S3 Appendix: Properties of the HHWMA *W* chart

This Appendix contains the derivations of the mean and variance of the HH_t statistic. The charting statistic of HH_t is given by

$$\begin{cases} H_t = \lambda_2 W_t + (1-\lambda_2)\bar{W}_{t-1} \\ HH_t = \lambda_1 H_t + (1-\lambda_1)\bar{H}_{t-1} \end{cases} \tag{C.1}$$

where

$$\bar{W}_{t-1} = \frac{\sum_k^{t-1} W_k}{t-1} \text{ and } \bar{H}_{t-1} = \frac{\sum_k^{t-1} H_k}{t-1}.$$

- For $t = 1$,

$$HH_1 = \lambda_1 H_1 + (1-\lambda_1)\bar{H}_0 = \lambda_1[\lambda_2 W_1 + (1-\lambda_2)\bar{W}_0] + (1-\lambda_1)\bar{H}_0. \tag{C.2}$$

Since $\bar{W}_0 = \bar{H}_0 = \mu_W$, when $t = 1$, Equation (C.2) becomes:

$$HH_1 = \lambda_1[\lambda_2 W_1 + (1-\lambda_2)\mu_W] + (1-\lambda_1)\mu_W = \lambda_1\lambda_2 W_1 + (1-\lambda_1\lambda_2)\mu_W. \tag{C.3}$$

Thus, the mean and variance of HH_1 are given by

$$E(HH_1) = \lambda_1\lambda_2 E(W_1) + (1-\lambda_1\lambda_2)\mu_W = \lambda_1\lambda_2\mu_W + \mu_W - \lambda_1\lambda_2\mu_W = \mu_W$$

and (C.4)

$$Var(HH_1) = Var(\lambda_1\lambda_2 W_1 + (1-\lambda_1\lambda_2)\mu_W) = \lambda_1^2\lambda_2^2 \mathrm{Var}(W_1) = \lambda_1^2\lambda_2^2\sigma_W^2,$$

respectively.

For $t = 2$,

$$\begin{aligned} HH_2 &= \lambda_1 H_2 + (1-\lambda_1)\bar{H}_1 \\ &= \lambda_1[\lambda_2 W_2 + (1-\lambda_2)\bar{W}_1] + (1-\lambda_1)\bar{H}_1 \\ &= \lambda_1\lambda_2 W_2 + \lambda_1(1-\lambda_2)W_1 + (1-\lambda_1)[\lambda_2 W_1 + (1-\lambda_2)\bar{W}_0] \\ &= \lambda_1\lambda_2 W_2 + \lambda_1(1-\lambda_2)W_1 + \lambda_2(1-\lambda_1)W_1 + (1-\lambda_1)(1-\lambda_2)\mu_W \\ &= \lambda_1\lambda_2 W_2 + (\lambda_1+\lambda_2-2\lambda_1\lambda_2)W_1 + (1-\lambda_1)(1-\lambda_2)\mu_W. \end{aligned}$$

Thus, the expression of HH_2 is given by

$$HH_2 = \lambda_1\lambda_2 W_2 + (\lambda_1+\lambda_2-2\lambda_1\lambda_2)W_1 + (1-\lambda_1)(1-\lambda_2)\mu_W\,. \tag{C.5}$$

From Equation (C.5), when $t = 2$, the mean of HH_t can be derived as follows:

$$\begin{aligned} E(HH_2) &= \lambda_1\lambda_2 \mathrm{E}(W_1) + (\lambda_1+\lambda_2-2\lambda_1\lambda_2)\mathrm{E}(W_1) + (1-\lambda_1)(1-\lambda_2)\mu_W \\ &= [\lambda_1\lambda_2 + \lambda_1 + \lambda_2 - 2\lambda_1\lambda_2 + 1 - \lambda_2 - \lambda_1 + \lambda_1\lambda_2]\mu_W \\ &= \mu_W. \end{aligned}$$

From Equation (C.5), when $t = 2$, the variance of HH_t can be derived as follows:

$$Var(HH_2) = \lambda_1^2\lambda_2^2 Var(W_2) + (\lambda_1+\lambda_2-2\lambda_1\lambda_2)^2 Var(W_1) = [\lambda_1^2\lambda_2^2 + (\lambda_1+\lambda_2-2\lambda_1\lambda_2)^2]\sigma_W^2.$$

Thus, the mean and variance of DH_2 are given by

$$E(HH_2) = \mu_W$$

and (C.6)

$$Var(HH_2) = [\lambda_1^2\lambda_2^2 + (\lambda_1+\lambda_2-2\lambda_1\lambda_2)^2]\sigma_W^2,$$

respectively.

- For $t > 2$,

$$
\begin{aligned}
HH_t &= \lambda_1 H_t + (1-\lambda_1)\bar{H}_{t-1} \\
&= \lambda_1[\lambda_2 W_t + (1-\lambda_2)\bar{W}_{t-1}] + \frac{(1-\lambda_1)}{t-1}\sum_{k=1}^{t-1} H_t \\
&= \lambda_1\lambda_2 W_t + \lambda_1(1-\lambda_2)\bar{W}_{t-1} + \frac{(1-\lambda_1)}{t-1}\sum_{k=1}^{t-1}[\lambda_2 W_k + (1-\lambda_2)\bar{W}_{k-1}] \\
&= \lambda_1\lambda_2 W_t + \lambda_1(1-\lambda_2)\bar{W}_{t-1} + \frac{\lambda_2(1-\lambda_1)}{t-1}\sum_{k=1}^{t-1} W_k + \frac{(1-\lambda_1)(1-\lambda_2)}{t-1}\sum_{k=1}^{t-1}\bar{W}_{k-1} \\
&= \lambda_1\lambda_2 W_t + (\lambda_1+\lambda_2-2\lambda_1\lambda_2)W_{t-1} + \frac{(1-\lambda_1)(1-\lambda_2)}{t-1}\sum_{k=0}^{t-2}\bar{W}_k \\
&= \lambda_1\lambda_2 W_t + (\lambda_1+\lambda_2-2\lambda_1\lambda_2)\bar{W}_{t-1} + \frac{(1-\lambda_1)(1-\lambda_2)}{t-1}\sum_{k=1}^{t-2}\bar{W}_k + \frac{(1-\lambda_1)(1-\lambda_2)}{t-1}\mu_W \\
&= \lambda_1\lambda_2 W_t + (\lambda_1+\lambda_2-2\lambda_1\lambda_2)\bar{W}_{t-1} + \frac{(1-\lambda_1)(1-\lambda_2)}{t-1}\sum_{u=1}^{t-2}\frac{1}{k}\sum_{k=u}^{t-2} W_u + \frac{(1-\lambda_1)(1-\lambda_2)}{t-1}\mu_W \\
&= \lambda_1\lambda_2 W_t + (\lambda_1+\lambda_2-2\lambda_1\lambda_2)\bar{W}_{t-1} + \frac{(1-\lambda_1)(1-\lambda_2)}{t-1}\sum_{u=1}^{t-2}\left(\sum_{k=u}^{t-2}\frac{1}{k}\right)W_u + \frac{(1-\lambda_1)(1-\lambda_2)}{t-1}\mu_W \\
&= \lambda_1\lambda_2 W_t + \frac{(\lambda_1+\lambda_2-2\lambda_1\lambda_2)}{t-1}W_{t-1} + \frac{(\lambda_1+\lambda_2-2\lambda_1\lambda_2)}{t-1}\sum_{u=1}^{t-2} W_u + \frac{(1-\lambda_1)(1-\lambda_2)}{t-1}\sum_{u=1}^{t-2}\left(\sum_{k=u}^{t-2}\frac{1}{k}\right)W_u \\
&\qquad + \frac{(1-\lambda_1)(1-\lambda_2)}{t-1}\mu_W \\
&= \lambda_1\lambda_2 W_t + \frac{(\lambda_1+\lambda_2-2\lambda_1\lambda_2)}{t-1}W_{t-1} + \frac{1}{t-1}\sum_{u=1}^{t-2}\left[(\lambda_1+\lambda_2-2\lambda_1\lambda_2) + (1-\lambda_1)(1-\lambda_2)\sum_{k=u}^{t-2}\frac{1}{k}\right]W_u \\
&\qquad + \frac{(1-\lambda_1)(1-\lambda_2)}{t-1}\mu_W.
\end{aligned}
$$

Thus, the expression of HH_t is given by

$$
\begin{aligned}
HH_t = \lambda_1\lambda_2 W_t + \frac{(\lambda_1+\lambda_2-2\lambda_1\lambda_2)}{t-1}W_{t-1} + \frac{1}{t-1}\sum_{u=1}^{t-2}\left[(\lambda_1+\lambda_2-2\lambda_1\lambda_2) + (1-\lambda_1)(1-\lambda_2)\sum_{k=u}^{t-2}\frac{1}{k}\right]W_u \\
+ \frac{(1-\lambda_1)(1-\lambda_2)}{t-1}\mu_W.
\end{aligned} \tag{C.7}
$$

From Equation (C.7), when $t > 2$, the mean of HH_t can be derived as follows:

$$
\begin{aligned}
E(HH_t) &= \left[\lambda_1\lambda_2 + \frac{(\lambda_1+\lambda_2-2\lambda_1\lambda_2)}{t-1} + \frac{1}{t-1}\sum_{u=1}^{t-2}\left[(\lambda_1+\lambda_2-2\lambda_1\lambda_2) + (1-\lambda_1)(1-\lambda_2)\sum_{k=u}^{t-2}\frac{1}{k}\right]\right. \\
&\qquad \left. + \frac{(1-\lambda_1)(1-\lambda_2)}{t-1}\right]\mu_W \\
&= \left[\lambda_1\lambda_2 + \frac{(\lambda_1+\lambda_2-2\lambda_1\lambda_2)}{t-1} + \frac{(\lambda_1+\lambda_2-2\lambda_1\lambda_2)(t-2)}{t-1} + \frac{(1-\lambda_1)(1-\lambda_2)}{t-1}\sum_{u=1}^{t-2}\left[+\sum_{k=u}^{t-2}\frac{1}{k}\right]\right. \\
&\qquad \left. + \frac{(1-\lambda_1)(1-\lambda_2)}{t-1}\right]\mu_W = [\lambda_1\lambda_2 + \lambda_1 + \lambda_2 - 2\lambda_1\lambda_2 + (1-\lambda_1)(1-\lambda_2)]\mu_W \\
&= \mu_W.
\end{aligned}
$$

From Equation (C.7), when $t > 2$, the variance of HH_t can be derived as follows:

$$
\begin{aligned}
Var(HH_t) &= Var\left[\lambda_1\lambda_2 W_t + \frac{(\lambda_1+\lambda_2-2\lambda_1\lambda_2)}{t-1}W_{t-1}\right. \\
&\qquad \left. + \frac{1}{t-1}\sum_{u=1}^{t-2}\left[(\lambda_1+\lambda_2-2\lambda_1\lambda_2) + (1-\lambda_1)(1-\lambda_2)\sum_{k=u}^{t-2}\frac{1}{k}\right]W_u + \frac{(1-\lambda_1)(1-\lambda_2)}{t-1}\mu_W\right] \\
&= \left[\lambda_1^2\lambda_2^2 + \frac{(\lambda_1+\lambda_2-2\lambda_1\lambda_2)^2}{(t-1)^2}\right. \\
&\qquad \left. + \frac{1}{(t-1)^2}\sum_{u=1}^{t-2}\left(\lambda_1+\lambda_2-2\lambda_1\lambda_2 + (1-\lambda_1)(1-\lambda_2)\sum_{k=u}^{t-2}\frac{1}{k}\right)^2\right]\sigma_W^2.
\end{aligned}
$$

Thus, when $t > 2$, the mean and variance of HH_t are given by

$$E(HH_t) = \mu_W$$

and

$$Var(HH_t) = \left[\lambda_1^2\lambda_2^2 + \frac{(\lambda_1+\lambda_2-2\lambda_1\lambda_2)^2}{(t-1)^2} + \frac{1}{(t-1)^2}\sum_{u=1}^{t-2}\left(\lambda_1+\lambda_2-2\lambda_1\lambda_2+(1-\lambda_1)(1-\lambda_2)\sum_{k=u}^{t-2}\frac{1}{k}\right)^2\right]\sigma_W^2, \quad \text{(C.8)}$$

respectively.

Therefore, at the sampling time t, the mean and variance of HH_t statistic are defined by

$$E(HH_t) = \mu_W$$

and

$$Var(HH_t) = \begin{cases} \lambda_1^2\lambda_2^2\sigma_W^2 & \text{for } t=1 \\ (\lambda_1^2\lambda_2^2 + (\lambda_1+\lambda_2-2\lambda_1\lambda_2)^2)\sigma_W^2 & \text{for } t=2 \\ \left[\lambda_1^2\lambda_2^2 + \frac{(\lambda_1+\lambda_2-2\lambda_1\lambda_2)^2}{(t-1)^2} + \frac{1}{(t-1)^2}\sum_{u=1}^{t-2}\left(\lambda_1+\lambda_2-2\lambda_1\lambda_2+(1-\lambda_1)(1-\lambda_2)\sum_{k=u}^{t-2}\frac{1}{k}\right)^2\right]\sigma_W^2 & \text{for } t>2 \end{cases} \quad \text{(C.9)}$$

respectively. Finally, note that when $\lambda_1 = \lambda_2$, then the results are the same as that of DH_t statistic.