Time consistent mean-variance asset allocation for a DC plan with regime switching under a jump-diffusion model

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Abstract. In this paper, we study a time consistent solution for a defined contribution pension plan under a mean-variance criterion with regime switching in a jump-diffusion setup, during the accumulation phase. We consider a market consisting of a risk-free asset and a geometric jump-diffusion risky asset process. Our solution allows the fund manager to incorporate a clause which allows for the distribution of a member's premiums to his surviving dependents, should the member die before retirement. Applying the extended Hamilton-Jacobi-Bellman (HJB) equation, we derive the explicit time consistent equilibrium strategy and the value function. We then provide some numerical simulations to illustrate our results.

1. Introduction

The investment allocation problem for pension funds is becoming a very important area of research. One possible reason for this is the need to have sufficient funds at retirement, for post retirement living expenses, taking into account the financial risks that accompany fund members' investments. There are two types of pension plans: a Defined Benefit (DB) plan, where the benefits are known in advance and the contributions are adjusted in time to ensure that the fund remains in balance and a Defined Contribution (DC) plan, where the contributions are defined in advance and the benefits depend on the return of the fund, with the investment risks taken by the plan members. For a thorough discussion on the theory of pension funds see e.g. [1], [12] and references therein. Since most developed and developing countries have moved or are moving from DB to DC plans, where the employee is directly exposed to the financial risks, the study of optimization problems in the context of pension funds is of particular interest. The solution of such problems will help the pension fund members or the pension fund managers who act in their behalf, in

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Key words and phrases. DC pension plan, Mean-variance, Stochastic income, regime-switching, extended HJB, mortality risks.

the allocation of funds in different assets in order to achieve the best retirement savings, even during periods of market fluctuations, jumps or lack of information.

Since the classical results on the mean-variance formulation for the portfolio allocation problem proposed by Markowitz [22] in 1952, this approach has become an important tool to study the pension fund asset allocation problem. Li *et. al.* [20] study a DC investment problem under a constant elasticity of variance model with stochastic salary. Using the extended dynamic programming principle, they obtain an explicit optimal strategy before and after retirement. He and Liang [17] consider the optimal investment strategy for a DC pension plan with mortality risks, allowing the return of the premium to protect the rights of a member who die before retirement. Sun *et. al.* [25] study in a mean-variance framework, a pre-commitment and equilibrium investment strategy for a DC pension pension plan under a jump-diffusion model, with deterministic income and mortality risks.

Under the expected utility framework, Chen *et. al.* [8] consider a DC asset allocation with loss aversion and minimum performance. Sun *et. al.* [26] consider a robust portfolio choice for a DC pension plan with stochastic income and interest rate. Sun *et. al.* [25] study a jump diffusion case of a DC investment plan with mortality risks. Other references include [3], [13] and [15], [16].

In all the above references, the basic assumption is that the associated parameters follow a Markovian structure, then applying the extended HJB-equation for solving the corresponding problem. However, it is well known that real market may not be following a Markovian one. For a non-markovian structure of the time-consistent for mean-variance problem, one may consider more general approaches (the backward stochastic differential equation (BSDE) approach or the semi-martingale method). For a BSDE approach, we refer, for instance to Hu *et. al.* [18], [19], Sun and Guo [24], Yan and Wong [28]. For the semi-martingale approach we refer to Czichowsky [11]. Note that the recent work by Sun and Guo [24] consider a jump-diffusion model similar to our current work for the case of not regime-switching.

Most of the above mentioned papers consider pension funds investment problems in a diffusion setup; however, as is well known the real market contains some fluctuations, discontinuities or sudden changes in the evolution of the price process. In order to characterize the dynamics of such markets, we consider a jump-diffusion modeling setup, which is a valuable extension of an existing work in a diffusion framework. Moreover, we assume that the market is described with regime switching, which helps to reflect the economic trends, such as political situations, natural catastrophes or change of law. For a market with two regimes, one can consider Regime 1, as a market with a declining market index (bear market) and a market in Regime 2, representing a growing market index (bull market). We consider a financial market comprising a risk-free asset and a risky asset derived by a jump-diffusion process with regime switching. To protect the rights of a member who dies before retirement, we introduce a clause which allows his/her dependents to withdraw his/her premiums. Moreover, we assume that the evolution of the income of the pension members follows a regime switching jump-diffusion process.

The aim of the fund manager is to maximize the pension fund size and minimize the volatility of the accumulation. We then formulate the problem in a continuous time mean-variance stochastic control setup with regime switching. Our main contribution is to consider a jump-diffusion framework in the presence of regime switching and mortality risk. In a discrete approach, Bian *et. al.* [4] study the pre-commitment and equilibrium strategies for a DC pension plan with regime switching, return premium clause and deterministic income salary. Our problem can also be related to the mean-variance asset-liability management with regime switching in [10], [27].

We then solve the problem via a time consistent dynamic programming principle and solve the extended Hamilton-Jacobi-Bellman (HJB) system of equations as in [5]. This is motivated by the fact that the preferences of an individual changes as times goes on. Therefore the mean variance problem can be viewed as a game problem, where every time $t \in [0, T]$ is a player who chooses a strategy $\pi(t, x, \ell, j)$ at time t. This game theoretic framework has been widely studied in the literature. See, for instance, [6], [21], [20], [25], and references therein.

The rest of the paper is organized as follows: in Section 2, we introduce the DC pension fund mean variance problem under study. In Section 3, we derive the system of extended HJB equations for our regime switching mean variance problem with jumps. Section 3.1 provides the equilibrium strategies and the corresponding value functions for our DC problem. Finally, we give a numerical example to illustrate our results in Section 4.

2. The model formulation

Let $T > 0$ be the investment horizon of a DC pension fund, with retirement date denoted by $t_0 + T$ and $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ a complete filtered probability space. We define on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ a two dimensional Brownian motions $\{W(t)\,,\ W_1(t),\ 0 \le t \le T\}$ and a one dimensional Lévy measure $\nu(\cdot)$, with a Poisson random measure $N(t, \cdot)$. The compensated Poisson random measure is given by

$$\tilde{N}(dt, d\zeta) := N(dt, d\zeta) - \nu(d\zeta)dt\,.$$

Furthermore, consider on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$, a continuous time-homogeneous Markov chain $\{\alpha(t),\ t \in [0, T]\}$ with a finite state regime space $\mathcal{S} = \{1, 2, \dots, D\}$ and a transition rate matrix $\mathbb{A} = \{a_{ij}\}_{i,j\in\mathcal{S}}$ with transition probability matrix (See [23], Theorem 2.1.1)

$$P(t) := (p_{ij}(t))_{i,j\in\mathcal{S}}\,,$$

where

$$p_{ij}(t) := \mathbb{P}(\alpha(t) = j \mid \alpha(0) = i)$$

and

$$\frac{d}{dt}P(t) = \mathbb{A}P(t), \quad P(0) = I\,.$$

Let $M_{ij}(t)$ denote the counting process defined by

$$M_{ij} = \sum_{0<s\le t} \chi_{\{\alpha(s-)=i\}}\chi_{\{\alpha(s)=j\}}\,,$$

where χ_A denotes the indicator function of a set A. The process M_{ij} gives the number of jumps of the Markov process α from state i to state j up to time t. We introduce the martingale process $\tilde{M}_{ij}$ given by

$$\tilde{M}_{ij}(t) = M_{ij}(t) - \int_0^t a_{ij}\chi_{\{\alpha(s-)=i\}} ds \,.$$

For simplicity, we assume throughout the paper that the three stochastic processes $W(t)$, $W_1(t)$, $\tilde{N}(t,\cdot)$ and $\tilde{M}(t)$ are independent.

We assume the existence of a financial market composed by two assets: a bank account, and a risky asset. The bank account has price $B(t)$ defined by

$$dB(t) = r(t, \alpha(t))B(t)dt \,, \tag{2.1}$$

where $r(t,j) \in \mathbb{R}_0^+$, for any $j \in \mathcal{S}$ are the risk-free interest rates corresponding to different market regimes. The risky asset $S(t)$ is defined by the following geometric jump-diffusion process

$$dS(t) = S(t-)\Big[\mu(t, \alpha(t-))dt + \sigma(t, \alpha(t-))dW(t) + \int_{\mathbb{R}} \gamma_S(t, \alpha(t-), \zeta)\tilde{N}(dt, d\zeta)\Big] \,, \tag{2.2}$$

where, for every fixed $j \in \mathcal{S}$, $\mu(t,j)$, $\sigma(t,j)$, $\gamma_S(t,j,\cdot)$ are deterministic continuous functions on the interval $t \in [0,T]$, representing the appreciation rates, volatilities and jump rates at different regimes, respectively. To ensure that the risky asset remains positive, we also assume that $\gamma_S(t,j,\cdot)$ is bounded bellow by -1.

We suppose that a pension member has a stochastic income during the contribution period driven by

$$\begin{aligned} d\ell(t) &= \ell(t-)\Big[\kappa(t, \alpha(t-))dt + \sigma_1(t, \alpha(t-))dW(t) + \sigma_2(t, \alpha(t-))dW_1(t) \\ &\quad + \int_{\mathbb{R}} \gamma_\ell(t, \alpha(t-), \zeta)\tilde{N}(dt, d\zeta)\Big] \,, \end{aligned} \tag{2.3}$$

where $\kappa(t,j)$ are the expected growth rates of the income, while $\sigma_1(t,j)$, $\sigma_2(t,j)$ and $\gamma_\ell(t,j,\cdot)$ represent the volatilities and jump rates of the income at different regimes, respectively. It is assumed that the parameters are also deterministic continuous functions.

Moreover, suppose that the pension member contributes an amount of $\delta\ell(t)$, at time t, where $\delta \in (0,1)$ is the proportion of the salary contributed to the pension plan. Note that this proportion of the salary is constant. We assume that the accumulation period of the fund starts from age $t_0 > 0$ of the member, until the retirement age $t_0 + T$. In order to protect the rights of the plan members who die before retirement, we adopt the withdrawal of the premiums for the member who dies, as in He and Liang [17]. In our DC investment problem, we assume that the employee is concerned about the accumulated amount at the retirement time for consumption after retirement and not the consumption during the accumulation period. For DC investment-consumption problem, we refer, for instance, to [14].

Let M_0 be the number of members who are still alive in the pension at time t, with age t_0+t. Then, the expected number of members who will die during the time interval $(t, t+\Delta t)$ is $M_0 P^{\Delta t}_{t_0+t}$, where $P^{\Delta t}_{t_0+t}$ is the probability that a person alive at the age t_0+t will die in the following time period of length Δt.

Let $Z(t)$ be the total accumulated premiums up to time t. Then, $Z(t)$ follows the dynamics

$$dZ(t) = \delta \ell(t) dt\,, \tag{2.4}$$

with the initial condition $Z(0)=z_0$ ($z_0 \geq 0$). Note that in real practice, we can assume $z_0=0$. Hence, the premium returned to the dependants or estate of a deceased member from time t to $t+\Delta t$ is $Z(t)P^{\Delta t}_{t_0+t}$. After returning the premium, the difference between the accumulation and the return is equally distributed to the surviving members. The expected number of members who are alive at time $t+\Delta t$ is $M_0(1-P^{\Delta t}_{t_0+t})$, which is a deterministic function of time.

Based on He and Liang [17], we adopt the de Moivre mortality model, i.e., the deterministic force of mortality $\beta_{t_0}(t)=\frac{1}{\tau-(t_0+t)}$, where $\tau>0$ is the maximal age of the life table. Then,

$$P^{\Delta t}_{t_0+t} = 1-\exp\Big\{-\int_t^{t+\Delta t}\beta_{t_0}(u)du\Big\} = \frac{\Delta t}{\tau - t_0 - t}\,, \quad 0 \leq \Delta t \leq \tau - t_0 - t.$$

Suppose that the value amount invested in the risky asset at time t is denoted by $\pi(t)$ and $X^\pi(t)$ the corresponding wealth process of the pension plan member. Similar to [17], [25] and [16], we adopt a return premium clause, when a pension member dies during the accumulation phase. Thus, after deducting the expected return of the premiums for the members who died during the time interval $(t, t+\Delta t)$, the total wealth of the pension members is given by

$$\begin{aligned}\hat{X}^\pi(t+\Delta t) &= M_0\Big(1-P^{\Delta t}_{t_0+t}\Big)\Big[(X^\pi(t)-\pi(t))\frac{B(t+\Delta t)}{B(t)} + \pi(t)\frac{S(t+\Delta t)}{S(t)} + \delta\ell(t)\Delta t\Big] \\ &\quad + M_0 P^{\Delta t}_{t_0+t} X^\pi(t) - \varepsilon M_0 Z(t) P^{\Delta t}_{t_0+t}\,.\end{aligned}$$

We assume that ε is a parameter with values 0 or 1. If $\varepsilon=0$, the pension member obtains nothing during the accumulation phase, while if $\varepsilon=1$, the premiums are returned to the member when he dies. Then, the total wealth is equally distributed to the surviving members and each of them has the pension wealth of

$$\begin{aligned}X^\pi(t+\Delta t) &= \frac{\hat{X}^\pi(t+\Delta t)}{M_0(1-P^{\Delta t}_{t_0+t})} \\ &= (X^\pi(t)-\pi(t))\frac{B(t+\Delta t)}{B(t)} + \pi(t)\frac{S(t+\Delta t)}{S(t)} + \delta\ell(t)\Delta t \\ &\quad -\beta_{t_0}(t)\Delta t[\varepsilon Z(t) - X^\pi(t)] + o(\Delta t)\,.\end{aligned}$$

Dividing by Δt and taking the limit, when $\Delta t \to 0$, we have the following wealth process in continuous time:

$$\begin{aligned} dX^\pi(t) &= \Big[X^\pi(t-)(r(t,\alpha(t-)) + \beta_{t_0}(t)) + (\mu(t,\alpha(t-)) - r(t,\alpha(t-)))\pi(t) + \delta\ell(t) \\ &\quad -\varepsilon\beta_{t_0}(t)Z(t)\Big]dt + \pi(t)\sigma(t,\alpha(t-))dW(t) + \pi(t)\int_{\mathbb{R}}\gamma_S(t,\alpha(t-),\zeta)\tilde{N}(dt,d\zeta)\,. \end{aligned} \tag{2.5}$$

Definition 2.1. We define $\mathcal{A}$ as the set of measurable (deterministic) functions $(t,x,\ell,z,j) \to \pi(t,x,\ell,z,j) \in \mathbb{R}$ such that, for each (x,ℓ,z,j), the closed-loop system (2.5) above, with the initial condition $(X^\pi(0),\ell(0),Z(0),\alpha(0)) = (x,\ell,z,j)$ has a unique strong solution.

We will then formulate the pension fund mean variance investment problem without pre-commitment. In order to understand such kind of problems, we first define the pre-commitment mean variance optimization problem. This problem can be described as the maximization of the following functional:

$$J(0,x,\pi) = \mathbb{E}_{0,x,\ell,z,j}[X^\pi(T)] - \frac{\xi(j)}{2}\mathsf{Var}_{0,x,\ell,z,j}[X^\pi(T)]\,,$$

over all admissible strategies $\pi \in \mathcal{A}$.

As we can see in the above functional, we fix the initial point $(0,X^\pi(0))$ and try to find the control π^* which maximizes $J(0,X^\pi(0),\pi)$, that is, there is no update on the optimal strategy $\pi*$ and future dates $(t,X^\pi(t))$. However, for the optimization problem without pre-commitment, the DC investment manager updates the investment strategy at each state $(t,X^\pi(t))$, i.e., the value function is given by

$$\sup_{\pi\in\mathcal{A}}\Big\{\mathbb{E}_{t,x,\ell,z,j}[X^\pi(T)] - \frac{\xi(j)}{2}\mathsf{Var}_{t,x,\ell,z,j}[X^\pi(T)]\Big\}\,. \tag{2.6}$$

Here, $\xi(j)$ are the risk aversion coefficients under the market regimes $j \in \mathcal{S}$ and $\mathbb{E}_{t,x,\ell,z,j}[\cdot]$, $\mathsf{Var}_{t,x,\ell,z,j}[\cdot]$ are the expectation and variance conditioned on the event $[X(t) = x, \ell(t) = \ell, Z(0) = z, \alpha(t) = j]$ respectively.

Assume that, for any $(t,x,\ell,z,j) \in [0,T] \times \mathbb{R}^3 \times \mathcal{S}$,

$$F(x,j) = x - \frac{\xi(j)}{2}x^2 \;\text{ and }\; \Gamma(x,j) = \frac{\xi(j)}{2}x^2\,.$$

We define the following functional

$$\Phi(t,x,\ell,z,j,\pi) := \mathbb{E}_{t,x,\ell,z,j}[F(X^\pi(T),j)] + \Gamma(\mathbb{E}_{t,x,\ell,z,j}[X^\pi(T)],j)\,.$$

Then, the mean-variance optimization problem (2.6) becomes

$$\sup_{\pi\in\mathcal{A}}\Phi(t,x,\ell,z,j,\pi)\,. \tag{2.7}$$

Note that Γ is a nonlinear function acting on the conditional expectation, which leads to a time-inconsistent optimization problem in (2.7) as was pointed out by [5] and [6]. Therefore, an optimal strategy at time t does not guarantee the optimality of Φ at subsequent moments $s > t$. However, since the time horizon T of a pension fund is very long, the plan member's preference may change over time, then it becomes very important to formulate the time-consistent optimal investment problem for the DC pension fund. To that end, we follow the game theoretic approach of our problem as in [5], [6] and define an equilibrium strategy which is consistent with time change, i.e., the optimal strategy derived at time t should agree with the optimal strategy at time $t+\epsilon$, $\epsilon > 0$.

Definition 2.2. For any fixed $(t, x, \ell, z, j) \in [0, T] \times \mathbb{R}^3 \times \mathcal{S}$, a control law π^* is an equilibrium control if for every admissible control π and $\epsilon > 0$, the collection of strategies π_ϵ, defined by

$$\pi_\epsilon(s, x, \ell, z, j) = \begin{cases} \pi(s, x, \ell, z, j), & \text{for } t \le s < t+\epsilon \\ \pi^*(s, x, \ell, z, j), & \text{for } t+\epsilon \le s \le T\,, \end{cases}$$

satisfies the property

$$\liminf_{\epsilon \to 0} \frac{\Phi(t, x, \ell, z, j, \pi^*) - \Phi(t, x, \ell, z, j, \pi_\epsilon)}{\epsilon} \ge 0\,.$$

The above definition implies that the equilibrium control law π^* is time consistent, i.e., if all the players $t+\epsilon \le s \le T$, choose the strategy π^* as their optimal, it remains optimal for the players $t \le s < t+\epsilon$.

In order to ensure the existence of the equilibrium value function and the equilibrium control strategy, it is sufficient to assume the uniform boundedness and non-degeneracy conditions on the associated parameters:

- $(\mathbf{A_1})$ $r(t,j)$, $\mu(t,j)$, $\sigma(t,j)$, $\gamma_S(t,j,\cdot)$, $\kappa(t,j)$, $\sigma_1(t,j)$ and $\gamma_\ell(t,j,\cdot)$ are uniformly bounded on $[0,T]$, for each $j \in \mathcal{S}$. And the risk aversion coefficient $\xi(j)$ is non-degenerate, that is, there exists $\epsilon > 0$, such that $\xi(j) > \epsilon$, for any $j \in \mathcal{S}$.

In order to solve the optimization problem (2.7), we will apply the game theoretic framework as in [5] for our mean-variance DC problem with regime switching and jumps problem. For simplicity of notation, we denote $r(t,j), \mu(t,j), \sigma(t,j), \gamma_S(t,j,\cdot), \kappa(t,j), \sigma_1(t,j), \sigma_2(t,j)$, $\gamma_\ell(t,j,\cdot)$, $\xi(j)$ by $r_j, \mu_j, \sigma_j, \gamma_{Sj}, \kappa_j, \sigma_{1j}, \sigma_{2j}, \gamma_{\ell j}$, ξ_j, respectively.

Applying the Itô's formula for Markov regime switching jump diffusion process (see [9], Theorem 3.1.), for any $\phi(t,x,\ell,z,j) \in \mathcal{C}^{1,2,2,1}([0,T] \times \mathbb{R}^3 \times \mathcal{S})$, with $\pi \in \mathcal{A}$ and x, ℓ, z in (2.5), (2.3) and (2.4), the infinitesimal generator operator is defined by

$$\begin{aligned} & \mathcal{L}\phi(t,x,\ell,z,j) \qquad (2.8) \\ = \;& \phi_t(t,x,\ell,z,j) + \Big[x(r(t,j)+\beta_{t_0}(t)) + (\mu(t,j)-r(t,j))\pi(t) + \delta\ell - \varepsilon\beta_{t_0}(t)z\Big]\phi_x(t,x,\ell,z,j) \\ & + \kappa(t,j)\ell\phi_\ell(t,x,\ell,z,j) + \delta\ell\phi_z(t,x,\ell,z,j) + \frac{1}{2}(\sigma_1^2(t,j)+\sigma_2^2(t,j))\ell^2\phi_{\ell\ell}(t,x,\ell,z,j) \\ & + \frac{1}{2}\pi^2(t)\sigma^2(t,j)\phi_{xx}(t,x,\ell,z,j) + \pi(t)\ell\sigma(t,j)\sigma_1(t,j)\phi_{x\ell}(t,x,\ell,z,j) \\ & + \sum_{\alpha\in\mathcal{S},\alpha\ne j} a_{j\alpha}(\phi(t,x,\ell,z,\alpha) - \phi(t,x,\ell,z,j)) \end{aligned}$$

$$+\int_{\mathbb{R}_0}\Big[\phi(t,x+\pi(t)\gamma_S(t,j,\zeta),\ell(1+\gamma_\ell(t,j,\zeta)),z,j)-\phi(t,x,\ell,z,j)$$
$$-\pi(t)\gamma_S(t,j,\zeta)\phi_x(t,x,\ell,z,j)-\ell\gamma_\ell(t,j,\zeta)\phi_\ell(t,x,\ell,z,j)\Big]\nu(d\zeta)\,.$$

3. The extended HJB system

In this section, we derive the extended HJB system for our mean variance problem with regime switching and the corresponding verification theorem. Our aim is to establish explicitly the admissible equilibrium control for our problem according to Definition 2.2. For more details we refer to [5], [27], [20], and references therein.

Theorem 3.1. *Suppose that there exist functions $\Psi,\varphi:[0,T]\times\mathbb{R}^3\times\mathcal{S}\to\mathbb{R}$ and $\psi:[0,T]\times\mathbb{R}^3\times\mathcal{S}\times[0,T]\times\mathcal{S}\to\mathbb{R}$ satisfying the following system of equations:*

$$\sup_{\pi\in\mathbb{R}}\Big\{\mathcal{L}^\pi\Psi(t,x,\ell,z,j)-\mathcal{L}^\pi\psi(t,x,\ell,z,j,j')+\mathcal{L}^\pi\psi^{j'}(t,x,\ell,z,j)-\mathcal{L}^\pi(\Gamma\circ\varphi)(t,x,\ell,z,j)$$
$$+\mathcal{M}^\pi\varphi(t,x,\ell,z,j)\Big\}=\,0\,,\tag{3.1}$$
$$\mathcal{L}^{\pi^*}\psi^{j'}(t,x,\ell,z,j)\,=\,0\,,\tag{3.2}$$
$$\mathcal{L}^{\pi^*}\varphi(t,x,\ell,z,j)\,=\,0\,,\tag{3.3}$$
$$\Psi(T,x,\ell,z,j)\,=\,x\,,$$
$$\psi^{j'}(T,x,\ell,z,j)\,=\,x-\frac{\xi(j')}{2}x^2\,,$$
$$\varphi(T,x,\ell,z,j)\,=\,x\,,$$

where $\psi^{j'}(t,x,\ell,z,j):=\psi(t,x,\ell,z,j,t,j)$, $(\Gamma\circ\varphi)(t,x,\ell,z,j):=\Gamma(t,\varphi(t,x,\ell,z,j),j)=\frac{\xi(j)}{2}\varphi(t,x,\ell,z,j)^2$, and $\mathcal{M}^\pi\varphi(t,x,\ell,z,j):=\Gamma_y(t,\varphi(t,x,\ell,z,j),j)\times\mathcal{L}^\pi\varphi(t,x,\ell,z,j)$, for any $j'\in\mathcal{S}$.

Then π^* is an equilibrium control law and Ψ is the corresponding value function. Furthermore, ψ and φ have the following probabilistic representation:

$$\psi(t,x,\ell,z,j,j')=\mathbb{E}_{t,x,\ell,z,j}[F(X^{\pi^*}(T),j)]\qquad\text{and}\qquad\varphi(t,x,\ell,z,j)=\mathbb{E}_{t,x,\ell,z,j}[X^{\pi^*}(T)]\,.$$

Proof. See Appendix. □

Note that from the probabilistic representation of ψ and φ, Ψ can clearly be written as

$$\Psi(t,x,\ell,z,j)=\psi(t,x,\ell,z,j,j')+\Gamma(t,\varphi(t,x,\ell,z,j),j)\,.\tag{3.4}$$

Hence, the first equation in Theorem 3.1 is simplified to

$$\sup_{\pi\in\mathbb{R}}\Big\{\mathcal{L}^\pi\psi^{j'}(t,x,\ell,z,j)+\mathcal{M}^\pi\varphi(t,x,\ell,z,j)\Big\}=\,0\,.$$

Using the generator (2.8), the extended HJB system can be written as

$$\sup_{\pi\in\mathbb{R}}\Big\{\psi_t^{j'}(t,x,\ell,z,j)+\xi_j\varphi(t,x,\ell,z,j)\varphi_t(t,x,\ell,z,j)+[x(r_j+\beta_{t_0})+(\mu_j-r_j)\pi$$

$$+\delta\ell-\varepsilon\beta_{t_0}z][\psi_x^{j'}(t,x,\ell,z,j)+\xi_j\varphi(t,x,\ell,z,j)\varphi_x(t,x,\ell,z,j)]$$
$$+\kappa_j\ell[\psi_\ell^{t,j}(t,x,\ell,z,j)+\xi_j\varphi(t,x,\ell,z,j)\varphi_\ell(t,x,\ell,z,j)]$$
$$+\delta\ell[\psi_z^{t,j}(t,x,\ell,z,j)+\xi_j\varphi(t,x,\ell,z,j)\varphi_z(t,x,\ell,z,j)]$$
$$+\frac{1}{2}\sigma_j^2\pi^2[\psi_{xx}^{j'}(t,x,\ell,z,j)+\xi_j\varphi(t,x,\ell,z,j)\varphi_{xx}(t,x,\ell,z,j)]$$
$$+\frac{1}{2}(\sigma_{1j}^2+\sigma_{2j}^2)\ell^2[\psi_{\ell\ell}^{j'}(t,x,\ell,z,j)+\xi_j\varphi(t,x,\ell,z,j)\varphi_{\ell\ell}(t,x,\ell,z,j)]$$
$$+\sigma_j\sigma_{1j}\ell\pi[\psi_{x\ell}^{j'}(t,x,\ell,z,j)+\xi_j\varphi(t,x,\ell,z,j)\varphi_{x\ell}(t,x,\ell,z,j)]$$
$$+\sum_{\alpha\in\mathcal{S},\alpha\neq j}a_{j\alpha}[\psi^{j'}(t,x,\ell,z,\alpha)+\xi_\alpha\varphi(t,x,\ell,z,\alpha)\varphi(t,x,\ell,z,\alpha)]$$
$$-\sum_{\alpha\in\mathcal{S},\alpha\neq j}a_{j\alpha}[\psi^{j'}(t,x,\ell,z,j)+\xi_j\varphi(t,x,\ell,z,j)\varphi(t,x,\ell,z,j)]$$
$$+\int_{\mathbb{R}_0}\Big[\psi^{j'}(t,x+\pi\gamma_{Sj}(\zeta),\ell(1+\gamma_{\ell j}(\zeta)),z,j)-\psi^{j'}(t,x,\ell,z,j)$$
$$+\xi_j\varphi(t,x,\ell,z,j)\varphi(t,x+\pi\gamma_{Sj}(\zeta),\ell(1+\gamma_{\ell j}(\zeta)),z,j)-\varphi(t,x,\ell,z,j)$$
$$-\pi\gamma_{Sj}(\zeta)[\psi_x^{j'}(t,x,\ell,z,j)+\xi_j\varphi(t,x,\ell,z,j)\varphi_x(t,x,\ell,z,j)]$$
$$-\ell\gamma_{\ell j}(\zeta)[\psi_\ell^{j'}(t,x,\ell,z,j)+\xi_j\varphi(t,x,\ell,z,j)\varphi_\ell(t,x,\ell,z,j)]\Big]\nu(d\zeta)\Big\}=0\,, \tag{3.5}$$

$$\psi_t^{j'}(t,x,\ell,z,j)+[x(r_j+\beta_{t_0})+(\mu_j-r_j)\pi^*+\delta\ell-\varepsilon\beta_{t_0}z]\psi_x^{j'}(t,x,\ell,z,j)$$
$$+\kappa_j\ell\psi_\ell^{j'}(t,x,\ell,z,j)+\delta\ell\psi_z^{j'}(t,x,\ell,z,j)+\frac{1}{2}\sigma_j^2(\pi^*)^2\psi_{xx}^{j'}(t,x,\ell,z,j)$$
$$+\frac{1}{2}(\sigma_{1j}^2+\sigma_{2j}^2)\ell^2\psi_{\ell\ell}^{j'}(t,x,\ell,z,j)+\sigma_j\sigma_{1j}\ell\pi^*\psi_{x\ell}^{j'}(t,x,\ell,z,j)$$
$$+\sum_{\alpha\in\mathcal{S},\alpha\neq j}a_{j\alpha}[\psi^{j'}(t,x,\ell,z,\alpha)-\psi^{j'}(t,x,\ell,z,j)]$$
$$+\int_{\mathbb{R}_0}\Big[\psi^{j'}(t,x+\pi^*\gamma_{Sj}(\zeta),\ell(1+\gamma_{\ell j}(\zeta)),z,j)-\psi^{j'}(t,x,\ell,z,j)$$
$$-\pi^*\gamma_{Sj}(\zeta)\psi_x^{j'}(t,x,\ell,z,j)-\ell\gamma_{\ell j}(\zeta)\psi_\ell^{j'}(t,x,\ell,z,j)\Big]\nu(d\zeta)=0\,, \tag{3.6}$$

$$\varphi_t(t,x,\ell,z,j)+[x(r_j+\beta_{t_0})+(\mu_j-r_j)\pi^*+\delta\ell-\varepsilon\beta_{t_0}z]\varphi_x(t,x,\ell,z,j)$$
$$+\kappa_j\ell\varphi_\ell(t,x,\ell,z,j)+\delta\ell\varphi_z(t,x,\ell,z,j)+\frac{1}{2}\sigma_j^2(\pi^*)^2\varphi_{xx}(t,x,\ell,z,j)$$
$$+\frac{1}{2}(\sigma_{1j}^2+\sigma_{2j}^2)\ell^2\varphi_{\ell\ell}(t,x,\ell,z,j)+\sigma_j\sigma_{1j}\ell\pi^*\varphi_{x\ell}(t,x,\ell,z,j)$$
$$+\sum_{\alpha\in\mathcal{S},\alpha\neq j}a_{j\alpha}[\varphi(t,x,\ell,z,\alpha)-\varphi(t,x,\ell,z,j)]$$

$$+\int_{\mathbb{R}_0}\Big[\varphi(t,x+\pi^*\gamma_{Sj}(\zeta),\ell(1+\gamma_{\ell j}(\zeta)),z,j)-\varphi(t,x,\ell,z,j)$$
$$-\pi^*\gamma_{Sj}(\zeta)\varphi_x(t,x,\ell,z,j)-\ell\gamma_{\ell j}(\zeta)\varphi_\ell(t,x,\ell,z,j)\Big]\nu(d\zeta)\;=\;0\,, \tag{3.7}$$

$$\psi^{j'}(T,x,\ell,z,j)\;=\;x-\frac{\xi_j}{2}x^2\,, \tag{3.8}$$

$$\varphi(T,x,\ell,z,j)\;=\;x\,. \tag{3.9}$$

If the expression in the sup function (3.5) is concave in π, i.e.,

$$\sigma_j^2[\psi^{j'}_{xx}(t,x,\ell,z,j)+\xi_j\varphi(t,x,\ell,z,j)\varphi_{xx}(t,x,\ell,z,j)]$$
$$+\int_{\mathbb{R}_0}\Big[\psi^{j'}_{\pi\pi}(t,x+\pi\gamma_{Sj}(\zeta),\ell(1+\gamma_{\ell j}(\zeta)),z,j)$$
$$+\xi_j\varphi(t,x,\ell,z,j)\varphi_{\pi\pi}(t,x+\pi\gamma_{Sj}(\zeta),\ell(1+\gamma_{\ell j}(\zeta)),z,j)\Big]\nu(d\zeta)\;<\;0\,, \tag{3.10}$$

then the equilibrium control π^* solves the following equation

$$(\mu_j-r_j)[\psi^{j'}_x(t,x,\ell,z,j)+\xi_j\varphi(t,x,\ell,z,j)\varphi_x(t,x,\ell,z,j)]+\sigma_j\sigma_{1j}\ell[\psi^{j'}_{x\ell}(t,x,\ell,z,j)$$
$$+\xi_j\varphi(t,x,\ell,z,j)\varphi_{x\ell}(t,x,\ell,z,j)]+\sigma_j^2\pi^*[\psi^{j'}_{xx}(t,x,\ell,z,j)+\xi_j\varphi(t,x,\ell,z,j)\varphi_{xx}(t,x,\ell,z,j)]$$
$$+\int_{\mathbb{R}_0}\Big[\psi^{j'}_{\pi}(t,x+\pi^*\gamma_{Sj}(\zeta),\ell(1+\gamma_{\ell j}(\zeta)),z,j)$$
$$+\xi_j\varphi(t,x,\ell,z,j)\varphi_{\pi}(t,x+\pi^*\gamma_{Sj}(\zeta),\ell(1+\gamma_{\ell j}(\zeta)),z,j)$$
$$-\gamma_{Sj}(\zeta)[\psi^{j'}_x(t,x,\ell,z,j)+\xi_j\varphi(t,x,\ell,z,j)\varphi_x(t,x,\ell,z,j)]\Big]\nu(d\zeta)\;=\;0\,. \tag{3.11}$$

In the following section, we derive the solution for the equilibrium control and the corresponding value function.

3.1. **Solution of the time-consistent mean-variance DC problem.**

We attempt to solve the extended HJB system (3.5)-(3.9) and obtain the explicit equilibrium control $\pi^*\in\mathcal{A}$. As in [27], we conjecture the solutions of the following form:

$$\varphi(t,x,\ell,j)\;=\;b(t,j)x+c(t,j)\ell+h(t,j)z+e(t,j)\,, \tag{3.12}$$

$$\begin{aligned}\psi(t,x,\ell,j,j')\;=\;&b(t,j)x+c(t,j)\ell+h(t,j)z+e(t,j)-\frac{\xi(j)}{2}\Big[A(t,j)x^2\\&+B(t,j)\ell^2+P(t,j)z^2+2C(t,j)x\ell+2R(t,j)xz+2K(t,j)z\ell\\&+2M(t,j)x+2N(t,j)\ell+2H(t,j)z+Q(t,j)\Big]\,,\end{aligned} \tag{3.13}$$

with the terminal conditions $b(T,j)=A(T,j)=1$ and $c(T,j)=h(t,j)=e(T,j)=B(T,j)=C(T,j)=M(T,j)=N(T,j)=H(t,j)=P(T,j)=R(T,j)=K(T,j)=Q(T,j)=0$. For the concavity condition (3.10) to be satisfied, we assume that $A(t,j)>0$, for all $(t,j)\in[0,T]\times\mathcal{S}$. For simplicity, we adopt the following notation b_j,c_j,h_j,e_j,

$A_j, B_j, P_j, C_j, M_j, N_j, H_j, R_j, K_j$ and Q_j for $b(t,j), c(t,j), h(t,j), e(t,j), A(t,j), B(t,j), C(t,j), M(t,j), N(t,j), H(t,j), P(t,j), R(t,j), K(t,j)$ and $Q(t,j)$ respectively.

From the conjecture (3.12)-(3.13) and equation (3.11), we can easily see that the equilibrium control law π^* is given by

$$\begin{aligned}\pi^*(t,x,\ell,z,j) \;=\; & \frac{(\mu_j - r_j)(b_j^2 - A_j)}{A_j\Big(\sigma_j^2 + \int_{\mathbb{R}_0}\gamma_{Sj}^2\nu(d\zeta)\Big)}x + \frac{(\mu_j - r_j)b_j h_j}{A_j\Big(\sigma_j^2 + \int_{\mathbb{R}_0}\gamma_{Sj}^2\nu(d\zeta)\Big)}z \\ & + \frac{(\mu_j - r_j)(b_j c_j - C_j) - C_j\Big(\sigma_j\sigma_{1j} + \int_{\mathbb{R}_0}\gamma_{Sj}\gamma_{\ell j}\nu(d\zeta)\Big)}{A_j\Big(\sigma_j^2 + \int_{\mathbb{R}_0}\gamma_{Sj}^2\nu(d\zeta)\Big)}\ell \\ & + \frac{(\mu_j - r_j)(b_j + \xi_j(b_j e_j - M_j))}{\xi_j A_j\Big(\sigma_j^2 + \int_{\mathbb{R}_0}\gamma_{Sj}^2\nu(d\zeta)\Big)}. \end{aligned} \tag{3.14}$$

Based on (3.12), (3.13) and (3.14), the expressions (3.6)-(3.7) can be written as

$$\begin{aligned}
&-\frac{\xi_j}{2}\Big[\dot{A}_j + 2(r_j + \beta_{t_0})A_j + \frac{(\mu_j - r_j)^2(b_j^2 - A_j)}{\sigma_j^2 + \int_{\mathbb{R}_0}\gamma_{Sj}^2\nu(d\zeta)}\Big(\frac{b_j^2}{A_j} + 1\Big) + \sum_{\alpha\in\mathcal{S},\alpha\neq j} a_{j\alpha}(A_\alpha - A_j)\Big]x^2 \\
&-\frac{\xi_j}{2}\Big\{\dot{B}_j + \Big(2\kappa_j + \sigma_{1j}^2 + \int_{\mathbb{R}_0}\gamma_{\ell j}^2\nu(d\zeta)\Big)B_j + 2\delta(C_j + K_j) \\
&+2\frac{(\mu_j - r_j)(b_j c_j - C_j) - C_j\Big(\sigma_j\sigma_{1j} + \int_{\mathbb{R}_0}\gamma_{Sj}\gamma_{\ell j}\nu(d\zeta)\Big)}{A_j\Big(\sigma_j^2 + \int_{\mathbb{R}_0}\gamma_{Sj}^2\nu(d\zeta)\Big)}\Big(\mu_j - r_j + \int_{\mathbb{R}_0}\gamma_{Sj}\gamma_{\ell j}\nu(d\zeta)\Big)C_j \\
&+\frac{\Big[(\mu_j - r_j)(b_j c_j - C_j) - C_j\Big(\sigma_j\sigma_{1j} + \int_{\mathbb{R}_0}\gamma_{Sj}\gamma_{\ell j}\nu(d\zeta)\Big)\Big]^2}{A_j\Big(\sigma_j^2 + \int_{\mathbb{R}_0}\gamma_{Sj}^2\nu(d\zeta)\Big)} + \sum_{\alpha\in\mathcal{S},\alpha\neq j} a_{j\alpha}(B_\alpha - B_j)\Big\}\ell^2 \\
&-\frac{\xi_j}{2}\Big[\dot{P}_j - 2\varepsilon\beta_{t_0}R_j + \frac{(\mu_j - r_j)^2 b_j h_j}{A_j\Big(\sigma_j^2 + \int_{\mathbb{R}_0}\gamma_{Sj}^2\nu(d\zeta)\Big)}\Big(2R_j + b_j h_j\Big) + \sum_{\alpha\in\mathcal{S},\alpha\neq j} a_{j\alpha}(P_\alpha - P_j)\Big]z^2 \\
&-\xi_j\Big\{\dot{C}_j + (r_j + \beta_{t_0} + \kappa_j)C_j + \frac{(\mu_j - r_j)(b_j^2 - A_j)C_j}{A_j\Big(\sigma_j^2 + \int_{\mathbb{R}_0}\gamma_{Sj}^2\nu(d\zeta)\Big)}\Big(\mu_j - r_j + \sigma_j\sigma_{1j} \\
&+\int_{\mathbb{R}_0}\gamma_{Sj}\gamma_{\ell j}\nu(d\zeta)\Big) + \frac{(\mu_j - r_j)\Big[(\mu_j - r_j)(b_j c_j - C_j) - C_j\Big(\sigma_j\sigma_{1j} + \int_{\mathbb{R}_0}\gamma_{Sj}\gamma_{\ell j}\nu(d\zeta)\Big)\Big]b_j^2}{A_j\Big(\sigma_j^2 + \int_{\mathbb{R}_0}\gamma_{Sj}^2\nu(d\zeta)\Big)} \\
&+\delta(A_j + R_j) + \sum_{\alpha\in\mathcal{S},\alpha\neq j} a_{j\alpha}(C_\alpha - C_j)\Big\}x\ell - \xi_j\Big[\dot{R}_j + (r_j + \beta_{t_0})R_j - \varepsilon\beta_{t_0}A_j
\end{aligned}$$

$$
\begin{aligned}
&+\frac{(\mu_j-r_j)(A_jb_jh_j+(R_j+b_jh_j)(b_j^2-A_j))}{A_j\Big(\sigma_j^2+\int_{\mathbb{R}_0}\gamma_{Sj}^2\nu(d\zeta)\Big)}+\sum_{\alpha\in\mathcal{S},\alpha\neq j}a_{j\alpha}(R_\alpha-R_j)\Big]xz\\
&-\xi_j\Big\{\dot{K}_j+\kappa K_j+2\delta(R_j+P_j)-\varepsilon\beta_{t_0}C_j+\frac{(\mu_j-r_j)(\mu_j-r_j+\sigma_j\sigma_{1j})b_jh_jC_j}{A_j\Big(\sigma_j^2+\int_{\mathbb{R}_0}\gamma_{Sj}^2\nu(d\zeta)\Big)}\\
&+\frac{(\mu_j-r_j)(R_j+b_jh_j)\Big[(\mu_j-r_j)(b_jc_j-C_j)-C_j\Big(\sigma_j\sigma_{1j}+\int_{\mathbb{R}_0}\gamma_{Sj}\gamma_{\ell j}\nu(d\zeta)\Big)\Big]}{A_j\Big(\sigma_j^2+\int_{\mathbb{R}_0}\gamma_{Sj}^2\nu(d\zeta)\Big)}\\
&+\sum_{\alpha\in\mathcal{S},\alpha\neq j}a_{j\alpha}(K_\alpha-K_j)\Big\}z\ell-\xi_j\Big\{\dot{M}_j+(r_j+\beta_{t_0})M_j\\
&+\frac{(\mu_j-r_j)^2(b_j^2-A_j)}{A_j\Big(\sigma_j^2+\int_{\mathbb{R}_0}\gamma_{Sj}^2\nu(d\zeta)\Big)}M_j+\frac{(\mu_j-r_j)^2(b_j+\xi_j(b_je_j-M_j))}{\xi_jA_j\Big(\sigma_j^2+\int_{\mathbb{R}_0}\gamma_{Sj}^2\nu(d\zeta)\Big)}b_j^2+\sum_{\alpha\in\mathcal{S},\alpha\neq j}a_{j\alpha}(M_\alpha-M_j)\Big\}x\\
&-\xi_j\Big\{\dot{N}_j+\kappa_jN_j+\delta(M_j+H_j)+\sigma_{Sj}\sigma_{1j}C_j+\sum_{\alpha\in\mathcal{S},\alpha\neq j}a_{j\alpha}(N_\alpha-N_j)\\
&+\frac{(\mu_j-r_j)(b_j+\xi_j(b_je_j-M_j))}{\xi_jA_j\Big(\sigma_j^2+\int_{\mathbb{R}_0}\gamma_{Sj}^2\nu(d\zeta)\Big)}\Big(\mu_j-r_j+\int_{\mathbb{R}_0}\gamma_{Sj}\gamma_{\ell j}\nu(d\zeta)\Big)C_j\\
&+\frac{(\mu_j-r_j)(b_jc_j-C_j)-C_j\Big(\sigma_j\sigma_{1j}+\int_{\mathbb{R}_0}\gamma_{Sj}\gamma_{\ell j}\nu(d\zeta)\Big)}{\xi_jA_j\Big(\sigma_j^2+\int_{\mathbb{R}_0}\gamma_{Sj}^2\nu(d\zeta)\Big)}(\mu_j-r_j)(1+\xi_je_j)b_j\Big\}\ell\\
&-\xi_j\Big[\dot{H}_j-\varepsilon\beta_{t_0}M_j+\frac{(\mu_j-r_j)^2b_jh_jM_j}{A_j\Big(\sigma_j^2+\int_{\mathbb{R}_0}\gamma_{Sj}^2\nu(d\zeta)\Big)}+\frac{(\mu_j-r_j)^2(b_j+\xi_j(b_je_j-M_j))(R_j+b_jh_j)}{\xi_jA_j\Big(\sigma_j^2+\int_{\mathbb{R}_0}\gamma_{Sj}^2\nu(d\zeta)\Big)}\\
&+\sum_{\alpha\in\mathcal{S},\alpha\neq j}a_{j\alpha}(H_\alpha-H_j)\Big]z\\
&-\frac{\xi_j}{2}\Big\{\dot{Q}_j+\frac{(\mu_j-r_j)^2(b_j+\xi_j(b_je_j-M_j))}{\xi_j^2A_j\Big(\sigma_j^2+\int_{\mathbb{R}_0}\gamma_{Sj}^2\nu(d\zeta)\Big)}(\xi_jM_j+(1+\xi_je_j)b_j)+\sum_{\alpha\in\mathcal{S},\alpha\neq j}a_{j\alpha}(Q_\alpha-Q_j)\Big\}\\
&\Big[\dot{b}_j+(r_j+\beta_{t_0})b_j+\frac{(\mu_j-r_j)^2(b_j^3-A_jb_j)}{A_j\Big(\sigma_j^2+\int_{\mathbb{R}_0}\gamma_{Sj}^2\nu(d\zeta)\Big)}+\sum_{\alpha\in\mathcal{S},\alpha\neq j}a_{j\alpha}(b_\alpha-b_j)\Big]x+\Big[\dot{c}_j+\kappa_jc_j\\
&+\delta(b_j+h_j)+\frac{(\mu_j-r_j)(b_jc_j-C_j)-C_j\Big(\sigma_j\sigma_{1j}+\int_{\mathbb{R}_0}\gamma_{Sj}\gamma_{\ell j}\nu(d\zeta)\Big)}{A_j\Big(\sigma_j^2+\int_{\mathbb{R}_0}\gamma_{Sj}^2\nu(d\zeta)\Big)}(\mu_j-r_j)b_j\\
&+\sum_{\alpha\in\mathcal{S},\alpha\neq j}a_{j\alpha}(c_\alpha-c_j)\Big]\ell+\Big[\dot{h}_j-\varepsilon\beta_{t_0}b_j+\frac{(\mu_j-r_j)^2b_jh_j}{A_j\Big(\sigma_j^2+\int_{\mathbb{R}_0}\gamma_{Sj}^2\nu(d\zeta)\Big)}+\sum_{\alpha\in\mathcal{S},\alpha\neq j}a_{j\alpha}(h_\alpha-h_j)\Big]z
\end{aligned}
$$

$$+\Big[\dot{e}_j + \frac{(\mu_j - r_j)^2 (b_j^2 + \xi_j (b_j^2 e_j - M_j b_j))}{\xi_j A_j \Big(\sigma_j^2 + \int_{\mathbb{R}_0} \gamma_{Sj}^2 \nu(d\zeta)\Big)} + \sum_{\alpha \in \mathcal{S}, \alpha \neq j} a_{j\alpha}(e_\alpha - e_j)\Big] = 0\,,$$

and

$$\Big[\dot{b}_j + (r_j + \beta_{t_0}) b_j + \frac{(\mu_j - r_j)^2 (b_j^3 - A_j b_j)}{A_j \Big(\sigma_j^2 + \int_{\mathbb{R}_0} \gamma_{Sj}^2 \nu(d\zeta)\Big)} + \sum_{\alpha \in \mathcal{S}, \alpha \neq j} a_{j\alpha}(b_\alpha - b_j)\Big] x + \Big[\dot{c}_j + \kappa_j c_j$$

$$+\delta(b_j + h_j) + \frac{(\mu_j - r_j)(b_j c_j - C_j) - C_j \Big(\sigma_j \sigma_{1j} + \int_{\mathbb{R}_0} \gamma_{Sj} \gamma_{\ell j} \nu(d\zeta)\Big)}{A_j \Big(\sigma_j^2 + \int_{\mathbb{R}_0} \gamma_{Sj}^2 \nu(d\zeta)\Big)} (\mu_j - r_j) b_j$$

$$+ \sum_{\alpha \in \mathcal{S}, \alpha \neq j} a_{j\alpha}(c_\alpha - c_j)\Big] \ell + \Big[\dot{h}_j - \varepsilon \beta_{t_0} b_j + \frac{(\mu_j - r_j)^2 b_j h_j}{A_j \Big(\sigma_j^2 + \int_{\mathbb{R}_0} \gamma_{Sj}^2 \nu(d\zeta)\Big)} + \sum_{\alpha \in \mathcal{S}, \alpha \neq j} a_{j\alpha}(h_\alpha - h_j)\Big] z$$

$$+\Big[\dot{e}_j + \frac{(\mu_j - r_j)^2 (b_j^2 + \xi_j (b_j^2 e_j - M_j b_j))}{\xi_j A_j \Big(\sigma_j^2 + \int_{\mathbb{R}_0} \gamma_{Sj}^2 \nu(d\zeta)\Big)} + \sum_{\alpha \in \mathcal{S}, \alpha \neq j} a_{j\alpha}(e_\alpha - e_j)\Big] = 0\,,$$

where $\dot{y}$ denotes the first derivative of y with respect to time t. Due to the arbitrariness of the variables x, ℓ and z, we deduce the following system of ordinary differential equations (ODEs):

$$\dot{A}_j + 2(r_j + \beta_{t_0}) A_j + \frac{(\mu_j - r_j)^2 (b_j^2 - A_j)}{\sigma_j^2 + \int_{\mathbb{R}_0} \gamma_{Sj}^2 \nu(d\zeta)} \Big(\frac{b_j^2}{A_j} + 1\Big) + \sum_{\alpha \in \mathcal{S}, \alpha \neq j} a_{j\alpha}(A_\alpha - A_j) = 0\,; \quad (3.15)$$

$$\dot{B}_j + \Big(2\kappa_j + \sigma_{1j}^2 + \int_{\mathbb{R}_0} \gamma_{\ell j}^2 \nu(d\zeta)\Big) B_j + 2\delta(C_j + K_j) \quad (3.16)$$

$$+2 \frac{(\mu_j - r_j)(b_j c_j - C_j) - C_j \Big(\sigma_j \sigma_{1j} + \int_{\mathbb{R}_0} \gamma_{Sj} \gamma_{\ell j} \nu(d\zeta)\Big)}{A_j \Big(\sigma_j^2 + \int_{\mathbb{R}_0} \gamma_{Sj}^2 \nu(d\zeta)\Big)} \Big(\mu_j - r_j + \int_{\mathbb{R}_0} \gamma_{Sj} \gamma_{\ell j} \nu(d\zeta)\Big) C_j$$

$$+ \frac{\Big[(\mu_j - r_j)(b_j c_j - C_j) - C_j \Big(\sigma_j \sigma_{1j} + \int_{\mathbb{R}_0} \gamma_{Sj} \gamma_{\ell j} \nu(d\zeta)\Big)\Big]^2}{A_j \Big(\sigma_j^2 + \int_{\mathbb{R}_0} \gamma_{Sj}^2 \nu(d\zeta)\Big)} + \sum_{\alpha \in \mathcal{S}, \alpha \neq j} a_{j\alpha}(B_\alpha - B_j) = 0\,;$$

$$\dot{P}_j - 2\varepsilon \beta_{t_0} R_j + \frac{(\mu_j - r_j)^2 b_j h_j}{A_j \Big(\sigma_j^2 + \int_{\mathbb{R}_0} \gamma_{Sj}^2 \nu(d\zeta)\Big)} \Big(2R_j + b_j h_j\Big) + \sum_{\alpha \in \mathcal{S}, \alpha \neq j} a_{j\alpha}(P_\alpha - P_j) = 0; \quad (3.17)$$

$$\dot{C}_j + (r_j + \beta_{t_0} + \kappa_j) C_j + \frac{(\mu_j - r_j)(b_j^2 - A_j) C_j}{A_j \Big(\sigma_j^2 + \int_{\mathbb{R}_0} \gamma_{Sj}^2 \nu(d\zeta)\Big)} \Big(\mu_j - r_j + \sigma_j \sigma_{1j} + \int_{\mathbb{R}_0} \gamma_{Sj} \gamma_{\ell j} \nu(d\zeta)\Big)$$

$$+\frac{(\mu_j - r_j)\Big[(\mu_j - r_j)(b_j c_j - C_j) - C_j\Big(\sigma_j\sigma_{1j} + \int_{\mathbb{R}_0}\gamma_{Sj}\gamma_{\ell j}\nu(d\zeta)\Big)\Big]b_j^2}{A_j\Big(\sigma_j^2 + \int_{\mathbb{R}_0}\gamma_{Sj}^2\nu(d\zeta)\Big)}$$

$$+\delta(A_j + R_j) + \sum_{\alpha\in\mathcal{S},\alpha\neq j} a_{j\alpha}(C_\alpha - C_j) \;=\; 0\,; \tag{3.18}$$

$$\dot{R}_j + (r_j + \beta_{t_0})R_j - \varepsilon\beta_{t_0}A_j + \frac{(\mu_j - r_j)(A_j b_j h_j + (R_j + b_j h_j)(b_j^2 - A_j))}{A_j\Big(\sigma_j^2 + \int_{\mathbb{R}_0}\gamma_{Sj}^2\nu(d\zeta)\Big)}$$

$$+\sum_{\alpha\in\mathcal{S},\alpha\neq j} a_{j\alpha}(R_\alpha - R_j) \;=\; 0; \tag{3.19}$$

$$\dot{K}_j + \kappa K_j + 2\delta(R_j + P_j) - \varepsilon\beta_{t_0}C_j + \frac{(\mu_j - r_j)(\mu_j - r_j + \sigma_j\sigma_{1j})b_j h_j C_j}{A_j\Big(\sigma_j^2 + \int_{\mathbb{R}_0}\gamma_{Sj}^2\nu(d\zeta)\Big)}$$

$$+\frac{(\mu_j - r_j)(R_j + b_j h_j)\Big[(\mu_j - r_j)(b_j c_j - C_j) - C_j\Big(\sigma_j\sigma_{1j} + \int_{\mathbb{R}_0}\gamma_{Sj}\gamma_{\ell j}\nu(d\zeta)\Big)\Big]}{A_j\Big(\sigma_j^2 + \int_{\mathbb{R}_0}\gamma_{Sj}^2\nu(d\zeta)\Big)}$$

$$+\sum_{\alpha\in\mathcal{S},\alpha\neq j} a_{j\alpha}(K_\alpha - K_j) \;=\; 0; \tag{3.20}$$

$$\dot{M}_j + (r_j + \beta_{t_0})M_j + \frac{(\mu_j - r_j)^2(b_j^2 - A_j)}{A_j\Big(\sigma_j^2 + \int_{\mathbb{R}_0}\gamma_{Sj}^2\nu(d\zeta)\Big)}M_j$$

$$+\frac{(\mu_j - r_j)^2(b_j + \xi_j(b_j e_j - M_j))}{\xi_j A_j\Big(\sigma_j^2 + \int_{\mathbb{R}_0}\gamma_{Sj}^2\nu(d\zeta)\Big)}b_j^2 + \sum_{\alpha\in\mathcal{S},\alpha\neq j} a_{j\alpha}(M_\alpha - M_j) \;=\; 0 \tag{3.21}$$

$$\dot{N}_j + \kappa_j N_j + \delta(M_j + H_j) + \sigma_{Sj}\sigma_{1j}C_j + \sum_{\alpha\in\mathcal{S},\alpha\neq j} a_{j\alpha}(N_\alpha - N_j)$$

$$+\frac{(\mu_j - r_j)(b_j + \xi_j(b_j e_j - M_j))}{\xi_j A_j\Big(\sigma_j^2 + \int_{\mathbb{R}_0}\gamma_{Sj}^2\nu(d\zeta)\Big)}\Big(\mu_j - r_j + \int_{\mathbb{R}_0}\gamma_{Sj}\gamma_{\ell j}\nu(d\zeta)\Big)C_j$$

$$+\frac{(\mu_j - r_j)(b_j c_j - C_j) - C_j\Big(\sigma_j\sigma_{1j} + \int_{\mathbb{R}_0}\gamma_{Sj}\gamma_{\ell j}\nu(d\zeta)\Big)}{\xi_j A_j\Big(\sigma_j^2 + \int_{\mathbb{R}_0}\gamma_{Sj}^2\nu(d\zeta)\Big)}(\mu_j - r_j)(1 + \xi_j e_j)b_j \;=\; 0\,; \tag{3.22}$$

$$\dot{H}_j - \varepsilon\beta_{t_0}M_j + \frac{(\mu_j - r_j)^2 b_j h_j M_j}{A_j\Big(\sigma_j^2 + \int_{\mathbb{R}_0}\gamma_{Sj}^2\nu(d\zeta)\Big)} + \frac{(\mu_j - r_j)^2(b_j + \xi_j(b_j e_j - M_j))(R_j + b_j h_j)}{\xi_j A_j\Big(\sigma_j^2 + \int_{\mathbb{R}_0}\gamma_{Sj}^2\nu(d\zeta)\Big)}$$

$$+\sum_{\alpha\in\mathcal{S},\alpha\neq j} a_{j\alpha}(H_\alpha - H_j) \;=\; 0; \tag{3.23}$$

$$\dot{Q}_j + \frac{(\mu_j - r_j)^2 (b_j + \xi_j (b_j e_j - M_j))}{\xi_j^2 A_j \Big(\sigma_j^2 + \int_{\mathbb{R}_0} \gamma_{Sj}^2 \nu(d\zeta) \Big)} (\xi_j M_j + (1 + \xi_j e_j) b_j) + \sum_{\alpha \in \mathcal{S}, \alpha \neq j} a_{j\alpha} (Q_\alpha - Q_j) \,=\, 0\,; \tag{3.24}$$

$$\dot{b}_j + (r_j + \beta_{t_0}) b_j + \frac{(\mu_j - r_j)^2 (b_j^3 - A_j b_j)}{A_j \Big(\sigma_j^2 + \int_{\mathbb{R}_0} \gamma_{Sj}^2 \nu(d\zeta) \Big)} + \sum_{\alpha \in \mathcal{S}, \alpha \neq j} a_{j\alpha} (b_\alpha - b_j) \,=\, 0\,; \tag{3.25}$$

$$\dot{h}_j - \varepsilon \beta_{t_0} b_j + \frac{(\mu_j - r_j)^2 b_j h_j}{A_j \Big(\sigma_j^2 + \int_{\mathbb{R}_0} \gamma_{Sj}^2 \nu(d\zeta) \Big)} + \sum_{\alpha \in \mathcal{S}, \alpha \neq j} a_{j\alpha} (h_\alpha - h_j) \,=\, 0\,; \tag{3.26}$$

$$\dot{c}_j + \kappa_j c_j + \delta(b_j + h_j) + \sum_{\alpha \in \mathcal{S}, \alpha \neq j} a_{j\alpha} (c_\alpha - c_j) + \frac{(\mu_j - r_j)(b_j c_j - C_j) - C_j \Big(\sigma_j \sigma_{1j} + \int_{\mathbb{R}_0} \gamma_{Sj} \gamma_{\ell j} \nu(d\zeta) \Big)}{A_j \Big(\sigma_j^2 + \int_{\mathbb{R}_0} \gamma_{Sj}^2 \nu(d\zeta) \Big)} (\mu_j - r_j) b_j \,=\, 0\,; \tag{3.27}$$

$$\dot{e}_j + \frac{(\mu_j - r_j)^2 (b_j^2 + \xi_j (b_j^2 e_j - M_j b_j))}{\xi_j A_j \Big(\sigma_j^2 + \int_{\mathbb{R}_0} \gamma_{Sj}^2 \nu(d\zeta) \Big)} + \sum_{\alpha \in \mathcal{S}, \alpha \neq j} a_{j\alpha} (e_\alpha - e_j) \,=\, 0\,; \tag{3.28}$$

with the terminal conditions $b(T,j) = A(T,j) = 1$ and $c(T,j) = h(T,j) = e(T,j) = B(T,j) = C(T,j) = P(T,j) = M(T,j) = N(T,j) = K(T,j) = R(T,j) = H(T,j) = Q(T,j) = 0$. Note that although these equations (3.15)-(3.28) look very complicated, they can be solved one by one following similar techniques as in [27]. First, from the fact that $a_{jj} = -\sum_{j \neq \alpha} a_{\alpha j}$, the solutions of the equations (3.15) and (3.25) are given by

$$A_j(t) = \exp\Big(2 \int_t^T (r_j(s) + \beta_{t_0}(s)) ds \Big) \quad \text{ and } \quad b_j(t) = \exp\Big(\int_t^T (r_j(s) + \beta_{t_0}(s)) ds \Big)\,,$$

for all $j \in \mathcal{S}$.

Moreover, the solution of the equation (3.26) is given by

$$h_j(t) = \exp\Big\{ \int_t^T \Big[\frac{(\mu_j(s) - r_j(s))^2}{b_j(s) \Big(\sigma_j^2(s) + \int_{\mathbb{R}_0} \gamma_{Sj}^2(s,\zeta) \nu(d\zeta) \Big)} - \varepsilon \beta_{t_0}(s) \Big] ds \Big\} - 1\,.$$

Based on the equations (3.18) and (3.27), and the explicit solutions for A_j and b_j above, we deduce that $C_j(t) = b_j(t) c_j(t)$. Hence, the ODE (3.27) can be simplified to the following first order linear system of ODEs.

$$\dot{c}_j + \Big[\kappa_j - \frac{(\mu_j - r_j)}{\sigma_j^2 + \int_{\mathbb{R}_0} \gamma_{Sj}^2 \nu(d\zeta)} \Big(\sigma_j \sigma_{1j} + \int_{\mathbb{R}_0} \gamma_{Sj} \gamma_{\ell j} \nu(d\zeta) \Big) \Big] c_j + \sum_{\alpha \in \mathcal{S}, \alpha \neq j} a_{j\alpha} (c_\alpha - c_j) + \delta(b_j + h_j) \,=\, 0\,.$$

Therefore, its solution exists and is well studied in the literature. Similarly, from equations (3.21) and (3.28), we deduce that $M_j(t) = b_j(t)e_j(t)$. Then, the ODE (3.28) can be simplified to the following first order linear system of ODEs:

$$\dot{e_j} + \sum_{\alpha\in\mathcal{S},\alpha\neq j} a_{j\alpha}(e_\alpha - e_j) + \frac{(\mu_j - r_j)^2}{\xi_j\Big(\sigma_j^2 + \int_{\mathbb{R}_0}\gamma_{Sj}^2\nu(d\zeta)\Big)} \;=\; 0\,.$$

Finally, since the explicit solutions for c_j and e_j can be obtained, combining with the solutions for A_j, b_j and h_j obtained above, the equations (3.16), (3.17), (3.19), (3.20), (3.22), (3.23) and (3.24) become the system of linear ODEs for B_j, P_j, R_j, K_j, N_j, H_j and Q_j that can be solved explicitly.

Therefore, our main result is given by the following theorem.

Theorem 3.2. *The equilibrium control for the mean-variance defined contribution optimization problem (2.7) is given by*

$$\pi^*(t,x,\ell,z,j) = \frac{(\mu_j - r_j)(1+\xi_j h_j z) - \xi_j\ell\Big(\sigma_j\sigma_{1j} + \int_{\mathbb{R}_0}\gamma_{Sj}\gamma_{\ell j}\nu(d\zeta)\Big)c_j}{\xi_j\Big(\sigma_j^2 + \int_{\mathbb{R}_0}\gamma_{Sj}^2\nu(d\zeta)\Big)} \times b(t,j) \qquad (3.29)$$

and the corresponding equilibrium value function is given by

$$\begin{aligned}\Psi(t,x,\ell,z,j,j') \;=\;& b(t,j)x + c(t,j)\ell + h(t,j)z + e(t,j) + \frac{\xi(j)}{2}\Big[(B(t,j) - c^2(t,j))\ell^2\\ & + P(t,j)z^2 + 2R(t,j)xz + 2K(t,j)z\ell + 2(N(t,j) - c(t,j)e(t,j))\ell\\ & + 2H(t,j)z + (Q(t,j) - e^2(t,j))\Big]\,,\end{aligned}$$

where

$$b(t,j) = \exp\Big(\int_t^T (r_j(s) + \beta_{t_0}(s))ds\Big)\,,$$

$$h_j(t) = \exp\Big\{\int_t^T \Big[\frac{(\mu_j(s) - r_j(s))^2}{b_j(s)\Big(\sigma_j^2(s) + \int_{\mathbb{R}_0}\gamma_{Sj}^2(s,\zeta)\nu(d\zeta)\Big)} - \varepsilon\beta_{t_0}(s)\Big]ds\Big\} - 1$$

and $c(t,j)$, $e(t,j)$, $B(t,j)$, $N(t,j)$ and $Q(t,j)$ satisfy the following system of first order linear ODEs.

$$\begin{aligned}&\dot{c}(t,j) + \Big[\kappa_j - \frac{(\mu_j - r_j)}{\sigma_j^2 + \int_{\mathbb{R}_0}\gamma_{Sj}^2\nu(d\zeta)}\Big(\sigma_j\sigma_{1j} + \int_{\mathbb{R}_0}\gamma_{Sj}\gamma_{\ell j}\nu(d\zeta)\Big)\Big]c(t,j)\\ &+ \sum_{\alpha\in\mathcal{S},\alpha\neq j} a_{j\alpha}(c(t,\alpha) - c(t,j)) + \delta(b(t,j) + h(t,j)) \;=\; 0\,;\\ &c(T,j) \;=\; 0\,;\end{aligned}$$

$$\dot{e}(t,j) + \sum_{\alpha\in\mathcal{S},\alpha\neq j} a_{j\alpha}(e(t,\alpha)-e(t,j)) + \frac{(\mu_j-r_j)^2}{\xi_j\Big(\sigma_j^2+\int_{\mathbb{R}_0}\gamma_{Sj}^2\nu(d\zeta)\Big)} \;=\; 0\,;$$

$$e(T,j) \;=\; 0\,;$$

$$\dot{B}(t,j) + \Big(2\kappa_j+\sigma_{1j}^2+\int_{\mathbb{R}_0}\gamma_{\ell j}^2\nu(d\zeta)\Big)B(t,j) + \sum_{\alpha\in\mathcal{S},\alpha\neq j} a_{j\alpha}(B(t,\alpha)-B(t,j))$$

$$+2\delta(b(t,j)c(t,j)+K(t,j)) + \frac{\sigma_j\sigma_{1j}+\int_{\mathbb{R}_0}\gamma_{Sj}\gamma_{\ell j}\nu(d\zeta)}{\sigma_j^2+\int_{\mathbb{R}_0}\gamma_{Sj}^2\nu(d\zeta)}\Big(\sigma_j\sigma_{1j}-\mu_j+r_j\Big)e^2(t,j) \;=\; 0\,;$$

$$B(T,j) \;=\; 0\,;$$

$$\dot{R}(t,j) + (r_j+\beta_{t_0})R(t,j) - \varepsilon\beta_{t_0}A(t,j) + \frac{(\mu_j-r_j)b(t,j)h(t,j)}{\sigma_j^2+\int_{\mathbb{R}_0}\gamma_{Sj}^2\nu(d\zeta)}$$

$$+\sum_{\alpha\in\mathcal{S},\alpha\neq j} a_{j\alpha}(R(t,\alpha)-R(t,j)) \;=\; 0;$$

$$R(T,j) \;=\; 0\,;$$

$$\dot{P}(t,j) - 2\varepsilon\beta_{t_0}R(t,j) + \frac{(\mu_j-r_j)^2h(t,j)}{b(t,j)\Big(\sigma_j^2+\int_{\mathbb{R}_0}\gamma_{Sj}^2\nu(d\zeta)\Big)}\Big(2R(t,j)+b(t,j)h(t,j)\Big)$$

$$+\sum_{\alpha\in\mathcal{S},\alpha\neq j} a_{j\alpha}(P(t,\alpha)-P(t,j)) \;=\; 0;$$

$$P(T,j) \;=\; 0\,;$$

$$\dot{K}(t,j) + \kappa K(t,j) + 2\delta(R(t,j)+P(t,j)) - \varepsilon\beta_{t_0}b(t,j)c(t,j)$$

$$+\sum_{\alpha\in\mathcal{S},\alpha\neq j} a_{j\alpha}(K(t,\alpha)-K(t,j)) + \frac{(\mu_j-r_j)(\mu_j-r_j+\sigma_j\sigma_{1j})h(t,j)c(t,j)}{\sigma_j^2+\int_{\mathbb{R}_0}\gamma_{Sj}^2\nu(d\zeta)}$$

$$-\frac{(\mu_j-r_j)(R(t,j)+b(t,j)h(t,j))\Big(\sigma_j\sigma_{1j}+\int_{\mathbb{R}_0}\gamma_{Sj}\gamma_{\ell j}\nu(d\zeta)\Big)c(t,j)}{b(t,j)\Big(\sigma_j^2+\int_{\mathbb{R}_0}\gamma_{Sj}^2\nu(d\zeta)\Big)} \;=\; 0;$$

$$K(T,j) \;=\; 0\,;$$

$$\dot{N}(t,j) + \kappa_jN(t,j) + \sum_{\alpha\in\mathcal{S},\alpha\neq j} a_{j\alpha}(N(t,\alpha)-N(t,j)) + \delta(b(t,j)e(t,j)+H(t,j))$$

$$+\sigma_{Sj}\sigma_{1j}b(t,j)c(t,j) + \frac{\mu_j-r_j}{\xi_j\Big(\sigma_j^2+\int_{\mathbb{R}_0}\gamma_{Sj}^2\nu(d\zeta)\Big)}\Big(\mu_j-r_j+\int_{\mathbb{R}_0}\gamma_{Sj}\gamma_{\ell j}\nu(d\zeta)$$

$$-\Big(\sigma_j\sigma_{1j}+\int_{\mathbb{R}_0}\gamma_{Sj}\gamma_{\ell j}\nu(d\zeta)\Big)(1+\xi_je(t,j))\Big)c(t,j) \;=\; 0\,;$$

$$N(T,j) \;=\; 0\,;$$

$$
\begin{aligned}
&\dot{H}_j - \varepsilon\beta_{t_0} b(t,j)e(t,j) + \frac{(\mu_j - r_j)^2 h(t,j)e(t,j)}{\sigma_j^2 + \int_{\mathbb{R}_0}\gamma_{Sj}^2\nu(d\zeta)} + \frac{(\mu_j - r_j)^2 (R(t,j) + b(t,j)h(t,j))}{\xi_j b(t,j)\Big(\sigma_j^2 + \int_{\mathbb{R}_0}\gamma_{Sj}^2\nu(d\zeta)\Big)} \\
&+ \sum_{\alpha\in\mathcal{S},\alpha\neq j} a_{j\alpha}(H(t,\alpha) - H(t,j)) \;=\; 0; \\
&H(T,j) \;=\; 0\,; \\
&\dot{Q}(t,j) + \sum_{\alpha\in\mathcal{S},\alpha\neq j} a_{j\alpha}(Q(t,\alpha) - Q(t,j)) + \frac{(\mu_j - r_j)^2}{\xi_j^2\Big(\sigma_j^2 + \int_{\mathbb{R}_0}\gamma_{Sj}^2\nu(d\zeta)\Big)}(1 + 2\xi_j e(t,j)) \;=\; 0\,; \\
&Q(T,j) \;=\; 0\,.
\end{aligned}
$$

Remark. From the expression of the equilibrium control (3.29), we can see that it depends on the force of mortality β_{t_0} and the stochastic income ℓ. This latter case can be compared to the dependence of the control problem on the asset-liability management in [27]. Moreover, from $(\mathbf{A_1})$, the existence of finite $c(t,j)$, $e(t,j)$, $B(t,j)$, $P(t,j)$, $R(t,j)$, $K(t,j)$, $N(t,j)$, $H(t,j)$, and $Q(t,j)$ can essentially be guaranteed by the uniform boundedness condition on the coefficients.

4. Numerical illustration

In this section, we provide some numerical simulations for the equilibrium control strategy π^*, to illustrate our main results. We assume that $N(t,\cdot)$ is a Poisson process with a intensity $\nu_j > 0$, and all the market parameters are time-homogeneous, i.e., they only depend on the regime switching. For simplicity, we assume the existence of two market regimes $\mathcal{S} = \{1,2\}$, with Regime 1 corresponding to the economy in expansion and Regime 2, the economy in recession, respectively. We adopt the deterministic force of mortality $\beta_{t_0}(t) = \frac{1}{\tau-(t_0+t)}$, with the maximal age $\tau = 100$. Then, from Theorem 3.2,

$$
\pi^*(t,x,\ell,z,j) = \frac{(\mu_j - r_j)(1+\xi_j h_j z) - \xi_j \ell(\sigma_j\sigma_{1j} + \gamma_{Sj}\gamma_{\ell j}\nu)c_j}{\xi_j(\sigma_j^2 + \gamma_{Sj}^2\nu)} \times \frac{\tau - (t_0+t)}{\tau - (t_0+T)} e^{r_j(T-t)}\,,
$$

where

$$
h_j(t) = \exp\Big\{\frac{(\mu_j - r_j)^2}{\sigma_j^2 + \gamma_{Sj}^2\nu}\int_t^T \frac{\tau-(t_0+T)}{\tau-(t_0+s)} e^{-r_j(T-s)} ds\Big\} \times \Big(\frac{\tau-(t_0+T)}{\tau-(t_0+t)}\Big)^{\varepsilon} - 1
$$

and $c(t,j)$ solves the following linear ODE:

$$
\begin{aligned}
&\frac{\partial c}{\partial t}(t,j) + \Big[\kappa_j - \frac{(\mu_j - r_j)}{\sigma_j^2 + \gamma_{Sj}^2\nu_j}(\sigma_j\sigma_{1j} + \gamma_{Sj}\gamma_{\ell j}\nu_j)\Big] c(t,j) \\
&+ \sum_{\alpha\in\mathcal{S},\alpha\neq j} a_{j\alpha}(c(t,\alpha) - c(t,j)) + \delta\frac{\tau-(t_0+t)}{\tau-(t_0+T)} e^{r_j(T-t)} + \delta h(t,j)) \;=\; 0\,; \\
&c(T,j) \;=\; 0\,;
\end{aligned}
$$

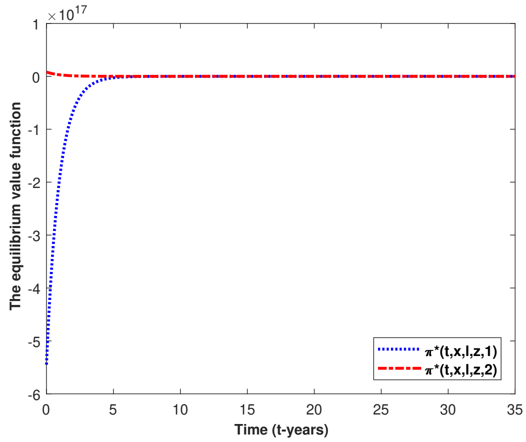

FIGURE 1. The equilibrium strategy for $\varepsilon = 1$.

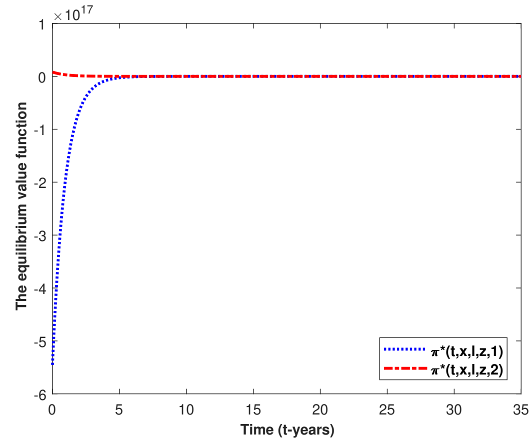

FIGURE 2. The equilibrium strategy for $\varepsilon = 0$.

Let $a_{11} = q_1,\ a_{22} = q_2,\ a_{12} = -q_1,\ a_{21} = -q_2$. We consider the following parameters (some of them are adapted from [27]):

Regime j	r_j	μ_j	σ_j	κ_j	σ_{1j}	γ_j	$\gamma_{\ell j}$	ν	ξ_j	t_0	τ	T	δ	q_j	ℓ
$j = 1$	0.06	0.2	0.13	0.09	0.03	0.1	0.1	1	0.6	25	100	35	0.1	-0.4	8
$j = 2$	0.04	0.05	0.3	0.03	0.07	0.2	0	2	0.5	25	100	35	0.1	-0.6	8

The Figures 1-2 show the effect of the market regimes on the equilibrium strategy for $\varepsilon = 1$, i.e., the premiums are returned to the member when he dies and $\varepsilon = 0$ (the pension member obtains nothing during the accumulation phase). We can see that in both cases, the pension manager makes a more conservative investment by short selling the stock to buy the risk free assets. This is because a time consistent pension investor, sacrifices the current happiness to ensure a consistent return during the investment period. We can see that in both cases, the amount dedicated to the stock, tends to stabilize to values approaching zero, after five years. It is consistent with the actuarial practices and DC pension funds regulations of many countries, where most of the pension investment models tend to put all the wealth into the risk-free asset as time of work goes.

Acknowledgment. The first author would like to express a deep gratitude to the University of Pretoria ABSA Chair in Actuarial Science for financial support.

We wish to extend our gratitude to the Editor and two anonymous reviewers, whose comments and suggestions have greatly improved this paper.

APPENDIX

Proof of the Theorem 3.1.

To prove the theorem, we follow [5], Theorem 7.1. We divide the proof in two parts, first we show that for the equilibrium control $\pi^* \in \mathcal{A}$, the value function $\Psi(t,x,\ell,z,j) = \Phi(t,x,\ell,z,j,\pi^*)$ and ψ and φ have the following probabilistic representation:

$$\psi(t,x,\ell,z,j,j') = \mathbb{E}_{t,x,\ell,z,j}[F(X^{\pi^*}(T),j)] \quad \text{and} \quad \varphi(\mathrm{t},\mathrm{x},\ell,\mathrm{z},\mathrm{j}) = \mathbb{E}_{\mathrm{t},\mathrm{x},\ell,\mathrm{z},\mathrm{j}}[\mathrm{X}^{\pi^*}(\mathrm{T})]\,.$$

In the second part, we prove that $\pi^* \in \mathcal{A}$ is indeed the equilibrium control strategy.

Let $h(t,x,\ell,z,j) \in \mathcal{C}^{1,2,2}([0,T]\times\mathbb{R}^2\times\mathcal{S})$, then by Itô's formula ([2], Lemma A1),

$$\begin{aligned}
& h(t,X^\pi(t),\ell(t),Z(t),j) \\
= \; & h(0,x,\ell,z,j) + \int_0^t \mathcal{L}h(s,X^\pi(s),\ell(s),Z(s),j)ds \\
& + \int_0^t \ell(s)\sigma_2 \frac{\partial h}{\partial \ell}(s,X^\pi(s),\ell(s),Z(s),j)dW_1(s) \\
& + \int_0^t \Big(\pi(s,X^\pi(s),\ell(s),Z(s),j)\sigma\frac{\partial h}{\partial x}(s,X^\pi(s),\ell(s),Z(s),j) \\
& \quad + \ell(s)\sigma_1\frac{\partial h}{\partial \ell}(s,X^\pi(s),\ell(s),Z(s),j)\Big)dW(s) \\
& + \int_0^t \sum_{\alpha\in\mathcal{S},\alpha\neq j}\Big(h(s,X^\pi(s),\ell(s),Z(s),\alpha) - h(s,X^\pi(s),\ell(s),Z(s),j)\Big)d\tilde{M}_{j\alpha} \\
& + \int_0^t\int_{\mathbb{R}}(h(s,X^\pi(s)+\pi(s,X^\pi(s),\ell(s),Z(s),j)\gamma_S,\ell(s),j) \\
& \qquad + h(s,X^\pi(s),\ell(s)(1+\sigma_1),Z(s),j) - 2h(s,X^\pi(s),\ell(s),Z(s),j))\tilde{N}(ds,d\zeta)\,,
\end{aligned}$$

where $\mathcal{L}h$ is given by (2.8). Since $W, \tilde{M}$ and $\tilde{N}$ are martingales, under the integrability condition $h \in L^2_T(X^{\pi^*})$, we have that

$$h(t,X^\pi(t),\ell(t),Z(s),\alpha(t)) - h(0,x,\ell,j) - \int_0^t \mathcal{L}^\pi h(s,X^\pi(s),\ell(s),Z(s),\alpha(s)ds$$

is a martingale. Thus, for $\Psi = h$, we have:

$$\begin{aligned}
& \mathbb{E}_{t,x,\ell,z,j}[\Psi(T,X(T),\ell(T),Z(T),\alpha(t)] \\
= \; & \Psi(t,x,\ell,z,j) + \mathbb{E}_{t,x,\ell,z,j}\Big[\int_t^T \mathcal{L}^{\pi^*}\Psi(s,X^{\pi^*}(s),\ell(s),Z(s),\alpha(t)ds\Big]\,.
\end{aligned} \tag{4.1}$$

In order to show that $\Psi(t,x,\ell,z,j) = \Phi(t,x,\ell,z,j,\pi^*)$, we use the Ψ-equation (3.1) to obtain:

$$\begin{aligned}
\mathcal{L}^{\pi^*}\Psi(t,x,\ell,z,j) \; = \; & \mathcal{L}^{\pi^*}\psi(t,x,\ell,z,j,j') - \mathcal{L}^{\pi^*}\psi^{j'}(t,x,\ell,z,j) \\
& + \mathcal{L}^{\pi^*}(\Gamma\circ\varphi)(t,x,\ell,z,j) - \mathcal{M}^{\pi^*}\varphi(t,x,\ell,z,j)\,.
\end{aligned}$$

Then, from (3.2)-(3.2), The relation (4.1) becomes:

$$\mathbb{E}_{t,x,\ell,z,j}[\Psi(T,X(T),\ell(T),Z(T),\alpha(t)]$$

$$= \Psi(t,x,\ell,z,j) + \mathbb{E}_{t,x,\ell,z,j}\Big[\int_t^T \Big(\mathcal{L}^{\pi^*}\psi(s,X(s),\ell(s),Z(s),\alpha(s),j') + \mathcal{L}^{\pi^*}(\Gamma\circ\varphi)(s,X(s),\ell(s),Z(s),\alpha(s))\Big)ds\Big].$$

Similarly,

$$\mathbb{E}_{t,x,\ell,z,j}\Big[\int_t^T \mathcal{L}^{\pi^*}\psi(s,X(s),\ell(s),Z(s),\alpha(s),j')ds\Big] = \mathbb{E}_{t,x,\ell,z,j}\Big[\psi(T,X(T),\ell(T),Z(T),\alpha(T),j')\Big] - \psi(t,x,\ell,z,j,j')$$

and

$$\mathbb{E}_{t,x,\ell,z,j}\Big[\int_t^T \mathcal{L}^{\pi^*}(\Gamma\circ\varphi)(s,X(s),\ell(s),Z(s),\alpha(s))ds\Big] = \mathbb{E}_{t,x,\ell,z,j}\Big[\Gamma(T,\varphi(T,X(T),\ell(T),Z(T),\alpha(T)),\alpha(t)\Big] - \Gamma(t,\varphi(t,x,\ell,z,j),j)\,.$$

Then, using the above relations, the boundary conditions and (3.4), we can easily conclude that

$$\begin{aligned}\Psi(t,x,\ell,z,j) &= \psi(t,x,\ell,z,j,j') + \Gamma(t,\varphi(t,x,\ell,z,j),j)\\ &= \mathbb{E}_{t,x,\ell,z,j}[F(X^{\pi^*}(T),\alpha(T)] + \mathbb{E}_{t,x,\ell,z,j}[X^{\pi^*}(T)]\\ &= \Phi(t,x,\ell,z,j,\pi^*)\,. \end{aligned} \tag{4.2}$$

In order to show that π^* is indeed an equilibrium control strategy, we construct, for any $\epsilon > 0$ and $\pi \in \mathcal{A}$, the control strategy π_ϵ defined in Definition 2.2. For any $s \in [t, t+\epsilon]$ we have: (See Lemma 2.2, [5])

$$\begin{aligned} &\Phi(t,x,\ell,z,j,\pi) \\ = &\ \mathbb{E}_{t,x,\ell,z,j}[\Psi(t+\epsilon,X^\pi_{t+\epsilon},\ell,\alpha)] \\ &-\Big\{\mathbb{E}_{t,x,\ell,z,j}[\psi(t+\epsilon,X^\pi_{t+\epsilon},\ell,Z,\alpha,j')] - \mathbb{E}_{t,x,\ell,z,j}[\psi(t+\epsilon,X^\pi_{t+\epsilon},\ell,Z,\alpha,j')]\Big\} \\ &-\Big\{\mathbb{E}_{t,x,\ell,z,j}[\Gamma(t+\epsilon,\varphi(t+\epsilon,X^\pi_{t+\epsilon},\ell,Z,\alpha),\alpha)] \\ &\quad -\Gamma(t+\epsilon,\mathbb{E}_{t,x,\ell,z,j}[\varphi(t+\epsilon,X^\pi_{t+\epsilon},\ell,Z,\alpha),\alpha])\Big\}\,. \end{aligned} \tag{4.3}$$

Moreover, for all $\pi \in \mathcal{A}$ and (3.1), we have

$$\mathcal{L}^\pi\Psi(t,x,\ell,z,j) - \mathcal{L}^\pi\psi(t,x,\ell,z,j,t,j) + \mathcal{L}^\pi\psi^{t,j}(t,x,\ell,z,j) - \mathcal{L}^\pi(\Gamma\circ\varphi)(t,x,\ell,z,j) + \mathcal{M}^\pi\varphi(t,x,\ell,z,j) \le 0\,.$$

Discretizing the above expression, we have

$$\mathbb{E}_{t,x,\ell,z,j}[\Psi(t+\epsilon,X^\pi_{t+\epsilon},\ell,Z,\alpha)] - \Psi(t,x,\ell,z,j) - \Big\{\mathbb{E}_{t,x,\ell,z,j}[\psi(t+\epsilon,X^\pi_{t+\epsilon},\ell,Z,\alpha,j')]$$

$$-\psi(t,x,\ell,z,j,j')\Big\}+\mathbb{E}_{t,x,\ell,z,j}[\psi(t+\epsilon,X^\pi_{t+\epsilon},\ell,Z,\alpha,j')]-\psi(t,x,\ell,z,j,j')$$
$$-\mathbb{E}_{t,x,\ell,z,j}[\Gamma(t+\epsilon,\varphi(t+\epsilon,X^\pi_{t+\epsilon},\ell,Z,\alpha),\alpha)]+\Gamma(t,\varphi(t,x,\ell,z,j),j)$$
$$+\Gamma(t+\epsilon,\mathbb{E}_{t,x,\ell,z,j}[\varphi(t+\epsilon,X^\pi_{t+\epsilon},\ell,Z,\alpha),\alpha])-\Gamma(t,\varphi(t,x,\ell,z,j),j)\ \le o(\epsilon)\,.$$

Hence

$$\begin{aligned}
&\Psi(t,x,\ell,z,j)\\
\ge\ &\mathbb{E}_{t,x,\ell,z,j}[\Psi(t+\epsilon,X^\pi_{t+\epsilon},\ell,Z,\alpha)]-\mathbb{E}_{t,x,\ell,z,j}[\psi(t+\epsilon,X^\pi_{t+\epsilon},\ell,Z,\alpha,j')]\\
&+\mathbb{E}_{t,x,\ell,z,j}[\psi(t+\epsilon,X^\pi_{t+\epsilon},\ell,Z,\alpha,j')]-\mathbb{E}_{t,x,\ell,z,j}[\Gamma(t+\epsilon,\varphi(t+\epsilon,X^\pi_{t+\epsilon},\ell,Z,\alpha),\alpha)]\\
&+\Gamma(t+\epsilon,\mathbb{E}_{t,x,\ell,z,j}[\varphi(t+\epsilon,X^\pi_{t+\epsilon},\ell,Z,\alpha),\alpha])+o(\epsilon)\,.
\end{aligned}$$

Therefore, from (4.2) and (4.3), we obtain

$$\Phi(t,x,\ell,z,j,\pi^*)-\Phi(t,x,\ell,z,j,\pi_\epsilon)\ge o(\epsilon)\,,$$

that is,

$$\liminf_{\epsilon\to 0}\frac{\Phi(t,x,\ell,z,j,\pi^*)-\Phi(t,x,\ell,z,j,\pi_\epsilon)}{\epsilon}\ge 0\,,$$

which completes the proof.

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