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Optimal inventory replenishment and shipment policies in a three-echelon supply chain for growing items with expiration dates

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Abstract The primary source of a vast majority of consumable food products is growing items such as crops or livestock. Most of these food products have specified maximum shelf lives or expiration dates. This implies that the products are no longer suitable for human consumption beyond their expiration dates. In addition, consumers rarely eat these products in their original form, this means that there are usually some forms of value-adding activities performed on the growing items in order to transform them into a consumable form, for instance, processing and packaging. Consequently, this study develops a model for managing inventory in a three-echelon supply chain for growing items with distinct farming, processing and retail operations. At the farming echelon, the growing items are reared but there is the possibility that some of them might die. The surviving items are then transferred to the processing echelon for slaughtering, processing and packaging. The processed inventory is then transferred to the retail echelon where consumer demand for consumable (i.e. processed and packaged) inventory is met under the assumption that the inventory has a specified expiration date. The proposed supply chain system is modelled as a cost minimisation problem. The benefits of integrating inventory replenishment decisions among all supply chain members are quantified through a numerical example.

Keywords Age-dependent deterioration · Expiration date · Growing items · Inventory management · Joint economic lot size · Supply chain management

1 Introduction

1.1 Context

One of the most fundamental changes to business management in recent years has been that businesses compete within supply chains, as opposed to competing as individual entities [15]. Businesses executives have realised that competitive advantages such as customer service, responsiveness and cost efficiency, among others, can be improved through collaboration with suppliers and customers. One form of collaboration is through the integration of inventory replenishment decisions with other supply chain members. A number of researchers have used this collaboration mechanism to develop numerous integrated inventory models in multi-echelon supply chain systems (i.e. supply chains with more than one party). Nonetheless, most of these models were developed specifically for either conventional or deteriorating items. While these two classes of items are important in inventory modelling, they are not the only groups of items that are important. Recently, a new class of items, namely, growing items, has been receiving attention from multiple researchers working in the field of inventory theory. Growing items, such as livestock, fish and crops, to name a few examples, represent an important class of items because they are the primary source of most food products.

1.2 Purpose

Given that growing items are rarely consumed in their original form (i.e. most of them are processed before being put on sale) and that there are usually multiple parties involved in the food production chain, growing items (in the context of inventory modelling) are the perfect candidates for an extension of the multi-echelon inventory model. Accordingly, this study is aimed at developing a coordinated model for managing inventory in a three-echelon supply chain for growing items while also taking into account the continuous deterioration of products until when it reaches its expiration date when its value becomes zero. The three echelons of the supply chain considered are the farming, processing and retail operations that are found in some food production systems. At the farming echelon, newborn items are procured and grown to maturity. It is assumed that some of the items do not survive at the farming operation due to factors like predators and illnesses. The mature items are then transferred to the processing echelon for slaughtering, processing and packaging. Following this, the processed inventory is delivered to the retail echelon where customer demand is met. At the the retail echelon, the processed inventory, which is now ready for human consumption, is sold to consumers. The processed inventory, as is the case with most food products, is perishable and consequently, it has an associated maximum shelf life or expiration date.

1.3 Relevance

The proposed inventory model accounts for a number of important issues in food production chains, namely the possibility of mortality (with reference to the live growing items), the integration of inventory replenishment decisions among multiple supply chain members and deterioration (of shelved stock items). Growing items are, like most living organisms, not immune to illnesses and various other health issues which might result in mortality. This makes item mortality an important consideration in the upstream portion of most food production chains. At the other end of the chain (downstream), shelf life becomes very critical because of government health and safety regulations regarding consumable food products.

This is because processed food products may no longer be safe for consumption after their expiration dates.

1.4 Organisation

Apart from the introduction, this paper has seven other sections. A review of previously published inventory models for growing items, deteriorating items with expiration dates and items in integrated production-inventory systems with multiple supply chain partners is provided in Section 2. The proposed inventory system is briefly outlined in Section 3 which also includes notations and assumptions utilised during the model development phase. The inventory system under consideration is then modelled as a cost minimisation problem in Section 4. This is followed by a derivation of a special case of the model and a proof of the model's optimality in Sections 5 and 6, respectively. Managerial insights are drawn from a numerical example presented in Section 7 which also shows the potential practical applications of the model. Concluding remarks and suggestions for future research are presented in Section 8.

2 Literature Review

The model presented in this study is based on three research stream within the field of inventory theory, whose roots lie in Harris' [9] classic economic order quantity (EOQ) model. These streams are growing items, deteriorating items with expiration dates and items in integrated production-inventory systems within multi-echelon supply chains.

2.1 Inventory models for growing items

The first inventory model to incorporate item growth is credited to Rezaei [23] who presented an EOQ model for items whose weight increases during the replenishment cycle. The model considered an inventory system with a single type of item that has the capability to grow, deterministic demand, and no shortages or quantity discounts. The total cost of managing inventory in the system was made up of setup, feeding and holding costs.

Rezaei's [23] model has received attention from numerous researchers who have extended the model to suit different practical situations. One such extension was developed by Nobil et al. [18] who relaxed the assumption that shortages are not permitted. Sebatjane and Adetunji [27] formulated a version of the model for a case where a random percentage of the items is of imperfect quality. The effects of suppliers (of the growing newborn items) offering quantity discounts were investigated by Sebatjane and Adetunji [28]. Gharaei and Almahware [6] proposed the so-called economic growing quantity (EGQ) model which considers item mortality, the utility of the growth function and a reorder point ordering policy. Breeding policies were incorporated by Pourmohammad-Zia and Karimi [20] through the development of a model for jointly optimising the replenishment and breeding under the assumption that the holding and breeding costs are dependent on the age of the growing items. Furthermore, Rezaei's [23] model has also been extended to multi-echelon supply chain systems. For instance, Malekitabar et al. [17] and Sebatjane and Adetunji [29] developed models for optimising inventory replenishment policies for growing items in two- and three-echelon supply chains, respectively. Sebatjane and Adetunji [31, 32] studied three-echelon supply chains for growing items with freshness- and price-dependent demand and freshness- and inventory level-dependent demand, respectively. A four-echelon supply chain system for growing items, with growing, processing, screening and retail echelons, was introduced by Sebatjane and Adetunji [30].

2.2 Inventory models for deteriorating items with expiration dates

Item deterioration was first incorporated into inventory theory by Ghare and Schrader [7] through the development of an EOQ-type model for an item experiencing constant deterioration. Covert and Phillip [3] generalised Ghare and Schrader [7]'s work by relaxing the constant deterioration rate assumption and considering a deterioration rate characterised by a Weibull distribution with two parameters. These two models have have spawned most of the literature on inventory management for deteriorating items. One of the most recent development in deteriorating inventory modelling is the incorporation of expiration dates, which in essence assumes that the items' deterioration rate is time dependent and consequently, the items have a maximum lifetime.

Hsu et al. [11] developed one of the first inventory models which consider expiration dates. In addition to expiration dates, their model also considered uncertain lead time and a seasonal demand pattern. Sarkar [26] proposed an EOQ model for a retailer selling items with an expiration date provided that the supplier permits the retailer to pay for the order at a later date (i.e. not at the time the order is delivered). Wang et al. [36] developed an extension of Sarkar [26]'s model which deemed the demand rate to be a function of the duration of the credit period. Based on the demand pattern of fresh produce, which is affected by factors such as freshness, expiration dates and the amount of stock displayed on shelves, Wu et al. [38] formulated an inventory model for an expiring product with a demand rate that depends on the product's freshness condition and it's stock level. Teng et al. [33] modelled an inventory situation where the supplier of deteriorating items with expiration dates, such as fresh produce, requires the retailer to pay for the inventory prior to its delivery. Retailers selling products with expiration dates often discount the products as the expiration dates approach. Using this logic, Banerjee and Agrawal [1] developed a model for optimising ordering, pricing and discounting policies for an inventory system consisting of expiring items with a demand rate that is dependent on the selling price. Khan et al. [13] proposed an EOQ-type model for deteriorating products with expiration dates when shortages are permitted and end user demand depends on the products' selling prices. Other latest development in deteriorating inventory systems modelling includes the incorporation of investment in preservation technology with pricing and marketing decisions [22], trade credit financing and successive quantity discounts [2], uncertainty in demand [25] and resource constraints using a case of the health-care industry [24].

2.3 Integrated inventory systems with multiple echelons

Supply chain management advocates for the coordination of various business decision, such as inventory replenishment and shipment policies, among the supply chain members. The joint economic lot size (JELS) model, credited to Goyal [8], is among the first inventory models aimed at reducing the costs of managing inventory through the cooperation of different supply chain members. The model was formulated for a single vendor supplying a single buyer with a single type of product.

Lu [16] extended the JELS model to a case where there are multiple buyers. Valentini and Zavanella [34] studied a vendor-buyer inventory system in which the vendor legally owns the stock and keeps it at the buyer's facility and developed a corresponding inventory model. Motivated by the fact that most supply chains are are complex networks with multiple chain members selling a variety of products, Khouja [14] formulated an inventory model for a complex three-echelon supply chain with multiple suppliers, multiple manufacturers and multiple customers. Ouyang et al. [19] relaxed the zero lead time and deterministic demand rate assumptions in the basic JELS model. Ho et al. [10] studied a coordinated vendor-buyer inventory system where the vendor permits the buyer to delay payment and simultaneously offers the buyer discounts for paying in cash. In addition, the buyer also permits their customer to delay payments. Geetha and Uthayakumar [5] developed a model for jointly optimising pricing and replenishment policies in a two echelon (vendor-buyer) supply chain whereby the vendor not only allows the buyer to pay for the inventory at a later date through trade credit financing, but also grants the buyer freight discounts for transporting the inventory based on the weight of the order. Priyan and Manivannan [21] studied a version of the JELS problem for a case where the vendor's production process produces some imperfect quality items that are screened out at the buyer's facility under the assumptions that the screening process is prone to errors and that the fraction of items that are of imperfect quality is a triangular fuzzy variable. Dey et al. [4] presented an inventory model for a vendor-buyer system with deteriorating items and investments in preservation technologies in an effort to slow down the deterioration process. Most of the research published on the JELS problem seldom accounts for the cost of transporting goods from the vendor to the buyer, Wangsa and Wee [37] developed a model which considered stochastic demand and compared two transportation modes, namely, less-than truck load (LTL) and truck load (TL) shipping. Vats et al. [35] investigated the effectiveness of a demand aggregation approach to inventory management in a supply chain with multiple distributors and multiple retailers under stochastic demand conditions. Islam and Hoque [12] proposed a model for optimising lot-sizing and shipment policies in a three-echelon agricultural supply chain with a single raw material supplier, a single manufacturer and multiple retailers.

2.4 Contribution

Table 1 provides a summary of the major contributions made by various published studies that are available in the current literature. Of particular interest are studies that pertain to growing items such as those by Gharaei and Almahdawe [6], Nobil et al. [18], Pourmohammad-Zia and Karimi [20], Rezaei [23], Sebatjane and Adetunji [27,28,29,30,31,32]. From those contributions, the main gap identified is that there seems to be no published inventory model involving growing items in a supply chain with multiple echelons that also accounts for a deterioration rate that is based on the expiration date of the processed inventory. While the studies by Sebatjane and Adetunji [31,32] also incorporate expiration dates (in addition to price-dependent demand and stock-dependent demand, respectively), they do so through a freshness index which is a linear function that relates the freshness condition of the inventory to the expiration date. In this study, the expiration date is used to express an age-dependent deterioration rate. In summary, the salient characteristics of the proposed model are the incorporation of an age-dependent deterioration rate at the retail echelon and item mortality at the farming echelon in a three-echelon supply chain involving growing items. Considering that most food production chains, for brevity, can be represented by the three echelons in the current model and that food items have shelf lives (i.e. expiration dates), the proposed supply chain model is more representative of inventory management in food production systems and the results and analysis can provide insights into the management of inventory in the food production industry.

Table 1: Contributions made by related inventory models in literature

References	**Types of items in the system** Conventional	Deteriorating	Growing	**Number of supply chain echelons**	**Additional inventory system characteristics**
Banerjee et al. [1]		✓		One	Discounting
Dey at al. [4]		✓		Two	Preservation efforts
Gharaei and Almahdawe [6]			✓	One	Item mortality
Hsu et al. [11]		✓		Three	Seasonal demand
Ho et al. [10]	✓			Two	Delayed payment
Islam and Hoque [12]	✓			Three	Agricultural products
Khan et al. [13]		✓		One	Shortages
Khouja [14]	✓			Three	Multiple members
Lu [16]	✓			Two	Multiple buyers
Malekitabar et al. [17]			✓	Two	Revenue-sharing
Nobil et al. [18]			✓	One	Shortages
Ouyang et al. [19]	✓			Two	Stochastic demand
Priyan and Manivannan [21]	✓			Two	Errors in inspection
Pourmohammad-Zia and Karimi [20]			✓	One	Breeding policy
Rezaei [23]			✓	One	-
Sarkar [26]		✓		One	Delayed payment
Sebatjane and Adetunji [27]			✓	One	Imperfect quality
Sebatjane and Adetunji [28]			✓	One	Quantity discounts
Sebatjane and Adetunji [29]			✓	Three	-
Sebatjane and Adetunji [31]		✓	✓	Three	Price-dependent demand
Sebatjane and Adetunji [32]		✓	✓	Three	Stock-dependent demand
Teng et al. [33]		✓		One	Advance payment
Valentini and Zavanella [34]	✓			Two	Consignment stock
Vats et al. [35]	✓			Two	Demand aggregation
Wang et al. [36]		✓		One	Default payment risk
Wangsa and Wee [37]	✓			Two	Transportation costs
Wu et al. [38]		✓		One	Item freshness
This study		✓	✓	Three	Item mortality

3 Problem description

The proposed supply chain inventory model consists of three echelons representing different stages of a typical food production chain, namely farming, processing and retail operations. At the farming echelon, newborn items are procured and reared until maturity. The items are declared mature once their weight reaches a predefined target. Following this, the live items are instantaneously transferred to the next echelon which is processing. The live items are slaughtered, prepared and packaged in preparation for consumption (or sale) at the processing plant. For convenience, all activities carried out at the processing plant are collectively called processing and they are carried out at a given processing rate. The processed inventory is delivered to the last echelon (i.e. retail) in a number of equally-sized shipments per processing run. At the retail outlet, the processed inventory is placed on shelves in order to meet consumer demand. However, the processed inventory can only be displayed on the shelves for a given amount of time. The inventory continuously loses some of its utility over time and at the end of the shelf life, often specified as an expiration date, it is no longer suitable for consumption.

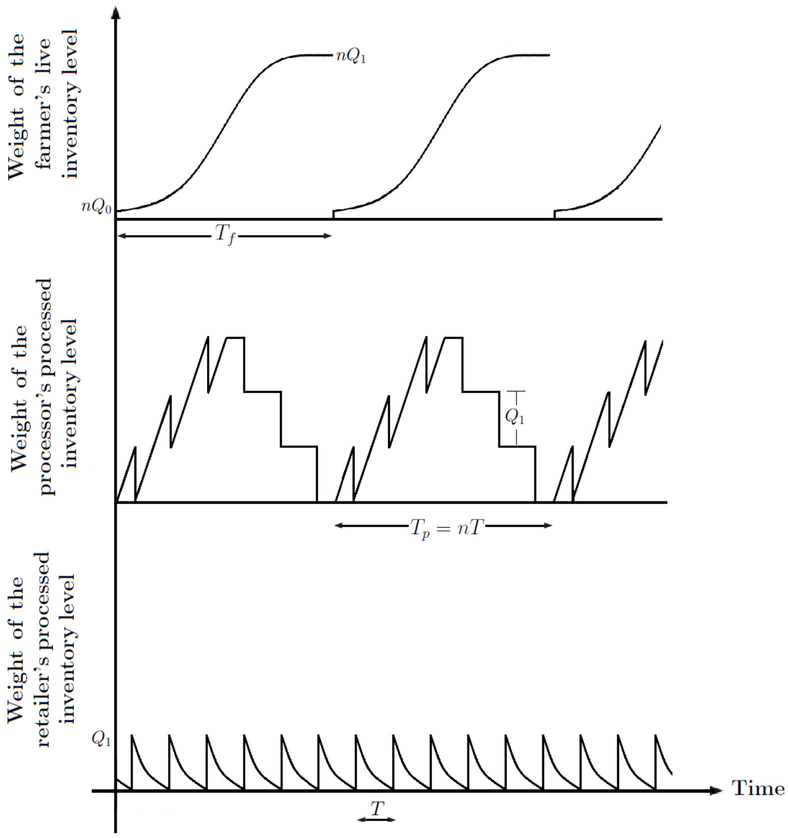


Fig. 1: Inventory system profile showing the weight of the live inventory at the farmer's growing facility and the weight of the processed inventory at the processor's and the retailer's facilities.

The inventory system profile for the problem at hand is depicted by Figure 1, which shows the behaviour (over time) of the farmer's live inventory, the processor's processed inventory and the retailer's processed and deteriorating inventory. The processor's cycle time is coordinated with the cycle times of the other two supply chain members on the basis of the behaviour of the inventory at that particular member's operations. For instance, the processor's cycle time is an integer multiple of the retailer's cycle time because the processed inventory is replenished frequently in an effort to keep it as fresh as possible due to its expiration dates. On the other hand, growing the newborn items requires a relatively longer period of time and a result the growing cycle is setup up such that when it ends a new processing cycle commences (as shown in Figure 1). In a nut shell, the farmer and the processor operate on a single setup-single delivery (SSSD) inventory replenishment policy while the processor and the retailer operate on a single setup-multiple delivery (SSMD) policy. The difference between these two policies lies in the number of shipments delivered by upstream supply chain member to the downstream member per cycle of the upstream member. For the SSSD replenishment policy, the farmer delivers one shipment of mature items to the processor during a single growing cycle while in the case of the SSMD policy, the processor delivers multiple shipments (an integer number) of processed inventory to the retailer during a single processing cycle.

The proposed inventory model is formulated as a cost minimisation problem aimed at determining the optimal number of newborn items that the farmer should order when a growing cycle commences (and by extension the processor and the retailer's order quantities and cycle times) and the optimal number of shipments processor should deliver to the retailer during a single processing run.

The following notations are used throughout this study:

$w(t)$	The weight of an item at time t
w_0	Newborn weight of each item
w_1	Maturity(i.e. target) weight of each item
x	Fraction of the live items which survive throughout the growth period
D	Demand rate for processed items in weight units per unit time
P	Processing rate in weight units per unit time
K_f	Farmer's setup cost per cycle
c_f	Farmer's feeding cost per weight unit per unit time
T_f	Duration of the farmer's growth period
K_p	Processor's setup cost per cycle
h_p	Processor's holding cost per weight unit per unit time
n	Number of shipments from the processor to the retailer per unit cycle of the processor
$I(t)$	The weight of the processed inventory at time t
$\theta(t)$	The age-dependent deterioration rate of the processed inventory at time t
L	The maximum lifetime (i.e. expiration date) of the processed inventory
T_p	Processor's cycle time
K_r	Retailer's ordering cost per cycle
h_r	Retailer's holding cost per weight unit per unit time

T	Retailer's cycle time
y	Retailer's lot size
Q_1	Weight of the retailer's lot size per shipment (i.e. $Q_1 = xyw_1$)
ny	Farmer's lot size (for live newborn items) per cycle
α	The items' asymptotic weight
β	Constant of integration
λ	Growth rate (exponential) of the items

The model representing the proposed inventory system is developed under the following assumptions:

- There is only one farmer, one processor and one retailer in the supply chain dealing in one type of growing item.
- A fraction of the ordered items dies before reaching maturity weight.
- The (processor's) processing rate is greater than the (retailer's) demand rate, both of which are deterministic constants.
- The arrival of successive shipments of processed inventory from the processor to the retailer is scheduled to occur when the previous shipment has just been depleted.
- The processor delivers processed inventory to the retailer just at the moment the processed inventory is enough to make up a batch size.
- The retailer's replenishment interval is an integer multiple of the processor's replenishment interval.
- The live inventory incurs feeding costs (during the growth period) while the processed inventory incurs holding costs (during the processing and selling periods).
- Once the processed inventory reaches the retailer's shelves, it has a specified shelf life (or maximum lifetime) indicated by an expiration date. Beyond this point, the inventory has lost all utility and it cannot be used to meet consumer demand.

4 Model development

The procurement of ny newborn items marks the start of the farmer's replenishment cycle. At the time they are procured, each of the newborn items weighs w_0. Multiplying the number of items procured by the weight of the items yields the weight of all the ordered items at the time of procurement (i.e. $nQ_0 = nyw_0$). The farmer feeds the live items throughout the growth cycle, of duration T_f, and stops only when the weight of each item increases to the target maturity weight of w_1. The live items have a survival rate of x [i.e during the growth period, $(1 - x)$ of the initially ordered newborn items die]. This implies that the weight of all the surviving ordered mature items (nQ_1) is therefore

$$nQ_1 = nxyw_1. \tag{1}$$

The logistic function, given by

$$w(t) = \frac{\alpha}{1 + \beta e^{-\lambda t}}, \tag{2}$$

is used to represent the items' growth function. It is chosen because of its distinctive "S"-shaped curve which is representative of the pattern of growth in most living organisms. The function describes the changes to the weight of items during the growth period and it makes use of three parameters, namely the items' asymptotic weight, the integration constant and the growth rate (represented by the symbols α, β and λ respectively). When the growth period is complete (i.e. when the weight of each has reached the target weight w_1 at time T_f), the items are delivered to the processor for slaughtering, preparation and packaging (i.e processing). Equation (2) can be rewritten in terms of the target weight and growth cycle duration as

$$T_f = -\frac{\ln\left[\frac{1}{\beta}\left(\frac{\alpha}{w_1} - 1\right)\right]}{\lambda}. \tag{3}$$

Since the processor and the retailer operate on a SSMD replenishment policy, for each processing run, the processor delivers an integer number (n) of equally-sized shipments of processed inventory to the retailer. The implications of this are that the retailer places n orders, of total weight Q_1, per processing setup. Likewise, the retailer's cycle time, T, and the processor's cycle time, T_p, are linked by the relation

$$T_p = nT. \tag{4}$$

This indicates that the processor should process live items of total weight nQ_1 per processing run. And since the farmer and the processor operate on a SSSD policy, the farmer should grow the same weight units of live inventory during each processing setup.

The total cost of managing inventory in the proposed three-echelon supply chain is made up of the individual inventory management costs incurred at each of the three echelons.

4.1 Retail operations

The inventory system profile for the retailer's processed and deteriorating inventory is illustrated by Figure 2. When a replenishment cycle commences, the retailer receives an order weighing Q_1 from the processor in order to meet consumer demand, with a rate D, for processed inventory. The retailer keeps the processed inventory on shelves and it deteriorates as a result. For health reasons, the processed inventory has a specified shelf life, L, beyond which it is no longer suitable for consumption.

The deterioration experienced by the retailer's processed inventory is age-dependent in the sense that the longer the items are on the shelf, the greater the deterioration. The rate of deterioration peaks (at 100%) at the expiration date. Beyond this point, the inventory is no longer useful in the sense that it can no longer be used to fulfil consumer demand. This type of deterioration is often associated with perishable food products such as fresh meat, fresh produce and milk, among other types of inventory items. Since growing items are the primary source of a vast majority of consumable food products, this type of deterioration is appropriate for the proposed inventory system. The defining characteristic of this type of deterioration is that the deterioration rate is dependent on the maximum lifetime (i.e expiration date) of the perishable product under study. In essence,

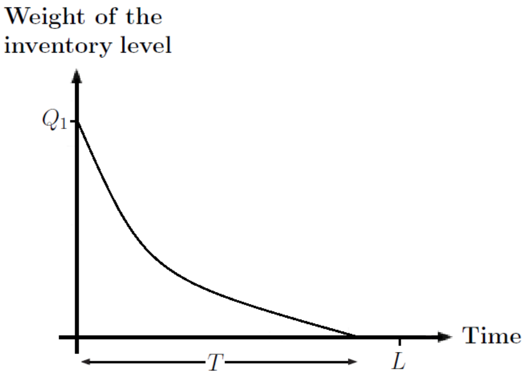


Fig. 2: Inventory system profile the retailer

the deterioration rate depends on the age of the product in the sense that as the product ages (i.e. closer to its expiration date), the rate of deterioration increases. The deterioration rate is given by

$$\theta(t) = \frac{1}{1+L-t}, \tag{5}$$

for $0 \leq t \leq T$ where L represents the maximum lifetime or expiration date of the processed inventory at the retail echelon. This deterioration rate is adopted from works such as Khan et al. [13], Sarkar [26] and Wang et al. [36], to name a few. Since the items' deterioration rate cannot exceeds 100% (i.e. $\theta(t) \leq 1$), Equation (5) implies that the retailer's replenishment cycle, T, cannot exceeds the expiration date of the processed inventory (i.e. $L > T$).

Throughout retailer's replenishment cycle, the weight of their inventory decreases due to consumer demand and deterioration. Accordingly, the changes to the weight of the retailer's processed inventory can be represented by the differential equation

$$\frac{dI(t)}{dt} = -D - \theta(t)I(t), \quad 0 \leq t \leq T. \tag{6}$$

Since the weight of the processed inventory is completely depleted at time T, Equation (6) has the boundary condition $I(T) = 0$. Using the boundary condition to solve Equation (6) results in

$$I(t) = D(1+L-t)\ln\left(\frac{1+L-t}{1+L-T}\right), \quad 0 \leq t \leq T. \tag{7}$$

The weight of the retailer's order quantity, computed by subsisting $t = 0$ in Equation (7), is therefore

$$Q_1 = I(0) = D(1+L)\ln\left(\frac{1+L}{1+L-T}\right). \tag{8}$$

Consequently, the corresponding number of items (or the retailer's lot size) is

$$y = \frac{1}{xw_1}\left[D(1+L)\ln\left(\frac{1+L}{1+L-T}\right)\right]. \tag{9}$$

The retailer incurs a cost of h_r for holding a single weight unit of the processed inventory per unit time. This cost is multiplied by the average weight of the processed inventory (i.e. the area under the graph in Figure 2 divided by the replenishment interval) in order to determine the holding cost per unit time as

$$HC_r = h_r \frac{\int_0^T I(t)\,dt}{T} = \frac{h_r D}{T}\left[\frac{(1+L)^2}{2}\ln\left(\frac{1+L}{1+L-T}\right) + \frac{T^2}{4} - \frac{(1+L)T}{2}\right]. \quad (10)$$

Furthermore, the retailer incurs a fixed ordering cost of K_r whenever a new order for processed inventory is placed. This means that the ordering cost per unit time is

$$KC_r = \frac{K_r}{T}. \quad (11)$$

The total cost (per unit time) of managing inventory at the retailer, computed by summing Equations (10) and (11), is thus

$$TC_r = \frac{K_r}{T} + \frac{h_r D}{T}\left[\frac{(1+L)^2}{2}\ln\left(\frac{1+L}{1+L-T}\right) + \frac{T^2}{4} - \frac{(1+L)T}{2}\right]. \quad (12)$$

4.2 Processing operations

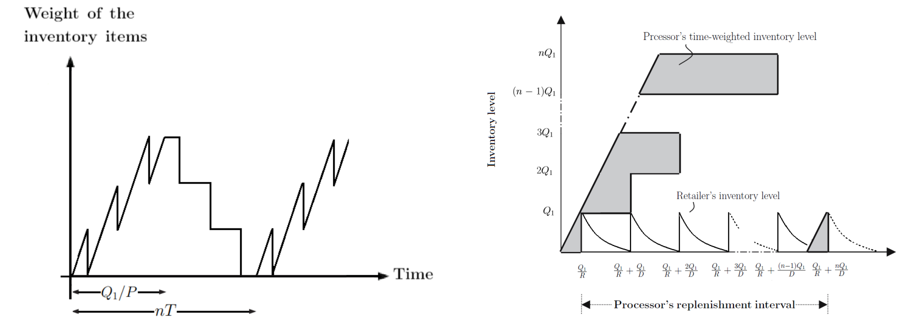
The total cost of managing inventory at the processing plant is made up of the holding and setup costs. The profile of the processor's inventory (in weight units) is represented by Figure 3. The processor receives one delivery from the farmer at the beginning of each replenishment cycle with a duration of $T_p = nT$. The weight of each shipment received is $nQ_1 = nxyw_1$. When a new replenishment cycle starts, the processor incurs a fixed cost of K_p for preparing the processing facility for slaughtering, preparation and packaging (all are collectively termed processing and they occur at a rate of P). This implies that the setup cost per unit time is

$$KC_p = \frac{K_p}{nT}. \quad (13)$$

In order to determine the processor's holding costs per unit time, the average weight of the processed inventory (in weight units) is multiplied by the holding cost (in weight units per unit time, i.e. h_p). It follows that

$$HC_p = \frac{h_p T D}{2}\left[\Big(n-1\Big)\Big(1-\frac{D}{P}\Big) + \frac{D}{P}\right]. \quad (14)$$

The average weight of the processed inventory is determined by dividing the area under the processed inventory level, as shown in Figure 3a, by the duration of the replenishment interval. So as to easily determine the area under Figure 3a, the figure is redrawn into Figure 3b. This approach is adapted from Yang et al.'s



(a) Original.

(b) Redrawn (modified from Yang et al. [39]).

Fig. 3: The processor's processed inventory system behaviour.

[39] JELS model and the resulting expression for the average inventory level is

$$\begin{aligned}
\text{Average inventory}_p &= \frac{\text{Processor's time-weighted inventory}}{\text{Processor's replenishment interval}} \\
&= \frac{\frac{nQ_1^2}{2P} + Q_1^2\left(\frac{1}{D}-\frac{1}{P}\right) + 2Q_1^2\left(\frac{1}{D}-\frac{1}{P}\right) + \cdots + (n-1)Q_1^2\left(\frac{1}{D}-\frac{1}{P}\right)}{nQ_1/D} \\
&= \frac{D}{nQ_1}\left[\frac{nQ_1^2}{2P} + \frac{n(n-1)Q_1^2}{2}\left(\frac{1}{D}-\frac{1}{P}\right)\right] \\
&= \frac{Q_1}{2}\left[\left(n-1\right)\left(1-\frac{D}{P}\right)+\frac{D}{P}\right].
\end{aligned} \tag{15}$$

The cost of managing the processor's inventory (per unit time) is therefore

$$TC_p = \frac{K_p}{nT} + \frac{h_p T D}{2}\left[\left(n-1\right)\left(1-\frac{D}{P}\right)+\frac{D}{P}\right]. \tag{16}$$

4.3 Farming operations

The farmer, whose inventory system profile is given by Figure 4, is responsible for rearing the live newborn items to maturity The items are deemed mature once they have grown to a pre-defined target weight. The farmer's total inventory management cost is comprised of the setup, feeding and mortality costs. In order for the retailer to meet a demand rate (for processed inventory) of D, the farmer delivers a shipment of processed inventory weighing nQ_1 to the processor, who in turn supplies the retailer with n shipments of processed inventory each weighing Q_1. Since the farmer and the processor operate on a SSSD replenishment policy, for each of the processor's processing setups the farmer starts a single replenishment cycle. Given that the farmer pays a fixed setup cost of K_f when a new replenishment cycle begins, the farmer's setup cost per unit time is therefore

$$KC_f = \frac{K_f}{nT}. \tag{17}$$

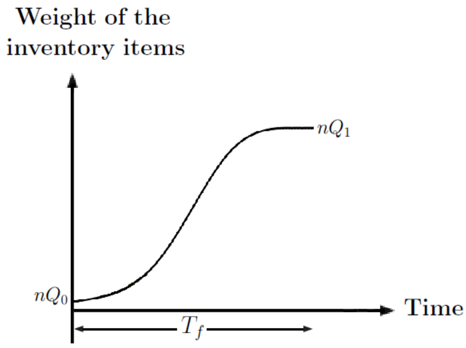


Fig. 4: Inventory system profile the farmer

With exception to the setup cost, all of the farmer's other cost components are dependent on the average weight of the farmer's live inventory. The average weight is determined by dividing the area under the inventory system graph, depicted by Figure 4, by the duration of the replenishment cycle and it is given by

$$\begin{aligned}\text{Average inventory}_f &= \frac{\text{Farmer's time-weighted inventory}}{\text{Farmer's replenishment interval}}\\ &= \frac{\int_0^{T_f} nyw(t)\,dt}{nT}\\ &= \frac{D(1+L)\ln\frac{1+L}{1+L-T}}{Txw_1}\left\{\alpha T_f + \frac{\alpha}{\lambda}\Big[\ln\left(1+\beta e^{-\lambda T_f}\right) - \ln\left(1+\beta\right)\Big]\right\}.\end{aligned} \tag{18}$$

The farmer incurs a cost associated with disposing the fraction of newborn items which do not survive until the end of the growing cycle. The farmer's mortality cost per unit time is computed as the product of the farmer's average inventory level, the fraction of items which do not not survive $(1-x)$ and the mortality cost per weight unit per unit time (m_f). The mortality cost per unit time is therefore

$$MC_f = \frac{m_f(1-x)D(1+L)\ln\frac{1+L}{1+L-T}}{Txw_1}\left\{\alpha T_f + \frac{\alpha}{\lambda}\Big[\ln\left(1+\beta e^{-\lambda T_f}\right) - \ln\left(1+\beta\right)\Big]\right\}. \tag{19}$$

Similarly, the farmer's feeding cost per unit time is determined as the product of the farmer's average inventory level, the fraction of items which survive (x) and the feeding cost per weight unit per unit time (c_f). It follows that

$$FC_f = \frac{c_f x D(1+L)\ln\frac{1+L}{1+L-T}}{Txw_1}\left\{\alpha T_f + \frac{\alpha}{\lambda}\Big[\ln\left(1+\beta e^{-\lambda T_f}\right) - \ln\left(1+\beta\right)\Big]\right\}. \tag{20}$$

The farmer's total cost per unit time is the sum of Equations (17), (19) and (20) and it is given by

$$TC_f = \frac{K_f}{nT} + \frac{c_f x + m_f(1-x)}{Txw_1}\left(\alpha T_f + \frac{\alpha}{\lambda}\Big[\ln\left(1+\beta e^{-\lambda T_f}\right) - \ln\left(1+\beta\right)\Big]\right)D(1+L)\ln\left(\frac{1+L}{1+L-T}\right). \tag{21}$$

4.4 Whole supply chain

4.4.1 Constraints governing the proposed inventory system

The feasibility and tractability for the proposed inventory model is dependent on the imposition of two constraints. Firstly, the number of shipments of processed inventory delivered to the retailer per processing setup (n) should be an integer. This makes the solution procedure tractable. Secondly, the duration of the farmer's growth period (T_f) should be less than or equal to the duration of the processor's cycle time ($T_p = nT$). This ensures that the solution to the problem is feasible by assuring that the weight of the live items has reached maturity at the start of the processing run.

4.4.2 Total inventory management cost across the whole supply chain

The total supply chain (inventory management) cost per unit time is determined by adding Equations (12), (16) and (21). Furthermore, the fraction of items items which survive throughout the farmer's replenishment cycle, x, is considered a random variable with a given probability density function $f(x)$. The two constraints and the expected value of the total supply chain cost per unit time are used to formulate the inventory problem at hand as

$$\begin{aligned}\text{Min.}\quad \Bigg\{ E[TC_{sc}] = \frac{K_r}{T} + \frac{h_r D}{T}\left[\frac{(1+L)^2}{2}\ln\left(\frac{1+L}{1+L-T}\right) + \frac{T^2}{4} - \frac{(1+L)T}{2}\right] + \frac{K_p}{nT} \\ + \frac{h_p T D}{2}\left[\Big(n-1\Big)\Big(1-\frac{D}{P}\Big) + \frac{D}{P}\right] + \frac{K_f}{nT} \\ + \left[\frac{c_f E[x] + m_f E[1-x]}{T E[x] w_1}\right]\left[\alpha T_f + \frac{\alpha}{\lambda}\Big[\ln\big(1+\beta e^{-\lambda T_f}\big) - \ln\big(1+\beta\big)\Big]\right]\left[D(1+L)\ln\left(\frac{1+L}{1+L-T}\right)\right]\Bigg\} \\ \text{s.t.}\quad n \in \mathbb{Z}, \quad T_f \le nT. \quad (22)\end{aligned}$$

4.4.3 Solution procedure

The following iterative procedure is followed when calculating the optimal values of n and T:

Step 1 Set n to 1.
Step 2 Find the value of T which minimises Equation (22).
Step 3 Increase n by 1 and find the value of T which minimises Equation (22). Carry on to Step 4.
Step 4 If the latest value of $E[TC_{sc}]$ decreases, go back to Step 3. If the value of $E[TC_{sc}]$ increases, the previously calculated value of $E[TC_{sc}$ (along with corresponding T and n values) is the best solution and in this case, carry on to Step 5.
Step 5 Verify the solution's feasibility with regard to the constraint $T_f \le nT$. T_f is calculated from Equation (2). If the solution is feasible, those values of n and T are optimal and if this is the case, carry on to Step 7. If the solution is not feasible, carry on to Step 6.

Step 6 If the constraint is violated, set T to T_f/n and use that T value to calculate $E[TC_{sc}]$ using Equation (22) and then carry on to Step 7.

Step 7 End.

5 Special case with no mortality nor deterioration

A special case of the proposed inventory model is derived by disregarding the possibility of some of the live items dying throughout the growing cycle and the fact that the processed inventory has a specified shelf life at the retail store. Before deriving this result, two scenarios (which aid in the derivation) are briefly discussed, namely one with no deterioration at the retail echelon and one with no mortality at the farming echelon.

5.1 Scenario I: Infinite shelf life (i.e. no deterioration)

Using the result

$$\lim_{L\to\infty}\left(\frac{1+L}{T}\right)\ln\left(\frac{1+L}{1+L-T}\right)=1, \tag{23}$$

from Wang et al. [36], the weight of the retailer's order for processed inventory, as given in Equation (8), can be rewritten in terms of the result in Equation (23) as

$$Q_1 = DT\left(\frac{1+L}{T}\right)\ln\left(\frac{1+L}{1+L-T}\right). \tag{24}$$

This means that when the processed inventory is assumed to have an infinite shelf life (i.e. in the absence of deterioration), the retailer's order for processed inventory weighs

$$Q_1 = DT, \tag{25}$$

as $L \to \infty$.

Likewise, the result

$$\lim_{L\to\infty}\left[\frac{(1+L)^2}{2}\ln\left(\frac{1+L}{1+L-T}\right)-\frac{(1+L)T}{2}\right]=\frac{T^2}{4}, \tag{26}$$

from Wang et al. [36] is used to evaluate the retailer's holding cost per unit time as given in Equation (10). This holding cost can be rewritten in terms of the result in Equation (26) as

$$HC_r = \frac{h_r D}{T}\left[\frac{(1+L)^2}{2}\ln\left(\frac{1+L}{1+L-T}\right)-\frac{(1+L)T}{2}+\frac{T^2}{4}\right]. \tag{27}$$

Consequently, when the processed inventory is assumed to have an infinite shelf life, the retailer's holding cost per unit time becomes

$$HC_r = \frac{h_r D}{T}\left(\frac{T^2}{4}+\frac{T^2}{4}\right)=\frac{h_r DT}{2}, \tag{28}$$

as $L \to \infty$.

Therefore, the retailer's total cost associated with managing the processed inventory per unit time becomes

$$TC_r = \frac{K_r}{T} + \frac{h_r DT}{2}. \tag{29}$$

Similarly, as $L \to \infty$, the farmer's total cost associated with managing the processed inventory per unit time becomes

$$TC_f = \frac{K_f}{nT} + \frac{\big[c_f x + m_f(1-x)\big]D}{xw_1}\left\{\alpha T_f + \frac{\alpha}{\lambda}\Big[\ln\big(1+\beta e^{-\lambda T_f}\big) - \ln\left(1+\beta\right)\Big]\right\}. \tag{30}$$

Since the processor's total cost function is not a function of L, it remains the same as the one given in Equation (16) even under this scenario.

5.2 Scenario II: No mortality

When all the live items are assumed to survive throughout the growing period (i.e. 100% survival rate or simply, $x = 1$), the farmer's total cost associated with managing the live inventory per unit time becomes

$$TC_f = \frac{K_f}{nT} + \frac{c_f}{Tw_1}\left\{\alpha T_f + \frac{\alpha}{\lambda}\Big[\ln\big(1+\beta e^{-\lambda T_f}\big) - \ln\left(1+\beta\right)\Big]\right\}D(1+L)\ln\left(\frac{1+L}{1+L-T}\right). \tag{31}$$

Under this scenario, the retailer and the processor's total cost function remain the same as those in Equations (12) and (16), respectively.

5.3 Special case

A special case of the proposed inventory model is derived by assuming that all the live ordered items survive (at the farming echelon) and that the processed inventory has an infinite shelf life (at the retail echelon). In essence, this special case is derived by letting $x = 1$ and $L \to \infty$ (simultaneously, as opposed to doing it separately as was the case in the two aforementioned scenarios). Since the processor's total cost function is not affected by neither x nor L, it remains the same as the one given in Equation (16). The retailer's total cost function is only affected by L and it becomes the same as the one given in Equation (29) which corresponds to a situation where $L \to \infty$. Since, the farmer's total cost function is affected by both x and L, the farmer's new total cost for the special case is determined by evaluating Equation (31) for $L \to \infty$. Equation (31) can be rewritten in terms of the result in Equation (23) as

$$\begin{aligned} TC_f &= \frac{K_f}{nT} + \frac{c_f}{Tw_1}\left\{\alpha T_f + \frac{\alpha}{\lambda}\Big[\ln\big(1+\beta e^{-\lambda T_f}\big) - \ln\left(1+\beta\right)\Big]\right\}DT\left(\frac{1+L}{T}\right)\ln\left(\frac{1+L}{1+L-T}\right) \\ &= \frac{K_f}{nT} + \frac{c_f D}{w_1}\left\{\alpha T_f + \frac{\alpha}{\lambda}\Big[\ln\big(1+\beta e^{-\lambda T_f}\big) - \ln\left(1+\beta\right)\Big]\right\}, \end{aligned} \tag{32}$$

as $L \to \infty$.

Consequently, the total inventory management cost across the supply chain becomes

$$TC_{sc} = \frac{K_r}{T} + \frac{h_r DT}{2} + \frac{K_p}{nT} + \frac{h_p TD}{2}\left[\left(n-1\right)\left(1-\frac{D}{P}\right)+\frac{D}{P}\right] + \frac{K_f}{nT} + \frac{c_f D}{w_1}\left\{\alpha T_f + \frac{\alpha}{\lambda}\left[\ln\left(1+\beta e^{-\lambda T_f}\right) - \ln\left(1+\beta\right)\right]\right\}. \quad (33)$$

The result in Equation (33) corresponds to the one in Sebatjane and Adetunji [29] who developed an inventory model for a three-echelon supply chain for growing items without considering the possibility of mortality at the farming echelon and the shelf life of the processed inventory at the retail echelon, making the model presented herein a generalisation of the one presented in that paper.

6 Proof of optimality

It is necessary to show that the objective function of the proposed inventory model has a unique solution that actually minimises the total cost function. Therefore, it suffices to prove that the function is convex. This is achieved through the following two theorems.

Theorem 1 *For a certain $n > 0$, $E[TC_{sc}]$ is a convex function of T and thus there exists a unique value of T which minimises $E[TC_{sc}]$.*

Proof The following auxiliary functions are derived by rewriting the objective function to be of the form $E[TC_{sc}] = \frac{g(T)}{h(T)}$

$$g(T) = K_r + h_r D\left[\frac{(1+L)^2}{2}\ln\left(\frac{1+L}{1+L-T}\right) + \frac{T^2}{4} - \frac{(1+L)T}{2}\right] + \frac{K_p}{n} + \frac{h_p T^2 D}{2}\left[\left(n-1\right)\left(1-\frac{D}{P}\right)+\frac{D}{P}\right] + \frac{K_f}{n} + \left[\frac{c_f E[x] + m_f E[1-x]}{E[x]w_1}\right]\left[\alpha T_f + \frac{\alpha}{\lambda}\left[\ln\left(1+\beta e^{-\lambda T_f}\right) - \ln\left(1+\beta\right)\right]\right]\left[D(1+L)\ln\left(\frac{1+L}{1+L-T}\right)\right], \quad (34)$$

and

$$h(T) = T. \quad (35)$$

Taking the first and the second derivatives of $g(T)$ with respect to T for any specified n results in

$$g'(T) = h_r D\left[\frac{(1+L)^2}{2(1+L-T)} + \frac{T}{2} - \frac{1+L}{2}\right] + \frac{h_p TD}{2}\left[\left(n-1\right)\left(1-\frac{D}{P}\right)+\frac{D}{P}\right] + \left[\frac{c_f E[x] + m_f E[1-x]}{E[x]w_1}\right]\left[\alpha T_f + \frac{\alpha}{\lambda}\left[\ln\left(1+\beta e^{-\lambda T_f}\right) - \ln\left(1+\beta\right)\right]\right]\left[\frac{D(1+L)}{(1+L-T)}\right]. \quad (36)$$

and

$$g''(T) = h_r D\left[\frac{(1+L)^2}{2(1+L-T)^2} + \frac{1}{2}\right] + \frac{h_p D}{2}\left[\Big(n-1\Big)\Big(1-\frac{D}{P}\Big) + \frac{D}{P}\right]$$
$$+\left[\frac{c_f E[x] + m_f E[1-x]}{E[x]w_1}\right]\left[\alpha T_f + \frac{\alpha}{\lambda}\left[\ln\left(1+\beta e^{-\lambda T_f}\right) - \ln\left(1+\beta\right)\right]\right]\left[\frac{D(1+L)}{(1+L-T)^2}\right]. \tag{37}$$

To show that $g(T)$ is strictly convex, it should be guaranteed that $\frac{(1+L)^2}{2(1+L-T)^2}$ and $\frac{D(1+L)}{(1+L-T)^2}$ are always positive. This achieved using Lemma 1. Consequently, $g''(T) > 0$ for all $T > 0$ and therefore $g(T)$ is a differentiable and positive convex function. Given that $h(T)$ is also differentiable and positive convex function, $E[TCU_{sc}]$ is a convex function of T for a given value of n and hence, there exists a unique optimal value of T.

Lemma 1 *$\frac{(1+L)^2}{2(1+L-T)^2}$ and $\frac{D(1+L)}{(1+L-T)^2}$ are always positive for all $T > 0$.*

Proof Let

$$\Delta_1(T) = \frac{(1+L)^2}{2(1+L-T)^2}, \tag{38}$$

$$\Delta_2(T) = \frac{D(1+L)}{(1+L-T)^2}. \tag{39}$$

Taking the first derivatives of $\Delta_1(T)$ and $\Delta_2(T)$ with respect to T results in

$$\Delta_1'(T) = \frac{(1+L)^2}{(1+L-T)^3}, \tag{40}$$

$$\Delta_2'(T) = \frac{2D(1+L)}{(1+L-T)^3}. \tag{41}$$

Since $\Delta_1'(T) > 0$ and $\Delta_2'(T) > 0$, $\Delta_1(T)$ and $\Delta_2(T)$ are increasing functions of T. Therefore, for all $T > 0$, $\frac{(1+L)^2}{2(1+L-T)^2}$ and $\frac{D(1+L)}{(1+L-T)^2}$ are always positive.

Theorem 2 *For all $T > 0$ values, $E[TC_{sc}]$ is a convex function of n and consequently, there exists a unique value of n which minimises $E[TC_{sc}]$.*

Proof Taking the first and the second derivatives of $E[TC_{sc}]$ with respect to n for any specified T results in

$$\frac{\partial E[TC_{sc}]}{\partial n} = -\frac{K_p}{n^2 T} + \frac{h_p T D}{2} - \frac{K_f}{n^2 T}, \tag{42}$$

$$\frac{\partial^2 E[TC_{sc}]}{\partial n^2} = \frac{K_p}{n^3 T} + \frac{K_f}{n^3 T}. \tag{43}$$

$E[TC_{sc}]$ is a convex function of n because $\frac{\partial^2 E[TC_{sc}]}{\partial n^2} > 0$. This indicates that a unique value of n which minimises $E[TC_{sc}]$ exists.

7 Numerical results

7.1 Base example

As a way of demonstrating the potential practical applications of the proposed inventory system, an example that considers a farmer, a processor and a retailer involved in the chicken production supply chain (at different stages) is studied. The retailer meets end consumer demand for processed chicken at a store and can only keep the chicken on shelves for a maximum of 4 days. The example considers the following input parameters: L=4 days; w_0= 0.06kg; w_1= 2kg; D=100 kg/day; P=150 kg/day; K_f=7 500 ZAR; c_f=1 ZAR/kg/day; m_f=2 ZAR/kg/day; K_p=5 000 ZAR; h_p=0.5 ZAR/kg/day; K_r=1 000 ZAR; h_r=1 ZAR/kg/day; α=6.87 kg; β=120; λ=0.11 /day. The fraction of items which survive throughout the farmer's growth period, x, is assumed to be a random variable that is uniformly distributed over [0.8, 1] with a probability density function given by

$$f(x) = \begin{cases} 5, & 0.8 \leq x \leq 1 \\ 1, & \text{otherwise.} \end{cases}$$

This means that

$$E[x] = \int_{0.8}^{1} 5x \quad dx = 5\left[\frac{(1^2 - 0.8^2)}{2}\right] = 0.9$$

The example is solved using Solver, a Microsoft Excel add-in, and the results are presented in Table 2.

Decision variables and objective function	**Quantity**
T^*	1.79 days
n^*	22 shipments
$E[TC_{sc}]^*$	2 909.78 ZAR/day

Table 2: Optimal number of shipments per processing run, cycle time and expected total profit

From those results, the optimal inventory replenishment and shipment policies for all three supply chain members are determined. The farmer should place an order for (ny =) 2 706 live newborn (day old) items. The weight of the all the ordered newborn items (nQ_0) would amount to 162 kg. When the growth period ends, ($E[x]$ =) 90% of the initially ordered items would have survived. This implies that the weight of the surviving mature items (nQ_1) would be 5 412 kg. The farmer should then transfer the entire lot to the processing plant. Throughout the processing cycle, the processor should deliver the items (now in a consumable form) to the retailer in (n =) 22 equally sized batches. Each batch that the retailer receives will weigh about (Q_1 =) 246 kg. The retailer should replenish their inventory every (T =) 1.79 days so that the processed items don't expire (after L = 4 days). The farmer and the processor should start new growing and processing cycles every (nT=) 39.6 days. By following this replenishment and shipment policy, the total costs of managing inventory in the supply will be minimised at 2 909.78 ZAR/day.

7.2 Sensitivity analysis

Table 3: Sensitivity analysis of various input parameters

Parameters	% change	**Retailer's cycle time (T^*)** days	% change	**Number of shipments (n^*)** shipments	% change	**Supply chain cost ($E[TC^*_{sc}]$)** ZAR/day	% change
Base example		**1.79**		**22**		**2 909.78**	
L	-50	1.34	-24.9	29	+31.8	3 183.07	+9.4
	-25	1.60	-10.6	24	+9.1	3 015.86	+3.6
	+25	1.96	+9.8	20	-9.1	2 835.93	-2.5
	+50	2.14	+19.4	18	-18.2	2 781.36	-4.4
K_r	-50	1.38	-23.1	28	+27.3	2 595.94	-10.8
	-25	1.61	-9.8	24	+9.1	2 763.34	-5.0
	+25	1.95	+9.1	20	-9.1	3 042.64	+4.6
	+50	2.08	+16.1	19	-13.6	3 166.25	+8.8
h_r	-50	1.87	+4.6	21	-4.5	2 855.84	-1.9
	-25	1.84	+3.0	21	-4.5	2 883.07	-0.9
	+25	1.76	-1.4	22	0	2 935.59	+0.9
	+50	1.74	-2.8	22	0	2 960.96	+1.8
K_p	-50	1.78	-0.4	20	-9.1	2 841.84	-2.3
	-25	1.81	+1.3	20	-9.1	2 876.62	-1.1
	+25	1.79	+0.1	23	+4.5	2 941.26	+1.1
	+50	1.79	+0.1	24	+9.1	2 971.36	+2.1
h_p	-50	1.81	+1.4	30	+36.4	2 713.09	-6.8
	-25	1.80	+0.7	25	+13.6	2 819.45	-3.1
	+25	1.78	-0.8	20	-9.1	2 989.93	+2.8
	+50	1.77	-1.0	20	-9.1	3 067.44	+5.4
K_f	-50	1.77	-1.0	20	-9.1	2 806.59	-3.5
	-25	1.80	+0.4	20	-9.1	2 859.30	-1.7
	+25	1.80	+0.8	23	+4.5	2 956.36	+1.6
	+50	1.79	0	25	+13.6	3 000.24	+3.1
c_f	-50	2.06	+14.9	19	-13.6	2 247.90	-22.7
	-25	1.92	+7.5	20	-9.1	2 582.00	-11.3
	+25	1.70	-5.0	23	+4.5	3 232.47	+11.1
	+50	1.62	-9.3	24	+9.1	3 551.25	+22.0
m_f	-50	1.85	+3.3	21	-4.5	2 764.71	-5.0
	-25	1.83	+2.4	21	-4.5	2 837.34	-2.5
	+25	1.77	-0.9	22	0	2 981.80	+2.5
	+50	1.76	-1.7	22	0	3 053.66	+4.9
$E[x]$	-50	1.29	-28.0	30	+36.4	5 680.03	+95.2
	-25	1.56	-12.9	25	+13.6	3 855.18	+32.5
	+25	2.03	+13.5	19	-13.6	2 322.78	-20.2
	+50	2.26	+26.2	17	-22.7	1 917.81	-34.1

A sensitivity analysis was performed on the major input parameters in the base example in order to observe the effect of changes (increases and decreases of 25% and 50%) to those parameters on the objective function ($E[TC_{sc}]$) and the two decision variables (T and n). The results are given in Table 3 and the following observations are note-worthy:

- While the shelf life (or expiration date) of the processed items affected the total inventory management cost across the supply chain, the effect was minimal when compared to those it had on the number of shipments and the retailer's cycle time. Case in point, a 50% reduction in the shelf life increased the cost by about 9%, increased the number of shipments by roughly 32% and reduced the retailer's cycle time by roughly 25%. The effects on the shipment and replenishment policy are not surprising considering that the retailer does not want to keep the products beyond their expiration dates. Consequently, when the shelf life of the products is reduced, the model's optimal solution recommends placing orders for relatively smaller lot sizes, but more frequently.

- The effects of changes to the retailer's ordering cost were also significant, but highly anticipated as well. When the cost of placing an order increases, the model's most obvious response is to let the retailer place orders less frequently (by increasing the lot size). However, this can have negative effects on the total cost, particularly due to the increased holding costs as a result of the larger lot size. For example, increasing the ordering cost by 50% increases the cycle time (and lot size) by about 17%. While this counters the effect of the increased ordering cost, it also increases the total cost by approximately 9%.
- When the retailer's holding costs are increased, the model's optimal solution responds by prompting the retailer to order less processed items. The main benefit of ordering smaller lot sizes is that they attract relatively lower holding costs, firstly, because there are fewer items that need to be kept in storage and secondly, because when the lot size is smaller, the retailer's cycle time is reduced which means that the processed inventory spends less time in storage and consequently, the holding costs are reduced. However, meeting a given demand rate from smaller lots leads to an increase in the number of shipments delivered to the retailer. This response is emulated by the results of the sensitivity analysis where, for example, a 50% increases in the retailer's holding costs resulted in a 3% reduction in the cycle time and a 2% increase in the shipments.
- Changes to the processor's setup and holding costs followed an identical response pattern to the retailer's holding and ordering costs. The major difference being that the response in the upstream members is not as sever as those shown by the downstream members. For example, when the processor and the retailer's fixed costs (i.e. setup and ordering costs, respectively) are reduced by 50%, the cycle time and total cost decreased by roughly 23% and 11% respectively in case of the retailer, while for the processor, the changes were about 1% and 2% respectively.
- A reduction in the farmer's feeding and mortality costs prompts the farmer to order more live newborn items. As a result, the processor and the retailer will receive relatively larger lots of mature items for processing and selling respectively. Consequently, the cycle time will increase and the number of shipments will decrease. The increased cycle time means that the processed inventory spends more time in holding and consequently, the total cost increase.
- As the fraction of live items which survive during the growing cycle increases, the number of shipments and the total cost decrease while the cycle time increases. The model's optimal solution responds this way because the a given demand rate for processed items can be met from a smaller lot size of newborn items since the survival rate has improved. While this increases the holding costs at the processor (since there are more mature surviving items that need to be processed), the reduced mortality costs at the farmer cushions against this and consequently, the total supply chain cost decreases despite the increased processor's costs.

7.3 Comparisons with alternative scenarios

The proposed inventory replenishment and shipment policy is compared with three alternative scenarios in order to investigate the benefits (or lack thereof) that might be reaped if those alternative scenarios occurred. The first of these alternative scenarios considers the shelf life of the processed items to be infinite (i.e. the items do not expire). The second alternative scenario considers a situation where the survival rate of the live items during the farmer's growth period is 100%, while the last scenario considers an independent replenishment policy where the replenishment decisions are made for the benefit of individual chain members as opposed to optimising costs for the whole supply chain. The results from the comparison are presented in Table 4.

For the first scenario, the processed items were assumed to have no expiration dates. When the items have an infinite shelf life at the retail store, the retailer's optimal cycle time increases significantly and the number of shipments delivered by the processor to the retailer decreases notably. This is achieved by ordering larger lot sizes. This scenario is actually beneficial to the whole supply chain because of the decreased fixed costs since fewer setups are required if the processed inventory can spends longer time periods in stock without expiring. In the example studied, the supply chain cost decreased by 21.6% under this scenario. While it might not be practical for management to infinitely increase the shelf life of the inventory, this result should motivate management to invest in better preservation technologies which have the potential to prolong the life time of the inventory.

Table 4: Performance of the proposed inventory control mechanism against various alternative scenarios

Variables	**Base Scenario (Proposed system)**	**Scenario 1 (Infinite shelf life)**		**Scenario 2 (No mortality)**		**Scenario 3 (Independent)**	
	Quantity	Quantity	% difference	Quantity	% difference	Quantity	% difference
T^* (days)	1.709	4.23	+136.3	1.91	+16.7	3.04	+70.0
n^* (shipments)	22	9	-59.1	20	-9.1	22	0
y^* (items)	123	235	+91.1	121	-1.6	260	+111.6
TC_r^* (ZAR/day)	663.18	447.89	-32.5	635.61	-4.2	540.02	-18.6
TC_p^* (ZAR/day)	469.90	483.98	+3.0	465.49	-0.9	657.54	+39.9
TC_f^* (ZAR/day)	1 777.70	1 350.25	-24.0	1 517.64	-14.6	2 086.33	+17.4
TC_{sc}^* (ZAR/day)	2 909.78	2 282.12	-21.6	2 618.74	-10.0	3 283.90	+12.9

For the second scenario, it was simply assumed that all (100%) of the initially ordered newborn items survive throughout the farmer's growth cycle. The increased inventory management resulting from mortality also show that item mortality is an important consideration in food production systems which derive most of their products primarily from growing items. Since growing items are living organisms, it is possible for the to die. The financial benefits of having no mortality are not only reaped by the farmer, they also trickle (albeit lower) downstream across the supply chain. The total cost of managing inventory across the supply chain decreased by 10% when the survival rate was 100% (compared to 90% in the base case). This result should motivate management to take measures aimed at increasing the survival rate of the items such as immunisations and inoculations.

For the scenario with an independent replenishment policy (i.e. third scenario), it was assumed that the retailer (who faces consumer demand for processed items) optimises their own inventory replenishment and shipment decisions (while ensuring that the lot does not expire) and these are passed down to the upstream chain members. This resulted in a sizeable reduction (of 18.6%) in the retailer's inventory management costs, however, the total costs of managing inventory across the supply chain increased (by 12.9%). The benefits of coordinating replenishment and shipment decisions with all supply chain members outweigh those achieved through individual optimisation. This emphasises the importance of one of the main objectives of supply chain management which collaborating will all chain members towards a common goal (for the benefit of all parties involved).

The cost differences between the proposed inventory system and the three alternative scenarios highlight the importance of the three major concepts incorporated in the proposed model, namely, item mortality, expiration dates and the integration of shipment and replenishment decisions with all supply chain members. It would, therefore, be advisable for procurement managers in food production systems with an inventory control setup similar to the proposed one to pay close attention to those three issues as they have sizable effects on the financial and operational performance of the supply chain.

8 Conclusion

Operations managers and inventory control specialists at various stages of food production systems are faced with a number of issues. For instance, at the down stream end of the supply chain, retailers are confronted with short product life cycles and their aim is to sell the inventory as fast as possible in order to avoid expiration and to reduce holding costs. One way of achieving this goal is to keep stock levels low but doing so puts the retailers at a higher risk of losing sales due to stock-outs. At the upstream end of the chain, item mortality is a threat to the livelihood of growing items. Another issue facing all supply chain members is deciding whether to make inventory replenishment decisions individually or jointly with other chain members.

This study take all these issues into consideration and develops a coordinated model for inventory management in a supply chain with distinct farming, processing and retail echelons. In addition to determining the optimal replenishment policies to be followed at each echelon, the model demonstrates the benefits (through cost savings) of supply chain integration as well as the drawbacks of item mortality which are not only detrimental at the farming echelon, but are also amplified across the entire supply chain.

Several assumptions, which have the potential to limit the practical applications of the model, were made during the model development process. These include, but are not limited to, deterministic demand and processing rates, one type of growing item in the inventory system and the absence of incentive policies between the supply chain members. Food production systems are not isolated from macroeconomic conditions and are therefore characterised by uncertainty and more often than not, retailers often stock multiple food products derived from growing items. Furthermore, incentive policies like quantity discounts, pre-payments and delayed payments are not uncommon in food production chains where margins are relatively low. Any of these factors, along with various popular EOQ extensions, can be used (solely or in combination with one another) to extend the proposed model.

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