CATEGORY MEASURES, THE DUAL OF $\mathrm{C}(K)^\delta$ AND HYPER-STONEAN SPACES

JAN HARM VAN DER WALT

Abstract. For a compact Hausdorff space K, we give descriptions of the dual of $\mathrm{C}(K)^\delta$, the Dedekind completion of the Banach lattice $\mathrm{C}(K)$ of continuous, real-valued functions on K. We characterize those functionals which are σ-order continuous and order continuous, respectively, in terms of Oxtoby's category measures. This leads to a purely topological characterization of hyper-Stonean spaces.

1. Introduction

It is well known that the Banach lattice $\mathrm{C}(K)$ of continuous, real-valued functions on a compact Hausdorff space K fails, in general, to be Dedekind complete. In fact, $\mathrm{C}(K)$ is Dedekind complete if and only if K is Stonean, i.e. the closure of every open set is open, see for instance [25]. A natural question is to describe the Dedekind completion of $\mathrm{C}(K)$; that is, the unique (up to linear lattice isometry) Dedekind complete Banach lattice containing $\mathrm{C}(K)$ as a majorizing, order dense sublattice. Several answers to this question have been given, see for instance [3, 8, 10, 14, 21, 22, 27] among others. A further question arises: what is the dual of $\mathrm{C}(K)^\delta$? As far as we are aware, no direct answer to this question has been stated explicitly in the literature.

We give two answers to this question; one in terms of Borel measures on the Gleason cover of K [13], and another in terms of measures on the category algebra of K [23, 24]. We use the latter description to obtain natural characterizations of strictly positive, σ-order continuous, and order continuous functionals on $\mathrm{C}(K)^\delta$. In particular, we show that $\mathrm{C}(K)^\delta$ admits a strictly positive, σ-order continuous functional if and only if K admits a category measure in the sense of Oxtoby [23]. Specialising to the case of a Stonean space K, we obtain a purely topological characterization of hyper-Stonean spaces. Although the class of hyper-Stonean spaces was introduced by Dixmier [11] in 1951, no such characterization has been given to date [7, page 197].

The paper is organised as follows. In Section 2 we introduce notation and recall definitions and results from the literature to be used throughout the rest of the paper. Section 3 contains two representations of $\mathrm{C}(K)^\delta$ which are amendable to the problem of characterizing the dual of this space; these characterizations of $(\mathrm{C}(K)^\delta)^*$ are given in Section 4. Strictly positive, σ-order continuous and order continuous functionals on $\mathrm{C}(K)^\delta$ are discussed in Section 5 which ends with a characterization of hyper-Stonean spaces.

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2. Preliminaries

2.1. **Vector and Banach lattices.** Let E be a real vector lattice. We assume that E is Archimedean; that is, $\inf\{\frac{1}{n}f \ : \ n \in \mathbb{N}\} = 0$ for every $f \geq 0$. Denote by E_+ the positive cone in E, $E_+ = \{f \in E \ : \ 0 \leq f\}$. For $f \in E$ the positive part, negative part and modulus of f is given by

$$f^+ = f \wedge 0, \ f^- = (-f) \wedge 0 \text{ and } |f| = f^+ + f^-,$$

respectively. For $D \subseteq E$ and $f \in E$ we write $D \downarrow f$ if D is downward directed and $\inf D = f$.

E is *Dedekind complete* if every subset of E which is bounded above (below) has a least upper bound (greatest lower bound). We call E *order separable* if every subset A of E contains a countable set with the same set of upper bounds as A.

Let F be a vector lattice subspace of E. F is *order dense* in E if for every $0 < f \in E$ there exists $g \in F$ so that $0 < g \leq f$. If for every $f \in E_+$ there exists $g \in F$ so that $f \leq g$ then F is *majorizing* in E. An *ideal* in E is a vector lattice subspace F of E so that if $f \in\in F$ and $|g| \leq |f|$ then $g \in F$. A σ-ideal in E is an ideal F in E so that $\sup A \in F$ whenever A is a countable subset of F and $\sup A$ exists in E. The *principal ideal* generated by $f \in E$ is $I_f = \{g \in E \ : \ |g| \leq \alpha |f| \textit{ for some } \alpha \in \mathbb{R}_+\}$.

An element e of E_+ is called an *order unit* if for every $f \in E_+$ there exits a real number $\alpha > 0$ so that $f \leq \alpha e$. A *weak order unit* is any $e \in E_+$ so that if $f \wedge e = 0$ then $f = 0$. Observe that $e \in E_+$ is an order unit if and only if $I_e = E$; e is a weak order unit if and only if I_e is order dense in E.

A linear functional φ on E is *order bounded* if it maps order bounded subsets of E to bounded subsets of $\mathbb{R}$, and *positive* if $\varphi(f) \geq 0$ whenever $f \in E_+$. Every positive functional on E is order bounded. Conversely, every order bounded functional is the difference of two positive functionals. The space $E^\sim$ of all order bounded functionals on E is a Dedekind complete vector lattice with respect to the ordering

$$\varphi \geq \psi \text{ if and only if } \varphi - \psi \text{ is positive.}$$

Therefore every $\varphi \in E^\sim$ can be expressed as the difference of positive functionals, $\varphi = \varphi^+ - \varphi^-$.

If E is Banach lattice then a functional φ on E is order bounded if and only if it is norm bounded. Therefore the norm dual E^* and order dual $E^\sim$ of E coincide and is itself a Banach lattice.

A functional $\varphi \in E^\sim$ is *σ-order continuous* if $\inf\{|\varphi(f_n)| \ : \ n \in \mathbb{N}\} = 0$ whenever $f_n \downarrow 0$ in E. If $\inf\{|\varphi(f)| \ : \ f \in D\} = 0$ for every $D \downarrow 0$ in E, we say that φ is *order continuous*. A functional φ is (σ-)order continuous if and only if $|\varphi|$ is (σ-)order continuous, if and only if both φ^+ and φ^- are (σ-)order continuous.

Lemma 2.1. *Let E be an Archimedean vector lattice and F an order dense, majorizing vector lattice subspace of E.*

(i) *If $\varphi \in E^\sim$ then φ is order continuous on E if and only if the restriction of φ to F is order continuous.*
(ii) *If $\varphi \in E^\sim$ is σ-order then the restriction of φ to F is σ-order continuous.*
(iii) *If $\varphi \in F^\sim$ is order continuous, then φ has a unique order continuous extension to E.*

Proof. We prove the results for positive functionals. The case of a general $\varphi \in E^\sim$ follows from the decomposition $\varphi = \varphi^+ - \varphi^-$.

Proof of (i). Let $D \subseteq F_+$. Because F is order dense in E, $D \downarrow 0$ in E if and only if $D \downarrow 0$ in F. Therefore, if φ is order continuous on E then its restriction to F is order continuous.

Conversely, assume that the restriction of φ to F is order continuous. Let $D \downarrow 0$ in E. For each $g \in D$ let $D_g = \{f \in F \ : \ g \leq f\}$, and let $D' = \bigcup\{D_g \ : \ g \in D\}$. The fact that F is majorizing in E guarantees that $D_g \neq \emptyset$ for each $g \in D$. We observe that $D \downarrow 0$ in E implies that D' is downward directed. In particular, if $f, h \in D'$ then $f \wedge h \in D'$. Because F is an order dense sublattice of E, $g = \inf D_g$ for every $g \in D$. Therefore $D' \downarrow 0$ in E, hence also in F. It follows from the positivity and order continuity of φ on F that $0 \leq \inf \varphi[D] \leq \inf \varphi[D'] = 0$. Therefore $\inf \varphi[D] = 0$.
Proof of (ii). The proof is similar to the proof of (i).
Proof of (iii). This is a special case of [2, Theorem 4.12]. □

2.2. $\mathrm{C}(K)$ **and its dual.** Let K be a compact Hausdorff space. $\mathrm{C}(K)$ is the real Banach lattice of continuous, real-valued functions on K; the algebraic operations and order relation are defined pointwise as always, and the norm on $\mathrm{C}(K)$ is $\|f\| = \max\{f(x)| \ : \ x \in K\}$. The symbols $\mathbf{0}$ and $\mathbf{1}$ denote the constant functions with values 0 and 1, respectively. For $A \subseteq K$, the characteristic function of A is denoted $\mathbf{1}_A$.

The space K satisfies the countable chain condition (ccc) if every pairwise disjoint collection of nonempty open sets in K is countable. $\mathrm{C}(K)$ is order separable if and only if K satisfies ccc [9, Theorem 10.3].

$\Sigma_K^{\mathcal{B}}$ denotes the σ-algebra of Borel sets in K and $\mathrm{M}(K)$ the space of regular signed Borel measures on K; that is, σ-additive, real valued regular measures on $\Sigma_K^{\mathcal{B}}$. As is well known, $\mathrm{M}(K)$ is a Banach lattice with respect to the total variation norm

$$\|\varphi\| = |\varphi|(K), \ \ \varphi \in \mathrm{M}(K)$$

and pointwise ordering. In fact, $\mathrm{M}(K)$ is isometrically lattice isomorphic to $\mathrm{C}(K)^*$. We will not make a notational distinction between a measure φ and the associated functional. A measure $\varphi \in \mathrm{M}(K)$ is called *normal* [7, 11] if the associated functional on $\mathrm{C}(K)$ is order continuous. We denote the set of normal measures on by $\mathrm{N}(K)$. Normal measures can be characterised as follows: $\varphi \in \mathrm{M}(K)$ is normal if and only if $|\varphi|(N) = 0$ for every meagre set $N \in \Sigma_K^{\mathcal{B}}$, see for instance [7, Theorem 7.4.7].

Recall that K is *Stonean* if the closure of every open set is open. As mentioned in the introduction, K is Stonean if and only if $\mathrm{C}(K)$ is Dedekind complete. If K is Stonean and, in addition, the union of the suppports of the normal measures on K is dense in K, we call K *hyper-Stonean*, see [11] and [7, Section 4.7]. We recall the following result which contains the main motivation for studying hyper-Stonean spaces, see for instance [7].

Theorem 2.2. *A space K is hyper-Stonean if and only if any one of the following conditions hold.*

(i) *For every $\mathbf{0} < f \in \mathrm{C}(K)$ there exists a positive order continuous functional φ on $\mathrm{C}(K)$ so that $\varphi(f) > 0$.*
(ii) $\mathrm{C}(K)$ *is isometrically isomorphic to the dual of a Banach space.*
(iii) $\mathrm{C}(K)$ *is isometrically lattice isomorphic to the dual of a Banach lattice.*
(iv) *The C^*-algebra of complex-valued continuous functions on K is a von Neumann algebra.*

2.3. **Category measures.** Recall that a subset A of K has the property of Baire if there exists an open set U and a meagre set N in K so that $A = U \Delta N$. Let

$$\begin{aligned} \mathcal{N}_K &= \{N \subset K \ : \ N \text{ is meagre}\}, \\ \Sigma_K^{\mathcal{C}} &= \{U \Delta N \ : \ U \text{ is open in } K, \ N \in \mathcal{N}\}. \end{aligned} \tag{2.1}$$

The collection $\Sigma_K^{\mathcal{C}}$ is a σ-algebra [24, Theorrem 4.3] and $\mathcal{N}_K$ is a σ-ideal in $\Sigma_K^{\mathcal{C}}$. In fact, $\mathcal{N}_K$ is a σ-ideal in the powerset of K. The quotient algebra $\Sigma_K^{\mathcal{C}}/\mathcal{N}_K$ is a complete Boolean algebra. In fact, $\Sigma_K^{\mathcal{C}}/\mathcal{N}_K$ is isomorphic to the algebra $\mathcal{R}_K$ if regular open subsets of K, see [21, 23].

A *category measure* on K is positive, σ-additive measure $\mu : \Sigma_K^{\mathcal{C}} \to \mathbb{R}$ with the property that for any $A \in \Sigma_K^{\mathcal{C}}$, $\mu(A) = 0$ if and only if $A \in \mathcal{N}_K$, see for instance [23]. There is a bijective correspondence between category measures on K and strictly positive, σ-additive measures on $\mathcal{R}_K$. Oxtoby [23, 24] investigated the problem of characterizing those topological spaces which admit a category measure. A solution to this problem was given by J. M. Ayerbe Toledano [26][1]. In order to formulate this result we recall the following definitions, see [4, 16, 26].

Definition 2.3. Let X be a nonempty set and $\mathcal{F}$ a collection of nonempty subsets of X. For any $n \in \mathbb{N}$ and $\bar{F} = \langle F_1, \ldots, F_n \rangle \in \mathcal{F}^n$ let

$$i(\bar{F}) = \max\{|J| \ : \ J \subseteq \{1, \ldots n\}, \ \bigcap_{j \in J} F_j \neq \emptyset\}.$$

The Kelley intersection number of $\mathcal{F}$ is defined as

$$k(\mathcal{F}) = \inf\{\frac{i(\bar{F})}{n} \ : \ n \in \mathbb{N}, \ \bar{F} \in \mathcal{F}^n\}.$$

Definition 2.4. Let K be a compact Hausdorff space and $\mathcal{T}$ the collection of nonempty, open subsets of K. We say that K satisfies property (***) if there exists a partition $\{\mathcal{T}_n \ : \ n \in \mathbb{N}\}$ of $\mathcal{T}$ such that the following conditions are satisfied for every $n \in \mathbb{N}$.

(i) $k(\mathcal{T}_n) > 0$.
(ii) If (U_m) is an increasing sequence of open sets so that $\bigcup\{U_m \ : \ m \in \mathbb{N}\} \in \mathcal{T}_n$ then there exists $m_0 \in \mathbb{N}$ so that $U_{m_0} \in \mathcal{T}_n$.
(iii) If $U \in \mathcal{T}_n$ and V is an open set so that $U \Delta V$ is meagre then $V \in \mathcal{T}_n$.

Ryll-Nardzewski obtained a necessary and sufficient condition of the existence of a σ-additive, strictly positive measure on a complete Boolean algebra, see [16, Adendum]. It is easy to see that property (***) is equivalent to $\mathcal{R}_K$ satisfying Ryll-Nardzewski's condition. The following theorem of Ayerbe Toledano [26] therefore follows immediately from Ryll-Nardzewski's result.

Theorem 2.5. *A compact Hausdorff space K admits a category measure if and only if it satisfies property (***).*

Remark 2.6. Cambern and Greim [6], see also [7], define a category measure to be a (not necessarily finite) positive Borel measure μ on a Stonean space K which satisfies the following conditions:

(i) μ is regular on closed sets of finite measure,
(ii) $\mu(N) = 0$ for every meagre Borel set N, and,

[1]We formulate the result for compact Hausdorff spaces, but it is valid for arbitrary Baire spaces.

(iii) for every nonempty clopen set A there exists a clopen set $A_0 \subseteq A$ so that $0 < \mu(A_0) < \infty$.

Such measures are called *perfect* in [5], and are distinct from the category measures discussed here.

3. Characterizations of $\mathrm{C}(K)^\delta$

3.1. **The Gleason cover and the Maeda-Ogasawara theorem.** Let K be a compact Hausdorff space. A seminal result of Gleason [13] associates, in a canonical way, with K a Stonean space G_K, its projective cover. In order to formulate this result we recall the following. If L is compact Hausdorff space, a continuous surjection $f : K \to L$ is called *irreducible* if $f[C] \neq L$ for every proper closed subset C of K.

Theorem 3.1. *Let K be a compact Hausdorff space. There exists a Stonean space G_K, unique up to homeomorphism, and an irreducible map $\pi_K : G_K \to K$.*

Because π_K is onto, the induced map $T_{\pi_K} : \mathrm{C}(K) \ni f \mapsto f \circ \pi_K \in \mathrm{C}(G_K)$ is an isometric linear lattice isomorphism onto a closed vector lattice subspace of $\mathrm{C}(G_K)$. Moreover, since π_K is irreducible, $\pi_K^*[\mathrm{C}(K)]$ is a order dense sublattice of $\mathrm{C}(G_K)$. Clearly, $\pi_K^*[\mathrm{C}(K)]$ is majorising in $\mathrm{C}(G_K)$, seeing as it contains the order unit $\mathbf{1}_{G_K}$ of $\mathrm{C}(G_K)$. Lastly we note that, G_K being Stonean, $\mathrm{C}(G_K)$ is Dedekind complete. Hence we have the following.

Theorem 3.2. *Let K be a compact Hausdorff space. Then $\mathrm{C}(K)^\delta$ is isometrically lattice isomorphic to $\mathrm{C}(G_K)$.*

Let $\bar{\mathbb{R}}$ be the two-point compactification of $\mathbb{R}$. Denote by $\mathrm{C}^\infty(K)$ the space of continuous functions $f : K \to \bar{\mathbb{R}}$ so that $f^{-1}[\mathbb{R}]$ is dense (hence open and dense) in K. If K is a Stonean space then $\mathrm{C}^\infty(K)$ is a universally complete vector lattice, see for instance [1, Theorem 7.27], and $\mathrm{C}(K)$ is the ideal in $\mathrm{C}^\infty(K)$ generated by $\mathbf{1}$.

A classical result in the representation theory for vector lattices is due to Maeda and Ogasawara [20], see for instance [1] for a more recent presentation.

Theorem 3.3. *Let E be an Archimedean vector lattice with weak order unit e. There exists a Stonean space Ω_E and a linear lattice isomorphism $T : E \to \mathrm{C}^\infty(\Omega_E)$ onto an order dense sublattice of $\mathrm{C}^\infty(\Omega_E)$ so that $Te = \mathbf{1}$.*

In Theorem 3.3, let $E = \mathrm{C}(K)$. Then $T(\mathbf{1}_K) = \mathbf{1}_{\Omega_E}$. For $f \in \mathrm{C}^\infty(\Omega_E)$, $f \in \mathrm{C}(\Omega_E)$ if and only if there exists $c > 0$ so that $|f| \leq c\mathbf{1}_{\Omega_E}$. Therefore $T[\mathrm{C}(K)]$ is an order dense and majorizing vector lattice subspace of $\mathrm{C}^\infty(\Omega_E)$. Hence we have the following.

Theorem 3.4. *Let K be a compact Hausdorff space. Then $\mathrm{C}(K)^\delta$ is isometrically lattice isomorphic to $\mathrm{C}(\Omega_{\mathrm{C}(K)})$.*

A compact Hausdorff space is uniquely determined, up to homeomorphism, by its lattice of real-valued continuous functions. The Dedekind completion of a vector lattice is unique up to a linear lattice isomorphism. Therefore $\Omega_{\mathrm{C}(K)}$ and G_K are homeomorphic. This can be seen directly by recalling the constructions of $\Omega_{\mathrm{C}(K)}$ and G_K, respectively. G_K may be constructed as the Stone space of $\mathcal{R}_K$. On the other hand, $\Omega_{\mathrm{C}(K)}$ is the Stone space of the Boolean algebra of bands in $\mathrm{C}(K)$ which is isomorphic to $\mathcal{R}_K$.

3.2. **Category measurable functions.** Denote by $\mathrm{B}(\Sigma_K^{\mathcal{C}})$ the Archimedean vector lattice consisting of all real-valued, bounded and $\Sigma_K^{\mathcal{C}}$-measurable functions on K. The subset

$$\mathrm{N}(\Sigma_K^{\mathcal{C}}, \mathcal{N}_K) = \{f \in \mathrm{B}(\Sigma_K^{\mathcal{C}}) \ : \ K \setminus f^{-1}[\{0\}] \in \mathcal{N}_K\}$$

of $\mathrm{B}(\Sigma_K^{\mathcal{C}})$ is a σ-ideal in $\mathrm{B}(\Sigma_K^{\mathcal{C}})$. Therefore

$$\mathrm{L}^\infty(\Sigma_K^{\mathcal{C}}, \mathcal{N}_K) = \mathrm{B}(\Sigma_K^{\mathcal{C}})/\mathrm{N}(\Sigma_K^{\mathcal{C}}, \mathcal{N}_K)$$

is an Archimedean vector lattice [19, Theorem 60.3]. For each $f \in \mathrm{B}(\Sigma_K^{\mathcal{C}})$ we denote by $\hat{f}$ the equivalence class in $\mathrm{L}^\infty(\Sigma_K^{\mathcal{C}}, \mathcal{N}_K)$ generated by f. We note that for $\hat{f}, \hat{g} \in \mathrm{L}^\infty(\Sigma_K^{\mathcal{C}}, \mathcal{N}_K)$,

$$\hat{f} \leq \hat{g} \text{ if and only if } \{x \in K \ : \ f(x) > g(x)\} \in \mathcal{N}_K. \tag{3.1}$$

For $\hat{f} \in \mathrm{L}^\infty(\Sigma_K^{\mathcal{C}}, \mathcal{N}_K)$ set

$$\|\hat{f}\|_\infty = \inf\{c \geq 0 \ : \ |\hat{f}| \leq c\hat{\mathbf{1}}\}.$$

Note that for $\hat{f} \in \mathrm{L}^\infty(\Sigma_K^{\mathcal{C}}, \mathcal{N}_K)$,

$$\|\hat{f}\|_\infty = \inf\{c \geq 0 \ : \ \{x \in K \ : \ |f(x)| > c\} \in \mathcal{N}_K\}.$$

With respect to this norm the vector lattice $\mathrm{L}^\infty(\Sigma_K^{\mathcal{C}}, \mathcal{N}_K)$ is a Banach lattice. Because $\Sigma_K^{\mathcal{C}}/\mathcal{N}_K$ is complete, $\mathrm{L}^\infty(\Sigma_K^{\mathcal{C}}, \mathcal{N}_K)$ is Dedekind complete, see for instance [12, 363M & 363N]. We remark that it is possible to prove this result directly. Every equivalence class in $\mathrm{L}^\infty(\Sigma_K^{\mathcal{C}}, \mathcal{N}_K)$ contains a bounded lower semi-continuous function, and, every bounded lower semi-continuous function belongs to $\mathrm{B}(\Sigma_K^{\mathcal{C}})$. Dedekind completeness of $\mathrm{L}^\infty(\Sigma_K^{\mathcal{C}}, \mathcal{N}_K)$ then follows from the fact that the pointwise supremum of any family of lower semi-continuous functions is again lower semi-continuous.

Theorem 3.5. *Let K be a compact Hausdorff space. Then $\mathrm{C}(K)^\delta$ is isometrically lattice isomorphic to $\mathrm{L}^\infty(\Sigma_K^{\mathcal{C}}, \mathcal{N}_K)$.*

Proof. $\mathrm{C}(K)$ is vector lattice subspace of $\mathrm{B}(\Sigma_K^{\mathcal{C}})$. Furthermore, for $\mathrm{C}(K) \cap \mathrm{N}(\Sigma_K^{\mathcal{C}}, \mathcal{N}_K) = \{\mathbf{0}\}$. Therefore the mapping $T : \mathrm{C}(K) \ni u \mapsto \hat{u} \in \mathrm{L}^\infty(\Sigma_K^{\mathcal{C}}, \mathcal{N}_K)$ is a linear lattice isomorphism onto a vector lattice subspace of $\mathrm{L}^\infty(\Sigma_K^{\mathcal{C}}, \mathcal{N}_K)$. Clearly T is an isometry and $T[\mathrm{C}(K)]$ is majorising in $\mathrm{L}^\infty(\Sigma_K^{\mathcal{C}}, \mathcal{N}_K)$.

Since $\mathrm{L}^\infty(\Sigma_K^{\mathcal{C}}, \mathcal{N}_K)$ is a Dedekind complete Banach lattice, it remains only to verify that $T[\mathrm{C}(K)]$ is order dense in $\mathrm{L}^\infty(\Sigma_K^{\mathcal{C}}, \mathcal{N}_K)$. To this end, consider $\hat{\mathbf{0}} < \hat{u} \in \mathrm{L}^\infty(\Sigma_K^{\mathcal{C}}, \mathcal{N}_K)$. There exists a real number $\epsilon > 0$, an open set U in K and $N \in \mathcal{N}_K$ so that $\epsilon < u(x)$ for every $x \in U \Delta N$. According to (3.1) there exists $M \in \mathcal{N}_K$ so that $0 \leq u(x)$ for all $x \in K \setminus M$. Consider a function $v \in \mathrm{C}(K)$ so that $\mathbf{0} < v \leq \epsilon \mathbf{1}_U$. Fix $x \in K \setminus (M \cup N)$. If $x \in U$ then $v(x) \leq \epsilon < u(x)$. If $x \in K \setminus U$ then $v(x) = 0 \leq u(x)$. Therefore $v(x) \leq u(x)$ for all $x \in K \setminus (M \cup N)$ so that $Tv = \hat{v} \leq \hat{u}$ by (3.1). □

4. The dual of $\mathrm{C}(K)^\delta$

From the results discussed in Section 3, in particular Theorems 3.4 and 3.5, we obtain immediately two characterizations of the dual of $\mathrm{C}(K)^\delta$. The first follows directly from the Riesz Representation Theorem for compact Hausdorff spaces.

Theorem 4.1. *Let K be a compact Hausdorff space. Then $(\mathrm{C}(K)^\delta)^*$ is isometrically lattice isomorphic to $\mathrm{M}(G_K)$.*

Theorem 3.5 yields a second characterization of $(\mathrm{C}(K)^\delta)^*$. Denote by $\mathrm{M}(\Sigma_K^{\mathcal{C}},\mathcal{N}_K)$ the set of bounded, finitely additive signed measures μ on $\Sigma_K^{\mathcal{C}}$ with the property that $\mu(N)=0$ for every $N\in\mathcal{N}_K$. The space $\mathrm{M}(\Sigma_K^{\mathcal{C}},\mathcal{N}_K)$ is a vector lattice [28, Section 1]. With respect to the total variation norm

$$\|\varphi\| = |\varphi|(K),\ \varphi\in \mathrm{M}(\Sigma_K^{\mathcal{C}},\mathcal{N}_K)$$

$\mathrm{M}(\Sigma_K^{\mathcal{C}},\mathcal{N}_K)$ is a Banach lattice, see for instance [18]. We now have the following.

Theorem 4.2. *Let K be a compact Hausdorff space. Then $(\mathrm{C}(K)^\delta)^*$ is isometrically lattice isomorphic to $\mathrm{M}(\Sigma_K^{\mathcal{C}},\mathcal{N}_K)$. In particular, for every $\varphi\in(\mathrm{C}(K)^\delta)^*$ there exists a unique $\mu_\varphi\in \mathrm{M}(\Sigma_K^{\mathcal{C}},\mathcal{N}_K)$ so that*

$$\varphi(\hat{u}) = \int_K u d\mu_\varphi,\ \hat{u}\in \mathrm{C}(K)^\delta,$$

and the mapping $S:(\mathrm{C}(K)^\delta)^*\ni\varphi\mapsto\mu_\varphi\in\mathrm{M}(\Sigma_K^{\mathcal{C}},\mathcal{N}_K)$ is a linear lattice isometry onto $\mathrm{M}(\Sigma_K^{\mathcal{C}},\mathcal{N}_K)$.

Proof. It follows immediately from [28, Theorem 2.3] that S is a linear isometry onto $\mathrm{M}(\Sigma_K^{\mathcal{C}},\mathcal{N}_K)$. That both S and its inverse are positive operators follows immediately from the construction of μ_φ from $\varphi\in(\mathrm{C}(K)^\delta)^*$ and the definition of the integral. Indeed, for $\varphi\in(\mathrm{C}(K)^\delta)^*_+$ and $\mu\in\mathrm{M}(\Sigma_K^{\mathcal{C}},\mathcal{N}_K)_+$

$$\mu_\varphi(B)=\varphi(\hat{\mathbf{1}}_B),\ B\in\Sigma_K^{\mathcal{C}}$$

and

$$\int_K f d\mu = \sup_{s\in S_f}\int_K s d\mu,\ f\in \mathrm{B}(\Sigma_K^{\mathcal{C}})_+$$

where S_f consists of all simple, positive functions dominated by f, see [18] and [28] for the details. □

Recall that $\Sigma_K^{\mathcal{C}}/\mathcal{N}_K$ is a complete Boolean algebra, isomorphic to $\mathcal{R}_K$. For $B\in\Sigma_K^{\mathcal{C}}$ let $\hat{B}$ denote the equivalence class in $\Sigma_K^{\mathcal{C}}/\mathcal{N}_K$ containing B. For $B_0,B_1\in\Sigma_K^{\mathcal{C}}$, $\hat{B}_0=\hat{B}_1$ if and only if $B_0\Delta B_1\in\mathcal{N}_K$. Denote by $\mathrm{M}(\mathcal{R}_K)$ the space of bounded finitely additive measures on $\mathcal{R}_K$. This space is a Banach lattice with respect to the pointwise order and variation norm, see for instance [12, 326Y (j)].

Let $\varphi\in\mathrm{M}(\Sigma_K^{\mathcal{C}},\mathcal{N}_K)$. Since $\varphi[\mathcal{N}_K]=\{0\}$, φ induces a (signed) finitely additive measure μ_φ on $\mathcal{R}_K$,

$$\mu_\varphi(\hat{B})=\varphi(B),\ B\in\Sigma_K^{\mathcal{C}}.$$

Conversely, every finitely additive measure μ on $\mathcal{R}_K$ induces a measure $\varphi^\mu\in\mathrm{M}(\Sigma_K^{\mathcal{C}},\mathcal{N}_K)$,

$$\varphi^\mu(B)=\mu(\hat{B}),\ B\in\Sigma_K^{\mathcal{C}}.$$

The maps $\mathrm{M}(\Sigma_K^{\mathcal{C}},\mathcal{N}_K)\ni\varphi\mapsto\mu_\varphi\in\mathrm{M}(\mathcal{R}_K)$ and $\mathrm{M}(\mathcal{R}_K)\ni\mu\mapsto\varphi^\mu\in\mathrm{M}(\Sigma_K^{\mathcal{C}},\mathcal{N}_K)$ are positive, linear isometries, and each is the inverse of the other. Therefore we have the following.

Corollary 4.3. *Let K be a compact Hausdorff space. The following statements are true.*

(i) *The following spaces are pairwise isometrically lattice isomorphic: $(\mathrm{C}(K)^\delta)^*$, $\mathrm{M}(\mathcal{R}_K)$, $\mathrm{M}(\Sigma_K^{\mathcal{C}},\mathcal{N}_K)$ and $\mathrm{M}(G_K)$.*

(iv) *If K is Stonean then* $\mathrm{C}(K)^*$, $\mathrm{M}(\mathcal{R}_K)$ *and* $\mathrm{M}(K)$ *pairwise isometrically lattice isomorphic.*

All this is well (but perhaps not widely) known, see for instance [12, Chapter 32]. That $\mathrm{M}(\Sigma_K^{\mathcal{C}}, \mathcal{N}_K)$ and $\mathrm{M}(G_K)$ are isometrically lattice isomorphic can also be derived from [28, Paragraph 4.5], and [12, 362A] may be used to show that $\mathrm{M}(\mathcal{R}_K)$ is isometrically lattice isomorphic to $\mathrm{M}(G_K)$.

For the remainder of the paper we will be primarily concerned with the representation of $(\mathrm{C}(K)^\delta)^*$ given in Theorem 4.2.

5. Order continuous elements of $(\mathrm{C}(K)^\delta)^*$ and hyper-Stonean spaces

The first result of this section is a characterisation of σ-order continuous functionals on $\mathrm{C}(K)^\delta$. In contrast with the classical representation theorem for functionals on $\mathrm{C}(K)$ in terms of Borel measures, there is an exact correspondence between countably additivity of a measure and σ-order continuity of the corresponding functional.

Theorem 5.1. *A measure* $\varphi \in (\mathrm{C}(K)^\delta)^*$ *is* σ*-order continuous if and only if it is countably additive on* $\Sigma_K^{\mathcal{C}}$*.*

Proof. Both the countably additive elements of $\mathrm{M}(\Sigma_K^{\mathcal{C}}, \mathcal{N}_K)$ and the σ-order continuous elements in $(\mathrm{C}(K)^\delta)^*$ are ideals in the respective ambient spaces. Therefore we may assume that $\varphi \geq 0$.

Assume that φ is σ-order continuous. Let (C_n) be a decreasing (with respect to inclusion) sequence of sets in Σ_C so that $\bigcap\{C_n \ : \ n \in \mathbb{N}\} = \emptyset$. For each $n \in \mathbb{N}$ let $f_n = \mathbf{1}_{C_n}$. Then $\hat{f}_n \downarrow \hat{\mathbf{0}}$ in $\mathrm{L}^\infty(\Sigma_K^{\mathcal{C}}, \mathcal{N}_K)$. Therefore

$$\varphi(C_n) = \int_K \hat{f}_n d\varphi \longrightarrow 0$$

so that φ is countably additive.

Conversely, assume that φ is countably additive. Consider a sequence $\hat{f}_n \downarrow \hat{\mathbf{0}}$ in $\mathrm{L}^\infty(\Sigma_K^{\mathcal{C}}, \mathcal{N}_K)$. Replacing each f_n with $(f_1 \wedge \ldots \wedge f_n) \vee \mathbf{0}$ if necessary and using (3.1) we may assume that the sequence (f_n) in $\mathrm{B}(\Sigma_K^{\mathcal{C}})$ is pointwise decreasing and bounded below by 0. Let $f : K \ni x \mapsto \inf_{n\in\mathbb{N}} f_n(x) \in \mathbb{R}$. Then f is $\Sigma_K^{\mathcal{C}}$-measurable and bounded on K; that is, $f \in \mathrm{B}(\Sigma_K^{\mathcal{C}})$. But $\hat{\mathbf{0}} \leq \hat{f} \leq \hat{f}_n$ for all $n \in N$. Therefore $\hat{f} = \hat{\mathbf{0}}$ so that $f^{-1}[\mathbb{R}_+] \in \mathcal{N}$. By the Lebesgue Dominated Convergence Theorem,

$$\varphi(\hat{f}_n) = \int_K f_n d\varphi \longrightarrow \int_K f d\varphi = 0.$$

so that φ is σ-order continuous. □

For a measure $\varphi \in \mathrm{M}(\Sigma_K^{\mathcal{C}}, \mathcal{N}_K)_+$ we define regularity in the same way as for Borel measures. That is, φ is *regular* if for every $B \in \Sigma_K^{\mathcal{B}}$,

$$\sup\{\mu(C) \ : \ C \subseteq B \text{ is compact}\} = \mu(B) = \inf\{\mu(U) \ : \ U \supseteq B \text{ is open}\}.$$

We note that each of the identities above implies the other. A general measure $\varphi \in \mathrm{M}(\Sigma_K^{\mathcal{C}}, \mathcal{N}_K)$ is regular if $|\varphi|$ is regular.

Theorem 5.2. *A measure* $\varphi \in (\mathrm{C}(K)^\delta)^*$ *is order continuous if and only if it is countably additive and regular on* $\Sigma_K^{\mathcal{C}}$*.*

Proof. Let $\varphi \geq 0$. Assume that φ is order continuous. By Theorem 5.1, φ is countably additive. Let $A \in \Sigma_K^{\mathcal{C}}$. Then $\hat{\mathbf{1}}_A \in \mathrm{L}^\infty(\Sigma_K^{\mathcal{C}}, \mathcal{N}_K)$. Fix $\epsilon > 0$ and let $D_\epsilon = \{\hat{f} \in \mathrm{L}^\infty(\Sigma_K^{\mathcal{C}}, \mathcal{N}_K) \ : \ f \in \mathrm{C}(K),\ (1+\epsilon/2)\mathbf{1}_A \leq f\}$. According to Theorem 3.5, $D_\epsilon \downarrow (1+\epsilon/2)\hat{\mathbf{1}}_A$ in $\mathrm{L}^\infty(\Sigma_K^{\mathcal{C}}, \mathcal{N}_K)$. By order continuity of φ,

$$(1+\epsilon/2)\varphi(A) = \int (1+\epsilon/2)\mathbf{1}_A d\phi = \inf_{\hat{f}\in D_\epsilon} \int_K f d\varphi.$$

Therefore there exists $\hat{f} \in D_\epsilon$ so that

$$\int_K f d\varphi < (1+\epsilon)\varphi(A).$$

Let $U = f^{-1}[(1,\infty)]$. Then $U \supseteq A$ is open and

$$\varphi(U) = \int_K \mathbf{1}_U d\varphi \leq \int_K f d\varphi < (1+\epsilon)\varphi(A).$$

Therefore $\varphi(A) \leq \inf\{\varphi(U) \ : \ U \supseteq A \text{ open}\} < (1+\epsilon)\varphi(A)$. This holds for all $\epsilon > 0$ so that $\varphi(A) = \inf\{\varphi(U) \ : \ U \supseteq A \text{ open}\}$. Therefore φ is outer regular, hence regular.

Conversely, assume that φ is countably additive and regular on $\Sigma_K^{\mathcal{C}}$. Then the restriction φ_0 of φ to $\Sigma_K^{\mathcal{B}}$ is countably additive and regular, and $\varphi_0(N) = 0$ for every meagre Borel set N. Therefore φ_0 is a normal Borel measure on K so that the restriction of the functional φ to $\mathrm{C}(K)$ is order continuous. By Lemma 2.1 (i), φ is order continuous on $\mathrm{C}(K)^\delta$. □

Proposition 5.3. *Let $\varphi \in (\mathrm{C}(K)^\delta)^*_+$. Then φ is strictly positive if and only if, for every every $A \in \Sigma_K^{\mathcal{C}}$, $\varphi(A) = 0$ if and only if $A \in \mathcal{N}_K$.*

Proof. Assume that φ is strictly positive. Let $A \in \Sigma_K^{\mathcal{C}} \setminus \mathcal{N}_K$. Then $\hat{\mathbf{0}} < \hat{\mathbf{1}}_A$ so that $\varphi(A) = \varphi(\hat{\mathbf{1}}_A) > 0$.

Assume that for every $A \in \Sigma_K^{\mathcal{C}}$, $\varphi(A) = 0$ if and only if $A \in \mathcal{N}_K$. Consider any $\hat{\mathbf{0}} < \hat{f} \in \mathrm{L}^\infty(\Sigma_K^{\mathcal{C}}, \mathcal{N}_K)$. Then there exists an $\epsilon > 0$ so that $A = f^{-1}[[\epsilon,\infty]] \in \Sigma_K^{\mathcal{C}} \setminus \mathcal{N}_K$. Consequently,

$$\varphi(\hat{f}) \geq \int_K \mathbf{1}_A d\varphi = \varphi(A) > 0.$$

Therefore φ is strictly positive. □

Theorems 2.5, 5.1 and 5.2, and Proposition 5.3 now yields the following result, establishing the relationship between order continuous functionals and category measures.

Corollary 5.4. *Let K be a compact Hausdorff space. Then the following statements are equivalent.*

(i) *K satisfies property (***).*
(ii) *K admits a category measure.*
(iii) *$\mathrm{C}(K)^\delta$ admits a strictly positive σ-order continuous linear functional.*
(iv) *$\mathrm{C}(K)$ admits a strictly positive σ-order continuous linear functional.*
(v) *$\mathrm{C}(K)$ admits a strictly positive order continuous linear functional.*
(vi) *$\mathrm{C}(K)^\delta$ admits a strictly positive order continuous linear functional.*

Proof. That (i) and (ii) are equivalent is Theorem 2.5. The equivalence of (ii) and (iii) follows immediately from the definition of a category measure, Theorem 5.1 and Proposition 5.3. That (iii) implies (iv) follows from Lemma 2.1 (ii).

To see that (iv) implies (v), assume that $\mathrm{C}(K)$ admits a strictly positive σ-order continuous linear functional φ. Then there exists a fully supported regular Borel measure on K, hence K satisfies cc, see for instance [7, Proposition 4.1.6]. Then $\mathrm{C}(K)$ is order separable so that φ is order continuous.

The equivalence of (v) and (vi) follows from Lemma 2.1 (i) and (ii). Lastly, that (vi) implies (iii) is obvious, which complete the proof. □

Remark 5.5. For the equivalence of (iii) to (vi) in Corollary 5.4 the assumption of strict positivity is essential. Indeed, there exists a compact Hausdorff space K and a σ-order continuous functional on $\mathrm{C}(K)$ which is not order continuous [7, Example 4.7.16]. Furthermore, the statement 'there exists a Stonean space K and a σ-order continuous functional on $\mathrm{C}(K)$ which is not order continuous' is equivalent to the existence of a measurable cardinal, see [17].

As an application of Corollary 5.4 we obtain the following topological characterization of hyper-Stonean spaces.

Theorem 5.6. *Let K be a Stonean space. Then K is hyper-Stonean if and only if there exists a collection $\mathcal{U}$ of clopen subsets of K so that $\bigcup\mathcal{U}$ is dense in K and every $U\in\mathcal{U}$ satisfies property (***).*

Proof. Assume that K is hyper-Stonean. For each normal measure μ on K, let S_μ denote the support of μ. Let $\mathcal{U}=\{S_\mu \ : \ \mu\in\mathrm{N}(K)\}$. By definition of a hyper-Stonean space $\bigcup\mathcal{U}$ is dense in K. Each $S_\mu\in\mathcal{U}$ is clopen, hence itself Stonean, and μ defines a strictly positive order continuous functional on $\mathrm{C}(S_\mu)$. Corollary 5.4 implies that each S_μ satisfies property (***).

Assume that there exists a collection $\mathcal{U}$ of clopen subsets of K so that $\bigcup\mathcal{U}$ is dense in K and every $U\in\mathcal{U}$ satisfies property (***). We claim that each $U\in\mathcal{U}$ is the support of a normal measure on K. Fix $U\in\mathcal{U}$. By Corollary 5.4 there exists a strictly positive order continuous functional φ on $\mathrm{C}(U)$. Consider the linear functional $\psi:\mathrm{C}(K)\ni f\mapsto\psi(f|U)\in\mathbb{R}$. Because U is clopen, hence regular closed in K, ψ is order continuous by [15, Theorem 3.4]. Therefore there exists a unique normal measure μ on K so that

$$\psi(u)=\int_K f d\mu,\ f\in\mathrm{C}(K).$$

For every $f\in\mathrm{C}(K)_+$, $f|S_\mu=\mathbf{0}$ if and only if $\psi(f)=0$, if and only if $f|U=\mathbf{0}$. Therefore $U=S_\mu$, which verifies our claim.

Since $\bigcup\mathcal{U}$ is dense in K, the union of the supports of the normal measures on K is dense in K; that is, K is hyper-Stonean. □

Remark 5.7. It is possible to obtain our main results, namely Theorem 4.2, from which the results in Section 5 follow in a less direct manner. We briefly recall how this may be achieved.

In [8], de Jonge and van Rooij give a construction of $\mathrm{C}(K)^\delta$ in terms of Borel measurable functions which is very similar to that given in Theorem 3.5. If B denotes the vector lattice of bounded Borel measurable functions on K and N the subspace of B consisting of these functions which vanish of a meagre Borel set, then $\mathrm{C}(K)^\delta$ can be identified with $\mathrm{D}(K)=\mathrm{B}/\mathrm{N}$. Dales et al. [7] call the space $\mathrm{D}(K)$

the Dixmier algebra of K. Using this construction our results can be obtained via the machinery of measures on Boolean algebras as set out, for instance, in [12]. We have opted for a direct and more transparent approach.

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Department of Mathematics and Applied Mathematics, University of Pretoria, Corner of Lynnwood Road and Roper Street, Hatfield 0083, Pretoria, South Africa