A simplified proof of CLT for convex bodies

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Abstract

We present a short proof of Klartag's central limit theorem for convex bodies, using only the most classical facts about log-concave functions. An appendix is included where we give the proof that thin shell implies CLT. The paper is accessible to anyone.

1 Introduction

The central limit theorem for convex bodies (Theorem 1 below) was conjectured by Brehm and Voigt [3] and independently (at about the same time) by Anttila, Ball and Perissinaki [1]. A 1998 preprint of [1] is cited in [2]. It took several years and various partial results before a full proof by Klartag emerged in [8] (see p95 for the history). A different proof was given soon afterwards by Fleury, Guédon, and Paouris [4]. Significantly improved quantitative bounds (from logarithmic to power type) were given by Klartag [9], followed by improved estimates by various authors on the related 'thin shell property' [5, 7, 11]. More information can be found in [5, 7, 8, 9, 10, 12].

We present a simple proof that is self-contained (except for very classical results such as the Prékopa-Leindler inequality) and is accessible to anyone. The bounds on ε_n and ω_n that this proof gives are poor; the contribution is simplicity. The methodology is a variation of that in Klartag's original proof and uses Fourier inversion; the main difference being that we apply concentration directly to the Fourier transform as opposed to the measure of half-spaces. The statement of Theorem 1 below is not identical to Theorem 1.1 in [8], however under log-concavity, a uniform estimate on the cumulative distribution gives an estimate on the total variation distance, so we do indeed recover Theorem 1.1 in [8]. The standard Euclidean norm and inner product on $\mathbb{R}^n$ are denoted as $|\cdot|$ and $\langle \cdot, \cdot \rangle$ respectively.

Theorem 1 *There exist sequences* $(\varepsilon_n)_1^\infty$ *and* $(\omega_n)_1^\infty$ *in* $(0,\infty)$ *with* $\lim_{n\to\infty} \varepsilon_n = \lim_{n\to\infty} \omega_n = 0$ *such that the following is true: Let* $n \in \mathbb{N}$*, let* X *be a random vector in* $\mathbb{R}^n$ *with* $\mathbb{E}X = 0$

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and $\operatorname{Cov}(X) = I_n$. *Assume that* X *has a density* $f = d\mu/dx$ *that is log-concave, i.e.* $f = e^{-g}$ *where* $g : \mathbb{R}^n \to (-\infty, \infty]$ *is convex. Then there exists a set* $\Theta \subset S^{n-1}$ *with* $\sigma_{n-1}(S^{n-1}) \geq 1 - \omega_n$ *such that for all* $\theta \in \Theta$,

$$\sup_{t \in \mathbb{R}} |\mathbb{P}\{\langle X, \theta \rangle \leq t\} - \Phi(t)| \leq \varepsilon_n$$

where σ_{n-1} *is Haar measure on* S^{n-1} *normalized so that* $\sigma_{n-1}(S^{n-1}) = 1$, *and* $\Phi(t) = (2\pi)^{-1/2} \int_{-\infty}^{t} \exp(-u^2/2)\, du$.

The proof uses two nontrivial properties of log-concave functions (see [8, 9, 10] for more details): with f as in Theorem 1,

• If $E \subset \mathbb{R}^n$ is any linear subspace of dimension $1 \leq k < n$, then the projection $P_E f : E \to [0, \infty)$ defined by

$$P_E f(x) = \int_{E^\perp} f(x + y)\, dy \tag{1}$$

is log-concave. Here integration is performed with respect to $n - k$ dimensional Lebesgue measure on $E^\perp$. This is a consequence of the Prékopa-Leindler inequality. Interpreting a convolution in terms of a projection of $\mathbb{R}^n \times \mathbb{R}^n$ onto $\mathbb{R}^n$, we see that if $\varphi : \mathbb{R}^n \to [0, \infty)$ is log-concave with $\int_{\mathbb{R}^n} \varphi(x) dx = 1$, then the convolution $f * \varphi$ is also log-concave.

• If X has the thin shell property, i.e.

$$\mathbb{P}\left\{\left|\frac{|X|}{R} - 1\right| < \varepsilon'\right\} > 1 - \varepsilon'$$

for some $\varepsilon', R > 0$ (here we can take $R = \sqrt{n}$), then the projection of X onto most one dimensional subspaces is approximately Gaussian, with estimates depending on ε'. Quantitative results of this type for log-concave measures can be found in [1, 2]. For completeness, we give a precise statement with proof in Section 3.

2 Proof of Theorem 1

The proof is in three main steps.

Step 1: Approximately spherically symmetric projections. The first step mimics Milman's proof of Dvoretzky's theorem [14], see for example [15], but in a different way to Klartag [8, Sections 3 and 4]. Let $Y = X + \sigma Z$ for some $\sigma > 0$, where Z has the standard normal distribution and is independent of X. The density of Y is $h = f * \phi_\sigma$, where $\phi_\sigma(x) = (2\pi\sigma^2)^{-n/2} \exp\left(-2^{-1}\sigma^{-2} |x|^2\right)$ and $*$ denotes convolution. Then $\widehat{h} = \widehat{f} \cdot \widehat{\phi}_\gamma$, where $\widehat{\cdot}$ denotes the Fourier transform,

$$\widehat{h}(\xi) = \int_{\mathbb{R}^n} \exp(-2\pi i \langle \xi, x \rangle)\, h(x)\, dx$$

and

$$\widehat{\phi}_\sigma(\xi) = \exp\left(-2\pi^2\sigma^2 |\xi|^2\right)$$

For any $\xi_1, \xi_2 \in \mathbb{R}^n$,

$$\begin{aligned}\left|\widehat{f}(\xi_1) - \widehat{f}(\xi_2)\right| &\leq \int_{\mathbb{R}^n} \left|\exp\left(-2\pi i \langle \xi_1, x\rangle\right) - \exp\left(-2\pi i \langle \xi_2, x\rangle\right)\right| f(x)dx \\ &\leq \int_{\mathbb{R}^n} 2\pi \left|\langle \xi_1, x\rangle - \langle \xi_2, x\rangle\right| f(x)dx \\ &= 2\pi \left|\xi_1 - \xi_2\right| \int_{\mathbb{R}^n} \left|\left\langle \frac{\xi_1 - \xi_2}{|\xi_1 - \xi_2|}, x\right\rangle\right| f(x)dx \\ &\leq 2\pi \left|\xi_1 - \xi_2\right| \left(\mathbb{E}\left|\left\langle \frac{\xi_1 - \xi_2}{|\xi_1 - \xi_2|}, X\right\rangle\right|^2\right)^{1/2}\end{aligned}$$

and we see that $\widehat{f}$ is 2π-Lipschitz on $\mathbb{R}^n$. Let $F \in G_{n,k}$ be any fixed subspace and U a random matrix uniformly distributed in $O(n)$ ($k < n$ to be determined later). Then $E = UF \in G_{n,k}$ is a random k-dimensional subspace uniformly distributed in $G_{n,k}$. Let $\varepsilon \in (0, 1/2)$ and let $\mathcal{N} \subset S_F = S^{n-1} \cap F$ be an ε-dense subset (i.e. for all $\theta \in S_F$ there exists $\omega \in \mathcal{N}$ such that $|\theta - \omega| < \varepsilon$. By considering the volume of disjoint balls, such a subset can be chosen with cardinality $|\mathcal{N}| \leq (3/\varepsilon)^k$. Assume that $k \leq c\left(\log \varepsilon^{-1}\right)^{-1} \delta n$. By Lévy's concentration inequality for Lipschitz functions on a sphere, see e.g. [10], and the union bound, with probability at least

$$1 - \sum_{m=0}^{\infty} \left(\frac{3}{\varepsilon}\right)^k \exp\left(-\left\{\sqrt{c\delta^2 + \frac{2\ln m}{n}}\right\}^2 n\right) \geq 1 - C\exp\left(-c\delta^2 n\right)$$

the following event occurs: for all $m \in \{0, 1, 2 \ldots\}$, and all $\theta \in \mathcal{N}$,

$$\left|\widehat{f}\left(U(1+\varepsilon)^m \sqrt{k}\sigma^{-1}\theta\right) - M\left((1+\varepsilon)^m \sqrt{k}\sigma^{-1}\right)\right| < C\left(\delta + \sqrt{\frac{\ln m}{n}}\right)(1+\varepsilon)^m \sigma^{-1}\sqrt{k}$$

where

$$M(t) = \int_{S^{n-1}} \widehat{f}(t\theta)\, d\sigma_{n-1}(\theta)$$

With the same probability, the same event holds with $(1+\varepsilon)^m$ replaced with $(1+\varepsilon)^{-m}$. Setting $\xi' = (1+\varepsilon)^{\pm m}\sqrt{k}\sigma^{-1}\theta$, making m the subject of the formula, and using the Lipschitz property of $\widehat{f}$, with high probability, for all $\xi \in F$,

$$\left|\widehat{f}(U\xi) - M(|\xi|)\right| < C\left(\delta + \varepsilon + \sqrt{\frac{\ln \varepsilon^{-1}}{n}} + \sqrt{\frac{1}{n}\ln\ln\max\left\{\frac{\sigma|\xi|}{\sqrt{k}}, \frac{\sqrt{k}}{\sigma|\xi|}\right\}}\right)|\xi|$$

Optimizing over ε we set $\varepsilon = \sqrt{(\ln n)/n}$. Let $P_E : \mathbb{R}^n \to E$ denote the orthogonal projection onto E, let $\mathcal{F}_{\mathbb{R}^n} : L^1(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n)$ denote the Fourier transform on $\mathbb{R}^n$ and let $\mathcal{F}_E : L^1(E) \to L^\infty(E)$ denote the Fourier transform on E (E as a Hilbert space in its own right). Recall the definition in (1). By Fubini's theorem, the function $P_E h$ is the density of the random vector $P_E X$ (with respect to k-dimensional Lebesgue measure in E). The Fourier transform works well with orthogonal projections, in particular

$$(\mathcal{F}_{\mathbb{R}^n} h)\,|_E = \mathcal{F}_E\,(P_E h)$$

where $(\mathcal{F}_{\mathbb{R}^n} f)\,|_E$ denotes the restriction of $\mathcal{F}_{\mathbb{R}^n} f$ to E. By Fourier inversion in E, for all $x \in E$,

$$P_E h\,(x) = \int_E \exp\left(2\pi i \langle x, \xi\rangle\right) \widehat{h}\,(\xi)\, d\xi$$

so for all $W \in O\,(E)$, (applying a change of variables)

$$\begin{aligned}
& |P_E h\,(x) - P_E h\,(Wx)| \\
\leq\ & \int_E \left|\widehat{h}\,(\xi) - \widehat{h}\,(W\xi)\right| d\xi \\
\leq\ & C\left(2\pi\sigma^2\right)^{-(k+1)/2} \int_E \left(\delta + \sqrt{\frac{\ln n}{n}} + \sqrt{\frac{1}{n}\ln\ln\max\left\{\frac{|y|}{\sqrt{2\pi k}}, \frac{\sqrt{2\pi k}}{|y|}\right\}}\right) e^{-\pi|y|^2}\,|y|\, dy \\
\leq\ & C\left(2\pi\sigma^2\right)^{-(k+1)/2}\left(\delta + \sqrt{\frac{\ln n}{n}}\right)\sqrt{k} \qquad (2)
\end{aligned}$$

Step 2: Behavior of $t \mapsto P_E h\,(t\theta)$ (in the spirit of Lemmas 4.3 and 4.4 in [8]). Consider any $x, y \in S_E = E \cap S^{n-1}$ and define $A, B : [0, \infty) \to \mathbb{R}$ by

$$P_E h\,(tx) = e^{-A(t)} \qquad P_E h\,(ty) = e^{-B(t)}$$

Since f and ϕ are log-concave, i.e. $-\log f$ and $-\log\phi$ are convex with values in $(-\infty, \infty]$, $h = f * \phi$ is also log-concave. It follows from the Prékopa-Leindler inequality (see for example the discussion in [8]) that $P_E h$ too is log-concave, and therefore A and B are convex. Since $P_E h = (P_E f) * (P_E \phi_\sigma)$, A and B are infinitely differentiable. In preparation for an integral over E in polar coordinates, we now study $t \mapsto t^{k-1} e^{-A(t)}$ and $t \mapsto t^{k-1} e^{-B(t)}$, $t \in [0, \infty)$. These functions are maximized at t_x,$t_y \in (0, \infty)$ that satisfy

$$A'\,(t_x)\,t_x = k - 1 \qquad B'\,(t_y)\,t_y = k - 1$$

Such numbers exist since $A'(t)t$ is continuous with limit 0 (resp. ∞) as $t \to 0$ (resp. $t \to \infty$), similarly for B. After a possible re-labeling of x and y we may assume that $t_x \leq t_y$. Our goal is to show that these numbers cannot be too far apart (in the sense that their ratio is close to 1). If $t_x = t_y$ there is nothing to show, so assume $t_x < t_y$. By convexity,

$$\begin{aligned}
A\,(t_y) - A\,(t_x) &\geq A'\,(t_x)\,(t_y - t_x) = (k-1)\left(\frac{t_y}{t_x} - 1\right) \\
B\,(t_y) - B\,(t_x) &\leq B'\,(t_y)\,(t_y - t_x) = (k-1)\left(1 - \frac{t_x}{t_y}\right)
\end{aligned}$$

and therefore

$$\sup_{t\in\{t_x,t_y\}}|A(t)-B(t)| \geq \frac{\{A(t_y)-A(t_x)\}-\{B(t_y)-B(t_x)\}}{2} = \frac{(k-1)(t_y-t_x)^2}{2t_xt_y} \quad (3)$$

Assume momentarily that there exists $t\in\{t_x,t_y\}$ such that $A(t)-B(t)\geq 1$. Since P_Eh is the log-concave density of a random vector in E with covariance $(1+\sigma^2)\,I$, it follows from Theorem 5.14 in [13] (see also (6) here) that $P_Eh(0)\geq 2^{-7k}(1+\sigma^2)^{-k/2}$. By convexity again,

$$\begin{aligned}\left|e^{-A(t)}-e^{-B(t)}\right| &= e^{-B(t)}\left|e^{B(t)-A(t)}-1\right| \geq \left(1-e^{-1}\right)e^{-B(t)}\\ &\geq \left(1-e^{-1}\right)\exp\left(-B(0)-tB'(t)\right)\\ &\geq \left(1-e^{-1}\right)P_Eh(0)\exp\left(-t_yB'(t_y)\right)\\ &\geq (e-1)\,2^{-7k}\left(1+\sigma^2\right)^{-k/2}\exp(-k)\end{aligned}$$

However, by (2),

$$\left|e^{-A(t)}-e^{-B(t)}\right| = |P_Eh(tx)-P_Eh(ty)| \leq C\left(2\pi\sigma^2\right)^{-(k+1)/2}\left(\delta+\sqrt{\frac{\ln n}{n}}\right)\sqrt{k}$$

We will choose the parameters δ, k, and σ so that the upper bound on $\left|e^{-A(t)}-e^{-B(t)}\right|$ is less than the lower bound, which implies that we may assume that $B(t)-A(t)>-1$ for all $t\in\{t_x,t_y\}$. Now let $t\in\{t_x,t_y\}$ such that

$$|A(t)-B(t)| = \sup_{u\in\{t_x,t_y\}}|A(u)-B(u)|$$

By (3),

$$\begin{aligned}\left|e^{-A(t)}-e^{-B(t)}\right| &= e^{-B(t)}\left|e^{B(t)-A(t)}-1\right|\\ &\geq \exp\left(-B(0)-B'(t)t\right)e^{-1}\left|B(t)-A(t)\right|\\ &\geq 2^{-7k}\left(1+\sigma^2\right)^{-k/2}e^{-k}\frac{(k-1)(t_y-t_x)^2}{2t_xt_y}\end{aligned}$$

so

$$\frac{t_y-t_x}{t_y} \leq \gamma := Ce^{ck}\left(1+\sigma^2\right)^{k/4}\sigma^{-(k+1)/2}\left(\delta^{1/2}+\left(\frac{\ln n}{n}\right)^{1/4}\right)$$

For an appropriate choice of parameters this will achieve our goal of showing that t_x and t_y cannot be too far apart (relatively). What this means is that in any direction $x\in S^{n-1}\cap E$, the function $t\mapsto t^{k-1}P_Eh(tx)$ achieves its peak in about the same place. Our next goal is to show that the mass in

$$\int_0^\infty t^{k-1}P_Eh(tx)\,dt$$

is concentrated around t_x. Since A lies above its tangent lines, defining q by

$$\begin{aligned} q(t) &= t^{k-1}e^{-A(t)} \leq \exp\left((k-1)\ln t - A(t_x) - (t-t_x)A'(t_x)\right) \\ &= \exp\left(-A(t_x) - \left(\frac{t}{t_x} - 1 - \ln\frac{t}{t_x} - \ln t_x\right)(k-1)\right) \\ &= \exp\left(-A(t_x) - \left(-\ln t_x + \sum_{j=2}^{\infty} j^{-1}\left(\frac{t}{t_x} - 1\right)^j\right)(k-1)\right) \\ &\leq t_x^{k-1}e^{-A(t_x)}\exp\left(-\frac{k-1}{3}\left(\frac{t}{t_x}-1\right)^2\right) \end{aligned}$$

provided $\left|\frac{t}{t_x} - 1\right| < 1/2$. We now translate this to tail probabilities. Fix any $t \in [t_x, 3t_x/2]$ and $s \geq t$. By log-concavity of q,

$$q(s) \leq \left[\left(\frac{q(t)}{q(t_x)}\right)^{1/(t-t_x)}\right]^{s-t} q(t) \leq \exp\left(-\frac{(k-1)(s-t)(t-t_x)}{3t_x^2}\right) q(t)$$

and therefore

$$\int_t^\infty q(s)ds \leq \frac{3t_x^2 q(t)}{(k-1)(t-t_x)}$$

On the other hand, for any $s \in [t_x, t]$,

$$q(s) \geq \left[\left(\frac{q(t_x)}{q(t)}\right)^{1/(t-t_x)}\right]^{t-s} q(t) \geq \exp\left(\frac{(k-1)(t-s)(t-t_x)}{3t_x^2}\right) q(t)$$

so

$$\int_0^\infty q(s)ds \geq \int_{t_x}^t q(s)ds \geq \frac{3t_x^2 q(t)}{(k-1)(t-t_x)}\left[\exp\left(\frac{(k-1)(t-t_x)^2}{3t_x^2}\right) - 1\right]$$

and

$$\int_t^\infty q(s)ds \leq \left[\exp\left(\frac{(k-1)(t-t_x)^2}{3t_x^2}\right) - 1\right]^{-1} \int_0^\infty q(s)ds$$

A similar bound holds for the left hand tail. Combining these,

$$\int_{(1-u)t_x}^{(1+u)t_x} q(s)ds \geq \left(1 - C\exp\left(-cku^2\right)\right)\left(\int_0^\infty q(s)ds\right) \tag{4}$$

provided $u \in [0, 1/2]$.

Step 3: Thin shell and small details. Now fix an arbitrary $x \in B_2^n \cap E$. By polar integration,

$$\mathbb{P}\left\{\left|\frac{|P_E Y|}{t_x} - 1\right| < C(u+\gamma)\right\} \geq 1 - C\exp\left(-cku^2\right) \tag{5}$$

which is the so called 'thin shell property' of $P_E Y$ in E (see Section 3 for more details), and by a result of Bobkov [2] (following Anttila, Ball and Perissinaki [1] in the symmetric case) this implies that with probability at least

$$1 - C\sqrt{k}\exp\left(-ck\left\{u + \gamma + \exp\left(-cku^2\right)\right\}^2\right)$$

a further random projection $P_{\theta'} P_E Y$ is approximately Gaussian (with mean zero and variance $1 + \sigma^2$), where θ' is uniformly distributed in S_E,

$$\left|\mathbb{P}\left\{\langle\theta', P_E Y\rangle \leq t\right\} - \Phi\left(\frac{t}{\sqrt{1+\sigma^2}}\right)\right| \leq C\left(u + \gamma + \exp\left(-cku^2\right)\right)$$

See Theorem 2. Now $\langle\theta', P_E Y\rangle = \langle\theta', P_E X\rangle + \langle\theta', \sigma P_E Z\rangle$, and $\langle\theta', P_E Z\rangle \sim N(0,1)$. Assume that $t \geq 0$ and $\sigma \leq 1$, and consider any $\nu \in (0,1)$. Since

$$\begin{aligned}
\left\{\langle\theta', P_E Y\rangle \leq t - \nu\right\} &\Rightarrow \left\{\langle\theta', P_E X\rangle \leq t\right\} \vee \left\{\langle\theta', \sigma P_E Z\rangle \leq -\nu\right\} \\
\left\{\langle\theta', P_E X\rangle \leq t\right\} &\Rightarrow \left\{\langle\theta', P_E Y\rangle \leq t + \nu\right\} \vee \left\{\langle\theta', \sigma P_E Z\rangle \geq \nu\right\}
\end{aligned}$$

by the union bound and (7), $\mathbb{P}\left\{\langle\theta', P_E X\rangle \leq t\right\}$ is bounded below by

$$\begin{aligned}
& \mathbb{P}\left\{\langle\theta', P_E Y\rangle \leq t - \nu\right\} - \mathbb{P}\left\{\langle\theta', \sigma P_E Z\rangle \leq -\nu\right\} \\
\geq\ & \Phi\left(\frac{t-\nu}{\sqrt{1+\sigma^2}}\right) - C\left(u + \gamma + \exp\left(-cku^2\right)\right) - C\exp\left(-c\sigma^{-2}\nu^2\right) \\
\geq\ & \Phi\left(t\right) - C\left(\nu + \sigma + u + \gamma + \exp\left(-cku^2\right) + \exp\left(-c\sigma^{-2}\nu^2\right)\right)
\end{aligned}$$

and above by

$$\begin{aligned}
& \mathbb{P}\left\{\langle\theta', P_E Y\rangle \leq t + \nu\right\} + \mathbb{P}\left\{\langle\theta', \sigma P_E Z\rangle \geq \nu\right\} \\
\leq\ & \Phi\left(t\right) + C\left(\nu + \sigma + u + \gamma + \exp\left(-cku^2\right) + \exp\left(-c\sigma^{-2}\nu^2\right)\right)
\end{aligned}$$

Choosing

$$\begin{aligned}
k &= \frac{c_1 \ln(n+1)}{\ln\ln(n+2)} \qquad \delta = \frac{\ln(n+1)}{\sqrt{n}} \qquad \sigma = \frac{1}{\ln(n+1)} \\
u &= \frac{C_2 \ln\ln(n+2)}{\sqrt{\ln(n+1)}} \qquad \nu = \frac{C_2}{\sqrt{\ln(n+1)}}
\end{aligned}$$

(a fairly arbitrary choice), where c_1 is chosen first to be small and then C_2 is chosen to be appropriately large, we get $\gamma \leq Cn^{-1/5}$ and the error bound reduces to

$$\left|\mathbb{P}\left\{\langle\theta', P_E X\rangle \leq t\right\} - \Phi(t)\right| \leq \delta_n := \frac{C\ln\ln(n+2)}{\sqrt{\ln(n+1)}}$$

the probability bound (of failure) reduces to

$$\omega_n \leq C\exp\left(-c\delta^2 n\right) + C\sqrt{k}\exp\left(-ck\left\{u + \gamma + \exp\left(-cku^2\right)\right\}^2\right) \leq C\left(\log n\right)^{-C_3}$$

where C_3 can be made arbitrarily large by taking C_2 large enough. The upper and lower bounds for $\left|e^{-A(t)} - e^{-B(t)}\right|$ earlier in the proof become (respectively) $Cn^{-1/2+0.1}$ and $Cn^{-0.1}$, which achieves the desired contradiction, and the required bound $k \leq c\delta^2 (\ln n)^{-1} n$ is satisfied. Note that $P_{\theta'} P_E = P_\theta$ where θ is uniformly distributed in S^{n-1}, so we have shown that the projection of X onto most one dimensional subspaces is approximately Gaussian, and Theorem 1 follows.

Note: Radius of the thin shell. When stating and applying the fact that the thin shell property implies CLT, it is convenient to replace t_x with $\sqrt{k}$ in (5). Let W_θ ($\theta \in S^{n-1} \cap E$) be a random variable with density proportional to $q_\theta(t) = t^{k-1} P_E h(t\theta)$, $t \geq 0$. From (4),

$$\begin{aligned}\mathbb{E}|W_\theta|^2 &= (\mathbb{E}|W_\theta|)^2 + \mathrm{Var}(W_\theta) \leq \left(t_\theta + Ck^{-1/2} t_\theta\right)^2 + \frac{Ct_\theta^2}{k} \\ &\leq t_x^2 (1 + C\gamma)\left(1 + Ck^{-1/2}\right) + \frac{Ct_x^2}{k}\end{aligned}$$

so

$$\begin{aligned}\mathbb{E}|P_E Y|^2 &= \mathrm{vol}_{k-1}\left(S^{k-1}\right) \int_{S^{n-1}\cap E} \left(\int_0^\infty t^2 q_\theta(t) \frac{dt}{\int_0^\infty q_\theta(s) ds} \right) \left(\int_0^\infty q_\theta(s) ds \right) d\sigma_{k-1}(\theta) \\ &\leq \left(1 + C\gamma + Ck^{-1/2}\right) t_x^2\end{aligned}$$

The last inequality follows since $\mathrm{vol}_{k-1}\left(S^{k-1}\right) \int_{S^{n-1}\cap E} \int_0^\infty q_\theta(s) ds d\sigma_{k-1}(\theta) = 1$. Similarly,

$$\mathbb{E}|P_E Y|^2 \geq \left(1 - C\gamma - Ck^{-1/2}\right) t_x^2$$

But $\mathbb{E}|P_E Y|^2 = k$, so

$$\left(1 - C\gamma - Ck^{-1/2}\right)\sqrt{k} \leq t_x \leq \left(1 + C\gamma + Ck^{-1/2}\right)\sqrt{k}$$

and (changing the constants involved) we may replace t_x with $\sqrt{k}$ in (5).

Note: Lower bound on $P_E f(0)$. To simplify notation we work with the original function $f : \mathbb{R}^n \to [0,\infty)$, but the corresponding result can then be applied to $P_E f : E \to [0,\infty)$ by replacing n with k. By log-concavity, $\{x \in \mathbb{R}^n : f(x) > f(0)\}$ is convex and there exists $\theta \in S^{n-1}$ such that $\langle \theta, x \rangle > 0$ implies $f(x) \leq f(0)$. It is an interesting exercise to show that for any log-concave random variable in $\mathbb{R}$ with zero mean and unit variance, such as $\langle \theta, X \rangle$, $\mathbb{P}\{\langle \theta, X \rangle > 0\} \geq \beta$ for some universal constant $\beta > 0$ (actually for $\beta = e^{-1}$). Now

$$n = \mathbb{E}|X|^2 \geq A^2 \alpha_n^2 \frac{n}{2\pi e} \mathbb{P}\left\{ |X| \geq A\alpha_n \sqrt{\frac{n}{2\pi e}} \right\} \geq A^2 \alpha_n^2 \frac{n}{2\pi e} \left(\beta - f(0) \frac{1}{2} \mathrm{vol}_n \left(A\alpha_n \sqrt{\frac{n}{2\pi e}} B_2^n \right) \right)$$

where α_n is such that $\mathrm{vol}_n\left(\alpha_n \sqrt{\frac{n}{2\pi e}}\right) = 1$ and $\alpha_n \to 1$ as $n \to \infty$ (and $B_2^n = \{x : |x| \leq 1\}$). Optimizing in A yields

$$f(0) \geq Cn^{-3/2} \left(e\sqrt{2\pi}\right)^{-n} \tag{6}$$

In the symmetric case one gets the optimal base $\sqrt{2\pi e}$. The estimate $f(0) \geq 2^{-7n}$ can be found, for example, in [13, Theorem 5.14].

3 Appendix: Thin shell implies CLT

For completeness we collect and prove various known results and tailor them to our specific use. We refer the reader to [2, Theorems 1.1 and 1.2, Eq. (1.7) Proposition 3.1] and [1] for a more extensive discussion. Our proof of Proposition 3.1 in [2] on the Lipschitz constant of $\theta \mapsto M(\theta, t)$ is slightly simplified.

Theorem 2 *Let $\varepsilon > 0$. Let μ be a probability measure on $\mathbb{R}^k$ with center of mass 0, identity covariance, and log-concave density $f = d\mu/dx$. If μ has the following thin shell property:*

$$\mu \left\{ x \in \mathbb{R}^k : \left| \frac{|x|}{\sqrt{k}} - 1 \right| > \varepsilon \right\} < \varepsilon$$

then there exists $\Theta \subset S^{k-1}$ with $\sigma_{n-1}(\Theta) \geq 1 - C\sqrt{k}\exp(-ck\varepsilon^2)$ such that for all $\theta \in \Theta$,

$$\sup_{t \in \mathbb{R}} \left| \Phi(t) - \mu\left\{ x \in \mathbb{R}^k : \langle x, \theta \rangle \leq t \right\} \right| \leq C\varepsilon$$

Proof. Write $M(\theta, t) = \mu\left\{ x \in \mathbb{R}^k : \langle x, \theta \rangle \leq t \right\}$. For any $\theta_1, \theta_2 \in S^{k-1}$ that are sufficiently close, say $|\theta_1 - \theta_2| < 1/10$,

$$|M(\theta_1, t) - M(\theta_2, t)| = \mu\left(M(\theta_1, t) \Delta M(\theta_2, t) \right)$$

where $A \Delta B = (A \backslash B) \cup (B \backslash A)$ denotes the symmetric difference of A and B. By projecting onto *span* $\{\theta_1, \theta_2\}$ and identifying *span* $\{\theta_1, \theta_2\}$ with $\mathbb{R}^2$, we conclude that

$$|M(\theta_1, t) - M(\theta_2, t)| = \int_{-\infty}^{t} \int_{(1 - x\cos\beta)/\sin\beta}^{\infty} q(x) dy dx + \int_{t}^{\infty} \int_{-\infty}^{(1 - x\cos\beta)/\sin\beta} q(x) dy dx$$

where q is the density of the measure projection of μ into E (identified with $\mathbb{R}^2$), see (1), and $\cos\beta = \langle \theta_1, \theta_2 \rangle$. By the Prékopa-Leindler inequality q is log-concave, and defines a probability measure with mean 0 and identity covariance. It is an elementary fact that for such a function, $q(x, y) \leq C\exp(-cx_1 - cx_2)$ with universal constants $C, c > 0$. By a change of variables (through translation),

$$|M(\theta_1, t) - M(\theta_2, t)| \leq 2C \int_{-\infty}^{0} \int_{t}^{-y\tan\beta} \exp(-c'x - c'y) \, dx dy \leq Ce^{-c|t|} |\theta_1 - \theta_2|$$

This implies that $M(\theta, t)$ is $Ce^{-c|t|}$-Lipschitz in θ. Now let $\theta \in S^{k-1}$ be chosen randomly, uniformly distributed on S^{k-1} and let $F(t) = \mathbb{E}M(\theta, t)$. By concentration on S^{n-1} (see e.g. [10]) and the union bound, with probability at least $1 - C\varepsilon^{-1}\exp(-cn\varepsilon^2) = 1 - C\sqrt{k}(k\varepsilon^2)^{-1/2}\exp(-ck\varepsilon^2)$, the following event occurs: for all $1 \leq j \leq m$, $|M(\theta, t_j) - F(t_j)| < \varepsilon$, where $m = \lfloor \varepsilon^{-1} \rfloor$ and $t_j = F^{-1}(j/m)$. Using monotonicity in t, we conclude that (with high probability) $|M(\theta, t) - F(t)| < C\varepsilon$ for all $t \in \mathbb{R}$. We now compare F to Φ. Let $\Phi_k(t) = \mathbb{P}\left\{ \sqrt{k}\theta_1 \leq t \right\}$, where θ is still uniform on S^{k-1}. Let X be a random vector in

$\mathbb{R}^k$ with distribution μ and independent of θ. The vector $Y = \left\langle \theta, k^{1/2} |X|^{-1} X \right\rangle$ is independent of $k^{-1/2} |X|$ and has the same distribution as θ_1. Using Fubini's theorem and independence, and assuming $t > 0$,

$$\begin{aligned}
F(t) &= \mathbb{P}\{\langle \theta, X\rangle \le t\} = \mathbb{P}\left\{ \frac{|X|}{\sqrt{k}} \left\langle \theta, \frac{\sqrt{k}X}{|X|} \right\rangle \le t \right\} = \mathbb{P}\left\{ Y \le \frac{t\sqrt{k}}{|X|} \right\} \\
&= \mathbb{P}\left\{ \left| \frac{|X|}{\sqrt{k}} - 1 \right| < \varepsilon \right\} \mathbb{P}\left\{ Y \le \frac{t\sqrt{k}}{|X|} : \left| \frac{|X|}{\sqrt{k}} - 1 \right| < \varepsilon \right\} \\
&\quad + \mathbb{P}\left\{ \left| \frac{|X|}{\sqrt{k}} - 1 \right| > \varepsilon \right\} \mathbb{P}\left\{ Y \le \frac{t\sqrt{k}}{|X|} : \left| \frac{|X|}{\sqrt{k}} - 1 \right| > \varepsilon \right\} \\
&\le 1 \cdot \Phi_k\left(\frac{t\sqrt{k}}{(1-\varepsilon)\sqrt{k}} \right) + \varepsilon \cdot 1
\end{aligned}$$

A similar lower bound holds. For any $\delta, x > 0$,

$$\Phi\left((1+\delta)x\right) - \Phi(x) \le \Phi'(x)\,\delta x \le C\delta \tag{7}$$

It follows from rotational invariance of the standard normal distribution and uniqueness of Haar measure that if Z is a standard normal vector in $\mathbb{R}^k$ then $\sqrt{k}\,|Z|^{-1} Z$ is uniformly distributed on $\sqrt{k}S^{k-1}$. Simulating $\theta = \sqrt{k}\,|Z|^{-1} Z$,

$$\Phi_k(t) - \Phi_k(-t) = \mathbb{P}\left\{ |Z_1| \le t k^{-1/2} |Z| \right\} = \mathbb{P}\left\{ |Z_1| \le t \left(1 - \frac{t^2}{k} \right)^{-1/2} \left(\frac{1}{k} \sum_{i=2}^{k} Z_i^2 \right) \right\}$$

which (after a bit of fiddling using (7) and Gaussian concentration of $|Z|$ about $k^{1/2}$) implies the well known estimate $|\Phi(t) - \Phi_k(t)| \le ck^{-1/2}$ for all $t \in \mathbb{R}$ (this can also be seen by considering the density Φ_k', similar details in [6, Section 3]). Putting all this together,

$$F(t) \le \Phi_k\left(\frac{t}{(1-\varepsilon)} \right) + \varepsilon \le \Phi\left(\frac{t}{(1-\varepsilon)} \right) + \frac{C}{\sqrt{k}} + \varepsilon \le \Phi(t) + C\varepsilon + \frac{C}{\sqrt{k}}$$

with a similar lower bound. Similarly, this also holds for $t < 0$. ■

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