal{F}^A)$ of type $(\mathcal{F},\rho)$ and every function $h : S \to A$, there exists a unique abstract algebra homomorphism*

$\bar{h} : T_{(\mathcal{F},\rho)}(S) \to A$ such that the following diagram commutes in **Set**.



That is to say, the pair $\left(T_{(\mathcal{F},\rho)}(S), j\right)$ is the free object over S of $\mathbf{AbsAlg}_{(\mathcal{F},\rho)}$. *Furthermore, the abstract algebra $\left(T_{(\mathcal{F},\rho)}(S), \mathcal{F}^{T_{(\mathcal{F},\rho)}(S)}\right)$ is generated by the subset $j[S]$.*

Our previous result is not yet enough for us to construct free objects in an algebraic category from Table 1 over a set. This is because we have thus far only formalised which operations are carried by an abstract algebra, which are encoded in its type. At this stage a free abstract algebra over a set will just be a set equipped with operations with no further information about how these operations interact with each other. We now turn to the missing ingredient: *identities*.

We will illustrate the idea of what it means for an abstract algebra to *satisfy an identity* by means of the following example: Consider an abstract algebra $(A, \mathcal{F}^A)$ of type $(\mathcal{F}, \rho)$. Let $f \in \mathcal{F}$ with $\rho(f) = 2$, thus we have a binary operation $f^A : A \times A \to A$, which we represent by $\oplus$ where $x \oplus y := f^A(x, y)$. We wish to express that the operation $\oplus$ is associative, i.e. for $x_1, x_2, x_3 \in A$,

$$x_1 \oplus (x_2 \oplus x_3) = (x_1 \oplus x_2) \oplus x_3.$$

Let S_ω denote a fixed countable set. By its construction, the abstract term algebra $T_{(\mathcal{F},\rho)}(S_\omega)$ contains all possible grammatical combinations of elements in S_ω with operations from $(\mathcal{F}, \rho)$. In particular, for $s_1, s_2, s_3 \in S_\omega$ we have the terms $fs_1fs_2s_3, ffs_1s_2s_3 \in T_{(\mathcal{F},\rho)}(S_\omega)$. The pair $(fs_1fs_2s_3, ffs_1s_2s_3) \in T_{(\mathcal{F},\rho)}(S_\omega) \times T_{(\mathcal{F},\rho)}(S_\omega)$ will be used to encode our prescription that the operation given by $f \in \mathcal{F}$ in an abstract algebra of type $(\mathcal{F}, \rho)$ needs to be associative. Similarly, for $g \in \mathcal{F}$ with $\rho(g) = 2$, the pair $(gs_1s_2, gs_2s_1) \in T_{(\mathcal{F},\rho)}(S_\omega) \times T_{(\mathcal{F},\rho)}(S_\omega)$ can be used to encode the prescription that the operation given by $g \in \mathcal{F}$ in an abstract algebra of type $(\mathcal{F}, \rho)$ needs to be commutative. The pairs $(t_1, t_2) \in T_{(\mathcal{F},\rho)}(S_\omega) \times T_{(\mathcal{F},\rho)}(S_\omega)$ will be called *identities of type* $(\mathcal{F}, \rho)$ and the set $T_{(\mathcal{F},\rho)}(S_\omega) \times T_{(\mathcal{F},\rho)}(S_\omega)$ then contains the collection of *all possible identities* for an abstract algebra of type $(\mathcal{F}, \rho)$. We denote a particular identity $(t_1, t_2) \in T_{(\mathcal{F},\rho)}(S_\omega) \times T_{(\mathcal{F},\rho)}(S_\omega)$ by writing $t_1 \approx t_2$.

How do we use this set of all possible identities of type $(\mathcal{F}, \rho)$ to express that the operation f^A of the particular abstract algebra $(A, \mathcal{F}^A)$ of type $(\mathcal{F}, \rho)$ is associative? This is given to us by the universal property of Theorem 3.2.13! Fix $s_1, s_2, s_3 \in S_\omega$. Then for every $x_1, x_2, x_3 \in A$ there is a map $h : S_\omega \to A$ where $h(s_i) = x_i$. By Theorem 3.2.13, there exists a unique abstract algebra homomorphism $\bar{h} : T_{(\mathcal{F},\rho)}(S_\omega) \to A$ extending h. Fixing the terms $t := fs_1fs_2s_3$ and $t' := ffs_1s_2s_3$ in $T_{(\mathcal{F},\rho)}(S_\omega)$, it follows that the operation f^A in $(A, \mathcal{F}^A)$ will satisfy associativity for a choice of three elements $x_1, x_2, x_3 \in A$ if and only if $\bar{h}(t) = \bar{h}(t')$ since

$$\bar{h}(t) = \bar{h}\left(f^A\left(s_1, f^A\left(s_2, s_3\right)\right)\right) = f^A\left(h(s_1), f^A\left(h(s_2), h(s_3)\right)\right) = (x_1 \oplus x_2) \oplus x_3.$$

and

$$\bar{h}(t') = \bar{h}\left(f^A\left(f^A\left(s_1, s_2\right), s_3\right)\right) = f^A\left(f^A\left(h(s_1), h(s_2)\right), h(s_3)\right) = (x_1 \oplus x_2) \oplus x_3.$$

This leads us to the following definition.

Definition 3.2.14. Let $(A, \mathcal{F}^A)$ be an abstract algebra of type $(\mathcal{F}, \rho)$. Consider terms $t_1, t_2 \in T_{(\mathcal{F},\rho)}(S_\omega)$. The abstract algebra $(A, \mathcal{F}^A)$ *satisfies* the identity $t_1 \approx t_2$ if $h(t_1) = h(t_2)$ for every abstract algebra homomorphism $h : T_{(\mathcal{F},\rho)}(S_\omega) \to A$.

For a collection of identities $\Sigma \subseteq T_{(\mathcal{F},\rho)}(S_\omega) \times T_{(\mathcal{F},\rho)}(S_\omega)$, we say A *satisfies* Σ when A satisfies $t_1 \approx t_2$ for every $(t_1, t_2) \in \Sigma$.

Further, if we consider any $\Sigma \subseteq T_{(\mathcal{F},\rho)}(S_\omega) \times T_{(\mathcal{F},\rho)}(S_\omega)$, then the collection of all abstract algebras $(A, \mathcal{F}^A)$ of type $(\mathcal{F}, \rho)$ which satisfy Σ is called the *equational class defined by* Σ.

Let $(\mathcal{F}, \rho)$ be a type and $\Sigma \subseteq T_{(\mathcal{F},\rho)}(S_\omega) \times T_{(\mathcal{F},\rho)}(S_\omega)$ a collection of identities. Define the category $\mathbf{AbsAlg}_{(\mathcal{F},\rho);\Sigma}$ whose objects are the abstract algebras in the equational class defined by Σ and whose morphisms are abstract algebra homomorphisms of type $(\mathcal{F}, \rho)$. It is clear that $\mathbf{AbsAlg}_{(\mathcal{F},\rho);\Sigma}$ forms a subcategory of $\mathbf{AbsAlg}_{(\mathcal{F},\rho)}$.

As one might expect, one can also force an abstract algebra to satisfy a particular identity by passing to a *quotient* of some kind. To formulate this precisely, we need the following definitions.

Definition 3.2.15 (Congruence relations). Let $(A, \mathcal{F}^A)$ be an abstract algebra of type $(\mathcal{F}, \rho)$. A relation $\theta \subseteq A \times A$ is called a *congruence relation on* A if θ is both an equivalence relation and satisfies the *substitution property*: For every $f \in \mathcal{F}$ with $\rho(f) \geq 1$ and for $x_1, \ldots, x_{\rho(f)} \in A$ and $y_1, \ldots, y_{\rho(f)} \in A$ such that $x_i \,\theta\, y_i$ for $1 \leq i \leq \rho(f)$ we have that

$$f^A\left(x_1, \ldots, x_{\rho(f)}\right) \theta\, f^A\left(y_1, \ldots, y_{\rho(f)}\right).$$

Definition 3.2.16 (Abstract quotient algebras). Let $(A, \mathcal{F}^A)$ be an abstract algebra of type $(\mathcal{F}, \rho)$ and θ a congruence relation on A. Denote by A/θ the collection of equivalence classes in A with respect to θ and let $q_\theta : A \to A/\theta$ denote the canonical map. For every $f \in \mathcal{F}$, define the following:

(i) If $\rho(f) = 0$, define $f^{A/\theta} \coloneqq q_\theta\left(f^A\right)$.

(ii) If $\rho(f) \geq 1$, define $f^{A/\theta}\left(q_\theta(x_1), \ldots, q_\theta(x_{\rho(f)})\right) \coloneqq q_\theta\left(f^A\left(x_1, \ldots, x_{\rho(f)}\right)\right)$ for $x_1, \ldots, x_{\rho(f)} \in A$.

Define $\mathcal{F}^{A/\theta} \coloneqq \{f^{A/\theta} : f \in \mathcal{F}\}$, then the pair $(A/\theta, \mathcal{F}^{A/\theta})$ forms an abstract algebra of type $(\mathcal{F}, \rho)$ which we call an *abstract quotient algebra*.

It follows precisely from the fact that θ satisfies the substitution property that the induced basic operations on A/θ from $(A, \mathcal{F}^A)$ are well-defined. The definition of the basic operations on A/θ also immediately implies that the canonical map $q_\theta : A \to A/\theta$ is an abstract algebra homomorphism.

We will use the next two results in the formulation of our main theorem for this section.

PROPOSITION 3.2.17. *Let $(A, \mathcal{F}^A)$ be an abstract algebra of type $(\mathcal{F}, \rho)$ and θ a congruence relation on A. If $X \subseteq A$ and $A = Sg^A(X)$, then $A/\theta = Sg^{A/\theta}(q_\theta[X])$.*

PROOF. By Theorem 3.2.10, we have that $A = \mathrm{Sg}^A(X) = \bigcup_{n\in\mathbb{N}_0} X_n$ where the sequence of sets $(X_n)_{n\in\mathbb{N}_0}$ is defined. Then

$$A/\theta = q_\theta[A] = q_\theta\left[\bigcup_{n\in\mathbb{N}_0} X_n\right] = \bigcup_{n\in\mathbb{N}_0} q_\theta[X_n].$$

Define the following sequence of subsets $(Y_n)_{n\in\mathbb{N}_0}$ of A/θ by recursion:

$$Y_0 \coloneqq q_\theta[X] \cup \left\{ f^{A/\theta} : \rho(f) = 0 \right\},$$
$$Y_{n+1} \coloneqq Y_n \cup \left\{ f^{A/\theta}\left(y_1, \dots, y_{\rho(f)}\right) \,:\, y_1, \dots, y_{\rho(f)} \in Y_n, f \in \mathcal{F} \text{ st. } \rho(f) \geq 1 \right\}.$$

By Theorem 3.2.10, we have that $\mathrm{Sg}^{A/\theta}(q_\theta[X]) = \bigcup_{n\in\mathbb{N}_0} Y_n$. The desired result will follow if we can verify that $Y_n = q_\theta[X_n]$. We prove this by induction. It follows directly from the definition of the basic operations on A/θ in Definition 3.2.16 that $Y_0 = q_\theta[X_0]$. Fix any $n \in \mathbb{N}$ and assume that $Y_n = q_\theta[X_n]$. Since

$$X_{n+1} \coloneqq X_n \cup \left\{ f^A\left(a_1, \dots, a_{\rho(f)}\right) \,:\, a_1, \dots, a_{\rho(f)} \in X_n, f \in \mathcal{F} \text{ st. } \rho(f) \geq 1 \right\}$$

from Definition 3.2.16 we have that $Y_{n+1} = q_\theta[X_{n+1}]$. Thus $A/\theta = \mathrm{Sg}^{A/\theta}(q_\theta[X])$. □

LEMMA 3.2.18 ([**27**, Lemma 5.2]). *Let $(A, \mathcal{F}^A)$ be an abstract algebra of type $(\mathcal{F}, \rho)$ and let θ be a congruence relation on A. The abstract algebra $(A/\theta, \mathcal{F}^{A/\theta})$ satisfies $t_1 \approx t_2$ if and only if $(h(t_1), h(t_2)) \in \theta$ for all abstract algebra homomorphisms $h : T_{(\mathcal{F},\rho)}(S_\omega) \to A$.*

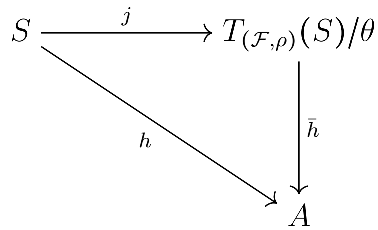
Our development of the theory above leads us next to the main result of this section, which states in precise language that for every type $(\mathcal{F}, \rho)$ and every set of identities $\Sigma \subseteq T_{(\mathcal{F},\rho)}(S_\omega) \times T_{(\mathcal{F},\rho)}(S_\omega)$ and every non-empty set S, there exists a free object over S in the category $\mathbf{AbsAlg}_{(\mathcal{F},\rho);\Sigma}$. This result is found in [**45**, Theorem 2.10, p. 82] and a proof is also given in [**27**, Theorem 5.4].

THEOREM 3.2.19. *Let $(\mathcal{F}, \rho)$ be a type and take $\Sigma \subseteq T_{(\mathcal{F},\rho)}(S_\omega) \times T_{(\mathcal{F},\rho)}(S_\omega)$. Let S be a non-empty set and consider the smallest congruence relation θ on $T_{(\mathcal{F},\rho)}(S)$ containing the pairs $(h(t_1), h(t_2))$ for all $(t_1, t_2) \in \Sigma$ and all abstract algebra homomorphisms $h : T_{(\mathcal{F},\rho)}(S_\omega) \to T_{(\mathcal{F},\rho)}(S)$. Then the following holds:*

(I) $T_{(\mathcal{F},\rho)}(S)/\theta \in \mathbf{AbsAlg}_{(\mathcal{F},\rho);\Sigma}$.

(II) Let $\iota : S \to T_{(\mathcal{F},\rho)}(S)$ denote the inclusion map and $q_\theta : T_{(\mathcal{F},\rho)}(S) \to T_{(\mathcal{F},\rho)}(S)/\theta$ the quotient map. Define $j = q_\theta \circ \iota$. The abstract algebra $T_{(\mathcal{F},\rho)}(S)/\theta$ is generated by the subset $j[S]$.

(III) For every $(A, \mathcal{F}^A) \in \mathbf{AbsAlg}_{(\mathcal{F},\rho);\Sigma}$ *and every function* $h : S \to A$*, there exists a unique abstract algebra homomorphism* $\bar{h} : T_{(\mathcal{F},\rho)}(S)/\theta \to A$ *such that the following diagram commutes in* **Set***.*



That is to say, the pair $\left(T_{(\mathcal{F},\rho)}(S)/\theta, j\right)$ *is the free object over* S *of* $\mathbf{AbsAlg}_{(\mathcal{F},\rho);\Sigma}$.

Using the content in Lemma 3.2.6 as a guidebook, the following can be done: Let $\mathbf{X}$ be any algebraic category in Table 1. We can encode the operations defined on the objects in $\mathbf{X}$ into a type $(\mathcal{F}, \rho)$ (if the objects in $\mathbf{X}$ have lattice structure, we use Lemma 3.2.5 to encode the lattice structure using the operations $\owedge$ and $\ovee$). Consulting the list of 21 identities in Lemma 3.2.6, we choose those identities that are appropriate to the category $\mathbf{X}$ to generate a formal set of identities $\Sigma \subseteq T_{(\mathcal{F},\rho)}(S_\omega) \times T_{(\mathcal{F},\rho)}(S_\omega)$. This gives us an associated category of abstract algebras $\mathbf{AbsAlg}_{(\mathcal{F},\rho);\Sigma}$. By following the example of Lemma 3.2.5, we can construct a bijection between the objects in $\mathbf{X}$ and the objects in $\mathbf{AbsAlg}_{(\mathcal{F},\rho);\Sigma}$ and this bijection will also preserve the morphisms in $\mathbf{X}$ and the morphisms in $\mathbf{AbsAlg}_{(\mathcal{F},\rho);\Sigma}$. Thus the categories $\mathbf{X}$ and $\mathbf{AbsAlg}_{(\mathcal{F},\rho);\Sigma}$ are *isomorphic*. By viewing each algebraic category $\mathbf{X}$ in Table 1 in this way, an application of Theorem 3.2.19 will give us the following free objects.

COROLLARY 3.2.20 ([**27**, Theorem 6.2]). *Let* S *be a set. The following free objects exist.*

(i) $(\mathrm{F}_{\mathbf{Set}}^{\mathbf{VS}}(S), j)$,

(ii) $(\mathrm{F}_{\mathbf{Set}}^{\mathbf{VL}}(S), j)$,

(iii) $\left(\mathrm{F}_{\mathbf{Set}}^{\mathbf{Alg}}(S), j\right)$,

(iv) $\left(\mathrm{F}_{\mathbf{Set}}^{\mathbf{Alg^1}}(S), j\right)$,

(v) $(\mathrm{F}_{\mathbf{Set}}^{\mathbf{VLA}}(S), j)$,

(vi) $\left(\mathrm{F}_{\mathbf{Set}}^{\mathbf{VLA^1}}(S), j\right)$,

(vii) $\left(\mathrm{F}_{\mathbf{Set}}^{\mathbf{VLA^{1+}}}(S), j\right)$.

Denote by F *any of the objects in (i)-(vii), then the subset* $j[S]$ *generates* F.

The free objects listed above are, of course, not new findings. The free vector space over a set is described in Section 1.1.2 and the free (unital) algebra over a set is obtained directly by taking the non-commutative version of the construction $(\mathbb{K}[S], j)$ in Section 1.1.2. For the free vector lattice over a set, both the direct construction and the universal algebra approach to existence has been known for a long time (see [**13**] and [**18**]). There is, however, no direct construction of the free ((positive) unital) vector lattice algebra over a set and the existence of these were first recorded in [**27**].

We have thus far proven the existence of free objects for the pairs $(\mathbf{Set}, \mathbf{X})$ where $\mathbf{X}$ is an algebraic category from Table 1. There are a further 18 non-trivial pairs of categories $(\mathbf{X}, \mathbf{Y})$ with $\mathbf{X} \supseteq \mathbf{Y}$ for which we can construct a free object. It turns out that it is not difficult to derive the existence of a free object between these pairs of categories $(\mathbf{X}, \mathbf{Y})$ by using the existence of a free object between $(\mathbf{Set}, \mathbf{Y})$. More details and examples are supplied in [**27**, Section 6], but the idea behind the general approach is illustrated well enough by means of an example. We will prove the existence of a free unital algebra over a vector space V. The free algebra over a vector space is usually called the *tensor algebra* of a vector space (see [**23**, Chapter III, Section 5]). However, our approach will prove the existence of $\mathrm{F}_{\mathbf{VS}}^{\mathbf{Alg^1}}(V)$ for a vector space V from the existence of $\mathrm{F}_{\mathbf{Set}}^{\mathbf{Alg^1}}(|V|)$ where $|V|$ denotes the underlying set of the vector space V.

Proposition 3.2.21. *Let V be a vector space. The free object $\left(\mathrm{F}_{\mathbf{VS}}^{\mathbf{Alg^1}}(V), j\right)$ exists.*

Proof. Let V be a vector space and denote by $|V|$ the underlying set of V without any operations or identities. We have already established in Corollary 3.2.20 that the free object $\left(\mathrm{F}_{\mathbf{Set}}^{\mathbf{Alg^1}}(|V|), j\right)$ exists. That is, for every unital algebra R^1 and every set map $\psi : |V| \to R^1$ there exists a unique unital algebra homomorphism $\bar{\psi} : \mathrm{F}_{\mathbf{Set}}^{\mathbf{Alg^1}}(|V|) \to R^1$ such that the following diagram commutes

$$\begin{array}{ccc} |V| & \xrightarrow{\ j\ } & \mathrm{F}_{\mathbf{Set}}^{\mathbf{Alg^1}}(|V|) \\ & \searrow^{\varphi} & \downarrow{\scriptstyle \bar{\varphi}} \\ & & R^1 \end{array}$$

For every $\lambda \in \mathbb{R}$ and $x, y \in V$, consider elements of the form

$$j(x+y) - (j(x) + j(y)),$$
$$j(\lambda x) - \lambda j(x).$$

Denote the above collection of elements by L and let I be the two-sided algebra ideal generated by L. The quotient $\mathrm{F}_{\mathbf{Set}}^{\mathbf{Alg^1}}(|V|)/I$ forms a unital algebra and the quotient map $q_I : \mathrm{F}_{\mathbf{Set}}^{\mathbf{Alg^1}}(|V|) \to \mathrm{F}_{\mathbf{Set}}^{\mathbf{Alg^1}}(|V|)/I$ is automatically a unital algebra homomorphism thanks to the definition of the operations on the quotient (Definition 3.2.16). The definition of the two-sided algebra ideal guarantees that the map $q_I \circ j : V \to \mathrm{F}_{\mathbf{Set}}^{\mathbf{Alg^1}}(|V|)/I$ is linear: For $x, y \in V$, we have

$$q_I \circ j\,(x+y) = q_I\,(j(x) + j(y)) = q_I \circ j(x) + q_I \circ j(y).$$

The analogous expression for scalar multiplication can also easily be verified. We verify that the pair $\left(\mathrm{F}_{\mathbf{Set}}^{\mathbf{Alg^1}}(|V|)/I, q_I \circ j\right)$ has the necessary universal property: Fix a unital algebra R^1 and fix a linear map $\varphi : V \to R^1$ and let $\bar{\varphi} : \mathrm{F}_{\mathbf{Set}}^{\mathbf{Alg^1}}(|V|) \to R^1$ be the unique extension of φ when φ is considered as a set map. For the element $j(x+y) - (j(x) + j(y)) \in L$, we have

$$\begin{aligned} &\bar{\varphi}(j(x+y) - (j(x)+j(y))) \\ =\ &\bar{\varphi}(j(x+y)) - \bar{\varphi}(j(x)) + \bar{\varphi}(j(y)) \\ =\ &\varphi(x+y) - \varphi(x) - \varphi(y) = 0. \end{aligned}$$

Similarly, one can verify that $\bar{\varphi}$ vanishes on elements of the form $j(\lambda x) - \lambda j(x) \in L$. Since L generates I and $\bar{\varphi}$ vanishes on L we conclude that $I \subseteq \ker \bar{\varphi}$. Now, define the map $\bar{\bar{\varphi}} : \mathrm{F}_{\mathbf{Set}}^{\mathbf{Alg^1}}(|V|)/I \to R^1$ where $\bar{\bar{\varphi}}([a]) = \bar{\varphi}(a)$. Since $I \subseteq \ker \bar{\varphi}$, the map $\bar{\bar{\varphi}}$ is well-defined and is a unital algebra homomorphism. The definition of $\bar{\bar{\varphi}}$ makes it clear that $\bar{\bar{\varphi}} \circ q_I = \bar{\varphi}$ and further $\bar{\bar{\varphi}}$ is unique with respect to this property since for any unital algebra homomorphism $\xi : \mathrm{F}_{\mathbf{Set}}^{\mathbf{Alg^1}}(|V|)/I \to R^1$ satisfying $\xi \circ q_I = \bar{\varphi}$, we have that

$$\xi([a]) = \xi \circ q_I(a) = \bar{\varphi}(a) = \bar{\bar{\varphi}} \circ q_I(a) = \bar{\bar{\varphi}}([a])$$

for $[a] \in \mathrm{F}_{\mathbf{Set}}^{\mathbf{Alg^1}}(|V|)/I$. Thus $\bar{\bar{\varphi}} = \xi$. As a result,

$$\bar{\bar{\varphi}} \circ (q_I \circ j) = (\bar{\bar{\varphi}} \circ q_I) \circ j = \bar{\varphi} \circ j = \varphi.$$

Let $\psi : \mathrm{F}_{\mathbf{Set}}^{\mathbf{Alg^1}}(|V|)/I \to R^1$ be any unital algebra homomorphism such that $\psi \circ (q_I \circ j) = \varphi$. Thus $(\psi \circ q_I) \circ j = \varphi$ and the uniqueness of the map $\bar{\varphi} : \mathrm{F}_{\mathbf{Set}}^{\mathbf{Alg^1}}(|V|) \to R^1$ implies that $\psi \circ q_I = \bar{\varphi}$ and the uniqueness of $\bar{\bar{\varphi}}$ implies that $\bar{\bar{\varphi}} = \psi$. Hence, $\left(\mathrm{F}_{\mathbf{Set}}^{\mathbf{Alg^1}}(|V|)/I, q_I \circ j\right)$ is the free object over V of $\mathbf{Alg^1}$. We note here that the construction of this free object as a quotient along with the fact that $j[|V|]$ generates $\mathrm{F}_{\mathbf{Set}}^{\mathbf{Alg^1}}(|V|)$ (Corollary 3.2.20 (iv)) and Proposition 3.2.17 implies that $q_I \circ j[V]$ does indeed generate $\mathrm{F}_{\mathbf{Set}}^{\mathbf{Alg^1}}(|V|)/I$. ☐

In summary, we have the following 25 non-trivial algebraic free objects.

THEOREM 3.2.22 ([**27**, Theorem 6.2]). *Let S be a set, V a vector space, E a vector lattice, R an algebra, R^1 a unital algebra, A a vector lattice algebra, A^1 a unital vector lattice algebra, and A^{1+} a positive unital vector lattice algebra. Let $\mathbf{Y}$ be an algebraic category from Table 1 and $\mathbf{X}$ any supercategory of $\mathbf{Y}$ from Table 1. Ranging over all valid choices of $\mathbf{X} \supseteq \mathbf{Y}$ generates the following table of existing free objects.*

	Set	**VS**	**VL**	**Alg**	**Alg¹**	**VLA**	**VLA¹**	**VLA¹⁺**
Set	S	$\mathrm{F}_{\mathbf{Set}}^{\mathbf{VS}}(S)$	$\mathrm{F}_{\mathbf{Set}}^{\mathbf{VL}}(S)$	$\mathrm{F}_{\mathbf{Set}}^{\mathbf{Alg}}(S)$	$\mathrm{F}_{\mathbf{Set}}^{\mathbf{Alg^1}}(S)$	$\mathrm{F}_{\mathbf{Set}}^{\mathbf{VLA}}(S)$	$\mathrm{F}_{\mathbf{Set}}^{\mathbf{VLA^1}}(S)$	$\mathrm{F}_{\mathbf{Set}}^{\mathbf{VLA^{1+}}}(S)$
VS		V	$\mathrm{F}_{\mathbf{VS}}^{\mathbf{VL}}(V)$	$\mathrm{F}_{\mathbf{VS}}^{\mathbf{Alg}}(V)$	$\mathrm{F}_{\mathbf{VS}}^{\mathbf{Alg^1}}(V)$	$\mathrm{F}_{\mathbf{VS}}^{\mathbf{VLA}}(V)$	$\mathrm{F}_{\mathbf{VS}}^{\mathbf{VLA^1}}(V)$	$\mathrm{F}_{\mathbf{VS}}^{\mathbf{VLA^{1+}}}(V)$
VL			E			$\mathrm{F}_{\mathbf{VL}}^{\mathbf{VLA}}(E)$	$\mathrm{F}_{\mathbf{VL}}^{\mathbf{VLA^1}}(E)$	$\mathrm{F}_{\mathbf{VL}}^{\mathbf{VLA^{1+}}}(E)$
Alg				R	$\mathrm{F}_{\mathbf{Alg}}^{\mathbf{Alg^1}}(R)$	$\mathrm{F}_{\mathbf{Alg}}^{\mathbf{VLA}}(R)$	$\mathrm{F}_{\mathbf{Alg}}^{\mathbf{VLA^1}}(R)$	$\mathrm{F}_{\mathbf{Alg}}^{\mathbf{VLA^{1+}}}(R)$
Alg¹					R^1		$\mathrm{F}_{\mathbf{Alg^1}}^{\mathbf{VLA^1}}(R^1)$	$\mathrm{F}_{\mathbf{Alg^1}}^{\mathbf{VLA^{1+}}}(R^1)$
VLA						A	$\mathrm{F}_{\mathbf{VLA}}^{\mathbf{VLA^1}}(A)$	$\mathrm{F}_{\mathbf{VLA}}^{\mathbf{VLA^{1+}}}(A)$
VLA¹							A^1	$\mathrm{F}_{\mathbf{VLA^1}}^{\mathbf{VLA^{1+}}}(A^1)$
VLA¹⁺								A^{1+}

TABLE 5. Table of algebraic free objects.

The top row and leftmost column of the table runs through all pairs of categories $(\mathbf{X},\mathbf{Y})$ from Table 1 with $\mathbf{X}$ chosen from the leftmost column and $\mathbf{Y}$ chosen from the top row. If a cell in the above table has an entry $\mathrm{F}_{\mathbf{X}}^{\mathbf{Y}}(X)$, then the free object $(\mathrm{F}_{\mathbf{X}}^{\mathbf{Y}}(X), j)$ exists. Entries along the diagonal correspond to the pairs of categories $(\mathbf{X},\mathbf{X})$ where the free object in $\mathbf{X}$ over an object in $\mathbf{X}$ is just the object itself. Entries below the diagonal are all blank since these pairs of categories $(\mathbf{X},\mathbf{Y})$ do not satisfy $\mathbf{X} \supseteq \mathbf{Y}$ as required in the definition of a free object. Blank entries above the diagonal correspond to those choice of categories that are incompatible.

3.3. Pseudo-solutions of free object problems

In this section, we will define and prove the existence of a substantial number of pseudo-solutions of free object problems. We will give a uniform approach for the construction of these objects: All these constructions start with the abstract existence of some algebraic free object (F, j) from Table 5. This algebraic free object is equipped with a seminorm ρ whose definition exploits the universal property of (F, j) to give this seminorm necessary properties (i.e. the Riesz property and/or submultiplicativity). By taking the quotient of (F, ρ) by $\ker\rho$, we end up with a normed structure that satisfies the necessary universal property of a pseudo-solution and we may pass to a completion if the pseudo-solution is required to be complete.

Beyond the routine verifications one needs to make in these constructions of pseudo-solutions, the most non-trivial step is the formulation of these seminorms mentioned above. Credit needs to be given to Mr. Mitchell Taylor and Prof. Marcel de Jeu for first developing the above approach which they used to prove the existence of, amongst others, pseudo-solutions to the free object problem between the categories **BLA** and **Set**.

We start with our definition of a pseudo-solution, for which we will need to introduce some notation.

DEFINITION 3.3.1. Let $\mathbf{X}$ be a category from Table 1, 2, or 3 and consider $X \in \mathbf{X}$. Define the collection of morphisms

$$\mathcal{M}(X) \coloneqq \{M \in \mathrm{Hom}_{\mathbf{Set}}(X, \mathbb{R}) : M(x) \geq 0 \ \forall\, x \in X\}.$$

Define the relation $\leq$ on $\mathcal{M}(X)$ where $M_1 \leq M_2$ if and only if $M_1(x) \leq M_2(x)$ for all $x \in X$. This makes $(\mathcal{M}(X), \leq)$ into an upwards directed partially ordered set. In particular, if $\mathbf{X}$ is a category of normed structures from Table 2 or 3, we may consider elements of $\mathcal{M}(X)$ of the form $M(x) := C\|x\|$ where $C > 0$. Denote this subcollection by $\mathcal{M}_b(X)$.

Definition 3.3.2. Let $\mathbf{Y}$ be any category of normed structures from Table 2 or 3 with $\mathbf{X}$ any category from Table 1, 2, or 3 such that $\mathbf{X} \supseteq \mathbf{Y}$. Fix $X \in \mathbf{X}$ and $M \in \mathcal{M}(X)$. For $Y \in \mathbf{Y}$, define

$$\Phi_M(X, Y) := \{\varphi \in \mathrm{Hom}_{\mathbf{X}}(X, Y) : \|\varphi(x)\|_Y \leq M(x)\ \forall\, x \in X\}.$$

Remark 3.3.3. Let $\mathbf{Y}$ be any category of normed structures in Table 2 or 3 with $\mathbf{X}$ any category from Table 1, 2, or 3 such that $\mathbf{X} \supseteq \mathbf{Y}$.

(i) Let $Y \in \mathbf{Y}$ and consider a morphism $\varphi : X \to Y$ in $\mathbf{X}$. If $\mathbf{X}$ is a category from Table 1, there exists $M_\varphi \in \mathcal{M}(X)$ such that for all $M \geq M_\varphi$ we have that $\varphi \in \Phi_M(X, Y)$, namely, $M_\varphi(x) := \|\varphi(x)\|$ for all $x \in X$. Otherwise, if $\mathbf{X}$ is a category from Table 2 or 3, then φ is a bounded morphism and there exists $C_\varphi > 0$ such that $\|\varphi(x)\| \leq C_\varphi\|x\|$ for all $x \in X$. In this case, we define $M_\varphi(x) := C_\varphi\|x\|$ then $\varphi \in \Phi_M(X, Y)$ for all $M \geq M_\varphi$. This simple observation will be crucial in the sequel.

(ii) Let $\mathbf{X}$ and $\mathbf{Y}$ be categories of normed structures, either from Table 2 or 3, such that $\mathbf{X} \supseteq \mathbf{Y}$. Fix $X \in \mathbf{X}$ and consider the constraining function $M \in \mathcal{M}(X)$ where $M(x) := \|x\|$. For every $Y \in \mathbf{Y}$ we have

$$\Phi_M(X, Y) = \mathrm{Hom}_{\mathbf{X_1}}(X, Y)$$

where $\mathbf{X_1}$ is the subcategory of $\mathbf{X}$ with contractive morphisms.

Definition 3.3.4. Let $\mathbf{Y}$ be any category of normed structures in Table 2 or 3 with $\mathbf{X}$ any category from Table 1, 2, or 3 such that $\mathbf{X} \supseteq \mathbf{Y}$. Let X be an object in $\mathbf{X}$. If $\mathbf{X}$ is a category from Table 1, consider any $M \in \mathcal{M}(X)$. Otherwise, if $\mathbf{X}$ is a category from Table 2 or 3, let $M \in \mathcal{M}_b(X)$. Consider an object $\mathrm{P}^{\mathbf{Y}}_{\mathbf{X}}(X, M)$ in $\mathbf{Y}$ and a morphism $j_M : X \to \mathrm{P}^{\mathbf{Y}}_{\mathbf{X}}(X, M)$ in $\mathbf{X}$ such that $\|j_M(x)\| \leq M(x)$ for all $x \in X$. The pair $(\mathrm{P}^{\mathbf{Y}}_{\mathbf{X}}(X, M), j_M)$ is called a *pseudo-free object over X of $\mathbf{Y}$ constrained by M* if it has the property that for every $Y \in \mathbf{Y}$ and every $\varphi \in \Phi_M(X, Y)$ there exists a unique *contractive* morphism $\bar{\varphi} : \mathrm{P}^{\mathbf{Y}}_{\mathbf{X}}(X, M) \to Y$ in $\mathbf{Y}$ such that the following diagram commutes in $\mathbf{X}$.

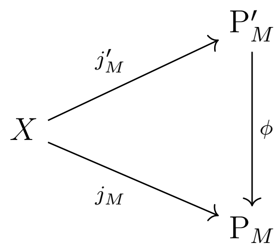
$$\begin{array}{ccc} X & \xrightarrow{\ j_M\ } & \mathrm{P}^{\mathbf{Y}}_{\mathbf{X}}(X, M) \\ & \overset{\varphi}{\searrow} & \big\downarrow{\scriptstyle \bar{\varphi}} \\ & & Y \end{array}$$

If the above holds, the pair $(\mathrm{P}^{\mathbf{Y}}_{\mathbf{X}}(X, M), j_M)$, or by abuse of notation just the object $\mathrm{P}^{\mathbf{Y}}_{\mathbf{X}}(X, M)$, is also called a *pseudo-solution for the free object problem over O_1 between the categories $\mathbf{X}$ and $\mathbf{Y}$ constrained by M.*

This definition will allow us to prove that pseudo-solutions are essentially unique in a very strong sense.

PROPOSITION 3.3.5. *Fix a category* $\mathbf{Y}$ *from Table 2 or 3 and a category* $\mathbf{X}$ *from Table 1, 2, or 3 with* $\mathbf{X} \supseteq \mathbf{Y}$. *Let* $X \in \mathbf{X}$ *and fix a constraining function* M *from either* $\mathcal{M}(X)$ *or* $\mathcal{M}_b(X)$ *as appropriate. Then the following holds:*

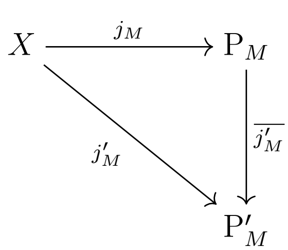
(i) *Pseudo-solutions for the free object problem over* X *between the categories* $\mathbf{X}$ *and* $\mathbf{Y}$ *constrained by* M *are unique up to a* compatible *isometric* $\mathbf{Y}$*-isomorphism, i.e. if pseudo-solutions* (P_M, j_M) *and* (P'_M, j'_M) *exist, then there is an isometric* $\mathbf{Y}$*-isomorphism* $\phi : \mathrm{P}'_M \to \mathrm{P}_M$ *making the following diagram commute in* $\mathbf{X}$.

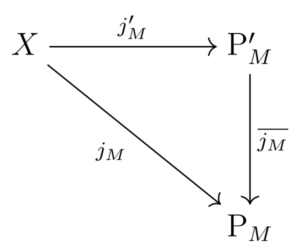


(ii) *Given a pseudo-solution* (P_M, j_M)*, the object in* $\mathcal{A}(\mathbf{Y})$ *generated by* $j_M[X]$ *is dense in* P_M.

(iii) *In light of (ii), pseudo-solutions for the free object problem over* X *between the categories* $\mathbf{X}$ *and* $\mathbf{Y}$ *constrained by* M *are unique up to* a unique compatible *isometric* $\mathbf{Y}$*-isomorphism, i.e. if pseudo-solutions* (P_M, j_M) *and* (P'_M, j'_M) *exist, then there is a* unique *isometric* $\mathbf{Y}$*-isomorphism* $\phi : \mathrm{P}'_M \to \mathrm{P}_M$ *making the diagram in (i) commute in* $\mathbf{X}$.

PROOF. We prove (i): Consider pseudo-solutions (P_M, j_M) and (P'_M, j'_M). By definition we have $j_M \in \Phi_M(X, \mathrm{P}_M)$ and $j'_M \in \Phi_M(X, \mathrm{P}'_M)$. Clearly the identity morphism $\mathbf{1}_{\mathrm{P}_M} : \mathrm{P}_M \to \mathrm{P}_M$ satisfies $\mathbf{1}_{\mathrm{P}_M} \circ j_M = j_M$ and is contractive. On the other hand, by the universal property of a pseudo-solution there exists unique contractive morphisms $\overline{j'_M} : \mathrm{P}_M \to \mathrm{P}'_M$ and $\overline{j_M} : \mathrm{P}'_M \to \mathrm{P}_M$ each uniquely satisfying their own diagram in $\mathbf{X}$.





Thus the morphisms $\overline{j_M} \circ \overline{j'_M} : \mathrm{P}_M \to \mathrm{P}_M$ and $\overline{j'_M} \circ \overline{j_M} : \mathrm{P}'_M \to \mathrm{P}'_M$ are contractive and also satisfy $\left(\overline{j_M} \circ \overline{j'_M}\right) \circ j_M = j_M$ and $\left(\overline{j'_M} \circ \overline{j_M}\right) \circ j'_M = j'_M$. By uniqueness, we have $\overline{j_M} \circ \overline{j'_M} = \mathbf{1}_{\mathrm{P}_M}$ and $\overline{j'_M} \circ \overline{j_M} = \mathbf{1}_{\mathrm{P}'_M}$, thus $\overline{j_M}$ is a $\mathbf{Y}$-isomorphism. To see that $\overline{j_M}$ is an isometry, for $x \in \mathrm{P}'_M$, note that since $\overline{j'_M}$ is contractive, we have

$$\|x\| = \left\|\overline{j'_M} \circ \overline{j_M}(x)\right\| \leq \left\|\overline{j'_M}\right\| \left\|\overline{j_M}(x)\right\| \leq \left\|\overline{j_M}(x)\right\|.$$

Since $\overline{j_M}$ is contractive, we conclude that $\overline{j_M}$ is an isometric $\mathbf{Y}$-isomorphism.

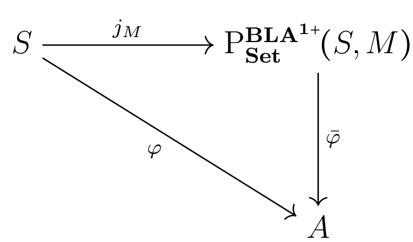
For (ii), consider a pseudo-solution (P_M, j_M). Let G denote the object in $\mathcal{A}(\mathbf{Y})$ generated by the subset $j_M[X]$ in P_M. Denote by H the closure of G in P_M and define the morphism $j' : X \to H$ where $j'(x) \coloneqq j(x)$. The proof of Proposition 3.1.3 is easily imitated to show that (H, j') is a pseudo-solution. By (i), there exists an isometric $\mathbf{Y}$-isomorphism $\phi : H \to \mathrm{P}_M$ such that $\phi \circ j = j_M$, hence G is dense in P_M.

For (iii), consider the pseudo-solutions (P_M, j_M) and (P'_M, j'_M) we had in (i) where we proved that $\overline{j_M} : \mathrm{P}'_M \to \mathrm{P}_M$ is an isometric $\mathbf{Y}$-isomorphism satisfying $\overline{j_M} \circ j'_M = j_M$. If $\phi : \mathrm{P}'_M \to \mathrm{P}_M$ is any isometric $\mathbf{Y}$-isomorphism satisfying $\phi \circ j'_M = j_M$, it follows immediately from the fact that the object in $\mathcal{A}(\mathbf{Y})$ generated by $j'_M[X]$ is dense in P'_M that $\phi = \overline{j_M}$. □

In the next result, we will construct the pseudo-solution to the free object problem over a set S between the categories **Set** and $\mathbf{BLA^{1+}}$ constrained by some $M \in \mathcal{M}(S)$. The proof will be structured in such a way to make it clear how the general approach can be modified to generate pseudo-solutions for different choices of categories. Our chosen example is amongst those with the most details to check and more examples will follow by omitting certain steps.

The next result also specifically addresses a part of [**67**, Problem 13]. The statement of this problem is quoted below in Question 3.3.11. As explained in the introductory remarks in Subsection 1.1.2, in the spirit of [**29**], this would be called the 'free positive unital Banach lattice algebra over a set'.

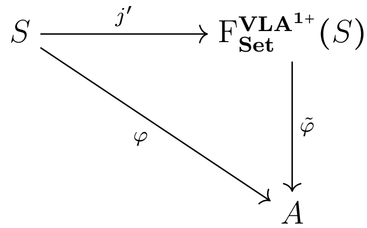
THEOREM 3.3.6. *Consider the categories* **Set** *and* $\mathbf{BLA^{1+}}$. *Fix* $S \in \mathbf{Set}$ *and* $M \in \mathcal{M}(S)$. *Then the pseudo-solution* $\left(\mathrm{P}^{\mathbf{BLA^{1+}}}_{\mathbf{Set}}(S, M), j_M\right)$ *exists. That is to say, for every* $A \in \mathbf{BLA^{1+}}$ *and every morphism* $\varphi \in \Phi_M(S, A)$, *there exists a unique contractive morphism* $\bar{\varphi} : \mathrm{P}^{\mathbf{BLA^{1+}}}_{\mathbf{Set}}(S, M) \to A$ *in* $\mathbf{BLA^{1+}}$ *such that the following diagram commutes in* **Set**.



PROOF. **Step I:** Existence of an algebraic free object.

Fix $S \in \mathbf{Set}$ and $M \in \mathcal{M}(S)$. Consider an arbitrary $A \in \mathbf{BLA^{1+}} \subseteq \mathbf{VLA^{1+}}$ and fix any morphism $\varphi \in \Phi_M(S, A)$. By Theorem 3.2.22, the free object $\left(\mathrm{F}^{\mathbf{VLA^{1+}}}_{\mathbf{Set}}(S), j'\right)$ exists and since $A \in \mathbf{VLA^{1+}}$ there exists a unique morphism $\tilde{\varphi} : \mathrm{F}^{\mathbf{VLA^{1+}}}_{\mathbf{Set}}(S) \to A$ in

$\mathbf{VLA^{1+}}$ such that the following diagram commutes in **Set**.



For brevity, denote $\mathrm{F}' \coloneqq \mathrm{F}^{\mathbf{VLA^{1+}}}_{\mathbf{Set}}(S)$ and denote by $\Phi'_M(S,A)$ the unique morphisms $\psi' : \mathrm{F}' \to A$ in $\mathbf{VLA^{1+}}$ that make the above diagram commute for morphisms $\psi \in \Phi_M(S,A)$.

Step II: Define a seminorm on F' using the above diagram.

Define the map $\tilde{\rho} : \mathrm{F}' \to [0,\infty]$ where

$$\tilde{\rho}(f') \coloneqq \sup\left\{\|\psi'(f')\|_A : A \in \mathbf{BLA^{1+}},\ \psi' \in \Phi'_M(S,A)\right\}$$

for $f' \in \mathrm{F}'$. Denote by G the collection of elements in F' that take on finite values under $\tilde{\rho}$. Define $\rho : G \to \mathbb{R}^+$ where $\rho(g) \coloneqq \tilde{\rho}(g)$ for $g \in G$.

Step III: G is a subvector lattice algebra of F' and ρ is a submultiplicative Riesz seminorm on G.

We prove G is a subalgebra of F': Fix $f, g \in G$. Since any $A \in \mathbf{BLA^{1+}}$ is equipped with a submultiplicative norm and all $\psi' \in \Phi'_M(S,A)$ are algebra homomorphisms, we have

$$\begin{aligned}\tilde{\rho}(fg) &= \sup\left\{\|\psi'(fg)\|_A : A \in \mathbf{BLA^{1+}},\ \psi' \in \Phi'_M(S,A)\right\}\\ &\le \sup\left\{\|\psi'(f)\|_A \|\psi'(g)\|_A : A \in \mathbf{BLA^{1+}},\ \psi' \in \Phi'_M(S,A)\right\}\\ &= \tilde{\rho}(f)\,\tilde{\rho}(g).\end{aligned}$$

Thus G is a subalgebra of F' and ρ is a submultiplicative seminorm on G. To see that G is a sublattice of F', fix $g \in G$. Since any $A \in \mathbf{BLA^{1+}}$ is equipped with a Riesz norm and all $\varphi' \in \Phi'_M(S,A)$ are lattice homomorphisms, we have

$$\begin{aligned}\tilde{\rho}(|g|) &= \sup\left\{\|\psi'(|g|)\|_A : A \in \mathbf{BLA^{1+}},\ \psi' \in \Phi'_M(S,A)\right\},\\ &= \sup\left\{\||\psi'(g)|\|_A : A \in \mathbf{BLA^{1+}},\ \psi' \in \Phi'_M(S,A)\right\},\\ &= \ \tilde{\rho}(g).\end{aligned}$$

Thus $|g| \in G$, which proves that G is a subvector lattice algebra of F'. It remains to show that ρ has the Riesz seminorm property: Let $g, g' \in G$ with $|g| \le |g'|$. Since all $\psi' \in \Phi'_M(S,A)$ are lattice homomorphisms, we have

$$|\psi'(g)| = \psi'(|g|) \le \psi'(|g'|) = |\psi'(g')|.$$

Fix any $\psi' \in \Phi'_M(S,A)$. Since all $A \in \mathbf{BLA^{1+}}$ are equipped with a Riesz norm, it follows that $\|\psi'(|g|)\|_A \le \|\psi'(|g'|)\|_A$ for all $A \in \mathbf{BLA^{1+}}$. This implies that $\rho(g) \le \rho(g')$.

Step IV: $G = \mathrm{F}'$

The collection $j'[S]$ is a subset of G: Fix $j'(s) \in j'[S]$. Then for each $A \in \mathbf{BLA^{1+}}$ and each $\psi' \in \Phi'_M(S, A)$, we have $\|\psi'(j'(s))\|_A = \|\psi(s)\|_A \le M(s)$. By Corollary 3.2.20 (vii), the collection $j'[S]$ generates F'. Thus $j'[S]$ cannot be contained in any proper subvector lattice algebra of F', hence $G = \mathrm{F}'$.

Step V: $\ker\rho$ is a bi-ideal in F'.

This claim follows easily from the fact that ρ is a submultiplicative Riesz seminorm on G.

Step VI: Construct a normed vector lattice algebra F'' from F' and ρ.

The quotient $\mathrm{F}'' \coloneqq \mathrm{F}'/\ker\rho$ is an object in $\mathbf{VLA^{1+}}$ when equipped with the standard quotient vector lattice algebra structure. Denote by $q : \mathrm{F}' \to \mathrm{F}''$ the quotient map and define $\|\cdot\| : \mathrm{F}'' \to \mathbb{R}^+$ where $\|[a]\| \coloneqq \rho(a)$ for $[a] \in \mathrm{F}''$. Then $(\mathrm{F}'', \|\cdot\|)$ is an object in $\mathbf{NVLA^{1+}}$ since the norm inherits all the necessary properties from ρ.

Step VII: Construct the unique morphism $\tilde{\tilde{\varphi}} : \mathrm{F}'' \to A$ in $\mathbf{NVLA^{1+}}$.

For $\psi' \in \Phi'_M(S, A)$, note that $\psi'(y) = 0$ for all $y \in \ker\rho$ since $0 \le \|\psi'(y)\|_A \le \rho(y) = 0$.

Now, we consider the morphism $\varphi \in \Phi_M(S, A)$ fixed in Step I above along with the unique morphism $\tilde{\varphi} : \mathrm{F}' \to A$ in $\mathbf{VLA^{1+}}$ making the diagram in Step I commute. Define the map $\tilde{\tilde{\varphi}} : \mathrm{F}'' \to A$ where $\tilde{\tilde{\varphi}}([a]) := \tilde{\varphi}(a)$ for $[a] \in \mathrm{F}''$. The previous observation shows that $\tilde{\tilde{\varphi}}$ is well-defined.

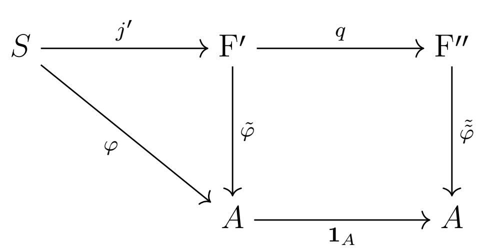
It is clear that $\tilde{\tilde{\varphi}} \circ q = \tilde{\varphi}$ and it is easy to see that $\tilde{\tilde{\varphi}}$ is a morphism in $\mathbf{VLA^{1+}}$. Further, for any $[a] \in \mathrm{F}''$ with $[a] \neq [0]$,

$$\frac{\|\tilde{\tilde{\varphi}}([a])\|_A}{\|[a]\|} = \frac{\|\tilde{\varphi}(a)\|_A}{\|[a]\|} \le 1$$

since $\|[a]\| \ge \|\tilde{\varphi}(a)\|_A$. Hence $\tilde{\tilde{\varphi}}$ is contractive. Now, for $s \in S$, we have

$$\tilde{\tilde{\varphi}} \circ (q \circ j')(s) = \tilde{\tilde{\varphi}}([j'(s)]) = \tilde{\varphi}(j'(s)) = \varphi(s).$$

Thus we have the following commutative diagram.



Note that the map $q \circ j' : S \to \mathrm{F}''$ has the property that for all $s \in S$,

$$\|q \circ j'(s)\| = \|[j'(s)]\| = \rho(j'(s)) \le M(s)$$

To show that $\tilde{\tilde{\varphi}}$ is the unique $\mathbf{NVLA^{1+}}$ morphism that satisfies $\tilde{\tilde{\varphi}} \circ (q \circ j') = \varphi$, let $\tilde{\tilde{\psi}} : \mathrm{F}'' \to A$ be any other $\mathbf{NVLA^{1+}}$ morphism that also satisfies this property. It follows that $\left(\tilde{\tilde{\psi}} \circ q\right) \circ j' = \varphi$. Since $\tilde{\varphi} : \mathrm{F}' \to A$ is the unique $\mathbf{VLA^{1+}}$ morphism that satisfies $\tilde{\varphi} \circ j' = \varphi$, we have that $\tilde{\tilde{\psi}} \circ q = \tilde{\varphi}$. Thus for every $[a] \in \mathrm{F}''$, we have

$$\tilde{\tilde{\psi}}([a]) = \tilde{\tilde{\psi}} \circ q(a) = \tilde{\varphi}(a) = \tilde{\tilde{\varphi}} \circ q(a) = \tilde{\tilde{\varphi}}([a]).$$

Thus $\tilde{\tilde{\psi}} = \tilde{\tilde{\varphi}}$.

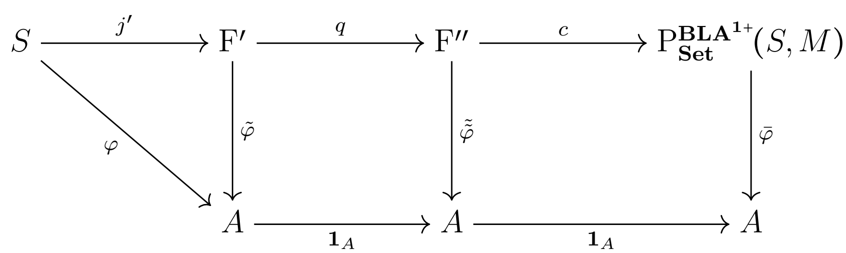
Step VIII: Extend to the completion of F''.

Denote by $\mathrm{P}^{\mathbf{BLA^{1+}}}_{\mathbf{Set}}(S, M)$ a completion of F'', which is an object in $\mathbf{BLA^{1+}}$. Denote by $c : \mathrm{F}'' \to \mathrm{P}^{\mathbf{BLA^{1+}}}_{\mathbf{Set}}(S, M)$ the isometric embedding and define the function $j_M : S \to \mathrm{P}^{\mathbf{BLA^{1+}}}_{\mathbf{Set}}(S, M)$ where $j_M := c \circ q \circ j'$. Since c is isometric, we have $\|j_M(s)\| = \|q \circ j'(s)\| \leq M(s)$ for all $s \in S$.

Further, since F'' is dense in its completion, there exists a unique contractive vector lattice algebra homomorphism $\bar{\varphi} : \mathrm{P}^{\mathbf{BLA^{1+}}}_{\mathbf{Set}}(S, M) \to A$ such that $\bar{\varphi} \circ c = \tilde{\tilde{\varphi}}$. It follows that

$$\bar{\varphi} \circ j_M = (\bar{\varphi} \circ c) \circ q \circ j' = \tilde{\tilde{\varphi}} \circ (q \circ j') = \varphi.$$

This gives us the following commutative diagram.



Finally, to see that $\bar{\varphi} : \mathrm{P}^{\mathbf{BLA^{1+}}}_{\mathbf{Set}}(S, M) \to A$ is the unique $\mathbf{BLA^{1+}}$ morphism with the property that $\bar{\varphi} \circ j_M = \varphi$, let $\bar{\psi} : \mathrm{P}^{\mathbf{BLA^{1+}}}_{\mathbf{Set}}(S, M) \to A$ be any other $\mathbf{BLA^{1+}}$ morphism with this property. Then

$$\bar{\psi} \circ j_M = \left(\bar{\psi} \circ c\right) \circ q \circ j' = \varphi.$$

Step VII shows that $\tilde{\tilde{\varphi}} : \mathrm{F}'' \to A$ is the unique morphism in $\mathbf{NVLA^{1+}}$ that satisfies $\tilde{\tilde{\varphi}} \circ (q \circ j') = \varphi$, hence $\bar{\psi} \circ c = \tilde{\tilde{\varphi}}$. Since $\bar{\varphi}$ is the unique extension of $\tilde{\tilde{\varphi}}$ from F'' to its completion, this implies that $\bar{\psi} = \bar{\varphi}$. □

The above approach is readily modified to prove the existence of pseudo-solutions to free object problems for pairs of categories $(\mathbf{X}, \mathbf{Y})$ where $\mathbf{X}$ ranges over Table 1 and $\mathbf{Y}$ ranges over Table 2 and 3 such that $\mathbf{X} \supseteq \mathbf{Y}$. For a particular choice of category $\mathbf{X}$ in Table 1, the corresponding row in Table 5 shows all possible choice of algebraic free objects that can be used as a 'foundation' in Step I of the above proof. The procedure can then either terminate with either Step VII or Step VIII depending on whether we are constructing a non-complete or complete pseudo-solution. Since there are 25 non-trivial algebraic free objects in Table 5 and every non-trivial algebraic free object can be used as a foundation for both a non-complete and complete pseudo-solution, the method outlined in Theorem 3.3.6 has delivered us 50 pseudo-solutions to free object problems.

THEOREM 3.3.7. *Consider categories* $\mathbf{X}$ *and* $\mathbf{Y}$ *with* $\mathbf{X}$ *ranging over Table 1 and* $\mathbf{Y}$ *ranging over Table 2 and 3 such that* $\mathbf{X} \supseteq \mathbf{Y}$. *Let* S *be a set,* V *a vector space,* E

a vector lattice, R an algebra, R^1 a unital algebra, A a vector lattice algebra, and A^1 a unital vector lattice algebra and let $X \in \mathbf{X}$ denote any one of these objects. For $M \in \mathcal{M}(X)$, the following pseudo-solutions to free object problems exist. In all cases, the morphism $j_M : X \to \mathrm{P}^{\mathbf{Y}}_{\mathbf{X}}(X, M)$ is a morphism in the category $\mathbf{X}$ satisfying $\|j_M(x)\| \le M(x)$ for all $x \in X$.

$\mathbf{Y}$ *from Table 2:*

(1) $\mathrm{P}^{\mathbf{NS}}_{\mathbf{Set}}(S, M)$
(2) $\mathrm{P}^{\mathbf{NVL}}_{\mathbf{Set}}(S, M)$
(3) $\mathrm{P}^{\mathbf{NVL}}_{\mathbf{VS}}(V, M)$
(4) $\mathrm{P}^{\mathbf{NA}}_{\mathbf{Set}}(S, M)$
(5) $\mathrm{P}^{\mathbf{NA}}_{\mathbf{VS}}(V, M)$
(6) $\mathrm{P}^{\mathbf{NA^1}}_{\mathbf{Set}}(S, M)$
(7) $\mathrm{P}^{\mathbf{NA^1}}_{\mathbf{VS}}(V, M)$
(8) $\mathrm{P}^{\mathbf{NA^1}}_{\mathbf{Alg}}(R, M)$
(9) $\mathrm{P}^{\mathbf{NVLA}}_{\mathbf{Set}}(S, M)$
(10) $\mathrm{P}^{\mathbf{NVLA}}_{\mathbf{VS}}(V, M)$
(11) $\mathrm{P}^{\mathbf{NVLA}}_{\mathbf{VL}}(E, M)$
(12) $\mathrm{P}^{\mathbf{NVLA}}_{\mathbf{Alg}}(R, M)$
(13) $\mathrm{P}^{\mathbf{NVLA^1}}_{\mathbf{Set}}(S, M)$
(14) $\mathrm{P}^{\mathbf{NVLA^1}}_{\mathbf{VS}}(V, M)$
(15) $\mathrm{P}^{\mathbf{NVLA^1}}_{\mathbf{VL}}(E, M)$
(16) $\mathrm{P}^{\mathbf{NVLA^1}}_{\mathbf{Alg}}(R, M)$
(17) $\mathrm{P}^{\mathbf{NVLA^1}}_{\mathbf{Alg^1}}(R^1, M)$
(18) $\mathrm{P}^{\mathbf{NVLA^1}}_{\mathbf{VLA}}(A, M)$
(19) $\mathrm{P}^{\mathbf{NVLA^{1+}}}_{\mathbf{Set}}(S, M)$
(20) $\mathrm{P}^{\mathbf{NVLA^{1+}}}_{\mathbf{VS}}(V, M)$
(21) $\mathrm{P}^{\mathbf{NVLA^{1+}}}_{\mathbf{VL}}(E, M)$
(22) $\mathrm{P}^{\mathbf{NVLA^{1+}}}_{\mathbf{Alg}}(R, M)$
(23) $\mathrm{P}^{\mathbf{NVLA^{1+}}}_{\mathbf{Alg^1}}(R^1, M)$
(24) $\mathrm{P}^{\mathbf{NVLA^{1+}}}_{\mathbf{VLA}}(A, M)$
(25) $\mathrm{P}^{\mathbf{NVLA^{1+}}}_{\mathbf{VLA^1}}(A^1, M)$

Moreover, the object $\mathrm{P}^{\mathbf{Y}}_{\mathbf{X}}(X, M)$ in the above list of 25 pseudo-solutions is generated by the subset $j_M[X]$.

$\mathbf{Y}$ *from Table 3:*

(1) $\mathrm{P}^{\mathbf{Ban}}_{\mathbf{Set}}(S, M)$
(2) $\mathrm{P}^{\mathbf{BL}}_{\mathbf{Set}}(S, M)$
(3) $\mathrm{P}^{\mathbf{BL}}_{\mathbf{VS}}(V, M)$
(4) $\mathrm{P}^{\mathbf{BA}}_{\mathbf{Set}}(S, M)$
(5) $\mathrm{P}^{\mathbf{BA}}_{\mathbf{VS}}(V, M)$
(6) $\mathrm{P}^{\mathbf{BA^1}}_{\mathbf{Set}}(S, M)$
(7) $\mathrm{P}^{\mathbf{BA^1}}_{\mathbf{VS}}(V, M)$
(8) $\mathrm{P}^{\mathbf{BA^1}}_{\mathbf{Alg}}(R, M)$
(9) $\mathrm{P}^{\mathbf{BLA}}_{\mathbf{Set}}(S, M)$
(10) $\mathrm{P}^{\mathbf{BLA}}_{\mathbf{VS}}(V, M)$
(11) $\mathrm{P}^{\mathbf{BLA}}_{\mathbf{VL}}(E, M)$
(12) $\mathrm{P}^{\mathbf{BLA}}_{\mathbf{Alg}}(R, M)$
(13) $\mathrm{P}^{\mathbf{BLA^1}}_{\mathbf{Set}}(S, M)$
(14) $\mathrm{P}^{\mathbf{BLA^1}}_{\mathbf{VS}}(V, M)$
(15) $\mathrm{P}^{\mathbf{BLA^1}}_{\mathbf{VL}}(E, M)$
(16) $\mathrm{P}^{\mathbf{BLA^1}}_{\mathbf{Alg}}(R, M)$
(17) $\mathrm{P}^{\mathbf{BLA^1}}_{\mathbf{Alg^1}}(R^1, M)$
(18) $\mathrm{P}^{\mathbf{BLA^1}}_{\mathbf{VLA}}(A, M)$
(19) $\mathrm{P}^{\mathbf{BLA^{1+}}}_{\mathbf{Set}}(S, M)$
(20) $\mathrm{P}^{\mathbf{BLA^{1+}}}_{\mathbf{VS}}(V, M)$
(21) $\mathrm{P}^{\mathbf{BLA^{1+}}}_{\mathbf{VL}}(E, M)$
(22) $\mathrm{P}^{\mathbf{BLA^{1+}}}_{\mathbf{Alg}}(R, M)$
(23) $\mathrm{P}^{\mathbf{BLA^{1+}}}_{\mathbf{Alg^1}}(R^1, M)$
(24) $\mathrm{P}^{\mathbf{BLA^{1+}}}_{\mathbf{VLA}}(A, M)$
(25) $\mathrm{P}^{\mathbf{BLA^{1+}}}_{\mathbf{VLA^1}}(A^1, M)$

For a pair of categories $(\mathbf{X}, \mathbf{Y})$ where $\mathbf{X}$ and $\mathbf{Y}$ both range over Table 2 and 3, the only modification we need to make to the approach laid out in Theorem 3.3.6 is that we only consider constraining functions in $\mathcal{M}_b(X)$ when $X \in \mathbf{X}$. This will ensure that the morphism $j_M : X \to \mathrm{P}^{\mathbf{Y}}_{\mathbf{X}}(X, M)$ is indeed bounded.

The following three tables show all valid choice of pairs of categories $(\mathbf{X}, \mathbf{Y})$ where $\mathbf{X}$ and $\mathbf{Y}$ both range over Table 2 and 3 such that $\mathbf{X} \supseteq \mathbf{Y}$. The entries in the top row in Table 6 - 8 fix a particular category $\mathbf{Y}$ and the categories in the corresponding column show all valid choices of $\mathbf{X}$ with $\mathbf{X} \supseteq \mathbf{Y}$.

NS	**NVL**	**NA**	$\mathbf{NA^1}$	**NVLA**	$\mathbf{NVLA^1}$	$\mathbf{NVLA^{1+}}$
	NS	**NS**	**NS**	**NS**	**NS**	**NS**
				NVL	**NVL**	**NVL**
			NA	**NA**	**NA**	**NA**
					$\mathbf{NA^1}$	$\mathbf{NA^1}$
					NVLA	**NVLA**
						$\mathbf{NVLA^1}$

TABLE 6. Valid pairs of categories from Table 2.

Ban	**BL**	**BA**	$\mathbf{BA^1}$	**BLA**	$\mathbf{BLA^1}$	$\mathbf{BLA^{1+}}$
	Ban	**Ban**	**Ban**	**Ban**	**Ban**	**Ban**
				BL	**BL**	**BL**
			BA	**BA**	**BA**	**BA**
					$\mathbf{BA^1}$	$\mathbf{BA^1}$
					BLA	**BLA**
						$\mathbf{BLA^1}$

TABLE 7. Valid pairs of categories from Table 3.

Ban	**BL**	**BA**	$\mathbf{BA^1}$	**BLA**	$\mathbf{BLA^1}$	$\mathbf{BLA^{1+}}$
	NS	**NS**	**NS**	**NS**	**NS**	**NS**
				NVL	**NVL**	**NVL**
			NA	**NA**	**NA**	**NA**
					$\mathbf{NA^1}$	$\mathbf{NA^1}$
					NVLA	**NVLA**
						$\mathbf{NVLA^1}$

TABLE 8. Valid pairs of categories from Table 2 and 3.

Given that the proof of the existence of the pseudo-solution $\left(\mathrm{P}^{\mathbf{BLA^{1+}}}_{\mathbf{Set}}(S, M), j_M\right)$ in Theorem 3.3.6 provides a detailed illustration of the general proof method, we will only give a brief outline of the proof of the existence of a pseudo-solution for one of the above pairs of categories. Fix $N \in \mathbf{NS}$ and $M \in \mathcal{M}_b(N)$ where $M(x) := C\,\|x\|$ for $x \in N$ and $C > 0$. We outline the existence of a pseudo-solution $\left(\mathrm{P}^{\mathbf{NA^1}}_{\mathbf{NS}}(N, M), j_M\right)$:

We start by fixing $B \in \mathbf{NA^1} \subseteq \mathbf{Alg^1}$ and $\varphi \in \Phi_M(N, B)$. The free object $\left(\mathrm{F}^{\mathbf{Alg^1}}_{\mathbf{VS}}(N), j'\right)$ exists and there exists a unique morphism $\tilde{\varphi} : \mathrm{F}^{\mathbf{Alg^1}}_{\mathbf{VS}}(N) \to B$ in $\mathbf{Alg^1}$ such that $\tilde{\varphi} \circ j' = \varphi$. We equip $\mathrm{F}^{\mathbf{Alg^1}}_{\mathbf{VS}}(N)$ with a submultiplicative seminorm ρ similar to that in Theorem 3.3.6. Since $\ker\rho$ is an order ideal, the seminorm ρ induces a norm on the quotient $\mathrm{P} := \mathrm{F}^{\mathbf{Alg^1}}_{\mathbf{VS}}(N)/\ker\rho$ and thus $(\mathrm{P}, \|\cdot\|)$ is an object in $\mathbf{NA^1}$.

Denote by $q : \mathrm{F}_{\mathbf{VS}}^{\mathbf{Alg^1}}(N) \to \mathrm{P}$ the quotient map and define $j_M := q \circ j'$. Since $\|j_M(x)\| \leq M(s) = C\|x\|$, it follows that j_M is a morphism in **NS**. It can then be shown that $\bar{\varphi} := \tilde{\varphi} \circ q$ is the unique morphism in $\mathbf{NA_1^1}$ such that $\bar{\varphi} \circ j_M = \varphi$. Additionally, the set $j_M[N]$ generates P since, as noted at the end of the proof of Proposition 3.2.21, $j'[N]$ generates $\mathrm{F}_{\mathbf{VS}}^{\mathbf{Alg^1}}(N)$.

The valid pairs of categories in Table 6 - 8 gives us another 54 pseudo-solutions.

THEOREM 3.3.8. *Consider categories* $\mathbf{X}$ *and* $\mathbf{Y}$ *ranging over Table 2 and 3 such that* $\mathbf{X} \supseteq \mathbf{Y}$. *Let* N *be a normed space,* X *a Banach space,* Y *a normed vector lattice,* Z *a Banach lattice,* U *a normed algebra,* W *a Banach algebra,* U^1 *a unital normed algebra,* W^1 *a unital Banach algebra,* L *a normed vector lattice algebra,* B *a Banach lattice algebra,* L^1 *a unital normed vector lattice algebra,* B^1 *a unital Banach lattice algebra and let* O *denote any one of these objects. For* $M \in \mathcal{M}_b(O)$, *the following pseudo-solutions to free object problems exist. In all cases, the morphism* $j_M : O \to \mathrm{P}_{\mathbf{X}}^{\mathbf{Y}}(O, M)$ *is a morphism in the category* $\mathbf{X}$ *satisfying* $\|j_M(x)\| \leq M(x)$ *for all* $x \in O$.

$\mathbf{X}$ *and* $\mathbf{Y}$ *from Table 2:*

(1) $\mathrm{P}_{\mathbf{NS}}^{\mathbf{NVL}}(N, M)$	*(7)* $\mathrm{P}_{\mathbf{NA}}^{\mathbf{NVLA}}(U, M)$	*(13)* $\mathrm{P}_{\mathbf{NS}}^{\mathbf{NVLA^{1+}}}(N, M)$
(2) $\mathrm{P}_{\mathbf{NS}}^{\mathbf{NA}}(N, M)$	*(8)* $\mathrm{P}_{\mathbf{NS}}^{\mathbf{NVLA^1}}(N, M)$	*(14)* $\mathrm{P}_{\mathbf{NVL}}^{\mathbf{NVLA^{1+}}}(Y, M)$
(3) $\mathrm{P}_{\mathbf{NS}}^{\mathbf{NA^1}}(N, M)$	*(9)* $\mathrm{P}_{\mathbf{NVL}}^{\mathbf{NVLA^1}}(Y, M)$	*(15)* $\mathrm{P}_{\mathbf{NA}}^{\mathbf{NVLA^{1+}}}(U, M)$
(4) $\mathrm{P}_{\mathbf{NA}}^{\mathbf{NA^1}}(U, M)$	*(10)* $\mathrm{P}_{\mathbf{NA}}^{\mathbf{NVLA^1}}(U, M)$	*(16)* $\mathrm{P}_{\mathbf{NA^1}}^{\mathbf{NVLA^{1+}}}(U^1, M)$
(5) $\mathrm{P}_{\mathbf{NS}}^{\mathbf{NVLA}}(N, M)$	*(11)* $\mathrm{P}_{\mathbf{NA^1}}^{\mathbf{NVLA^1}}(U^1, M)$	*(17)* $\mathrm{P}_{\mathbf{NVLA}}^{\mathbf{NVLA^{1+}}}(L, M)$
(6) $\mathrm{P}_{\mathbf{NVL}}^{\mathbf{NVLA}}(Y, M)$	*(12)* $\mathrm{P}_{\mathbf{NVLA}}^{\mathbf{NVLA^1}}(L, M)$	*(18)* $\mathrm{P}_{\mathbf{NVLA^1}}^{\mathbf{NVLA^{1+}}}(L^1, M)$

Moreover, the object $\mathrm{P}_{\mathbf{X}}^{\mathbf{Y}}(X, M)$ *in the above list of 18 pseudo-solutions is generated by the subset* $j_M[X]$.

$\mathbf{X}$ *and* $\mathbf{Y}$ *from Table 3:*

(1) $\mathrm{P}_{\mathbf{Ban}}^{\mathbf{BL}}(X, M)$	*(7)* $\mathrm{P}_{\mathbf{BA}}^{\mathbf{BLA}}(W, M)$	*(13)* $\mathrm{P}_{\mathbf{Ban}}^{\mathbf{BLA^{1+}}}(X, M)$
(2) $\mathrm{P}_{\mathbf{Ban}}^{\mathbf{BA}}(X, M)$	*(8)* $\mathrm{P}_{\mathbf{Ban}}^{\mathbf{BLA^1}}(X, M)$	*(14)* $\mathrm{P}_{\mathbf{BL}}^{\mathbf{BLA^{1+}}}(Z, M)$
(3) $\mathrm{P}_{\mathbf{Ban}}^{\mathbf{BA^1}}(X, M)$	*(9)* $\mathrm{P}_{\mathbf{BL}}^{\mathbf{BLA^1}}(Z, M)$	*(15)* $\mathrm{P}_{\mathbf{BA}}^{\mathbf{BLA^{1+}}}(W, M)$
(4) $\mathrm{P}_{\mathbf{BA}}^{\mathbf{BA^1}}(W, M)$	*(10)* $\mathrm{P}_{\mathbf{BA}}^{\mathbf{BLA^1}}(W, M)$	*(16)* $\mathrm{P}_{\mathbf{BA^1}}^{\mathbf{BLA^{1+}}}(W^1, M)$
(5) $\mathrm{P}_{\mathbf{Ban}}^{\mathbf{BLA}}(X, M)$	*(11)* $\mathrm{P}_{\mathbf{BA^1}}^{\mathbf{BLA^1}}(W^1, M)$	*(17)* $\mathrm{P}_{\mathbf{BLA}}^{\mathbf{BLA^{1+}}}(B, M)$
(6) $\mathrm{P}_{\mathbf{BL}}^{\mathbf{BLA}}(Z, M)$	*(12)* $\mathrm{P}_{\mathbf{BLA}}^{\mathbf{BLA^1}}(B, M)$	*(18)* $\mathrm{P}_{\mathbf{BLA^1}}^{\mathbf{BLA^{1+}}}(B^1, M)$

$\mathbf{X}$ *from Table 2 and* $\mathbf{Y}$ *from Table 3:*

(1) $\mathrm{P}^{\mathbf{BL}}_{\mathbf{NS}}(N,M)$
(2) $\mathrm{P}^{\mathbf{BA}}_{\mathbf{NS}}(N,M)$
(3) $\mathrm{P}^{\mathbf{BA^1}}_{\mathbf{NS}}(N,M)$
(4) $\mathrm{P}^{\mathbf{BA^1}}_{\mathbf{NA}}(U,M)$
(5) $\mathrm{P}^{\mathbf{BLA}}_{\mathbf{NS}}(N,M)$
(6) $\mathrm{P}^{\mathbf{BLA}}_{\mathbf{NVL}}(Y,M)$
(7) $\mathrm{P}^{\mathbf{BLA}}_{\mathbf{NA}}(U,M)$
(8) $\mathrm{P}^{\mathbf{BLA^1}}_{\mathbf{NS}}(N,M)$
(9) $\mathrm{P}^{\mathbf{BLA^1}}_{\mathbf{NVL}}(Y,M)$
(10) $\mathrm{P}^{\mathbf{BLA^1}}_{\mathbf{NA}}(U,M)$
(11) $\mathrm{P}^{\mathbf{BLA^1}}_{\mathbf{NA^1}}(U^1,M)$
(12) $\mathrm{P}^{\mathbf{BLA^1}}_{\mathbf{NVLA}}(L,M)$
(13) $\mathrm{P}^{\mathbf{BLA^{1+}}}_{\mathbf{NS}}(N,M)$
(14) $\mathrm{P}^{\mathbf{BLA^{1+}}}_{\mathbf{NVL}}(Y,M)$
(15) $\mathrm{P}^{\mathbf{BLA^{1+}}}_{\mathbf{NA}}(U,M)$
(16) $\mathrm{P}^{\mathbf{BLA^{1+}}}_{\mathbf{NA^1}}(U^1,M)$
(17) $\mathrm{P}^{\mathbf{BLA^{1+}}}_{\mathbf{NVLA}}(L,M)$
(18) $\mathrm{P}^{\mathbf{BLA^{1+}}}_{\mathbf{NVLA^1}}(L^1,M)$

Now that we have proven the existence of these pseudo-solutions, we can add the following remark on their structure.

REMARK 3.3.9. For a pseudo-solution $(\mathrm{P}^{\mathbf{Y}}_{\mathbf{X}}(X,M), j_M)$, we defined the morphism $j_M : X \to \mathrm{P}^{\mathbf{Y}}_{\mathbf{X}}(X,M)$ to satisfy the bound $\|j_M(x)\| \leq M(x)$ for all $x \in X$. We used this property in Proposition 3.3.5 to show that pseudo-solutions are unique in a strong sense.

In the case of the pseudo-solution $\left(\mathrm{P}^{\mathbf{BLA^{1+}}}_{\mathbf{Set}}(S,M), j_M\right)$, indeed, any of the above pseudo-solutions over sets, the constraining function M gives a 'sharp' bound for the norms $\|j_M(s)\|$ for $s \in S$: Since the constraining function $M : S \to \mathbb{R}$ constrains itself, i.e. $M \in \Phi_M(S,\mathbb{R})$ and $\mathbb{R}$ is an object in $\mathbf{BLA^{1+}}$, it follows by the universal property of the pseudo-solution $\left(\mathrm{P}^{\mathbf{BLA^{1+}}}_{\mathbf{Set}}(S,M), j_M\right)$ that there exists a unique contractive morphism $\bar{M} : \mathrm{P}^{\mathbf{BLA^{1+}}}_{\mathbf{Set}}(S,M) \to \mathbb{R}$ in $\mathbf{BLA^{1+}}$ such that

$$M(s) = |M(s)| = \left|\bar{M} \circ j_M(s)\right| \leq \|j_M(s)\|$$

for all $s \in S$. As a result, we have that $\|j_M(s)\| = M(s)$ for all $s \in S$.

Similar arguments will work for some pseudo-solutions, but not for others. We mention two more examples where the constraining function M gives a sharp bound: Consider the pseudo-solution $(\mathrm{P}^{\mathbf{BA}}_{\mathbf{VS}}(V,M), j_M)$. For every $v \in V$, an argument with basis vectors gives us a linear map $\varphi_v : V \to \mathbb{R}$ such that $\varphi_v(v) = M(v)$. By the universal property of the pseudo-solution, there exists a unique contractive morphism $\bar{\varphi}_v : \mathrm{P}^{\mathbf{BA}}_{\mathbf{VS}}(V,M) \to \mathbb{R}$ in $\mathbf{BA}$ such that

$$M(v) = |\bar{\varphi}_v\left(j_M(v)\right)| \leq \|\bar{\varphi}_v\| \, \|j_M(v)\| \leq \|j_M(v)\|.$$

By [**25**, Chapter III, Corollary 6.6], the same argument applies to $\left(\mathrm{P}^{\mathbf{BLA^{1+}}}_{\mathbf{NS}}(N,M), j_M\right)$ since for every $x \in N$ there exists a bounded linear map $\varphi_x : N \to \mathbb{R}$ such that $\varphi_x(x) = M(x)$.

In view of Remark 3.3.3 (ii), it is clear that the 54 pseudo-solutions for free object problems in Theorem 3.3.8 are in fact full solutions to free object problems when we restrict ourselves to categories with contractive morphisms.

COROLLARY 3.3.10. *Consider categories* $\mathbf{X}$ *and* $\mathbf{Y}$ *ranging over Table 2 and 3 such that* $\mathbf{X} \supseteq \mathbf{Y}$. *Let* N *be a normed space,* X *a Banach space,* Y *a normed vector lattice,* Z *a Banach lattice,* U *a normed algebra,* W *a Banach algebra,* U^1 *a unital normed algebra,* W^1 *a unital Banach algebra,* L *a normed lattice algebra,* B *a Banach lattice* *algebra, L^1 a unital normed lattice algebra, B^1 a unital Banach lattice algebra and let O denote any one of these objects. The following free objects exist. In all cases, the morphism $j : O \to \mathrm{F}_{\mathbf{X_1}}^{\mathbf{Y_1}}(O)$ is a morphism in the category $\mathbf{X_1}$.*

$\mathbf{X}$ *and* $\mathbf{Y}$ *from Table 2:*

(1) $\mathrm{F}_{\mathbf{NS_1}}^{\mathbf{NVL_1}}(N)$
(2) $\mathrm{F}_{\mathbf{NS_1}}^{\mathbf{NA_1}}(N)$
(3) $\mathrm{F}_{\mathbf{NS_1}}^{\mathbf{NA_1^1}}(N)$
(4) $\mathrm{F}_{\mathbf{NA_1}}^{\mathbf{NA_1^1}}(U)$
(5) $\mathrm{F}_{\mathbf{NS_1}}^{\mathbf{NA_1}}(N)$
(6) $\mathrm{F}_{\mathbf{NVL_1}}^{\mathbf{NVLA_1}}(Y)$
(7) $\mathrm{F}_{\mathbf{NA_1}}^{\mathbf{NVLA_1}}(U)$
(8) $\mathrm{F}_{\mathbf{NS_1}}^{\mathbf{NVLA_1^1}}(N)$
(9) $\mathrm{F}_{\mathbf{NVL_1}}^{\mathbf{NVLA_1^1}}(Y)$
(10) $\mathrm{F}_{\mathbf{NA_1}}^{\mathbf{NVLA_1^1}}(U)$
(11) $\mathrm{F}_{\mathbf{NA_1^1}}^{\mathbf{NVLA_1^1}}(U^1)$
(12) $\mathrm{F}_{\mathbf{NVLA_1}}^{\mathbf{NVLA_1^1}}(L)$
(13) $\mathrm{F}_{\mathbf{NS_1}}^{\mathbf{NVLA_1^{1+}}}(N)$
(14) $\mathrm{F}_{\mathbf{NVL_1}}^{\mathbf{NVLA_1^{1+}}}(Y)$
(15) $\mathrm{F}_{\mathbf{NA_1}}^{\mathbf{NVLA_1^{1+}}}(U)$
(16) $\mathrm{F}_{\mathbf{NA_1^1}}^{\mathbf{NVLA_1^{1+}}}(U^1)$
(17) $\mathrm{F}_{\mathbf{NVLA_1}}^{\mathbf{NVLA_1^{1+}}}(L)$
(18) $\mathrm{F}_{\mathbf{NVLA_1^1}}^{\mathbf{NVLA_1^{1+}}}(L^1)$

Moreover, the object $\mathrm{F}_{\mathbf{X_1}}^{\mathbf{Y_1}}(X)$ in the above list of 18 solutions to free object problems is generated by the subset $j[X]$.

$\mathbf{X}$ *and* $\mathbf{Y}$ *from Table 3:*

(1) $\mathrm{F}_{\mathbf{Ban_1}}^{\mathbf{BL_1}}(X)$
(2) $\mathrm{F}_{\mathbf{Ban_1}}^{\mathbf{BA_1}}(X)$
(3) $\mathrm{F}_{\mathbf{Ban_1}}^{\mathbf{BA_1^1}}(X)$
(4) $\mathrm{F}_{\mathbf{BA_1}}^{\mathbf{BA_1^1}}(W)$
(5) $\mathrm{F}_{\mathbf{Ban_1}}^{\mathbf{BLA_1}}(X)$
(6) $\mathrm{F}_{\mathbf{BL_1}}^{\mathbf{BLA_1}}(Z)$
(7) $\mathrm{F}_{\mathbf{BA_1}}^{\mathbf{BLA_1}}(W)$
(8) $\mathrm{F}_{\mathbf{Ban_1}}^{\mathbf{BLA_1^1}}(X)$
(9) $\mathrm{F}_{\mathbf{BL_1}}^{\mathbf{BLA_1^1}}(Z)$
(10) $\mathrm{F}_{\mathbf{BA_1}}^{\mathbf{BLA_1^1}}(W)$
(11) $\mathrm{F}_{\mathbf{BA_1^1}}^{\mathbf{BLA_1^1}}(W^1)$
(12) $\mathrm{F}_{\mathbf{BLA_1}}^{\mathbf{BLA_1^1}}(B)$
(13) $\mathrm{F}_{\mathbf{Ban_1}}^{\mathbf{BLA_1^{1+}}}(X)$
(14) $\mathrm{F}_{\mathbf{BL_1}}^{\mathbf{BLA_1^{1+}}}(Z)$
(15) $\mathrm{F}_{\mathbf{BA_1}}^{\mathbf{BLA_1^{1+}}}(W)$
(16) $\mathrm{F}_{\mathbf{BA_1^1}}^{\mathbf{BLA_1^{1+}}}(W^1)$
(17) $\mathrm{F}_{\mathbf{BLA_1}}^{\mathbf{BLA_1^{1+}}}(B)$
(18) $\mathrm{F}_{\mathbf{BLA_1^1}}^{\mathbf{BLA_1^{1+}}}(B^1)$

$\mathbf{X}$ *from Table 2 and* $\mathbf{Y}$ *from Table 3:*

(1) $\mathrm{F}_{\mathbf{NS_1}}^{\mathbf{BL_1}}(N)$
(2) $\mathrm{F}_{\mathbf{NS_1}}^{\mathbf{BA_1}}(N)$
(3) $\mathrm{F}_{\mathbf{NS_1}}^{\mathbf{BA_1^1}}(N)$
(4) $\mathrm{F}_{\mathbf{NA_1}}^{\mathbf{BA_1^1}}(U)$
(5) $\mathrm{F}_{\mathbf{NS_1}}^{\mathbf{BLA_1}}(N)$
(6) $\mathrm{F}_{\mathbf{NVL_1}}^{\mathbf{BLA_1}}(Y)$
(7) $\mathrm{F}_{\mathbf{NA_1}}^{\mathbf{BLA_1}}(U)$
(8) $\mathrm{F}_{\mathbf{NS_1}}^{\mathbf{BLA_1^1}}(N)$
(9) $\mathrm{F}_{\mathbf{NVL_1}}^{\mathbf{BLA_1^1}}(Y)$
(10) $\mathrm{F}_{\mathbf{NA_1}}^{\mathbf{BLA_1^1}}(U)$
(11) $\mathrm{F}_{\mathbf{NA_1^1}}^{\mathbf{BLA_1^1}}(U^1)$
(12) $\mathrm{F}_{\mathbf{NVLA_1}}^{\mathbf{BLA_1^1}}(L)$
(13) $\mathrm{F}_{\mathbf{NS_1}}^{\mathbf{BLA_1^{1+}}}(N)$
(14) $\mathrm{F}_{\mathbf{NVL_1}}^{\mathbf{BLA_1^{1+}}}(Y)$
(15) $\mathrm{F}_{\mathbf{NA_1}}^{\mathbf{BLA_1^{1+}}}(U)$
(16) $\mathrm{F}_{\mathbf{NA_1^1}}^{\mathbf{BLA_1^{1+}}}(U^1)$
(17) $\mathrm{F}_{\mathbf{NVLA_1}}^{\mathbf{BLA_1^{1+}}}(L)$
(18) $\mathrm{F}_{\mathbf{NVLA_1^1}}^{\mathbf{BLA_1^{1+}}}(L^1)$

We conclude this section with remarks on Problems 13 and 15 in [**67**], which we quote here for ease of reference.

QUESTION 3.3.11 ([**67**, Problem 13]). There is a theory of free Banach lattices, [**29**]. Is there a sensible notion of a free Banach lattice algebra? If so, what can be said about its representations?

Using the terminology from [**29**], in Theorem 3.3.6 we proved the existence of the free positive unital Banach lattice algebra over a set. In addition, in Theorems 3.3.7 and 3.3.8 we proved the existence of substantial number of free ((positive) unital) Banach lattice algebras over both algebraic and normed structures. As a result, we can consider the first part of Question 3.3.11 as settled. However, we have nothing to report at this stage regarding representations of these free Banach lattice algebras.

QUESTION 3.3.12 ([**67**, Problem 15]). If A is a non-unital Banach lattice algebra, can it be embedded in a unital Banach lattice superalgebra B in such a way that every lattice algebra homomorphism from A into a unital Banach lattice algebra C extends uniquely to a unital lattice and algebra homomorphism from B into C? If we restrict C to lie in the class of unital Banach lattice algebras with positive identities, can we make B have a positive identity?

We will use some of the pseudo-solutions obtained in this section to get rather close to a full answer for Question 3.3.12. Consider a Banach lattice algebra A. We use some observations from [**27**, Remark 2.3 and Lemma 2.4] in the sequel. It is routine to verify that the vector lattice direct sum $A \oplus \mathbb{R}$ supplied with the multiplication

$$(x,\alpha)\cdot(y,\beta) \coloneqq (\alpha y + \beta x, \alpha\beta)$$

and the norm $\|(x,\alpha)\| \coloneqq \|x\|_B + |\alpha|$ makes $A \oplus \mathbb{R}$ into a positive unital Banach lattice algebra. The inclusion $J : A \to A \oplus \mathbb{R}$ where $J(x) \coloneqq (x, 0)$ is clearly an isometric Banach lattice algebra embedding. For every constant $C \geq 1$, consider $M_C \in \mathcal{M}_b(A)$ where $M_C(x) \coloneqq C\,\|x\|_A$. Since $\|J\| \leq C$, by the universal property of the pseudo-solution $\left(\mathrm{P}^{\mathbf{BLA^{1+}}}_{\mathbf{BLA}}(A, M_C), j_{M_C}\right)$, there exists a unique contractive $\mathbf{BLA^{1+}}$-morphism $\bar{J} : \mathrm{P}^{\mathbf{BLA^{1+}}}_{\mathbf{BLA}}(A, M_C) \to A \oplus \mathbb{R}$ such that $\bar{J} \circ j_{M_C} = J$. Thus, for $x, y \in B$ with $x \neq y$ we have

$$\bar{J}\left(j_{M_C}(x)\right) = J(x) \neq J(y) = \bar{J}\left(j_{M_C}(y)\right)$$

which implies that $j_{M_C}(x) \neq j_{M_C}(y)$. As a result, for every $C \geq 1$, the morphism $j_{M_C} : A \to \mathrm{P}^{\mathbf{BLA^{1+}}}_{\mathbf{BLA}}(A, M_C)$ is an injective $\mathbf{BLA^{1+}}$-morphism where $\|J(x)\| \leq C\,\|x\|$. On the other hand, the definition of the norm in Theorem 3.3.6 implies that for every $x \in X$,

$$\|j_{M_C}(x)\| \geq \left\|\bar{J}\left(j_{M_C}(x)\right)\right\| = \|J(x)\| = \|x\|\,.$$

We conclude that if $C = 1$, then $j_{M_C} : A \to \mathrm{F}^{\mathbf{BLA_1^{1+}}}_{\mathbf{BLA_1}}(A)$ is an isometric $\mathbf{BLA}$-embedding, and if $C > 1$, then $j_{M_C} : A \to \mathrm{P}^{\mathbf{BLA^{1+}}}_{\mathbf{BLA}}(A, M_C)$ is an isomorphic $\mathbf{BLA}$-embedding.

Using the universal property of the free object

$$\left(\mathrm{F}^{\mathbf{BLA_1^{1+}}}_{\mathbf{BLA_1}}(A)\,,j\right)$$

we have the following. The Banach lattice algebra A is isometrically embedded in the positive unital Banach lattice algebra $\mathrm{F}^{\mathbf{BLA_1^{1+}}}_{\mathbf{BLA_1}}(A)$. In addition, for every positive unital Banach lattice algebra B^{1+} and every *contractive* **VLA**-morphism $\varphi : A \to B^{1+}$, there exists a unique *contractive* $\mathbf{VLA^1}$-morphism $\bar{\varphi} : \mathrm{F}^{\mathbf{BLA_1^{1+}}}_{\mathbf{BLA_1}}(A) \to B^{1+}$ extending φ.

Similarly, using the universal property of the pseudo-solutions

$$\left(\mathrm{P}^{\mathbf{BLA^{1+}}}_{\mathbf{BLA}}(A, M_C)\,, j_{M_C}\right)$$

we have the following. For every constant $C > 1$, the Banach lattice algebra A is isomorphically embedded in a positive unital Banach lattice algebra $\mathrm{P}^{\mathbf{BLA^{1+}}}_{\mathbf{BLA}}(A, M_C)$. In addition, for every positive unital Banach lattice algebra B^{1+} and every *bounded* **VLA**-morphism $\varphi : A \to B^{1+}$ such that $\|\varphi\| \le C$, there exists a unique *contractive* $\mathbf{VLA^1}$-morphism $\bar{\varphi} : \mathrm{P}^{\mathbf{BLA^{1+}}}_{\mathbf{BLA}}(A, M_C) \to B^{1+}$ extending φ.

If we consider a unital Banach lattice algebra A^1 along with the Banach lattice algebra A fixed above, one may readily verify that the argument we just performed can also be applied to pseudo-solutions of the following respective forms

$$\left(\mathrm{P}^{\mathbf{BLA^{1}}}_{\mathbf{BLA}}(A, M_C)\,, j_{M_C}\right) \qquad \left(\mathrm{P}^{\mathbf{BLA^{1+}}}_{\mathbf{BLA^1}}(A^1, M_C)\,, j_{M_C}\right)$$

The same argument also applies to the corresponding free objects for categories with contractive morphisms found in Corollary 3.3.10. We summarise our findings on unitisations and extensions of bounded vector lattice algebra homomorphisms in the following proposition.

PROPOSITION 3.3.13. *Let A be a Banach lattice algebra and A^1 a unital Banach lattice algebra. Then the following holds:*

Extensions of contractive *morphisms:*

- *(i) There exists $\mathrm{F}^1(A)$ in $\mathbf{BLA^1}$ into which A is* isometrically *embedded with the property that for every B^1 in $\mathbf{BLA^1}$ and every* contractive **VLA***-morphism $\varphi : A \to B^1$, there exists a unique* contractive $\mathbf{VLA^1}$*-morphism $\bar{\varphi} : \mathrm{F}^1(A) \to B^1$ extending φ.*
- *(ii) There exists $\mathrm{F}^{1+}(A^1)$ in $\mathbf{BLA^{1+}}$ into which A^1 is* isometrically *embedded with the property that for every B^{1+} in $\mathbf{BLA^{1+}}$ and every* contractive $\mathbf{VLA^1}$*-morphism $\varphi : A^1 \to B^{1+}$, there exists a unique* contractive $\mathbf{VLA^1}$*-morphism $\bar{\varphi} : \mathrm{F}^{1+}(A^1) \to B^{1+}$ extending φ.*
- *(iii) There exists $\mathrm{F}^{1+}(A)$ in $\mathbf{BLA^{1+}}$ into which A is* isometrically *embedded with the property that for every B^{1+} in $\mathbf{BLA^{1+}}$ and every* contractive **VLA***-morphism $\varphi : A \to B^{1+}$, there exists a unique* contractive $\mathbf{VLA^1}$*-morphism $\bar{\varphi} : \mathrm{F}^{1+}(A) \to B^{1+}$ extending φ.*

Extensions of bounded *morphisms:*

(i) *For every constant* $C > 1$, *there exists* $\mathrm{P}^1_C(A)$ *in* $\mathbf{BLA^1}$ *into which* A *is* isomorphically *embedded with the property that for every* B^1 *in* $\mathbf{BLA^1}$ *and every* bounded $\mathbf{VLA}$-*morphism* $\varphi : A \to B^1$ *such that* $\|\varphi\| \le C$, *there exists a unique* contractive $\mathbf{VLA^1}$-*morphism* $\bar{\varphi} : \mathrm{P}^1_C(A) \to B^1$ *extending* φ.

(ii) *For every constant* $C > 1$, *there exists* $\mathrm{P}^{1+}_C(A^1)$ *in* $\mathbf{BLA^{1+}}$ *into which* A^1 *is* isomorphically *embedded with the property that for every* B^{1+} *in* $\mathbf{BLA^{1+}}$ *and every* bounded $\mathbf{VLA^1}$-*morphism* $\varphi : A^1 \to B^{1+}$ *such that* $\|\varphi\| \le C$, *there exists a unique* contractive $\mathbf{VLA^1}$-*morphism* $\bar{\varphi} : \mathrm{P}^{1+}_C(A^1) \to B^{1+}$ *extending* φ.

(iii) *For every constant* $C > 1$, *there exists* $\mathrm{P}^{1+}_C(A)$ *in* $\mathbf{BLA^{1+}}$ *into which* A *is* isomorphically *embedded with the property that for every* B^{1+} *in* $\mathbf{BLA^{1+}}$ *and every* bounded $\mathbf{VLA}$-*morphism* $\varphi : A \to B^{1+}$ *such that* $\|\varphi\| \le C$, *there exists a unique* contractive $\mathbf{VLA^1}$-*morphism* $\bar{\varphi} : \mathrm{P}^{1+}_C(A) \to B^{1+}$ *extending* φ.

However, by [**71**, Theorem 18.4], we know that positive operators between Banach lattices are automatically bounded. Thus vector lattice algebra homomorphisms between Banach lattice algebras must also be automatically bounded. With this fact we can rearrange the findings in the last proposition to obtain something relatively close to a full answer for Question 3.3.12.

Corollary 3.3.14. *Let* A *be a Banach lattice algebra and* A^1 *a unital Banach lattice algebra. Then the following holds:*

(i) *Consider* B^1 *in* $\mathbf{BLA^1}$ *and* $\varphi : A \to B^1$ *a* $\mathbf{VLA}$-*morphism, then* φ *is automatically bounded and there exists a constant* $C \ge 1$ *such that* $\|\varphi\| \le C$.

  (a) *If* $C = 1$, *there exists* $\mathrm{F}^1(A)$ *in* $\mathbf{BLA^1}$ *into which* A *is* isometrically *embedded with the property that there exists a unique* contractive $\mathbf{VLA^1}$-*morphism* $\bar{\varphi} : \mathrm{F}^1(A) \to B^1$ *extending* φ.

  (b) *If* $C > 1$, *there exists* $\mathrm{P}^1_C(A)$ *in* $\mathbf{BLA^1}$ *into which* A *is* isomorphically *embedded with the property that there exists a unique* contractive $\mathbf{VLA^1}$-*morphism* $\bar{\varphi} : \mathrm{P}^1_C(A) \to B^1$ *extending* φ.

(ii) *Consider* B^{1+} *in* $\mathbf{BLA^{1+}}$ *and* $\varphi : A^1 \to B^{1+}$ *a* $\mathbf{VLA^1}$-*morphism, then* φ *is automatically bounded and there exists a constant* $C \ge 1$ *such that* $\|\varphi\| \le C$.

  (a) $C = 1$, *there exists* $\mathrm{F}^{1+}(A^1)$ *in* $\mathbf{BLA^{1+}}$ *into which* A^1 *is* isometrically *embedded with the property that there exists a unique* contractive $\mathbf{VLA^1}$-*morphism* $\bar{\varphi} : \mathrm{F}^{1+}(A^1) \to B^{1+}$ *extending* φ.

  (b) *If* $C > 1$, *there exists* $\mathrm{P}^{1+}_C(A^1)$ *in* $\mathbf{BLA^{1+}}$ *into which* A^1 *is* isomorphically *embedded with the property that there exists a unique* contractive $\mathbf{VLA^1}$-*morphism* $\bar{\varphi} : \mathrm{P}^{1+}_C(A^1) \to B^{1+}$ *extending* φ.

(iii) *Consider* B^{1+} *in* $\mathbf{BLA^{1+}}$ *and* $\varphi : A \to B^{1+}$ *a* $\mathbf{VLA}$-*morphism, then* φ *is automatically bounded and there exists a constant* $C \ge 1$ *such that* $\|\varphi\| \le C$.

(a) $C = 1$, *there exists* $\mathrm{F}^{1+}(A)$ *in* $\mathbf{BLA^{1+}}$ *into which* A *is* isometrically *embedded with the property that there exists a unique* contractive $\mathbf{VLA^1}$-*morphism* $\bar{\varphi} : \mathrm{F}^{1+}(A) \to B^{1+}$ *extending* φ.

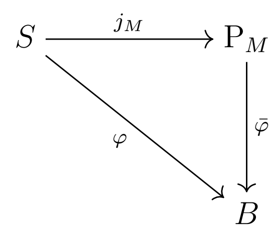
(b) *If* $C > 1$, *there exists* $\mathrm{P}^{1+}_C(A)$ *in* $\mathbf{BLA^{1+}}$ *into which* A *is* isomorphically *embedded with the property that there exists a unique* contractive $\mathbf{VLA^1}$-*morphism* $\bar{\varphi} : \mathrm{P}^{1+}_C(A) \to B^{1+}$ *extending* φ.

3.4. Two concrete families of pseudo-solutions

Using the method of constructing pseudo-solutions outlined in the previous section, we are able to give concrete descriptions of certain pseudo-solutions. This will be done in two cases where the underlying algebraic free object used in the construction of the pseudo-solution is known explicitly.

3.4.1. Pseudo-solution for the free object problem over a point between unital Banach algebras and sets. In this section we consider algebras over $\mathbb{C}$. Fix a one-point set $S := \{s\}$ and fix a constant $M > 0$. We will give a concrete description of the pseudo-solution $\left(\mathrm{P}^{\mathbf{BA^1}}_{\mathbf{Set}}(S, M), j_M\right)$. By omission of certain details, the work to follow will also give concrete descriptions of the pseudo-solutions $(\mathrm{P}^{\mathbf{BA}}_{\mathbf{Set}}(S, M), j_M)$, $\left(\mathrm{P}^{\mathbf{NA^1}}_{\mathbf{Set}}(S, M), j_M\right)$, and $(\mathrm{P}^{\mathbf{NA}}_{\mathbf{Set}}(S, M), j_M)$.

For the sake of brevity, we write $\mathrm{P}_M := \mathrm{P}^{\mathbf{BA^1}}_{\mathbf{Set}}(S, M)$. The pair (P_M, j_M) satisfies the following universal property: For every $B \in \mathbf{BA^1}$ and every morphism $\varphi : S \to B$ such that $\|\varphi(s)\| \leq M$ there exists a unique bounded unital algebra homomorphism $\bar{\varphi} : \mathrm{P}_M \to B$ such that the following diagram commutes in **Set**.



Recall that for a given $B \in \mathbf{BA^1}$, the collection of morphisms $\varphi : S \to B$ such that $\|\varphi(s)\| \leq M$ is denoted by $\Phi_M(S, B)$ and the associated collection of morphisms $\bar{\varphi}$ in $\mathbf{Alg^1}$ that uniquely factor through (P_M, j_M) is denoted by $\Phi'_M(S, B)$.

As outlined in Section 3.3, the object P_M is obtained by starting with the free unital algebra over a one-point set $(\mathbb{C}[X], j')$ where $\mathbb{C}[X]$ is the polynomial ring over $\mathbb{C}$ in one variable and $j'(s) := X$. As was done in Theorem 3.3.6, define $\rho_M : \mathbb{C}[X] \to \mathbb{R}$ where

$$\rho_M(p) := \sup\left\{\|\bar{\varphi}(p)\| \ : \ B \in \mathbf{BA^1},\ \bar{\varphi} \in \Phi'_M(S, B)\right\}.$$

Since the morphisms $\bar{\varphi} \in \Phi'_M(S, B)$ are unital algebra homomorphisms, it follows that for $p = \sum_{n=0}^{k} a_n X^n \in \mathbb{C}[X]$, we have

$$\|\bar{\varphi}(p)\| = \left\|\sum_{n=0}^{k} a_n \bar{\varphi}(X)^n\right\| \leq \sum_{n=0}^{k} |a_n| \, \|\varphi(s)\|^n \leq \sum_{n=0}^{k} |a_n| M^n.$$

Thus $\rho_M(p) \leq \sum_{n=0}^{k} |a_n| M^n$. We claim that this last inequality is in fact an equality for every $p = \sum_{n=0}^{k} a_n X^n \in \mathbb{C}[X]$. For the reverse inequality, consider the following:

Denote $\mathbb{N}_0 := \{0, 1, 2, \ldots\}$ and define the *weight* $w_M : \mathbb{N}_0 \to \mathbb{N}$ where $w_M(n) := M^n$ and consider the *weighted sequence space*

$$\ell^1(\mathbb{N}_0, w_M) := \left\{ (x_n) \in \mathbb{R}^{\mathbb{N}_0} \;:\; \sum_{n=0}^{\infty} |x_n| w_M(n) < \infty \right\}.$$

The space $\ell^1(\mathbb{N}_0, w_M)$ is made into a Banach space when equipped with the coordinate-wise vector space operations and the weighted norm

$$\|x\| := \sum_{n=0}^{\infty} |x_n| w_M(n), \qquad \left(x = (x_n) \in \ell^1(\mathbb{N}_0, w_M)\right).$$

For $n \in \mathbb{N}_0$, define the element $\delta_n \in \ell^1(\mathbb{N}_0, w_M)$ where

$$\delta_n(k) := \begin{cases} 1 \text{ if } k = n, \\ 0 \text{ if } k \neq n. \end{cases}$$

In addition, the space $\ell^1(\mathbb{N}_0, w_M)$ is made into a unital Banach algebra when equipped with the *Cauchy product*: For $x = (x_n) \in \ell^1(\mathbb{N}_0, w_M)$ and $y = (y_n) \in \ell^1(\mathbb{N}_0, w_M)$, the n^{th} index of the product xy is given by

$$(xy)(n) := \sum_{i=0}^{n} x_i y_{n-i}.$$

The definition of the Cauchy product makes it clear that the element δ_0 is the multiplicative identity of $\ell^1(\mathbb{N}_0, w_M)$. Further, we also have that $(\delta_1)^n = \delta_n$ and $\|\delta_n\| = M^n$ for $n \in \mathbb{N}_0$. A standard series rearrangement argument shows that $\|xy\| \leq \|x\| \, \|y\|$ for $x, y \in \ell^1(\mathbb{N}_0, w_M)$

Returning to the pseudo-solution (P_M, j_M), consider the morphism $\varphi : S \to \ell^1(\mathbb{N}_0, w_M)$ where $\varphi(s) := \delta_1$. Then $\|\varphi(s)\| = \|\delta_1\| = M$ and so there exists a unique morphism $\bar{\varphi} : \mathbb{C}[X] \to \ell^1(\mathbb{N}_0, w_M)$ in $\mathbf{Alg^1}$ factoring through $(\mathbb{C}[X], j')$. By the definition of ρ_M, for $p = \sum_{n=0}^{k} a_n X^n \in \mathbb{C}[X]$, we have

$$\begin{aligned} \rho_M(p) \geq \|\bar{\varphi}(p)\| &= \left\| \sum_{n=0}^{k} a_n (\bar{\varphi}(X))^n \right\| = \left\| \sum_{n=0}^{k} a_n (\delta_1)^n \right\| \\ &= \left\| \sum_{n=0}^{k} a_n \delta_n \right\| = \sum_{n=0}^{k} |a_n| M^n. \end{aligned}$$

The desired equality follows and it is easy to see that ρ_M actually forms a submultiplicative norm on $\mathbb{C}[X]$ and thus $(\mathbb{C}[X], \rho_M)$ is an object in $\mathbf{NA^1}$. Following the general approach in Section 3.3, it only remains to complete $(\mathbb{C}[X], \rho_M)$ in order to obtain our desired pseudo-solution.

Indeed, the weighted sequence space $\ell^1(\mathbb{N}_0, w_M)$ turns out to be a completion of the unital normed algebra $(\mathbb{C}[X], \rho_M)$ since the map $J : \mathbb{C}[X] \to \ell^1(\mathbb{N}_0, w_M)$ where $\sum_{n=0}^{k} a_n X^n \longmapsto \sum_{n=0}^{k} a_n (\delta_1)^n$ is an isometric unital algebra homomorphism and $J[\mathbb{C}[X]]$ is dense in $\ell^1(\mathbb{N}_0, w_M)$.

Define $j : S \to \ell^1(\mathbb{N}_0, w_M)$ where $j(s) := \delta_1$. Then the pair $(\ell^1(\mathbb{N}_0, w_M), j)$ is the pseudo-solution for the free object problem over S between the categories $\mathbf{BA^1}$ and $\mathbf{Set}$: Let $B \in \mathbf{BA^1}$ and $\varphi : S \to B$ a morphism in $\Phi_M(S, B)$, then there exists a unique morphism $\tilde{\varphi} : \mathbb{C}[X] \to B$ in $\mathbf{Alg^1}$ factoring through $(\mathbb{C}[X], j')$. By definition of the norm ρ_M, the morphism $\tilde{\varphi}$ is bounded. Since $\ell^1(\mathbb{N}_0, w_M)$ is a completion of $(\mathbb{C}[X], \rho_M)$, there exist a morphism $\bar{\varphi} : \ell^1(\mathbb{N}_0, w_M) \to B$ in $\mathbf{BA^1}$ uniquely extending $\tilde{\varphi}$ from $\mathbb{C}[X]$. The morphism $\bar{\varphi}$ satisfies the necessary universal property:

$$\bar{\varphi} \circ j(s) = \bar{\varphi}(\delta_1) = \bar{\varphi}(J(X)) = \tilde{\varphi}(X) = \tilde{\varphi} \circ j'(s) = \varphi(s).$$

Thus for $x = \sum_{n=0}^{\infty} x_n (\delta_1)^n \in \ell^1(\mathbb{N}_0, w_M)$, we have

$$\bar{\varphi}(x) = \sum_{n=0}^{\infty} x_n (\varphi(s))^n .$$

If we let $\bar{\psi} : \ell^1(\mathbb{N}_0, w_M) \to B$ be any other morphism in $\mathbf{BA^1}$ such that $\bar{\psi} \circ j = \varphi$, then it is easily seen that $\bar{\varphi}$ and $\bar{\psi}$ coincide on $J[\mathbb{C}[X]]$ and by density it follows that $\bar{\varphi} = \bar{\psi}$. Thus we have a concrete description of the pseudo-solution $\left(\mathrm{P}^{\mathbf{BA^1}}_{\mathbf{Set}}(S, M), j_M\right)$.

If we remove all mention of multiplicative identities above, the above derivation will show that $(\ell^1(\mathbb{N}, w_M), j)$ is also a concrete description of $(\mathrm{P}^{\mathbf{BA}}_{\mathbf{Set}}(S, M), j_M)$. Further, if we replace all instances of the category $\mathbf{BA^1}$ with $\mathbf{NA^1}$, the above argument will show that the pair $((\mathbb{C}[X], \rho_M), j')$ is a concrete description of $\left(\mathrm{P}^{\mathbf{NA^1}}_{\mathbf{Set}}(S, M), j_M\right)$. Lastly, if we take this last modification of the above argument and remove all mention of multiplicative identities, then the pair $((\mathbb{C}[X], \rho_M), j')$ is also a concrete description of $(\mathrm{P}^{\mathbf{NA}}_{\mathbf{Set}}(S, M), j_M)$.

3.4.2. Pseudo-solution for free object problem over a set between Banach spaces and sets. Let S be a set and denote by $\mathcal{M}_{>0}(S)$ the elements $M' \in \mathcal{M}(S)$ such that $M'(s) > 0$ for all $s \in S$. Fix some $M \in \mathcal{M}_{>0}(S)$. In this section we give a concrete description of the pseudo-solution $(\mathrm{P}^{\mathbf{Ban}}_{\mathbf{Set}}(S, M), j_M)$. Following the same procedure as in the previous section, we start with the free vector space (V_S, j') where V_S consists of functions $f : S \to \mathbb{C}$ with finite support and $j'(s) := e_s$ for all $s \in S$. The collection $\{e_s : s \in S\}$ is a basis for V_S. For every $X \in \mathbf{Ban}$ and every $\varphi \in \Phi_M(S, X)$ there exists a unique morphism $\bar{\varphi} : V_S \to X$ in $\mathbf{VS}$ factoring through (V_S, j'). Define $\rho_M : V_S \to \mathbb{R}$ where

$$\rho_M(f) := \sup\left\{\|\bar{\varphi}(f)\| \ : \ X \in \mathbf{Ban},\ \bar{\varphi} \in \Phi'_M(S, X)\right\}.$$

As was done before, we show that ρ_M is a norm on V_S. Take $f = \sum_{s\in F} \alpha_s e_s \in V_S$ where F is a finite subset of S. Then for any $\varphi \in \Phi_M(S, X)$ with unique associated morphism $\bar{\varphi} : V_S \to X$, we have

$$\|\bar{\varphi}(f)\| = \left\|\sum_{s\in F} \alpha_s \bar{\varphi}(e_s)\right\| \leq \sum_{s\in F} |\alpha_s| M(s).$$

Thus $\rho_M(f) \leq \sum_{s\in F} |\alpha_s| M(s)$. For the reverse inequality, consider the weighted ℓ^1 space

$$\ell^1(S, M) := \left\{ f \in \mathbb{R}^S \ : \ \sum_{s\in S} |f(s)| M(s) < \infty \right\}.$$

Denote by $\mathcal{F}(S)$ the collection of all finite subsets of S. Note that the quantity $\sum_{s\in S} |f(s)| M(s)$ is defined if the net $\left(\sum_{s\in F} |f(s)| M(s)\right)_{F\in\mathcal{F}(S)}$ converges in $\mathbb{R}$. For every $s \in S$, define the element $\delta_s \in \ell^1(S, M)$ where

$$\delta_s(t) \coloneqq \begin{cases} 1 \text{ if } t = s, \\ 0 \text{ if } t \neq s. \end{cases}$$

The space $\ell^1(S, M)$ is made into a Banach space when equipped with the pointwise vector space operations and the norm

$$\|f\|_M \coloneqq \sum_{s \in S} |f(s)| M(s), \qquad \left(f \in \ell^1(S, M)\right).$$

Indeed, it is clear that $\|\cdot\|_{M'}$ is a seminorm for any $M' \in \mathcal{M}(S)$ and $\|\cdot\|_{M'}$ is a norm if and only if $M' \in \mathcal{M}_{>0}(S)$. Now, define the morphism $\varphi : S \to \ell^1(S, M)$ where $\varphi(s) := \delta_s$ and let $\bar{\varphi} : V_S \to \ell^1(S, M)$ be the unique morphism in **VS** factoring through (V_S, j'). Then for every $f = \sum_{s \in F} \alpha_s e_s \in V_S$, we have

$$\rho_M(f) \geq \|\bar{\varphi}(f)\|_M = \left\| \sum_{s \in F} \alpha_s \bar{\varphi}(e_s) \right\|_M = \left\| \sum_{s \in F} \alpha_s \delta_s \right\|_M = \sum_{s \in F} |\alpha_s| M(s).$$

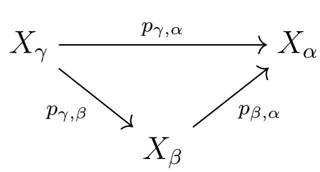
Thus the pair (V_S, ρ_M) is a normed space and the map $J : V_S \to \ell^1(S, M)$ where $\sum_{s \in F} \alpha_s e_s \longmapsto \sum_{s \in F} \alpha_s \delta_s$ is an isometry between normed spaces with dense image. Thus $\ell^1(S, M)$ is a completion of (V_S, ρ_M). Define $j : S \to \ell^1(S, M)$ where $j(s) \coloneqq \delta_s$. The argument in the previous section can also be used to conclude that $(\ell^1(S, M), j)$ is a concrete description of the pseudo-solution $(\mathrm{P}^{\mathbf{Ban}}_{\mathbf{Set}}(S, M), j_M)$. By replacing all instances of **Ban** with **NS** and omitting the completion, it will follow that the pair $((V_S, \rho_M), j')$ is a concrete description of the pseudo-solution $(\mathrm{P}^{\mathbf{NS}}_{\mathbf{Set}}(S, M), j_M)$.

3.5. Free objects and inverse limits

In Section 3.2, we gave an account of the algebraic free objects we have at our disposal. These were used in Section 3.3 to prove the existence of a number of pseudo-solutions to free object problems. In this section, we develop some general categorical machinery that we will use to construct free objects in categories of locally convex structures. The central part of this machinery is the notion of an *inverse limit* which we used extensively in Chapter 2.

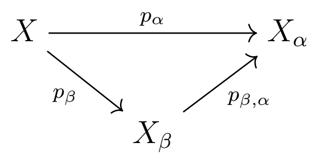
3.5.1. Inverse systems and limits. In order to make the reading of this chapter independent from Chapter 2, we recall the definitions of *inverse systems* and *inverse limits*.

Definition 3.5.1. Let **C** be any category and $(I, \preccurlyeq)$ an upwards directed set. Consider a family of objects $\{X_\alpha\}_{\alpha \in I}$ in **C**. For all $\beta \succcurlyeq \alpha$ in I, let $p_{\beta,\alpha} : X_\beta \to X_\alpha$ be a morphism in **C**. The pair $\mathcal{P} \coloneqq ((X_\alpha)_{\alpha \in I}, (p_{\beta,\alpha})_{\beta \succcurlyeq \alpha})$ is called an *inverse system* in **C** if, for all $\alpha \preccurlyeq \beta \preccurlyeq \gamma$ in I, the diagram



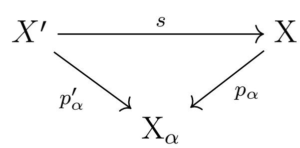
commutes in **C**.

Let X be an object in $\mathbf{C}$ and $p_\alpha : X \to X_\alpha$ a morphism in $\mathbf{C}$ for every $\alpha \in I$. The pair $(X, (p_\alpha)_{\alpha\in I})$ is called a *compatible system over* $\mathcal{P}$ *in* $\mathbf{C}$ if for all $\alpha \preccurlyeq \beta$ in I the diagram



commutes in $\mathbf{C}$.

Lastly, an *inverse limit of* $\mathcal{P}$ *in* $\mathbf{C}$ is a compatible system $(X, (p_\alpha)_{\alpha\in I})$ in $\mathbf{C}$ satisfying the property that for any compatible system $(X', (p'_\alpha)_{\alpha\in I})$ in $\mathbf{C}$ there exists a unique morphism $s : X' \to X$ in $\mathbf{C}$ so that for all $\alpha \in I$ the diagram



commutes in $\mathbf{C}$.

Recall the following result, proven in [**21**, Chapter III, §7.1], which we will make use of in the sequel.

THEOREM 3.5.2 (Canonical inverse limit in **Set**). *Let* $\mathcal{I} := ((X_\alpha)_{\alpha\in I}, (p_{\beta,\alpha})_{\alpha\preccurlyeq\beta})$ *a inverse system in* **Set**. *Define the set*

$$X := \left\{ (x_\alpha)_{\alpha\in I} \in \prod_{\alpha\in I} X_\alpha : p_{\beta,\alpha}(x_\beta) = x_\alpha, \ \forall \beta \in I, \beta \succcurlyeq \alpha \right\}.$$

For every $\alpha \in I$, *let* $\pi_\alpha : \prod_{\alpha\in I} X_\alpha \to X_\alpha$ *denote the coordinate projections and define* $p_\alpha := \pi_\alpha|_X$. *Then the pair* $(X, (p_\alpha)_{\alpha\in I})$ *is the inverse limit of* $\mathcal{I}$ *in the category* **Set**.

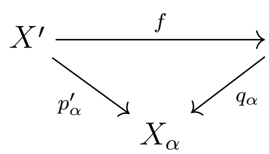
In Chapter 2, Section 2.4, we argued using the universal property of an inverse limit that an inverse limit of an inverse system is essentially unique in a similar sense as that for free objects. In the sequel, we will often have to consider the following set-up: Let $\mathbf{C}$ and $\mathbf{D}$ be categories with $\mathbf{C} \subseteq \mathbf{D}$. Consider an inverse system $\mathcal{I} := ((X_\alpha)_{\alpha\in I}, (p_{\beta,\alpha})_{\beta\succcurlyeq\alpha})$ in $\mathbf{C}$. Since the objects and morphisms in $\mathbf{C}$ are among those in $\mathbf{D}$, we can also consider $\mathcal{I}$ as an inverse system in $\mathbf{D}$. It may then happen that $\mathcal{S}_{\mathbf{C}} := (X, (p_\alpha)_{\alpha\in I})$ is an inverse limit of $\mathcal{I}$ in $\mathbf{C}$ and $\mathcal{S}_{\mathbf{D}} := (Y, (q_\alpha)_{\alpha\in I})$ is an inverse limit of $\mathcal{I}$ in $\mathbf{D}$.

These inverse limits need not coincide in general and it may also happen that the inverse limit of $\mathcal{I}$ exists in one of these categories but not in the other: If we suppose that an inverse limit $\mathcal{S}_{\mathbf{C}}$ of $\mathcal{I}$ exists in $\mathbf{C}$, then $\mathcal{S}_{\mathbf{C}}$ need not satisfy the universal property of an inverse limit with respect to all compatible systems over $\mathcal{I}$ in $\mathbf{D}$.

On the other hand, if an inverse limit $\mathcal{S}_{\mathbf{D}}$ of $\mathcal{I}$ exists in $\mathbf{D}$, it is the case that all compatible systems $(X', (p'_\alpha)_{\alpha\in I})$ over $\mathcal{I}$ in $\mathbf{C}$ are also compatible systems over $\mathcal{I}$ in

D, but $\mathcal{S}_{\mathbf{D}}$ need not necessarily be a compatible system over $\mathcal{I}$ in **C**, i.e. the object or the morphisms in $\mathcal{S}_{\mathbf{D}}$ are not found in the smaller category **C**.

Further, it may even be the case that an inverse limit $\mathcal{S}_{\mathbf{D}}$ of $\mathcal{I}$ in **D** is also a compatible system over $\mathcal{I}$ in **C** and yet $\mathcal{S}_{\mathbf{D}}$ need not be an inverse limit of $\mathcal{I}$ in **C**: For a compatible system $(X', (p'_\alpha)_{\alpha\in I})$ over $\mathcal{I}$ in **C**, there may be a unique morphism $s : X' \to Y$ *in* ***D*** making the diagram



commute *in* ***D*** for all $\alpha \in I$, but s may not be a morphism *in* ***C*** and thus $\mathcal{S}_{\mathbf{D}}$ would not be an inverse limit of $\mathcal{I}$ *in* ***C***.

In light of the above, when considering inverse systems residing in more than one category, it is vital to unambiguously state in which category a purported inverse limit exists. These observations along with the definition of a full subcategory (see Section 3.1.3) give us the following result.

PROPOSITION 3.5.3. *Let* **C** *and* **D** *be categories with* $\mathbf{C} \subseteq \mathbf{D}$ *and consider an inverse system* $\mathcal{I} := ((X_\alpha)_{\alpha\in I}, (p_{\beta,\alpha})_{\beta\geqslant\alpha})$ *in* **C** *with inverse limit* $\mathcal{S}_{\mathbf{D}}$ *in* **D**. *Assume that the following holds:*

(i) **C** *is a full subcategory in* **D**.

(ii) The inverse limit $\mathcal{S}_{\mathbf{D}}$ *in* **D** *is also a compatible system over* $\mathcal{I}$ *in* **C**.

Then $\mathcal{S}_{\mathbf{D}}$ *is an inverse limit of* $\mathcal{I}$ *in* **C**.

The following definition will be important in the sequel.

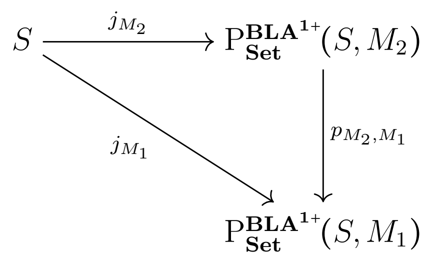
DEFINITION 3.5.4. Let **C** and **D** be categories with $\mathbf{C} \subseteq \mathbf{D}$. We assume that for every inverse system $\mathcal{I}$ in **C** an inverse limit $\mathcal{S}_{\mathcal{I}}$ of $\mathcal{I}$ exists in **D**. Denote by $X(\mathcal{S}_{\mathcal{I}}, \mathcal{I}) \in \mathbf{D}$ the object in an inverse limit $\mathcal{S}_{\mathcal{I}}$ of an inverse system $\mathcal{I}$ in **D**. Denote by $\mathbf{D}\text{-}\varprojlim \mathbf{C}$ the full subcategory of **D** whose objects are such $X(\mathcal{S}_{\mathcal{I}}, \mathcal{I})$ in **D** obtained from inverse limits of inverse systems $\mathcal{I}$ in **C**. The category $\mathbf{D}\text{-}\varprojlim \mathbf{C}$ will be called a *category of inverse limits*.

To avoid any possible ambiguity, the notation for a category of inverse limits explicitly notes both which category the concerned inverse systems come from and in what category the inverse limit is taken.

REMARK 3.5.5. Let **C** be any category with $X \in \mathbf{C}$. We note that the pair $\mathcal{T} := (X, \mathbf{1}_X)$ trivially satisfies the definition of an inverse system. Further, for any $A \in \mathbf{C}$ and any morphism $f : A \to X$ in **C**, the pair (A, f) is a compatible system over $\mathcal{T}$ in **C**. It is then easy to see that $\mathcal{T}$ is the inverse limit of itself in **C**. As a result, when considering categories **C** and **D** which satisfy the conditions in Definition 3.5.4, it follows that $\mathbf{C} \subseteq \mathbf{D}\text{-}\varprojlim \mathbf{C}$.

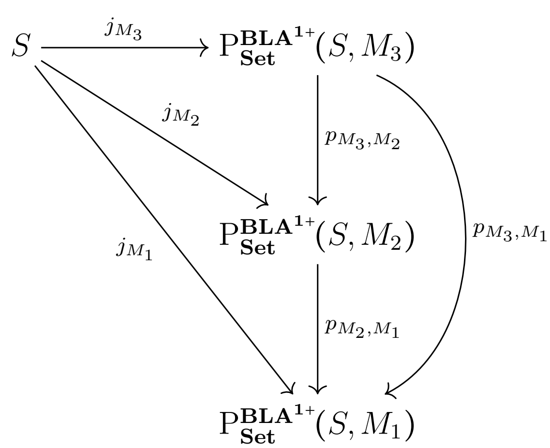
With these preliminaries in place, it is not difficult to use the universal property in Definition 3.3.4 to show that pseudo-solutions $(\mathrm{P}_{\mathbf{X}}^{\mathbf{Y}}(S,M), j_M)$ varying over $M \in \mathcal{M}(X)$ form an inverse system in $\mathbf{Y}$. We prove this for the particular case of $\mathbf{X} = \mathbf{Set}$ and $\mathbf{Y} = \mathbf{BLA^{1+}}$. The details remain the same for other choices of $\mathbf{X}$ and $\mathbf{Y}$ with $\mathbf{X} \supseteq \mathbf{Y}$.

Proposition 3.5.6. *Fix a set S and for every $M \in \mathcal{M}(S)$ consider the pair $\big(\mathrm{P}_{\mathbf{Set}}^{\mathbf{BLA^{1+}}}(S,M), j_M\big)$ constructed in Theorem 3.3.6. For every $M_1, M_2 \in \mathcal{M}(S)$ with $M_2 \geq M_1$, there exists a unique contractive morphism $p_{M_2,M_1} : \mathrm{P}_{\mathbf{Set}}^{\mathbf{BLA^{1+}}}(S,M_2) \to \mathrm{P}_{\mathbf{Set}}^{\mathbf{BLA^{1+}}}(S,M_1)$ in $\mathbf{BLA^{1+}}$ such that the following diagram commutes in $\mathbf{Set}$.*



Then $\mathcal{P} \coloneqq \big((\mathrm{P}_{\mathbf{Set}}^{\mathbf{BLA^{1+}}}(S,M))_{M\in\mathcal{M}(S)}, (p_{M_2,M_1})_{M_2\geq M_1}\big)$ is an inverse system in $\mathbf{BLA^{1+}}$.

Proof. Given $M_1, M_2 \in \mathcal{M}(S)$ with $M_2 \geq M_1$, the existence of the unique contractive morphism $p_{M_2,M_1} : \mathrm{P}_{\mathbf{Set}}^{\mathbf{BLA^{1+}}}(S,M_2) \to \mathrm{P}_{\mathbf{Set}}^{\mathbf{BLA^{1+}}}(S,M_1)$ in $\mathbf{BLA^{1+}}$ follows by Theorem 3.3.6 since $\|j_{M_1}(s)\| \leq M_1(s) \leq M_2(s)$ for all $s \in S$. Now, for $M_1, M_2, M_3 \in \mathcal{M}(S)$ with $M_3 \geq M_2 \geq M_1$ we have the following diagram.



It follows that

$$(p_{M_2,M_1} \circ p_{M_3,M_2}) \circ j_{M_3} = p_{M_2,M_1} \circ (p_{M_3,M_2} \circ j_{M_3}) = p_{M_2,M_1} \circ j_{M_2} = j_{M_1}.$$

Since p_{M_3,M_1} is the unique map satisfying $p_{M_3,M_1} \circ j_{M_3} = j_{M_1}$, we conclude that $p_{M_3,M_1} = p_{M_2,M_1} \circ p_{M_3,M_2}$. Thus $\mathcal{P} \coloneqq \big((\mathrm{P}_{\mathbf{Set}}^{\mathbf{BLA^{1+}}}(S,M))_{M\in\mathcal{M}(S)}, (p_{M_2,M_1})_{M_2\geq M_1}\big)$ is an inverse system in $\mathbf{BLA^{1+}}$. □

It is clear from the general argument displayed in Proposition 3.5.6 that the concrete descriptions of pseudo-solutions in Section 3.4 will also form inverse systems. In the following remark we characterise these inverse systems for use in the sequel.

REMARK 3.5.7. Let $S_0 := \{s_0\}$ be any one-point set. For $M' \geq M > 0$, consider the pseudo-solutions $(\ell^1(\mathbb{N}_0, w_{M'}), j_{M'})$ and $(\ell^1(\mathbb{N}_0, w_M), j_M)$ found in Section 3.4.1. We write $\delta_n^{(M)}$, for example, to denote that the sequence is considered as a member of $\ell^1(\mathbb{N}_0, w_M)$. In this case, $\left\|\delta_n^{(M)}\right\| = M^n$. Since $\|j_M(s_0)\| = \left\|\delta_1^{(M)}\right\| = M \leq M'$, there exists a unique morphism $p_{M',M} : \ell^1(\mathbb{N}_0, w_{M'}) \to \ell^1(\mathbb{N}_0, w_M)$ in $\mathbf{BA^1}$ such that $p_{M',M} \circ j_{M'} = j_M$. The argument in Proposition 3.5.6 will also show that

$$\left((\ell^1(\mathbb{N}_0, w_M))_{M \in \mathbb{R}^+ \setminus \{0\}}, (p_{M',M})_{M' \geq M}\right)$$

is an inverse system in $\mathbf{BA^1}$. We characterise the linking maps $p_{M',M}$: Fix $M' \geq M > 0$ and consider $x = \sum_{n=0}^{\infty} x_n \left(\delta_1^{(M')}\right)^n \in \ell^1(\mathbb{N}_0, w_{M'})$, then

$$p_{M',M}(x) = \sum_{n=0}^{\infty} x_n \left(p_{M',M}(j_{M'}(s_0))\right)^n = \sum_{n=0}^{\infty} x_n \left(\delta_1^{(M)}\right)^n = x.$$

Thus the linking maps $p_{M',M}$ are inclusions.

Similarly, fix any set S. For $M_1, M_2 \in \mathcal{M}_{>0}(S)$ with $M_2 \geq M_1$, consider the pseudo-solutions $(\ell^1(S, M_1), j_{M_1})$ and $(\ell^1(S, M_2), j_{M_2})$ found in Section 3.4.2. As above, there exists a unique morphism $p_{M_2,M_1} : \ell^1(S, M_2) \to \ell^1(S, M_1)$ in $\mathbf{Ban}$ satisfying $p_{M_2,M_1} \circ j_{M_2} = j_{M_1}$ giving us the inverse system

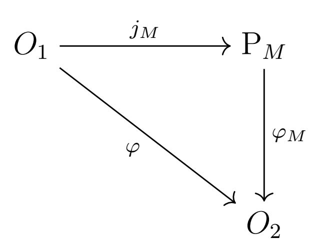
$$\left((\ell^1(S, M))_{M \in \mathcal{M}_{>0}(S)}, (p_{M_2,M_1})_{M_2 \geq M_1}\right)$$

in $\mathbf{Ban}$. Since the span of $\{\delta_s^{(M_2)} : s \in S\}$ is dense in $\ell^1(S, M_2)$, it follows by continuity that the linking maps p_{M_2,M_1} are also inclusion maps.

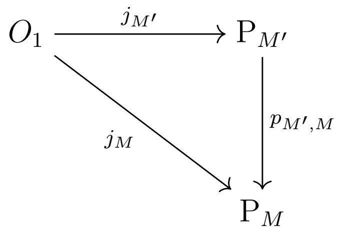
Having shown that our pseudo-solutions form inverse systems, the following purely categorical lemma will allow us to find free objects in categories of inverse limits.

LEMMA 3.5.8. *Let $\mathbf{C}_1$ and $\mathbf{C}_2$ be categories such that $\mathbf{C}_2 \subseteq \mathbf{C}_1$. Fix $O_1 \in \mathbf{C}_1$ and consider an upwards directed set $(\mathcal{M}, \leq)$. For every $M \in \mathcal{M}$, let P_M be an object in $\mathbf{C}_2$ and $j_M : O_1 \to \mathrm{P}_M$ a morphism in $\mathbf{C}_1$ such that the following holds:*

(i) For every $O_2 \in \mathbf{C}_2$ and every morphism $\varphi : O_1 \to O_2$ in $\mathbf{C}_1$, there exists $M_0 \in \mathcal{M}$ such that for all $M \geq M_0$ there exists a unique morphism $\varphi_M : \mathrm{P}_M \to O_2$ in $\mathbf{C}_2$ such that the following diagram commutes in $\mathbf{C}_1$.

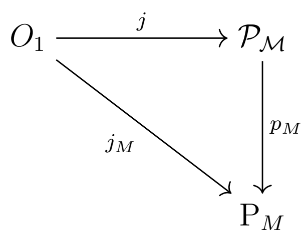


(ii) For every $M, M' \in \mathcal{M}$ *with* $M' \geq M$*, there exists a unique morphism* $p_{M',M} : \mathrm{P}_{M'} \to \mathrm{P}_M$ *in* $\mathbf{C}_2$ *such that the following diagram commutes in* $\mathbf{C}_1$*.*



Thus $\mathcal{P} \coloneqq ((\mathrm{P}_M)_{M \in \mathcal{M}}, (p_{M',M})_{M' \geq M})$ *is an inverse system in* $\mathbf{C}_2$ *and* $(O_1, (j_M)_{M \in \mathcal{M}})$ *is a compatible system over* $\mathcal{P}$ *in* $\mathbf{C}_1$*.*

(iii) Let $\mathbf{C}_3$ *be a category with* $\mathbf{C}_2 \subseteq \mathbf{C}_3 \subseteq \mathbf{C}_1$ *such that* $\mathcal{P}$ *has an inverse limit* $(\mathcal{P}_{\mathcal{M}}, (p_M)_{M \in \mathcal{M}})$ *in* $\mathbf{C}_3$ *where* $(\mathcal{P}_{\mathcal{M}}, (p_M)_{M \in \mathcal{M}})$ *is also an inverse limit of* $\mathcal{P}$ *in* $\mathbf{C}_1$*. Let* $j : O_1 \to \mathcal{P}_{\mathcal{M}}$ *be the unique morphism in* $\mathbf{C}_1$ *making the following diagram commute in* $\mathbf{C}_1$ *for all* $M \in \mathcal{M}$*.*

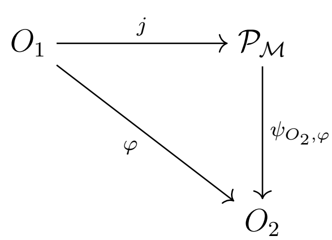


Then there exists a family of morphisms in $\mathbf{C}_3$

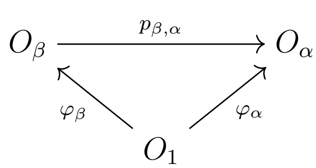
$$\Psi \coloneqq \{\psi_{O_2,\varphi} : \mathcal{P}_{\mathcal{M}} \to O_2 \ : \ O_2 \in \mathbf{C}_2, \ \varphi \in \mathrm{Hom}_{\mathbf{C}_1}(O_1, O_2)\}$$

satisfying the following two properties:

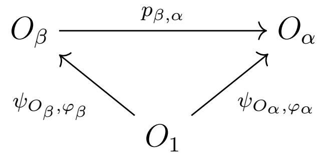
(a) For every $O_2 \in \mathbf{C}_2$ *and every morphism* $\varphi : O_1 \to O_2$ *in* $\mathbf{C}_1$*, the morphism* $\psi_{O_2,\varphi} : \mathcal{P}_{\mathcal{M}} \to O_2$ *in* $\mathbf{C}_3$ *makes the following diagram commutes in* $\mathbf{C}_1$*.*



(b) Let O_α *and* O_β *be objects in* $\mathbf{C}_2$ *with* $\varphi_\alpha : O_1 \to O_\alpha$ *and* $\varphi_\beta : O_1 \to O_\beta$ *morphisms in* $\mathbf{C}_1$*. If there exists a morphism* $p_{\beta,\alpha} : O_\beta \to O_\alpha$ *in* $\mathbf{C}_2$ *such that the following diagram commutes in* $\mathbf{C}_1$*,*

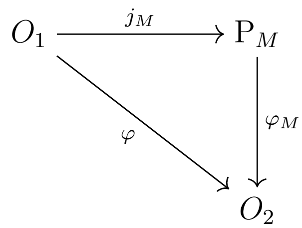


then the morphisms $\psi_{O_\alpha,\varphi_\alpha}, \psi_{O_\beta,\varphi_\beta} \in \Psi$ *make the following diagram commute in* $\mathbf{C}_3$.

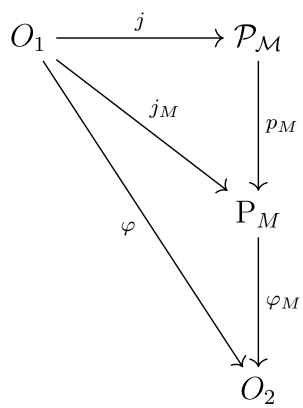


Proof. It follows by the same reasoning as in the proof of Proposition 3.5.6 that the pair $\mathcal{P} = ((\mathrm{P}_M)_{M\in\mathcal{M}}, (p_{M',M})_{M'\geq M})$ is an inverse system in $\mathbf{C}_2$ (thus also in $\mathbf{C}_3$ and $\mathbf{C}_1$). The diagram in (ii) above makes it clear that $(O_1, (j_M)_{M\in\mathcal{M}})$ is a compatible system over $\mathcal{P}$ in $\mathbf{C}_1$.

We first construct the family of morphisms Ψ: Fix an object O_2 in $\mathbf{C}_2$ and a morphism $\varphi : O_1 \to O_2$ in $\mathbf{C}_1$. By (i) above, there exists $M_0 \in \mathcal{M}$ such that for all $M \geq M_0$ there exists a unique morphism $\varphi_M : \mathrm{P}_M \to O_2$ in $\mathbf{C}_2$ such that the following diagram commutes in $\mathbf{C}_1$.



Fixing $M_0 \in \mathcal{M}$, the previous diagram may be extended for each $M \geq M_0$.



The large outer triangle commutes in $\mathbf{C}_1$ for each $M \geq M_0$ since $(\varphi_M \circ p_M) \circ j = \varphi_M \circ (p_M \circ j) = \varphi_M \circ j_M = \varphi$. Thus we may define $\psi_{O_2,\varphi} \coloneqq \varphi_M \circ p_M$ in $\mathbf{C}_3$ if it can be verified that this definition is independent of the choice of $M \in \mathcal{M}$ with $M \geq M_0$. Since $\mathcal{M}$ is upwards directed, it is sufficient to consider $M, M' \in \mathcal{M}$ with $M' \geq M \geq M_0$. Since $M' \geq M$ and $(\mathcal{P}_{\mathcal{M}}, (p_M)_{M\in\mathcal{M}})$ is a compatible system over $\mathcal{P}$ in $\mathbf{C}_3$ we have $p_{M',M} \circ p_{M'} = p_M$ in $\mathbf{C}_3$. Thus $\varphi_M \circ p_M = \varphi_M \circ (p_{M',M} \circ p_{M'}) = (\varphi_M \circ p_{M',M}) \circ p_{M'}$ in $\mathbf{C}_3$ where $\varphi_M \circ p_{M',M} : \mathrm{P}_{M'} \to O_2$ is a morphism in $\mathbf{C}_2$.

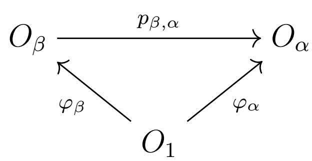
By (i) above, the morphism $\varphi_{M'} : \mathrm{P}_{M'} \to O_2$ in $\mathbf{C}_2$ is the unique morphism in $\mathbf{C}_2$ such that $\varphi_{M'} \circ j_{M'} = \varphi$ in $\mathbf{C}_1$. However, since $(O_1, (j_M)_{M\in\mathcal{M}})$ is a compatible system over $\mathcal{P}$ in $\mathbf{C}_1$ it follows that

$$(\varphi_M \circ p_{M',M}) \circ j_{M'} = \varphi_M \circ (p_{M',M} \circ j_{M'}) = \varphi_M \circ j_M = \varphi.$$

Thus we conclude that $\varphi_M \circ p_{M',M} = \varphi_{M'}$ which implies $\varphi_M \circ p_M = (\varphi_M \circ p_{M',M}) \circ p_{M'} = \varphi_{M'} \circ p_{M'}$. Hence the morphism $\psi_{O_2,\varphi}$ in $\mathbf{C}_3$ is well-defined. As a result, we have the desired family of $\mathbf{C}_3$-morphisms Ψ and the statement in (a) follows by the definition of the morphisms in Ψ.

For the statement in (b), fix objects O_α and O_β in $\mathbf{C}_2$ with $\varphi_\alpha : O_1 \to O_\alpha$ and $\varphi_\beta : O_1 \to O_\beta$ morphisms in $\mathbf{C}_1$. Consider a morphism $p_{\beta,\alpha} : O_\beta \to O_\alpha$ in $\mathbf{C}_2$ such that the following diagram commutes in $\mathbf{C}_1$.

(3.5.1)



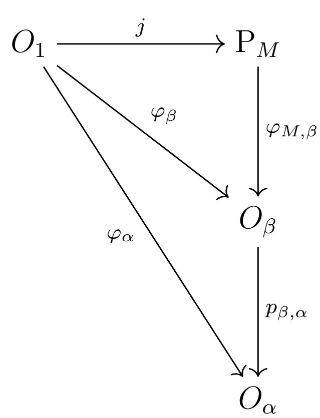
By (i) above and since $\mathcal{M}$ is directed, there exists $M_0 \in \mathcal{M}$ such that for all $M \geq M_0$ there exists unique morphisms $\varphi_{M,\alpha} : \mathrm{P}_M \to O_\alpha$ and $\varphi_{M,\beta} : \mathrm{P}_M \to O_\beta$ such that the following diagrams commute in $\mathbf{C}_1$.

(3.5.2)

By part (a), we have morphisms $\psi_{O_\alpha,\varphi_\alpha} : \mathcal{P}_\mathcal{M} \to O_\alpha$ and $\psi_{O_\beta,\varphi_\beta} : \mathcal{P}_\mathcal{M} \to O_\beta$ in Ψ where

$$\psi_{O_\alpha,\varphi_\alpha} = \varphi_{M,\alpha} \circ p_M, \qquad \psi_{O_\beta,\varphi_\beta} = \varphi_{M,\beta} \circ p_M.$$

The diagrams in (3.5.1) and (3.5.2) show that the following diagram commutes in $\mathbf{C}_1$.



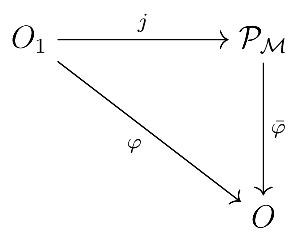
The uniqueness in (i) above implies that $p_{\beta,\alpha} \circ \varphi_{M,\beta} = \varphi_{M,\alpha}$ and thus

$$\psi_{O_\alpha,\varphi_\alpha} = \varphi_{M,\alpha} \circ p_M = (p_{\beta,\alpha} \circ \varphi_{M,\beta}) \circ p_M = p_{\beta,\alpha} \circ (\varphi_{M,\beta} \circ p_M) = p_{\beta,\alpha} \circ \psi_{O_\beta,\varphi_\beta}$$

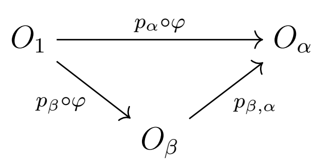
which proves the statement in (b). □

With many of the details relegated to the statement of the previous lemma, we are now able to state the following shorter result.

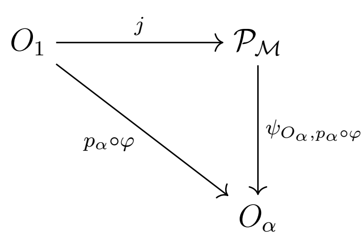
Corollary 3.5.9. *Let* $\mathbf{C}_1$*,* $\mathbf{C}_2$*, and* $\mathbf{C}_3$ *be categories such that* $\mathbf{C}_2 \subseteq \mathbf{C}_3 \subseteq \mathbf{C}_1$*. Fix* $O_1 \in \mathbf{C}_1$ *and consider an upwards directed set* $(\mathcal{M}, \leq)$*. For every* $M \in \mathcal{M}$*, let* P_M *be an object in* $\mathbf{C}_2$ *and* $j_M : O_1 \to \mathrm{P}_M$ *a morphism in* $\mathbf{C}_1$ *satisfying conditions (i)-(iii) in Lemma 3.5.8. Let* $\mathcal{I} \coloneqq ((O_\alpha)_{\alpha \in I}, (p_{\beta,\alpha})_{\beta \geq \alpha})$ *be an inverse system in* $\mathbf{C}_2$ *with inverse limit* $(O, (p_\alpha)_{\alpha \in I})$ *in* $\mathbf{C}_3$ *such that* $(O, (p_\alpha)_{\alpha \in I})$ *is also an inverse limit of* $\mathcal{I}$ *in* $\mathbf{C}_1$*. Let* $\varphi : O_1 \to O$ *be a morphism in* $\mathbf{C}_1$*, then there exists* $\bar{\varphi} : \mathcal{P}_\mathcal{M} \to O$ *in* $\mathbf{C}_3$ *such that the following diagram commutes in* $\mathbf{C}_1$*.*



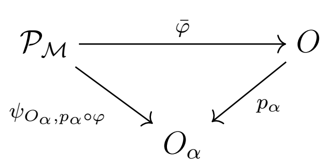
Proof. Since $(O, (p_\alpha)_{\alpha \in I})$ is a compatible system over $\mathcal{I}$ in $\mathbf{C}_3$, the following diagram commutes in $\mathbf{C}_1$ for all $\alpha, \beta \in I$ with $\beta \geq \alpha$.



Thus $(O_1, (p_\alpha \circ \varphi)_{\alpha \in I})$ is a compatible system over $\mathcal{I}$ in $\mathbf{C}_1$. For every $\alpha \in I$, consider the morphism $\psi_{O_\alpha, p_\alpha \circ \varphi} : \mathcal{P}_\mathcal{M} \to O_\alpha$ in $\mathbf{C}_3$ constructed in Lemma 3.5.8 (a) satisfying the following diagram in $\mathbf{C}_1$.



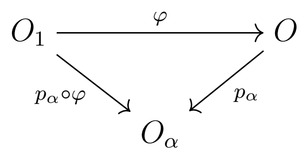
It follows directly from the fact that $(O_1, (p_\alpha \circ \varphi)_{\alpha \in I})$ is a compatible system over $\mathcal{I}$ in $\mathbf{C}_1$ and Lemma 3.5.8 (b) that $(\mathcal{P}_\mathcal{M}, (\psi_{O_\alpha, p_\alpha \circ \varphi})_{\alpha \in I})$ is a compatible system over $\mathcal{I}$ in $\mathbf{C}_3$. Since $(O, (p_\alpha)_{\alpha \in I})$ is an inverse limit of $\mathcal{I}$ in $\mathbf{C}_3$ there exists a unique morphism $\bar{\varphi} : \mathcal{P}_\mathcal{M} \to O$ such that the following diagram commutes in $\mathbf{C}_3$ for all $\alpha \in I$.



From the previous two diagrams, we see that the following holds in $\mathbf{C}_1$.

$$p_\alpha \circ (\bar{\varphi} \circ j) = (p_\alpha \circ \bar{\varphi}) \circ j = \psi_{O_\alpha, p_\alpha \circ \varphi} \circ j = p_\alpha \circ \varphi.$$

However, since $(O_1, (p_\alpha \circ \varphi)_{\alpha \in I})$ is a compatible system over $\mathcal{I}$ in $\mathbf{C}_1$ and $(O, (p_\alpha)_{\alpha \in I})$ is an inverse limit of $\mathcal{I}$ in $\mathbf{C}_1$, we conclude that the morphism $\varphi : O_1 \to O$ is the unique morphism in $\mathbf{C}_1$ making the following diagram commute in $\mathbf{C}_1$.



This uniqueness implies that $\bar{\varphi} \circ j = \varphi$, as required. □

With the progress we have made thus far, some observations are in order: Let $\mathbf{Y}$ be a category from Table 2 or 3 with $\mathbf{X}$ any category from Table 1 such that $\mathbf{X} \supseteq \mathbf{Y}$. Take $X \in \mathbf{X}$ and consider the pseudo-solutions $(\mathrm{P}^{\mathbf{Y}}_{\mathbf{X}}(X, M), j_M)$ for $M \in \mathcal{M}(X)$. In order to apply Corollary 3.5.9 to these pseudo-solutions, the conditions (i)-(iii) from Lemma 3.5.8 need to be in place: Condition (i) was addressed in Remark 3.3.3 (i) and Theorem 3.3.7. The work done previously in this section takes care of (ii). For (iii), we will need to show that our inverse system of pseudo-solutions $\mathcal{P}$ has an inverse limit $\mathcal{S}_{\mathcal{P}}$ in the associated category of locally convex structures $\mathbf{LCY}$ where $\mathcal{S}_{\mathcal{P}}$ is also an inverse limit in $\mathbf{X}$. The work in the next section will give us the information we need regarding inverse limits in categories of locally convex structures.

Lastly, we were not able to show from general categorical considerations in Corollary 3.5.9 that the morphism $\bar{\varphi} : \mathcal{P}_{\mathcal{M}} \to O$ is unique with respect to the diagram in Corollary 3.5.9. In the end, we will make use of the properties of the inverse limit $(\mathcal{P}_{\mathcal{M}}, (p_M)_{M \in \mathcal{M}})$ along with the properties of the pseudo-solutions derived in Section 3.3 to conclude uniqueness.

3.6. Locally convex spaces and inverse limits

In this section, we put the last few results in place that we need to finally conclude the existence of free objects in categories of locally convex structures in the next section. In particular, we will show that our categories of locally convex structures are closed under the formation of inverse limits. More specifically, this means that the inverse limits of normed structures exist in the relevant category of locally convex structures. Beyond this, we will also expand a known result to show that every object in a category of complete locally convex structures $\mathbf{ComLC\hat{Y}}$ is in fact an inverse limit of an inverse system in $\hat{\mathbf{Y}}$ with the inverse limit taken in $\mathbf{ComLC\hat{Y}}$.

We start our exposition with the following simple proposition for which we omit the proof.

Proposition 3.6.1. *Let I be an index set and for every $\alpha \in I$, let A_α be an object in* $\mathbf{ComLM\text{-}C\text{-}SVLA^{1+}}$. *Consider the product $\prod_{\alpha \in I} A_\alpha$ with the projections $\pi_\alpha : \prod_{\alpha \in I} A_\alpha \to A_\alpha$. Then the following holds:*

(i) *The product* $\prod_{\alpha\in I} A_\alpha$ *equipped with component-wise operations is an object in* $\mathbf{VLA^{1+}}$ *and for every* $\alpha \in I$*, the projection* $\pi_\alpha : \prod_{\alpha\in I} A_\alpha \to A_\alpha$ *is a* $\mathbf{VLA^{1+}}$*-morphism.*

(ii) *The product topology* τ *makes* $\prod_{\alpha\in I} A_\alpha$ *into an object in* **LM-C-SVLA**$^{\mathbf{1+}}$*.*

(iii) *If* A_α *is complete for every* $\alpha \in I$*, then* $(\prod_{\alpha\in I} A_\alpha, \tau)$ *is an object in* **ComLM-C-SVLA**$^{\mathbf{1+}}$*.*

As we saw for other cases in Chapter 2, the inverse limit of an inverse system of locally convex structures $\mathcal{I} := ((A_\alpha)_{\alpha\in I}, (p_{\beta,\alpha})_{\beta\succcurlyeq\alpha})$ is easily obtained by taking the inverse limit of the underlying sets in $\mathcal{I}$ and inducing the natural algebraic and topological structure on this inverse limit of sets from the product.

Proposition 3.6.2. *Let* $\mathcal{I} := ((A_\alpha)_{\alpha\in I}, (p_{\beta,\alpha})_{\beta\succcurlyeq\alpha})$ *be an inverse system in* **LM-C-SVLA**$^{\mathbf{1+}}$*. Define the set*

$$A := \left\{ (x_\alpha)_{\alpha\in I} \in \prod_{\alpha\in I} A_\alpha : p_{\beta,\alpha}(x_\beta) = x_\alpha, \forall\, \beta \succcurlyeq \alpha \right\}$$

and define $p_\alpha : A \to A_\alpha$ *where* $p_\alpha := \pi_\alpha|_A$ *for* $\alpha \in I$*. Equipping* A *with the subspace topology from* $(\prod_{\alpha\in I} A_\alpha, \tau)$ *makes the pair* $(A, (p_\alpha)_{\alpha\in I})$ *an inverse limit of* $\mathcal{I}$ *in* **LM-C-SVLA**$^{\mathbf{1+}}$*. In addition, if* $\mathcal{I}$ *is an inverse system in* **ComLM-C-SVLA**$^{\mathbf{1+}}$*, then* $(A, (p_\alpha)_{\alpha\in I})$ *is an inverse limit* $\mathcal{I}$ *in* **ComLM-C-SVLA**$^{\mathbf{1+}}$*.*

Proof. It follows directly from the definitions in Section 3.1.3 along with the definitions of the product and subspace topology that A is an object in **LM-C-SVLA**$^{\mathbf{1+}}$ and that $p_\alpha : A \to A_\alpha$ are morphisms in **LM-C-SVLA**$^{\mathbf{1+}}$. Further, it is immediate from the definition of the set A that $(A, (p_\alpha)_{\alpha\in I})$ is a compatible system over $\mathcal{I}$ in **LM-C-SVLA**$^{\mathbf{1+}}$.

Let $\left(\tilde{A}, (\tilde{p}_\alpha)_{\alpha\in I}\right)$ be any compatible system over $\mathcal{I}$ in **LM-C-SVLA**$^{\mathbf{1+}}$. Define the $\mathbf{VLA^{1+}}$-morphism $s : \tilde{A} \to A$ where $s(y) := (\tilde{p}_\alpha(y))_{\alpha\in I}$ for $y \in \tilde{A}$. It follows precisely from the compatibility of $\left(\tilde{A}, (\tilde{p}_\alpha)_{\alpha\in I}\right)$ that $s\left[\tilde{A}\right] \subseteq A$ and it follows by the continuity of the $\tilde{p}_\alpha$ morphisms that s is continuous, hence s is a **LM-C-SVLA**$^{\mathbf{1+}}$-morphism. Let $s' : \tilde{A} \to A$ be any morphism in **LM-C-SVLA**$^{\mathbf{1+}}$ such that $p_\alpha \circ s' = \tilde{p}_\alpha$ for all $\alpha \in I$. Then $p_\alpha(s'(x)) = p_\alpha(s(x))$ which implies that $s' = s$. Thus $(A, (p_\alpha)_{\alpha\in I})$ is an inverse limit of $\mathcal{I}$ in **LM-C-SVLA**$^{\mathbf{1+}}$.

For the case where $\mathcal{I}$ is an inverse system in **ComLM-C-SVLA**$^{\mathbf{1+}}$, by Proposition 3.6.1 (iii), the product $\prod_{\alpha\in I} A_\alpha$ is complete and since A is a closed subvector lattice algebra of $\prod_{\alpha\in I} A_\alpha$, it follows that A is an object in **ComLM-C-SVLA**$^{\mathbf{1+}}$. The rest of the proof for this case proceeds in the same manner as the above. □

In the above result, since every A_α is a vector space and every morphism $p_{\beta,\alpha}$ is a linear map, it follows that $0 \in A$ and A is thus never empty. As a result, the inverse limit of an inverse system of locally convex structures is never empty.

Remark 3.6.3. Let **Y** be a category of normed structures from Table 2 or 3 with **LCY** the associated category of locally convex structures. The above proof is easily repurposed to show that the category **LCY** is also closed under the formation of inverse limits of inverse systems in **LCY**. In particular, since $\mathbf{Y} \subseteq \mathbf{LCY}$ it is then also clear that the inverse limits of inverse systems in **Y** always exist in **LCY**.

Further, if we consider an algebraic category $\mathbf{X} \subseteq \mathcal{A}(\mathbf{Y}) \subseteq \mathbf{Y}$ and an inverse system $\mathcal{I} \coloneqq ((A_\alpha)_{\alpha \in I}, (p_{\beta,\alpha})_{\beta \geqslant \alpha})$ in **Y**, since the objects A_α and the morphisms $p_{\beta,\alpha}$ are also in **X**, it follows that $(A, (p_\alpha)_{\alpha \in I})$ as constructed above is both an inverse limit in **LCY** and in **X**.

With these results on the existence of inverse limits in categories of locally convex structures in mind, we are now in a position to use Definition 3.5.4 to form categories of inverse limits. We list this new collection of categories.

Proposition 3.6.4. *The following list of categories defined using Definition 3.5.4 exist. By definition, for each category* **Y** *from Table 2 or 3, the category of inverse limits* $\mathbf{LCY}\text{-}\varprojlim \mathbf{Y}$ *is a full subcategory in* **LCY***.*

Categories of inverse limits of normed structures:

(i) $\mathbf{LCS}\text{-}\varprojlim \mathbf{NS}$,

(ii) $\mathbf{LC\text{-}SVL}\text{-}\varprojlim \mathbf{NVL}$,

(iii) $\mathbf{LM\text{-}CA}\text{-}\varprojlim \mathbf{NA}$,

(iv) $\mathbf{LM\text{-}CA^1}\text{-}\varprojlim \mathbf{NA^1}$,

(v) $\mathbf{LM\text{-}C\text{-}SVLA}\text{-}\varprojlim \mathbf{NVLA}$,

(vi) $\mathbf{LM\text{-}C\text{-}SVLA^1}\text{-}\varprojlim \mathbf{NVLA^1}$,

(vii) $\mathbf{LM\text{-}C\text{-}SVLA^{1+}}\text{-}\varprojlim \mathbf{NVLA^{1+}}$.

Categories of inverse limits of complete normed structures:

(i) $\mathbf{ComLCS}\text{-}\varprojlim \mathbf{Ban}$,

(ii) $\mathbf{ComLC\text{-}SVL}\text{-}\varprojlim \mathbf{BL}$,

(iii) $\mathbf{ComLM\text{-}CA}\text{-}\varprojlim \mathbf{BA}$,

(iv) $\mathbf{ComLM\text{-}CA^1}\text{-}\varprojlim \mathbf{BA^1}$,

(v) $\mathbf{ComLM\text{-}C\text{-}SVLA}\text{-}\varprojlim \mathbf{BLA}$,

(vi) $\mathbf{ComLM\text{-}C\text{-}SVLA^1}\text{-}\varprojlim \mathbf{BLA^1}$,

(vii) $\mathbf{ComLM\text{-}C\text{-}SVLA^{1+}}\text{-}\varprojlim \mathbf{BLA^{1+}}$.

For the next result, we need to recall some terminology. Let X be a locally convex space. The family of seminorms P generating the topology on X is *saturated* if for every finite collection $\{\rho_1, \cdots, \rho_n\} \subseteq P$ the seminorm $\rho_1 \vee \cdots \vee \rho_n : X \to \mathbb{R}$ is also in P where $(\rho_1 \vee \cdots \vee \rho_n)(x) \coloneqq \rho_1(x) \vee \cdots \vee \rho_n(x)$.

For a family of seminorms P, the collection P' obtained by adding all of the elements of the form above is called the *saturation of* P. It is clear that if P is a family of submultiplicative (Riesz) seminorms then the elements in P' will also be submultiplicative (Riesz) seminorms.

It is known that the objects in **ComLCS** are precisely the inverse limits of objects in **Ban** with the inverse limit taken in **ComLCS**, see [**60**, Chapter II, Theorem 5.4]. It is also known that the objects in **ComLM-CA** are precisely the inverse limits of objects in **BA** with the inverse limit taken in **ComLM-CA**, see [**14**, Theorem 4.5.3]. In the language we have developed this is stated as

$$\mathbf{ComLCS}\text{-}\varprojlim \mathbf{Ban} = \mathbf{ComLCS}, \qquad \mathbf{ComLM\text{-}CA}\text{-}\varprojlim \mathbf{BA} = \mathbf{ComLM\text{-}CA}.$$

We adapt the proof in [**14**, Theorem 4.5.3] to obtain the following result.

THEOREM 3.6.5. *The following equality of categories holds.*

$$\mathbf{ComLM\text{-}C\text{-}SVLA^{1+}} = \mathbf{ComLM\text{-}C\text{-}SVLA^{1+}}\text{-}\varprojlim \mathbf{BLA^{1+}}$$

PROOF. Since $\varprojlim \mathbf{BLA^{1+}}$ is a full subcategory of $\mathbf{ComLM\text{-}C\text{-}SVLA^{1+}}$, it is enough for us to verify that the object classes of these two categories are equal. Fix an object $Y \in \mathbf{ComLM\text{-}C\text{-}SVLA^{1+}}$, then there exists a separating family of submultiplicative Riesz seminorms $P' := \{\rho_\alpha : \alpha \in I\}$ generating the topology on Y. We assume without loss of generality that the family P' is saturated. Define a partial order $\preccurlyeq$ on I where $\alpha \preccurlyeq \beta$ if and only if $\rho_\alpha(x) \leq \rho_\beta(x)$ for all $x \in X$. The pair $(I, \preccurlyeq)$ is upwards directed since P' is saturated.

For every $\alpha \in I$, consider the quotient $\tilde{Y}_\alpha := Y / \ker \rho_\alpha$ which is made into a positive unital normed vector lattice algebra when equipped with the standard quotient vector lattice algebra structure and the norm $\|\cdot\|_\alpha : \tilde{Y}_\alpha \to \mathbb{R}$ where $\|[x]_\alpha\|_\alpha := \rho_\alpha(x)$. For every $\alpha \in I$, the quotient map $\tilde{q}_\alpha : Y \to \tilde{Y}_\alpha$ is a continuous unital vector lattice algebra homomorphism. For every $\alpha \in I$, let $(Y_\alpha, \|\cdot\|_\alpha)$ denote a completion of $(\tilde{Y}_\alpha, \|\cdot\|_\alpha)$. Thus $(Y_\alpha, \|\cdot\|_\alpha)$ is an object in $\mathbf{BLA^{1+}}$ for all $\alpha \in I$. Clearly for $\alpha, \beta \in I$ with $\beta \succcurlyeq \alpha$, we have $\ker \rho_\alpha \subseteq \ker \rho_\beta$. For $\beta \succcurlyeq \alpha$, define the unital vector lattice algebra homomorphism $\tilde{p}_{\beta,\alpha} : \tilde{Y}_\beta \to \tilde{Y}_\alpha$ where $[x]_\beta \mapsto [x]_\alpha$. It is easy to verify that for all $\beta \succcurlyeq \alpha$ in I we have

$$\tilde{p}_{\beta,\alpha} \circ \tilde{q}_\beta = \tilde{q}_\alpha. \tag{3.6.1}$$

Since $\ker \rho_\alpha \subseteq \ker \rho_\beta$ it is clear that $\tilde{p}_{\beta,\alpha}$ is contractive and thus the maps $\tilde{p}_{\beta,\alpha} : \tilde{Y}_\beta \to \tilde{Y}_\alpha \subseteq Y_\alpha$ extend uniquely to contractive morphisms $p_{\beta,\alpha} : Y_\beta \to Y_\alpha$ in $\mathbf{BLA^{1+}}$. Hence, the pair $\tilde{\mathcal{I}} := \big((\tilde{Y}_\alpha)_{\alpha\in I}, (\tilde{p}_{\beta,\alpha})_{\beta\succcurlyeq\alpha}\big)$ forms an inverse system in $\mathbf{NVLA^{1+}}$ and a standard density argument shows that the pair $\mathcal{I} := ((Y_\alpha)_{\alpha\in I}, (p_{\beta,\alpha})_{\beta\succcurlyeq\alpha})$ forms an inverse system in $\mathbf{BLA^{1+}}$. Now, define the positive unital vector lattice algebra

$$\tilde{Y} := \left\{ ([y_\alpha]_\alpha)_{\alpha\in I} \in \prod_{\alpha\in I} \tilde{Y}_\alpha : \tilde{p}_{\beta,\alpha}([y_\beta]_\beta) = [x_\alpha]_\alpha, \forall\, \beta \geq \alpha \right\}.$$

Define the family of seminorms $Q := \{\sigma_\gamma : \gamma \in I\}$ where for each $\gamma \in I$,

$$\sigma_\gamma\left(([y_\alpha]_\alpha)_{\alpha\in I}\right) := \|[y_\gamma]_\gamma\|_\gamma = \rho_\gamma(y_\gamma). \tag{3.6.2}$$

Define the projections $\tilde{p}_\alpha : \tilde{Y} \to \tilde{Y}_\alpha$ for $\alpha \in I$. By Proposition 3.6.2, the pair $\big(\tilde{Y}, (\tilde{p}_\alpha)_{\alpha\in I}\big)$ is the inverse limit of $\tilde{\mathcal{I}}$ in $\mathbf{LM\text{-}C\text{-}SVLA^{1+}}$.

By (3.6.1), it is clear that the pair $(Y, (\tilde{q}_\alpha)_{\alpha\in I})$ is a compatible system over $\tilde{\mathcal{I}}$ in $\mathbf{LM\text{-}C\text{-}SVLA^{1+}}$ and so there exists a unique morphism $\Phi : Y \to \tilde{Y}$ in $\mathbf{LM\text{-}C\text{-}SVLA^{1+}}$ where $y \mapsto (\tilde{q}_\alpha(y))_{\alpha\in I}$ for $y \in Y$. We show that Φ is an isomorphism in $\mathbf{LM\text{-}C\text{-}SVLA^{1+}}$, i.e. a bijective bicontinuous unital vector lattice algebra homomorphism. Continuity and the homomorphism property has already been given universal property of the inverse limit of $\tilde{\mathcal{I}}$. For injectivity, since P' is a separating family seminorms, we have that $\bigcap_{\alpha\in I} \ker \rho_\alpha = \{0\}$. If $y \in Y$ is such that $\Phi(y) = (q_\alpha(y))_{\alpha\in I} = 0$, then $y \in \bigcap_{\alpha\in I} \ker \rho_\alpha$. Hence $y = 0$.

To show surjectivity, fix $([u_\alpha]_\alpha)_{\alpha\in I} \in \tilde{Y}$. Denote by $\mathcal{F}(I)$ the collection of all finite subsets of I, which set inclusion makes into a partially ordered upwards directed set. Note that for every $S \in \mathcal{F}(I)$ there exists $\beta_S \in I$ such that $\beta_S \succcurlyeq \alpha$ for all $\alpha \in S$. For every $S \in \mathcal{F}(I)$, fix such a $\beta_S \in I$. Since all the quotient maps $\{\tilde{q}_\alpha : \alpha \in I\}$ are surjective, for every $S \in \mathcal{F}(I)$ with fixed associated upper bound $\beta_S \in I$, fix $x_S \in Y$ such that $\tilde{q}_{\beta_S}(x_S) = [u_{\beta_S}]_{\beta_S}$. This gives us a net $(x_S)_{S\in\mathcal{F}(I)}$ in Y. By (3.6.1) above, for every $S \in \mathcal{F}(I)$ and all $\alpha \in S$ we have

$$\tilde{q}_\alpha(x_S) = \tilde{p}_{\beta_S,\alpha} \circ \tilde{q}_{\beta_S}(x_S) = \tilde{p}_{\beta_S,\alpha}([u_{\beta_S}]_{\beta_S}) = [u_\alpha]_\alpha.$$

Consider $S_1, S_2 \in \mathcal{F}(I)$ with $S_1 \cap S_2 \neq \varnothing$. It follows from the previous observation that

$$\tilde{q}_\alpha(x_{S_1}) = [u_\alpha]_\alpha = \tilde{q}_\alpha(x_{S_2})$$

for all $\alpha \in S_1 \cap S_2$, which implies that $x_{S_1} - x_{S_2} \in \ker \rho_\alpha \subseteq \{y \in Y : \rho_\alpha(y) < \epsilon\}$ for arbitrary $\epsilon > 0$. It is now easy to see that $(x_S)_{S\in\mathcal{F}(I)}$ is a Cauchy net in Y: Fix an open neighbourhood U of zero, there exists a basis element of the form $\bigcap_{j=1}^{k} \{y \in Y : \rho_{\alpha_j}(y) < \epsilon_j\} \subseteq U$. Define $S_0 := \{\alpha_1, \cdots, \alpha_k\}$, then for all $S, T \supseteq S_0$ we have that $x_S - x_T \in \ker \rho_{\alpha_j} \subseteq \{y \in Y : \rho_{\alpha_j}(y) < \epsilon_j\}$. Since Y is complete there exists $x \in Y$ such that $(x_S)_{S\in\mathcal{F}(I)} \to x$. Fix any $\alpha \in I$. Since $\tilde{q}_\alpha$ is continuous, we have that $\tilde{q}_\alpha(x_S) \to \tilde{q}_\alpha(x)$ but the net $(\tilde{q}_\alpha(x_S))_{S\in\mathcal{F}(I)}$ is eventually constant since $\tilde{q}_\alpha(x_S) = [u_\alpha]_\alpha$ for all $S \supseteq \{\alpha\}$. It follows that $\Phi(x) = ([u_\alpha]_\alpha)_{\alpha\in I}$.

The definition of the seminorms on $\tilde{Y}$ given in (3.6.2) makes it clear that for a net $(\Phi(y_j))_{j\in J}$ in $\tilde{Y}$ and $\Phi(y) \in \tilde{Y}$, if $(\Phi(y_j))_{j\in J} \to \Phi(y)$ in $\tilde{Y}$, then $(y_j)_{j\in J} \to y$ in Y. Thus the map $\Phi : Y \to \tilde{Y}$ is a $\mathbf{LM\text{-}C\text{-}SVLA^{1+}}$-isomorphism, which implies that $\tilde{Y}$ is complete.

Now, consider the inverse system $\mathcal{I} := ((Y_\alpha)_{\alpha\in I}, (p_{\beta,\alpha})_{\beta\succcurlyeq\alpha})$ in $\mathbf{BLA^{1+}}$ defined above. Define the set

$$\bar{Y} := \left\{ (\hat{y}_\alpha)_{\alpha\in I} \in \prod_{\alpha\in I} Y_\alpha : p_{\beta,\alpha}(\hat{y}_\beta) = \hat{y}_\alpha, \forall\, \beta \geq \alpha \right\}$$

By inducing structure on $\bar{Y}$ as in Proposition 3.6.2 and defining the projections $p_\alpha : \bar{Y} \to Y_\alpha$, we obtain the inverse limit $\big(\bar{Y}, (p_\alpha)_{\alpha\in I}\big)$ of $\mathcal{I}$ in $\mathbf{ComLM\text{-}C\text{-}SVLA^{1+}}$.

Given that $\tilde{Y}_\alpha$ forms a subvector lattice algebra in Y_α for all $\alpha \in I$ and the morphisms $\{p_{\beta,\alpha} : Y_\beta \to Y_\alpha\}_{\beta\succcurlyeq\alpha}$ are extensions of the morphisms $\{\tilde{p}_{\beta,\alpha} : \tilde{Y}_\beta \to \tilde{Y}_\alpha\}_{\beta\succcurlyeq\alpha}$, we may identify $\tilde{Y}$ with a subvector lattice algebra of $\bar{Y}$ and since $\tilde{Y}$ is complete, we conclude that $\tilde{Y}$ is closed in $\bar{Y}$. To conclude the result, we will show that $\tilde{Y}$ is dense in $\bar{Y}$:

Fix any $\hat{y} := (\hat{y}_\alpha)_{\alpha\in I} \in \bar{Y}$. We construct a net in $\tilde{Y}$ converging to $\hat{y}$ in $\bar{Y}$. For every $S \in \mathcal{F}(I)$, fix $\beta_S \in I$ such that $\beta_S \succcurlyeq \alpha$ for all $\alpha \in S$. Since the quotient map $\tilde{q}_{\beta_S} : Y \to \tilde{Y}_{\beta_S}$ is surjective and $\tilde{Y}_{\beta_S}$ is dense in Y_{β_S}, for every $\epsilon > 0$ there exists $x_{S,\epsilon} \in Y$ such that

$$\|\tilde{q}_{\beta_S}(x_{S,\epsilon}) - \hat{y}_{\beta_S}\| < \epsilon.$$

Since the linking maps $p_{\beta,\alpha}$ in $\mathcal{I}$ extend the contractive linking maps $\tilde{p}_{\beta,\alpha}$ in $\tilde{\mathcal{I}}$, for all $\alpha \in S$ we have

$$\begin{aligned}\|\tilde{q}_\alpha(x_{S,\epsilon}) - \hat{y}_\alpha\| &= \|\tilde{p}_{\beta_S,\alpha} \circ \tilde{q}_{\beta_S}(x_{S,\epsilon}) - p_{\beta_S,\alpha}(\hat{y}_{\beta_S})\| \\ &= \|p_{\beta_S,\alpha} \circ \tilde{q}_{\beta_S}(x_{S,\epsilon}) - p_{\beta_S,\alpha}(\hat{y}_{\beta_S})\| \\ &\le \|\tilde{q}_{\beta_S}(x_{S,\epsilon}) - \hat{y}_{\beta_S}\| < \epsilon.\end{aligned}$$

Denote $\mathbb{R}_{>0} := \mathbb{R}^+\backslash\{0\}$ and define the ordering $\le$ on $\mathcal{F}(I)\times\mathbb{R}_{>0}$ where $(S_1, r_1) \le (S_2, r_2)$ if and only if $S_1 \subseteq S_2$ and $r_2 \le r_1$. It follows that the net $(\tilde{q}_\alpha(x_{S,\epsilon}))_{(S,\epsilon)\in\mathcal{F}(I)\times\mathbb{R}_{>0}}$ converges to $\hat{y}_\alpha \in Y_\alpha$ and since $\prod_{\alpha\in I} Y_\alpha$ possesses the product topology it follows that $(\Phi(x_{S,\epsilon}))_{(S,\epsilon)\in\mathcal{F}(I)\times\mathbb{R}_{>0}}$ converges to $\hat{y} \in \bar{Y}$. Hence $\tilde{Y} = \bar{Y}$.

Thus we have the isomorphism $\Phi : Y \to \bar{Y}$ in $\mathbf{ComLM\text{-}C\text{-}SVLA^{1+}}$. Since $\left(\bar{Y}, (p_\alpha)_{\alpha\in I}\right)$ is an inverse limit of $\mathcal{I}$ in $\mathbf{ComLM\text{-}C\text{-}SVLA^{1+}}$, it follows that $(Y, (p_\alpha \circ \Phi)_{\alpha\in I})$ is also an inverse limit of $\mathcal{I}$ in $\mathbf{ComLM\text{-}C\text{-}SVLA^{1+}}$. Hence $Y \in \varprojlim \mathbf{BLA^{1+}}$. □

Let $\hat{\mathbf{Y}}$ be a category of complete normed structures from Table 3. The proof of Theorem 3.6.5 can be modified to show that $\mathbf{ComLC\hat{Y}}\text{-}\varprojlim \hat{\mathbf{Y}} = \mathbf{ComLC\hat{Y}}$. The crucial steps in the above proof is showing that the map $\Phi : Y \to \tilde{Y}$ is an isomorphism and showing that $\tilde{Y}$ is a dense subspace in $\bar{Y}$. These steps rely on topological arguments and the universal property of compatible systems and will therefore remain essentially the same when verifying the analogous statement for the category $\hat{\mathbf{Y}}$. Modifying the other parts of the proof is routine and largely amount to dealing with different underlying algebraic structures. Thus we have the following equalities of categories.

Corollary 3.6.6. *The following equalities hold:*

(i) $\mathbf{ComLCS}\text{-}\varprojlim \mathbf{Ban} = \mathbf{ComLCS}$.

(ii) $\mathbf{ComLC\text{-}SVL}\text{-}\varprojlim \mathbf{BL} = \mathbf{ComLC\text{-}SVL}$.

(iii) $\mathbf{ComLM\text{-}CA}\text{-}\varprojlim \mathbf{BA} = \mathbf{ComLM\text{-}CA}$.

(iv) $\mathbf{ComLM\text{-}CA^1}\text{-}\varprojlim \mathbf{BA^1} = \mathbf{ComLM\text{-}CA^1}$.

(v) $\mathbf{ComLM\text{-}C\text{-}SVLA}\text{-}\varprojlim \mathbf{BLA} = \mathbf{ComLM\text{-}C\text{-}SVLA}$.

(vi) $\mathbf{ComLM\text{-}C\text{-}SVLA^1}\text{-}\varprojlim \mathbf{BLA^1} = \mathbf{ComLM\text{-}C\text{-}SVLA^1}$.

(vii) $\mathbf{ComLM\text{-}C\text{-}SVLA^{1+}}\text{-}\varprojlim \mathbf{BLA^{1+}} = \mathbf{ComLM\text{-}C\text{-}SVLA^{1+}}$.

We close this section with a few observations. For a category of complete normed structures $\hat{\mathbf{Y}}$, the equality $\mathbf{ComLC\hat{Y}}\text{-}\varprojlim \hat{\mathbf{Y}} = \mathbf{ComLC\hat{Y}}$ tells us that every object in $\mathbf{ComLC\hat{Y}}$ is the inverse limit of *some* inverse system $\hat{\mathbf{Y}}$ (with the inverse limit taken in $\mathbf{ComLC\hat{Y}}$). However, it is implicit in the proof of Theorem 3.6.5 that every object in $\mathbf{ComLC\hat{Y}}$ is in fact an inverse limit of an inverse system in $\hat{\mathbf{Y}}$ with *contractive* linking maps.

As a separate note, consider a category of normed structures $\mathbf{Y}$ and an inverse system $\mathcal{I}$ in $\mathbf{Y}$. Since an inverse limit $\mathcal{S}_{\mathcal{I}}$ exists in $\mathbf{LCY}$, the category $\mathbf{LCY}\text{-}\varprojlim \mathbf{Y}$ is a full subcategory of $\mathbf{LCY}$, and $\mathcal{S}_{\mathcal{I}}$ is a compatible system over $\mathcal{I}$ in $\mathbf{LCY}\text{-}\varprojlim \mathbf{Y}$, it follows by Proposition 3.5.3 that $\mathcal{S}_{\mathcal{I}}$ is an inverse limit of $\mathcal{I}$ in $\mathbf{LCY}\text{-}\varprojlim \mathbf{Y}$.

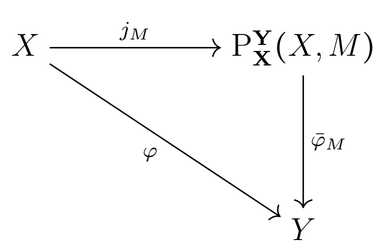
3.7. Free objects in categories of inverse limits

With the results from the last section in place, we are now in a position to apply Corollary 3.5.9 to the pseudo-solutions listed in Theorem 3.3.7 in order to obtain free objects in categories of locally convex structures.

The formulation of Lemma 3.5.8 uses categories $\mathbf{C}_1$, $\mathbf{C}_2$, and $\mathbf{C}_3$ satisfying $\mathbf{C}_2 \subseteq \mathbf{C}_3 \subseteq \mathbf{C}_1$. In the rest of this section, we will only consider categories $\mathbf{X}$ and $\mathbf{Y}$ with $\mathbf{Y}$ from Table 2 or 3 and $\mathbf{X}$ from Table 1 such that $\mathbf{X} \supseteq \mathbf{Y}$. We note that for any choice of category $\mathbf{X}$ from Table 1 we have $\mathbf{Y} \subseteq \mathbf{LCY}\text{-}\varprojlim \mathbf{Y} \subseteq \mathcal{A}(\mathbf{Y}) \subseteq \mathbf{X}$. Thus the categories $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{LCY}\text{-}\varprojlim \mathbf{Y}$ may take the role of $\mathbf{C}_1$, $\mathbf{C}_2$, and $\mathbf{C}_3$ in Lemma 3.5.8 respectively. We now state once and for all that the conditions (i)-(iii) in Lemma 3.5.8 are satisfied for these choices of categories $\mathbf{X}$ and $\mathbf{Y}$.

PROPOSITION 3.7.1. *Let $\mathbf{Y}$ be a category from Table 2 or 3 and $\mathbf{X}$ a category from Table 1 such that $\mathbf{X} \supseteq \mathbf{Y}$. Fix $X \in \mathbf{X}$ and for every $M \in \mathcal{M}(X)$ consider the pseudo-solution $(\mathrm{P}^{\mathbf{Y}}_{\mathbf{X}}(X, M), j_M)$. Then the following holds:*

(i) For every $Y \in \mathbf{Y}$ and every morphism $\varphi : X \to Y$, there exists $M_\varphi \in \mathcal{M}(X)$ such that for all $M \geq M_\varphi$ there exists a unique contractive morphism $\bar{\varphi}_M : \mathrm{P}^{\mathbf{Y}}_{\mathbf{X}}(X, M) \to Y$ such that the following diagram commutes in $\mathbf{X}$.



(ii) For every $M, M' \in \mathcal{M}(X)$ with $M' \geq M$, there exists a unique contractive morphism $p_{M',M} : \mathrm{P}^{\mathbf{Y}}_{\mathbf{X}}(X, M') \to \mathrm{P}^{\mathbf{Y}}_{\mathbf{X}}(X, M)$ in $\mathbf{Y}$ such that the following

diagram commutes in $\mathbf{X}$*.*

$$\begin{array}{ccc} X & \xrightarrow{\;j_{M'}\;} & \mathrm{P}^{\mathbf{Y}}_{\mathbf{X}}(X,M') \\ & \searrow^{j_M} & \downarrow{\scriptstyle p_{M',M}} \\ & & \mathrm{P}^{\mathbf{Y}}_{\mathbf{X}}(X,M) \end{array}$$

Thus $\mathcal{P} := \big((\mathrm{P}^{\mathbf{Y}}_{\mathbf{X}}(X,M))_{M\in\mathcal{M}(X)}, (p_{M',M})_{M'\geq M}\big)$ *is an inverse system in* $\mathbf{Y}$ *and* $\big(X, (j_M)_{M\in\mathcal{M}(X)}\big)$ *is a compatible system over* $\mathcal{P}$ *in* $\mathbf{X}$*.*

(iii) *The inverse system* $\mathcal{P}$ *has an inverse limit* $\big(\mathcal{P}_{\mathcal{M}(X)}, (p_M)_{M\in\mathcal{M}}\big)$ *in* $\mathbf{LCY}\text{-}\varprojlim\mathbf{Y}$ *which is also an inverse limit of* $\mathcal{P}$ *in* $\mathbf{X}$*. Since* $\big(X, (j_M)_{M\in\mathcal{M}(X)}\big)$ *is a compatible system over* $\mathcal{P}$ *in* $\mathbf{X}$*, there exists a unique morphism* $j : X \to \mathcal{P}_{\mathcal{M}(X)}$ *in* $\mathbf{X}$ *such that the following diagram commutes in* $\mathbf{X}$ *for all* $M \in \mathcal{M}(X)$*.*

$$\begin{array}{ccc} X & \xrightarrow{\;j\;} & \mathcal{P}_{\mathcal{M}(X)} \\ & \searrow^{j_M} & \downarrow{\scriptstyle p_M} \\ & & \mathrm{P}^{\mathbf{Y}}_{\mathbf{X}}(X,M) \end{array}$$

Proof. Condition (i) is addressed in Remark 3.3.3 and Theorem 3.3.7. For (ii), it follows from the proof for Proposition 3.5.6 that $\mathcal{P}$ is an inverse system in $\mathbf{Y}$ and the fact that $\big(X, (j_M)_{M\in\mathcal{M}(X)}\big)$ is a compatible system over $\mathcal{P}$ in $\mathbf{X}$ is seen from the diagram in condition (ii). The discussion in Remark 3.6.3 shows that the pair $\big(\mathcal{P}_{\mathcal{M}(X)}, (p_M)_{M\in\mathcal{M}}\big)$ constructed in Proposition 3.6.2 is both an inverse limit of $\mathcal{P}$ in $\mathbf{LCY}$ and $\mathbf{X}$. It follows from the fact that $\mathbf{LCY}\text{-}\varprojlim\mathbf{Y}$ is a full subcategory in $\mathbf{LCY}$ and Proposition 3.5.3 that $\big(\mathcal{P}_{\mathcal{M}(X)}, (p_M)_{M\in\mathcal{M}}\big)$ is an inverse limit of $\mathcal{P}$ in $\mathbf{LCY}\text{-}\varprojlim\mathbf{Y}$. □

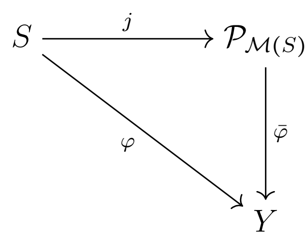
Applying Corollary 3.5.9 to Proposition 3.7.1, we almost have a new collection of free objects.

Corollary 3.7.2. *Let* $\mathbf{Y}$ *be a category from Table 2 or 3 and* $\mathbf{X}$ *a category from Table 1 such that* $\mathbf{X} \supseteq \mathbf{Y}$ *and fix* $X \in \mathbf{X}$*. Then there exists an object* $\mathcal{P}_{\mathcal{M}(X)}$ *in* $\mathbf{LCY}\text{-}\varprojlim\mathbf{Y}$ *and a morphism* $j : X \to \mathcal{P}_{\mathcal{M}(X)}$ *in* $\mathbf{X}$ *such that for every object* O *in* $\mathbf{LCY}\text{-}\varprojlim\mathbf{Y}$ *and every morphism* $\varphi : X \to O$ *in* $\mathbf{X}$ there exists $\bar{\varphi} : \mathcal{P}_{\mathcal{M}(X)} \to O$ *such that the following diagram commutes in* $\mathbf{X}$*.*

$$\begin{array}{ccc} X & \xrightarrow{\;j\;} & \mathcal{P}_{\mathcal{M}(X)} \\ & \searrow^{\varphi} & \downarrow{\scriptstyle \bar{\varphi}} \\ & & O \end{array}$$

It remains to establish that the morphism $\bar{\varphi}$ factoring through $\left(\mathcal{P}_{\mathcal{M}(X)}, j\right)$ is unique. For a given pseudo-solution $(\mathrm{P}_{\mathbf{X}}^{\mathbf{Y}}(X, M), j_M)$, it was shown in Proposition 3.3.5 that the object in $\mathcal{A}(\mathbf{Y})$ generated by the subset $j_M[X]$ is dense in $\mathrm{P}_{\mathbf{X}}^{\mathbf{Y}}(X, M)$. Using this fact along with the construction of the alleged free object as an inverse limit of pseudo-solutions will be enough for us to show uniqueness of the factoring morphisms. We will show how this is done in a particular case.

Proposition 3.7.3. *Consider the categories* $\mathbf{ComLM\text{-}C\text{-}SVLA^{1+}}$ *and* $\mathbf{Set}$ *and fix* $S \in \mathbf{Set}$. *By Corollary 3.5.9, there exists an object* $\mathcal{P}_{\mathcal{M}(S)}$ *in* $\mathbf{ComLM\text{-}C\text{-}SVLA^{1+}}$ *and a morphism* $j : S \to \mathcal{P}_{\mathcal{M}(S)}$ *such that for every* Y *in* $\mathbf{ComLM\text{-}C\text{-}SVLA^{1+}}$ *and every morphism* $\varphi : S \to Y$ *in* $\mathbf{Set}$ *there exists a morphism* $\bar{\varphi} : \mathcal{P}_{\mathcal{M}(S)} \to Y$ *in* $\mathbf{ComLM\text{-}C\text{-}SVLA^{1+}}$ *such that the following diagram commutes in* $\mathbf{Set}$.



In addition, the object in $\mathbf{VLA^{1+}}$ *generated by the subset* $j[S]$ *in* $\mathcal{P}_{\mathcal{M}(S)}$ *is dense in* $\mathcal{P}_{\mathcal{M}(S)}$. *As a result, the pair* $\left(\mathcal{P}_{\mathcal{M}(S)}, j\right)$ *is the free object over* S *of the category* $\mathbf{ComLM\text{-}C\text{-}SVLA^{1+}}$.

Proof. Let Y be an object in $\mathbf{ComLM\text{-}C\text{-}SVLA^{1+}}$ and $\varphi : S \to Y$ in $\mathbf{Set}$. The existence of the morphism $\bar{\varphi} : \mathcal{P}_{\mathcal{M}(S)} \to Y$ satisfying the above diagram follows by Corollary 3.7.2.

Denote by G the sub-vector lattice algebra in $\mathcal{P}_{\mathcal{M}(S)}$ generated by the subset $j[S]$. We show that G is dense in $\mathcal{P}_{\mathcal{M}(S)}$. For every $M \in \mathcal{M}(S)$, denote by $\|\cdot\|_M$ the norm of the pseudo-solution $\mathrm{P}_{\mathbf{Set}}^{\mathbf{BLA^{1+}}}(S, M)$ and consider the family of seminorms $\{\rho_M : \mathcal{P}_{\mathcal{M}(S)} \to \mathbb{R}\}_{M \in \mathcal{M}(S)}$ where $\rho_M(x) := \|\pi_M(x)\|_M$ generating the locally m-convex-solid topology on $\mathcal{P}_{\mathcal{M}(S)}$. Fix $y_0 \in \mathcal{P}_{\mathcal{M}(S)}$ and consider the basis element

$$B := \bigcap_{i=1}^{n} \left\{x \in \mathcal{P}_{\mathcal{M}(S)} : \rho_{M_i}(x - y_0) < \epsilon_i\right\}$$

where $n \in \mathbb{N}$ and $\{\epsilon_1, \ldots, \epsilon_n\} \subseteq \mathbb{R}^+$. Define $\epsilon^* := \min\{\epsilon_1, \ldots, \epsilon_n\}$. Since $\mathcal{M}(S)$ is upwards directed, there exists $M^* \geq M_i$ for all $1 \leq i \leq n$. Denote by G_{M^*} the sub-vector lattice algebra generated by $j_{M^*}[S]$ in $\mathrm{P}_{\mathbf{Set}}^{\mathbf{BLA^{1+}}}(S, M^*)$. By Theorem 3.3.7, G_{M^*} is dense in $\mathrm{P}_{\mathbf{Set}}^{\mathbf{BLA^{1+}}}(S, M^*)$, thus there exists $s_1, \ldots, s_k \subseteq S$ for some $k \in \mathbb{N}$ and an element $\tilde{p} := P(j_{M^*}(s_1), \ldots, j_{M^*}(s_k)) \in G_{M^*}$ where P is some combination of the operations defined on $\mathrm{P}_{\mathbf{Set}}^{\mathbf{BLA^{1+}}}(S, M^*)$ such that

$$\|\tilde{p} - \pi_{M^*}(y_0)\|_{M^*} < \epsilon^*.$$

Consider the element $p := P(j(s_1), \ldots, j(s_k)) \in G$ where $\pi_{M^*}(p) = \tilde{p}$. Since $M^* \geq M_i$ for all $1 \leq i \leq n$, by the properties of the inverse limit $\left(\mathcal{P}_{\mathcal{M}(S)}, (p_M)_{M \in \mathcal{M}(S)}\right)$ and the fact that the morphisms in the inverse system $\mathcal{P}$ are contractive, we have that for every $1 \le i \le n$,

$$\begin{aligned}\rho_{M_i}(p - y_0) = \ &\|\pi_{M_i}(p) - \pi_{M_i}(y_0)\|_{M_i} = \|p_{M^*,M_i}(\pi_{M^*}(p - y_0))\|_{M_i} \\ \le \ &\|\pi_{M^*}(p) - \pi_{M^*}(y_0)\|_{M^*} = \|\tilde{p} - \pi_{M^*}(y_0)\|_{M^*} < \epsilon^* \le \epsilon_i.\end{aligned}$$

Thus $B \cap G \neq \varnothing$ and we conclude that G is dense in $\mathcal{P}_{\mathcal{M}(S)}$.

Now, let $\bar{\psi} : \mathcal{P}_{\mathcal{M}(S)} \to X$ be any morphism in $\mathbf{ComLM\text{-}C\text{-}SVLA^{1+}}$ such that $\bar{\psi} \circ j = \varphi$. Since $\bar{\varphi}$ and $\bar{\psi}$ coincide on $j[S]$, it follows by Proposition 3.2.11 that $\bar{\varphi}$ and $\bar{\psi}$ coincide on G since $\bar{\varphi}$ and $\bar{\psi}$ are morphisms in $\mathbf{VLA^{1+}}$. Since $\bar{\varphi}$ and $\bar{\psi}$ are continuous, we conclude that $\bar{\varphi} = \bar{\psi}$. ☐

This informs the following more general result.

Proposition 3.7.4. *Let $\mathbf{Y}$ be a category from Table 2 or 3 and $\mathbf{X}$ a category from Table 1 such that $\mathbf{X} \supseteq \mathbf{Y}$ and fix $X \in \mathbf{X}$. Consider the pair $\left(\mathcal{P}_{\mathcal{M}(X)}, j\right)$ constructed in Corollary 3.7.2, then the object in $\mathcal{A}(\mathbf{Y})$ generated by the subset $j[X]$ in $\mathcal{P}_{\mathcal{M}(X)}$ is dense in $\mathcal{P}_{\mathcal{M}(X)}$.*

Proof. We treat two separate cases. For a category of complete normed structures $\hat{\mathbf{Y}}$ from Table 3, it is clear that the proof of Proposition 3.7.3 can be adapted to obtain the desired result since the proof only relies on the topological structure on $\mathcal{P}_{\mathcal{M}(X)}$, the density of G_M in $\mathrm{P}^{\hat{\mathbf{Y}}}_{\mathbf{X}}(X, M)$ for $M \in \mathcal{M}(X)$ and the properties of inverse limits.

On the other hand, the proof of Proposition 3.7.3 is adjusted in the following way for categories of normed structures $\mathbf{Y}$ from Table 2: Fix $y_0 \in \mathcal{P}_{\mathcal{M}(X)}$ and consider the basis element

$$B \coloneqq \bigcap_{i=1}^{n} \left\{x \in \mathcal{P}_{\mathcal{M}(X)} : \rho_{M_i}(x - y_0) < \epsilon_i\right\}.$$

Fix $M^* \ge M_i$ for all $1 \le i \le n$. Since $\mathrm{P}^{\mathbf{Y}}_{\mathbf{X}}(X, M^*)$ is generated by the subset $j_{M^*}[S]$, there exists $x_1, \dots, x_k \subseteq X$ for some $k \in \mathbb{N}$ such that $\pi_{M^*}(y_0) = P(j_{M^*}(x_1), \dots, j_{M^*}(x_k))$ where P is some combination of the operations defined on $\mathrm{P}^{\mathbf{Y}}_{\mathbf{X}}(X, M^*)$. Define $p \coloneqq P(j(x_1), \dots, j(x_k))$ satisfying $\pi_{M^*}(p) = \pi_{M^*}(y_0)$. This implies that $\rho_{M_i}(y_0 - p) = 0$ for all $1 \le i \le n$. Thus $p \in B$ and it follows that G is dense in $\mathcal{P}_{\mathcal{M}(X)}$. ☐

It is clear that the work in Section 3.3 is of independent interest given the connections that have been made to the existing literature, however, all the work after Section 3.4 has led up to the following list of free objects in categories of locally convex structures. Their existence follows immediately from Corollary 3.7.2 and Proposition 3.7.4.

The notation we have developed does not lead to the most aesthetic list of answers, but this is preferable over ambiguous notation.

Theorem 3.7.5. *Let S be a set, V a vector space, E a vector lattice, R an algebra, R^1 a unital algebra, A a vector lattice algebra, and A^1 a unital vector lattice algebra* *and let X denote any one of these objects. The following free objects exist. In all cases, the morphism $j : X \to \mathrm{F}^{\mathbf{LCY}\text{-}\varprojlim\mathbf{Y}}_{\mathbf{X}}(X)$ is a morphism in the category $\mathbf{X}$.*

Categories of inverse limits of normed structures:

(1) $\mathrm{F}^{\mathbf{LCS}\text{-}\varprojlim\mathbf{NS}}_{\mathbf{Set}}(S)$

(2) $\mathrm{F}^{\mathbf{LC}\text{-}\mathbf{SVL}\text{-}\varprojlim\mathbf{NVL}}_{\mathbf{Set}}(S)$

(3) $\mathrm{F}^{\mathbf{LC}\text{-}\mathbf{SVL}\text{-}\varprojlim\mathbf{NVL}}_{\mathbf{VS}}(V)$

(4) $\mathrm{F}^{\mathbf{LM}\text{-}\mathbf{CA}\text{-}\varprojlim\mathbf{NA}}_{\mathbf{Set}}(S)$

(5) $\mathrm{F}^{\mathbf{LM}\text{-}\mathbf{CA}\text{-}\varprojlim\mathbf{NA}}_{\mathbf{VS}}(V)$

(6) $\mathrm{F}^{\mathbf{LM}\text{-}\mathbf{CA}^1\text{-}\varprojlim\mathbf{NA}^1}_{\mathbf{Set}}(S)$

(7) $\mathrm{F}^{\mathbf{LM}\text{-}\mathbf{CA}^1\text{-}\varprojlim\mathbf{NA}^1}_{\mathbf{VS}}(V)$

(8) $\mathrm{F}^{\mathbf{LM}\text{-}\mathbf{CA}^1\text{-}\varprojlim\mathbf{NA}^1}_{\mathbf{Alg}}(R)$

(9) $\mathrm{F}^{\mathbf{LM}\text{-}\mathbf{C}\text{-}\mathbf{SVLA}\text{-}\varprojlim\mathbf{NVLA}}_{\mathbf{Set}}(S)$

(10) $\mathrm{F}^{\mathbf{LM}\text{-}\mathbf{C}\text{-}\mathbf{SVLA}\text{-}\varprojlim\mathbf{NVLA}}_{\mathbf{VS}}(V)$

(11) $\mathrm{F}^{\mathbf{LM}\text{-}\mathbf{C}\text{-}\mathbf{SVLA}\text{-}\varprojlim\mathbf{NVLA}}_{\mathbf{VL}}(E)$

(12) $\mathrm{F}^{\mathbf{LM}\text{-}\mathbf{C}\text{-}\mathbf{SVLA}\text{-}\varprojlim\mathbf{NVLA}}_{\mathbf{Alg}}(R)$

(13) $\mathrm{F}^{\mathbf{LM}\text{-}\mathbf{C}\text{-}\mathbf{SVLA}^1\text{-}\varprojlim\mathbf{NVLA}^1}_{\mathbf{Set}}(S)$

(14) $\mathrm{F}^{\mathbf{LM}\text{-}\mathbf{C}\text{-}\mathbf{SVLA}^1\text{-}\varprojlim\mathbf{NVLA}^1}_{\mathbf{VS}}(V)$

(15) $\mathrm{F}^{\mathbf{LM}\text{-}\mathbf{C}\text{-}\mathbf{SVLA}^1\text{-}\varprojlim\mathbf{NVLA}^1}_{\mathbf{VL}}(E)$

(16) $\mathrm{F}^{\mathbf{LM}\text{-}\mathbf{C}\text{-}\mathbf{SVLA}^1\text{-}\varprojlim\mathbf{NVLA}^1}_{\mathbf{Alg}}(R)$

(17) $\mathrm{F}^{\mathbf{LM}\text{-}\mathbf{C}\text{-}\mathbf{SVLA}^1\text{-}\varprojlim\mathbf{NVLA}^1}_{\mathbf{Alg}^1}(R^1)$

(18) $\mathrm{F}^{\mathbf{LM}\text{-}\mathbf{C}\text{-}\mathbf{SVLA}^1\text{-}\varprojlim\mathbf{NVLA}^1}_{\mathbf{VLA}}(A)$

(19) $\mathrm{F}^{\mathbf{LM}\text{-}\mathbf{C}\text{-}\mathbf{SVLA}^{1+}\text{-}\varprojlim\mathbf{NVLA}^{1+}}_{\mathbf{Set}}(S)$

(20) $\mathrm{F}^{\mathbf{LM}\text{-}\mathbf{C}\text{-}\mathbf{SVLA}^{1+}\text{-}\varprojlim\mathbf{NVLA}^{1+}}_{\mathbf{VS}}(V)$

(21) $\mathrm{F}^{\mathbf{LM}\text{-}\mathbf{C}\text{-}\mathbf{SVLA}^{1+}\text{-}\varprojlim\mathbf{NVLA}^{1+}}_{\mathbf{VL}}(E)$

(22) $\mathrm{F}^{\mathbf{LM}\text{-}\mathbf{C}\text{-}\mathbf{SVLA}^{1+}\text{-}\varprojlim\mathbf{NVLA}^{1+}}_{\mathbf{Alg}}(R)$

(23) $\mathrm{F}^{\mathbf{LM}\text{-}\mathbf{C}\text{-}\mathbf{SVLA}^{1+}\text{-}\varprojlim\mathbf{NVLA}^{1+}}_{\mathbf{Alg}^1}(R^1)$

(24) $\mathrm{F}^{\mathbf{LM}\text{-}\mathbf{C}\text{-}\mathbf{SVLA}^{1+}\text{-}\varprojlim\mathbf{NVLA}^{1+}}_{\mathbf{VLA}}(A)$

(25) $\mathrm{F}^{\mathbf{LM}\text{-}\mathbf{C}\text{-}\mathbf{SVLA}^{1+}\text{-}\varprojlim\mathbf{NVLA}^{1+}}_{\mathbf{VLA}^1}(A^1)$

Categories of complete locally convex structures:

(1) $\mathrm{F}^{\mathbf{ComLCS}}_{\mathbf{Set}}(S)$

(2) $\mathrm{F}^{\mathbf{ComLC}\text{-}\mathbf{SVL}}_{\mathbf{Set}}(S)$

(3) $\mathrm{F}^{\mathbf{ComLC}\text{-}\mathbf{SVL}}_{\mathbf{VS}}(V)$

(4) $\mathrm{F}^{\mathbf{ComLM}\text{-}\mathbf{CA}}_{\mathbf{Set}}(S)$

(5) $\mathrm{F}^{\mathbf{ComLM}\text{-}\mathbf{CA}}_{\mathbf{VS}}(V)$

(6) $\mathrm{F}^{\mathbf{ComLM}\text{-}\mathbf{CA}^1}_{\mathbf{Set}}(S)$

(7) $\mathrm{F}^{\mathbf{ComLM}\text{-}\mathbf{CA}^1}_{\mathbf{VS}}(V)$

(8) $\mathrm{F}^{\mathbf{ComLM}\text{-}\mathbf{CA}^1}_{\mathbf{Alg}}(R)$

(9) $\mathrm{F}^{\mathbf{ComLM}\text{-}\mathbf{C}\text{-}\mathbf{SVLA}}_{\mathbf{Set}}(S)$

(10) $\mathrm{F}^{\mathbf{ComLM}\text{-}\mathbf{C}\text{-}\mathbf{SVLA}}_{\mathbf{VS}}(V)$

(11) $\mathrm{F}^{\mathbf{ComLM}\text{-}\mathbf{C}\text{-}\mathbf{SVLA}}_{\mathbf{VL}}(E)$

(12) $\mathrm{F}^{\mathbf{ComLM}\text{-}\mathbf{C}\text{-}\mathbf{SVLA}}_{\mathbf{Alg}}(R)$

(13) $\mathrm{F}^{\mathbf{ComLM}\text{-}\mathbf{C}\text{-}\mathbf{SVLA}^1}_{\mathbf{Set}}(S)$

(14) $\mathrm{F}^{\mathbf{ComLM}\text{-}\mathbf{C}\text{-}\mathbf{SVLA}^1}_{\mathbf{VS}}(V)$

(15) $\mathrm{F}^{\mathbf{ComLM}\text{-}\mathbf{C}\text{-}\mathbf{SVLA}^1}_{\mathbf{VL}}(E)$

(16) $\mathrm{F}^{\mathbf{ComLM}\text{-}\mathbf{C}\text{-}\mathbf{SVLA}^1}_{\mathbf{Alg}}(R)$

(17) $\mathrm{F}^{\mathbf{ComLM}\text{-}\mathbf{C}\text{-}\mathbf{SVLA}^1}_{\mathbf{Alg}^1}(R^1)$

(18) $\mathrm{F}^{\mathbf{ComLM}\text{-}\mathbf{C}\text{-}\mathbf{SVLA}^1}_{\mathbf{VLA}}(A)$

(19) $\mathrm{F}^{\mathbf{ComLM}\text{-}\mathbf{C}\text{-}\mathbf{SVLA}^{1+}}_{\mathbf{Set}}(S)$

(20) $\mathrm{F}^{\mathbf{ComLM}\text{-}\mathbf{C}\text{-}\mathbf{SVLA}^{1+}}_{\mathbf{VS}}(V)$

(21) $\mathrm{F}^{\mathbf{ComLM}\text{-}\mathbf{C}\text{-}\mathbf{SVLA}^{1+}}_{\mathbf{VL}}(E)$

(22) $\mathrm{F}^{\mathbf{ComLM}\text{-}\mathbf{C}\text{-}\mathbf{SVLA}^{1+}}_{\mathbf{Alg}}(R)$

(23) $\mathrm{F}^{\mathbf{ComLM}\text{-}\mathbf{C}\text{-}\mathbf{SVLA}^{1+}}_{\mathbf{Alg}^1}(R^1)$

(24) $\mathrm{F}_{\mathbf{VLA}}^{\mathbf{ComLM\text{-}C\text{-}SVLA^{1+}}}(A)$ *(25)* $\mathrm{F}_{\mathbf{VLA^1}}^{\mathbf{ComLM\text{-}C\text{-}SVLA^{1+}}}(A^1)$

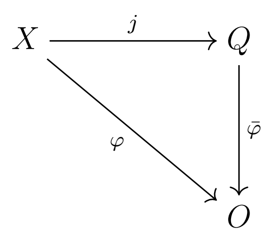
It is interesting to note that although the notion of an inverse limit is used in a fundamental sense in the construction of these free objects (both in constructing a candidate free object F from pseudo-solutions and showing that the image of the accompanying morphism j is dense in F), for the above collection of free objects in categories of complete locally convex structures there is no explicit mention of an inverse limit in sight anymore.

The above list of free objects contains all possible solutions we could obtain using the pseudo-solutions in Theorem 3.3.7 as building blocks in our procedure. Can the same not be done with the pseudo-solutions in Theorem 3.3.8? The pseudo-solutions in Theorem 3.3.8 are between categories $\mathbf{X}$ and $\mathbf{Y}$, both ranging over Table 2 and 3 such that $\mathbf{X} \supseteq \mathbf{Y}$. Unfortunately, the formulation of Lemma 3.5.8 and Corollary 3.5.9 does not allow for this: The categories $\mathbf{X}$ and $\mathbf{Y}$ need to satisfy the inclusions $\mathbf{Y} \subseteq \mathbf{LCY}\text{-}\varprojlim \mathbf{Y} \subseteq \mathbf{X}$ in order for us to apply Lemma 3.5.8. However, the inclusion $\mathbf{LCY}\text{-}\varprojlim \mathbf{Y} \subseteq \mathbf{X}$ will not hold in general for such categories $\mathbf{X}$ and $\mathbf{Y}$ since the objects in $\mathbf{LCY}\text{-}\varprojlim \mathbf{Y}$ are locally convex structures and the objects in $\mathbf{X}$ are normed structures.

It is possible to address this problem by introducing a category $\mathbf{Z}$ such that $\mathbf{X} \subseteq \mathbf{Z}$ and $\mathbf{LCY} \subseteq \mathbf{Z}$ and requiring that the inverse system

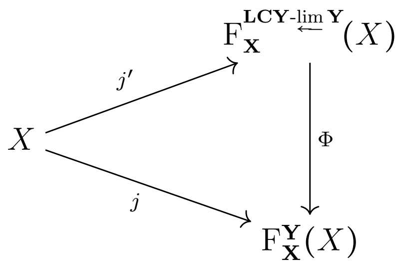
$$\mathcal{P} \coloneqq \big((\mathrm{P}_{\mathbf{X}}^{\mathbf{Y}}(X, M))_{M \in \mathcal{M}_b(X)}, (p_{M',M})_{M' \geq M}\big)$$

has an inverse limit $\big(\mathcal{P}_{\mathcal{M}(X)}, (p_M)_{M \in \mathcal{M}_b(X)}\big)$ in both $\mathbf{LCY}\text{-}\varprojlim \mathbf{Y}$ and $\mathbf{Z}$. One can reformulate the results in Lemma 3.5.8 and Corollary 3.5.9 where this 'upper bound' category $\mathbf{Z}$ is included. In the end, the following can be proven for categories $\mathbf{X}$ and $\mathbf{Y}$, both ranging over Table 2 and 3 such that $\mathbf{X} \supseteq \mathbf{Y}$: Fix an object $X \in \mathbf{X}$ and consider an upper bound category $\mathbf{Z}$ for $\mathbf{X}$ and $\mathbf{Y}$. Then there exists an object Q in $\mathbf{LCY}\text{-}\varprojlim \mathbf{Y}$ and a morphism $j : X \to Q$ in $\mathbf{Z}$ such that for every O in $\mathbf{LCY}\text{-}\varprojlim \mathbf{Y}$ and every morphism $\varphi : X \to O$ in $\mathbf{Z}$ there exists a unique morphism $\bar{\varphi} : Q \to O$ in $\mathbf{LCY}\text{-}\varprojlim \mathbf{Y}$ such that the following diagram commutes in $\mathbf{Z}$.



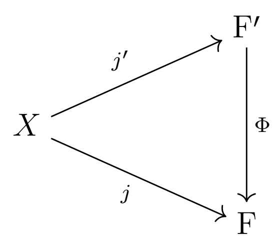
For example, we can take $\mathbf{Y} = \mathbf{BL}$ and $\mathbf{X} = \mathbf{Ban}$ then $\mathbf{LCY}\text{-}\varprojlim \mathbf{Y} = \mathbf{ComLC\text{-}SVL}$. We may take $\mathbf{ComLCS}$ as upper bound category $\mathbf{Z}$ for $\mathbf{ComLC\text{-}SVL}$ and $\mathbf{Ban}$. For $X \in \mathbf{Ban}$, there exists an object Q in $\mathbf{ComLC\text{-}SVL}$ and a morphism $j : X \to Q$ in $\mathbf{ComLCS}$ such that for every O in $\mathbf{ComLC\text{-}SVL}$ and every morphism $\varphi : X \to O$ in $\mathbf{ComLCS}$ (i.e. a continuous linear map) there exists a unique morphism $\bar{\varphi} : Q \to O$ in $\mathbf{ComLC\text{-}SVL}$ (i.e. a continuous vector lattice homomorphism) such that $\bar{\varphi} \circ j = \varphi$ holds in $\mathbf{ComLCS}$. Given that the pair (Q, j) satisfies a universal property similar to that of a free object, one might want to call the pair (Q, j) a *quasi-free object* *over* X *of* $\mathbf{LCY}\text{-}\varprojlim \mathbf{Y}$ *via* $\mathbf{Z}$. The category $\mathbf{Z}$ needs to be noted since the choice of $\mathbf{Z}$ affects the universal property we obtain in the diagram above.

Returning to our genuine free objects in Theorem 3.7.5, we can ask the following question: It may happen when we consider a particular choice of categories $\mathbf{X}$ and $\mathbf{Y}$ and $X \in \mathbf{X}$ that the free object $(\mathrm{F}_{\mathbf{X}}^{\mathbf{Y}}(X), j)$ does indeed exist. The procedure outlined in this chapter will still deliver us a free object $\left(\mathrm{F}_{\mathbf{X}}^{\mathbf{LCY}\text{-}\varprojlim \mathbf{Y}}(X), j'\right)$ in the larger category $\mathbf{LCY}\text{-}\varprojlim \mathbf{Y}$. One would hope that our procedure for constructing free objects would be able to recover this prior answer, in the sense that there is an isomorphism $\Phi : \mathrm{F}_{\mathbf{X}}^{\mathbf{LCY}\text{-}\varprojlim \mathbf{Y}}(X) \to \mathrm{F}_{\mathbf{X}}^{\mathbf{Y}}(X)$ in $\mathbf{Y}$ such that the following diagram commutes in $\mathbf{X}$.



This would mean that $\mathrm{F}_{\mathbf{X}}^{\mathbf{LCY}\text{-}\varprojlim \mathbf{Y}}(X)$ is in fact a normed space and the pair $(\mathrm{F}_{\mathbf{X}}^{\mathbf{Y}}(X), j)$ is not just a free object over X of the category of normed structures $\mathbf{Y}$, but is also a free object over X of the larger category of inverse limits of normed structures $\mathbf{LCY}\text{-}\varprojlim \mathbf{Y}$. The following result shows that this is indeed the case.

PROPOSITION 3.7.6. *Let* $\mathbf{Y}$ *be a category from Table 2 or 3 and* $\mathbf{X}$ *a category from Table 1 such that* $\mathbf{X} \supseteq \mathbf{Y}$ *and fix* $X \in \mathbf{X}$. *Assume that the free object* (F, j) *over* X *of* $\mathbf{Y}$ *exists and consider the free object* (F', j') *over* X *of* $\mathbf{LCY}\text{-}\varprojlim \mathbf{Y}$ *from Theorem 3.7.5. Then there exists a unique isomorphism* $\Phi : \mathrm{F}' \to \mathrm{F}$ *in* $\mathbf{Y}$ *such that the following diagram commutes in* $\mathbf{X}$.

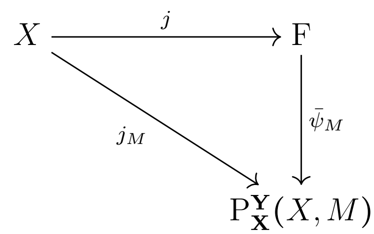


PROOF. We note that since a linear map between normed spaces is continuous if and only if it is bounded, it follows that $\mathbf{Y}$ is a full subcategory in $\mathbf{LCY}\text{-}\varprojlim \mathbf{Y}$. Since $F \in \mathbf{Y} \subseteq \mathbf{LCY}\text{-}\varprojlim \mathbf{Y}$, it follows by the universal property of (F', j') that there exists a unique morphism $\Phi : \mathrm{F}' \to \mathrm{F}$ in $\mathbf{LCY}\text{-}\varprojlim \mathbf{Y}$ such that $\Phi \circ j' = j$. We will show that Φ is in fact an isomorphism in $\mathbf{LCY}\text{-}\varprojlim \mathbf{Y}$.

Since Φ would then be a homeomorphism between a locally convex structure and a normed structure, it would follow by [**25**, Chapter IV, Proposition 2.6] that $\mathrm{F}' \in \mathbf{Y}$. Since $\mathbf{Y}$ is a full subcategory in $\mathbf{LCY}\text{-}\varprojlim \mathbf{Y}$, it would follow that Φ is the unique isomorphism in $\mathbf{Y}$ such that $\Phi \circ j' = j$.

It remains to construct an inverse morphism of $\Phi : \mathrm{F} \to \mathrm{F}'$ in $\mathbf{LCY}\text{-}\varprojlim \mathbf{Y}$: For every $M \in \mathcal{M}(X)$, the pseudo-solution $\mathrm{P}^{\mathbf{Y}}_{\mathbf{X}}(X, M)$ is an object in $\mathbf{Y}$ with a morphism $j_M : X \to \mathrm{P}^{\mathbf{Y}}_{\mathbf{X}}(X, M)$. By the universal property of (F, j), for every $M \in \mathcal{M}(X)$ there exists a unique morphism $\bar{\psi}_M : \mathrm{F} \to \mathrm{P}^{\mathbf{Y}}_{\mathbf{X}}(X, M)$ in $\mathbf{Y}$ such that the following diagram commutes in $\mathbf{X}$.

(3.7.1)



We claim that the pair $\big(\mathrm{F}, (\bar{\psi}_M)_{M\in\mathcal{M}(X)}\big)$ is a compatible system over the inverse system $\mathcal{P} := \big((\mathrm{P}^{\mathbf{Y}}_{\mathbf{X}}(X, M))_{M\in\mathcal{M}(X)}, (p_{M',M})_{M'\geq M}\big)$ in $\mathbf{LCY}\text{-}\varprojlim \mathbf{Y}$: For $M \in \mathcal{M}(X)$, denote by G_M the object in $\mathcal{A}(\mathbf{Y})$ generated by the subset $j_M[X]$ in $\mathrm{P}^{\mathbf{Y}}_{\mathbf{X}}(X, M)$. By Theorem 3.3.7, the subset G_M is at least dense in $\mathrm{P}^{\mathbf{Y}}_{\mathbf{X}}(X, M)$ for $M \in \mathcal{M}(X)$. Fix $M' \geq M$ in $\mathcal{M}(X)$ and take $j(x_0) \in j[X]$. By (3.7.1), we have

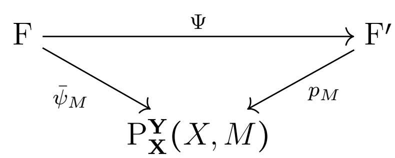
$$\bar{\psi}_M\,(j(x_0)) = j_M(x_0)$$

and,

$$p_{M',M} \circ \bar{\psi}_{M'}\,(j(x_0)) = p_{M',M}\,(j_{M'}(x_0)) = j_M(x_0).$$

Thus $p_{M',M} \circ \bar{\psi}_{M'}$ and $\bar{\psi}_M$ coincide on $j[X]$. By Proposition 3.1.3, the object G in $\mathcal{A}(\mathbf{Y})$ generated by the subset $j[X]$ is dense in F. Since $p_{M',M} \circ \bar{\psi}_{M'}$ and $\bar{\psi}_M$ are morphisms in $\mathbf{Y}$, it follows that $p_{M',M} \circ \bar{\psi}_{M'} = \bar{\psi}_M$. Hence, $\big(\mathrm{F}, (\bar{\psi}_M)_{M\in\mathcal{M}(X)}\big)$ is a compatible system over $\mathcal{P}$ in $\mathbf{LCY}\text{-}\varprojlim \mathbf{Y}$. Since $\big(\mathrm{F}', (p_M)_{M\in\mathcal{M}(X)}\big)$ is an inverse limit of $\mathcal{P}$ in $\mathbf{LCY}\text{-}\varprojlim \mathbf{Y}$, there exists a unique morphism $\Psi : \mathrm{F} \to \mathrm{F}'$ in $\mathbf{LCY}\text{-}\varprojlim \mathbf{Y}$ such that the following diagram commutes in $\mathbf{LCY}\text{-}\varprojlim \mathbf{Y}$.

(3.7.2)



It remains to verify that Φ and Ψ are inverse morphisms in $\mathbf{LCY}\text{-}\varprojlim \mathbf{Y}$: By (3.7.1) and (3.7.2), for every $M \in \mathcal{M}(X)$, we have

$$p_M \circ (\Psi \circ j) = (p_M \circ \Psi) \circ j = \bar{\psi}_M \circ j = j_M.$$

By Proposition 3.7.1 (iii), since $j' : X \to \mathrm{F}'$ is the unique morphism in $\mathbf{X}$ such that $p_M \circ j' = j_M$, it follows that $\Psi \circ j = j'$. Consider the morphism $\Psi \circ \Phi : \mathrm{F}' \to \mathrm{F}'$, then for $j'(x_0) \in j'[X]$, we have

$$(\Psi \circ \Phi)\,(j'(x_0)) = \Psi \circ j(x_0) = j'(x_0).$$

Thus $\Psi \circ \Phi$ and $\mathbf{1}_{\mathrm{F}'}$ coincide on $j'[X]$. By Proposition 3.7.4, we know that the object G' in $\mathcal{A}(\mathbf{Y})$ generated by the subset $j'[X]$ is dense in F′ and thus $\Psi \circ \Phi = \mathbf{1}_{\mathrm{F}'}$. Since $\Phi \circ j' = j$ and $\Psi \circ j = j'$ it follows by a similar argument that $\Phi \circ \Psi = \mathbf{1}_{\mathrm{F}}$. ☐

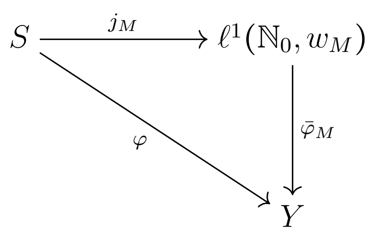
In the last two sections, we apply the free object machinery we have developed to the concrete pseudo-solutions in Section 3.4 to obtain concrete descriptions of two of the free objects whose existence was derived above.

3.8. The free complete locally m-convex algebra over a point

In this section, we give a concrete description of the free complete locally convex unital algebra over a one-point set by applying the inverse limits construction to the pseudo-solutions found in Section 3.4.1. We only consider algebras over $\mathbb{C}$ in this section.

Fix a one-point set $S := \{s_0\}$. For every $M > 0$, consider the pseudo-solution $(\ell^1(\mathbb{N}_0, w_M), j_M)$ where $j_M(s_0) \coloneqq \delta_1^{(M)}$ found in Section 3.4.1. We note that conditions (i)-(iii) in Lemma 3.5.8 are satisfied:

(i) For every $Y \in \mathbf{BA^1}$ and every morphism $\varphi : S \to Y$ in **Set** there exists $M_0 > 0$ such that for all $M \geq M_0$ there exists a unique morphism $\bar{\varphi}_M : \ell^1(\mathbb{N}_0, w_M) \to Y$ in $\mathbf{BA^1}$ such that the following diagram commutes in **Set**.



(ii) As shown in Remark 3.5.7, the pair $\mathcal{P} \coloneqq ((\ell^1(\mathbb{N}_0, w_M))_{M>0}, (p_{M_2,M_1})_{M_2 \geq M_1})$ is an inverse system in $\mathbf{BA^1}$ and the linking maps p_{M_2,M_1} are inclusions. Further, the pair $(S, (j_M)_{M>0})$ is a compatible system over $\mathcal{P}$ in **Set**.

(iii) For every $M > 0$, denote by $\|\cdot\|_M$ the norm on $\ell^1(\mathbb{N}_0, w_M)$. Define the set

$$E \coloneqq \left\{ (f_M) \in \prod_{M>0} \ell^1(\mathbb{N}_0, w_M) \ : \ p_{M_2,M_1}(f_{M_2}) = f_{M_1}, \forall\, M_2 \geq M_1 \right\}.$$

For $M > 0$, define $p_M : E \to \ell^1(\mathbb{N}_0, w_M)$ where $p_M \coloneqq \pi_M|_E$ and equip E with the topology τ_0 generated by the seminorms induced by the component norms. The results in Section 3.6 show that (E, τ_0) is a complete unital locally m-convex algebra and the pair $((E, \tau_0), (p_M)_{M>0})$ is an inverse limit of $\mathcal{P}$ in both **Set** and $\mathbf{ComLM\text{-}CA^1}$. Since $(S, (j_M)_{M>0})$ is a compatible system over $\mathcal{P}$ in **Set**, there exists a unique morphism $j' : S \to E$ in **Set** such that $p_M \circ j' = j_M$ holds in **Set** for all $M > 0$.

Thus we know from our previous work that $((E, \tau_0), j')$ is the free complete unital locally m-convex algebra over S. We consider a related inverse system to show that this free object has a more familiar guise.

For every $M > 0$, denote $D_M \coloneqq \{z \in \mathbb{C} : |z| < M\}$ and define the collection of functions

$$H_M \coloneqq \left\{ f : \bar{D}_M \to \mathbb{C} \ : \ f = \sum_{k=0}^{\infty} a_k z^k, \ \sum_{k=0}^{\infty} |a_k| M^k < \infty \right\}.$$

It is clear that each element in H_M is holomorphic on D_M and continuous on $\bar{D}_M$. Adding the standard vector space and multiplication operations along with the norm $\|\cdot\|_{H_M} : H_M \to \mathbb{R}$ where

$$\|f\|_{H_M} := \sum_{k=0}^{\infty} |a_k| M^k$$

makes H_M into a unital Banach algebra. For a given $M > 0$ and every $x = (a_k) \in \ell^1(\mathbb{N}_0, w_M)$ define the associated power series $f_x \in H_M$ where $f_x(z) := \sum_{k=0}^{\infty} a_k z^k$ for $z \in \bar{D}_M$. It is easy to see that the map $T_M : \ell^1(\mathbb{N}_0, w_M) \to H_M$ where $x = (a_k) \longmapsto f_x$ is an isometric unital algebra isomorphism. For $M_2 \geq M_1 > 0$ denote by $r_{M_2,M_1} : H_{M_2} \to H_{M_1}$ the restriction map. It is clear that the pair $\mathcal{H} := ((H_M)_{M>0}, (r_{M_2,M_1})_{M_2 \geq M_1})$ forms an inverse system in $\mathbf{BA^1}$ where the linking maps r_{M_2,M_1} are contractive $\mathbf{BA^1}$-morphisms. Define

$$H := \left\{ (g_M) \in \prod_{M>0} H_M \;:\; r_{M_2,M_1}(g_{M_2}) = g_{M_1}, \forall\, M_2 \geq M_1 \right\}.$$

For $M > 0$, define $r_M : H \to H_M$ where $r_M := \pi_M|_H$ and equip H with the topology τ generated by the family of seminorms $\{\eta_M : H \to \mathbb{R} \;:\; M > 0\}$ where $\eta_M(x) := \|\pi_M(x)\|_{H_M}$ for $x \in H$. As before, the pair $((H,\tau), (p_M)_{M>0})$ is an inverse limit of $\mathcal{H}$ in $\mathbf{ComLM\text{-}CA^1}$. For every $M > 0$, define $I_M \in H_M$ as the function $I_M(z) := z$ for $z \in \bar{D}_M$ and define the morphism $j'' : S \to H$ in $\mathbf{Set}$ where $j''(s_0) := (I_M)_{M>0}$. We note that the following diagram commutes in $\mathbf{BA^1}$ for all $M_2 \geq M_1 > 0$.

$$\begin{array}{ccc}
\ell^1(\mathbb{N}_0, w_{M_2}) & \xrightarrow{T_{M_2}} & H_{M_2} \\
\Big\downarrow{\scriptstyle p_{M_2,M_1}} & & \Big\downarrow{\scriptstyle r_{M_2,M_1}} \\
\ell^1(\mathbb{N}_0, w_{M_1}) & \xrightarrow[T_{M_1}]{} & H_{M_1}
\end{array}$$

Since T_M is an $\mathbf{BA^1}$-isomorphism for every $M > 0$, it follows by Proposition 2.4.2 in Chapter 2 that there exists a unique $\mathbf{BA^1}$-isomorphism $T : E \to H$ such that the following diagram commutes in $\mathbf{BA^1}$ for all $M > 0$.

$$\begin{array}{ccc}
E & \xrightarrow{T} & H \\
\Big\downarrow{\scriptstyle p_M} & & \Big\downarrow{\scriptstyle r_M} \\
\ell^1(\mathbb{N}_0, w_{M_1}) & \xrightarrow[T_M]{} & H_M
\end{array}$$

which implies that

$$r_M \circ T \circ j'(s_0) = T_M \circ p_M \circ j'(s_0) = T_M\left(\delta_1^{(M)}\right) = I_M = r_M \circ j''(s_0).$$

It follows that $T \circ j' = j''$ and thus by the essential uniqueness of free objects we conclude that $((H,\tau), j'')$ is also a free complete unital locally m-convex algebra over S.

Now, consider the unital algebra of entire functions $H(\mathbb{C})$. Denote by $I \in H(\mathbb{C})$ the function $I(z) := z$ for $z \in \mathbb{C}$ and define $j : S \to H(\mathbb{C})$ where $j(s_0) := I$. Since every entire function can be represented as a power series $f = \sum_{k=0} a_k z^k$ converging everywhere in $\mathbb{C}$ one sees that the map $R : H(\mathbb{C}) \to H$ where $f \longmapsto \left(f|_{\bar{D}_M}\right)_{M>0}$ is a unital algebra isomorphism. For $M > 0$, define the seminorm $\rho_M : H(\mathbb{C}) \to \mathbb{R}$ where $\rho_M := \eta_M \circ R$ and denote by τ_M the topology generated by this family of seminorms. By [**57**, Theorem 5.7.3], we have that (H, τ) and $(H(\mathbb{C}), \tau_M)$ are isomorphic via the $\mathbf{ComLM\text{-}CA^1}$-isomorphism R and since $R \circ j = j''$ it follows that $((H(\mathbb{C}), \tau_M), j)$ is also a free complete unital locally m-convex algebra over S.

We claim that $H(\mathbb{C})$ equipped with the τ_M topology derived from the pseudo-solutions construction is homeomorphic to $H(\mathbb{C})$ equipped with the familiar topology of uniform convergence on compact sets: Denote by $\mathcal{K}(\mathbb{C})$ the collection of compact subsets of $\mathbb{C}$ and for every $K \in \mathcal{K}(\mathbb{C})$, define the seminorm $p_K : H(\mathbb{C}) \to \mathbb{R}$ where $p_K(f) := \max_{z \in K} |f(z)|$. The family of seminorms $\{p_K : K \in \mathcal{K}(\mathbb{C})\}$ generates the topology of uniform convergence on compact sets, which we denote as τ_K. First, we show that the identity map $J_1 : (H(\mathbb{C}), \tau_M) \longrightarrow (H(\mathbb{C}), \tau_K)$ is continuous: Fix $K \in \mathcal{K}(\mathbb{C})$, then there exists $M > 0$ such that $|z| \le M$ for all $z \in K$. Then for every $f = \sum_{k=0}^{\infty} a_k z^k \in H(\mathbb{C})$ and all $z \in K$, we have

$$|f(z)| \le \sum_{k=0}^{\infty} |a_k| |z|^k \le \sum_{k=0}^{\infty} |a_k| M^k = \rho_M(f)$$

which implies that $p_K \le \rho_M$ and thus J_1 is continuous. Next, we show that the identity map $J_2 : (H(\mathbb{C}), \tau_K) \longrightarrow (H(\mathbb{C}), \tau_M)$ is continuous: Fix $M > 0$. By the Cauchy integral formula [**62**, Chapter 2, Corollary 4.2], for every $f = \sum_{k=0}^{\infty} a_k z^k \in H(\mathbb{C})$, we have

$$a_k = \frac{f^{(k)}(0)}{k!} = \frac{1}{2\pi i} \oint_{|z|=2M} \frac{f(z)}{z^{k+1}}\, dz.$$

Denote $C := \{z \in \mathbb{C} : |z| = 2M\}$. This gives the upper estimate

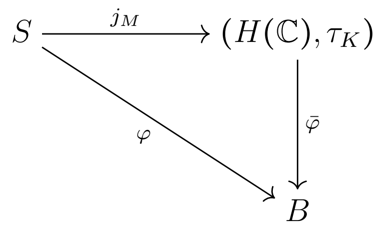
$$|a_k| \le \frac{4\pi M}{2\pi} \max_{z \in C} \left|\frac{f(z)}{z^{k+1}}\right| = \frac{1}{(2M)^k} \max_{z \in C} |f(z)|.$$

As a result

$$\rho_M(f) = \sum_{k=0}^{\infty} |a_k| M^k \le \sum_{k=0}^{\infty} \left(\frac{1}{(2M)^{k+1}} \max_{z \in C} |f(z)|\right) M^k = \sum_{k=0}^{\infty} \frac{1}{2^k} \max_{z \in C} |f(z)| = 2p_C(f).$$

This implies that J_2 is continuous and we conclude that $(H(\mathbb{C}), \tau_M)$ and $(H(\mathbb{C}), \tau_K)$ are indeed homeomorphic. This gives us the concrete description $((H(\mathbb{C}), \tau_K), j)$ for the free complete unital locally m-convex algebra over S.

Now that we have calculated the answer $((H(\mathbb{C}), \tau_K), j)$ as the solution to the free object problem between the categories $\mathbf{ComLM\text{-}CA^1}$ and **Set**, it is not difficult to verify that this pair is a solution by showing directly that the necessary universal property is satisfied: Fix $B \in \mathbf{ComLM\text{-}CA^1}$ and a morphism $\varphi : S \to B$ in **Set**. We show that there exists a unique morphism $\bar{\varphi} : H(\mathbb{C}) \to B$ in $\mathbf{ComLM\text{-}CA^1}$ such that the following diagram commutes in **Set**.



Define the map $\bar{\varphi} : H(\mathbb{C}) \to B$ where

$$\bar{\varphi}(f) := \sum_{k=0}^{\infty} a_k \left(\varphi(s_0)\right)^k, \qquad \left(f = \sum_{k=0}^{\infty} a_k z^k \in H(\mathbb{C})\right).$$

We first need to verify that $\bar{\varphi}$ is well-defined: Fix $f = \sum_{k=0}^{\infty} a_k z^k \in H(\mathbb{C})$. We show that $\left(\sum_{k=0}^{n} a_k \left(\varphi(s_0)\right)^k\right)_{n \in \mathbb{N}_0}$ is a Cauchy sequence. Denote by P_B the family of seminorms defining the topology on B and take $p \in P_B$. For $m > n$ in $\mathbb{N}_0$, we have

$$p\left(\sum_{k=0}^{m} a_k \left(\varphi(s_0)\right)^k - \sum_{k=0}^{n} a_k \left(\varphi(s_0)\right)^k\right) \leq \sum_{k=n+1}^{m} |a_k| p(\varphi(s_0))^k.$$

Since $f \in H(\mathbb{C})$, it follows that $\sum_{k=0}^{\infty} |a_k| p(\varphi(s_0))^k < \infty$ for every $p \in P_M$ which implies that $\left(\sum_{k=0}^{n} a_k \left(\varphi(s_0)\right)^k\right)_{n \in \mathbb{N}_0}$ is indeed a Cauchy sequence. We conclude from the completeness of B that $\bar{\varphi}$ is indeed well-defined.

Denote by $P(\mathbb{C})$ the collection of polynomials on $\mathbb{C}$, which is dense in $H(\mathbb{C})$. To verify that $\bar{\varphi}$ is continuous let $p \in P_B$ and $f = \sum_{k=0}^{\infty} a_k z^k \in H(\mathbb{C})$, then

$$p\left(\bar{\varphi}(f)\right) = p\left(\lim_{n \to \infty} \sum_{k=0}^{n} a_k \left(\varphi(s_0)\right)^k\right) = \lim_{n \to \infty} p\left(\sum_{k=0}^{n} a_k \left(\varphi(s_0)\right)^k\right) \leq \lim_{n \to \infty} \sum_{k=0}^{n} |a_k| p\left(\varphi(s_0)\right)^k.$$

For every $M > 0$ with $M \geq p\left(\varphi(s_0)\right)$, it follows that $p\left(\bar{\varphi}(f)\right) \leq \rho_M(f)$. Since $\tau_M = \tau_K$, this implies that $\bar{\varphi} : (H(\mathbb{C}), \tau_K) \to B$ is continuous. It is then easy to verify using the density of $P(\mathbb{C})$ in $H(\mathbb{C})$ and the continuity of $\bar{\varphi}$ that $\bar{\varphi}$ is linear on $H(\mathbb{C})$ and since $\bar{\varphi}$ is multiplicative on $P(\mathbb{C})$ it also follows by continuity that $\bar{\varphi}$ is multiplicative on $H(\mathbb{C})$. Thus $\bar{\varphi}$ is a **ComLM-CA$^{\mathbf{1}}$**-morphism satisfying $\bar{\varphi} \circ j = \varphi$.

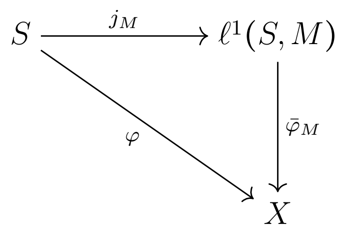
Let $\bar{\psi} : (H(\mathbb{C}), \tau_K) \to B$ be any **ComLM-CA$^{\mathbf{1}}$**-morphism satisfying $\bar{\psi} \circ j = \varphi$. Since the function $I \in H(\mathbb{C})$ generates the sub-algebra $P(\mathbb{C})$ in $H(\mathbb{C})$ and both $\bar{\varphi}$ and $\bar{\psi}$ are unital algebra homomorphisms it follows that $\bar{\varphi}$ and $\bar{\psi}$ coincide on $P(\mathbb{C})$ and by density of $P(\mathbb{C})$ in $H(\mathbb{C})$ we conclude that $\bar{\varphi} = \bar{\psi}$. Thus $((H(\mathbb{C}), \tau_K), j)$ does indeed satisfy the desired universal property.

3.9. The free complete locally convex space over a set

Lastly, we make use of the concrete descriptions of pseudo-solutions found in Section 3.4.2 to construct the free (complete) locally convex space over an arbitrary set S. It turns out that this free object can be viewed both as an inverse limit and a direct limit.

3.9.1. Inverse limit construction. Fix any set S. It is important to note that in Section 3.4.2, we only gave concrete descriptions of the pseudo-solutions $(\mathrm{P}^{\mathbf{Ban}}_{\mathbf{Set}}(S,M), j_M)$ as $(\ell^1(S,M), j_M)$ for $M \in \mathcal{M}_{>0}(S)$. However, thanks to the formulation of Lemma 3.5.8 and Corollary 3.5.9 as well as the fact that $\mathcal{M}_{>0}(S)$ forms a confinal subset in $\mathcal{M}(S)$, this will be sufficient for us to construct a concrete description of the free complete locally convex space over S. We note that conditions (i)-(iii) in Lemma 3.5.8 are satisfied in this situation:

(i) It is clear that $\mathcal{M}_{>0}(S)$ is upwards directed and cofinal in $\mathcal{M}(S)$. As a result, from Section 3.4.2 we know that for every $X \in \mathbf{Ban}$ and every morphism $\varphi : S \to X$ in $\mathbf{Set}$ there exists $M_0 \in \mathcal{M}_{>0}(S)$ such that for all $M \geq M_0$ in $\mathcal{M}_{>0}(S)$ there exists a unique morphism $\bar{\varphi}_M : \ell^1(S,M) \to X$ in $\mathbf{Ban}$ such that the following diagram commutes in $\mathbf{Set}$.



(ii) As shown in Remark 3.5.7, the pair $\mathcal{P} \coloneqq \big((\ell^1(S,M))_{M\in\mathcal{M}_{>0}(S)}, (p_{M_2,M_1})_{M_2 \geq M_1}\big)$ is an inverse system in $\mathbf{Ban}$ and the linking maps p_{M_2,M_1} are inclusions. Further, the pair $\big(S, (j_M)_{M\in\mathcal{M}_{>0}(S)}\big)$ is a compatible system over $\mathcal{P}$ in $\mathbf{Set}$.

(iii) For every $M \in \mathcal{M}_{>0}(S)$, denote by $\|\cdot\|_M$ the norm on $\ell^1(S,M)$. Define the set

$$F \coloneqq \left\{ (f_M) \in \prod_{M\in\mathcal{M}_{>0}(S)} \ell^1(S,M) \ : \ p_{M_2,M_1}(f_{M_2}) = f_{M_1}, \forall\, M_2 \geq M_1 \right\}.$$

For $M \in \mathcal{M}_{>0}(S)$, define $p_M : F \to \ell^1(S,M)$ where $p_M \coloneqq \pi_M|_F$ and equip F with the topology τ generated by the family of seminorms

$$\{\eta_M : F \to \mathbb{R} \ : \ M \in \mathcal{M}_{>0}(S)\}$$

where $\eta_M(x) := \|\pi_M(x)\|_M$ for $x \in F$. From the considerations in Section 3.6, we know that (F,τ) is a complete locally convex space and the pair $\big(F, (p_M)_{M\in\mathcal{M}_{>0}(S)}\big)$ is an inverse limit of $\mathcal{P}$ in both $\mathbf{Set}$ and $\mathbf{ComLCS}$. Since $\big(S, (j_M)_{M\in\mathcal{M}_{>0}(S)}\big)$ is a compatible system over $\mathcal{P}$ in $\mathbf{Set}$, there exists a unique morphism $j' : S \to F$ in $\mathbf{Set}$ such that $p_M \circ j' = j_M$ holds in $\mathbf{Set}$ for all $M \in \mathcal{M}_{>0}(S)$.

Thus from the procedure we have developed we know that $((F,\tau), j')$ is the free complete locally convex space over S. However, more can be said about the structure of this free object.

We claim that F is isomorphic as a vector space to the free vector space V_S over S: Consider the map $T : V_S \to F$ where $f \longmapsto (f)_{M\in\mathcal{M}_{>0}(S)}$. It is clear that T is an injective linear map. Fix $(f_M)_{M\in\mathcal{M}_{>0}(S)} \in F$ and fix indices $M_1, M_2 \in \mathcal{M}_{>0}(S)$. Then there exists $M^* \geq M_1, M_2$ in $\mathcal{M}_{>0}(S)$ such that $p_{M^*,M_1}(f_{M^*}) = f_{M_1}$ and $p_{M^*,M_2}(f_{M^*}) = f_{M_2}$.

Since the linking maps in $\mathcal{P}$ are the inclusions maps it follows that $f_{M_1} = f_{M^*} = f_{M_2}$. This implies that $f_{M_1} \in \ell^1(S, M)$ for all $M \geq M_1$ in $\mathcal{M}_{>0}(S)$. However, it is clear that the quantity $\sum_{s\in S} |f_{M_1}(s)|M(s)$ is not defined for sufficiently large $M \geq M_1$ if the set $\{s \in S : f_{M_1}(s) \neq 0\}$ is infinite. Thus $f_{M_1} \in V_S$ and this shows that T is indeed a vector space isomorphism.

Denote by τ_M the topology on V_S generated by the family of seminorms

$$\{\rho_M : V_S \to \mathbb{R} \ : \ M \in \mathcal{M}_{>0}(S)\}$$

where $\rho_M(f) \coloneqq \sum_{s\in S} |\alpha_s|M(s)$ for $f = \sum_{s\in S} \alpha_s e_s \in V_S$. We note that ρ_M is in fact a norm for every $M \in \mathcal{M}_{>0}(S)$. Since $T : V_S \to F$ is a vector space isomorphism and $\rho_M \circ T = \eta_M$ for all $M \in \mathcal{M}_{>0}(S)$ it is clear that (F, τ) and (V_S, τ_M) are isomorphic as locally convex spaces. Define $j : S \to V_S$ where $j(s) \coloneqq e_s \in V_S$. Using the universal property of $((F, \tau), j')$ and the fact that $T \circ j = j'$, it is not difficult to show that $((V_S, \tau_M), j)$ is also a free complete locally convex space over S. However, this also follows by direct computation: Let Y be a complete locally convex space and denote by P_Y the collection of seminorms generating the topology on Y. Let $\varphi : S \to Y$ be any morphism in **Set**. By the universal property of the free vector space (V_S, j) over S, there exists a unique linear morphism $\bar{\varphi} : V_S \to Y$. For every $p \in Y_P$ and $f = \sum_{s\in S} \alpha_s e_s \in V_S$, we have

$$p(\bar{\varphi}(f)) \leq \sum_{s\in S} |\alpha_s| p(\varphi(s)).$$

Define $\tilde{M} : S \to \mathbb{R}$ where $\tilde{M}(s) \coloneqq p(\varphi(s))$ and choose $M \in \mathcal{M}_{>0}(S)$ such that $M \geq \tilde{M}$, then we have $p \circ \bar{\varphi} \leq \rho_M$ and by [**57**, Theorem 5.7.3] it follows that $\bar{\varphi} : V_S \to Y$ is continuous. Since $\bar{\varphi}$ is uniquely determined on $j[S]$, our direct computation has also verified that $((V_S, \tau_M), j)$ is free complete locally convex space over S. Since the completeness of Y is not used in the above verification, it follows that $((V_S, \tau_M), j)$ is also the free locally convex space over S.

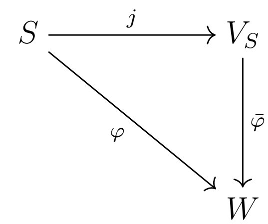
If S is a finite set with $|S| = n$, it is clear that (V_S, τ_M) is homeomorphic to $(\mathbb{R}^n, \|\cdot\|_1)$ since all norms on a finite-dimensional vector space are equivalent. Thus $((V_S, \tau_M), j)$ is also the free Banach space over S when S is finite.

There is yet another construction of the free locally convex space over an arbitrary set S: Consider the free vector space (V_S, j) over S and equip V_S with the strongest locally convex topology on V_S (i.e. the topology generated by the collection of all seminorms on V_S) which we denote as τ_{s}. The locally convex space (V_S, τ_{s}) has the property that every linear map $T : V_S \to W$ where W is a locally convex space is automatically continuous: Let W be a locally convex space and denote by P_W the collection of seminorms generating the topology τ_W on W. Fix $p \in P_W$ and let $T : V_S \to W$ be a linear map. It is clear that $p' \coloneqq p \circ T : V_S \to \mathbb{R}$ is a seminorm on V_S and since $p \circ T \leq p'$ it follows that $T : (V_S, \tau_{\mathrm{s}}) \to (W, \tau_W)$ is continuous. Thus the universal property of the free vector space over S implies that $((V_S, \tau_{\mathrm{s}}), j)$ is the free locally convex space over S. We know by the essential uniqueness of free objects that (V_S, τ_M) and (V_S, τ_{s}) are homeomorphic, but it is also not difficult to show this directly: Let $\rho : V_S \to \mathbb{R}$ be a seminorm and $f = \sum_{s\in S} \alpha_s e_s \in V_S$, then

$$\rho(f) \leq \sum_{s\in S} |\alpha_s| \rho(e_s).$$

Define $M_\rho : S \to \mathbb{R}$ where $M_\rho(s) \coloneqq \rho(e_s)$ and choose $M \in \mathcal{M}_{>0}(S)$ such that $M \geq M_\rho$. Then $\rho \leq \rho_M$. It follows by [**57**, Exercise 5.203 (c), p. 152] that (V_S, τ_M) and (V_S, τ_s) are homeomorphic.

The pair $((V_S, \tau_s), j)$ is also a 'natural guess' answer to the question of what is the free locally convex space over S: We start with the free vector space (V_S, j) over S. What locally convex topology τ can we place on V_S such that for every locally convex space W and every morphism $\varphi : S \to W$ in **Set** there exists a unique continuous linear map $\bar{\varphi} : (V_S, \tau) \to W$ making the following diagram commute in **Set**?



It would have to be τ_s since for every linear map $T : V_S \to W$ there exists $\varphi_T : S \to W$ in **Set** where $\varphi_T(s) := T(s)$. The morphism φ_T needs to extend uniquely to a continuous linear map $\bar{\varphi}_T : (V_S, \tau) \to W$ and since $\bar{\varphi}_T = T$ it is clear that every such T needs to be continuous with respect to the locally convex topology τ. Thus τ can only be τ_s.

It is satisfactory to see that our approach to the construction of free objects via inverse limits of pseudo-solutions delivers us an answer which is easily reconciled with this 'natural guess' answer.

3.9.2. Direct limit construction. Fix any set S. In addition to the previous section, it turns out that it is also possible to construct the free locally convex space over S as a direct limit.

If we consider the set S equipped with the discrete topology, we see that the free vector space V_S may be identified with $\mathrm{C_c}(S)$, the compactly supported continuous functions $f : S \to \mathbb{K}$. Denote by $\mathcal{K}(S)$ the collection of all compact (i.e. finite) subsets of S. For every $K \in \mathcal{K}(S)$, denote by $\mathrm{C_c}(S; K)$ the subspace of $\mathrm{C_c}(S)$ consisting of functions $f : S \to \mathbb{K}$ where $\{s \in S : f(s) \neq 0\} \subseteq K$. For compact subsets $K, K_1, K_2 \in \mathcal{K}(S)$ with $K_1 \subseteq K_2$ we have the inclusion maps

$$e_{K_1,K_2} : \mathrm{C_c}(S; K_1) \to \mathrm{C_c}(S; K_2)$$

and,

$$e_K : \mathrm{C_c}(S; K) \to \mathrm{C_c}(S).$$

Using Definition 2.3.1, we see that $\mathcal{D} \coloneqq \big((\mathrm{C_c}(S;K))_{K \in \mathcal{K}(S)}, (e_{K_1,K_2})_{K_1 \subseteq K_2}\big)$ is a direct system in **VS** and $\big(\mathrm{C_c}(S), (e_K)_{K \in \mathcal{K}(S)}\big)$ is the direct limit of $\mathcal{D}$ in **VS**. Further, equipping each finite-dimensional vector space $\mathrm{C_c}(S; K)$ for $K \in \mathcal{K}(S)$ with its unique linear topology makes $\mathcal{D}$ into a direct system in **LCS**. Using the approach in [**22**, Chapter II, § 4, No. 4] we construct the *direct limit topology* on $\mathrm{C_c}(S)$:

Let $\mathcal{B}$ be the collection of all absorbing convex balanced sets V in $\mathrm{C_c}(S)$ such that $V \cap \mathrm{C_c}(S; K)$ is a neighbourhood of zero in $\mathrm{C_c}(S; K)$ for all $K \in \mathcal{K}(S)$. Then $\mathcal{B}$ is a neighbourhood base of a locally convex topology τ_{dl} on $\mathrm{C_c}(S)$. The locally convex space $(\mathrm{C_c}(S), \tau_{dl})$ has the following properties:

(i) τ_{dl} is the strongest locally convex topology on $\mathrm{C_c}(S)$ such that the inclusions $\{e_K : K \in \mathcal{K}(S)\}$ are all continuous.

(ii) Let F be a locally convex space and $T : \mathrm{C_c}(S) \to F$ a linear map. Then T is continuous with respect to τ_{dl} if and only if $T \circ e_K : \mathrm{C_c}(S;K) \to F$ is continuous for all $K \in \mathcal{K}(S)$.

Using the fact that $\big(\mathrm{C_c}(S), (e_K)_{K \in \mathcal{K}(S)}\big)$ is the direct limit of $\mathcal{D}$ in **VS** and the above properties of direct limit topology on $\mathrm{C_c}(S)$, we conclude that $\big((\mathrm{C_c}(S), \tau_{dl}), (e_K)_{K \in \mathcal{K}(S)}\big)$ is the direct limit of $\mathcal{D}$ in **LCS**. Since V_S is identified with $\mathrm{C_c}(S)$, we may consider the identity map

$$I : (\mathrm{C_c}(S), \tau_{dl}) \to (\mathrm{C_c}(S), \tau_M)$$

where τ_M was defined in the previous section. By property (ii) of the direct limit topology, the identity map I is continuous if and only if $I \circ e_K : \mathrm{C_c}(S;K) \to (\mathrm{C_c}(S), \tau_M)$ is continuous for all $K \in \mathcal{K}(S)$. Denote by P_M the family of norms defining the locally convex topology τ_M on $\mathrm{C_c}(S)$ and take $\rho_M \in P_M$. Then, for every $K \in \mathcal{K}(S)$,

$$\rho_M \circ (I \circ e_K) \leq \rho_M.$$

It is clear that ρ_M is a norm on $\mathrm{C_c}(S;K)$ and thus generates the unique linear topology on $\mathrm{C_c}(S;K)$. By [**57**, Theorem 5.7.3], it follows that $I \circ e_K$ is continuous for all $K \in \mathcal{K}(S)$ and thus I is continuous. This implies that $\tau_M \subseteq \tau_{dl}$, but the topology τ_M was identified in the previous section as the strongest locally convex topology on $\mathrm{C_c}(S)$. Thus $(\mathrm{C_c}(S), \tau_M)$ and $(\mathrm{C_c}(S), \tau_{dl})$ are homeomorphic which implies that $((\mathrm{C_c}(S), \tau_{dl}), j)$ is the free locally convex space over S.

APPENDIX A

Measures and functionals

A.1. A Riesz Representation Theorem

In this section, we prove a Riesz Representation Theorem for $\mathrm{C}(X)$ with X realcompact which is used in Section 2.9 and in the work of Xiong in [**69**]. Some of these results are also found in [**40**]. However, our first result, which is Theorem 3.1 in [**40**], avoids an inconsistency occurring between the notation introduced before Definition 3.1 in [**40**] and the statement of Theorem 3.1 in [**40**]. This modification ends up making the proof a little simpler.

THEOREM A.1.1. *Let X be a Tychonoff space and let $\phi \in \mathrm{C}(X)^\sim$. For every $f \in \mathrm{C}(X)^+$ there exists $N \in \mathbb{N}$ such that if $n \geq N$ then*

$$\phi(f \wedge n\mathbf{1}_X) = \phi(f).$$

PROOF. We first prove the statement for $0 \leq \phi \in \mathrm{C}(X)^\sim$. Denote by $\alpha\mathbb{R}$ the one-point compactification of $\mathbb{R}$. Fix $f \in \mathrm{C}(X)^+ \subseteq \mathrm{C}(X, \alpha\mathbb{R})$ and consider its unique extension $\bar{f} \in \mathrm{C}(\beta X, \alpha\mathbb{R})$. For every $n \in \mathbb{N}$, define $h_n := f - f \wedge n\mathbf{1}_X \in \mathrm{C}(X)$. Note that since $f \wedge n\mathbf{1}_X \leq f$ for all $n \in \mathbb{N}$, we have that $h_n \geq 0$ for all $n \in \mathbb{N}$. Suppose for the sake of a contradiction that $\epsilon_n := \phi(h_n) > 0$ for all $n \in \mathbb{N}$. For $n \in \mathbb{N}$, define the closed sets

$$K_n := \left\{x \in \beta X : \bar{f}(x) \geq n\right\}, \qquad G_n := \left\{x \in \beta X : \bar{f}(x) \leq n-1\right\}.$$

By Urysohn's Lemma, for every $n \in \mathbb{N}$ define $\bar{p}_n \in \mathrm{C}(\beta X)$ where the range of $\bar{p}_n$ is contained in $[0, 1/\epsilon_n]$ and where $\bar{p}_n[G_n] = 0$ and $\bar{p}_n[K_n] = 1/\epsilon_n$. For every $n \in \mathbb{N}$, consider the function $p_n \in \mathrm{C_b}(X)$ where $p_n = \bar{p}_n|_X$. Define the function $g : X \to \mathbb{R}$ where

$$g(x) := \sum_{n \in \mathbb{N}} p_n(x) f(x).$$

We verify that g is real-valued and continuous: Fix $x \in X$; then there exists $m_0 \in \mathbb{N}$ such that $f(x) < m_0 - 1$. Since f is continuous, there exists an open neighbourhood U_x of x such that $f(y) < m_0 - 1$ for all $y \in U_x$. Since $f = \bar{f}|_X$, it follows that $U_x \subseteq G_m$ for all $m \geq m_0$. Thus $p_m(y) = 0$ for all $y \in U_x$, which implies

$$g(y) = \sum_{n=1}^{m_0 - 1} p_n(y) f(y)$$

for all $y \in U_x$. Thus g is real-valued and since g coincides with a continuous function on an open neighbourhood U_x of every point $x \in X$, we conclude that $g \in \mathrm{C}(X)$.

We claim that $p_n f \geq h_n/\epsilon_n$ for all $n \in \mathbb{N}$: Fix $n \in \mathbb{N}$ and $x \in X$. We either have $f(x) \leq n$ or $f(x) > n$. If $f(x) \leq n$, then $h_n(x) = f(x) - f(x) \wedge n = 0$ and thus

$p_n(x)f(x) \geq 0 = h_n(x)/\epsilon_n$. On the other hand, if $f(x) > n$ then $x \in K_n$ and it follows that

$$p_n(x)f(x) = 1/\epsilon_n f(x) \geq 1/\epsilon_n\,(f(x) - n) = h_n(x)/\epsilon_n.$$

Thus the claim is verified. However, since $g \geq \sum_{n=1}^{N} p_n f$ for every $N \in \mathbb{N}$, this implies that

$$\phi(g) \geq \phi\left(\sum_{n=1}^{N} p_n \cdot f\right) = \sum_{n=1}^{N} \phi\,(p_n \cdot f) \geq \sum_{n=1}^{N} \phi\,(h_n/\epsilon_n) = N,$$

which is impossible. We conclude that there exists $N \in \mathbb{N}$ such that $\phi(h_n) = 0$ for all $n \geq N$.

Now, consider $\psi \in \mathrm{C}(X)^{\sim}$ and $u \in \mathrm{C}(X)^{+}$. There exists $0 \leq \psi^{+}, \psi^{-} \in \mathrm{C}(X)^{\sim}$ such that $\psi = \psi^{+} - \psi^{+}$. From the work above, there exist $N \in \mathbb{N}$ such that if $n \geq N$, then

$$\psi^{+}\,(u) = \psi^{+}\,(u \wedge n\mathbf{1}_X)\,, \quad \text{and} \quad \psi^{-}\,(u) = \psi^{-}\,(u \wedge n\mathbf{1}_X)\,.$$

Therefore $\psi\,(u) = (\psi^{+} - \psi^{+})\,(u) = (\psi^{+} - \psi^{+})\,(u \wedge n\mathbf{1}_X) = \psi\,(u \wedge n\mathbf{1}_X)$. □

COROLLARY A.1.2. *Let X be a Tychonoff space and let $\phi \in \mathrm{C}(X)^{\sim}$. The following statements are true.*

(i) If $\phi(g) = 0$ for all $g \in \mathrm{C_b}(X)$, then $\phi = 0$.

(ii) The restriction operator $R : \mathrm{C}(X)^{\sim} \to \mathrm{C_b}(X)^{\sim}$ is a vector lattice embedding.

PROOF. The statement in (i) follows directly from Theorem A.1.1. For (ii), since $\mathrm{C_b}(X)$ is an ideal in $\mathrm{C}(X)$, it follows by [**49**, p. 84-85] that R is a lattice homomorphism and the injectivity of R is given by (i). □

The following result follows directly from the previous result, the fact that $\mathrm{C_b}(X)^{\sim}$ and $\mathrm{C}(\beta X)^{\sim}$ are lattice isomorphic, and the Riesz Representation Theorem [**61**, Theorem 18.4.1] applied to $\mathrm{C}(\beta X)^{\sim} = \mathrm{C}(\beta X)^{\star}$.

COROLLARY A.1.3. *Let X be a Tychonoff space and let $\phi \in \mathrm{C}(X)^{\sim}$. There exists a unique Radon measure $\mu_\phi \in \mathrm{M}\,(\beta X)$ such that*

$$\phi(g) = \int_{\beta X} g\; d\mu_\phi, \qquad g \in \mathrm{C_b}(X) \cong \mathrm{C}(\beta X).$$

In addition, the map $\Phi : \mathrm{C}(X)^{\sim} \to \mathrm{M}\,(\beta X)$ where $\phi \longmapsto \mu_\phi$ is a vector lattice embedding.

In the sequel, we will make no distinction between elements in $\mathrm{C_b}(X)$ and their uniquely associated elements in $\mathrm{C}(\beta X)$. The following characterisations of realcompact topological spaces are found in [**36**, Section 3.11].

PROPOSITION A.1.4. *Let X be a Tychonoff space. Then the following statements are equivalent.*

(i) X is realcompact.

(ii) X is homeomorphic to a closed subspace of $\mathbb{R}^m$ for some cardinal m.

(iii) *For every point* $x \in \beta X \setminus X$*, there exists* $h : \beta X \to [0,1]$ *with* $h(x_0) = 0$ *and* $h(x) > 0$ *for all* $x \in X$*.*

PROPOSITION A.1.5. *Let* X *be a realcompact space. Let* $\phi \in \mathrm{C}(X)^\sim$ *with* $\mu_\phi \in \mathrm{M}(\beta X)$ *the unique Radon measure identified with* ϕ *in Corollary A.1.3. Then* $S_{\mu_\phi} \subseteq X$*.*

PROOF. Fix $x_0 \in \beta X \setminus X$. We show that $x_0 \in \beta X \setminus S_{\mu_\phi}$. By Proposition A.1.4 (iii), there exists a continuous function $h : \beta X \to [0,1]$ such that $h(x_0) = 0$ and $h(x) > 0$ for all $x \in X$. Define $u \in \mathrm{C}(X)$ where $u(x) \coloneqq 1/h(x)$ and define

$$V_n \coloneqq \{x \in \beta X : h(x) \in [0, 1/n)\}.$$

Then $x_0 \in V_n$ for all $n \in \mathbb{N}$ and $u(x) > n$ if and only if $x \in V_n \cap X$. By Theorem A.1.1 and Corollary A.1.3, there exists $N_0 \in \mathbb{N}$ such that for all $n \geq N_0$, we have

$$\phi(u) = \phi(u \wedge n\mathbf{1}_X) = \int_{\beta X} u \wedge n\mathbf{1}_X \; d\mu_\phi$$

Therefore

$$\int_{\beta X} u \wedge (N_0 + 1)\mathbf{1}_X - u \wedge N_0 \mathbf{1}_X \; d\mu_\phi = 0.$$

However, for all $x \in V_{N_0+1} \cap X$, we have

$$u(x) \wedge (N_0 + 1) - u(x) \wedge N_0 = N_0 + 1 - N_0 = 1.$$

As a result

$$\begin{aligned} 0 \leq \mu_\phi(V_{N_0+1}) &= \int_{V_{N_0+1}} \mathbf{1}_X \; d\mu_\phi \\ &= \int_{V_{N_0+1}} u \wedge (N_0 + 1)\mathbf{1}_X - u \wedge N_0 \mathbf{1}_X \; d\mu_\phi \\ &\leq \int_{\beta X} u \wedge (N_0 + 1)\mathbf{1}_X - u \wedge N_0 \mathbf{1}_X \; d\mu_\phi = 0. \end{aligned}$$

Therefore $x_0 \in \beta X \setminus S_{\mu_\phi}$ and we conclude that $S_{\mu_\phi} \subseteq X$. □

COROLLARY A.1.6. *Let* X *be a realcompact space. Let* $\phi \in \mathrm{C}(X)^\sim$ *with* $\mu_\phi \in \mathrm{M}(\beta X)$ *the unique Radon measure identified with* ϕ *in Corollary A.1.3. Then*

$$\phi(f) = \int_{S_{\mu_\phi}} f \; d\mu_\phi, \qquad f \in \mathrm{C}(X).$$

PROOF. Fix $0 \leq \phi \in \mathrm{C}(X)^\sim$ and $f \in \mathrm{C}(X)$. By Theorem A.1.1 and Corollary A.1.3, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have

$$\phi(f) = \phi(f \wedge n\mathbf{1}_X) = \int_{S_{\mu_\phi}} f \wedge n\mathbf{1}_X \; d\mu_\phi.$$

By Proposition A.1.5, since $S_{\mu_\phi} \subseteq X \subseteq \beta X$, and S_{μ_ϕ} is closed in βX, it follows that S_{μ_ϕ} is a compact subset of X. Therefore there exists $m \geq N$ such that $f(x) < m$ for all $x \in S_{\mu_\phi}$. Therefore,

$$\phi(f) = \int_{S_{\mu_\phi}} f \wedge m\mathbf{1}_X \; d\mu_\phi = \int_{S_{\mu_\phi}} f \; d\mu_\phi. \qquad \square$$

Denote by $\mathrm{M}_0(\beta X)$ the collection of measures $\mu \in \mathrm{M}(\beta X)$ such that $S_\mu \subseteq X$.

THEOREM A.1.7. *Let X be a realcompact space. For every $\phi \in \mathrm{C}(X)^{\sim}$, denote by $\mu_\phi \in \mathrm{M}_0(\beta X)$ the unique Radon measure identified with ϕ in Corollary A.1.3. The map $\Phi : \mathrm{C}(X)^{\sim} \to \mathrm{M}_0(\beta X)$ where $\phi \longmapsto \mu_\phi$ is a lattice isomorphism.*

PROOF. By Corollary A.1.3, the map $\Phi : \mathrm{C}(X)^{\sim} \to \mathrm{M}(\beta X)$ where $\phi \longmapsto \mu_\phi$ is a vector lattice embedding and by Proposition A.1.5, $\Phi[\mathrm{C}(X)^{\sim}] \subseteq \mathrm{M}_0(\beta X)$. Take $\nu \in \mathrm{M}_0(\beta X)$. Since S_ν is a compact subset of X, the map $\psi : \mathrm{C}(X) \to \mathbb{R}$ where

$$\psi(f) := \int_{S_\nu} f \, d\nu, \qquad f \in \mathrm{C}(X)$$

defines an order bounded functional on $\mathrm{C}(X)$. By Corollary A.1.3 and Proposition A.1.5, there exists a unique $\nu_\psi \in \mathrm{M}_0(\beta X)$ such that

$$\psi(g) = \int_{\beta X} g \, d\nu = \int_{\beta X} g \, d\nu_\psi$$

for all $g \in \mathrm{C_b}(X)$, which implies that $\Phi(\psi) = \nu_\psi = \nu$. Thus $\Phi[\mathrm{C}(X)^{\sim}] = \mathrm{M}_0(\beta X)$, which proves the theorem. □

Recall that $\mathrm{M}_c(X)$ denotes the space of compactly supported Radon measures on X.

PROPOSITION A.1.8. *Let X be a realcompact space. The vector lattices $\mathrm{M}_0(\beta X)$ and $\mathrm{M}_c(X)$ are isomorphic.*

PROOF. Let $\nu \in \mathrm{M}_0(\beta X)$. For every $B \in \mathfrak{B}_X$, there exists $B' \in \mathfrak{B}_{\beta X}$ such that $B' \cap X = B$ [**40**, p. 108]. If we have $B', B'' \in \mathfrak{B}_{\beta X}$ so that $B' \cap X = B'' \cap X$ then $\nu(B') = \nu(B' \cap S_\nu) = \nu(B'' \cap S_\nu) = \nu(B'')$. Define $\nu^* \in \mathrm{M}_c(X)$ where

$$\nu^*(B) := \nu(B') \text{ with } B' \in \mathfrak{B}_{\beta X} \text{ so that } B' \cap X = B.$$

It follows from the previous observation that ν^* is well-defined and it is clear that the map $T : \mathrm{M}_0(\beta X) \to \mathrm{M}_c(X)$ where $\nu \longmapsto \nu^*$ is an injective positive linear map. To show surjectivity, let $\mu \in \mathrm{M}_c(X)$. For every $B \in \mathfrak{B}_{\beta X}$ let $\nu(B) := \mu(B \cap X)$. Then $\nu \in \mathrm{M}_0(\beta X)$ and $\nu^* = \mu$. Thus T is a bijective bipositive linear map, hence a lattice isomorphism. □

Combining Theorem A.1.7 and Proposition A.1.8 with Corollary A.1.6 gives us the following result.

COROLLARY A.1.9. *Let X be a realcompact space. There is a lattice isomorphism $\Psi : \mathrm{C}(X)^{\sim} \to \mathrm{M}_c(X)$ where $\phi \longmapsto \nu_\phi$ so that for every $\phi \in \mathrm{C}(X)^{\sim}$,*

$$\phi(f) = \int_{S_{\nu_\phi}} f \, d\nu_\phi, \qquad f \in \mathrm{C}(X).$$

Recall from Section 2.1.2 that a measure $\mu \in \mathrm{M}_c(X)$ is *normal* if $|\mu|(L) = 0$ for all closed nowhere dense sets L in X. We denote by $\mathrm{N}_c(X)$ the space of compactly supported normal measures on X.

LEMMA A.1.10. *Let X be a realcompact space with $\mu \in \mathrm{N}_c(X)$. Then S_μ is regular closed.*

PROOF. Since $S_\mu = S_{|\mu|}$, there is no loss of generality in assuming that $\mu \geq 0$. Note that $S_\mu = \overline{\mathrm{int} S_\mu} \cup S_\mu \setminus \overline{\mathrm{int} S_\mu}$. The result will follow if $S_\mu \setminus \overline{\mathrm{int} S_\mu}$ is shown to be empty. The set $S_\mu \setminus \mathrm{int} S_\mu$ is closed nowhere dense since $\mathrm{int}\,(S_\mu \setminus \mathrm{int} S_\mu) \subseteq \mathrm{int} S_\mu \setminus \mathrm{int} S_\mu = \varnothing$. Thus $\mu(S_\mu \setminus \mathrm{int} S_\mu) = 0$. Further note that $S_\mu \setminus \overline{\mathrm{int} S_\mu} \subseteq S_\mu \setminus \mathrm{int} S_\mu$, which implies that $\mu(S_\mu \setminus \overline{\mathrm{int} S_\mu}) = 0$.

Now, assume for the sake of a contradiction that there exists $x \in S_\mu \setminus \overline{\mathrm{int} S_\mu}$. Since X is a regular topological space and $\overline{\mathrm{int} S_\mu}$ is closed, there exists an open neighbourhood V of x such that $V \cap \overline{\mathrm{int} S_\mu} = \varnothing$. Now $\mu(V) = \mu(V \cap S_\mu) > 0$ since $x \in S_\mu$. However, since $S_\mu = S_\mu \setminus \overline{\mathrm{int} S_\mu} \cup \overline{\mathrm{int} S_\mu}$, it follows that $V \cap S_\mu = V \cap (S_\mu \setminus \overline{\mathrm{int} S_\mu}) \subseteq S_\mu \setminus \overline{\mathrm{int} S_\mu}$. Hence $\mu(V) = \mu(V \cap S_\mu) \leq \mu(S_\mu \setminus \overline{\mathrm{int} S_\mu}) = 0$, a contradiction. □

The following result shows that we also have a Riesz Representation Theorem for order continuous functionals on $\mathrm{C}(X)$ for X realcompact.

THEOREM A.1.11. *Let X be a realcompact space. Consider the lattice isomorphism $\Psi : \mathrm{C}(X)^\sim \to \mathrm{M}_c(X)$ where $\phi \longmapsto \nu_\phi$ defined in Corollary A.1.9. Then $\Psi\left[\mathrm{C}(X)^\sim_{\mathrm{n}}\right] = \mathrm{N}_c(X)$.*

PROOF. First, take $0 \leq \mu \in \mathrm{M}_c(X)$ which is not normal. Then there exists a closed nowhere dense L in X such that $\mu(L) > 0$. Since $\mu\,(X \setminus S_\mu) = 0$, we may assume that $L \subseteq S_\mu \subseteq X \subseteq \beta X$. Since S_μ is compact in X, it follows that S_μ is closed in βX. The properties of topological interiors and closures imply that L is a closed nowhere dense in βX. By Urysohn's Lemma, there exists a collection $(g_\alpha)_{\alpha \in I} \downarrow 0$ in $\mathrm{C}(\beta X)$ with $g_\alpha[L] = 1$ for all $\alpha \in I$. Since $\mathrm{C}(\beta X)$ is lattice isomorphic to $\mathrm{C_b}(X)$ and $\mathrm{C_b}(X)$ is an ideal in $\mathrm{C}(X)$, thus a regular sublattice in $\mathrm{C}(X)$, it follows that $(g_\alpha)_{\alpha \in I} \downarrow 0$ in $\mathrm{C}(X)$. Let $\phi \in \mathrm{C}(X)^\sim$ such that $\Psi(\phi) = \mu$. Then

$$\phi\,(g_\alpha) = \int_X g_\alpha \, d\mu \geq \int_L g_\alpha \, d\mu = \mu(L) > 0.$$

Which implies that $\phi \notin \mathrm{C}(X)^\sim_{\mathrm{n}}$. Thus $\Psi\left[\mathrm{C}(X)^\sim_{\mathrm{n}}\right] \subseteq \mathrm{N}_c(X)$.

Conversely, take $\nu \in \mathrm{N}_c(X)$ with $\psi \in \mathrm{C}(X)^\sim$ the associated functional. We show that ψ is order continuous. Consider $(f_\alpha)_{\alpha \in I} \downarrow 0$ in $\mathrm{C}(X)$. By Lemma A.1.10, S_ν is a regular closed subset of X and by [**48**, Theorem 3.4], we have that $\left(f_\alpha|_{S_\nu}\right)_{\alpha \in I} \downarrow 0$ in $\mathrm{C}(S_\nu)$. Define $\nu_* \in \mathrm{M}_c(S_\nu)$ where $\nu_*(B) \coloneqq \nu(B)$ for $B \in \mathfrak{B}_{S_\nu} \subseteq \mathfrak{B}_X$ [**20**, Vol. II, Lemma 6.2.4]. Since every subset of S_ν that is closed nowhere dense is also closed nowhere dense in X, it follows that $\nu_* \in \mathrm{N}_c(S_\nu)$. Since S_ν is compact, it follows by [**26**, Definition 4.7.1, Theorem 4.7.4] that

$$\psi(f_\alpha) = \int_{S_\nu} f_\alpha \, d\nu = \int_{S_\nu} f_\alpha|_{S_\nu} \, d\nu_* \longrightarrow 0.$$

Thus $\psi \in \mathrm{C}(X)^\sim_{\mathrm{n}}$. □

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