ON THE SENSITIVITY ANALYSIS OF ENERGY QUANTO OPTIONS

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Abstract. In recent years there has been an advent of quanto options in energy markets. The structure of the payoff is rather a different type from other markets since it is written as a product of an underlying energy index and a measure of temperature. In the Heath-Jarrow-Morton (HJM) framework, by adopting the futures energy dynamics and model with stochastic volatility, we use the Malliavin calculus to derive the energy delta, temperature delta and cross-gamma formulae. The results reveal that these quantities are expressed in terms of expectations of the payoff and a random variable only depending on the underlying dynamics. This work can be viewed as a generalization of the work done, for example, by Benth et al. (2015).

1. Introduction

The paper investigates hedging of the energy quanto options using the Malliavin Calculus approach by Nualart (2006). This method has shown that it outperforms the finite difference approach when it comes to discontinuous payoffs, see Benth et al. (2010). Quanto options in the equity market differ from those designed for energy markets by the structure of their payoffs. The energy quanto option has a product payoff which is structured in such a way that it takes advantage of the high correlation between energy consumption and certain weather conditions thereby enabling price and weather risk to be controlled simultaneously, refer Caporin et al. (2012). On the other hand, the equity quanto has a normal structure. Ho et al.

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(1995) noted that quanto options, in general, are better hedgers than a simple combination of plain vanilla options. In energy markets, they give exposure to the volumetric risk input of weather conditions on energy prices, see Zhang (2001).

Heath et al. (1992) introduced the so-called HJM approach in fixed income markets where the dynamics of the forward rates are directly specified, see Benth et al. (2008). The fact that most contracts in energy markets are settled in futures and forward, the framework was later on in 2000 adopted in this market by Clewlow & Strickland (2000). There have been few papers in literature analysing the hedging of this quanto option product. Benth et al. (2015) recently studied the pricing and hedging of quanto energy options in this framework basing on both the spot and the futures products as the underlying processes. The authors derived analytic expressions for the energy delta, temperature delta, and cross-gamma hedging formulae using direct differentiation of the price of quanto options. If the payoff functions are discontinuous then their approach method fails.

The Malliavin calculus technique has been used by several authors to obtain *Greeks* (partial derivatives of option prices with respect to underlying parameters) in equity derivative products, see for example, Benth et al. (2008), Benth et al. (2003), Di Nunno et al. (2009), Fournié et al. (1999, 2001), Ocone & Karatzas (1991), Mhlanga (2011), Mhlanga & Becker (2013). In all these references the methods were not applied in a product payoff structure such as ours and with an interval delivering period. Our results can be viewed as a generalization of Benth et al. (2015) in the sense that our approach allows for discontinuous payoff functionals.

Besides Malliavin calculus, finite difference, pathwise differentiation and likelihood ratio approaches are used to derive the derivative free formulae. The finite difference approach involves simulating the derivative prices at two or more values of the underlying parameter and then estimate the derivative free formulae by taking difference quotients between these values. Finite difference approach is easy to implement, however it is prone to large bias and large variance especially when dealing with discontinuous payoff functions, as in the case of a digital type and a barrier type quanto option (Jackel (2003)). The pathwise method computes the derivative of the payoff function with respect to the parameter of interest. This method only works for specific payoff functions, hence we cannot generalize the implementation of this approach. When it is applicable, the method gives unbiased results (Glasserman (2004), page 386). However, the pathwise approach cannot be applied to non-differentiable payoff functions as in the case of barrier type and digital type quanto options. The likelihood ratio method assumes that the probability density function of the price is explicitly known and depends on the parameter of interest. The derivative free formulae is then computed by computing the derivative of the probability density of the underlying variable rather than the derivative of the payoff function (Broadie & Glaserman (1996)). The likelihood ratio method is restricted by requiring an explicit knowledge of the density of the underlying model, for example the probability density function for Asian type quanto options is not known.

Comparing with the mentioned approaches, Malliavin calculus approach has several advantages as it avoids the need to differentiate payoff functions and does not require explicit knowledge of the density of the underlying asset. However, the Malliavin calculus method is not reported to be better than the finite difference method when dealing with smoother payoff functionals such as the vanilla options (Benth et al. (2010)). The mathematical challenge arises from payoff functions which tend to be discontinuous, non-differentiable or even more complicated. Typical energy quanto options have payoff functions that are determined by the level of both the energy price and an index related to weather (temprature) (Benth et al.

(2015)). Typically, the energy quanto options have a general payoff, factorising in functionals of the two underlying, that is,

$$f(x, y) = g(x)h(y) \tag{1.1}$$

for some measurable functions g and h. The energy quanto options considered so far in literature have a continuous payoff of European call type

$$g(x)h(y) = \max\{x - K^E, 0\} \times \max\{y - K^I, 0\} \tag{1.2}$$

where $K^E, K^I > 0$ are strike prices (Benth et al., 2015). However, one or both of the two payoff functions (either g, h or both) could be discontinuous. A typical example is the so-called digital type (or binary options) energy quanto option, that is,

$$g(x) = 1_{\{x \geq K^E\}} \text{ and/or } h(y) = 1_{\{y \geq K^I\}}. \tag{1.3}$$

Since the energy quanto options should hedge the joint price and volume risk, this paper considers, for example, a discontinuous payoff structure of the form

$$\max\{x - K^E, 0\} \times 1_{\{y \geq K^I\}} \tag{1.4}$$

which hedges the price risk, if and only if the average temperature during a certain period is too high. For such type of options one cannot use the pathwise or the likelihood ratio methods. This is where Malliavin calculus approach is most suited.

The purpose of the current paper is to derive the so-called derivative free hedging formulae using a much more powerful tool: the Malliavin calculus. In particular, we derive the energy delta, temperature delta and the cross-gamma hedging expectation formulae. Other hedging formulae are derived by using a similar approach. The energy delta is defined as the partial derivative of the price of quanto option with respect to the energy future price. The temperature delta is defined as the partial derivative of the price of quanto option with respect to the temperature index futures. The cross-gamma is defined as the second partial derivative of the price of quanto option with respect to both the energy future price and the temperature index futures. The contribution of this study is to derive applicable formulae in the context of Malliavin calculus. In passing, we generalize the calculations of Greeks in Benth et al. (2015). The use of Malliavin calculus allows us to obtain Greeks which are suitable for Monte Carlo simulation. Information about Greeks is useful for constructing replicating portfolios to protect the portfolio against possible changes related to certain risk factors.

The paper is organised as follows. Section 2, present the structure of the quanto option and formulate the pricing problem as in Benth et al. (2015). In addition, we present the futures asset dynamics general diffusion models under the HJM framework. In Section 3, we review the necessary tools from Malliavin calculus to be applied in our proofs. In Section 4, we derive formulae for the energy delta, temperature delta and cross-gamma when the energy and temperature are independent while in Section 5, we derive the same when there is correlation between energy and temperature. Examples for both the independence case and correlation case are provided in Section 6. Section 7 is devoted quanto options with stochastic volatility. In Section 8, we discuss the residual risk. Finally, in Section 9, we conclude.

2. The Contract Structure and Pricing of Quanto Options

In this section, we review the commodity quanto pricing, see, for example, Benth et al. (2015) and in particular, we follow their notations therein. Benth et al. (2015) have considered the quanto option with a payoff function S given by:

$$S = (T_{\text{var}} - T_{\text{fix}})^+ \times (E_{\text{var}} - E_{\text{fix}})^+,$$

with $x^+ = \max(x, 0)$ where T_{var} represents some variable temperature measure, T_{fix} represents some fixed temperature measure, and $E_{\text{var}}, E_{\text{fix}}$ are the variable and fixed energy price, respectively. To avoid the downside risk on this quanto contract it has been reported in Benth et al. (2015) that for hedging purposes, it is reasonable to buy a contract with optionality. In the temperature market of Chicago Mercantile Exchange (CME), the contracts are written on the aggregated amount of heating-degree days (HDD) and cooling-degree days (CDD). The temperature index is used as the underlying. The HDD (similarly the CDD) over a measurement period $[\tau_1, \tau_2]$ is defined by:

$$HDD(t) := \max\{c - T(t), 0\}, \tag{2.1}$$

where $T(t)$ is the mean temperature on day t, and c is the pre-specified temperature threshold (eg., 65^0F or 18^0C). If the contract is specified as the accumulated HDD over $[\tau_1, \tau_2]$ we have

$$\sum_{t=\tau_1}^{\tau_2} HDD(t) = \sum_{t=\tau_1}^{\tau_2} \max\{c - T(t), 0\}, \tag{2.2}$$

analogously for CDD.
We note that quanto options have a payoff function that is a function of two underlying assets, temperature and price. We focus on quanto options with payoff function $f(E, I)$ where E is an index of the energy price and I is an index of temperature. The energy index E over a period $[\tau_1, \tau_2]$ with $\tau_1 < \tau_2$ is defined to be the average spot price namely:

$$E = \frac{1}{\tau_2 - \tau_1} \sum_{u=\tau_1}^{\tau_2} S_u, \tag{2.3}$$

where S_u is the energy spot price. In addition, we assume that the temperature index is defined by

$$I = \sum_{u=\tau_1}^{\tau_2} g(T_u), \tag{2.4}$$

where T_u denotes the temperature at time u and g some function. For example, for a quanto option involving the HDD index, we choose $g(x) = \max\{x - 18, 0\}$. To price the quanto option exercised at the time τ_2, its arbitrage-free price at time $t \le \tau_2$ becomes

$$C_t = e^{-r(\tau_2 - t)} \mathbb{E}_t^{\mathbb{Q}} \left[f\left(\frac{1}{\tau_2 - \tau_1} \sum_{u=\tau_1}^{\tau_2} S_u, \ \sum_{u=\tau_1}^{\tau_2} g(T_u) \right) \right], \tag{2.5}$$

where $r > 0$ represents a constant risk-free interest rate and $\mathbb{E}_t^{\mathbb{Q}}$ is the expectation operator with respect to $\mathbb{Q}$, conditioned on the market information at time t given by the filtration $\mathcal{F}_t$. Following Benth et al. (2015)'s argument on the relationship between the quanto option and the futures contract on the energy and temperature indexes E and I, we note that the price at time $t \le \tau_2$ of a futures contract written on some energy price with delivery period $[\tau_1, \tau_2]$

is given by

$$F^E(t;\tau_1,\tau_2) = \mathbb{E}_t^{\mathbb{Q}}\left[\frac{1}{\tau_2-\tau_1}\sum_{u=\tau_1}^{\tau_2} S_u\right]. \tag{2.6}$$

At $t=\tau_2$ we have:

$$F^E(\tau_2;\tau_1,\tau_2) = \frac{1}{\tau_2-\tau_1}\sum_{u=\tau_1}^{\tau_2} S_u\,. \tag{2.7}$$

This means that the future price is exactly equal to what is being delivered. Applying the same argument to the temperature index, with price dynamics denoted by $F^I(t,\tau_1,\tau_2)$, the quanto option price C_t can be written as:

$$\begin{aligned} C_t &= e^{-r(\tau_2-t)}\mathbb{E}_t^{\mathbb{Q}}\left[f\left(\frac{1}{\tau_2-\tau_1}\sum_{u=\tau_1}^{\tau_2} S_u,\ \sum_{u=\tau_1}^{\tau_2} g(T_u)\right)\right] \\ &= e^{-r(\tau_2-t)}\mathbb{E}_t^{\mathbb{Q}}\left[f\left(F^E(\tau_2;\tau_1,\tau_2), F^I(\tau_2;\tau_1,\tau_2)\right)\right]. \end{aligned} \tag{2.8}$$

The advantage of writing the quanto option price as in Eq.(2.8) is that futures are traded financial assets. Let $\bar{K}_E, \bar{K}_I$ denote the high strikes for the energy and temperature indexes, respectively and $\underline{K}_E, \underline{K}_I$ denote the low strikes for the energy and temperature indexes, respectively. Now we can define the payoff function

$$p(F^E(\tau_2;\tau_1,\tau_2), F^I(\tau_2;\tau_1,\tau_2), \bar{K}_E, \bar{K}_I, \underline{K}_E, \underline{K}_I) := p$$

so that

$$\begin{aligned} p &= \alpha\times[\max\{F^E(\tau_2;\tau_1,\tau_2)-\bar{K}_E, 0\}\times 1_{\{F^I(\tau_2;\tau_1,\tau_2)\geq \bar{K}_I\}} \\ &+ \max\{\underline{K}_E - F^E(\tau_2;\tau_1,\tau_2), 0\}\times 1_{\{F^I(\tau_2;\tau_1,\tau_2)\geq \bar{K}_I\}}], \end{aligned}$$

where α is the contractual volume adjustment factor. As an example as in Benth et al. (2015), we consider the product call structure with the volume adjuster α normalized to 1, that is, we consider the price of an option with the following payoff function:

$$\hat{p} = \max\{F^E(\tau_2;\tau_1,\tau_2)-\bar{K}_E, 0\}\times 1_{\{F^I(\tau_2;\tau_1,\tau_2)\geq \bar{K}_I\}}, \tag{2.9}$$

and the quanto option at time t is given by:

$$C_t = e^{-r(\tau_2-t)}\mathbb{E}_t^{\mathbb{Q}}\left[\hat{p}(F^E(\tau_2;\tau_1,\tau_2), F^I(\tau_2;\tau_1,\tau_2), \bar{K}_E, \bar{K}_I)\right].$$

2.1. **The Asset Dynamics.** We use the HJM risk-neutral dynamics of the forward contract at time t. Consider the general diffusion futures model under the risk-neutral measure $\mathbb{Q}$ be given as :

$$dF^E(t;\tau_1,\tau_2) = \sigma_E(t, F^E(t;\tau_1,\tau_2))dW^E(t), \tag{2.10}$$

$$dF^I(t;\tau_1,\tau_2) = \sigma_I(t, F^I(\tau;\tau_1,\tau_2))dW^I(t)\,. \tag{2.11}$$

with $F^E(0;\tau_1,\tau_2) > 0$ and $F^I(0;\tau_1,\tau_2) > 0$ where σ_E, σ_I are deterministic volatilities and W^E, W^I are correlated Brownian motions with a correlation parameter $\rho\in(-1,1)$. The process F^E is the option price of a future contact written on some energy price and F^I is the option price of a future contact written on some temperature price.

Given an arbitrary W^E, there exists $\widetilde{W}^I$ which is independent of W^E and W^I. Then, we can express W^I as follows

$$W^I = \rho W^E + \sqrt{1-\rho^2}\widetilde{W}^I. \tag{2.12}$$

Thus we have

$$\begin{aligned} dF^E(t;\tau_1,\tau_2) &= \sigma_E(t, F^E(t;\tau_1,\tau_2))dW^E(t), && (2.13)\\ dF^I(t;\tau_1,\tau_2) &= \rho\sigma_I(t, F^I(t;\tau_1,\tau_2))dW^E(t) + \sigma_I(t, F^I(t;\tau_1,\tau_2))\sqrt{1-\rho^2}d\widetilde{W}^I(t)\,. && (2.14)\end{aligned}$$

The above equations can be written in matrix form as follows:

$$\begin{pmatrix} dF^E \\ dF^I \end{pmatrix} = \begin{pmatrix} \sigma_E(t,F^E) & 0 \\ \rho\sigma_I(t,F^I) & \sigma_I(t,F^I)\sqrt{1-\rho^2} \end{pmatrix} \begin{pmatrix} dW^E \\ d\widetilde{W}^I \end{pmatrix}.$$

We can write this as:

$$d\bar{F} = a(t, F^I, F^E)d\bar{W}\,, \tag{2.15}$$

where the matrix $a : ([0,\tau_2]\times\mathbb{R}^2) \to \mathcal{M}_2$, satisfies the growth and Lipschitz conditions. We can write (2.15) as:

$$\bar{F} = \bar{F}_0 + \int_0^t a(t, F^I, F^E)d\bar{W}\,,\ \bar{F}_0 > \mathbf{0}\,. \tag{2.16}$$

For example, given this dynamics the quanto option becomes :

$$C_t = \mathbb{E}^{\mathbb{Q}}\left[\tilde{g}\left(\int_0^{\tau_2}\sigma_E(t,F^E)dW^E\right)\tilde{h}\left(\int_0^{\tau_2}\sigma_I(t,F^I)dW^E, \int_0^{\tau_2}\sigma_I(t,F^I)d\widetilde{W}^I\right)\right], \tag{2.17}$$

where $\tilde{g}(x) = \max\{x-K^E, 0\}$ and $\tilde{h}(x,y) = \rho\max\{x-K^E,0\}\times\sqrt{1-\rho^2}1_{\{y\geq K^I\}}$ are measurable functions.

3. A Primer on the Malliavin Derivative Properties

In this section, we review the necessary Malliavin derivative properties. These properties were also highlighted in Fournié et al. (1999) and Mhlanga (2011) and the proofs can be found in Nualart (2006). Let $\{W(t),\ 0\leq t\leq\tau_2\}$ be an n-dimensional Brownian motion defined on a complete probability space $(\Omega,\mathcal{F},\mathbb{F},\mathbb{Q})$. Let S denote the class of random variables of the form

$$F = f\left(\int_0^{\tau_2} h_1(t)dW(t),\cdots,\int_0^{\tau_2} h_n(t)dW(t)\right),\quad f\in C^\infty(\mathbb{R}^n)\,,$$

where $h_1,\cdots,h_n \in L^2([0,\tau_2])$.

For $F\in S$, the Malliavin derivative DF of F is defined as the process $\{D_tF,\ t\in[0,\tau_2]\}$ in $L^2([0,\tau_2])$ by :

$$D_tF = \sum_{i=1}^n \frac{\partial f}{\partial x_i}\left(\int_0^{\tau_2} h_1(t)dW(t),\cdots,\int_0^{\tau_2} h_n(t)dW(t)\right)h_i(t)\,,\quad t\geq 0\ a.s.$$

On $L^2([0,\tau_2])$ define the norm as :

$$||F||_{1,2} := \left(\mathbb{E}^{\mathbb{Q}}|F|^2 + \mathbb{E}^{\mathbb{Q}}[\int_0^{\tau_2}|D_tF|^2dt]\right)^{\frac{1}{2}}.$$

The chain rule holds for the Malliavin derivative in the following form.

Property P1. Let $F = (F_1, \dots, F_n) \in \mathbb{D}^{1,2}$ and let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function with bounded partial derivatives. Then $\varphi(F) \in \mathbb{D}^{1,2}$ and

$$D_t\varphi(F) = \sum_{i=1}^{n} \frac{\partial\varphi}{\partial x_i}(F) D_t F_i,\ t \geq 0 \ a.s. \tag{3.1}$$

Property P2. Let $\{X_t,\ t \geq 0\}$ be an $\mathbb{R}^n$ valued Itô process whose dynamics are governed by the stochastic differential equation

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t, \tag{3.2}$$

where b and σ are supposed to be continuously differentiable functionals with bounded derivatives and $\sigma(x) \neq 0$ for all $x \in \mathbb{R}^n$. Let $\{Y_t,\ t \geq 0\}$ be the associated first variation process given by the stochastic differential equation

$$dY_t = b'(X_t)Y_t dt + \sum_{i=1}^{n} \sigma_i'(X_t)Y_t dW_t^i,\ Y_0 = I_n, \tag{3.3}$$

where I_n is the identity matrix of $\mathbb{R}^n$, primes denote derivatives and σ_i is the i-th column vector of σ. The the process $\{X_t,\ t \geq 0\}$ belongs to $\mathbb{D}^{1,2}$ and its Malliavin derivative is given by

$$D_r X_t = Y_t Y_r^{-1} \sigma(X_r) 1_{\{r \leq t\}},\ r \geq 0 \ a.s., \tag{3.4}$$

which is equivalent to

$$Y_t = D_r X_t \sigma^{-1}(X_r) Y_r 1_{\{r \leq t\}} \ a.s. \tag{3.5}$$

The Malliavin derivative has an adjoint operator called Skorohod integral (also known as the divergence operator δ). We shall denote the domain of the adjoint operator δ by $\mathrm{Dom}(\delta)$.

Property P3. Let $u \in L^2(\Omega \times [0, \tau_2])$. Then u belongs to the domain $\mathrm{Dom}(\delta)$ of δ if for all $F \in \mathbb{D}^{1,2}$ we have

$$\mid \mathbb{E}\left[\langle DF, u\rangle_{L^2(\Omega)}\right] \mid = \mid \mathbb{E}\left[\int_0^{\tau_2} D_t F u(t) dt\right] \mid \leq c \parallel F \parallel_{L^2(\Omega)} \tag{3.6}$$

where c is some constant depending on u. If u belongs to $\mathrm{Dom}(\delta)$, then

$$\delta(u) = \int_0^{\tau_2} u_t \delta W_t \tag{3.7}$$

is the element of $L^2(\Omega)$ such that the integration by parts formula holds:

$$\mathbb{E}\left[\left(\int_0^{\tau_2} D_t F u_t dt\right)\right] = \mathbb{E}[F\delta(u)] \text{ for all } F \in \mathbb{D}^{1,2}. \tag{3.8}$$

An important property of the Skorohod integral δ is that its domain $\mathrm{Dom}(\delta)$ contains all adapted stochastic processes which belong to $L^2(\Omega \times [0, \tau_2])$. For such processes the Skorohod integral δ coincides with the Itô stochastic integral.

Property P4. If u is an adapted process belonging to $L^2(\Omega \times [0, \tau_2])$, then

$$\delta(u) = \int_0^{\tau_2} u(t) dW_t. \tag{3.9}$$

Further, if the random variable F is $\mathcal{F}_{\tau_2}$-adapted and belongs to $\mathbb{D}^{1,2}$ then, for any u in $Dom(\delta)$, the random variable Fu will be Skorohod integrable.

Property P5. Let F belongs to $\mathbb{D}^{1,2}$ and $u \in Dom(\delta)$ such that $\mathbb{E}[\int_0^{\tau_2} F^2 u_t^2 dt] < \infty$. Then $Fu \in Dom(\delta)$ and

$$\delta(Fu) = F\delta(u) - \int_0^{\tau_2} D_t F u_t dt, \tag{3.10}$$

whenever the right hand side belongs to $L^2(\Omega)$. In particular, if u is moreover adapted, we have

$$\delta(Fu) = F\int_0^{\tau_2} u_t dW_t - \int_0^{\tau_2} D_t F u_t dt. \tag{3.11}$$

4. Computation of Greeks: The Independent Case

From the diffusion stochastic differential equation (2.13) with $\rho = 0$, consider the following HJM risk-neutral dynamics of the forward contract at time t. We call this the 'independent case'. Let the future price processes under the risk-neutral measure $\mathbb{Q}$ be given as

$$dF^i(t;\tau_1,\tau_2) = \sigma_i(t;\tau_1,\tau_2)F^i(t;\tau_1,\tau_2)dW^i(\tau),\ F^i(0;\tau_1,\tau_2) > 0, \tag{4.1}$$

for $E, I = i$. The function $F^i(0;\tau_1,\tau_2)$ represents today's forward price. We call this, the independent case since $\rho = 0$. Explicitly this can be written as:

$$\begin{aligned} F^E(\tau_2;\tau_1,\tau_2) &= F^E(0;\tau_1,\tau_2)\exp\left(-\frac{1}{2}\int_0^{\tau_2}\sigma_E^2(u;\tau_1,\tau_2)du + \int_0^{\tau_2}\sigma_E(u;\tau_1,\tau_2)dW^E(u)\right) \\ F^I(\tau_2;\tau_1,\tau_2) &= F^I(0;\tau_1,\tau_2)\exp\left(-\frac{1}{2}\int_0^{\tau_2}\sigma_I^2(u;\tau_1,\tau_2)du + \int_0^{\tau_2}\sigma_I(u;\tau_1,\tau_2)dW^I(u)\right), \end{aligned}$$

where W^E and W^I are Brownian motions. Here, $\int_0^{\tau_2}\sigma_i^2(\tau;\tau_1,\tau_2)d\tau < \infty$ meaning $\tau \mapsto F^i(\tau;\tau_1,\tau_2)$ is a martingale. Introduce $g : \mathbb{R} \mapsto \mathbb{R}$ and $h : \mathbb{R} \mapsto \mathbb{R}$ measurable functions. The payoff structure of a quanto option on the forwards with maturity at time τ_2 given by

$$C = \mathbb{E}^{\mathbb{Q}}[g(F^E(\tau_2;\tau_1,\tau_2))h(F^I(\tau_2;\tau_1,\tau_2))], \tag{4.2}$$

where $g(x) = \max\{x - K^E, 0\}$ and $h(y) = 1_{\{y \geq K^I\}}$ and the risk-free interest rate $r = 0$. We assume the following integrability conditions:

$$\mathbb{E}[g^2(F^E(\tau_2;\tau_1,\tau_2))] < \infty, \quad \mathbb{E}[h^2(F^I(\tau_2;\tau_1,\tau_2))] < \infty. \tag{4.3}$$

At several places, we will require the diffusion matrix σ_i, $i = E, I$ to satisfy the following condition:

$$\exists \eta > 0\ \xi^*\sigma_i^*(t;\tau_1,\tau_2)\sigma_i(t;\tau_1,\tau_2)\xi > \eta \mid \xi \mid^2 \text{ for all } \xi \in \mathbb{R}^n,\ t \in [\tau_1,\tau_2] \text{ with } \xi \neq 0. \tag{4.4}$$

where $i = E, I$ and ξ^* denotes the transpose of ξ. This is called the *uniform ellipticity condition.*

The weight function obtained when computing Greeks using the integration by parts formula should not degenerate with probability one, otherwise the computation will not be valid. To avoid this degeneracy we introduce the set Υ_n (see Mhlanga (2011)) defined by

$$\Upsilon_n = \{a \in L^2([0,\tau_2]) \mid \int_0^{t_i} a(t)dt = 1 \text{ for all } i = 1,\dots,n\}. \tag{4.5}$$

We need the following lemma.

Lemma 4.1. *Let a_n and b_n be two convergent sequences such that $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$. Then the sequence $a_n b_n$ is convergent and $\lim_{n\to\infty} a_n b_n = ab$.*

Finally, we state our results.

Proposition 4.2. *Assume that the diffusion matrix σ_E is uniformly elliptic. Then for all $a \in \Upsilon_n$, the energy delta is given by :*

$$\Delta_E = \mathbb{E}^{\mathbb{Q}}[g(F^E(\tau_2;\tau_1,\tau_2))h(F^I(\tau_2;\tau,\tau_2))\pi^{\Delta_E}], \tag{4.6}$$

where the Malliavin weight π^{Δ_E} is

$$\pi^{\Delta_E} = \int_0^{\tau_2} a(t)\left(\sigma_E^{-1}(t;\tau_1,\tau_2)Y_E(t;\tau_1,\tau_2)\right)^* dW^E(t).$$

Proof. Let g be a continuously differentiable function with bounded derivatives. Introduce

$$Y_E(t;\tau_1,\tau_2) = \exp\left(-\frac{1}{2}\int_0^t \sigma_E^2(u;\tau_1,\tau_2)du + \int_0^t \sigma_E(u;\tau_1,\tau_2)dW^E(u)\right).$$

This implies that

$$F^E(\tau;\tau_1,\tau_2) = F^E(0;\tau_1,\tau_2)Y_E(t;\tau_1,\tau_2).$$

An application of *Property P*2 shows that $F^E(\tau_2;\tau_1,\tau_2)$ belongs to $\mathbb{D}^{1,2}$ and we have:

$$D_tF^E(\tau_2;\tau_1,\tau_2) \quad = \quad Y_E(\tau_2;\tau_1,\tau_2)Y_E^{-1}(t;\tau_1,\tau_2)\sigma_E(t;\tau_1,\tau_2)1_{t<\tau_2}.$$

This is equivalent to

$$Y_E(\tau_2;\tau_1,\tau_2)1_{t<\tau_2} \quad = \quad D_tF^E(\tau_2;\tau_1,\tau_2)\sigma_E^{-1}(t;\tau_1,\tau_2)Y_E(t;\tau_1,\tau_2).$$

Multiply both sides by a square function which integrates to 1 on $[0,\tau_2]$

$$Y_E(\tau_2;\tau_1,\tau_2) = \int_0^{\tau_2} D_tF^E(\tau_2;\tau_1,\tau_2)a(t)\sigma_E^{-1}(t;\tau_1,\tau_2)Y_E(t;\tau_1,\tau_2)dt.$$

Now

$$\begin{aligned}
\Delta_E \quad &:= \quad \frac{\partial}{\partial F^E(0;\tau_1,\tau_2)}\mathbb{E}^{\mathbb{Q}}[g(F^E(\tau_2;\tau_1,\tau_2))h(F^I(\tau_2;\tau_1,\tau_2))] \\
&= \quad \mathbb{E}^{\mathbb{Q}}[g'(F^E(\tau_2;\tau_1,\tau_2))h(F^I(\tau_2;\tau_1,\tau_2))\frac{\partial F^E(\tau_2;\tau_1,\tau_2)}{\partial F^E(0;\tau_1,\tau_2)}] \\
&= \quad \mathbb{E}^{\mathbb{Q}}[g'(F^E(\tau_2;\tau_1,\tau_2))h(F^I(\tau_2;\tau_1,\tau_2))Y_E(\tau_2;\tau_1,\tau_2)] \\
&= \quad \mathbb{E}^{\mathbb{Q}}\left[\int_0^{\tau_2} g'(F^E(\tau_2;\tau_1,\tau_2))h(F^I(\tau_2;\tau_1,\tau_2))\right. \\
&\qquad \times D_tF^E(\tau_2;\tau_1,\tau_2)a(t)\sigma_E^{-1}(t;\tau_1,\tau_2)Y_E(t;\tau_1,\tau_2)dt] \\
&= \quad \mathbb{E}^{\mathbb{Q}}\left[h(F^I(\tau_2;\tau_1,\tau_2))\int_0^{\tau_2} D_tg(F^E(\tau_2;\tau_1,\tau_2))a(t)\sigma_E^{-1}(t;\tau_1,\tau_2)Y_E(t;\tau_1,\tau_2)dt\right] \\
&= \quad \mathbb{E}^{\mathbb{Q}}\left[g(F^E(\tau_2;\tau_1,\tau_2))h(F^I(\tau_2;\tau_1,\tau_2))\int_0^{\tau_2} a(t)\left(\sigma_E^{-1}(t;\tau_1,\tau_2)Y_E(t;\tau_1,\tau_2)\right)^* dW^E(t)\right]
\end{aligned}$$

where g' denotes the derivative of g with respect to $F^E(0;\tau_1,\tau_2)$. Here, we have used the chain rule property, (*Property P*1), the integration by parts formula (*Property P*3), and the fact that the Skorohod integral coincides with the Itô stochastic integral (*Property P*4).
Now consider the general case $g(F^E(\tau_2;\tau_1,\tau_2))h(F^I(\tau_2;\tau,\tau_2)) \in L^2(\Omega)$. Since smooth functions with compact support $C_K^\infty(\mathbb{R})$ is dense in $L^2(\mathbb{R})$, we can find sequences of functions $g_n \in C_K^\infty(\mathbb{R})$ and $h_n \in C_K^\infty(\mathbb{R})$ such that $g_n \to g$ and $h_n \to h$ in $L^2(\mathbb{R})$. An application of

Lemma 4.1 shows that $(g_n h_n)_{n\in\mathbb{N}}$ converges uniformly on compact sets to gh as $n\to\infty$, that is,

$$\lim_{n\to\infty}\mathbb{E}^{\mathbb{Q}}\left[\left(g_n(F^E(\tau_2;\tau_1,\tau_2))h_n(F^I(\tau_2;\tau,\tau_2)) - g(F^E(\tau_2;\tau_1,\tau_2))h(F^I(\tau_2;\tau,\tau_2))\right)^2\right]\to 0. \tag{4.7}$$

Denote by

$$f(x) = \mathbb{E}^{\mathbb{Q}}\left[g(F^E(\tau_2;\tau_1,\tau_2))h(F^I(\tau_2;\tau,\tau_2))\int_0^{\tau_2} a(t)\left(\sigma_E^{-1}(t;\tau_1,\tau_2)Y_E(t;\tau_1,\tau_2)\right)^* dW^E(t)\right].$$

Using the obtained result for g_n, together with the Cauchy-Schwartz inequality and the Itô isometry, we obtain

$$\begin{aligned}
&\left|\frac{\partial}{\partial F^E(0;\tau_1,\tau_2)}\mathbb{E}^{\mathbb{Q}}[g_n(F^E(\tau_2;\tau_1,\tau_2))h_n(F^I(\tau_2;\tau,\tau_2))] - f(x)\right| \\
= &\left|\mathbb{E}^{\mathbb{Q}}\left[\left(g_n(F^E(\tau_2;\tau_1,\tau_2))h_n(F^I(\tau_2;\tau,\tau_2)) - g(F^E(\tau_2;\tau_1,\tau_2))h(F^I(\tau_2;\tau,\tau_2))\right)\right.\right. \\
&\left.\left.\times\int_0^{\tau_2} a(t)\left(\sigma_E^{-1}(t;\tau_1,\tau_2)Y_E(t;\tau_1,\tau_2)\right)^* dW^E(t)\right]\right| \\
\leq &\ \mathbb{E}^{\mathbb{Q}}\left[\left(g_n(F^E(\tau_2;\tau_1,\tau_2))h_n(F^I(\tau_2;\tau,\tau_2)) - g(F^E(\tau_2;\tau_1,\tau_2))h(F^I(\tau_2;\tau,\tau_2))\right)^2\right]^{\frac{1}{2}} \\
&\times\mathbb{E}^{\mathbb{Q}}\left[\int_0^{\tau_2}\left|a(t)\left(\sigma_E^{-1}(t;\tau_1,\tau_2)Y_E(t;\tau_1,\tau_2)\right)^*\right|^2 dt\right]^{\frac{1}{2}}
\end{aligned}$$

It follows from (4.7) that the first expression on the right hand side of the above inequality converges uniformly on compact sets to 0. The second expression is a continuous function, hence it is bounded on any compact set. Therefore, we have

$$\frac{\partial}{\partial F^E(0;\tau_1,\tau_2)}\mathbb{E}^{\mathbb{Q}}\left[g_n(F^E(\tau_2;\tau_1,\tau_2))h_n(F^I(\tau_2;\tau,\tau_2))\right]\to f(x)$$

uniformly on compact sets. Hence, we conclude that $\mathbb{E}^{\mathbb{Q}}\left[g_n(F^E(\tau_2;\tau_1,\tau_2))h_n(F^I(\tau_2;\tau,\tau_2))\right]$ is differentiable with respect to $F^E(0;\tau_1,\tau_2)$ and that the derivative is given by

$$\begin{aligned}
&\frac{\partial}{\partial F^E(0;\tau_1,\tau_2)}\mathbb{E}^{\mathbb{Q}}[g_n(F^E(\tau_2;\tau_1,\tau_2))h_n(F^I(\tau_2;\tau,\tau_2))] \\
&= \mathbb{E}^{\mathbb{Q}}\left[g(F^E(\tau_2;\tau_1,\tau_2))h(F^I(\tau_2;\tau,\tau_2))\int_0^{\tau_2} a(t)\left(\sigma_E^{-1}(t;\tau_1,\tau_2)Y_E(t;\tau_1,\tau_2)\right)^* dW^E(t)\right]
\end{aligned}$$

which completes the proof. □

Similarly, we obtain the following result.

Proposition 4.3. *Assume that the diffusion matrix σ_I is uniformly elliptic. Then for all $a\in\Upsilon_n$, the temperature delta is given by:*

$$\Delta_I = \mathbb{E}^{\mathbb{Q}}[g(F^E(\tau_2;\tau_1,\tau_2))h(F^I(\tau_2;\tau,\tau_2))\pi^{\Delta_I}],$$

where the Malliavin weight π^{Δ_I} is

$$\pi^{\Delta_I} = \int_0^{\tau_2} a(t)\left(\sigma_I^{-1}(t;\tau_1,\tau_2)Y_I(t;\tau_1,\tau_2)\right)^* dW^I(t).$$

Proof. Follows along the lines of the proof of Proposition of 4.2. □

The following result gives the cross-gamma hedge in the independent case:

Proposition 4.4. *Assume that the diffusion matrices σ_i, $i = E, I$ are uniformly elliptic. Then for all $a \in \Upsilon_n$, the cross-gamma is given by:*

$$\Delta_{EI} = \mathbb{E}^{\mathbb{Q}}[g(F^E(\tau_2;\tau_1,\tau_2))h(F^I(\tau_2;\tau,\tau_2))\pi^{\Delta_{EI}}],$$

where the Malliavin weight $\pi^{\Delta_{EI}}$ is

(4.8)

$$\pi^{\Delta_{EI}} = \int_0^{\tau_2} a(t)\left(\sigma_E^{-1}(t;\tau_1,\tau_2)Y_E(t;\tau_1,\tau_2)\right)^* dW^E(t) \int_0^{\tau_2} a(t)\left(\sigma_I^{-1}(t;\tau_1,\tau_2)Y_I(t;\tau_1,\tau_2)\right)^* dW^I(t).$$

Proof. We first assume that g and h are continuously differentiable with bounded derivatives. From Proposition 4.2 we have

$$\Delta_E = \mathbb{E}^{\mathbb{Q}}[g(F^E(\tau_2;\tau_1,\tau_2))h(F^I(\tau_2;\tau,\tau_2))\pi^{\Delta_E}].$$

As in Proposition 4.2, we introduce

$$Y_I(t;\tau_1,\tau_2) = \exp\left(-\frac{1}{2}\int_0^t \sigma_I^2(u;\tau_1,\tau_2)du + \int_0^t \sigma_I(u;\tau_1,\tau_2)dW^I(u)\right).$$

This implies that

$$F^I(\tau;\tau_1,\tau_2) = F^I(0;\tau_1,\tau_2)Y_I(t;\tau_1,\tau_2).$$

An application of *Property* $P2$ shows that $F^I(\tau_2;\tau_1,\tau_2)$ belongs to $\mathbb{D}^{1,2}$ and we have:

$$D_t F^I(\tau_2;\tau_1,\tau_2) \quad = \quad Y_I(\tau_2;\tau_1,\tau_2)Y_I^{-1}(t;\tau_1,\tau_2)\sigma_I(t;\tau_1,\tau_2)1_{t<\tau_2}.$$

This is equivalent to

$$Y_I(\tau_2;\tau_1,\tau_2)1_{t<\tau_2} \quad = \quad D_t F^I(\tau_2;\tau_1,\tau_2)\sigma_I^{-1}(t;\tau_1,\tau_2)Y_I(t;\tau_1,\tau_2).$$

Multiply both sides by a square function which integrates to 1 on $[0,\tau_2]$

$$Y_I(\tau_2;\tau_1,\tau_2) = \int_0^{\tau_2} D_t F^I(\tau_2;\tau_1,\tau_2)a(t)\sigma_I^{-1}(t;\tau_1,\tau_2)Y_I(t;\tau_1,\tau_2)dt.$$

Now

$$\begin{aligned}
\Delta_{EI} \quad &:= \quad \frac{\partial^2}{\partial F^E(0;\tau_1,\tau_2)\partial F^I(0;\tau_1,\tau_2)}\mathbb{E}^{\mathbb{Q}}\left[g(F^E(\tau_2;\tau_1,\tau_2))h(F^I(\tau_2;\tau,\tau_2))\right] \\
&= \quad \frac{\partial}{\partial F^I(0;\tau_1,\tau_2)}\Delta_E \\
&= \quad \frac{\partial}{\partial F^I(0;\tau_1,\tau_2)}\left[\mathbb{E}^{\mathbb{Q}}\left[g(F^E(\tau_2;\tau_1,\tau_2))h(F^I(\tau_2;\tau,\tau_2))\pi^{\Delta_E}\right]\right] \\
&= \quad \mathbb{E}^{\mathbb{Q}}\left[g(F^E(\tau_2;\tau_1,\tau_2))\pi^{\Delta_E}h'(F^I(\tau_2;\tau,\tau_2))\frac{\partial F^I(\tau_2;\tau_1,\tau_2)}{\partial F^I(0;\tau_1,\tau_2)}\right] \\
&= \quad \mathbb{E}^{\mathbb{Q}}\left[g(F^E(\tau_2;\tau_1,\tau_2))\pi^{\Delta_E}h'(F^I(\tau_2;\tau,\tau_2))Y_I(\tau_2;\tau_1,\tau_2)\right] \\
&= \quad \mathbb{E}^{\mathbb{Q}}\left[\int_0^{\tau_2} g(F^E(\tau_2;\tau_1,\tau_2))h'(F^I(\tau_2;\tau_1,\tau_2))\pi^{\Delta_E}\right. \\
&\qquad \left.\times D_t F^I(\tau_2;\tau_1,\tau_2)a(t)\sigma_I^{-1}(t;\tau_1,\tau_2)Y_I(t;\tau_1,\tau_2)dt\right] \\
&= \quad \mathbb{E}^{\mathbb{Q}}\left[g(F^E(\tau_2;\tau_1,\tau_2))\pi^{\Delta_E}\right.
\end{aligned}$$

$$\begin{aligned}
&\quad \times \int_0^{\tau_2} D_t(h(F^I(\tau_2;\tau_1,\tau_2)))a(t)\sigma_I^{-1}(t;\tau_1,\tau_2)Y_I(t;\tau_1,\tau_2)dt\Big] \\
&= \mathbb{E}^{\mathbb{Q}}\left[g(F^E(\tau_2;\tau_1,\tau_2))h(F^I(\tau_2;\tau_1,\tau_2))\pi^{\Delta_E}\right. \\
&\quad \left.\times \int_0^{\tau_2} a(t)\left(\sigma_I^{-1}(t;\tau_1,\tau_2)Y_I(t;\tau_1,\tau_2)\right)^* dW^I(t)\right].
\end{aligned}$$

Here, we have used the chain rule property, (*Property* $P1$), the integration by parts formula (*Property* $P3$), and the fact that the Skorohod integral coincides with the Itô stochastic integral (*Property* $P4$).
Now consider the general case $g(F^E(\tau_2;\tau_1,\tau_2))h(F^I(\tau_2;\tau,\tau_2)) \in L^2(\Omega)$. Since the set $C_K^\infty(\mathbb{R})$ of infinitely differentiable functions with compact support is dense in $L^2(\mathbb{R})$, there exist sequences of functions $g_n \in C_K^\infty(\mathbb{R})$ and $h_n \in C_K^\infty(\mathbb{R})$ such that $g_n \to g$ and $h_n \to h$ in $L^2(\mathbb{R})$. By Lemma 4.1, $(g_n h_n)_{n\in\mathbb{N}}$ converges uniformly on compact sets to gh as $n \to \infty$, that is,
(4.9)

$$\lim_{n\to\infty} \mathbb{E}^{\mathbb{Q}}\left[\left(g_n(F^E(\tau_2;\tau_1,\tau_2))h_n(F^I(\tau_2;\tau,\tau_2)) - g(F^E(\tau_2;\tau_1,\tau_2))h(F^I(\tau_2;\tau,\tau_2))\right)^2\right] \to 0.$$

Denote by

$$f(x) = \mathbb{E}^{\mathbb{Q}}\left[g(F^E(\tau_2;\tau_1,\tau_2))h(F^I(\tau_2;\tau,\tau_2))\pi^{\Delta_{EI}}\right]$$

with $\pi^{\Delta_{EI}}$ given by (4.8). Using the first step

$$\frac{\partial^2}{\partial F^E(0;\tau_1,\tau_2)\partial F^I(0;\tau_1,\tau_2)}\mathbb{E}^{\mathbb{Q}}\left[g(F^E(\tau_2;\tau_1,\tau_2))h(F^I(\tau_2;\tau,\tau_2))\right]$$

exists and

$$\begin{aligned}
&\left|\frac{\partial^2}{\partial F^E(0;\tau_1,\tau_2)\partial F^I(0;\tau_1,\tau_2)}\mathbb{E}^{\mathbb{Q}}\left[g(F^E(\tau_2;\tau_1,\tau_2))h(F^I(\tau_2;\tau,\tau_2))\right] - f(x)\right| \\
= &\left|\mathbb{E}^{\mathbb{Q}}\left[\left(g_n(F^E(\tau_2;\tau_1,\tau_2))h_n(F^I(\tau_2;\tau,\tau_2)) - g(F^E(\tau_2;\tau_1,\tau_2))h(F^I(\tau_2;\tau,\tau_2))\right)\pi^{\Delta_{EI}}\right]\right| \\
\leq &\ \mathbb{E}^{\mathbb{Q}}\left[\left(g_n(F^E(\tau_2;\tau_1,\tau_2))h_n(F^I(\tau_2;\tau,\tau_2)) - g(F^E(\tau_2;\tau_1,\tau_2))h(F^I(\tau_2;\tau,\tau_2))\right)^2\right]^{\frac{1}{2}} \\
&\times \mathbb{E}^{\mathbb{Q}}\left[\left|\pi^{\Delta_{EI}}\right|^2\right]^{\frac{1}{2}}
\end{aligned}$$

where we have used the Cauchy-Schwartz inequality. It follows from (4.9) that the first expression on the right hand side of the above inequality converges uniformly on compact sets to 0. Using the Cauchy-Schwartz inequality and Itô isometry one can show that the second expression is finite. Therefore, we have

$$\frac{\partial^2}{\partial F^E(0;\tau_1,\tau_2)\partial F^I(0;\tau_1,\tau_2)}\mathbb{E}^{\mathbb{Q}}\left[g(F^E(\tau_2;\tau_1,\tau_2))h(F^I(\tau_2;\tau,\tau_2))\right] \to f(x)$$

uniformly on compact sets. Hence, we conclude that $\mathbb{E}^{\mathbb{Q}}\left[g_n(F^E(\tau_2;\tau_1,\tau_2))h_n(F^I(\tau_2;\tau,\tau_2))\right]$ is twice differentiable with respect to $F^E(0;\tau_1,\tau_2)$ and $F^I(0;\tau_1,\tau_2)$, and that the desired formula holds. The proof is complete. □

5. Computation of Greeks: The Correlation Case

We consider the following HJM

$$dF^E(t;\tau_1,\tau_2) = \sigma_E(t,F^E(t;\tau_1,\tau_2))dW^E(t), \tag{5.1}$$

$$dF^I(t;\tau_1,\tau_2) = \rho\sigma_I(t,F^I(t;\tau_1,\tau_2))dW^E(t)+\sigma_I(t,F^I(t;\tau_1,\tau_2))\sqrt{1-\rho^2}d\widetilde{W}^I(t)\,. \tag{5.2}$$

That is, we consider the case when there is correlation between F^E and F^I. Suppose the Brownian motions B_1 and B_2 are independent. Let $W_1 = B_1$ and $W_2 = \rho B_1 + \sqrt{1-\rho^2}B_2$. This implies that

$$g(W_1)h(W_2) = g(B_1)h(\rho B_1 + \sqrt{1-\rho^2}B_2).$$

In this setting, we have the following quanto option structure:

$$C = \mathbb{E}^{\mathbb{Q}}[g(F^E(\tau_2;\tau_1,\tau_2))h(\rho F^E(\tau_2;\tau_1,\tau_2)+\sqrt{1-\rho^2}F^I(\tau_2;\tau_1,\tau_2))]\,.$$

Now we derive the energy delta.

Proposition 5.1. *Assume that the diffusion matrix σ_E uniformly elliptic. Then for all $a \in \Upsilon_n$, the energy delta is given by:*

$$\Delta_E = \mathbb{E}^{\mathbb{Q}}[g(F^E(\tau_2;\tau_1,\tau_2))h(\rho F^E(\tau_2;\tau_1,\tau_2)+\sqrt{1-\rho^2}F^I(\tau_2;\tau_1,\tau_2))\pi^{\Delta_E}(1+\rho)]\,, \tag{5.3}$$

where the Malliavin weight π^{Δ_E} is

$$\pi^{\Delta_E} = \int_0^{\tau_2} a(t)\left(\sigma_E^{-1}(t;\tau_1,\tau_2)Y_E(t;\tau_1,\tau_2)\right)^* dW^E(t).$$

Proof. Let g be a continuously differentiable function with bounded derivatives. As in Proposition 4.2, introduce

$$Y_E(t;\tau_1,\tau_2) = \exp(-\frac{1}{2}\int_0^t \sigma_E^2(u;\tau_1,\tau_2)du + \int_0^t \sigma_E(u;\tau_1,\tau_2)dW^E(u)).$$

This implies that

$$F^E(t;\tau_1,\tau_2) = F^E(0;\tau_1,\tau_2)Y_E(t;\tau_1,\tau_2).$$

An application of *Property P*2 shows that $F^E(\tau_2;\tau_1,\tau_2)$ belongs to $\mathbb{D}^{1,2}$ and we have:

$$D_tF^E(\tau_2;\tau_1,\tau_2) = Y_E(\tau_2;\tau_1,\tau_2)Y_E^{-1}(t;\tau_1,\tau_2)\sigma_E(t;\tau_1,\tau_2)1_{t<\tau_2}.$$

This is equivalent to

$$Y_E(\tau_2;\tau_1,\tau_2)1_{t<\tau_2} = D_tF^E(\tau_2;\tau_1,\tau_2)\sigma_E^{-1}(t;\tau_1,\tau_2)Y_E(t;\tau_1,\tau_2).$$

Multiply both sides by a square function which integrates to 1 on $[0,\tau_2]$

$$Y_E(\tau_2;\tau_1,\tau_2) = \int_0^{\tau_2} D_tF^E(\tau_2;\tau_1,\tau_2)a(t)\sigma_E^{-1}(t;\tau_1,\tau_2)Y_E(t;\tau_1,\tau_2)dt.$$

Now

$$\begin{aligned}
\Delta_E &= \mathbb{E}^{\mathbb{Q}}\Big[g'(F^E(\tau_2;\tau_1,\tau_2))h(\rho F^E(\tau_2;\tau_1,\tau_2)+\sqrt{1-\rho^2}F^I(\tau_2;\tau_1,\tau_2))Y_E(\tau_2;\tau_1,\tau_2) \\
&\quad +g(F^E(\tau_2;\tau_1,\tau_2))h'(\rho F^E(\tau_2;\tau_1,\tau_2)+\sqrt{1-\rho^2}F^I(\tau_2;\tau_1,\tau_2))\rho Y_E(\tau_2;\tau_1,\tau_2)\Big] \\
&= \mathbb{E}^{\mathbb{Q}}\Big[\int_0^{\tau_2} g'(F^E(\tau_2;\tau_1,\tau_2))h(\rho F^E(\tau_2;\tau_1,\tau_2)+\sqrt{1-\rho^2}F^I(\tau_2;\tau_1,\tau_2)) \\
&\quad \times D_tF^E(\tau_2;\tau_1,\tau_2)a(t)\sigma_E^{-1}(t;\tau_1,\tau_2)Y_E(t;\tau_1,\tau_2)dt
\end{aligned}$$

$$
\begin{aligned}
& + \int_0^{\tau_2} g(F^E(\tau_2;\tau_1,\tau_2))h'(\rho F^E(\tau_2;\tau_1,\tau_2) + \sqrt{1-\rho^2}F^I(\tau_2;\tau_1,\tau_2)) \\
& \times D_t F^E(\tau_2;\tau_1,\tau_2)a(t)\sigma_E^{-1}(t;\tau_1,\tau_2)Y_E(t;\tau_1,\tau_2)dt\big] \\
= \; & \mathbb{E}^{\mathbb{Q}}\Big[h(\rho F^E(\tau_2;\tau_1,\tau_2) + \sqrt{1-\rho^2}F^I(\tau_2;\tau_1,\tau_2)) \\
& \times \int_0^{\tau_2} D_t g(F^E(\tau_2;\tau_1,\tau_2))a(t)\sigma_E^{-1}(t;\tau_1,\tau_2)Y_E(t;\tau_1,\tau_2)dt + \rho g(F^E(\tau_2;\tau_1,\tau_2)) \\
& \times \int_0^{\tau_2} D_t h(\rho F^E(\tau_2;\tau_1,\tau_2) + \sqrt{1-\rho^2}F^I(\tau_2;\tau_1,\tau_2))a(t)\sigma_E^{-1}(t;\tau_1,\tau_2)Y_E(t;\tau_1,\tau_2)dt\Big] \\
= \; & \mathbb{E}^{\mathbb{Q}}\Big[h(\rho F^E(\tau_2;\tau_1,\tau_2) + \sqrt{1-\rho^2}F^I(\tau_2;\tau_1,\tau_2)) \\
& \times g(F^E(\tau_2;\tau_1,\tau_2)) \int_0^{\tau_2} a(t)\left(\sigma_E^{-1}(t;\tau_1,\tau_2)Y_E(t;\tau_1,\tau_2)\right)^* dW^E(t) \\
& + \rho g(F^E(\tau_2;\tau_1,\tau_2))h(\rho F^E(\tau_2;\tau_1,\tau_2) + \sqrt{1-\rho^2}F^I(\tau_2;\tau_1,\tau_2)) \\
& \times \int_0^{\tau_2} a(t)\left(\sigma_E^{-1}(t;\tau_1,\tau_2)Y_E(t;\tau_1,\tau_2)\right)^* dW^E(t)\Big].
\end{aligned}
$$

As in the proof of Proposition 4.2, we have used the chain rule property, (*Property* $P1$), the integration by parts formula (*Property* $P3$), and the fact that the Skorohod integral coincides with the Itô stochastic integral (*Property* $P4$).
The proof for the general case follow similar arguments as in the proof of Proposition 4.2 with minor modifications. We omit the details. □

Now we derive the temperature delta in the correlation case.

Proposition 5.2. *Assume that the diffusion matrix σ_I uniformly elliptic. Then for all $a \in \Upsilon_n$, the temperature delta is given by:*

$$
\Delta_I = \sqrt{1-\rho^2}\mathbb{E}^{\mathbb{Q}}[g(F^E(\tau_2;\tau_1,\tau_2))h(\rho F^E(\tau_2;\tau_1,\tau_2) + \sqrt{1-\rho^2}F^I(\tau_2;\tau_1,\tau_2))\pi^{\Delta_I}].
$$

where the Malliavin weight π^{Δ_I} is

$$
\pi^{\Delta_I} = \int_0^{\tau_2} a(t)\left(\sigma_I^{-1}(t;\tau_1,\tau_2)Y_I(t;\tau_1,\tau_2)\right)^* dW^I(t).
$$

Proof. The proof is similar to that of Proposition 5.1 with minor modifications. □

The following result gives the cross-gamma hedge in the correlation case.

Proposition 5.3. *Assume that the diffusion matrices σ_i, $i = E, I$ are uniformly elliptic. Then for all $a \in \Upsilon_n$, the cross-gamma is given by:*

$$
\begin{aligned}
\Delta_{EI} = \; & \sqrt{1-\rho^2}\mathbb{E}^{\mathbb{Q}}[g(F^E(\tau_2;\tau_1,\tau_2))h(\rho F^E(\tau_2;\tau_1,\tau_2) + \sqrt{1-\rho^2}F^I(\tau_2;\tau_1,\tau_2))\pi^{\Delta_{EI}} \\
& + \rho\sqrt{1-\rho^2}g(F^E(\tau_2;\tau_1,\tau_2))h(\rho F^E(\tau_2;\tau_1,\tau_2) + \sqrt{1-\rho^2}F^I(\tau_2;\tau_1,\tau_2))\pi^{\Delta_{EI}}]
\end{aligned}
$$

where the Malliavin weight $\pi^{\Delta_{EI}}$ is:

$$
\pi^{\Delta_{EI}} = \int_0^{\tau_2} a(t)\left(\sigma_E^{-1}(t;\tau_1,\tau_2)Y_E(t;\tau_1,\tau_2)\right)^* dW^E(t) \int_0^{\tau_2} a(t)\left(\sigma_I^{-1}(t;\tau_1,\tau_2)Y_I(t;\tau_1,\tau_2)\right)^* dW^I(t).
$$

Proof. The proof follows the same line of argument as in the proof of Proposition 4.4 with minor modifications. The details are omitted. □

6. Examples

We will provide Malliavin weights in the case where the quanto option payoff functions depend on the terminal value, that is, $\tau_2 = T$

6.1. **The independent case.**

We consider the following stochastic differential equations to describe the energy price F^E and the temperature price F^I dynamics

$$\frac{dF^E}{F^E} = \sigma_E dW_t^E,\ F^E(0) > 0 \tag{6.1}$$

$$\frac{dF^I}{F^I} = \sigma_I dW_t^I,\ F^I(0) > 0, \tag{6.2}$$

where σ_E, σ_I are deterministic volatilities and W^E, W^I are independent Brownian motions. The quanto option pricing formula is then expressed as

$$C_t = \mathbb{E}^{\mathbb{Q}}[g(F^E)h(F^I)], \tag{6.3}$$

with $g(x) = \max\{x - K^E, 0\}$ and $h(y) = 1_{\{y \geq K^I\}}$. By using the general formulae developed in the previous sections, we are able to compute analytically the values of different Malliavin weights. Here we set $a(t) = \frac{1}{T}$. We have

$$\pi^{\Delta_E} = \frac{1}{F^E(0)T}\int_0^T \frac{1}{\sigma_E} dW^E(t).$$

$$\pi^{\Delta_I} = \frac{1}{F^I(0)T}\int_0^T \frac{1}{\sigma_I} dW^I(t).$$

$$\pi^{\Delta_{EI}} = \frac{1}{F^E(0)F^I(0)T^2}\left(\int_0^T \frac{1}{\sigma_E} dW^E(t)\right)\left(\int_0^T \frac{1}{\sigma_I} dW^I(t)\right).$$

6.2. **The correlation case.**

Again, we consider the following stochastic differential equations to describe the energy price F^E and the temperature price F^I dynamics

$$\frac{dF^E}{F^E} = \sigma_E dW_t^E,\ F^E(0) > 0 \tag{6.4}$$

$$\frac{dF^I}{F^I} = \rho\sigma_I dW_t^I + \sigma_I\sqrt{1-\rho^2}\, d\widetilde{W}^I,\ F^I(0) > 0, \tag{6.5}$$

where W_t^E, W_t^I are correlated Brownian motions with correlation parameter $\rho \in (-1, 1)$.

The system of stochastic differential equations can be written in a matrix form

$$\begin{pmatrix} \frac{dF^E}{F^E} \\ \frac{dF^I}{F^I} \end{pmatrix} = \begin{pmatrix} \sigma_E & 0 \\ \rho\sigma_I & \sigma_I\sqrt{1-\rho^2} \end{pmatrix}\begin{pmatrix} dW^E \\ d\widetilde{W}^I \end{pmatrix}.$$

The inverse matrix of

$$\begin{pmatrix} \sigma_E & 0 \\ \rho\sigma_I & \sigma_I\sqrt{1-\rho^2} \end{pmatrix}$$

is calculated as

$$\frac{1}{\sigma_E\sigma_I\sqrt{1-\rho^2}}\begin{pmatrix}\sigma_I\sqrt{1-\rho^2} & 0\\ -\rho\sigma_I & \sigma_E\end{pmatrix}=\begin{pmatrix}\frac{1}{\sigma_E} & 0\\ -\frac{\rho}{\sigma_E\sqrt{1-\rho^2}} & \frac{1}{\sigma_I\sqrt{1-\rho^2}}\end{pmatrix}.$$

The quanto option pricing formula, in this setting, is given by

$$C_t=\mathbb{E}^{\mathbb{Q}}[g(F^E)h(\rho F^E+\sqrt{1-\rho^2}F^I)], \tag{6.6}$$

with $g(x)=\max\{x-K^E,0\}$ and $h(x,y)=\rho\max\{x-K^E,0\}\times\sqrt{1-\rho^2}1_{\{y\geq K^I\}}$. By using the general formulae developed in the previous sections, we are able to compute analytically the values of different Malliavin weights. Here we set $a(t)=\frac{1}{T}$. We have

$$\pi^{\Delta_E} = \frac{1}{F^E(0)T}\int_0^T\frac{1}{\sigma_E}dW^E(t)-\frac{1}{F^E(0)T}\int_0^T\frac{\rho}{\sigma_E\sqrt{1-\rho^2}}d\widetilde{W}^I(t).$$

$$\pi^{\Delta_I} = \frac{1}{F^I(0)T}\int_0^T\frac{1}{\sigma_I\sqrt{1-\rho^2}}d\widetilde{W}^I(t).$$

$$\begin{aligned}\pi^{\Delta_{EI}} &= \frac{1}{F^E(0)F^I(0)T^2}\left\{\left(\int_0^T\frac{1}{\sigma_E}dW^E(t)\right)\left(\int_0^T\frac{1}{\sigma_I\sqrt{1-\rho^2}}d\widetilde{W}^I(t)\right)\right.\\ &\quad\left.-\int_0^T\frac{\rho}{\sigma_E\sigma_I(1-\rho^2)}dt\right\}.\end{aligned}$$

7. Application to stochastic volatility models

Stochastic volatility models describe the joint evolution of the futures prices (i.e. the energy futures price F_t^E and the temperature future price F_t^I) and their corresponding variances.

7.1. The independent case. We consider the following dynamics

$$\frac{dF^E}{F^E} = \sqrt{\sigma_E}dW_t^E,\quad d\sigma_E=-k_1\sigma_Edt+\eta_1\sqrt{\sigma_E}dB_t^E, \tag{7.1}$$

$$\frac{dF^I}{F^I} = \sqrt{\sigma_I}dW_t^I,\quad d\sigma_I=-k_2\sigma_Idt+\eta_2\sqrt{\sigma_I}dB_t^I. \tag{7.2}$$

Here, W_t^E, B_t^E, W_t^I and B_t^I are independent standard Brownian motions and κ_1, κ_2, η_1 and η_2 are constants. $\sigma_E(t)$ and $\sigma_I(t)$ are the volatility processes.

$$\begin{pmatrix}\frac{dF^E}{F^E}\\ d\sigma_E\\ \frac{dF_I}{F^I}\\ d\sigma_I\end{pmatrix}=\begin{pmatrix}0\\ -\kappa_1\sigma_E\\ 0\\ -\kappa_2\sigma_I\end{pmatrix}dt+\begin{pmatrix}\sqrt{\sigma_E} & 0 & 0 & 0\\ 0 & \eta_1\sqrt{\sigma_E} & 0 & 0\\ 0 & 0 & \sqrt{\sigma_I} & 0\\ 0 & 0 & 0 & \eta_2\sqrt{\sigma_I}\end{pmatrix}\begin{pmatrix}dW^E\\ d\sigma_E\\ dW^I\\ dB^I\end{pmatrix}.$$

The inverse of the diffusion matrix is

$$\begin{pmatrix}\frac{1}{\sqrt{\sigma_E}} & 0 & 0 & 0\\ 0 & \frac{1}{\eta_1\sqrt{\sigma_E}} & 0 & 0\\ 0 & 0 & \frac{1}{\sqrt{\sigma_I}} & 0\\ 0 & 0 & 0 & \frac{1}{\eta_2\sqrt{\sigma_I}}\end{pmatrix}.$$

The quanto option pricing formula is then expressed as

$$C_t = \mathbb{E}^{\mathbb{Q}}[g(F^E)h(F^I)], \tag{7.3}$$

with $g(x) = \max\{x - K^E, 0\}$ and $h(y) = 1_{\{y \geq K^I\}}$. By using the general formulae developed in the previous sections, we are able to compute analytically the values of different Malliavin weights. Here we set $a(t) = \frac{1}{T}$. We have

$$\pi^{\Delta_E} = \frac{1}{F^E(0)T}\int_0^T \frac{1}{\sqrt{\sigma_E(t)}}dW^E(t).$$

$$\pi^{\Delta_I} = \frac{1}{F^I(0)T}\int_0^T \frac{1}{\sqrt{\sigma_I(t)}}dW^I(t).$$

$$\pi^{\Delta_{EI}} = \frac{1}{F^E(0)F^I(0)T^2}\left(\int_0^T \frac{1}{\sqrt{\sigma_E(t)}}dW^E(t)\right)\left(\int_0^T \frac{1}{\sqrt{\sigma_I(t)}}dW^I(t)\right).$$

7.2. **The correlation case.** We consider the following dynamics

$$\frac{dF^E}{F^E} = \sqrt{\sigma_E}dW_t^E, \quad d\sigma_E = -k_1\sigma_E dt + \eta_1\sqrt{\sigma_E}dB_t^E, \tag{7.4}$$

$$\frac{dF^I}{F^I} = \sqrt{\sigma_I}dW_t^I, \quad d\sigma_I = -k_2\sigma_I dt + \eta_2\sqrt{\sigma_I}dB_t^I. \tag{7.5}$$

Here, W_t^E, B_t^E, W_t^I and B_t^I are correlated standard Brownian motions and $\kappa_1,\ \kappa_2,\ \eta_1$ and η_2 are constants. $\sigma_E(t)$ and $\sigma_I(t)$ are the volatility processes.

For constants $\rho,\ \rho_1,\ \rho_2$ the following hold:

$$\begin{aligned} W_t^E &= W_t^{(1)}, \quad B_t^E = \rho W_t^{(1)} + \sqrt{1-\rho^2}W_t^{(2)}, \\ W_t^I &= \rho_1 W_t^{(1)} + \sqrt{1-\rho_1^2}W_t^{(3)}, \quad B_t^I = \rho_1\rho_2 W_t^{(1)} + \rho_2\sqrt{1-\rho_1^2}W_t^{(3)} + \sqrt{1-\rho_2^2}W_t^{(4)}, \end{aligned}$$

where $W_t^{(1)}$, $W_t^{(2)}$, $W_t^{(3)}$ and $W_t^{(4)}$ are mutually independent standard Brownian motions.

The system of stochastic differential equations (7.4) - (7.5) can be written in a matrix form

$$\begin{pmatrix} \frac{dF^E}{F^E} \\ d\sigma_E \\ \frac{dF_I}{F^I} \\ d\sigma_I \end{pmatrix} = \begin{pmatrix} 0 \\ -\kappa_1\sigma_E \\ 0 \\ -\kappa_2\sigma_I \end{pmatrix} dt + \begin{pmatrix} \sqrt{\sigma_E} & 0 & 0 & 0 \\ \rho\eta_1\sqrt{\sigma_E} & \eta_1\sqrt{\sigma_E}\sqrt{1-\rho^2} & 0 & 0 \\ \rho_1\sqrt{\sigma_I} & 0 & \sqrt{\sigma_I}\sqrt{1-\rho_1^2} & 0 \\ \rho_1\rho_2\eta_2\sqrt{\sigma_I} & 0 & \rho_2\eta_2\sqrt{\sigma_I}\sqrt{1-\rho_1^2} & \eta_2\sqrt{\sigma_I}\sqrt{1-\rho_2^2} \end{pmatrix} \begin{pmatrix} dW_t^{(1)} \\ dW_t^{(2)} \\ dW_t^{(3)} \\ dW_t^{(4)} \end{pmatrix}.$$

The inverse of the diffusion matrix is

$$\begin{pmatrix} \frac{1}{\sqrt{\sigma_E}} & 0 & 0 & 0 \\ -\frac{\rho}{\sqrt{\sigma_E}\sqrt{1-\rho^2}} & \frac{1}{\eta_1\sqrt{\sigma_E}\sqrt{1-\rho^2}} & 0 & 0 \\ -\frac{\rho_1}{\sqrt{\sigma_E}\sqrt{1-\rho_1^2}} & 0 & \frac{1}{\sqrt{\sigma_I}\sqrt{1-\rho_1^2}} & 0 \\ 0 & 0 & -\frac{\rho_2}{\sqrt{\sigma_I}\sqrt{1-\rho_1^2}} & \frac{1}{\eta_2\sqrt{\sigma_I}\sqrt{1-\rho_2^2}} \end{pmatrix}.$$

The quanto option pricing formula, in this setting, is given by

$$C_t = \mathbb{E}^{\mathbb{Q}}[g(F^E)h(\rho F^E + \sqrt{1-\rho^2}F^I)], \tag{7.6}$$

with $g(x) = \max\{x - K^E, 0\}$ and $h(x, y) = \rho \max\{x - K^E, 0\} \times \sqrt{1-\rho^2}1_{\{y \geq K^I\}}$. By using the general formulae developed in the previous sections, we are able to compute analytically the values of different Malliavin weights. Choosing $a(t) = \frac{1}{T}$ and making use of the matrix property $(AB)^* = B^*A^*$, for A and B matrices, where $*$ denotes the transpose, we have

$$\begin{aligned} \pi^{\Delta_E} &= \frac{1}{F^E(0)T}\left(\int_0^T \frac{1}{\sqrt{\sigma_E(t)}}dW^{(1)}(t) - \frac{\rho}{\sqrt{1-\rho^2}}\int_0^T \frac{1}{\sqrt{\sigma_E(t)}}dW^{(2)}(t)\right. \\ &\quad \left. - \frac{\rho_1}{\sqrt{1-\rho_1^2}}\int_0^T \frac{1}{\sqrt{\sigma_E(t)}}dW^{(3)}(t)\right). \end{aligned}$$

$$\pi^{\Delta_I} = \frac{1}{F^I(0)T}\left(\frac{1}{\sqrt{1-\rho_1^2}}\int_0^T \frac{1}{\sqrt{\sigma_I(t)}}dW^{(3)}(t) - \frac{\rho_2}{\sqrt{1-\rho_2^2}}\int_0^T \frac{1}{\sqrt{\sigma_I(t)}}dW^{(4)}(t)\right).$$

$$\begin{aligned} \pi^{\Delta_{EI}} &= \frac{1}{F^E(0)F^I(0)T^2}\left\{\left(\int_0^T \frac{1}{\sqrt{\sigma_E(t)}}dW_t^{(1)} - \frac{\rho}{\sqrt{1-\rho^2}}\int_0^T \frac{1}{\sqrt{\sigma_E(t)}}dW_t^{(2)}\right)\cdot\right. \\ &\quad \left.\left(\frac{1}{\sqrt{1-\rho_1^2}}\int_0^T \frac{1}{\sqrt{\sigma_I(t)}}dW_t^{(3)}\right) - \frac{\rho_1}{(1-\rho_1^2)}\int_0^T \frac{1}{\sqrt{\sigma_E(t)\sigma_I(t)}}dt\right\} \\ &\quad - \frac{\rho_2\pi^{\Delta_E}}{F^I(0)T\sqrt{1-\rho_2^2}}\int_0^T \frac{1}{\sqrt{\sigma_E(t)}}dW_t^{(4)}. \end{aligned}$$

8. The residual risk

If we take the independent delta of energy Δ^{Ind}, say, as the benchmark value and the correlated case as Δ^{Corr}. Then the residual risk is determined by the difference between the independent energy delta and the correlated case as follows:

$$|\Delta_E^{Corr} - \Delta_E^{Ind}|,$$

for each ρ. The same analysis goes for the temperature delta and cross-gamma hedging formulae.

We can state the following robust result.

Proposition 8.1. *Let Δ_E^{Corr} be given by (5.3) and Δ_E^{Ind} be given by (4.6). Then*

$$\lim_{\rho \to 0} \Delta_E^{Corr} = \Delta_E^{Ind}.$$

Proof. The application of Cauchy-Schwartz inequality isometry gives

$$\begin{aligned}
&\left|\Delta_E^{Corr} - \Delta_E^{Ind}\right| \\
\leq\ & \Big|\mathbb{E}^{\mathbb{Q}}\left[g(F^E(\tau_2;\tau_1,\tau_2))h(\rho F^E(\tau_2;\tau_1,\tau_2) + \sqrt{1-\rho^2}F^I(\tau_2;\tau_1,\tau_2))\pi^{\Delta_E}(1+\rho)\right] \\
&-\mathbb{E}^{\mathbb{Q}}\left[g(F^E(\tau_2;\tau_1,\tau_2))h(F^I(\tau_2;\tau,\tau_2))\pi^{\Delta_E}\right]\Big| \\
\leq\ & \mathbb{E}^{\mathbb{Q}}\Big[\Big|g(F^E(\tau_2;\tau_1,\tau_2))\Big\{h(\rho F^E(\tau_2;\tau_1,\tau_2) + \sqrt{1-\rho^2}F^I(\tau_2;\tau_1,\tau_2))(1+\rho) \\
&-h(F^I(\tau_2;\tau,\tau_2))\Big\}\pi^{\Delta_E}\Big|\Big] \\
\leq\ & \mathbb{E}^{\mathbb{Q}}\left[\left|g(F^E(\tau_2;\tau_1,\tau_2))\right|^2\right]^{\frac{1}{2}}\mathbb{E}^{\mathbb{Q}}\left[\left|\pi^{\Delta_E}\right|^2\right]^{\frac{1}{2}} \\
&\times\mathbb{E}^{\mathbb{Q}}\left[\left|\left\{h(\rho F^E(\tau_2;\tau_1,\tau_2) + \sqrt{1-\rho^2}F^I(\tau_2;\tau_1,\tau_2))(1+\rho) - h(F^I(\tau_2;\tau,\tau_2))\right\}\right|^2\right]^{\frac{1}{2}}.
\end{aligned}$$

It follows from the integrability conditions (4.3) that the first expression on the right hand side of the above inequality is finite. Using the Itô isometry it follows that

$$\mathbb{E}^{\mathbb{Q}}\left[\left|\pi^{\Delta_E}\right|^2\right] = \mathbb{E}^{\mathbb{Q}}\left[\int_0^{\tau_2}\left|a(t)\left(\sigma_E^{-1}(t;\tau_1,\tau_2)Y_E(t;\tau_1,\tau_2)\right)^*\right|^2 dt\right] < \infty.$$

The last expression on the right hand side of the above inequality can be bounded uniformly in ρ, and hence the result follows by the dominated convergence. □

Remark. Following similar arguments, we can obtain robust results for the temperature delta and cross-gamma hedging formulae.

9. Concluding remarks

In this paper, we have derived the energy delta, the temperature delta and the cross-gamma formulae of the quanto energy option written on a forward contract under the HJM framework. Our results shows that the hedging formulae can be expressed as the expectation of payoff functionals multiplied y the weight functions. The weight functions are independent of the payoff functionals. The Malliavin calculus approach increases the efficiency when dealing with discontinuous payoff functionals as compared to finite difference approximation. In addition, no extra computation is required for other payoff functionals as long as the payoff functional is a function of the same points of the Brownian motion trajectory. The weighting function smoothen the function in the case of the energy delta and the temperature delta to simulate as the payoff functional does not require to be numerically differentiated. In the case of the cross-gamma it smoothen twice the payoff functional as it reduces a second order partial differentiation to no partial differentiation. This leads to high efficiency for the simulation of the cross-gamma.

We have considered the independent and the correlation cases to facilitate the residual risk analysis. Our results generalise the work in Benth et al. (2015) in the sense that they can be to discontinuous payoff functions. In addition, we have also considered application to stochastic volatility models. In Benth et al. (2010), the authors analysed a volatility model for a different payoff structure to the one considered in this paper.

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