Beyond the *tanh* method - looking for explicit travelling wave solutions to partial differential equations

by

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Declaration

I, Khanyisani Alfred Maqele declare that the dissertation, which I hereby submit for the degree Master of Science at the University of Pretoria, is my own work and has not previously been submitted by me for a degree at this or any other tertiary institution.

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Abstract

In this work, we focus on a general procedure for finding exact travelling wave solutions for evolution equations with polynomial nonlinearites. Mathematically, looking for travelling wave solutions is asking the question whether a given PDE has solutions invariant under a Galilean transformation; in such a case, it can be reduced to an ODE. We discuss the existence of travelling wave solutions by using phase plane analysis. We show that popular methods such as the tanh-method, G'/G-method and many more are special cases of the presented approach. Analytical solutions to several examples of nonlinear equations are illustrated. In the application, we use the Maple program to compute solutions to nonlinear systems of equations.

Acknowledgments

I would like to thank Prof. Jacek Banasiak, my supervisor, for his guidance and financial support he provided to me. I would also like to thank the administrative staff of the Mathematics and Applied Mathematics department: Ms Rachel Combrink and Ms Lorelle September for the support they provided.

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Chapter 1

Introduction

The study of non-linear partial differential equations has been conducted in many fields of study by various researchers. The fields like Applied Mathematics, Theoretical Physics and Engineering utilize partial differential equations (PDEs) to solve their daily problems [1]. These equations are classified as linear PDEs or nonlinear PDEs. PDEs may be divided into three categories, which are elliptic equations, parabolic equations and hyperbolic equations. The Laplace equation is the example of an elliptic equation, heat equations are parabolic equation and wave equations are hyperbolic equations. Both linear and non-linear PDEs often appear in various fields, like Mathematical Biology [2], Physics [28], Plasma Physics [4], Solid-state Physics [5], and Chemistry [6]. Equations like reaction-diffusion equations are used to model the spread of populations and tumors in mathematical biology, and they are also used in chemistry to study heat conduction problems. Physicists use the Schrödinger equation in quantum mechanics and Maxwell's equation in electrodynamics. The mathematical methods developed to deal with linear and nonlinear equations are different. The solution space for a linear, homogeneous PDE is a vector space and the linear structure of that space can be used in constructing solutions with desired properties that can meet diverse boundaries and initial conditions. This is not the case for nonlinear equations [1]. Nonlinear PDEs are more useful to solve world problems. The solution methods for nonlinear equations are perturbation methods, similarity methods, transformation, numerical methods and travelling wave solutions. In current mathematics, the theory of travelling wave solutions of nonlinear PDEs is applied to describe various situations in ecology [7], farming [8], forestry [9], cell structure [10], etc. The motive of the presented study is to find exact solutions to nonlinear PDEs using an appropriate method. Methods like variation iteration [18], tanh method [23], G'/G-expansion method [24], extended G'/G-expansion method [22], etc., have been employed to generate exact traveling wave solutions of nonlinear PDEs. Some other researchers claim that their methods produce travelling wave solutions when, in fact, this is not true. For instance, by the definition of the travelling wave solution to be introduced, the modified $\tanh - \coth$ method [25] does not generate travelling wave solutions. It must be noted that no single method mentioned above can be utilized to solve all types of nonlinear PDE's. The hyperbolic tangent method (tanh-method) is mentioned and tested as a potent method helps in finding exact traveling waves solutions of nonlinear PDEs [23]. The tanh method was first introduced by W. Malfliet. Its solutions are functions of a hyperbolic tangent.

In this thesis, we focus on a general procedure for finding exact travelling wave solutions of nonlinear evolution equations with polynomial nonlinearities. The structure of the thesis is as follows. In Chapter 2, we introduce the theory behind travelling wave solutions. In Chapter 3 we give theory for systems of differential equations. The chapter then proceeds to the phase plane analysis showing the existence of travelling wave solutions for non-integrable equations (equations like reaction-diffusion, one of them is the Fisher equation). Chapter 4 gives an overview of the method used to find travelling wave solutions. In Chapter 5, we apply the method outlined in Chapter 4. In Chapter 6, we present the second approach (slightly different from the one presented in Chapter 4) to finding exact travelling wave solutions and their application. Chapter 7 summarizes the results we obtained with the method developed in the thesis and compares the results with previous results by other authors.

Chapter 2

Travelling wave solutions

The travelling wave is a wave that moves in a certain direction and keeps the same shape as it moves. Its travelling speed remains constant for the entire motion. In real life, travelling waves describe movement processes. The normal case is a movement from one equilibrium to another. The travelling wave solutions of Reaction-Diffusion PDE can be obtained using the tanh-method, G'/G-method and many more. We want to show that all these methods are examples of one general approach. The objective of the thesis is to describe the theory behind method of finding travelling wave solutions and later demonstrate its application by giving examples. The number of natural processes involve mechanisms of both diffusion and reaction [16], and such problems are often modelled with equations of the form

$$\frac{\partial u}{\partial t} = d_u \frac{\partial^2 u}{\partial x^2} + f(u). \tag{2.1}$$

In this equation, f is a nonlinear function (which describes reaction processes) and d_u is a diffusion constant. Our interest is to investigate the existence of travelling wave solutions to Eq.(2.1). The method of travelling wave solutions discussed in this work is based on the assumption that the solution of a PDE is a function of a linear combination of the space variable x and time t, $z(x,t) = k(x - ct)$. Then the solutions of the form $u(x,t) = \mathcal{U}(k(x - ct)) = \mathcal{U}(z)$, where the function $\mathcal{U}(z)$ has finite limits as $z \to \pm\infty$, are called travelling wave solutions. The constants, k and c, are called the wave number and wave speed, respectively. The wave front solution is described as a travelling wave solution with different constant states at $\pm\infty$, that is,

$$\lim_{z\to-\infty} \mathcal{U}(z) = u_1 \quad \text{and} \quad \lim_{z\to+\infty} \mathcal{U}(z) = u_2. \tag{2.2}$$

Figure 2.1 shows the profile of a travelling wave front. Here, $\mathcal{U}(z)$ decreases monotonically with z from the constant value u_1 to the constant value u_2. The wave front occurs in waves of combustion, waves in chemical kinetics, etc. If $u_1 = u_2$, then the travelling wave is called a pulse. The wave form $\mathcal{U}$, k and c are unknown. The function $\mathcal{U}$ is called a standing wave solution when c is equal to zero. If Eq.2.1 transforms from two independent variables to a single independent variable $u(x,t) = \mathcal{U}(z)$ and use the Chain Rule, we obtain

$$\frac{d\mathcal{U}}{dz}\frac{\partial z}{\partial t} = d_u \frac{\partial z}{\partial x}\left[\frac{\partial z}{\partial x}\frac{d^2\mathcal{U}}{dz^2} + \frac{d\mathcal{U}}{dz}\frac{d}{dz}\left(\frac{\partial z}{\partial x}\right)\right] + f(\mathcal{U}). \tag{2.3}$$

Hence, Eq.2.1 can be written as the ordinary differential equation (ODE)

$$-ck\frac{d\mathcal{U}}{dz} = d_u k^2 \frac{d^2\mathcal{U}}{dz^2} + f(\mathcal{U}). \tag{2.4}$$

The canonical form of Eq.2.4 is

$$\frac{d\mathcal{U}}{dz} = h\frac{d^2\mathcal{U}}{dz^2} + \eta f(\mathcal{U}), \tag{2.5}$$

where $h = \frac{d_u k}{c}$ and $\eta = \frac{1}{ck}$. To further illustrate the various forms of travelling

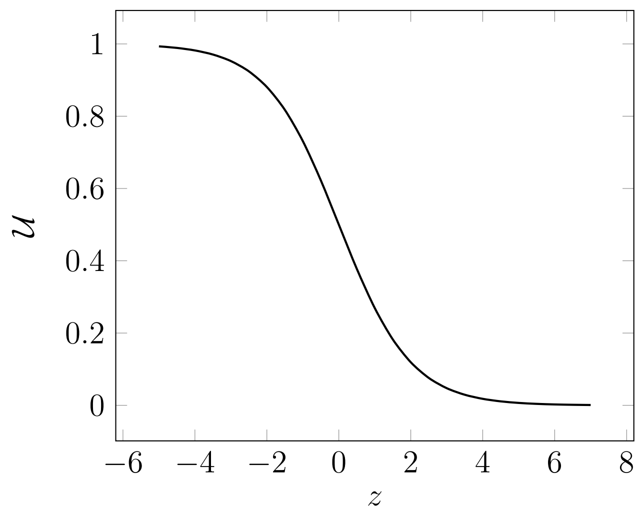


Figure 2.1: Travelling wave: wave front.

wave solutions, we present the travelling wave solutions of two equations , the Burgers equation and the KdV equation.

2.1 Burgers equation

As the first example, we consider the Burgers equation arising in fluid dynamics,

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - d\frac{\partial^2 u}{\partial x^2} = 0, \tag{2.6}$$

where d is a constant. It is an equation consisting of three terms, time evolution $\frac{\partial u}{\partial t}$, nonlinear convection term $u\frac{\partial u}{\partial x}$ and linear diffusion term $\frac{\partial^2 u}{\partial x^2}$. Using the travelling wave transformation, we get

$$-ck\frac{d\mathcal{U}}{dz} + k\mathcal{U}\frac{d\mathcal{U}}{dz} - dk^2\frac{d^2\mathcal{U}}{dz^2} = 0. \tag{2.7}$$

Integrating Eq.2.7 with respect to z yields

$$-ck\mathcal{U} + \frac{k}{2}\mathcal{U}^2 - dk^2\frac{d\mathcal{U}}{dz} + A_2 = 0, \tag{2.8}$$

where A_2 is a constant of integration. After rearranging, we have

$$dk^2\frac{d\mathcal{U}}{dz} = -ck\mathcal{U} + \frac{k}{2}\mathcal{U}^2 + A_2. \tag{2.9}$$

Factorizing the right hand side of Eq.2.9, we obtain

$$-ck\mathcal{U} + \frac{k}{2}\mathcal{U}^2 + A_2 = (\mathcal{U} - u_1)(\mathcal{U} - u_2), \tag{2.10}$$

where

$$u_1 = ck + \sqrt{c^2k^2 - 2kA_2} \quad \text{and} \quad u_2 = ck - \sqrt{c^2k^2 - 2kA_2}, \tag{2.11}$$

provided $c^2k^2 > 2kA_2$. Hence, the wave speed

$$c = \frac{u_1 + u_2}{2k}. \tag{2.12}$$

Eq.2.9 turns to

$$dk^2\frac{d\mathcal{U}}{dz} = (\mathcal{U} - u_1)(\mathcal{U} - u_2). \tag{2.13}$$

We assume that $u_1, u_2 \in \mathbb{R}$ and $u_1 > u_2$. Introducing the integral in Eq.2.13 results in

$$\int \frac{1}{(\mathcal{U} - u_1)(\mathcal{U} - u_2)}d\mathcal{U} = \frac{1}{dk^2}(z + A_3). \tag{2.14}$$

Integration gives

$$\frac{-1}{u_2 - u_1}\int \frac{1}{\mathcal{U} - u_1}d\mathcal{U} + \frac{1}{u_2 - u_1}\int \frac{1}{\mathcal{U} - u_2}d\mathcal{U} = \frac{1}{dk^2}(z + A_3). \tag{2.15}$$

$$\frac{1}{u_1 - u_2}\ln\left|\frac{\mathcal{U} - u_1}{\mathcal{U} - u_2}\right| = \frac{1}{dk^2}(z + A_3). \tag{2.16}$$

In Eq.2.16, we only consider the case where $u_2 < \mathcal{U} < u_1$ (travelling wave solution exists only in between this interval; more details are given in the next section on the theory for systems of ODEs). This leads us to

$$\ln\left(\frac{u_1 - \mathcal{U}}{\mathcal{U} - u_2}\right) = \frac{u_1 - u_2}{dk^2}\left(z + A_3\right). \tag{2.17}$$

Hence, the wave form is given by

$$\mathcal{U}(z) = \frac{u_1 + u_2 e^{\frac{u_1 - u_2}{dk^2}(z + A_3)}}{1 + e^{\frac{u_1 - u_2}{dk^2}(z + A_3)}}. \tag{2.18}$$

From Eq.2.18, we notice that indeed

$$\lim_{z \to -\infty} \mathcal{U}(z) = u_1 \quad \text{and} \quad \lim_{z \to +\infty} \mathcal{U}(z) = u_2. \tag{2.19}$$

The plot of Eq.2.18 takes the same shape as in Figure.2.1, $\mathcal{U}(z)$ decreases monotonically with z from the constant value u_1 to u_2. The wave profile travels from left to the right at a constant speed $c = \frac{u_1 + u_2}{2k}$. The presence of the diffusion term in this problem prevents formation of the shock wave. The diffusion coefficient d changes the shape of the wave profile. Large d has a significant diffusive effect and the wave profile has a shallow gradient. If d is small, the gradient becomes steeper.

2.2 Korteweg-de Vries equation.

The KdV equation is

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0. \tag{2.20}$$

It is derived from fluid mechanics to describe shallow water waves in a rectangular channel [1]. Rewriting Eq.2.20 in terms of z by setting $u(x,t) =$

$\mathcal{U}(k(x-ct)) = \mathcal{U}(z)$, we get the ODE

$$-ck\frac{d\mathcal{U}}{dz} + k\mathcal{U}\frac{d\mathcal{U}}{dz} + k^3\frac{d^3\mathcal{U}}{dz^3} = 0. \tag{2.21}$$

We can integrate Eq.2.21 to get

$$-ck\mathcal{U} + k\frac{\mathcal{U}^2}{2} + k^3\frac{d^2\mathcal{U}}{dz^2} + A_1 = 0, \tag{2.22}$$

where A_1 is a constant of integration. Multiplying Eq.2.22 by $\frac{d\mathcal{U}}{dz}$, we get

$$-ck\mathcal{U}\frac{d\mathcal{U}}{dz} + \frac{k}{2}\mathcal{U}^2\frac{d\mathcal{U}}{dz} + k^3\frac{d\mathcal{U}}{dz}\frac{d^2\mathcal{U}}{dz^2} + A_1\frac{d\mathcal{U}}{dz} = 0. \tag{2.23}$$

After integrating Eq.2.23, we obtain

$$\frac{k^3}{2}\left(\frac{d\mathcal{U}}{dz}\right)^2 = -\frac{k}{6}\mathcal{U}^3 + ck\mathcal{U}^2 + A_1\mathcal{U} + A_2, \tag{2.24}$$

hence

$$\frac{d\mathcal{U}}{dz} = \sqrt{-\frac{1}{3k^2}\mathcal{U}^3 + \frac{2c}{k^2}\mathcal{U}^2 + \frac{2}{k^3}A_1\mathcal{U} + \frac{2}{k^3}A_2} = \sqrt{M(\mathcal{U})}, \tag{2.25}$$

where A_2 is the constant of integration and we use plus sign in the square root. Imposing boundary conditions $\lim_{z\to-\infty}\mathcal{U}(z) = u_1$ and $\lim_{z\to+\infty}\mathcal{U}(z) = u_2$, we get

$$\begin{aligned} A_1 &= \frac{k}{6}u_1^2 - kcu_1 - A_2u_1, \\ A_1 &= \frac{k}{6}u_2^2 - kcu_2 - A_2u_2. \end{aligned} \tag{2.26}$$

From Eq.2.26, we determine the wave speed to be

$$c = \frac{1}{6}(u_2 + u_1) - \frac{A_2}{k}. \tag{2.27}$$

Eq.2.25 is separable and its expression on the right-hand side is a cubic in $\mathcal{U}$. To factorize the cubic expression, different cases must be considered depending on the number of real roots. The cases lead to solitary waves and cnoidal waves. The cnoidal wave case is when the solution oscillates between two values $h_1 < h_2$ (which we can assume are also roots of $M(\mathcal{U})$ without loss of generality). In this study, we are interested in the case when one is a double root (solitary wave), hence

$$-\frac{1}{3k^2}\mathcal{U}^3 + \frac{2c}{k^2}\mathcal{U}^2 + \frac{2}{k^3}A_1\mathcal{U} + \frac{2}{k^3}A_2 = (\mathcal{U} - \alpha_1)^2(\alpha_2 - \mathcal{U}) \quad , \quad 0 < \alpha_1 < \alpha_2. \tag{2.28}$$

After arranging Eq.2.25 and integrating, we get

$$\int \frac{1}{\sqrt{(\mathcal{U} - \alpha_1)^2(\alpha_2 - \mathcal{U})}} d\mathcal{U} = \int dz + A_3, \tag{2.29}$$

where A_3 is a constant of integration. If we let

$$\mathcal{U} = \alpha_1 + (\alpha_2 - \alpha_1)\text{sech}^2 w,$$

then

$$d\mathcal{U} = -2(\alpha_2 - \alpha_1)\tanh w \text{sech}^2 w dw, \tag{2.30}$$

and

$$\begin{aligned}(\mathcal{U} - \alpha_1)^2(\alpha_2 - \mathcal{U}) &= (\alpha_2 - \alpha_1)^2\text{sech}^4 w\left((\alpha_2 - \alpha_1) - (\alpha_2 - \alpha_1)\text{sech}^2 w\right), \\ &= (\alpha_2 - \alpha_1)^3\text{sech}^4 w\left(1 - \text{sech}^2 w\right), \\ &= (\alpha_2 - \alpha_1)^3\text{sech}^4 w \tanh^2 w.\end{aligned} \tag{2.31}$$

Substituting Eq.2.31 and Eq.2.30 into Eq.2.29, we get

$$\begin{aligned}\int \frac{-2(\alpha_2 - \alpha_1)\tanh w\text{sech}^2 w}{\sqrt{(\alpha_2 - \alpha_1)^3\text{sech}^4 w \tanh^2 w}} dw &= \int dz + A_3, \\ \int \frac{-2(\alpha_2 - \alpha_1)\tanh w\text{sech}^2 w}{(\alpha_2 - \alpha_1)\text{sech}^2 w \tanh w\sqrt{\alpha_2 - \alpha_1}} dw &= \int dz + A_3.\end{aligned} \tag{2.32}$$

Eq.2.32 becomes

$$\int \frac{-2}{\sqrt{\alpha_2 - \alpha_1}} dw = z + A_3. \tag{2.33}$$

Evaluating the integral, we get

$$w = \frac{\sqrt{\alpha_2 - \alpha_1}}{-2}(z + A_3). \tag{2.34}$$

Hence, the travelling wave solution is given by

$$\begin{aligned} \mathcal{U}(z) &= \alpha_1 + (\alpha_2 - \alpha_1)\text{sech}^2 \frac{\sqrt{\alpha_2 - \alpha_1}}{2}(z + A_3), \\ &= \alpha_1 + (\alpha_2 - \alpha_1)\left(1 - \tanh^2 \frac{\sqrt{\alpha_2 - \alpha_1}}{2}(z + A_3)\right) \\ &= \alpha_1 + (\alpha_2 - \alpha_1)\left(\frac{4}{e^{\frac{\sqrt{\alpha_2-\alpha_1}}{2}(z+A_3)} + 2 + e^{\frac{-\sqrt{\alpha_2-\alpha_1}}{2}(z+A_3)}}\right). \end{aligned} \tag{2.35}$$

2.3 Solving PDEs

Many equations of mathematical physics present pulse-type solutions, e.g., the Boussinesq equation, the Sine-Godon equation, the Born-Infeld equation, and the nonlinear Schrodinger equation [28]. The methods used in Example 1 and Example 2 can not solve more challenging PDEs problems. Reaction-diffusion equations can not be solved in closed form. In the following Chapter we give a geometric approach for solving reaction-diffusion equations. Before we introduce reaction-diffusion travelling wave solution theory, we start by giving a relevant theory for ODEs which might be useful in determining the existence of travelling waves for R-D equations.

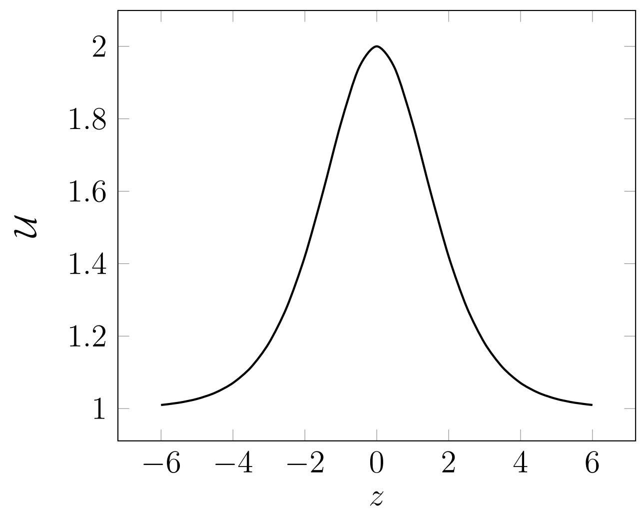


Figure 2.2: Pulse wave profile (KdV equation profile), where $\alpha_1 = 1$, $\alpha_2 = 2$.

Chapter 3

Theory for systems of differential equations

In this chapter we introduce the general theory of one and two dimensional (scalar) autonomous equations and use it to find qualitative information about their solutions.

3.1 Equilibria and their stability

Consider the system of autonomous equations

$$\frac{d\mathbf{U}}{dz} = \mathbf{P}(\mathbf{U}), \tag{3.1}$$

where $\mathbf{P} : \mathbb{R}^n \to \mathbb{R}^n$, $n \geq 1$. Eq.3.1 is called an autonomous system because function $\mathbf{P}(\mathbf{U})$ does not depend explicitly on z. A point $\mathbf{U}^*$ is an equilibrium point of Eq.3.1 if

$$\mathbf{P}(\mathbf{U}^*) = 0.$$

Then, the equilibrium solution is

$$\mathbf{U}(z) = \mathbf{U}^*, \ \forall z.$$

The stability definition states that the solution $\mathbf{U}(z)$ of Eq.3.1 is stable if every other solution $\mathbf{U}(z_0) = \mathbf{U_0}$ (for different choice of initial data) that starts sufficiently close to $\mathbf{U}(z)$ will remain close to it for all times and asymptotically stable if it returns to equilibrium point. The precise definition of stability of equilibrium points is:

Definition 3.1.1 *The equilibrium point $\mathbf{U}^*$ is stable if for any given $\epsilon > 0$ there exists a $\delta > 0$ such that if $\mathbf{U_0}$ satisfies*

$$||\mathbf{U_0} - \mathbf{U}^*|| < \delta,$$

then the solution $\mathbf{U}(z)$ satisfies

$$||\mathbf{U}(z) - \mathbf{U}^*|| < \epsilon, \ \forall z > 0.$$

If $\mathbf{U}^*$ is not stable, then it is called unstable. Additionally, $\mathbf{U}^*$ is called attracting if there is δ^* such that if

$$||\mathbf{U_0} - \mathbf{U}^*|| < \delta^*,$$

then

$$\lim_{z \to \infty} \mathbf{U}(z, \mathbf{U_0}) = \mathbf{U}^*.$$

If $\delta^* = \infty$, then $\mathbf{U}^*$ is called a globally attracting equilibrium. Furthermore, the equilibrium $\mathbf{U}^*$ is called asymptotically stable if it is both stable and attracting. If $\mathbf{U}^*$ is globally attracting and stable, then it is called globally asymptotically stable equilibrium.

3.2 Scalar equations

Consider a one dimensional scalar autonomous equation

$$\frac{dU}{dz} = P(U), \tag{3.2}$$

where $P(U)$ is a nonlinear function. All autonomous equations are separable and hence can be solved by quadrature. Although the explicit formula for the solution is often impossible or difficult to derive, can we use graphical methods to represent and better understand it. As an example, we consider $P(U) = \rho U^2 + \gamma U + \sigma$, then

$$\frac{dU}{dz} = \rho U^2 + \gamma U + \sigma. \tag{3.3}$$

In this case, $P(U) = 0$ implies

$$U_0 = \frac{-\gamma + \sqrt{\gamma^2 - 4\rho\sigma}}{2\rho} \quad or \quad U_1 = \frac{-\gamma - \sqrt{\gamma^2 - 4\rho\sigma}}{2\rho}$$

where $\gamma^2 - 4\rho\sigma > 0$. These points represent what we call equilibrium solutions to Eq.3.3. These are solutions of the form $U(z) = U_0$ and $U(z) = U_1$. An equilibrium solution means that, if we start the system from the initial state being an equilibrium point, the physical system does not move. The next step is to present a solution of Eq.3.3 by constructing a phase diagram. For simplicity, we let $\rho = -1$, $\gamma = 1$ and $\sigma = 0$, then we plot the polynomial function $P(U) = U - U^2$. Hence, the equilibrium points are $U_0 = 0$ and $U_1 = 1$. In the interval $P > 0$, we have an increasing solution (that is indicated by right arrow) and if $P < 0$, the solution decreases (shown by left arrow), as shown in Figure 3.1. The phase diagram of Eq.3.3 is shown in Figure 3.1. Equilibrium point $(1, 0)$ is an attractor (stable), point $(0, 0)$ is a

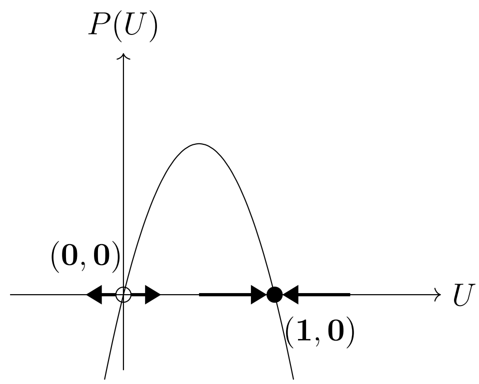


Figure 3.1: The graph of $P(U) = U - U^2$.

repeller (unstable). We also have semi-stable (but not graphically represented here) if and only if the solution flows towards the equilibrium point on one side and flows away from the other side.

Theorem 3.2.1 (Stability of nonlinear equation) *Let U_0 be an equilibrium point of Eq.3.2 such that $P'(U_0) \neq 0$.*

- *The equilibrium point U_0 is stable if and only if $P'(U_0) < 0$.*
- *The equilibrium point U_0 is unstable if and only if $P'(U_0) > 0$.*

The last step is to plot solutions on the Uz-plane using information in Figure 3.1. Here, we take the horizontal axis of Figure 3.1 to be the vertical axis of the Uz-plane. In Figure 3.2, the z-dependent solutions are in red or green and the equilibrium solution in blue. Equilibrium points in Figure 3.2 are called equilibrium solutions in the Uz-plane. The equilibrium solution $U = 1$ is stable (represented by a blue solid line) and the equilibrium solution $U = 0$ is unstable (represented by a blue dotted line). Next we find the analytical

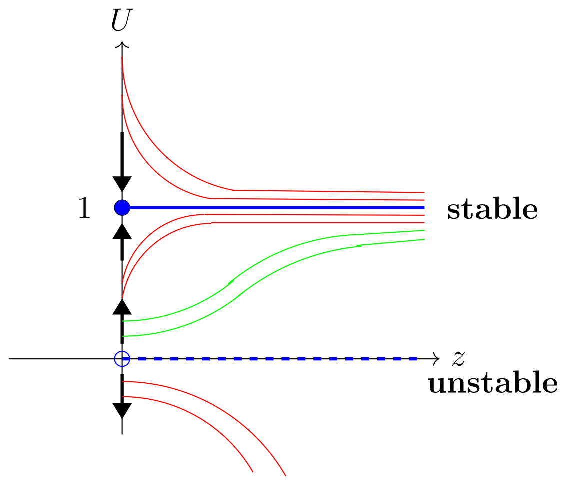


Figure 3.2: The graph of solution U for different $U(0)$.

solutions for Eq.3.3

$$\frac{dU}{dz} = \rho U^2 + \gamma U + \sigma.$$

Eq.3.3 is a separable first order ODE which can be solved by integrating both sides, that is,

$$\int \frac{1}{\rho U^2 + \gamma U + \sigma} dU = \int dz + A, \quad \textit{if} \quad \rho U^2 + \gamma U + \sigma \neq 0, \tag{3.4}$$

where A is the integration constant. The integral of this kind can give us three different solutions under different cases of the discriminant of $\rho U^2 + \gamma U + \sigma$, which are $\gamma^2 - 4\rho\sigma > 0$(we get two real roots), $\gamma^2 - 4\rho\sigma = 0$ (one real root) and $\gamma^2 - 4\rho\sigma < 0$ (no real roots). In this study we only consider the case

$$\gamma^2 - 4\rho\sigma > 0. \tag{3.5}$$

Factorizing the polynomial, we get

$$\begin{aligned}\rho U^2 + \gamma U + \sigma &= \left[U - \left(\frac{-\gamma + \sqrt{\Delta}}{2\rho}\right)\right]\left[U - \left(\frac{-\gamma - \sqrt{\Delta}}{2\rho}\right)\right] \\ &= \frac{1}{4\rho}(2\rho U + \gamma - \sqrt{\Delta})(2\rho U + \gamma - \sqrt{\Delta}),\end{aligned} \tag{3.6}$$

where $\Delta = \gamma^2 - 4\rho\sigma$. The equilibrium points of Eq.3.3 are

$$U = \frac{-\gamma + \sqrt{\Delta}}{2\rho}$$

and

$$U = \frac{-\gamma - \sqrt{\Delta}}{2\rho}.$$

We then rewrite our integral as

$$4\rho \int \frac{1}{(2\rho U + \gamma - \sqrt{\Delta})(2\rho U + \gamma + \sqrt{\Delta})} dU = \int z + A. \tag{3.7}$$

After introducing partial fractions on the left hand side of Eq.3.7, we get

$$\frac{2\rho}{\sqrt{\Delta}} \int \frac{1}{2\rho U + \gamma - \sqrt{\Delta}} dU - \frac{2\rho}{\sqrt{\Delta}} \int \frac{1}{2\rho U + \gamma + \sqrt{\Delta}} dU = \int z + A. \tag{3.8}$$

Substituting $x = 2\rho U + \gamma - \sqrt{\Delta}$ and $y = 2\rho U + \gamma + \sqrt{\Delta}$, we get

$$\begin{aligned}\frac{1}{\sqrt{\Delta}} \int \frac{1}{x} dx - \frac{1}{\sqrt{\Delta}} \int \frac{1}{y} dy &= z + A, \\ \frac{1}{\sqrt{\Delta}} \ln \left| \frac{2\rho U + \gamma - \sqrt{\Delta}}{2\rho U + \gamma + \sqrt{\Delta}} \right| &= z + A.\end{aligned} \tag{3.9}$$

An absolute value can be eliminated by considering three cases:

$$\ln \left| \frac{2\rho U + \gamma - \sqrt{\Delta}}{2\rho U + \gamma + \sqrt{\Delta}} \right| = \begin{cases} \ln\left(\frac{2\rho U + \gamma - \sqrt{\Delta}}{2\rho U + \gamma + \sqrt{\Delta}}\right) & \text{if } (2\rho U + \gamma) > \sqrt{\Delta}, \\ \ln\left(\frac{2\rho U + \gamma - \sqrt{\Delta}}{2\rho U + \gamma + \sqrt{\Delta}}\right) & \text{if } (2\rho U + \gamma) < -\sqrt{\Delta}, \\ \ln\left(\frac{\sqrt{\Delta} - (2\rho U + \gamma)}{2\rho U + \gamma + \sqrt{\Delta}}\right) & \text{if } -\sqrt{\Delta} < (2\rho U + \gamma) < \sqrt{\Delta}. \end{cases}$$

Due to its implications for travelling wave solutions, described later, we shall focus on the case which lies between equilibrium points. Then the third case $-\sqrt{\Delta} < (2\rho U + \gamma) < \sqrt{\Delta}$ leads us to

$$\frac{1}{\sqrt{\Delta}} \ln \left(\frac{\sqrt{\Delta} - (2\rho U + \gamma)}{2\rho U + \gamma + \sqrt{\Delta}} \right) = z + A. \tag{3.10}$$

Solving Eq.3.10, we get U as an exponential or hyperbolic function

$$\begin{aligned}
\frac{\sqrt{\Delta} - (2\rho U + \gamma)}{2\rho U + \gamma + \sqrt{\Delta}} &= e^{\sqrt{\Delta}(z+A)} \\
2\rho U e^{\sqrt{\Delta}(z+A)} + 2\rho U &= \sqrt{\Delta} - \gamma - \gamma e^{\sqrt{\Delta}(z+A)} - \sqrt{\Delta} e^{\sqrt{\Delta}(z+A)} \\
U(2\rho(1 + e^{\sqrt{\Delta}(z+A)})) &= \sqrt{\Delta}(1 - e^{\sqrt{\Delta}(z+A)}) - \gamma(1 + e^{\sqrt{\Delta}(z+A)}) \\
U(z) &= \frac{\sqrt{\Delta}(1 - e^{\sqrt{\Delta}(z+A)}) - \gamma(1 + e^{\sqrt{\Delta}(z+A)})}{2\rho(1 + e^{\sqrt{\Delta}(z+A)})} \\
&= \frac{-\gamma}{2\rho} + \frac{\sqrt{\Delta}}{2\rho} \left(\frac{1 - e^{\sqrt{\Delta}(z+A)}}{1 + e^{\sqrt{\Delta}(z+A)}} \right).
\end{aligned} \tag{3.11}$$

Furthermore,

$$\tanh\left(\frac{-\sqrt{\Delta}}{2}(z+A) \right) = \frac{e^{\frac{-\sqrt{\Delta}}{2}(z+A)} - e^{\frac{\sqrt{\Delta}}{2}(z+A)}}{e^{\frac{-\sqrt{\Delta}}{2}(z+A)} + e^{\frac{\sqrt{\Delta}}{2}(z+A)}}.$$

If we multiply $\tanh\left(\frac{-\sqrt{\Delta}}{2}(z+A) \right)$ by $\frac{e^{\frac{\sqrt{\Delta}}{2}(z+A)}}{e^{\frac{\sqrt{\Delta}}{2}(z+A)}}$ we get

$$\tanh\left(\frac{-\sqrt{\Delta}}{2}(z+A) \right) = \left(\frac{1 - e^{\sqrt{\Delta}(z+A)}}{e^{\sqrt{\Delta}(z+A)} + 1} \right).$$

Substituting $\tanh\left(\frac{-\sqrt{\Delta}}{2}(z+A) \right)$ in Eq.3.11, we get

$$U(z) = \frac{-\gamma}{2\rho} + \frac{\sqrt{\gamma^2 - 4\rho\sigma}}{2\rho} \tanh\left(\frac{-1}{2}\left(\sqrt{\gamma^2 - 4\rho\sigma}\,(z+A) \right) \right). \tag{3.12}$$

3.3 Two-dimensional nonlinear system

Consider a two-dimensional autonomous system

$$\begin{aligned} \frac{dx_1}{dt} &= g_1(x_1, x_2), \\ \frac{dx_2}{dt} &= g_2(x_1, x_2), \end{aligned} \tag{3.13}$$

where functions $g_1(x_1, x_2)$ and $g_2(x_1, x_2)$ are nonlinear. The solution of Eq.3.13 is graphically represented in the x_1x_2 plane, called phase plane. The solution curve is called an orbit, path, or trajectory of Eq.3.13. The function $g_1(x_1, x_2)$ determines the motion of the solutions in a phase plane in the x_1 direction at location (x_1, x_2). Similarly, $g_2(x_1, x_2)$ determines the motion in x_2 direction at location (x_1, x_2). The point (ξ_1, ξ_2) is called equilibrium point if and only if

$$g_1(\xi_1, \xi_2) = 0, \qquad g_2(\xi_1, \xi_2) = 0. \tag{3.14}$$

The trajectories of Eq.3.13 and equilibrium points graphed in the phase plane are called the phase diagram. We then assume that g_1, g_2 have Taylor expansion at equilibrium point ξ. That is,

$$g_1(x_1, x_2) = g_1(\xi_1, \xi_2) + \frac{\partial g_1}{\partial x_1}(\xi_1, \xi_2)(x_1 - \xi_1) + \frac{\partial g_1}{\partial x_2}(\xi_1, \xi_2)(x_2 - \xi_2) + \cdots ,$$

$$g_2(x_1, x_2) = g_2(\xi_1, \xi_2) + \frac{\partial g_2}{\partial x_1}(\xi_1, \xi_2)(x_1 - \xi_1) + \frac{\partial g_2}{\partial x_2}(\xi_1, \xi_2)(x_2 - \xi_2) + \cdots .$$

Let $X_1 = (x_1 - \xi_1)$, $X_2 = (x_2 - \xi_2)$ and $G_1 = g_1(\xi_1, \xi_2)$, $G_2 = g_2(\xi_1, \xi_2)$. Therefore, we get

$$\begin{aligned} g_1(x_1, x_2) &= G_1 + \frac{\partial G_1}{\partial x_1}X_1 + \frac{\partial G_1}{\partial x_2}X_2 + \cdots , \\ g_2(x_1, x_2) &= G_2 + \frac{\partial G_2}{\partial x_1}X_1 + \frac{\partial G_2}{\partial x_2}X_2 + \cdots . \end{aligned} \tag{3.15}$$

Looking back, we have $x_1 = X_1 + \xi_1$ and $x_2 = X_2 + \xi_1$ and equilibrium points (ξ_1, ξ_2) are constants, that means

$$\frac{dx_1}{dt} = \frac{dX_1}{dt} \quad and \quad \frac{dx_2}{dt} = \frac{dX_2}{dt}. \tag{3.16}$$

If we substitute Eq.3.16 and Eq.3.15 into Eq.3.13, we get

$$\begin{aligned} \frac{dX_1}{dt} &= G_1 + \frac{\partial G_1}{\partial x_1}X_1 + \frac{\partial G_1}{\partial x_2}X_2 + \cdots, \\ \frac{dX_2}{dt} &= G_2 + \frac{\partial G_2}{\partial x_1}X_1 + \frac{\partial G_2}{\partial x_2}X_2 + \cdots. \end{aligned} \tag{3.17}$$

If we neglect second order terms in X_1, X_2 and use $G_1 = G_2 = 0$ since ξ_1, ξ_2 are equilibrium points, then we obtain linearization of Eq.3.17 at (ξ_1, ξ_2)

$$\begin{aligned} \frac{dX_1}{dt} &= \frac{\partial G_1}{\partial x_1}X_1 + \frac{\partial G_1}{\partial x_2}X_2, \\ \frac{dX_2}{dt} &= \frac{\partial G_2}{\partial x_1}X_1 + \frac{\partial G_2}{\partial x_2}X_2, \end{aligned} \tag{3.18}$$

which can be written as

$$\frac{d}{dt}\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} \frac{\partial G_1}{\partial x_1} & \frac{\partial G_1}{\partial x_2} \\ \frac{\partial G_2}{\partial x_1} & \frac{\partial G_2}{\partial x_2}, \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \tag{3.19}$$

or, in compact form,

$$\frac{dX(t)}{dt} = GX(t), \tag{3.20}$$

where G is the Jacobi matrix at (ξ_1, ξ_2). The equilibrium points of two dimensional nonlinear systems are classified in line with the eigenvalues of their corresponding linearization. The eigenvalues λ of a square matrix $G(\xi_1, \xi_2)$ are the solutions of

$$det(G(\xi_1, \xi_2) - \lambda I) = 0. \tag{3.21}$$

Classification of an equilibria of linear 2-dimensional systems.

Definition 3.3.1 *[16] The equilibrium point* (ξ_1, ξ_2) *of a square matrix (2X2) is:*

- *a* ***stable node*** *if* $\lambda_2 < \lambda_1 < 0$*, both eigenvalues of* G *are negative;*
- *a* ***unstable node*** *if* $0 < \lambda_1 < \lambda_2$*, both eigenvalues of* G *are positive;*
- *a* ***saddle*** *if* $\lambda_1 < 0 < \lambda_2$ *or* $\lambda_2 < 0 < \lambda_1$*, one eigenvalue of* G *is positive and the other is negative;*
- *a* ***stable degenerate node*** *if* $\lambda_1 = \lambda_2 = \lambda < 0$*, equal negative eigenvalues;*
- *a* ***unstable degenerate node*** *if* $\lambda_1 = \lambda_2 = \lambda > 0$*, equal positive eigenvalues;*
- *a* ***center*** *if* $\lambda_1 = v + iw$*,* $\lambda_2 = v - iw$ *and* $v = 0$*, both eigenvalues of* G *are pure imaginary;*
- *a* ***stable focus*** *if* $\lambda_1 = v + iw$*,* $\lambda_2 = v - iw$ *and* $v < 0$*;*
- *a* ***unstable focus*** *if* $\lambda_1 = v + iw$*,* $\lambda_2 = v - iw$ *and* $v > 0$*.*

Equilibrium point (ξ_1, ξ_2) is sometimes called hyperbolic if real part of $\lambda_{1,2} \neq 0$.

Theorem 3.3.1 *[16] Suppose that* g *is a differentiable function in some neighborhood of the equilibrium point* (ξ_1, ξ_2)*. Then,*

- *The equilibrium point* (ξ_1, ξ_2) *is asymptotically stable if all the eigenvalues of the matrix* G *have negative real parts, that is, if the equilibrium* *solution $X(t) = 0$ of the linearized system is asymptotically stable. In particular, for sufficiently small initial conditions the solutions are defined for all t.*

- *The equilibrium point (ξ_1, ξ_2) is unstable if at least one eigenvalue has a positive real part.*

- *If all the eigenvalues of G have non-negative real part but at least one of them has real part equal to 0, then the stability of the equilibrium point (ξ_1, ξ_2) of the Eq.3.20 can not be determined from the stability of its linearization.*

In summary, a phase diagram can be determined by finding all equilibrium points, analyzing their nature and stability and examining global behavior and structure of the trajectories. If a trajectory connects two equilibrium points in the system, that connection is called a heteroclinic trajectory. The trajectory connecting an equilibrium point to itself is called homoclinic trajectory.

3.4 Reaction-diffusion travelling wave solutions: theory

In this subsection, we give a procedure to understand the travelling wave solutions of reaction-diffusion equations and we also perform phase plane analysis. We then consider the reaction-diffusion equation,

$$\frac{\partial u}{\partial t} = d_u \frac{\partial^2 u}{\partial x^2} + f(u), \qquad x \in \mathbb{R}, \tag{3.22}$$

where $f(u)$ represents reaction and it is a nonlinear function of u. Applying $u(x,t) = \mathcal{U}(z)$ in Eq.2.1, we get

$$-ck\frac{d\mathcal{U}}{dz} = d_u k^2 \frac{d^2\mathcal{U}}{dz^2} + f(\mathcal{U}), \qquad z \in \mathbb{R}. \tag{3.23}$$

We now have a second order ODE of a single independent variable. The equation can't be solved explicitly. To understand its solution, we write the equation as a first order system. The system turns out to be an autonomous system. A phase plane can then be used to study it. Transforming Eq.3.23 into a first order ODEs of two unknowns, $\mathcal{U}$ and $\frac{d\mathcal{U}}{dz} = \mathcal{V}$, we get

$$\begin{aligned} \frac{d\mathcal{U}}{dz} &= \mathcal{V}, \\ \frac{dV}{dz} &= -\frac{c}{d_u k}\mathcal{V} - \frac{1}{d_u k^2} f(\mathcal{U}). \end{aligned} \tag{3.24}$$

To continue with our investigation, let us assume that $f(\mathcal{U}) = \mathcal{U}(1-\mathcal{U})$. Then the equation becomes the well known Fisher equation. Eq.3.24 becomes

$$\begin{aligned} \frac{d\mathcal{U}}{dz} &= \mathcal{V}, \\ \frac{dV}{dz} &= -\frac{c}{d_u k}\mathcal{V} - \frac{1}{d_u k^2}\mathcal{U}(1-\mathcal{U}). \end{aligned} \tag{3.25}$$

We are interested in travelling wave solutions, i.e., the solutions $\mathcal{U}$ such that

$$\lim_{z\to\pm\infty} \mathcal{U}(z) = u_{\pm\infty}.$$

It can be proved that

$$\lim_{z\to\pm\infty} \frac{d\mathcal{U}}{dz}(z) = \lim_{z\to\pm\infty} \mathcal{V}(z) = 0.$$

So, we are interested in solution such that

$$\lim_{z\to\pm\infty} (\mathcal{U}, \mathcal{V}) = (u_{\pm\infty}, 0).$$

But points $(u_{\pm\infty}, 0)$ are the equilibria of Eq.3.22. So, looking for travelling wave solutions for Eq.3.22 is equivalent to looking for solutions of Eq.3.22 joining equilibrium points. Points $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$ are equilibrium points of Eq.3.25 if

$$\begin{aligned} g_1(\mathcal{U},\mathcal{V}) =& \mathcal{V} = 0, \\ g_2(\mathcal{U},\mathcal{V}) =& \frac{-c\mathcal{V}}{d_u k} - \frac{1}{d_u k^2}(\mathcal{U} - \mathcal{U}^2) = 0. \end{aligned} \tag{3.26}$$

Hence, we get two equilibrium points, $\boldsymbol{\xi}_1 = (0,0)$ and $\boldsymbol{\xi}_2 = (1,0)$. The Jacobi matrix of the linearized system is

$$G(\mathcal{U},\mathcal{V}) = \begin{pmatrix} 0 & 1 \\ \frac{2\mathcal{U}-1}{d_u k^2} & \frac{-c}{d_u k} \end{pmatrix}. \tag{3.27}$$

Jacobi matrix of the linearized system at the equilibrium point θ_1 is

$$G(0,0) = \begin{pmatrix} 0 & 1 \\ \frac{-1}{d_u k^2} & \frac{-c}{d_u k} \end{pmatrix}. \tag{3.28}$$

By using Eq.3.21, we get two eigenvalues,

$$\lambda_1 = \frac{-c - \sqrt{c^2 - 4d_u}}{2d_u k} \quad \text{and} \quad \lambda_2 = \frac{-c + \sqrt{c^2 - 4d_u}}{2d_u k}. \tag{3.29}$$

If $c^2 \geq 4d_u$, the eigenvalues are real and both are negative (thus point $(0,0)$ is a stable node) and if $c^2 < 4d_u$, $(0,0)$ is a stable spiral (the eigenvalues are complex with negative real part). The second equilibrium point, $\boldsymbol{\xi}_2$ leads us to

$$G(1,0) = \begin{pmatrix} 0 & 1 \\ \frac{1}{d_u k^2} & \frac{-c}{d_u k}. \end{pmatrix} \tag{3.30}$$

and the eigenvalues are

$$\lambda_1 = \frac{-c - \sqrt{c^2 + 4d_u}}{2d_u k} \quad \text{and} \quad \lambda_2 = \frac{-c + \sqrt{c^2 + 4d_u}}{2d_u k}. \tag{3.31}$$

Here λ_1, λ_2 are real numbers of opposite signs. It means $(1, 0)$ is a saddle point. Point $(0, 0)$ is stable and point $(1, 0)$ is unstable and they are separated. For the existence of a travelling wave, there must be a path connecting two points: $(0, 0)$ and $(1, 0)$. As indicated by arrows, the path only connects from unstable point $(1, 0)$ to stable point $(0, 0)$. Figure 3.3 shows how we connected the stable and unstable points using dashed lines. Assuming that the path connecting $(0, 0)$ and $(1, 0)$ exists, then the path is expressed by $\mathcal{U}, V$ with boundary conditions

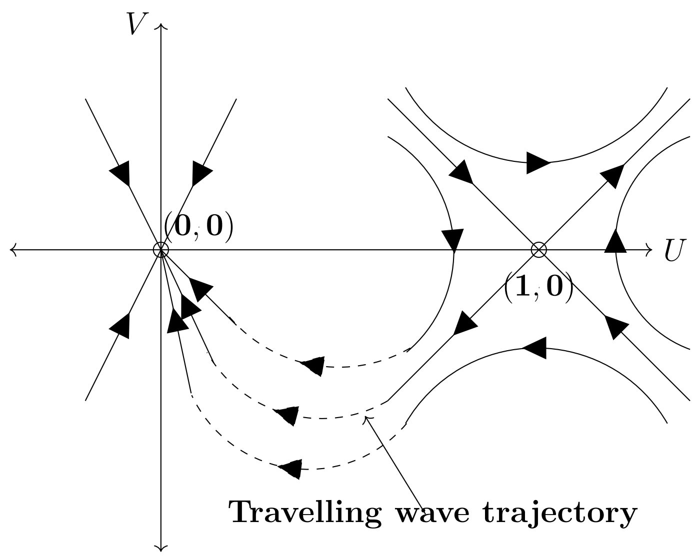


Figure 3.3: The Fisher equation phase plane trajectories.

$$\lim_{z \to -\infty} \mathcal{U}(z) = 1 \quad and \quad \lim_{z \to +\infty} \mathcal{U}(z) = 0. \tag{3.32}$$

The path connecting $(0, 0)$ and $(1, 0)$ represents a monotone decreasing travelling wave solution (Figure.3.4). The existence of a travelling wave solution is based on the existence of a path connecting two equilibrium points in the

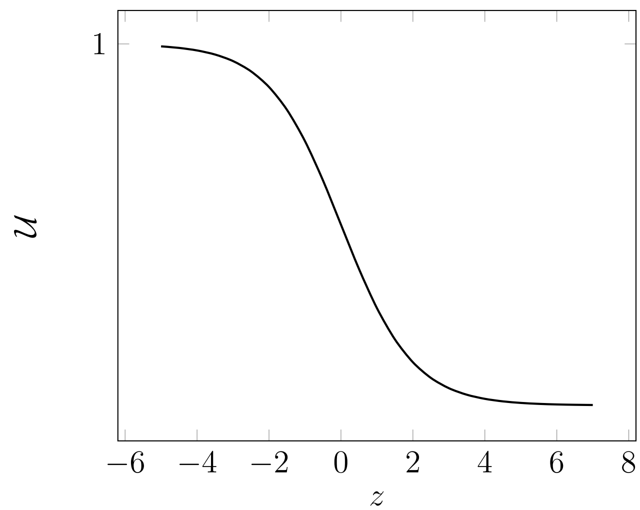


Figure 3.4: Travelling wave solution to Fisher's equation when $c^2 \geq 4d_u$

phase plane.

Chapter 4

Beyond the *'tanh'* expansion

The method is based on a method proposed by [15]. In this section we propose a more systematic way of that extent to find exact solutions of nonlinear equations with polynomial nonlinearities. The examples include equations such as the KdV equation, Burgers Equation, as well as Reaction Diffusion equations. We focus on nonlinear PDEs of only two independent variables, space x and time t of the form

$$\frac{\partial u}{\partial t} = u^{\tau}\frac{\partial u}{\partial x} + d_u\frac{\partial^{\omega} u}{\partial x^{\omega}} + \Phi(u), \tag{4.1}$$

where

$$\Phi(u) = \sum_{j=0}^{\beta} c_j u^j, \tag{4.2}$$

and τ, ω,β are natural numbers. As explained earlier, we seek travelling wave solutions using travelling wave coordinates, $u(x,t) = \mathcal{U}(z)$, where $z = k(x - ct)$, $c > 0$. Then Eq.4.1 turns to an ODE in terms of z,

$$ck\frac{d\mathcal{U}}{dz} + k\mathcal{U}^{\tau}\frac{d\mathcal{U}}{dz} + d_u k^{\omega}\frac{d^{\omega}\mathcal{U}}{dz^{\omega}} + \Phi(\mathcal{U}) = 0. \tag{4.3}$$

We presume that the solution of Eq.4.3 can be written as

$$\mathcal{U}(z) = \sum_{i=1}^{m} a_i f_i, \tag{4.4}$$

where $a_i = a_1, a_2, \cdots, a_m$, m, are constants to be determined and $f_1, \cdots, f_m$ are functions from a given set $\mathcal{F} = \{f_1, f_2, \cdots, f_m, \cdots\}$. We denote the set of linear combinations of f_i's by $\mathcal{L}in\mathcal{F}$. We assume that $\mathcal{F}$ satisfies the following assumptions

A1. $\mathcal{F} = \{f_1, f_2, \cdots, f_m, \cdots\}$ is a given linearly independent set of functions.

The infinite set of functions $\mathcal{F} = \{f_1, f_2, \cdots, f_m, \cdots\}$ is linearly independent if and only if every finite subset $\mathcal{G} = \{f_1, f_2, \cdots, f_m\}$ of $\mathcal{F}$ is linearly independent. A subset $\{f_1, f_2, \cdots, f_m\}$ is linearly independent if $\{f_1, \cdots, f_m\}$ of functions

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_m f_m x = 0, \qquad \forall x, \tag{4.5}$$

implies that $c_1 = c_2 = \cdots = c_m = 0$, where $c_1, c_2, \cdots, c_n$ are constants. Sometimes it is not easy to just use definition to show that a given set of functions is linearly independent. We find Wronskian as a powerful tool to determine independence of functions. The Wronskian matrix of $\{f_1, f_2, \cdots, f_m\}$ is

$$W(f_1, f_2, \cdots, f_m) = \begin{pmatrix} f_1 & f_2 & \cdots & f_m \\ f_1' & f_2' & \cdots & f_m' \\ \vdots & \vdots & \vdots & \vdots \\ f_1^{(m-1)} & f_2^{(m-1)} & \cdots & f_m^{(m-1)} \end{pmatrix}. \tag{4.6}$$

Then, the set $\{f_1, f_2, \cdots, f_m\}$ is linearly independent if and only if $det(W(f_1, f_2, \cdots, f_m)) \neq 0$.

A2. The product of any two functions $f_i, f_j \in \mathcal{F}$ satisfies $f_i * f_j \in \mathcal{F}$.

Lemma 4.0.1 *For any $f_i, f_j \in \mathcal{L}in\mathcal{F}$, $f_i * f_j \in \mathcal{L}in\mathcal{F}$.*

Proof of Lemma 4.0.1: Let $g = \sum_{i=0}^{m_1} a_i f_i$ and $h = \sum_{j=0}^{m_2} b_j f_j$ be two of functions in $\mathcal{L}in\mathcal{F}$. Then,

$$g * h = \sum_{i=0}^{m_1} \sum_{j=0}^{m_2} a_i b_j f_i f_j = \sum_{s=0}^{m_3} c_s f_s,$$

where $a_i * b_j = c_s$ and $f_i * f_j = f_s$, $s \in \mathbb{N}$. By A2, $f_s \in \mathcal{F}$ and we conclude that $f_i * f_j \in \mathcal{L}in\mathcal{F}$.

A3. If $f_j \in \mathcal{F}$, then $\frac{d}{dz} f_j \in \mathcal{L}in\mathcal{F}$.

Lemma 4.0.2 *(a) If $g \in \mathcal{L}in\mathcal{F}$, then $\frac{d}{dz} g \in \mathcal{L}in\mathcal{F}$.*

(b) If $g \in \mathcal{L}in\mathcal{F}$, $\frac{d^n}{dz^n} g \in \mathcal{L}in\mathcal{F}$,$n \in \mathbb{N}$.

(c) $g * \frac{d^n}{dz^n} h \in \mathcal{L}in\mathcal{F}$ for any $g, h \in \mathcal{L}in\mathcal{F}$.

Proof of lemma 4.0.2.

(a) Let $g \in \mathcal{L}in\mathcal{F}$, then

$$\begin{aligned} \frac{d}{dz} g &= \sum_{i=0}^{m_1} a_i \frac{d}{dz} f_i \\ &= \sum_{i=0}^{m_1} a_i \sum_{j=0}^{m_2} b_j f_j \qquad (4.7) \\ &= \sum_{i=0}^{m_{1,2}} c_i f_i, \end{aligned}$$

for $f_i \in \mathcal{F}$, hence $\frac{dg}{dz} \in \mathcal{L}in\mathcal{F}$ by A3. (b) Consider higher derivative, from proof (a), let $\frac{d}{dz}g = g_1 \in \mathcal{L}in\mathcal{F}$, then

$$\begin{aligned} g_1 &= \sum_{i=0}^{m_1} a_i \frac{d}{dz} f_i \\ &= \sum_{j=0}^{m_3} \alpha_j f_j, \end{aligned} \tag{4.8}$$

and

$$\begin{aligned} \frac{d}{dz} g_1 &= \sum_{i=0}^{m_4} \alpha_i \frac{d}{dz} f_i \\ &= \sum_{i=0}^{m_4} \alpha_j \sum_{j=0}^{m_5} \beta_j f_j \\ &= \sum_{i=0}^{m_{4,5}} \gamma_i f_i \end{aligned} \tag{4.9}$$

for $f_i \in \mathcal{F}$, thus $\frac{d}{dz}g_1 \in \mathcal{L}in\mathcal{F}$. It follows the same trend for the third, fourth, derivative. Therefore

$$\frac{d^n}{dz^n} g \in \mathcal{L}in\mathcal{F}, \tag{4.10}$$

where n is a natural number.

(c) If $g \in Lin\mathcal{F}$, then from (b) $\frac{d^n}{dz^n}h \in Lin\mathcal{F}$, therefore

$$\begin{aligned} g * \frac{d^n}{dz^n} h &= \sum_{i=0}^{m_1} a_i f_i \sum_{j=0}^{m_2} b_j \frac{d^n}{dz^n} f_j \\ &= \sum_{i=0}^{m_1} a_i f_i \sum_{j=0}^{m_2} b_j \sum_{k=0}^{m_3} c_k f_k \\ &= \sum_{i,j,k=0}^{m_6} a_i b_j c_k f_i * f_k. \end{aligned} \tag{4.11}$$

From **A**2 $f_i * f_k \in Lin\mathcal{F} = f_s$, then

$$g * \frac{d^n}{dz^n} h = \sum_{i=0}^{m_7} \beta_i f_s.$$

Therefore $g * \frac{d^n}{dz^n} h \in \mathcal{L}in\mathcal{F}$ for any $g, h \in \mathcal{L}in\mathcal{F}$.

In the next subsections, we introduce four different examples of functions that can be used to express a nonlinear PDE solution. In each example, we also demonstrate that the introduced conditions are satisfied.

4.1 Example 1

As an example, we consider a function $U(z)$. We define

$$\mathcal{F} = \{1, U, U^2, \cdots\}. \tag{4.12}$$

We assume that the solution of Eq.4.3 is expressed as

$$\mathcal{U}(U) = \sum_{i=0}^{m} a_i U^i, \tag{4.13}$$

with $U = U(z)$ satisfies Eq.3.2. We then introduce the Wronskian $(W(f_1, \cdots, f_n))$ to test independence. Any finite subset $\{1, U, U^2, \cdots\}$ is contained in $\{1, U, U^2, \cdots, U^n\}$ for some n. Then, from the definition of linear independence of functions, the set $\{1, U, U^2, \cdots\}$ is linearly independent iff for any n, the only solution to Eq.4.14 is $0 = c_n = \cdots = c_{n+1} = 0$.

$$\begin{pmatrix} 1 & U & U^2 & \cdots & U^n \\ 0 & 1 & 2U & \cdots & nU^{n-1} \\ 0 & 0 & 2 & \cdots & n(n-1)U^{n-2} \\ 0 & 0 & 0 & \cdots & n(n-1)(n-2)U^{n-3} \\ \vdots & & & \ddots & \vdots \\ & & & \cdots & n! \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_{n+1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \quad (4.14)$$

For condition **A**2, we let

$$f_i = U^i, \qquad f_i \in \mathcal{F}, \qquad i \in \mathbb{N}, \quad (4.15)$$

and

$$f_j = U^j, \qquad f_j \in \mathcal{F}, \qquad j \in \mathbb{N}. \quad (4.16)$$

Then,

$$f_i * f_j = U^{i+j}. \quad (4.17)$$

Therefore, $f_i * f_j \in \mathcal{F}$. Conditions **A**1 and **A**2 are satisfied. to prove that **A**3 to satisfied, we use $f_i = U^i$. Then

$$\frac{d}{dz} f_i = iU^{i-1}\frac{dU}{dz}.$$

We see that for **A**3 to be satisfied, $\frac{dU}{dz}$ must be a polynomial in U, hence we must have $\frac{dU}{dz} = P(U)$, see section 3.2. Postulate is that

$$\lim_{z\to\pm\infty} \mathcal{U} = u_{\pm\infty}.$$

This means that

$$\lim_{z\to\pm\infty} U = \bar{u}_{\pm\infty},$$

where $U(z)$ is given by Eq.3.12.

4.2 Example 2

Here, we consider function U^α,with $0 < \alpha < 1$, we define $\mathcal{F}$ as

$$\mathcal{F} = \{1, U^\alpha, (U^\alpha)^2, \cdots\}. \tag{4.18}$$

We then assume that the solution is expressed as

$$\mathcal{U}(U) = \sum_{i=0}^{m} a_i (U^\alpha)^i. \tag{4.19}$$

In this study, we assume that $\alpha = \frac{1}{k}$ for $k \in \mathbb{N}$. By following the same procedure as in **Example 1**, we can easily conclude that all three conditions are satisfied if $\mathcal{F} = \{1, U^\alpha, (U^\alpha)^2, \cdots\}$.

4.3 Example 3

Now consider the extended-**Example 1** method. This extended method is derived from the so-called the extended-G'/G method. The G'/G method was introduced by Wang, Li and Zhang [17] to search for travelling wave solutions of nonlinear evolution equations. The main step of this method is to assume that the solutions of Eq.4.3 can be expressed in the form

$$\mathcal{U} = \sum_{i=0}^{N} a_i \left(\frac{G'}{G}\right)^i, \tag{4.20}$$

where $G = G(z)$ satisfies the differential equation

$$G'' + \lambda G' + \mu G = 0. \tag{4.21}$$

Later, Shimin Guo and Yubin Zhou expanded the so-called G'/G method. They formed the method called the extended G'/G-expansion method. The method is applied to find travelling wave solutions of Whitham-Broer-Kaup-Like equations and coupled Hirota-Satsuma KdV equations [22]. In the extended method, we assume that the solution of the nonlinear ODE is expressed as

$$\mathcal{U}(z) = a_0 + \sum_{i=1}^{n} a_i \left(\frac{G'}{G}\right)^i + \sum_{i=1}^{n} b_i \left(\frac{G'}{G}\right)^{i-1} \sqrt{\sigma\left(1+\frac{1}{\mu}\left(\frac{G'}{G}\right)^2\right)}, \quad (4.22)$$

where a_0, a_i, b_i are constants to be computed; $\sigma = \pm 1$, n is a positive integer and $G = G(z)$ satisfies

$$G'' + \mu G = 0, \quad (4.23)$$

where μ is a constant. Simplifying and modifying the method, we let $G'/G = U$ and obtain

$$\mathcal{U}(z) = a_0 + \sum_{i=1}^{n} a_i (U)^i + \sum_{i=1}^{n} b_i (U)^{i-1} \sqrt{\rho U^2 + \gamma U + \sigma}, \quad (4.24)$$

where $U = U(z)$ satisfies the first order ODE

$$\frac{dU}{dz} = -(\rho U^2 + \gamma U + \sigma) = -P(U). \quad (4.25)$$

Here (Example 3), $\mathcal{F}$ is a set given by

$$\mathcal{F} = \{1, U, \sqrt{P(U)}, U^2, U\sqrt{P(U)}, \cdots\}. \quad (4.26)$$

We assume that the solution is expressed as

$$\mathcal{U}(U) = a_0 + \sum_{i=1}^{m} a_i U^i + \sum_{i=1}^{m} b_i U^{i-1} \sqrt{P(U)}. \quad (4.27)$$

To show that the product of two functions belongs to $\mathcal{F}$, we let

$$g_i = U^i + U^{i-1}\sqrt{\rho U^2 + \gamma U + \sigma}, \qquad g_i \in \mathcal{F}, \qquad i \in \mathbb{N}, \quad (4.28)$$

and

$$g_j = U^j + U^{j-1}\sqrt{\rho U^2 + \gamma U + \sigma}, \qquad g_j \in \mathcal{F}, \qquad j \in \mathbb{N}. \tag{4.29}$$

Then,

$$g_i * g_j = U^{i+j} + \rho U^{i+j} + \gamma U^{i+j-1} + \sigma U^{i+j-2} + 2U^{i+j-1}\sqrt{P}. \tag{4.30}$$

Hence, $g_i * g_j \in \mathcal{F}$. For the third condition, we only prove that $g_{i2} = U^{i-1}\sqrt{P}$ satisfies condition **A**3 because we already proved that $g_{i1} = U^i$ satisfies all conditions in Example 1. Then,

$$\begin{aligned}\frac{dg_i}{dz} =&(i-1)U^{i-2}\frac{dU}{dz}\sqrt{P} + \frac{U^{i-1}}{2\sqrt{P}}\frac{dP}{dU}\frac{dU}{dz} \\ =&(i-1)\left(\rho U^i\sqrt{P} + \gamma U^{i-1}\sqrt{P} + \sigma U^{i-2}\sqrt{P}\right) - \\ &\rho U^i\sqrt{P} - \frac{\gamma}{2}U^{i-1}\sqrt{P}.\end{aligned} \tag{4.31}$$

Hence, $\frac{dg_i}{dz} \in \mathcal{F}$, $\frac{d^n g_i}{dz^n} \in \mathcal{F}$ and $g_j\frac{d^n g_i}{dz^n} \in \mathcal{F}$.

4.3.1 Description of method

Summary of main steps of finding travelling wave solutions by using **Example 1-3**.

Step 1. We transform the nonlinear PDE to a nonlinear ODE using travelling wave coordinates $z = k(x - ct)$ such that

$$u(x,t) = \mathcal{U}(z).$$

Step 2. We assume that the solution of a nonlinear ODE can be expressed by one of the following (depending on the type of equation under study):

$$\begin{aligned}\mathcal{U}(U) =& \sum_{i=0}^{m} a_i U^i,\\ \mathcal{U}(U) =& \sum_{i=0}^{m} a_i (U^\alpha)^i,\\ \mathcal{U}(U) =& a_0 + \sum_{i=1}^{m} a_i U^i + \sum_{i=1}^{m} b_i U^{i-1}\sqrt{P(U)},\end{aligned} \tag{4.32}$$

where $U = U(z)$ satisfies the first order linear ODE in form

$$\frac{dU}{dz} = \rho U^2 + \gamma U + \sigma'.$$

The positive integer m can be determined by balancing the highest degree terms in U. It can be achieved by substituting one of E.q4.32 together with a linear ODE into a nonlinear ODE.

Step 3. After determining the value of m, we then substitute one of Eq.4.32 along with a linear ODE into a nonlinear ODE. We collect all terms of the same order of U, $U\sqrt{P(U)}$, $U \ln U$ together and equate their coefficients to zero.

Step 4. We then solve for the unknown constants $(a_0, \cdots, a_m)$, $(b_1, \cdots, b_m)$, k and c. Substituting these constants and the general solution Eq.3.12 of a linear ODE, into one of Eq.4.32, we can obtain a travelling wave solution.

4.3.2 Balancing exponents

In this study we consider the equations containing the following elements: linear diffusion, nonlinear advection and nonlinear reaction. Such an equation is given by Eq.4.1. By reconsidering an ODE of Eq.4.1, we assume that the solution of Eq.4.3 has the form:

$$\mathcal{U} = \sum_{i=0}^{m} a_i f_i,$$

where f_i is any function in **Examples 1-3** and $U = U(z)$ satisfies first order ODE $dU/dz = \rho U^2 + \gamma U + \sigma$. Here, we want to determine the parameter m by balancing highest degree terms in U. In giving the formula of m for balancing exponents of Eq.4.3, we consider three cases or equations that can come from Eq.4.1. The Eq.4.1 is a nonlinear advection reaction diffusion equation. For equations of this type, we can mention the Burgers like-Huxly equation, and many more. The second equation arises by dropping the advection term to get a diffusion equation with nonlinear source terms, normally called the reaction-diffusion equation,

$$\frac{\partial u}{\partial t} = d_u \frac{\partial^\omega u}{\partial x^\omega} + \Phi(u). \tag{4.33}$$

There are many equations of this form; equations like the Fisher equation, the Zeldovich equation, the Nagumo equation, the Newell-Whitehed equation, ect. The third case is obtained if the reaction term does not form part of Eq.4.1. We get a nonlinear advection diffusion equation

$$\frac{\partial u}{\partial t} = u^\tau \frac{\partial u}{\partial x} + d_u \frac{\partial^\omega u}{\partial x^\omega}. \tag{4.34}$$

Equations of this kind are the Burgers equation, KdV equation, ect. So, Eq.4.1 accommodates three different equations. The next step is to give formulas of the value of m for different functions, based on **Examples 1-3**. We write the terms with the highest powers coming from each term in Eq.4.3.

4.3.2.1 Example 1, balancing

In Example 1, by writing the term with highest powers, we assume that the solution is expressed as

$$\mathcal{U} = a_m U^m + \cdots , \tag{4.35}$$

where $a_m \neq 0$ and $U = U(z)$ satisfies Eq.3.2. Taking derivatives of Eq.4.35, we get

$$\begin{aligned}
\frac{d\mathcal{U}}{dz} =& m\rho a_m U^{m+1} + m\gamma a_m U^m + m\sigma a_m U^{m-1} + \cdots ,\\
\frac{d^2\mathcal{U}}{dz^2} =& m(m-1)p a_m \left(\rho U^{m+2} + \gamma U^{m+1} + \sigma U^m\right) + \cdots ,\\
&\vdots\\
\frac{d^\omega\mathcal{U}}{dz^\omega} =& (m^2-m)p a_m \left(\rho U^{m+\omega} + \gamma U^{m+\omega-1} + \sigma U^{m+\omega-2}\right) + \cdots ,
\end{aligned} \tag{4.36}$$

$$\mathcal{U}^\tau \frac{d\mathcal{U}}{dz} = m\rho a_m^2 U^{m+m\tau+1} + m\gamma a_m^2 U^{m+m\tau} + m\sigma a_m^2 U^{m+m\tau-1} + \cdots . \tag{4.37}$$

The polynomial $\Phi(\mathcal{U})$ becomes

$$\begin{aligned}
\Phi(u) &= \sum_{j=2}^{\beta} c_j u^j,\\
&= \sum_{j=2}^{\beta} c_j \left(\sum_{i=0}^{m} a_i U^i\right)^j ,\\
&= c_\beta a_m^\beta U^{m\beta} + \cdots .
\end{aligned} \tag{4.38}$$

The highest power of $\Phi(\mathcal{U})$ is $U^{m\beta}$, $\mathcal{U}^\tau \frac{d\mathcal{U}}{dz}$ is $U^{m\tau+m+1}$, $\frac{d^\omega\mathcal{U}}{dz^\omega}$ is $U^{m+\omega}$, and $\frac{d\mathcal{U}}{dz}$ is U^{m+1}. Equating the exponents of U, we have

$$m = \frac{\omega}{\beta - 1}, \tag{4.39}$$

where balancing is between diffusion term and reaction term. If the balancing is between nonlinear advection term and diffusion term, we have

$$m = \frac{\omega - 1}{\tau}. \tag{4.40}$$

To determine the term with the highest exponents between diffusion, advection and reaction terms, we start by using both Eq.4.39 and Eq.4.40. After comparing the two values, we then use the lowest value of m. If the obtained m value is found to be a fraction, we then use **Example 2** and Table 2 to address the problem. We only use Eq.4.40 to determine the value of m for the

Table 4.1: Possible values of ω, β, τ and m.

Cases	ω	β	τ	m
$\omega \geq \beta - 1$	1	2		1
	2	2		2
	2	3		1
$\beta \geq 2$	3	2		3
	3	4		1
	4	2		4
	4	3		2
	4	5		1
$\omega > \tau$ and $\omega + \tau = odd$	2		1	1
$\omega \geq 2$	3		2	1
$\tau \geq 1$	4		1	3
	4		3	1

KdV equation, Burger equations and many more if the following conditions are satisfied:

$$\omega \leq \beta - 1, \qquad \omega, \beta \geq 2, \tag{4.41}$$

and

$$\omega + \tau = 2\mathbb{N} + 1, \qquad \omega > \tau, \qquad \omega, \tau \geq 2, \tag{4.42}$$

where $\mathbb{N}$ is a natural number. Table 1 shows the possible values of β, ω, τ and m. Balancing the reaction diffusion equation with no advection term, we use Eq.4.39. If the mentioned conditions are not satisfied, we use **Example 2**.

Table 4.2: Possible values of ω, β, τ, α and m.

Cases	ω	β	τ	m	α	Results (m)
$\omega < \beta - 1$	2	5		$1/2\alpha$	$1/2$	1
$\omega \geq 2$	2	4		$2/3\alpha$	$2/3$	2
$\beta > 2$	3	5		$3/4\alpha$	$3/4$	3
$\omega > \beta - 1$	3	3		$3/2\alpha$	$3/2$	3
$\beta > 2$, $\omega > 2$	4	4		$4/3\alpha$	$4/3$	4
	5	3		$5/2\alpha$	$5/2$	5
$\omega > \tau$ and $\omega + \tau = even$	5		3	$4/3\alpha$	$4/3$	4
	4		2	$3/2\alpha$	$3/2$	3
	8		2	$7/2\alpha$	$7/2$	7
$\omega - 1 \leq \tau$	2		2	$1/2\alpha$	$1/2$	1
	2		3	$1/2\alpha$	$1/2$	1
	2		4	$1/4\alpha$	$1/4$	1

4.3.2.2 Example 2, balancing

In Example 2, we have $\mathcal{U} = \sum_{i=0}^{m} a_i(U^\alpha)^i$. Balancing terms with highest powers lead us to

$$m = \frac{\omega}{\alpha(\beta - 1)}, \tag{4.43}$$

if the balancing is between a nonlinear advection term and diffusion term, but

$$m = \frac{\omega - 1}{\alpha\tau}, \tag{4.44}$$

if the balancing is between diffusion term and the reaction term. Table 2 gives possible values of α, ω, τ and m.

4.3.2.3 Example 3, balancing

$\mathcal{U}(U) = a_0 + \sum_{i=1}^{m} a_i U^i + \sum_{i=1}^{m} b_i U^{i-1}\sqrt{P(U)}$, we obtain same results as of $\mathcal{U}(U) = \sum_{i=0}^{m} a_i U^i$.

Chapter 5

Application

5.1 Whitham-Broer-Kaup equation

In this chapter, we have studied the 1-dimensional Whitham-Broer-Kaup equation, Fisher equation and Burger Fisher equation by using **Example 1-3** to find exact travelling wave solutions. The Whitham-Broer-Kaup equation is an important equation in the field of mathematical physics. The Whitham-Broer-Kaup model is generally used to study tsunami waves. The model describes the tsunami wave dynamics under gravity [26]. It was developed based on the fluid mechanics assumption, which says that fluid is incomprehensible and irrotational. The Whitham-Broer-Kaup model can be written as

$$\begin{aligned} \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \hbar\frac{\partial h}{\partial x} + b\frac{\partial^2 u}{\partial x^2} &= 0, \\ \frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu) + a\frac{\partial^3 u}{\partial x^3} - b\frac{\partial^2 h}{\partial x^2} &= 0, \end{aligned} \tag{5.1}$$

where $\hbar$, a, b, are constants. The main variables u describe the horizontal velocity and h is the height deviating from the equilibrium position of the liquid. After the transformation of $u(x,t) = \mathcal{U}(z)$ and $h(x,t) = \mathcal{H}(z)$, we get

$$\begin{aligned} -ck\frac{d\mathcal{U}}{dz} + k\mathcal{U}\frac{d\mathcal{U}}{dz} + k\hbar\frac{d\mathcal{H}}{dz} + k^2 b\frac{d^2\mathcal{U}}{dz^2} &= 0, \\ -ck\frac{d\mathcal{H}}{dz} + k\frac{d}{dz}(\mathcal{H}\mathcal{U}) + k^3 a\frac{d^3\mathcal{U}}{dz^3} - k^2 b\frac{d^2\mathcal{H}}{dz^2} &= 0. \end{aligned} \tag{5.2}$$

Integrating Eq.5.2 with respect to z, we get

$$\begin{aligned} -ck\mathcal{U} + \frac{1}{2}k\mathcal{U}^2 + k\hbar\mathcal{H} + k^2 b\frac{d\mathcal{U}}{dz} + A_1 &= 0, \\ -ck\mathcal{H} + k\mathcal{H}\mathcal{U} + k^3 a\frac{d^2\mathcal{U}}{dz^2} - k^2 b\frac{d\mathcal{H}}{dz} + A_2 &= 0, \end{aligned} \tag{5.3}$$

where A_1 and A_2 are integration constants. By **Example 3**, we assume that the solution of Eq.5.3 is expressed as

$$\begin{aligned} \mathcal{U}(z) &= a_0 + \sum_{i=1}^{n} a_i U^i + \sum_{i=1}^{n} b_i U^{i-1}\sqrt{U - U^2}, \\ \mathcal{H}(z) &= c_0 + \sum_{i=1}^{n_1} c_i U^i + \sum_{i=1}^{n_1} d_i U^{i-1}\sqrt{U - U^2}, \end{aligned} \tag{5.4}$$

where $U = U(z)$ satisfies a first order ODE

$$\frac{dU}{dz} = U^2 - U.$$

The highest power of the product of $\mathcal{H}$ and $\mathcal{U}$ is given by

$$\mathcal{H}\mathcal{U} = a_n c_{n_1} U^{n_1+w} + \cdots . \tag{5.5}$$

Balancing the highest exponents between $\frac{d^2\mathcal{U}}{dz^2}$ and $\mathcal{H}\mathcal{U}$ in the second equation of Eq.5.3, we have $n + 2$ and $n + n_1$ respectively. Hence, $n_1 = 2$. The value of n in the first equation can be determined using table 4.1. That is, $n = 1$. Therefore

$$\begin{aligned}
\mathcal{U}(z) &= a_0 + a_1 U + b_1\sqrt{U - U^2}, \\
\mathcal{H}(z) &= c_0 + c_1 U + c_2 U^2 + d_1\sqrt{U - U^2} + d_2 U\sqrt{U - U^2}.
\end{aligned} \tag{5.6}$$

Substituting Eq.5.6 in Eq.5.3 and equating coefficients of powers of U^i and $U^i\sqrt{U - U^2}$ $(i = 0, 1, \cdots)$ to zero, we obtain the following:

$$\begin{aligned}
U^0 &: -2cka_0 + ka_0^2 + 2\hbar kc_0 + 2A_1 = 0, \\
U^1 &: -2cka_1 + 2ka_0a_1 + kb_1^2 + 2k\hbar c_1 - 2k^2ba_1 = 0, \\
U^2 &: ka_1^2 - kb_1^2 + 2k\hbar c_2 + 2k^2ba_1 = 0, \\
U^0\sqrt{U - U^2} &: -2ckb_1 + 2a_0b_1 + 2k\hbar d_1 - k^2bb_1 = 0, \\
U\sqrt{U - U^2} &: ka_1b_1 + k\hbar d_2 + k^2bb_1 = 0, \\
U^0 &: ka_0c_0 - ckc_0 + A2 = 0, \\
U^1 &: k^2aa_1 + k^2bc_1 + ka_0c_1 + ka_1c_0 + kb_1d_1 - ckc_1 = 0, \\
U^2 &: -3k^3aa_1 - k^2bc_1 + 2k^2bc_2 + ka_0c_2 + ka_1c_1 - kb_1d_1 + \\
&\quad kb_1d_2 - ckc_2 = 0, \\
U^3 &: 2k^3aa_1 - 2k^2bc_2 + ka_1c_2 - kb_1d_2 = 0, \\
U^0\sqrt{U - U^2} &: k^3ab_1 + 2k^2bd_1 + 4ka_0d_1 + 4kb_1c_0 - 4ckd_1 = 0, \\
U\sqrt{U - U^2} &: -4k^3ab_1 - 2k^2bd_1 + 3k^2bd_2 + 2ka_1d_1 + 2kb_1c_1 - \\
&\quad 2ckd_2 + 2ka_0d_2 = 0, \\
U^2\sqrt{U - U^2} &: 2k^3ab_1 - 2k^2bd_2 + ka_1d_2 + kb_1c_2 = 0.
\end{aligned} \tag{5.7}$$

After solving the unknowns of Eq.5.7, we get the following list of solutions.

Solution 1:

$$
\begin{aligned}
&A_1 = \frac{1}{2}kc^2 - \frac{1}{16}b_1^2, A_2 = 0, a = \frac{-d_2^2 - c_2^2}{2k^2c_2}, a_0 = c, a_1 = 0, \\
&b = \frac{-b_1d_2}{2c_2k}, b_1 = b_1, c = c, c_0 = \frac{c_2}{8}, c_1 = -c_2, c_2 = c_2, \\
&d_1 = \frac{-1}{2}d_2, d_2 = d_2, \hbar = \frac{b_1^2}{2c_2}, k = k
\end{aligned}
$$

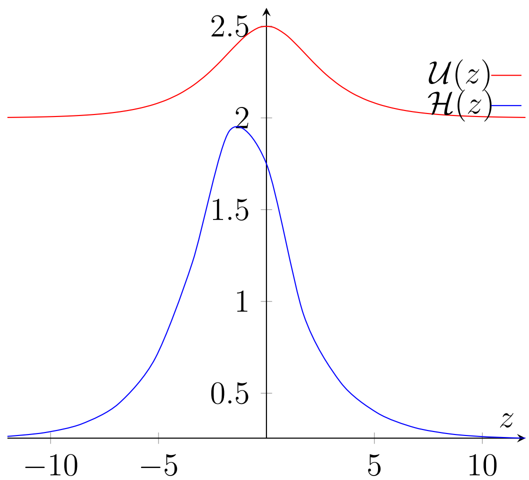
Solution 2:

$$
\begin{aligned}
&A_1 = -\frac{1}{2}ka_0^2 + cka_0, A_2 = 0, a = \frac{-b^2k^2 + a_0^2 - 2a_0c + c^2}{k^2\hbar}, a_0 = a_0, \\
&a_1 = -2a_0 + 2c, b = b, b_1 = 0, c = c, c_0 = 0, c_1 = \frac{(2a_0 - 2c)(-bk + a_0 - c)}{\hbar}, \\
&c_2 = \frac{(2c - 2a_0)(-bk + a_0 - c)}{\hbar}, d_1 = 0, d_2 = 0, \hbar = \hbar, k = k.
\end{aligned}
\tag{5.8}
$$

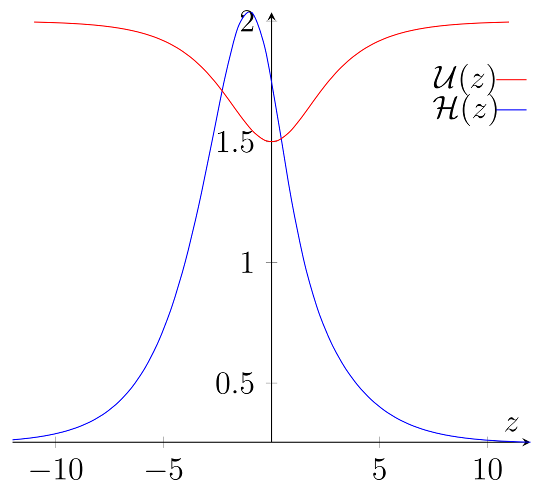
We get the following exact solutions of the systems,

Solution 1:

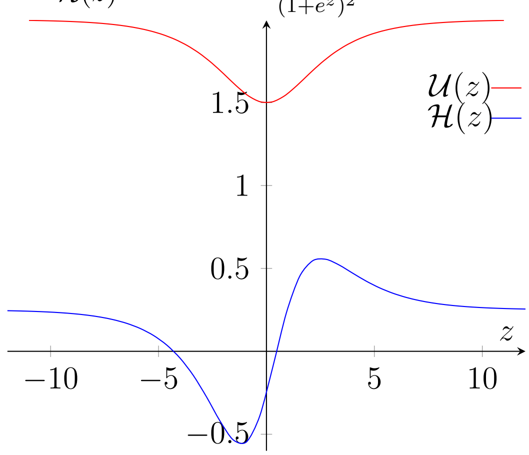
$$
\begin{aligned}
&\mathcal{U}(U) = c \pm \sqrt{\frac{-k^2}{2ak^2+1}}\sqrt{U - U^2}, \\
&\mathcal{H}(U) = \frac{-k^2}{4\hbar(2ak^2+1)} + \frac{2k^2}{\hbar(2ak^2+1)}U + \frac{-2k^2}{\hbar(2ak^2+1)}U^2 + \\
&\qquad - \frac{2bk^3}{\hbar}\frac{1}{\pm\sqrt{-2ak^4 - k^2}}\sqrt{U - U^2} + \frac{4bk^3}{\hbar}\frac{1}{\pm\sqrt{-2ak^4 - k^2}}U\sqrt{U - U^2}.
\end{aligned}
\tag{5.9}
$$



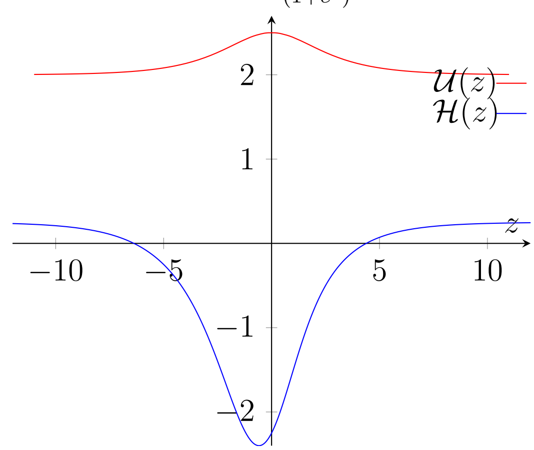
(a) $\mathcal{U}(z) = 2 + \frac{\sqrt{e^z}}{1+e^z}$, $\mathcal{H}(z) = \frac{(1+e^z)(4e^z+2)+2+2\sqrt{e^z}(3+e^z)}{(1+e^z)^2}$



(b) $\mathcal{U}(z) = 2 - \frac{\sqrt{e^z}}{1+e^z}$, $\mathcal{H}(z) = \frac{(1+e^z)(4e^z+2)+2+2\sqrt{e^z}(3+e^z)}{(1+e^z)^2}$



(c) $\mathcal{U}(z) = 2 - \frac{\sqrt{e^z}}{1+e^z}$, $\mathcal{H}(z) = \frac{(1+e^z)(4e^z+2)+2+2\sqrt{e^z}(e^z-3)}{(1+e^z)^2}$



(d) $\mathcal{U}(z) = 2 + \frac{\sqrt{e^z}}{1+e^z}$, $\mathcal{H}(z) = \frac{(1+e^z)(4e^z+2)+2-2\sqrt{e^z}(5+e^z)}{(1+e^z)^2}$

Figure 5.2: Solution 1:The pulse wave profiles of Witham-Broer-Kaup equation where $\hbar_1 = 1$, $c = 2$, $b = 1$, and $a = -1$.

By substituting Eq.3.12 into Eq.5.9 yields a pulse wave solution

$$\begin{aligned}
\mathcal{U}(z) &= c \pm \sqrt{\frac{-k^2}{2ak^2+1}}\left(\frac{\sqrt{e^z}}{1+e^z}\right), \\
\mathcal{H}(z) &= \frac{-k^2}{4\hbar(2ak^2+1)} + \frac{2k^2}{\hbar(2ak^2+1)}\left(\frac{1}{1+e^z}\right) + \\
&\quad \frac{-2k^2}{\hbar(2ak^2+1)}\left(\frac{1}{1+e^z}\right)^2 + \\
&\quad -\frac{2bk^3}{\hbar}\frac{1}{\pm\sqrt{-2ak^4-k^2}}\left(\frac{\sqrt{e^z}}{1+e^z}\right) + \\
&\quad \frac{4bk^3}{\hbar}\frac{1}{\pm\sqrt{-2ak^4-k^2}}\left(\frac{\sqrt{e^z}}{(1+e^z)^2}\right),
\end{aligned} \tag{5.10}$$

or

$$\begin{aligned}
u(x,t) &= c \pm \sqrt{\frac{-k^2}{2ak^2+1}}\left(\frac{\sqrt{e^{k(x-ct)}}}{1+e^{k(x-ct)}}\right), \\
h(x,t) &= \frac{-k^2}{4\hbar(2ak^2+1)} + \frac{2k^2}{\hbar(2ak^2+1)}\left(\frac{1}{1+e^{k(x-ct)}}\right) + \\
&\quad \frac{-2k^2}{\hbar(2ak^2+1)}\left(\frac{1}{1+e^{k(x-ct)}}\right)^2 - \\
&\quad \frac{2bk^3}{\hbar}\frac{1}{\pm\sqrt{-2ak^4-k^2}}\left(\frac{\sqrt{e^{k(x-ct)}}}{1+e^{k(x-ct)}}\right) + \\
&\quad \frac{4bk^3}{\hbar}\frac{1}{\pm\sqrt{-2ak^4-k^2}}\left(\frac{\sqrt{e^{k(x-ct)}}}{(1+e^{k(x-ct)})^2}\right),
\end{aligned} \tag{5.11}$$

Solution 2:

$$\begin{aligned}
\mathcal{U}(U) &= c \pm \sqrt{ak^2\hbar + b^2k^2} \pm 2\sqrt{ak^2\hbar + b^2k^2}U, \\
\mathcal{H}(U) &= \frac{\pm\sqrt{ak^2\hbar + b^2k}\left(bk \pm \sqrt{ak^2\hbar + b^2k^2}\right)}{\hbar}U + \\
&\quad \frac{\pm\sqrt{ak^2\hbar + b^2k}\left(-bk \pm \sqrt{ak^2\hbar + b^2k^2}\right)}{\hbar}U^2.
\end{aligned} \tag{5.12}$$

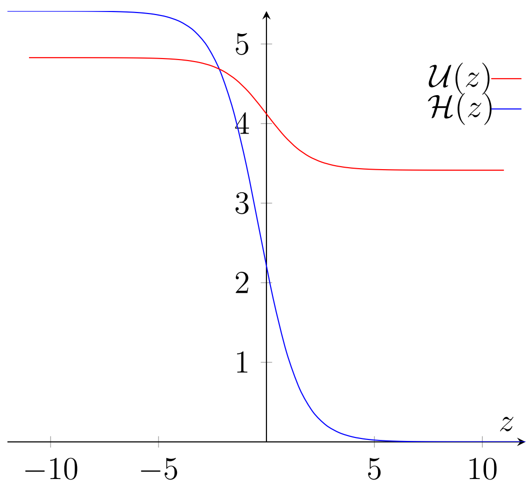
Again, substituting Eq.3.12 in Eq.5.12, we get an exact travelling wave solution

$$\begin{aligned}
\mathcal{U}(z) &= c \pm \sqrt{ak^2h + b^2k^2} \pm 2\sqrt{ak^2h + b^2k^2}\left(\frac{1}{1+e^z}\right), \\
\mathcal{H}(z) &= \frac{\pm\sqrt{ak^2\hbar + b^2k}\left(bk \pm \sqrt{ak^2\hbar + b^2k^2}\right)}{\hbar}\left(\frac{1}{1+e^z}\right) +, \\
&\frac{\pm\sqrt{ak^2\hbar + b^2k}\left(-bk \pm \sqrt{ak^2\hbar + b^2k^2}\right)}{\hbar}\left(\frac{1}{1+e^z}\right)^2,
\end{aligned} \tag{5.13}$$

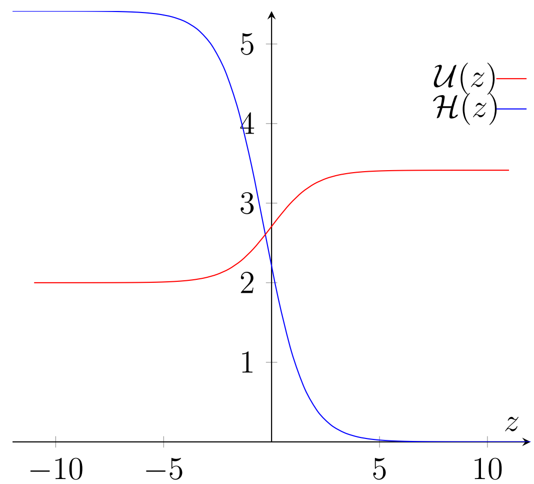
or

$$\begin{aligned}
u(x,t) &= c \pm \sqrt{ak^2h + b^2k^2} \pm 2\sqrt{ak^2h + b^2k^2}\left(\frac{1}{1+e^{k(x-ct)}}\right), \\
h(x,t) &= \frac{\pm\sqrt{ak^2\hbar + b^2k}\left(bk \pm \sqrt{ak^2\hbar + b^2k^2}\right)}{\hbar}\left(\frac{1}{1+e^{k(x-ct)}}\right) +, \\
&\frac{\pm\sqrt{ak^2\hbar + b^2k}\left(-bk \pm \sqrt{ak^2\hbar + b^2k^2}\right)}{\hbar}\left(\frac{1}{1+e^{k(x-ct)}}\right)^2,
\end{aligned} \tag{5.14}$$

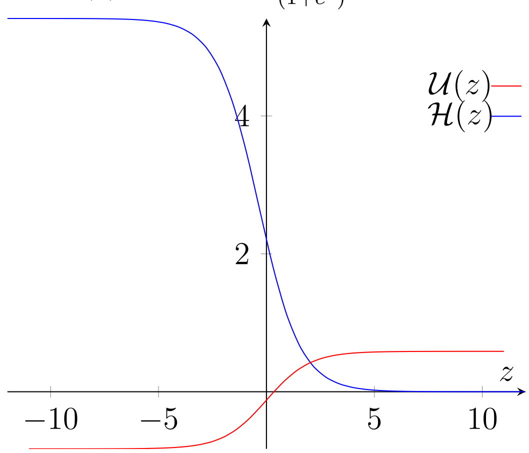
In this example, we applied **Example 3** to find exact travelling wave solutions of the Whitham-Broer-Kaup equation. Comparing the solutions reported by [22], we notice that we obtained the same wave speed in Solution 1 and parameter d_1 as an arbitrary parameter, while in their case $d_1 = 0$. They also wrongly claimed that by setting $A_1 = 0$, $A_2 \neq 0$ you get a pulse wave solution. In Figure 5.2, pulse wave solutions b, c, and d might not be used in application of the study of tsunami waves because they are negative pulse wave solutions. Solution 2 is the travelling wave front of Eq.5.1. The travelling wave profile is shown by Figure 5.4. Application of Example 1 has successfully lead us to new pulse wave solutions. The Example 1 method is very powerful and that is emphasized by some similarities in wave profiles obtained by [26]. By using **Example 3**, we more clearly illustrate the method's utility, simplicity, and briefness in comparison to the extended G'/G method for finding travelling solutions for Whitham-Broer-Kaup equations.



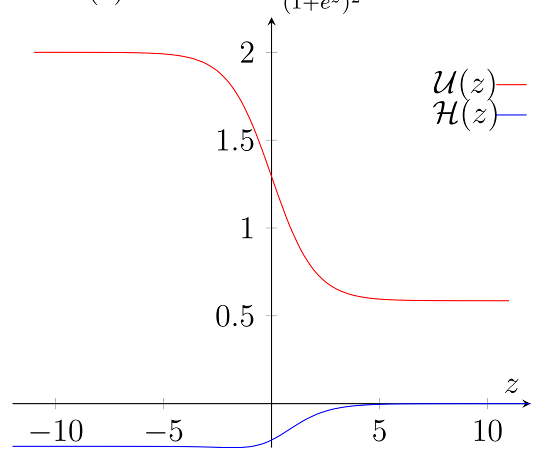
(a) $\mathcal{U}(z) = \sqrt{2} + 2 + \frac{\sqrt{2}}{1+e^z}$,
$\mathcal{H}(z) = \frac{\sqrt{2}(1+\sqrt{2})(1+e^z)+2\sqrt{2}(\sqrt{2}-1)}{(1+e^z)^2}$



(b) $\mathcal{U}(z) = \sqrt{2} + 2 - \frac{\sqrt{2}}{1+e^z}$,
$\mathcal{H}(z) = \frac{\sqrt{2}(1+\sqrt{2})(1+e^z)+2\sqrt{2}(\sqrt{2}-1)}{(1+e^z)^2}$



(c) $\mathcal{U}(z) = -\sqrt{2} + 2 - \frac{\sqrt{2}}{1+e^z}$,
$\mathcal{H}(z) = \frac{\sqrt{2}(1+\sqrt{2})(1+e^z)+2\sqrt{2}(\sqrt{2}-1)}{(1+e^z)^2}$



(d) $\mathcal{U}(z) = -\sqrt{2} + 2 + \frac{\sqrt{2}}{1+e^z}$,
$\mathcal{H}(z) = \frac{\sqrt{2}(1-\sqrt{2})(1+e^z)-2\sqrt{2}(\sqrt{2}-1)}{(1+e^z)^2}$

Figure 5.4: Solution 2:The travelling wave profiles of Witham-Broer-Kaup equation where $\hbar_1 = 1$, $c = 2$, $b = 1$, $k = 1$ and $a = 1$.

5.2 Fisher equation

The Fisher's equation [2],

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(1-u),$$

is the model that describes the evolution of a population density function $u(x,t)$ at time t and location x. It is a model of diffusion of a species in a one dimensional habitat. As in the previous example, we first transform Eq.6.15 to get

$$ck\frac{d\mathcal{U}}{dz} + k^2\frac{d^2\mathcal{U}}{dz^2} + \mathcal{U}(1-\mathcal{U}) = 0. \tag{5.15}$$

We introduced earlier the **Example 1**, hence we assume that the solution of Eq.5.15 is expressed as Eq.4.13 with $U = U(z)$ satisfies Eq.3.2. From Table 1, we can see that $m = 2$. Therefore, we have

$$\mathcal{U}(z) = a_0 + a_1 U + a_2 U^2. \tag{5.16}$$

Substituting Eq.5.16 into Eq.5.15 with Eq.3.2 and equating coefficients of powers of U to zero, we obtain the following system of nonlinear equations:

$$\begin{aligned}
&U^0 : a_0 - a_0^2 + ck\sigma a_1 + k^2\gamma\sigma a_1 + 2k^2\sigma^2 a_2 = 0,\\
&U^1 : a_1 - 2a_0a_1 + ck\gamma a_1 + 2ck\sigma a_2 + 2k^2\rho\sigma a_1 +\\
&\quad k^2\gamma^2 a_1 + 6k^2\gamma\sigma a_2 = 0,\\
&U^2 : a_2 - 2a_0a_2 - a_1^2 + ck\rho a_1 + 2ck\gamma a_2 +\\
&\quad 3k^2\rho\gamma a_1 + 8k^2\rho\sigma a_2 + 4k^2\gamma^2 a_2 = 0,\\
&U^3 : -2a_1a_2 + 2ck\rho a_2 + 2k^2\rho^2 a_1 + 10k^2\rho\gamma a_2 = 0,\\
&U^4 : 6k^2\rho^2 a_2 - a_2^2 = 0.
\end{aligned} \tag{5.17}$$

In solving the above algebraic equations with the aid of Maple, we get

$$
\begin{aligned}
&a_0 = \frac{-3}{2}k^2\gamma^2 + \frac{1}{2}(6k\gamma \pm \sqrt{6})k\gamma + \frac{1}{4}, k = k, \\
&a_1 = (6k\gamma \pm \sqrt{6})\rho k, a_2 = 6k^2\rho^2, c = \pm\frac{5}{6}\sqrt{6}, \\
&\rho = \rho, \gamma = \gamma, \sigma = \frac{1}{24}\frac{6k^2\gamma^2 - 1}{k^2\rho}.
\end{aligned}
\tag{5.18}
$$

Hence, putting what we obtained in Eq.5.18 into Eq.5.16 we get

$$
\begin{aligned}
\mathcal{U}(z) =& \frac{-3}{2}k^2\gamma^2 + \frac{1}{2}(6k\gamma \pm \sqrt{6})k\gamma + \frac{1}{4} + \\
&(6k\gamma \pm \sqrt{6})\rho k U + 6k^2\rho^2 U^2.
\end{aligned}
\tag{5.19}
$$

By substituting Eq.3.12 into Eq.5.19, we obtain the final solution of the Fisher equation to be

$$
\begin{aligned}
u(x,t) =& \frac{1}{4} \pm \frac{k\sqrt{6}}{2}\sqrt{\gamma^2 - 4\rho\sigma}\tanh\left(-\frac{\sqrt{\gamma^2 - 4\rho\sigma}}{2}\left(kx \pm \frac{5k}{\sqrt{6}}t + A\right)\right) + \\
&\frac{3k^2\gamma^2}{2}\tanh^2\left(-\frac{\sqrt{\gamma^2 - 4\rho\sigma}}{2}\left(kx \pm \frac{5k}{\sqrt{6}}t + A\right)\right) - \\
&6k^2\sigma\rho\tanh^2\left(-\frac{\sqrt{\gamma^2 - 4\rho\sigma}}{2}\left(kx \pm \frac{5k}{\sqrt{6}}t + A\right)\right).
\end{aligned}
\tag{5.20}
$$

It is the travelling wave moving to the left or right if Eq.2.2 is satisfied. Hence, that can be achieved by considering different values of ρ, σ and γ. Here, we consider few cases:

- case 1: If $\rho = -1$, $\gamma = 0$, $\sigma = 1$, then

$$\begin{aligned} i)\ u(x,t) =& \frac{1}{4} - \frac{1}{2}\tanh\left(\pm\frac{1}{2\sqrt{6}}\left(x \pm \frac{5}{\sqrt{6}}t\right) + A\right) + \\ & \frac{1}{4}\tanh^2\left(\pm\frac{1}{2\sqrt{6}}\left(x \pm \frac{5}{\sqrt{6}}t\right) + A\right), \\ ii)\ u(x,t) =& \frac{1}{4} + \frac{1}{2}\tanh\left(\pm\frac{1}{2\sqrt{6}}\left(x \pm \frac{5}{\sqrt{6}}t\right) + A\right) + \\ & \frac{1}{4}\tanh^2\left(\pm\frac{1}{2\sqrt{6}}\left(x \pm \frac{5}{\sqrt{6}}t\right) + A\right). \end{aligned} \tag{5.21}$$

- case 2: On condition that $\rho = 1$, $\gamma = 0$, $\sigma = -1$ we get

$$\begin{aligned} i)\ u(x,t) =& \frac{1}{4} - \frac{1}{2}\tanh\left(\pm\frac{1}{2\sqrt{6}}\left(x \pm \frac{5}{\sqrt{6}}t\right) + A\right) + \\ & \frac{1}{4}\tanh^2\left(\pm\frac{1}{2\sqrt{6}}\left(x \pm \frac{5}{\sqrt{6}}t\right) + A\right), \\ ii)\ u(x,t) =& \frac{1}{4} + \frac{1}{2}\tanh\left(\pm\frac{1}{2\sqrt{6}}\left(x \pm \frac{5}{\sqrt{6}}t\right) + A\right) + \\ & \frac{1}{4}\tanh^2\left(\pm\frac{1}{2\sqrt{6}}\left(x \pm \frac{5}{\sqrt{6}}t\right) + A\right). \end{aligned} \tag{5.22}$$

- case 3: Given that $\rho = 4$, $\gamma = 3$, $\sigma = -1$, we get

$$\begin{aligned} i)\ u(x,t) =& \frac{1}{4} - \frac{1}{2}\tanh\left(\pm\frac{1}{2\sqrt{6}}\left(x \pm \frac{5}{\sqrt{6}}t\right) + A\right) + \\ & \frac{1}{4}\tanh^2\left(\pm\frac{1}{2\sqrt{6}}\left(x \pm \frac{5}{\sqrt{6}}t\right) + A\right), \\ ii)\ u(x,t) =& \frac{1}{4} + \frac{1}{2}\tanh\left(\pm\frac{1}{2\sqrt{6}}\left(x \pm \frac{5}{\sqrt{6}}t\right) + A\right) + \\ & \frac{1}{4}\tanh^2\left(\pm\frac{1}{2\sqrt{6}}\left(x \pm \frac{5}{\sqrt{6}}t\right) + A\right). \end{aligned} \tag{5.23}$$

- case 4: For $\rho = -1$, $\gamma = 1$, $\sigma = 0$, we obtain

$$\begin{aligned} i)\ u(x,t) =& \frac{1}{4} - \frac{1}{2}\tanh\left(\pm\frac{1}{2\sqrt{6}}\left(x \pm \frac{5}{\sqrt{6}}t\right) + A\right) + \\ & \frac{1}{4}\tanh^2\left(\pm\frac{1}{2\sqrt{6}}\left(x \pm \frac{5}{\sqrt{6}}t\right) + A\right), \\ ii)\ u(x,t) =& \frac{1}{4} + \frac{1}{2}\tanh\left(\pm\frac{1}{2\sqrt{6}}\left(x \pm \frac{5}{\sqrt{6}}t\right) + A\right) + \\ & \frac{1}{4}\tanh^2\left(\pm\frac{1}{2\sqrt{6}}\left(x \pm \frac{5}{\sqrt{6}}t\right) + A\right). \end{aligned} \tag{5.24}$$

In each case we got two travelling waves solutions, propagating in different directions. Therefore, Eq.5.20 becomes a travelling wave solution if

$$\gamma^2 - 4\rho\sigma > 0.$$

The four cases demonstrate that we get same result for $\gamma^2 - 4\rho\sigma > 0$.

5.3 Burgers-Fisher equation

The Burgers-Fisher equation is normally used to model fluid dynamics, number theory, heat conduction, elasticity and many more [27], [28]. The tanh method for generalized forms of Burgers-Fisher equations was presented by [28]. As an example, we provide solution of the generalized Burgers-Fisher equation by using Example 2. The generalized Burgers-Fisher equation is given by

$$\frac{\partial u}{\partial t} + u^2\frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x^2} = u(1-u^2). \tag{5.25}$$

Transforming to the z variable yields

$$ck\frac{d\mathcal{U}}{dz} - k\mathcal{U}^2\frac{d\mathcal{U}}{dz} + k^2\frac{d^2\mathcal{U}}{dz^2} + \mathcal{U} - \mathcal{U}^3 = 0. \tag{5.26}$$

After comparing the values of m using Eq.4.39 and Eq.4.40, we obtained m to be a fraction. Hence, we move to **Example 2** and Table 2 to find that $m = 1$ and $\alpha = 1/2$. Therefore, we can seek the solution of Eq.5.26 in the form

$$\mathcal{U}(U) = \sum_{i=0}^{1} a_i (U^{1/2})^i, \tag{5.27}$$

where $U = U(z)$ satisfies

$$\frac{dU}{dz} = U^2 - U.$$

Putting Eq.5.27 into Eq.5.26 together with Eq.3.3, collecting all terms with like powers U^j and $U^{1/j}$ and setting them to zero, we get the system of six equations,

$$\begin{aligned}
U^0 \,&: a_0 - a_0^3 = 0,\\
U^1 \,&: k a_0 a_1^2 - 3 a_0 a_1^2 = 0,\\
U^2 \,&: -k a_0 a_1^2 = 0,\\
U^{1/2} \,&: -2cka_1 + k^2 a_1 + 2k a_1 a_0^2 - 12 a_0^2 a_1 + 4a_1 = 0,\\
U^{3/2} \,&: 2cka_1 - 4k^2 a_1 - 2k a_1 a_0^2 + 2k a_1^3 - 4a_1^3 = 0,\\
U^{5/2} \,&: 3k^2 a_1 - 2k a_1^3 = 0.
\end{aligned} \tag{5.28}$$

Solving the above equations, we get the following set of solutions:

$$\begin{aligned}
&\left\{a_0 = 0, a_1 = 1, c = \frac{10}{3}, k = \frac{2}{3}\right\},\\
&\left\{a_0 = 0, a_1 = -1, c = \frac{10}{3}, k = \frac{2}{3}\right\}.
\end{aligned} \tag{5.29}$$

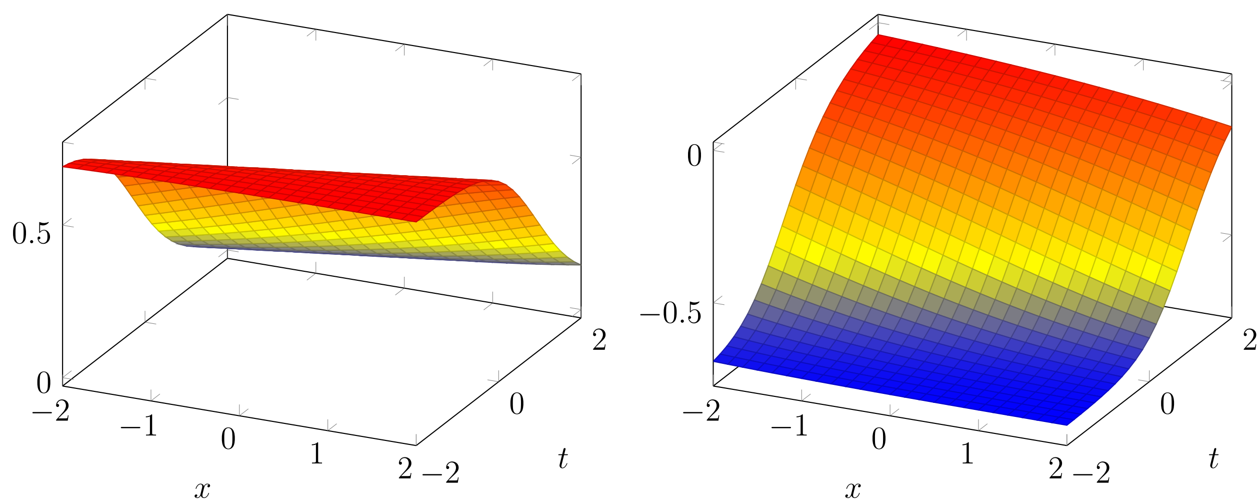
Substituting the above set into Eq.5.27, we get

$$\mathcal{U}(U) = \pm U^{1/2}. \tag{5.30}$$

Finally, after substituting Eq.3.12 into Eq.5.30, we get two travelling wave solutions:

$$\begin{aligned} u(x,t) &= -\frac{1}{2}\left(1+\tanh\left(\frac{x}{3}-\frac{10}{9}t\right)+\frac{A}{2}\right)^{1/2}, \\ u(x,t) &= \frac{1}{2}\left(1+\tanh\left(\frac{x}{3}-\frac{10}{9}t\right)+\frac{A}{2}\right)^{1/2}. \end{aligned} \tag{5.31}$$

Figure 5.5 shows the shape of two exact travelling solutions, in which both waves are travelling at the same speed but in opposite directions. The same results have been obtained by many researchers. In [28] work, they found m



(a) Burger Fisher solution, $u(x,t)=\frac{1}{2}\left(1+\tanh\left(\frac{x}{3}-\frac{10}{9}t\right)\right)^{1/2}$, $A=0$
.

(b) Burger Fisher solution, $u(x,t)=-\frac{1}{2}\left(1+\tanh\left(\frac{x}{3}-\frac{10}{9}t\right)\right)^{1/2}$, $A=0$
.

Figure 5.5: Burger Fisher solution profiles

to be a fraction. Hence, they utilize substitution to get rid of that fraction. Here, we demonstrate a simple, general and straightforward method that can be used to solve problems that involve m as a fraction. **Example 2** together with Table 1 and Table 2 proved to be an effective way of solving such problems without any tedious calculations. Our results are totally different to what [25] obtained. The modified $\tanh - \coth$ method utilized in [25] does not produce travelling wave solutions.

Chapter 6

Second approach of finding exact travelling wave solutions

In Chapter 2, we stated that travelling wave solutions occur between two equilibrium. We also assume that $P(U)$ is a quadratic polynomial. Here, we want to show that $P(U)$ can be $U\ln U - U$. Consider a first order autonomous equation

$$\frac{dU}{dz} = U\ln U - U \quad , \quad U > 0. \tag{6.1}$$

We then find the exact expression for the solution to Eq.6.1. An ODE is separable. By arranging and integrating both sides, we get

$$\int \frac{1}{U(\ln U - 1)} dU = \int dz + A_1. \tag{6.2}$$

Let $h = \ln U - 1$, so $dh = \frac{1}{U}dU$, implies

$$\int \frac{1}{h} dU = \int dz + A_1, \tag{6.3}$$

where A_1 is the integration constant. By using partial fractions, we get

$$\begin{aligned} \ln|\ln U - 1| &= z + A_1, \\ |\ln U - 1| &= e^{z+A_1}, \end{aligned} \tag{6.4}$$

$$|\ln U - 1| = \ln U - 1, \quad \text{if} \quad U > e. \tag{6.5}$$

Finally, we have

$$U(z) = e^{e^{z+A_1}+1}. \tag{6.6}$$

In this case, Eq.6.6 does not need to satisfy Eq.2.2 but Eq.2.2 needs to be satisfied when Eq.6.6 is substituted into the following assumption. Let us assume that the solution of an ODE is of the form:

$$\mathcal{U}(z) = a_0 + \sum_{i=1}^{m}\sum_{j=1}^{n} a_i b_j U^{-i+1}(\ln U)^{-j}, \tag{6.7}$$

where $U = U(z)$ satisfies the first order nonlinear ODE:

$$\frac{dU}{dz} = U\ln U - U \quad , \quad U > 0. \tag{6.8}$$

We then demonstrate the application of the method by solving two well known equations, the Fisher equation and Korteweg-de Vries equation.

6.1 Korteweg-de Vries equation

The Korteweg-de Vries equation is a third order nonlinear partial differential equation. It was derived from fluid mechanics to describe shallow water waves in a rectangular channel. The equation is written as

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + b\frac{\partial^3 u}{\partial x^3} = 0, \tag{6.9}$$

where constant $b > 0$. After transformation and integrating once, we obtain

$$-ck\mathcal{U} + k\frac{1}{2}\mathcal{U}^2 + bk^2\frac{d^2\mathcal{U}}{dt^2} - B_1 = 0, \tag{6.10}$$

where B_1 is the integration constant. After balancing exponents, we again have $m = 1$ and $n = 2$. Then, Eq.6.7 leads us to

$$\mathcal{U}(z) = a_0 + a_1b_1(\ln U)^{-1} + a_1b_2(\ln U)^{-2} \tag{6.11}$$

Substituting Eq.6.11 with Eq.6.6 into Eq.6.10 and collecting and equating coefficients of $U^{-i+1}(\ln U)^{-j}$ to zero, we obtain

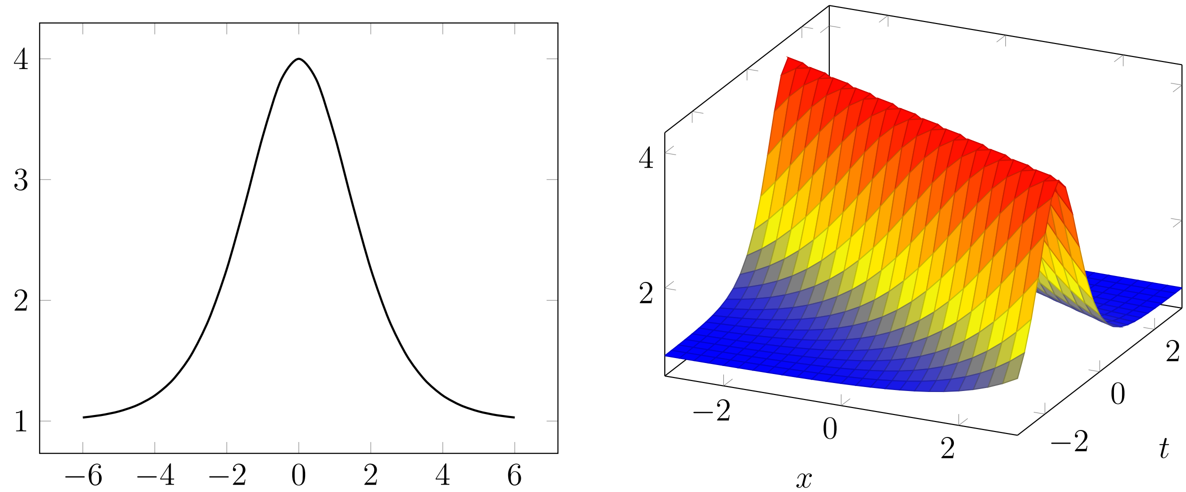
$$\begin{aligned}
U^0(\ln U)^0 &: -2cka_0 + ka_0^2 - 2B_1 = 0,\\
U^0(\ln U)^{-1} &: -cka_1b_1 + ka_0a_1b_1 + bk^2a_1b_1 = 0,\\
U^0(\ln U)^{-2} &: -2cka_1b_2 + 2ka_0a_1b_2 + ka_1^2b_1^2 - 6bk^2a_1b_1 + 8bk^2a_1b_2 = 0,\\
U^0(\ln U)^{-3} &: ka_1^2b_1b_2 + 2bk^2a_1b_1 - 10bk^2a_1b_2 = 0,\\
U^0(\ln U)^{-4} &: ka_1^2b_2^2 + 12bk^2a_1b_2 = 0.
\end{aligned} \tag{6.12}$$

Solving the above equations, we have

$$\begin{aligned}
&\Big\{B_1 = \frac{1}{2}b^2k^3 - \frac{1}{2}c^2k, a_0 = -bk + c, a_1 = -\frac{12bk}{b_2},\\
&b = b, b_1 = -b_2, b_2 = b_2, c = c, k = k\Big\}.
\end{aligned} \tag{6.13}$$

Hence, Eq.6.11 becomes

$$\begin{aligned}
\mathcal{U}(U(z)) &= (-bk + c) + 12bk(\ln U)^{-1} - 12bk(\ln U)^{-2},\\
\mathcal{U}(z) &= (-bk + c) + 12bk\left(\frac{e^{z+A_1}}{(e^{z+A_1} + 1)^2}\right)\\
&= (-bk + c) + 3bk\left(\frac{4}{e^{z+A} + 2 + e^{-z-A}}\right).
\end{aligned} \tag{6.14}$$



(a) $U(z) = 1 + \frac{12e^{z}}{(e^{z}+1)^2}$, $A_1 = 0$.

(b) $u(x,t) = 1 + \frac{12e^{x-2t}}{(e^{x-2t}+1)^2}$, $A = 0$.

Figure 6.1: Korteweg-de Vries equation profiles, where $A_1 = 0$, $c = 2$, $k = 1$, $b = 1$.

Comparing these results with the results we obtained in Chapter 2 (Eq.2.35), we have same results.

6.2 Fisher equation

Reconsider the Fisher equation,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(1-u), \tag{6.15}$$

After balancing the exponents, we get $m = 1$ and $n = 2$. Following all steps as in previous examples, we get the system of equations

$$\begin{aligned}
U^0(\ln U)^0 :& a_0 - a_0^2 = 0,\\
U^0(\ln U)^{-1} :& a_1b_1 - 2a_0a_1b_1 - cka_1b_1 + k^2a_1b_1 = 0,\\
U^0(\ln U)^{-2} :& a_1b_2 - 2a_0a_1b_2 - a_1^2b_1^2 + cka_1b_1 - 2cka_1b_2 - 3k^2a_1b_1 + 4k^2a_1b_2 = 0,\\
U^0(\ln U)^{-3} :& -2a_1^2b_1b_2 + 2cka_1b_2 + 2k^2a_1b_1 - 10k^2a_1b_2 = 0,\\
U^0(\ln U)^{-4} :& -a_1^2b_2^2 + 6k^2a_1b_2 = 0.
\end{aligned} \tag{6.16}$$

From the above equations, we get the following results,

$$\begin{aligned}
&\left\{a_0 = 0, a_1 = \frac{1}{b_2}, b_1 = 0, b_2 = b_2, c = \frac{5}{6}\sqrt{6}, k = \frac{1}{6}\sqrt{6}\right\},\\
&\left\{a_0 = 0, a_1 = \frac{1}{b_2}, b_1 = 0, b_2 = b_2, c = \frac{-5}{6}\sqrt{6}, k = -\frac{1}{6}\sqrt{6}\right\},\\
&\left\{a_0 = 1, a_1 = \frac{1}{b_2}, b_1 = -2b_2, b_2 = b_2, c = \frac{-5}{6}\sqrt{6}, k = \frac{1}{6}\sqrt{6}\right\},\\
&\left\{a_0 = 1, a_1 = \frac{1}{b_2}, b_1 = -2b_2, b_2 = b_2, c = \frac{5}{6}\sqrt{6}, k = -\frac{1}{6}\sqrt{6}\right\},
\end{aligned} \tag{6.17}$$

The the following exact travelling wave solutions are produced,

$$\begin{aligned}
u(x,t) &= \frac{1}{\left(e^{\frac{\sqrt{6}}{6}\left(x - \frac{5}{6}\sqrt{6}t\right) + A_1} + 1\right)^2},\\
u(x,t) &= \frac{1}{\left(e^{-\frac{\sqrt{6}}{6}\left(x + \frac{5}{6}\sqrt{6}t\right) + A_1} + 1\right)^2},
\end{aligned} \tag{6.18}$$

and

$$\begin{aligned}
u(x,t) &= 1 - \frac{2}{e^{\frac{\sqrt{6}}{6}\left(x - \frac{5}{6}\sqrt{6}t\right) + A_1} + 1} + \frac{1}{(e^{\frac{\sqrt{6}}{6}\left(x - \frac{5}{6}\sqrt{6}t\right) + A_1} + 1)^2},\\
u(x,t) &= 1 - \frac{2}{e^{\frac{-\sqrt{6}}{6}\left(x + \frac{5}{6}\sqrt{6}t\right) + A_1} + 1} + \frac{1}{(e^{-\frac{\sqrt{6}}{6}\left(x + \frac{5}{6}\sqrt{6}t\right) + A_1} + 1)^2}.
\end{aligned} \tag{6.19}$$

The presented solution of the Fisher equation solutions shows the same result as the previous method where $P(U)$ was treated as a second order polynomial. The intention of presenting the second approach was to emphasize that any function $P(U)$ (either a polynomial or not, but it must have more than one root) can be utilized to produce exact travelling wave solutions. That goal has been achieved; the Fisher equation and KdV equation produced travelling wave solutions in both cases. The application to established PDEs, demonstrated that the method technique is simple and capable of finding analytical travelling wave solutions for nonlinear evolution equations.

Chapter 7

Discussion

The general presented method of finding travelling wave solutions for nonlinear PDEs as linear combinations of functions satisfying certain assumptions provides a straightforward algorithm to work out nonlinear PDEs exact travelling wave solutions. It is shown that the methods like the G'/G, tanh-method and many more, are examples of th general method introduced in the dissertation. The main procedure of the method lies in the fact that $\mathcal{U}(z)$ must approach constant states, that is $\mathcal{U}(z \to -\infty) = u_1$ and $\mathcal{U}(z \to +\infty) = u_2$, which must be at equilibria of $P(U)$, hence $P(U)$ must have at least two roots.

Bibliography

[1] Logan J.D., *An introduction to nonlinear partial differential equations*, John Wiley & Sons, New Jersey and Canada, 2008, Vol.89, 2nd ed.

[2] Murray J.D., *Mathematical Biology*, Springer, Berlin, 1989, Vol.19 of Biomathematics.

[3] Whitham G.B., *Linear and Nonlinear Waves*, John Wiley & Sons, New York, 1974.

[4] Davidson R.C., *Methods in Nonlinear Plasma Theory*, New York, 1972.

[5] Toda M., Studies of a Nonlinear Lattice, Springer, Berlin, 1981.

[6] Gray P., Scott S., *Chemical Oscillations and Instabilities*, Oxford, 1990.

[7] Leung A.W., Hou X., Li, Yi., *Exclusive travelling waves for competitive reaction diffusion systems and their stabilities*, *J. Math. Anal. Appl.* 2008, Vol.338, No.2, pp.902-924.

[8] Wakano J., *A mathematical analysis on public goods games in the continuous space*, *Math. Biosci.* 2006, Vol.201, No.1-2, pp.72-89.

[9] Brown S., Dockery J., Pernarowski M., *Travelling wave solutions of a reaction diffusion model for competing pioneer and climax species. Math. Biosci* 2005, Vol.194, No.1, pp.21-36.

[10] Feltham D.L., Chaplain A.J., Mark A.J., *Travelling waves in a model of species migration*, *Appl. Math. Lett.* 2000, Vol.13, No.7, pp.67-73.

[11] Kudryashov N.A., *Exact Soliton solutions of the generalized evolution equation of wave dynamics*, *J. Apple. Math. Mech.* 1988, Vol.52, No.3, pp.361-365.

[12] Kan-on Y., *Parameter dependence of propagation speed of travelling waves for competition-diffusion equations*, *SIAM J. Math. Anal.* 1995, Vol.26, No.2, pp.340–363.

[13] Kan-on Y., Fang Q., *Stability of monotone travelling waves for competition-diffusion equations*, *Japan J. Indust. Appl. Math.*1996, Vol.13, No.2, pp.343–349.

[14] Y. Kan-on Y., *Existence of standing waves for competition-diffusion equations*, *Japan J. Indust. Appl. Math.* 1996, Vol.13, No.1, pp.117–133.

[15] Rodrigo M., Mimura M., *Exact solutions of a competition-diffusion system*, *Hiroshima Math. J.* 2000, Vol.30, No.2, pp.257-270.

[16] Banasiak J., *Models and methods in mathematical epidemiology*, Lecture notes, WTW850, University of Pretoria, 2019.

[17] Wang M., Li X., Zhang J., *The G'/G-expansion method and travelling wave solutions of nonlinear evolution equations in mathematical physics*, *Phys. Lett. A 372*, 2008, Vol.372, No.4, pp.417-423.

[18] Soliman A.A., *Numerical simulation of the generalized regularized long wave equation by He's variational iteration method*, *Math. Comput. Simulation*,2005, Vol.70, No. 2, pp.119–124.

[19] Rashidi M.M., Domairry G., Dinarvand S., *Approximate solutions for the Burger and regularized long wave equations by means of the homotopy analysis method*, *Communications in Nonlinear Science and Numerical Simulation*, 2009, Vol.14, No.3, pp.708–717.

[20] Gardner L.R.T., Gardner G.A, Dag I., *AB-spline finite element method for the regularized long wave equation*, *Comm. Numer.*, 1995, Vol.11, No.1, pp.59–68.

[21] Ozis T., Agırseven D., *He's homotopy perturbation method for solving heat-like and wave-like equations with variable coefficients*, *Phys. Lett. A*, 2008, Vol.372, No.38, pp.5944–595

[22] Guo S., Zhou Y., *The extended G'/G-expansion method and its applications to the Whitham–Broer–Kaup–Like equations and coupled Hirota–Satsuma*, *School of Mathematics and Statistics*, 2010, Vol.215, No.9, pp.3214-3221.

[23] Hereman W., Malfliet W., *The Tanh Method: A Tool to Solve Nonlinear Partial Differential Equations with Symbolic Software*, Proceedings

9th World Multi-Conference on Systemics, *Cybernetics and Informatics*, Orlando, FL, 1996.

[24] Özis T., Aslan I., *Application of the G'/G-expansion method to Kawahara type equations using symbolic computation*, *Applied Math. and computation*, 2010, Vol.216, No.8, pp.2360-2365.

[25] Wazzan L., *A modified tanh–coth method for solving the general Burgers–Fisher and the Kuramoto–Sivashinsky equations*, *Comm. in Nonlinear Science and Numerical Simulation*, 2009, Vol.14, No.6, pp.2642-2652.

[26] Wang Y., XU H., Sun Q., *New group of solutions to the Whitham-Broer-Kaup equation*, *Applied Mathematics and Mechanics*, 2020, Vol.41, No.11, pp.1735-1746.

[27] Ismail H.N.A., Raslan K., Rabboh A.A.A., *Adomian decomposition method for Burgers' Huxley and Burgers-Fisher equations*, *Appl.Math. Comput.*, 2004, Vol.159, No.1, pp.291-301.

[28] Wazwaz A.M., *The tanh method for generalized forms of nonlinear heat conduction and Burgers-Fisher equations*, *Appl. Math. Comput.*, 2005, Vol.169, No.1, pp.321-338.