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A BSD formula for high-weight modular forms

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Abstract

The Birch and Swinnerton-Dyer conjecture – which is one of the seven million-dollar Clay Mathematics Institute Millennium Prize Problems – and its generalizations are a significant focus of number theory research.

A 2017 article of Jetchev, Skinner and Wan proved a Birch and Swinnerton-Dyer formula at a prime p for certain rational elliptic curves of rank 1. We generalize and adapt that article's arguments to prove an analogous formula for certain modular forms. For newforms f of even weight higher than 2 with Galois representation V containing a Galois-stable lattice T, let $W = V/T$ and let K be an imaginary quadratic field in which the prime p splits. Our main result is that under some conditions, the p-index of the size of the Shafarevich-Tate group of W with respect to the Galois group of K is twice the p-index of a logarithm of the Abel-Jacobi map of a Heegner cycle defined by Bertolini, Darmon and Prasanna.

Significant original adaptations we make to the 2017 arguments are (1) a generalized version of a previous calculation of the size of the cokernel of a localization-modulo-torsion map, and (2) a comparison of different Heegner cycles.

Keywords: Birch and Swinnerton-Dyer, Heegner cycle, modular form, number theory, Shafarevich-Tate

MSC2020 classification codes: 11G40, 11F23, 11F80, 11F85, 11R23

Declarations of interest: none

Contents

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1. Introduction, main result and outline of proof

A major theme in modern number-theoretic research is that analytic objects (like L-functions) yield information about algebraic or geometric objects (like Galois characters and groups of rational points on elliptic curves). A famous example of a result expected to be true is the Birch and Swinnerton-Dyer (BSD) conjecture:

Conjecture 1.1 (BSD)**.** *Suppose an elliptic curve $E/\mathbb{Q}$ is given. Let the analytic rank of E be the order of the zero of $L(E,s)$ at $s=1$. Then the analytic rank of E equals the (algebraic) rank of the finitely generated abelian group $E(\mathbb{Q})$, and*

$$\frac{1}{R(E/\mathbb{Q})\Omega_E}\cdot\lim_{s\to 1}\frac{L(E,s)}{(s-1)^{\mathrm{rank}\ E(\mathbb{Q})}}=\frac{\#\text{Ш}(E/\mathbb{Q})\cdot\prod_{\ell\nmid\infty}c_\ell}{(\#E(\mathbb{Q})_{\mathrm{tor}})^2}$$

where the regulator $R(E/\mathbb{Q})$ is defined as in [33] except that the height pairing in that source is to be doubled, and where the period Ω_E, Shafarevich-Tate group Ш$(E/\mathbb{Q})$ and Tamagawa numbers c_ℓ are defined as in [33].

So far, the main progress on BSD has been for analytic and algebraic rank 0 and rank 1 cases.

In the recent paper [15] of Jetchev, Skinner and Wan, the following "BSD formula at a prime p" was proved. We write $\mathrm{ind}_p x$ for the p-index of x; for example, $\mathrm{ind}_p(p^n)=n$ for $n\in\mathbb{Z}$.

Theorem 1.2. *[15, Theorem 1.2.1] Assume that* *(i) the elliptic curve $E/\mathbb{Q}$ is semistable,*

(ii) the rational prime p is odd and does not divide the conductor of E,

(iii) the Galois representation $E[p]$ of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ over $\mathbb{F}_p$ is irreducible,

(iv) E has analytic rank 1, and

(v) if E has supersingular reduction at p, then $a_p(E) = 0$.

Then

$$\mathrm{ind}_p\left(\frac{L'(E,1)}{R(E/\mathbb{Q})\Omega_E}\right) = \mathrm{ind}_p\left(\#\text{Ш}(E/\mathbb{Q}) \cdot \prod_\ell c_\ell\right).$$

The proof's broad structure was as follows. For suitable auxiliary imaginary quadratic fields K', K'', the following results were obtained.

(a) A theorem obtained from Brooks, linked to work of Bertolini, Darmon and Prasanna [15, Proposition 5.1.7]: For a certain Heegner point $z_{K'} \in E(K')$, a certain differential form ω_E on E, and a certain L-function L_{BDP} of Bertolini, Darmon and Prasanna, we have

$$2\mathrm{ind}_p \log_{\omega_E}(z_{K'}) + 2\mathrm{ind}_p((1 - a_p(E) + p)/p) = \mathrm{ind}_p L_{BDP}(1).$$

(b) Interpolating and comparing L-functions [15, Corollary 5.3.2]: For a certain L-function L_{Wan} of Wan, we have $\mathrm{ind}_p L_{BDP}(1) = \mathrm{ind}_p L_{Wan}(1)$.

(c) Iwasawa theory [15, Proposition 6.2.1], relying on a result of Wan that is half of an Iwasawa main conjecture: For a certain cohomology-related quantity $C(E[p^\infty])$, we have

$$\mathrm{ind}_p L_{Wan}(1) \le \mathrm{ind}_p(C(E[p^\infty])\#H^1_{ac}(K', E[p^\infty])).$$

(d) Galois cohomology [15, (3.5d)]: We have

$$\begin{aligned}&\mathrm{ind}_p(C(E[p^\infty])\#H^1_{ac}(K', E[p^\infty]))\\ &\quad = \mathrm{ind}_p(\#\text{Ш}(E/K')) + 2\mathrm{ind}_p \log_{\omega_E}(z_{K'}) + 2\mathrm{ind}_p((1 - a_p(E) + p)/p)\\ &\qquad -2\mathrm{ind}_p(E(K') : \mathbb{Z}z_{K'}) + (p\text{-indices of Tamagawa factors}).\end{aligned}$$

Points (a) to (d) yield

$$2\mathrm{ind}_p(E(K') : \mathbb{Z}z_{K'}) - (p\text{-indices of Tamagawa factors}) \le \mathrm{ind}_p(\#\text{Ш}(E/K')[p^\infty]).$$

(e) Euler systems [15, Theorem 4.4.1], relying on a result of Nekovář: We have

$$\mathrm{ind}_p(\#\text{Ш}(E/K'')[p^\infty]) \le 2\mathrm{ind}_p(E(K'') : \mathbb{Z}z_{K''}).$$

Applying Gross-Zagier formulas for the Heegner points $z_{K'}$, $z_{K''}$, re-writing the Shafarevich-Tate groups $Ш(E/K')$, $Ш(E/K'')$ in terms of $Ш(E/\mathbb{Q})$ and Ш of quadratic twists of E, and applying a previously known rank 0 case of the BSD conjecture produced Theorem 1.2.

This article replaces E with a modular form f of weight larger than 2, adapting [15]'s arguments. Analogously to the intermediate results of Jetchev, Skinner and Wan mentioned above, our main result (Theorem 11.1) says that the p-index of a certain Shafarevich-Tate group is twice the p-index of the logarithm of the Abel-Jacobi map of a Heegner cycle.

First, section 2 sets some notation and underlying assumptions. Sections 3 to 5 then review some background on class field theory, modular forms, algebraic geometry and cohomology. Finally, sections 6 to 11 prove Theorem 11.1. The basic structure of our argument is as follows; note the similarity with [15].

(a) First, a formula of Bertolini, Darmon and Prasanna [1] links the logarithm of the Abel-Jacobi map of a Heegner cycle to a p-adic L-function.

(b) Second, p-adic L-functions are interpolated and compared.

(c) Third, half of an Iwasawa main conjecture links a p-adic L-function to Galois cohomology.

(d) Fourth, Galois cohomology is linked to the Shafarevich-Tate group of f.

(e) Fifth, an Euler-system-related result links Sha to the Abel-Jacobi image of a Heegner cycle of Masoero.

(f) Sixth, Masoero's Heegner cycle is compared with the Heegner cycle from the first step.

Combining these six steps, we get a chain of inequalities $x_1 \leq x_2 \leq \cdots \leq x_6 \leq x_1$, so all x_i are equal, and this yields the final result.

2. Notation and setup

2.1. Notation

For $n \in \mathbb{Z}_{>0}$, S_n is the symmetric group of bijections from $\{1, 2, \ldots, n\}$ to itself. Let G_{tor} be the torsion subgroup of an abelian group G; write $G/\text{tor} := G/G_{\text{tor}}$. Let M_{div} be the maximal p-divisible subgroup of a $\mathbb{Z}_p$-module M.

For a rational prime p, let $\widehat{\overline{\mathbb{Q}}}_p$ be the completion of the algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$. Write $\text{ind}_p : \widehat{\overline{\mathbb{Q}}}_p^{\times} \to \mathbb{Q}_{>0}$ for the multiplicative p-adic valuation with $\text{ind}_p(p^n) = n$ for $n \in \mathbb{Z}$. For a finite-degree field extension $L/\mathbb{Q}_p$, let O_L be the ring of integers of L, with maximal ideal m_L, and let $\widehat{O}_L^{ur}$ be the ring of integers of the completion $\widehat{L^{ur}}$ of the maximal unramified extension L^{ur} of L.

For a number field F, let O_F be the ring of integers of F and, for each place v of F, take the v-adic completion F_v. For finite v, F_v has ring of integers $O_{F,v}$, and we abuse notation by denoting the maximal ideal of $O_{F,v}$, and that ideal's intersection with O_F, as v. Let the Hilbert class field, class group and class number of F be respectively F_1, Cl_F and h_F.

The spaces of adeles, finite adeles, ideles and finite ideles over F are written $\mathbb{A}_F$ (as in [28, section VI.1]), $\mathbb{A}_{F,f}$, $\mathbb{A}_F^\times$, $\mathbb{A}_{F,f}^\times$ respectively, with elements $z = (z_v)_v$ where each $z_v \in F_v$.

For finite v and a character (that is, a continuous group homomorphism) $\eta : F_v^\times \to \mathbb{C}^\times$, the conductor of η is denoted $C(\eta)$, and η is called unitary when its image is in $\{z \in \mathbb{C}^\times : |z| = 1\}$.

For a Hecke character $\chi : F^\times \backslash \mathbb{A}_F^\times \to \mathbb{C}^\times$, the conductor of χ is denoted $C(\chi)$, and χ_v is the restriction of $\chi : \mathbb{A}_F^\times \to \mathbb{C}^\times$ to $F_v^\times$. For a fractional ideal $\mathfrak{a} = \prod_{v \nmid \infty} v^{a(v)}$ of F with each $a(v) \in \mathbb{Z}$, if $\mathfrak{a}$ is coprime to $C(\chi)$, then write $\chi(\mathfrak{a})$ for the value of χ at an idele $z \in \mathbb{A}_{F,f}^\times$ with $zO_F = \mathfrak{a}$ and $z_v = 1$ for $v \mid C(\chi)$.

For a number field F, as in [1], let the Hecke character $\mathrm{N} : F^\times \backslash \mathbb{A}_F^\times \to \mathbb{C}^\times$ of conductor O_F be such that for F's fractional ideals $\mathfrak{a}$, the positive element of $\mathbb{Q}$ that generates the fractional ideal $N_{F/\mathbb{Q}}\mathfrak{a}$ of $\mathbb{Q}$ as a $\mathbb{Z}$-module is $\mathrm{N}(\mathfrak{a})$. For an integral ideal $\mathfrak{a}$ of F, we have $\mathrm{N}(\mathfrak{a}) = (O_F : \mathfrak{a})$.

We use the following notation from [35, section 2.1]. For a number field F, the extension F_Σ/F and the Galois groups G_F, $G_{F,\Sigma} = \mathrm{Gal}(F_\Sigma/F)$, $G_{F,v}$ and $I_{F,v}$ are defined in the standard way. For an imaginary quadratic extension $K/\mathbb{Q}$, let K_∞/K, K_∞^+/K and K_∞^-/K be the $\mathbb{Z}_p^2$-extension, the cyclotomic extension and the anticyclotomic extension respectively of K, and write $\Gamma_K = \mathrm{Gal}(K_\infty/K)$, $\Gamma_K^+ = \mathrm{Gal}(K_\infty^+/K)$ and $\Gamma_K^- = \mathrm{Gal}(K_\infty^-/K)$.

For a cusp form $f = \sum_{n=1}^\infty a(n,f)q^n$, let $\mathbb{Q}(f) = \mathbb{Q}(a(n,f) : n \in \mathbb{Z}_{>0})$ be the number field generated over $\mathbb{Q}$ by all the $a(n,f)$, and let $O(f) = \mathbb{Z}[a(n,f) : n \in \mathbb{Z}_{>0}]$ be the ring generated as a $\mathbb{Z}$-algebra by all the $a(n,f)$.

2.2. Assumptions

The following assumptions apply throughout. In this paper's final theorem, the hypotheses will be these assumptions, plus some additional technical statements to be described later.

A prime p is fixed, together with an isomorphism $\widehat{\overline{\mathbb{Q}}}_p \cong \mathbb{C}$. Fix an unramified finite-degree field extension $L/\mathbb{Q}_p$.

Let the imaginary quadratic field extension $K/\mathbb{Q}$ (with complex conjugation c) have squarefree discriminant $D_K \equiv 1 \bmod 4$ with $D_K < -3$. Assume $K_1 \subseteq L$.

Let $f = \sum_{n=1}^\infty a(n,f)q^n \in S_k(\Gamma_0(N))$ be a non-CM newform of conductor N with $a(1,f) = 1$ such that $N \geq 5$ is an odd integer, $k > 2$ is an even integer, and $k/2$ is not congruent to 0 or 1 modulo $p-1$. Assume $\mathbb{Q}(f) \subseteq L$.

Let the representations T_f, V_f, W_f be as defined in subsection 5.4. Assume $T_f/m_L T_f$ is an irreducible G_K-representation of dimension ≥ 2.

Assume the following Heegner hypothesis: each prime factor of N splits or ramifies in K, at least one rational prime factor of N ramifies in K, and every prime $q \mid N$ ramifying in K is such that $q^2 \nmid N$. This implies that there is an ideal $\mathfrak{C}$ of O_K for which the inclusion $\mathbb{Z} \hookrightarrow O_K$ induces an isomorphism $\mathbb{Z}/N\mathbb{Z} \cong O_K/\mathfrak{C}$; fix such an ideal $\mathfrak{C}$.

Let the prime p split in K as $p = v_0\overline{v}_0$. Define the set $N_p = \{v_0, \overline{v}_0\}$.

For some representatives $\mathfrak{a}$ of the class group of K, assume that the norms $\mathrm{N}(\mathfrak{a})$ are p-adic units when viewed as elements of $\widehat{\overline{\mathbb{Q}}}_p$.

Assume that $p \geq k/2$, the Fourier coefficient $a(p, f)$ is a p-adic unit, and the prime p does not divide $(k-2)! \cdot 6N\phi(N)D_K h_K \cdot (O_{\mathbb{Q}(f)} : O(f))$.

3. Class field theory

This section briefly reviews class field theory and Hecke characters. We use and adapt notation from [10, 11, 35]. For this section, take a discrete valuation ring O with $\overline{\mathbb{Q}}_p \supseteq O \supseteq \mathbb{Z}_p$.

3.1. Class field theory and Galois extensions

For $M \in \mathbb{Z}_{>0}$, the ray class group modulo $Mp^\infty\infty$ over $\mathbb{Q}$ is

$$Z(M) = \mathbb{Q}_{>0}^\times \backslash \mathbb{A}_{\mathbb{Q},f}^\times / U_{\mathbb{Q}}(Mp^\infty) \cong \widehat{\mathbb{Z}}^\times / U_{\mathbb{Q}}(Mp^\infty)$$

where

$$U_{\mathbb{Q}}(Mp^\infty) = \{z \in \widehat{\mathbb{Z}}^\times : z_p = 1, z_\ell \in 1 + M\mathbb{Z}_\ell \text{ for finite } \ell \neq p\}.$$

For $p \nmid M$, we identify $\mathbb{Z}_p^\times \times (\mathbb{Z}/M\mathbb{Z})^\times \cong Z(M)$ in the standard way.

For all $M \in \mathbb{Z}_{>0}$, the cyclotomic character $\epsilon : Z(M) \to \mathbb{Z}_p^\times$ is identified via geometrically normalized Artin reciprocity with the Galois character describing the Galois action on roots of unity with order a power of p [35, section 2.2.3]. For a p-adic Galois representation U and an integer n, let $U(\epsilon^n) := U \otimes \epsilon^n$ be the twist of U by ϵ^n. (We write $U(\epsilon^n)$ instead of $U(n)$ to keep the notation uniform and make the choice of normalization for ϵ clear.)

The classical Teichmüller character $\omega : \mathbb{Z}_p^\times \to \mathbb{Z}_p^\times$ satisfies $\omega(y)^p = \omega(y) \equiv y \bmod p\mathbb{Z}_p$ for $y \in \mathbb{Z}_p^\times$. Define a Teichmüller character $\omega : Z(M) \to \mathbb{Z}_p^\times$, with the same image as the previous ω, so that

(a) If the embedding $\mathbb{Z}_p^\times \hookrightarrow \mathbb{A}_{\mathbb{Q},f}^\times$ sends $y \in \mathbb{Z}_p^\times$ to $y_p \in \mathbb{A}_{\mathbb{Q},f}^\times$ in the equivalence class $[y_p] \in Z(M)$, then $\omega([y_p]) = \omega(y)$ (so $\epsilon([y_p]) \equiv \omega(y) \bmod p\mathbb{Z}_p$); and

(b) Each element $(1, y_M \bmod M\mathbb{Z}) \in \mathbb{Z}_p^\times \times (\mathbb{Z}/M\mathbb{Z})^\times$ corresponds to an element of $Z(M)$ in the kernel of $\omega : Z(M) \to \mathbb{Z}_p^\times$.

(Under geometrically normalized reciprocity, this ω corresponds to the inverse of the character denoted ω in both [35, section 2.2.4] and [37, Theorems 1.1-1.2].)

The embeddings $\mathbb{Z}_p \hookrightarrow O_{K,v_0}$ and $\mathbb{Z}_p \hookrightarrow O_{K,\overline{v}_0}$ are isomorphisms. Let the Teichmüller characters $\omega_{v_0} : O_{K,v_0}^\times \to \mathbb{Z}_p^\times$ and $\omega_{\overline{v}_0} : O_{K,\overline{v}_0}^\times \to \mathbb{Z}_p^\times$ send the embeddings of $y \in \mathbb{Z}_p$ in respectively O_{K,v_0} and $O_{K,\overline{v}_0}$ to $\omega(y)$.

Write

$$Z_K(\mathfrak{C}) = K^\times \backslash \mathbb{A}_{K,f}^\times / U_K(\mathfrak{C}p^\infty) \cong K^\times \backslash \mathbb{A}_K^\times / (U_K(\mathfrak{C}p^\infty) K_\infty^\times)$$

where

$$U_K(\mathfrak{C}p^\infty) = \{z \in \widehat{O}_K^\times : z_{v_0} = z_{\overline{v}_0} = 1, z_v \in 1 + \mathfrak{C}O_{K,v} \text{ for finite } v \nmid p\}.$$

Since $\mathfrak{C}$ is relatively prime to $p = v_0\overline{v}_0$ and $(O_K : \mathfrak{C}) > 1$ is odd, we have a standard group isomorphism

$$O_{K,v_0}^\times \times O_{K,\overline{v}_0}^\times \times Cl_K \times ((O_K/\mathfrak{C})^\times/\{\pm 1\}) \cong Z_K(\mathfrak{C}) \tag{1}$$

which is the product of an isomorphism from

$$\omega_{v_0}[O_{K,v_0}^\times] \times \omega_{\overline{v}_0}[O_{K,\overline{v}_0}^\times] \times Cl_K \times ((O_K/\mathfrak{C})^\times/\{\pm 1\}) \tag{2}$$

to $Z_K(\mathfrak{C})_{\mathrm{tor}}$, and an isomorphism

$$i : (1 + v_0 O_{K,v_0}) \times (1 + \overline{v}_0 O_{K,\overline{v}_0}) \overset{\cong}{\to} \Gamma_K$$

which is the composition of group maps

$$(1 + v_0 O_{K,v_0}) \times (1 + \overline{v}_0 O_{K,\overline{v}_0}) \hookrightarrow O_{K,v_0}^\times \times O_{K,\overline{v}_0}^\times \hookrightarrow \mathbb{A}_K^\times \twoheadrightarrow Z_K(\mathfrak{C})/\mathrm{tor} \cong \Gamma_K$$

using geometrically normalized reciprocity and Galois theory in the usual way to identify $Z_K(\mathfrak{C})/\mathrm{tor}$ and Γ_K. (There is no p-part in $(O_K/\mathfrak{C})^\times/\{\pm 1\}$ or Cl_K since $p \nmid \phi(N)h_K$.)

In K_∞/K, the maximum extension unramified at $\overline{v}_0$ (respectively, v_0) is the extension K_{v_0}/K (respectively, $K_{\overline{v}_0}/K$) such that K_{v_0} is the fixed field of $i[\{1\} \times (1+\overline{v}_0 O_{K,\overline{v}_0})]$ (respectively, $K_{\overline{v}_0}$ is the fixed field of $i[(1+v_0 O_{K,v_0}) \times \{1\}]$) in K_∞. The standard quotient map $pr_{v_0} : \Gamma_K \twoheadrightarrow \mathrm{Gal}(K_{v_0}/K)$ sends $i(y_{v_0}, y_{\overline{v}_0}) \in \Gamma_K$ to the class of $i(y_{v_0}, 1)$ in $\mathrm{Gal}(K_{v_0}/K) \cong \Gamma_K/\mathrm{Gal}(K_\infty/K_{v_0})$.

The embeddings $\mathbb{Z}_p \hookrightarrow O_{K,v_0}$ and $\mathbb{Z}_p \hookrightarrow O_{K,\overline{v}_0}$ yield

$$(1 + p\mathbb{Z}_p)^2 \cong (1 + v_0 O_{K,v_0}) \times (1 + \overline{v}_0 O_{K,\overline{v}_0}).$$

The group Γ_K^+ (respectively, Γ_K^-) is topologically generated by the element $\gamma_+ = i((1+p)^{1/2}, (1+p)^{1/2})$ (respectively, $\gamma_- = i((1+p)^{1/2}, (1+p)^{-1/2})$), or more precisely, by the class of that element in the appropriate quotient of Γ_K. The standard quotient map $pr_{ac} : \Gamma_K \twoheadrightarrow \Gamma_K^-$ sends $g = i(y_{v_0}, y_{\overline{v}_0}) \in \Gamma_K$ to the class of $(gg^{-c})^{1/2} = i(y_{v_0}^{1/2} y_{\overline{v}_0}^{-1/2}, y_{v_0}^{-1/2} y_{\overline{v}_0}^{1/2})$ in $\Gamma_K^- \cong \Gamma_K/\mathrm{Gal}(K_\infty/K_\infty^-)$.

Define the squaring maps $sq : Z_K(\mathfrak{C}) \to Z_K(\mathfrak{C})$, $sq : \Gamma_K \to \Gamma_K$ and $sq : \mathrm{Gal}(K_{v_0}/K) \to \mathrm{Gal}(K_{v_0}/K)$ given by $g \mapsto g^2$.

Define the O-algebra maps $pr_{ac} : O[[\Gamma_K]] \twoheadrightarrow O[[\Gamma_K^-]]$, $sq : O[[Z_K(\mathfrak{C})]] \to O[[Z_K(\mathfrak{C})]]$ and $sq : O[[\Gamma_K]] \to O[[\Gamma_K]]$ by extending O-linearly and continuously.

Let $c : z \mapsto \overline{z}$ be the conjugation map on $\mathbb{C}$ (or on any subfield of $\mathbb{C}$ stable under conjugation). The group $\mathrm{Gal}(K/\mathbb{Q}) = \{1, c\}$ acts on Γ_K via conjugation (c sends $g \in \Gamma_K$ to $cgc^{-1} \in \Gamma_K$); c acts on Γ_K^+, Γ_K^- as 1, -1 respectively.

3.2. Complex and p-adic Hecke characters

For a Hecke character $\chi : K^\times \backslash \mathbb{A}_K^\times \to \mathbb{C}$ with $\chi(z_\infty) = z_\infty^t \overline{z}_\infty^u$ identically for some $t, u \in \mathbb{Z}$, the p-adic avatar of χ is a p-adic Hecke character $\widetilde{\chi} : K^\times \backslash \mathbb{A}_{K,f}^\times \to \overline{\mathbb{Q}}_p^\times$ satisfying

$$\chi(z) = (z_\infty^t \overline{z}_\infty^u) \cdot (z_{v_0}^{-t} z_{\overline{v}_0}^{-u}) \widetilde{\chi}(z_f) \tag{3}$$

(use $\widehat{\overline{\mathbb{Q}}}_p \cong \mathbb{C}$ to view $(z_{v_0}^{-t} z_{\overline{v}_0}^{-u})\widetilde{\chi}(z_f) \in \overline{\mathbb{Q}}_p^\times$ as belonging to $\mathbb{C}^\times$). Write $\chi = \widetilde{\chi}^{alg}$. The corresponding Galois character $\sigma_\chi : G_K \to \overline{\mathbb{Q}}_p^\times$ sends the geometric Frobenius at any $v \nmid pC(\chi)$ to χ_v of a uniformizer at v [35, section 2.2.1].

Recall the identification $Z_K(\mathfrak{C}) \cong \Gamma_K \times Z_K(\mathfrak{C})_{\mathrm{tor}}$. Characters $P : \Gamma_K \to \overline{\mathbb{Q}}_p^\times$ and $\psi : Z_K(\mathfrak{C})_{\mathrm{tor}} \to \overline{\mathbb{Q}}_p^\times$, respectively, can be precomposed with the projections $Z_K(\mathfrak{C}) \twoheadrightarrow \Gamma_K$ and $Z_K(\mathfrak{C}) \twoheadrightarrow Z_K(\mathfrak{C})_{\mathrm{tor}}$ to yield characters P and ψ from $Z_K(\mathfrak{C})$ to $\overline{\mathbb{Q}}_p^\times$, whose product $P\psi : Z_K(\mathfrak{C}) \to \overline{\mathbb{Q}}_p^\times$ sends $(\sigma, \zeta) \in \Gamma_K \times Z_K(\mathfrak{C})_{\mathrm{tor}} \cong Z_K(\mathfrak{C})$ to $P(\sigma)\psi(\zeta)$. Precomposing with $K^\times \backslash \mathbb{A}_{K,f}^\times \twoheadrightarrow Z_K(\mathfrak{C})$ gives a p-adic Hecke character $P\psi : K^\times \backslash \mathbb{A}_{K,f}^\times \to \overline{\mathbb{Q}}_p^\times$.

A continuous character $P_{ac} : \Gamma_K^- \to \overline{\mathbb{Q}}_p^\times$ gives a character $P = P_{ac} \circ pr_{ac} : \Gamma_K \to \overline{\mathbb{Q}}_p^\times$. A character $\psi : Z_K(\mathfrak{C})_{\mathrm{tor}} \to O^\times$ yields a continuous O-algebra map $\psi_\pm : O[[Z_K(\mathfrak{C})]] \to O[[\Gamma_K]]$ (respectively, $\psi_{ac} : O[[Z_K(\mathfrak{C})]] \twoheadrightarrow O[[\Gamma_K^-]]$) which restricts to the identity (respectively, pr_{ac}) on Γ_K and which restricts to ψ on $Z_K(\mathfrak{C})_{\mathrm{tor}}$.

4. Modular forms

This section briefly reviews modular forms while fixing notation. From now on, let O be any ring with $O_L \subseteq O \subseteq \overline{\mathbb{Q}}_p \subseteq \widehat{\overline{\mathbb{Q}}}_p \cong \mathbb{C}$.

4.1. p-adic modular forms

Let $\overline{S}_k(M, O)$ be the space of p-adic cusp forms of level M and weight k with Fourier coefficients in O, let $h_k(M, O)$ be its Hecke algebra, and let their nearly ordinary parts be $\overline{S}_k^{ord}(M, O)$ and $h_k^{ord}(M, O)$ respectively. (To be precise: in [10, 11], these correspond to $\overline{S}_{k,w}(V_1(M)(p^\infty), O)$, $h_{k,w}(V_1(M)(p^\infty), O)$, $\overline{S}_{k,w}^{n,ord}(V_1(M)(p^\infty), O)$ and $h_{k,w}^{n,ord}(V_1(M)(p^\infty), O)$ for a suitable choice of w, e.g., $w = k/2$ for k even.) Write the Fourier expansion of a p-adic cusp form $f \in \overline{S}_k(M, O)$ as $f = \sum_{n=1}^\infty a(n, f)q^n$. Let e be the ordinary projector.

There is a continuous multiplicative map $Z(M) \to h_k(M, O) : z \mapsto \langle z \rangle$ (see [10, sections 2-3] and [11, p. 334]), and for $a \in \mathbb{Z}_p^\times$ yielding $a_p \in \mathbb{A}_{\mathbb{Q},f}^\times$, there is a Hecke operator $\mathbf{T}(a_p) \in h_k(M, O)$ [11, pp. 330-332].

The perfect pairing

$$\overline{S}_k(M, O) \times h_k(M, O) \to O : (f, H) \mapsto a(1, f|H)$$

yields isomorphisms between each of its arguments and $\mathrm{Hom}_O(\cdot, O)$ of the other ([11, Theorem 3.1]; see also [10, Theorem 5.3]). Applying $\otimes_O \overline{\mathbb{Q}}_p$ yields a perfect pairing over $\overline{\mathbb{Q}}_p$ given by the same formula with each O replaced by $\overline{\mathbb{Q}}_p$.

4.2. Hida families and parameterizations

This subsection introduces Hida families of modular forms, following [11, pp. 335-337].

Let the $O[[\Gamma_K]]$-algebra I be contained in the integral closure of $O[[\Gamma_K]]$ in a finite-degree field extension of the quotient field of $O[[\Gamma_K]]$.

Let $\lambda : h_k(M, O) \to I$ be an O-algebra map such that for $\sigma \in 1 + p\mathbb{Z}_p$ corresponding to $z = [\sigma_p^{-1}] \in Z(M)$, the map λ sends $\langle z \rangle$ to $i(\sigma, \sigma) \in \Gamma_K$, and if $\sigma \in 1 + p\mathbb{Z}_p \cong 1 + \overline{v}_0 O_{K,\overline{v}_0}^\times$, then λ sends $\mathbf{T}(\sigma_p^{-1})$ to $i(1, \sigma) \in \Gamma_K$.

Let $P : I \to \overline{\mathbb{Q}}_p$ be an O-algebra map so that for some finite-order multiplicative characters $\varepsilon_P : 1 + p\mathbb{Z}_p \to \overline{\mathbb{Q}}_p^\times$ and $\varepsilon'_P : 1 + p\mathbb{Z}_p \to \overline{\mathbb{Q}}_p^\times$, and for some $w \in \mathbb{Z}$, we have

(a) $P(i(\sigma, \sigma)) = \sigma^{k-2w} \varepsilon_P(\sigma)$ for $\sigma \in 1 + p\mathbb{Z}_p$, and

(b) $P(i(1, \sigma)) = \sigma^{1-w} \varepsilon'_P(\sigma)$ for $\sigma \in 1 + p\mathbb{Z}_p$.

Call such P arithmetic, following Hida [11, pp. 316, 335-337] as well as Skinner and Urban [35, section 3.3.8]. Write $k(P) := k$ and $w(P) := w$.

From λ and P, we obtain the $\overline{\mathbb{Q}}_p$-algebra map $\lambda(P) : h_k(M, \overline{\mathbb{Q}}_p) \to \overline{\mathbb{Q}}_p$ as the composite

$$h_k(M, \overline{\mathbb{Q}}_p) \xrightarrow{e} h_k^{ord}(M, \overline{\mathbb{Q}}_p) \hookrightarrow h_k(M, \overline{\mathbb{Q}}_p) \xrightarrow{(P \circ \lambda) \otimes \overline{\mathbb{Q}}_p} \overline{\mathbb{Q}}_p .$$

Define the finite-order characters $\psi_P : Z(M) \to \overline{\mathbb{Q}}_p^\times$ and $\psi'_P : \mathbb{Z}_p^\times \to \overline{\mathbb{Q}}_p^\times$ by

$$\begin{aligned} \psi_P(\zeta[\sigma_p^{-1}]) &= \epsilon(\zeta)^{k-2w} \cdot \lambda(P)(\langle \zeta \rangle) \cdot \varepsilon_P(\sigma) \\ \psi'_P(\zeta' \sigma) &= (\zeta')^{w-1} \cdot \lambda(P)(\mathbf{T}((\zeta')_p^{-1})) \cdot \varepsilon'_P(\sigma) \end{aligned}$$

for $\zeta \in Z(M)_{\mathrm{tor}}$, $\zeta' \in (\mathbb{Z}_p^\times)_{\mathrm{tor}}$ and $\sigma \in 1 + p\mathbb{Z}_p$.

Via Hecke algebra duality, $\lambda(P)$ yields an eigenform $F(\lambda, P) \in \overline{S}_k(M, \overline{\mathbb{Q}}_p)$ such that $a(1, F(\lambda, P)) = 1$ and, for each element H of the Hecke algebra, $F(\lambda, P)|H = \lambda(P)(H) \cdot F(\lambda, P)$.

The map λ is a cuspidal Hida family; it corresponds to the collection of ordinary normalized eigenforms $F(\lambda, P)$ ranging over the arithmetic points $P : I \to \overline{\mathbb{Q}}_p$.

4.3. Theta series

In this subsection, we describe classical theta series and fit them into a Hida family. See [9, p. 257], [12, pp. 234-238] and [17, sections 5.1-5.2].

Let $\chi : K^\times \backslash \mathbb{A}_K^\times \to \mathbb{C}^\times$ be a Hecke character so that for some $n \in \mathbb{Z}_{>0}$ and some finite-order character $\psi : (O_K/C(\chi))^\times \to \mathbb{C}^\times$, for all $a \in O_K$ coprime to $C(\chi)$, we have $\chi(aO_K) = a^n \psi(a)^{-1}$. Then the theta series of χ is $\theta_\chi = \sum_{\mathfrak{a}} \chi(\mathfrak{a}) q^{N_{K/\mathbb{Q}} \mathfrak{a}}$ with L-series $L(s, \theta_\chi) = L(s, \chi) = \sum_{\mathfrak{a}} \chi(\mathfrak{a})(N_{K/\mathbb{Q}} \mathfrak{a})^{-s}$ summing over nonzero integral ideals $\mathfrak{a}$ of O_K coprime to $C(\chi)$. If

$$\varphi_K : (\mathbb{Z}/|D_K|\mathbb{Z})^\times \to \{\pm 1\}$$

is the Legendre-symbol character of $K/\mathbb{Q}$ with $\varphi_K(\ell) = \left(\frac{D_K}{\ell}\right)$ for odd rational primes ℓ, then $\theta_\chi \in S_{n+1}(|D_K|(N_{K/\mathbb{Q}} C(\chi)), \varepsilon)$ for the character

$$\varepsilon : (\mathbb{Z}/|D_K|(N_{K/\mathbb{Q}} C(\chi))\mathbb{Z})^\times \to \mathbb{C}^\times$$

with $\varepsilon(m) = \varphi_K(m)\psi^{-1}(m)$ for $m \in \mathbb{Z}_{>0}$.

The modular form θ_χ will now be fit into a Hida family.

Let the character $P_{-n,0} : \Gamma_K \cong Z_K(\mathfrak{C})/Z_K(\mathfrak{C})_{\mathrm{tor}} \to \overline{\mathbb{Q}}_p^\times$ satisfy $P_{-n,0}(i(y_{v_0}, y_{\overline{v}_0})) = y_{v_0}^{-n}$ for $y_{v_0} \in 1 + p\mathbb{Z}_p \cong 1 + v_0 O_{K,v_0}$ and $y_{\overline{v}_0} \in 1 + p\mathbb{Z}_p \cong 1 + \overline{v}_0 O_{K,\overline{v}_0}$.

Interpreting a character $\psi : Z_K(\mathfrak{C})_{\mathrm{tor}} \to \overline{\mathbb{Q}}_p^\times$ as a finite-order character $\mathbb{A}_{K,f}^\times \to \overline{\mathbb{Q}}_p^\times$ whose restriction to $\widehat{O}_K^\times$ corresponds to a Dirichlet character $(O_K/C(\psi))^\times \to \overline{\mathbb{Q}}_p^\times$, we have $C(\psi) \mid \mathfrak{C}v_0\overline{v}_0$, because ψ factors through

$$\begin{aligned} &\omega_{v_0}[O_{K,v_0}^\times] \times \omega_{\overline{v}_0}[O_{K,\overline{v}_0}^\times] \times Cl_K \times (O_K/\mathfrak{C})^\times/\{\pm 1\} \\ &\cong (O_K/v_0)^\times \times (O_K/\overline{v}_0)^\times \times Cl_K \times (O_K/\mathfrak{C})^\times/\{\pm 1\} \end{aligned}$$

and $C(\psi)$ is determined by the restriction of ψ to $\widehat{O}_K^\times$.

For each ideal $\mathfrak{a}$ of O_K coprime to $C(\psi)p$, let

$$[\mathfrak{a}] \in Z_K(\mathfrak{C}) \cong K^\times\backslash\mathbb{A}_{K,f}^\times/U_K(\mathfrak{C}p^\infty)$$

be the class of some $z \in \mathbb{A}_{K,f}^\times$ with $zO_K = \mathfrak{a}$ and $z_v = 1$ for $v \mid C(\psi)$. For an ideal $\mathfrak{a}$ of O_K not coprime to $C(\psi)p$, let $[\mathfrak{a}] = 0 \in O[[Z_K(\mathfrak{C})]]$.

In this paragraph, assume $C(\omega_{v_0}^{-n}\psi) = \mathfrak{C}v_0\overline{v}_0$. The character $\widetilde{\chi} = P_{-n,0}\omega_{v_0}^{-n}\psi : K^\times\backslash\mathbb{A}_{K,f}^\times \to \overline{\mathbb{Q}}_p^\times$ is the p-adic avatar of a Hecke character $\chi : K^\times\backslash\mathbb{A}_K^\times \to \mathbb{C}^\times$ such that for $a \in O_K$ coprime to $\mathfrak{C}v_0\overline{v}_0$, we have $\chi(aO_K) = a^n\psi^{-1}(a)$ (viewing $\chi(aO_K)$ as the value of χ at $a_f/a_{\mathfrak{C}v_0\overline{v}_0}$). Furthermore, as in [12, pp. 234-238], consider the O-algebra map $\lambda_1 : h_{2n+1,1}(pN|D_K|, O) \to O[[Z_K(\mathfrak{C})]]$ with $\lambda_1(T(\ell)) = \sum_{\mathfrak{a}:N_{K/\mathbb{Q}}\mathfrak{a}=\ell}[\mathfrak{a}]$. Compose λ_1 with $(\omega_{v_0}^{-n}\psi)_\pm : O[[Z_K(\mathfrak{C})]] \to O[[\Gamma_K]]$ to obtain a Hida family $\lambda : h_{2n+1,1}(N|D_K|p, O) \to O[[\Gamma_K]]$. Then $\theta_\chi = F(\lambda, P_{-n,0})$, so the Hida family λ interpolates θ_χ.

In this paragraph, assume $C(\omega_{v_0}^{-n}\psi) = \mathfrak{C}v_0$. As before, $P_{-n,0}\omega_{v_0}^{-n}\psi : Z_K(\mathfrak{C}) \to \overline{\mathbb{Q}}_p^\times$ yields a character $\widetilde{\chi} = P_{-n,0}\omega_{v_0}^{-n}\psi : K^\times\backslash\mathbb{A}_{K,f}^\times \to \overline{\mathbb{Q}}_p^\times$, which is the p-adic avatar of a Hecke character $\chi : K^\times\backslash\mathbb{A}_K^\times \to \mathbb{C}^\times$ such that for $a \in O_K$ coprime to $\mathfrak{C}$, we have $\chi(aO_K) = a^n\psi^{-1}(a)$. As before, compose λ_1 with the map $\psi_\pm : O[[Z_K(\mathfrak{C})]] \to O[[\Gamma_K]]$ to obtain the Hida family $\lambda : h_{2n+1,1}(N|D_K|, O) \to O[[\Gamma_K]]$. On the complex upper half plane, define the function $U\theta_\chi$ so that $U\theta_\chi(s) = \theta_\chi(s) - \chi(\overline{v}_0)\theta_\chi(ps)$; then $U\theta_\chi$ has q-expansion

$$U\theta_\chi = \sum_{n=1}^{\infty} \left\{ \begin{array}{cc} a(n, \theta_\chi) & p \nmid n \\ 0 & p \mid n \end{array} \right\} q^n$$

and $U\theta_\chi = F(\lambda, P_{-n,0})$. This p-stabilized Hida family appears in p-adic L-function interpolation formulas later in the article.

5. Algebraic geometry and cohomology

5.1. Kuga-Sato varieties and projections

We refer to [1, 21, 23] as references.

Let $Y(N)$, $Y_1(N)$, $Y_0(N)$ be the open modular curves over $\mathbb{Q}$, and let $X(N)$, $X_1(N)$, $X_0(N)$ be the complete modular curves over $\mathbb{Q}$ (see, e.g., [5]).

Let $j : Y(N) \hookrightarrow X(N)$ be the standard inclusion map. Let $\pi : \mathcal{E}_Y(\Gamma(N)) \to Y(N)$ be the universal elliptic curve. Let $\mathcal{E}_X(\Gamma(N)) \to X(N)$ be the universal generalized elliptic curve. For positive $r \in \mathbb{Z}$, the rth power of $\mathcal{E}_X(\Gamma(N))$ over $X(N)$ has as its standard desingularization the Kuga-Sato variety $\widetilde{\mathcal{E}}^r(\Gamma(N))$. Similarly define $j_1 : Y_1(N) \hookrightarrow X_1(N)$, $\pi_1 : \mathcal{E}_Y(\Gamma_1(N)) \to Y(N)$, $\mathcal{E}_X(\Gamma_1(N)) \to X_1(N)$ and $\widetilde{\mathcal{E}}^r(\Gamma_1(N))$. Fixing the "forget a $(\mathbb{Z}/N\mathbb{Z})$-basis vector" map $X(N) \to X_1(N)$, we get a map $\mathcal{E}_X(\Gamma(N)) \to \mathcal{E}_X(\Gamma_1(N))$, which yields maps $P_r : \widetilde{\mathcal{E}}^r(\Gamma(N)) \to \widetilde{\mathcal{E}}^r(\Gamma_1(N))$.

Noting $p \nmid N\phi(N)$, we have the projection operators

$$\pi_B = (1/\#(\Gamma_0(N)/\Gamma(N))) \textstyle\sum_{\sigma \in \Gamma_0(N)/\Gamma(N)} \sigma \in \mathbb{Z}_p\left[\Gamma_0(N)/\Gamma(N)\right]$$
$$\pi_{B,1} = (1/\#(\Gamma_0(N)/\Gamma_1(N))) \textstyle\sum_{\sigma \in \Gamma_0(N)/\Gamma_1(N)} \sigma \in \mathbb{Z}_p\left[\Gamma_0(N)/\Gamma_1(N)\right].$$

For $t \in \{0,1,2\}$ and $r \in \mathbb{Z}_{>0}$, define the group $G(t,r) = ((\mathbb{Z}/N\mathbb{Z})^t \rtimes \{\pm 1\})^r \rtimes S_r$. The group $G(0,r)$ acts on the rth power A^r of any elliptic curve A (see [1, p. 1052]; A^r can be viewed as a total space with fiber A^r over a base space consisting of one point), $G(1,r)$ acts on $\widetilde{\mathcal{E}}^r(\Gamma_1(N))$ (see [1, pp. 1056-1057]) and $G(2,r)$ acts on $\widetilde{\mathcal{E}}^r(\Gamma(N))$ (see [23, section 2] and [31, section 1.1]): the subgroup S_r permutes fiber components, then the subgroups $\{\pm 1\}$ multiply fiber components by ± 1, then the subgroups $(\mathbb{Z}/N\mathbb{Z})^t$ translate fiber components by sections of order dividing N.

For $t \in \{0,1,2\}$, let the group map $c_t : G(t,r) \to \{\pm 1\}$ be 1 on each $(\mathbb{Z}/N\mathbb{Z})^t$ factor, the identity on each $\{\pm 1\}$ factor, and the sign map on S_r. Define

$$\pi_{t,r} = (1/\#G(t,r)) \textstyle\sum_{\sigma \in G(t,r)} c_t(\sigma) \cdot \sigma \in \mathbb{Q}[G(t,r)].$$

Since $p \nmid (k-2)!$, we have $\pi_{2,k-2} \in \mathbb{Z}_p[G(2,k-2)]$.

Take a field $F \supseteq K_1$. For an elliptic curve A defined over F, where A has complex multiplication by O_K, we may choose F-vector space generators ω_A, η_A of $H^{1,0}_{dR}(A/F)$, $H^{0,1}_{dR}(A/F)$ respectively; then $\pi_{0,r} H^*_{dR}(A^r/F) = \mathrm{Sym}^r H^1_{dR}(A/F)$ is generated as an F-vector space by the $r+1$ elements

$$\omega_A^j \eta_A^{r-j} = \binom{r}{j}^{-1} \sum_{S \subseteq \{1,2,\ldots,r\}} \left(\left(\bigwedge_{s \in S} pr_s^* \omega_A \right) \wedge \left(\bigwedge_{s \in \{1,2,\ldots,r\} - S} pr_s^* \eta_A \right) \right)$$

with $j \in \{0,1,\ldots,r\}$; see [1, section 1.4]. Also, we have an isomorphism [1, Proposition 2.5]

$$S_{r+2}(\Gamma_1(N), F) \otimes \mathrm{Sym}^r H^1_{dR}(A/F) \cong \mathrm{Fil}^{r+1} \pi_{1,r} \pi_{0,r} H^{2r+1}_{dR}(\widetilde{\mathcal{E}}^r(\Gamma_1(N)) \times A^r/F)$$

sending $f \otimes \eta$ to $\omega_f \wedge \eta$ for the differential form ω_f corresponding to f as in [1, section 1.1].

5.2. Chow groups and Heegner cycles

This subsection gives definitions related to Chow groups and defines Heegner cycles that will be used later in the paper.

For an algebraic variety U defined over a field F, $CH^a(U/F)$ is the Chow group of codimension a cycles in U defined over F up to rational equivalence, and the subgroup $CH^a_0(U/F)$ is the group of such classes of cycles homologically equivalent to zero up to torsion (see [8, p. 426] and [26, section 1]).

See [21, section 4.1]. Recall the ideal $\mathfrak{C}$ of O_K and the isomorphism $\mathbb{Z}/N\mathbb{Z} \cong O_K/\mathfrak{C}$ from subsection 2.2. The isogeny $\mathbb{C}/O_K \to \mathbb{C}/\mathfrak{C}^{-1}$ gives $x_0 \in X_0(N)(K)$ by CM theory. Choose an $x \in X(N)$ that is sent to x_0 under the standard map $X(N) \to X_0(N)$; the fiber E_x for $\mathcal{E}_X(\Gamma(N)) \to X(N)$ at x (which is also the fiber E_{x_1} for $\mathcal{E}_X(\Gamma_1(N)) \to X_1(N)$ at the image x_1 of x in $X_1(N)$) is an elliptic curve with complex multiplication by O_K, so the variety $\mathrm{Graph}(\sqrt{D_K})$ exists in E_x^2. We have an embedding $i_x : E_x^{k-2} \hookrightarrow \widetilde{\mathcal{E}}^{k-2}(\Gamma(N))$. Define the Heegner cycle

$$\Delta_{Ma19} = \pi_B \pi_{2,k-2}(i_x)_*(\mathrm{Graph}(\sqrt{D_K})^{k/2-1}) \in CH^{k/2}(\widetilde{\mathcal{E}}^{k-2}(\Gamma(N))/K_1) \otimes_{\mathbb{Z}} \mathbb{Z}_p$$

and let

$$Z_{Ma19} = N_{K_1/K}\Delta_{Ma19} \in N_{K_1/K}(CH^{k/2}(\widetilde{\mathcal{E}}^{k-2}(\Gamma(N))/K_1) \otimes_{\mathbb{Z}} \mathbb{Z}_p)$$

be the image of Δ_{Ma19} under the norm map $N_{K_1/K} = \sum_{g \in \mathrm{Gal}(K_1/K)} g$.

See [2, section 3]. Similarly, with an embedding $i_{x_1} : E_{x_1}^{k-2} \hookrightarrow \widetilde{\mathcal{E}}^{k-2}(\Gamma_1(N))$, we define

$$\Delta_{Ca13} = \pi_{B,1}\pi_{1,k-2}(i_{x_1})_*(\mathrm{Graph}(\sqrt{D_K})^{k/2-1}) \in CH^{k/2}(\widetilde{\mathcal{E}}^{k-2}(\Gamma_1(N))/K_1) \otimes_{\mathbb{Z}} \mathbb{Z}_p$$

as a Heegner cycle. Also, for an ideal $\mathfrak{a}$ of O_K, define the modified Heegner cycle

$$\Delta_{Ca13,\mathfrak{a}} = \pi_{B,1}\pi_{1,k-2}(\mathrm{Graph}(\sqrt{D_K})^{k/2-1}_{E_{x_1}/E_{x_1}[\mathfrak{a}]}) \in CH^{k/2}(\widetilde{\mathcal{E}}^{k-2}(\Gamma_1(N))/K_1) \otimes_{\mathbb{Z}} \mathbb{Z}_p.$$

For an isogeny $\varphi : A \to A'$ between elliptic curves A and A', where A' has $\Gamma_1(N)$ structure, consider

$$\mathrm{Graph}(\varphi)^r \subseteq (A')^r \times A^r \subseteq \widetilde{\mathcal{E}}^r(\Gamma_1(N)) \times A^r$$

(embedding in the fiber in $\widetilde{\mathcal{E}}^r(\Gamma_1(N))$ at the point linked to A') and the corresponding Heegner cycle $\Delta_\varphi = \pi_{1,r}\pi_{0,r}(\mathrm{Graph}(\varphi)^r)$ in $\widetilde{\mathcal{E}}^r(\Gamma_1(N)) \times A^r$ (where $\pi_{1,r}$, $\pi_{0,r}$ act on $\widetilde{\mathcal{E}}^r(\Gamma_1(N))$, A^r respectively). For each nonzero integral ideal $\mathfrak{a}$ of O_K and elliptic curve A, we have a "modulo $\mathfrak{a}$-torsion" isogeny $\varphi(A, \mathfrak{a}) : A \to A/A[\mathfrak{a}]$ (see [1, formula 1.4.7]).

Choose representatives $\mathfrak{a}$ of the class group of K so that the numbers $\mathrm{N}(\mathfrak{a})$ seen as elements of $\widehat{\overline{\mathbb{Q}}}_p$ are p-adic units. Then, taking a sum over the classes $[\mathfrak{a}]$ of the class group of K, define

$$Z_{BeDaPr13} = \frac{1}{(k/2-1)!} \sum_{[\mathfrak{a}]} \frac{1}{\mathrm{N}^{k/2-1}(\mathfrak{a})} \cdot \Delta_{\varphi(E_{x_1}, \mathfrak{a})} \in CH(\widetilde{\mathcal{E}}^{k-2}(\Gamma_1(N)) \times E_{x_1}^{k-2}) \otimes_{\mathbb{Z}} \mathbb{Z}_p.$$

5.3. Cohomology

For a topological group G and a G-module U, see [29] for the definitions of the continuous cohomology groups $H^n(G,U)$, the restriction and corestriction maps $\mathrm{res}_{G/H} : H^n(G,U) \to H^n(H,U)$ and $\mathrm{cor}_{G/H} : H^n(H,U) \to H^n(G,U)$, and the conjugation maps $g_* : H^n(H,U) \to H^n(H,U)$ for $g \in G$ and certain subgroups H of G. For additional background, see [30, Appendix B] and [36].

If G acts on finitely generated free R-modules U_1, U_2 for a commutative ring R with 1, then $H^1(G, \mathrm{Hom}_R(U_2,U_1)) \cong \mathrm{Ext}^1(U_2,U_1)$ (see [38, Proposition 4] for the case $U_1 = U_2$); for subsection 5.4's representation V_f, this yields an isomorphism between $H^1_f(K_{v_0}, V_f)$ and the group $\mathrm{Ext}^1_{cris}(\mathbb{Q}_p, V_f)$ of crystalline extensions $V_f \hookrightarrow E \twoheadrightarrow \mathbb{Q}_p$ of $G_{K_{v_0}}$-modules over $\mathbb{Q}_p$ [26, section 3.4].

Suppose G acts linearly and continuously over a finitely-generated O_L-module U, and B is an O_L-submodule of $H^n(G,U)$. For an element $c \in B$ that is not in B_{tor}, define $\mathrm{ind}_p(c,B)$ to be the maximum of the set

$$\{M \in \mathbb{Z} : M \geq 0 \text{ and there is } c' \in B \text{ such that } c - p^M c' \in B_{\mathrm{tor}}\}.$$

Intuitively, just as the p-index ind_p of a positive integer is the number of factors of p in the prime factorization of that integer, so $\mathrm{ind}_p(c,B)$ can be viewed as the number of factors of p in the class c thought of as an element of B/B_{tor}.

5.4. Galois representations

Recall subsection 5.1's projectors π_B, $\pi_{B,1}$, $\pi_{2,k-2}$.

The Galois representation T_p linked to f can be defined as follows [21, 23]: For the p-adic sheaf $\mathcal{F} = \varprojlim_n \mathcal{F}_n$ over $Y(N)$ with the sheaves

$$\mathcal{F}_n = \mathrm{Sym}^{k-2}(R^1\pi_*(\mathbb{Z}/p^n)_{\mathcal{E}_Y(\Gamma(N))})$$

over $Y(N)$, define the Galois representations

$$J_p = \pi_B H^1_{et}(X(N) \otimes \overline{\mathbb{Q}}, j_*\mathcal{F})(\epsilon), \; T_p = \{x \in J_p : I_f x = 0\}$$

where I_f is the kernel of the $O(f)$-algebra map from the Hecke algebra with coefficients in $\mathbb{Z}$ to $O_{\mathbb{Q}(f)}$ sending $T(\ell)$ to $a(\ell, f)$. As mentioned in [23, p. 102], because f is a newform, a map $R : J_p \to T_p$ exists such that R respects Hecke operators, R is $G_{\mathbb{Q}}$-equivariant and for some non-negative integer c, the restriction of R to T_p is multiplication by p^c. By [23, Proposition 2.1] (which comes from [31, Theorem 1.2.1]) and [23, Lemma 2.2], $H^1_{et}(X(N) \otimes \overline{\mathbb{Q}}, j_*\mathcal{F})$ is torsion free (this is nontrivial) and there are isomorphisms

$$\begin{aligned}&\pi_{2,k-2} H^*_{et}(\widetilde{\mathcal{E}}^{k-2}(\Gamma(N)) \otimes \overline{\mathbb{Q}}, \mathbb{Z}/p^n)(\epsilon^{k/2-1}) \\ &\cong H^1_{et}(X(N) \otimes \overline{\mathbb{Q}}, j_*\mathcal{F}_n) \cong H^1_{et}(X(N) \otimes \overline{\mathbb{Q}}, j_*\mathcal{F})/p^n\end{aligned}$$

so that identifying π_B with a projection on $\pi_{2,k-2} H^*_{et}(\widetilde{\mathcal{E}}^{k-2}(\Gamma(N)) \otimes \overline{\mathbb{Q}}, \mathbb{Z}_p)$ yields

$$J_p \cong \pi_B \pi_{2,k-2} H^*_{et}(\widetilde{\mathcal{E}}^{k-2}(\Gamma(N)) \otimes \overline{\mathbb{Q}}, \mathbb{Z}_p)(\epsilon^{k/2}).$$

Using the standard map $O_{\mathbb{Q}(f)} \otimes \mathbb{Z}_p \twoheadrightarrow O_{\mathbb{Q}(f),\varpi_{\mathbb{Q}(f)}} \hookrightarrow O_L$ for the first tensor product below, define

$$T_f = T_p \otimes_{O_{\mathbb{Q}(f)} \otimes \mathbb{Z}_p} O_L, \ V_f = T_f \otimes_{O_L} L, \ W_f = T_f \otimes_{O_L} (L/O_L).$$

The usual short exact sequence $T_f \hookrightarrow V_f \twoheadrightarrow W_f$ and maps $p^{-n} : T_f \twoheadrightarrow W_f[p^n]$ for $n \in \mathbb{Z}_{>0}$ exist, as in [30, sections 1.1-1.2].

Let V_f^a (respectively, V_f^g) be the Deligne/Scholl representations over L, pure[2] of weight $1-k$ (respectively, $k-1$), with $\det(xI - F) = x^2 - a_\ell x + \ell^{k-1}$ the characteristic polynomial of arithmetic (respectively, geometric) Frobenius F at $\ell \nmid Np$ [4, section 12.5]. Then:

(a) $V_f^g = \mathrm{Hom}_L(V_f^a, L)$.

(b) $V_f^g(\epsilon^{k/2})$ is self-dual by a Poincare duality map $V_f^g(\epsilon^{k/2}) \times V_f^g(\epsilon^{k/2}) \to L(\epsilon)$ [27, section 1.3], so $V_f^g(\epsilon^{k/2}) \cong V_f^a(\epsilon^{1-(k/2)})$.

(c) $V_f \cong V_f^g(\epsilon^{k/2}) \cong V_f^a(\epsilon^{1-(k/2)})$ is pure of weight -1 and $\mathrm{Hom}_L(V_f, L(\epsilon)) \cong V_f$.

Let $T_f^g := T_f(\epsilon^{-k/2})$, so that $T_f^g \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong V_f^g$ as Galois representations.

For the p-adic sheaf $\mathcal{F}^1 = \mathrm{Sym}^{k-2}(R^1\pi_*(\mathbb{Z}_p)_{\mathcal{E}_Y(\Gamma_1(N))})$ over $Y_1(N)$, we similarly have an isomorphism (see [32, section 2.8])

$$\pi_{1,k-2} H^*_{et}(\widetilde{\mathcal{E}}^{k-2}(\Gamma_1(N)) \otimes \overline{\mathbb{Q}}, \mathbb{Z}_p) \cong H^1_{et}(X_1(N) \otimes \overline{\mathbb{Q}}, j_{1,*}\mathcal{F}^1)$$

and we define

$$J_p^1 = \pi_{B,1} H^1_{et}(X_1(N) \otimes \overline{\mathbb{Q}}, j_{1,*}\mathcal{F}^1)(\epsilon) \cong \pi_{B,1}\pi_{1,k-2} H^*_{et}(\widetilde{\mathcal{E}}^{k-2}(\Gamma_1(N)) \otimes \overline{\mathbb{Q}}, \mathbb{Z}_p)(\epsilon)$$

after identifying $\pi_{B,1}$ with a projector on $\pi_{1,k-2} H^*_{et}(\widetilde{\mathcal{E}}^{k-2}(\Gamma_1(N)) \otimes \overline{\mathbb{Q}}, \mathbb{Z}_p)$. Note that $\mathcal{F}^1 = \pi_{Y(N)\to Y_1(N),*}\mathcal{F}$ (where $\pi_{Y(N)\to Y_1(N),*}$ has the obvious meaning); in fact,

$$J_p^1 = \pi_{B,1} H^1_{et}(X_1(N) \otimes \overline{\mathbb{Q}}, j_{1,*}\mathcal{F}^1)(\epsilon) \cong \pi_B H^1_{et}(X(N) \otimes \overline{\mathbb{Q}}, j_*\mathcal{F})(\epsilon) = J_p$$

(see [25, section II.2.5] for the analogous result with $X_0(N)$ instead of $X_1(N)$).

5.5. *Selmer group conditions*

We define the Bloch-Kato Selmer groups following [21, section 2.2]. (Note the small difference in principle between this and [23], which relaxes all conditions for places over N.) For a number field F, letting I_v be the inertia group in G_{L_v} for a place $v \nmid \infty$ of F, define

$$H^1_f(F_v, V_f) = \left\{ \begin{array}{cc} \ker(H^1(F_v, V_f) \to H^1(I_v, V_f)) & v \nmid p \\ \ker(H^1(F_v, V_f) \to H^1(F_v, V_f \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}})) & v \mid p \end{array} \right\}$$

[2] "Pure of weight w" means that the eigenvalues of geometric Frobenius at v have absolute value $(\mathrm{N}v)^{w/2}$.

and let $H^1_f(F_v, T_f)$ (respectively, $H^1_f(F_v, W_f[p^n])$) be the inverse image (respectively, image) of $H^1_f(F_v, V_f)$ under the standard map $H^1(F_v, T_f) \to H^1(F_v, V_f)$ (respectively, $p^{-n} : H^1(F_v, T_f) \to H^1(F_v, W_f[p^n])$). Define the global Bloch-Kato Selmer groups

$$\begin{aligned} H^1_f(F, T_f) &= \{c \in H^1(F, T_f) : \forall \text{ places } v \text{ of } F : c_v \in H^1_f(F_v, T_f)\} \\ H^1_f(F, V_f) &= \{c \in H^1(F, V_f) : \forall \text{ places } v \text{ of } F : c_v \in H^1_f(F_v, V_f)\} \\ H^1_f(F, W_f) &= \{c \in H^1(F, W_f) : \forall \text{ places } v \text{ of } F : c_v \in H^1_f(F_v, W_f)\}. \end{aligned}$$

The Shafarevich-Tate group is

$$Ш_f(F, W_f) := H^1_f(F, W_f)/H^1_f(F, W_f)_{\text{div}} \tag{4}$$

which has finite cardinality since the O_L-module $H^1_f(F, W_f)$ has finite corank.

As in [15, section 2.3.4], we define the anticyclotomic Selmer groups

$$H^1_{ac}(K_v, V_f) = \left\{ \begin{array}{cl} \ker(H^1(K_v, V_f) \to H^1(I_v, V_f)) & \text{split } v \nmid p\infty \\ H^1(K_v, V_f) & v = \overline{v}_0 \\ 0 & \text{otherwise} \end{array} \right\}$$

$$H^1_{ac}(K, V_f) = \{c \in H^1(K, V_f) : \forall \text{ places } v \text{ of } K : c_v \in H^1_{ac}(K_v, V_f)\}.$$

Define the local cohomology groups $H^1_{ac}(K_v, T_f)$, $H^1_{ac}(K_v, W_f)$ by taking preimages and images of $H^1_{ac}(K_v, V_f)$, and define the cohomology groups $H^1_{ac}(K, T_f)$, $H^1_{ac}(K, W_f)$ as the groups of global elements localizing to elements of $H^1_{ac}(K_v, T_f)$, $H^1_{ac}(K_v, W_f)$ respectively at all v.

5.6. The p-adic Abel-Jacobi map

As in [26, section 1], for any smooth proper variety U defined over a field F and any $n \in \mathbb{Z}_{\geq 0}$, there is a p-adic Abel-Jacobi map

$$AJ^U_F : CH^n_0(U/F) \to H^1(F, H^{2n-1}(\overline{U}_{et}, \mathbb{Z}_p))(\epsilon^n)$$

coming from the cycle class map and Hochschild-Serre spectral sequence. The Abel-Jacobi map is Galois equivariant (see [21, section 3.2] and [23, Proposition 4.2]) and commutes with pushforwards and pullbacks of correspondences ([23, proof of Proposition 4.2]; see [16, section 2] for the complex algebraic geometry version of this result).

We consider the p-adic Abel-Jacobi map in the following three different settings.

As described by [21, sections 3.1-3.3] and [23, chapters 2-4], for any field F containing $\mathbb{Q}$, there is a p-adic Abel-Jacobi map (extending the map $\Phi_{p,L}$ in [21] by $\mathbb{Z}_p$-linearity)

$$\Phi : CH^{k/2}_0(\widetilde{\mathcal{E}}^{k-2}(\Gamma(N))/F) \otimes_{\mathbb{Z}} \mathbb{Z}_p \to H^1(F, H^{k-1}_{et}(\widetilde{\mathcal{E}}^{k-2}(\Gamma(N)) \otimes \overline{F}, \mathbb{Z}_p(\epsilon^{k/2}))).$$

Composing Φ with the map that is $H^1(F, \cdot)$ of the composite

$$H^{k-1}_{et}(\widetilde{\mathcal{E}}^{k-2}(\Gamma(N)) \otimes \overline{F}, \mathbb{Z}_p(\epsilon^{k/2})) \xrightarrow{\pi_B \pi_{2,k-2}} J_p \xrightarrow{\quad R \quad} T_p$$

and then applying $\otimes O_L$ or $\otimes L$ yields compatible Abel-Jacobi maps

$$\begin{array}{rcl} AJ_F : CH_0^{k/2}(\widetilde{\mathcal{E}}^{k-2}(\Gamma(N))/F) \otimes O_L & \to & H^1_f(F, T_f) \\ AJ_F : CH_0^{k/2}(\widetilde{\mathcal{E}}^{k-2}(\Gamma(N))/F) \otimes L & \to & H^1_f(F, V_f). \end{array}$$

See [2]. Similarly,

$$\Phi_{Ca13} : CH_0^{k/2}(\widetilde{\mathcal{E}}^{k-2}(\Gamma_1(N))/F) \otimes_{\mathbb{Z}} \mathbb{Z}_p \to H^1(L, H^{k-1}_{et}(\widetilde{\mathcal{E}}^{k-2}(\Gamma_1(N)) \otimes \overline{F}, \mathbb{Z}_p(\epsilon^{k/2})))$$

is a p-adic Abel-Jacobi map which, together with the composition of maps

$$H^{k-1}_{et}(\widetilde{\mathcal{E}}^{k-2}(\Gamma_1(N)) \otimes \overline{F}, \mathbb{Z}_p(\epsilon^{k/2})) \xrightarrow{\pi_{B,1}\pi_{1,k-2}} J^1_p \cong J_p \xrightarrow{R} T_p$$

and the application of $\otimes O_L$, yields an Abel-Jacobi map

$$AJ^1_F : CH_0^{k/2}(\widetilde{\mathcal{E}}^{k-2}(\Gamma_1(N))/F) \otimes O_L \to H^1(F, T_f).$$

See [1, sections 3.1-3.4], taking that paper's r to be $k-2$. Let F be a field containing K_1, let A and A' be elliptic curves over $\mathbb{Q}$, let A' have $\Gamma_1(N)$ structure, and let $\varphi : A \to A'$ be an isogeny for which Δ_φ is defined over F. (Note that $\widetilde{\mathcal{E}}^r(\Gamma_1(N)) \times A^r$ is defined over $\mathbb{Q}$ [1, p. 1056].) Define

$$\begin{array}{rcl} J_{BeDaPr} & = & \pi_{1,k-2}\pi_{0,k-2} H^{2k-3}_{et}(\overline{\widetilde{\mathcal{E}}^{k-2}(\Gamma_1(N)) \times A^{k-2}}, \mathbb{Q}_p(\epsilon^{k-1})) \\ J^{\mathbb{Z}_p}_{BeDaPr} & = & \pi_{1,k-2}\pi_{0,k-2} H^{2k-3}_{et}(\overline{\widetilde{\mathcal{E}}^{k-2}(\Gamma_1(N)) \times A^{k-2}}, \mathbb{Z}_p(\epsilon^{k-1})) \\ J^{dR}_{BeDaPr} & = & \pi_{1,k-2}\pi_{0,k-2} H^{2k-3}_{dR}((\widetilde{\mathcal{E}}^{k-2}(\Gamma_1(N)) \times A^{k-2})/F)(\epsilon^{k-2}). \end{array}$$

There are compatible p-adic Abel-Jacobi maps

$$\begin{array}{rcl} AJ^{1,A}_F : CH_0^{k-1}((\widetilde{\mathcal{E}}^{k-2}(\Gamma_1(N)) \times A^{k-2})/F) \otimes_{\mathbb{Z}} \mathbb{Z}_p & \to & H^1_f(F, J^{\mathbb{Z}_p}_{BeDaPr}) \\ AJ^{1,A}_F : CH_0^{k-1}((\widetilde{\mathcal{E}}^{k-2}(\Gamma_1(N)) \times A^{k-2})/F) \otimes_{\mathbb{Z}} \mathbb{Q}_p & \to & H^1_f(F, J_{BeDaPr}). \end{array}$$

5.7. Shafarevich-Tate-like groups

Following [21, section 3.3], we define a variant of the Shafarevich-Tate group which will be used in section 8.

For each positive $n \in \mathbb{Z}$, the map $p^{-n} : T_f \to W_f[p^n]$ and the inclusion $W_f[p^n] \hookrightarrow W_f[p^{n+1}]$ yield maps on cohomology

$$\begin{array}{rcl} p^{-n} : H^1(K, T_f) & \to & H^1(K, W_f[p^n]) \\ 1 : H^1(K, W_f[p^n]) & \to & H^1(K, W_f[p^{n+1}]). \end{array}$$

These maps combine to form a commutative diagram

$$
\begin{array}{ccc}
H^1(K,T_f) & \xrightarrow{p^{-1}} & H^1(K,W_f[p]) \\
\downarrow{\scriptstyle p} & & \downarrow{\scriptstyle 1} \\
\vdots & & \vdots \\
\downarrow{\scriptstyle p} & & \downarrow{\scriptstyle 1} \\
H^1(K,T_f) & \xrightarrow{p^{-n}} & H^1(K,W_f[p^n]) \\
\downarrow{\scriptstyle p} & & \downarrow{\scriptstyle 1} \\
H^1(K,T_f) & \xrightarrow{p^{-(n+1)}} & H^1(K,W_f[p^{n+1}]) \\
\downarrow{\scriptstyle p} & & \downarrow{\scriptstyle 1} \\
\vdots & & \vdots
\end{array}
$$

from which the direct limit of the maps $p^{-n} : H^1(K,T_f) \to H^1(K,W_f[p^n])$ is a map

$$H^1(K,T_f) \otimes_{O_L} (L/O_L) \to H^1(K,W_f). \tag{5}$$

Define $Ш_{p^n}(K,W_f)$ to be the quotient of $H^1_f(K,W_f[p^n])$ by the image under the map $p^{-n} : H^1(K,T_f) \to H^1(K,W_f[p^n])$ of $(\text{im } AJ_K)/p^n(\text{im } AJ_K)$. Similarly, define the Shafarevich-Tate-like group $Ш(K,W_f)$ to be the quotient of $H^1_f(K,W_f)$ by the image under the map (5) of $(\text{im } AJ_K) \otimes_{O_L} (L/O_L)$. Then we have a commutative diagram of short exact sequences

$$
\begin{array}{ccccc}
(\text{im } AJ_K)/p(\text{im } AJ_K) & \xhookrightarrow{p^{-1}} & H^1_f(K,W_f[p]) & \twoheadrightarrow & Ш_p(K,W_f) \\
\downarrow{\scriptstyle p} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} \\
\vdots & & \vdots & & \vdots \\
\downarrow{\scriptstyle p} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} \\
(\text{im } AJ_K)/p^n(\text{im } AJ_K) & \xhookrightarrow{p^{-n}} & H^1_f(K,W_f[p^n]) & \twoheadrightarrow & Ш_{p^n}(K,W_f) \\
\downarrow{\scriptstyle p} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} \\
(\text{im } AJ_K)/p^{n+1}(\text{im } AJ_K) & \xhookrightarrow{p^{-(n+1)}} & H^1_f(K,W_f[p^{n+1}]) & \twoheadrightarrow & Ш_{p^{n+1}}(K,W_f) \\
\downarrow{\scriptstyle p} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} \\
\vdots & & \vdots & & \vdots
\end{array}
$$

in which each term has finite size, and these sequences' direct limit is

$$(\text{im } AJ_K) \otimes_{O_L} (L/O_L) \hookrightarrow H^1_f(K,W_f) \twoheadrightarrow Ш(K,W_f)$$

which yields a surjection $Ш(K,W_f) \twoheadrightarrow Ш_f(K,W_f)$ by (4), since $(\text{im } AJ_K) \otimes_{O_L} (L/O_L)$ is divisible. Note that $Ш_f(K,W_f)$ has finite cardinality.

5.8. Logarithms on local cohomology

In this subsection, we define logarithm maps on local cohomology groups.

Recall that $H^1(K_{v_0}, V_f) \cong \mathrm{Ext}^1(\mathbb{Q}_p, V_f)$ via an isomorphism which takes the subgroup $H^1_f(K_{v_0}, V_f)$ to $\mathrm{Ext}^1_{cris}(\mathbb{Q}_p, V_f)$ (see [26, section 3.4]).

Define

$$\widetilde{V} := \pi_B \pi_{2,k-2} H^{k-1}_{dR}(\widetilde{\mathcal{E}}^{k-2}(\Gamma(N))/L)(\epsilon^{k/2})$$

and let $\mathrm{Ext}^1_{ffm}(L, \widetilde{V})$ consist of the classes of extensions $\widetilde{V} \hookrightarrow E \twoheadrightarrow L$ of filtered Frobenius modules [1, sections 3.2-3.3]. As in [1, sections 3.2-3.3] and [24, Proposition 1.21 and Corollary 1.22], since the extension $L/\mathbb{Q}_p$ is unramified, an etale-versus-de-Rham-cohomology comparison theorem of Faltings [7, Theorem 5.6] yields a map

$$\mathrm{comp} : \mathrm{Ext}^1_{cris}(\mathbb{Q}_p, J_p \otimes L) \overset{\cong}{\to} \mathrm{Ext}^1_{ffm}(L, \widetilde{V}) \cong \widetilde{V}/\mathrm{Fil}^0\widetilde{V}.$$

From the inclusion $T_p \hookrightarrow J_p$, we obtain a map

$$H^1_f(K_{v_0}, \mathrm{Hom}(\mathbb{Q}_p, V_f)) \cong H^1_f(K_{v_0}, \mathrm{Hom}(\mathbb{Q}_p, T_p \otimes L)) \to H^1_f(K_{v_0}, \mathrm{Hom}(\mathbb{Q}_p, J_p \otimes L))$$

which, because of the isomorphism between $H^1_f(K_{v_0}, \cdot)$ and $\mathrm{Ext}^1_{cris}(\mathbb{Q}_p, \cdot)$, is identified with a map

$$J' : \mathrm{Ext}^1_{cris}(\mathbb{Q}_p, V_f) \cong \mathrm{Ext}^1_{cris}(\mathbb{Q}_p, T_p \otimes L) \to \mathrm{Ext}^1_{cris}(\mathbb{Q}_p, J_p \otimes L)$$

of which the image is sent by comp to a module which we'll call $\widetilde{U} \subseteq \widetilde{V}/\mathrm{Fil}^0\widetilde{V}$.

Define $\widetilde{\widetilde{V}}$ to be the annihilator of $\mathrm{Fil}^0\widetilde{V}$ with respect to the Poincare duality map $\widetilde{V} \times \widetilde{V} \to L$ (see also [27, section 1.3.4]); then that duality gives an isomorphism $J : \widetilde{V}/\mathrm{Fil}^0\widetilde{V} \cong \mathrm{Hom}_L(\widetilde{\widetilde{V}}, L)$.

We define the logarithm as the composite

$$\log : J \circ \mathrm{comp} \circ J' : H^1_f(K_{v_0}, V_f) \cong \mathrm{Ext}^1_{cris}(\mathbb{Q}_p, V_f) \to \mathrm{Hom}_L(\widetilde{\widetilde{V}}, L)$$

and, for a differential form $\eta \in \widetilde{\widetilde{V}}$, we define $\log_\eta : H^1_f(K_{v_0}, V_f) \to L$ as the map sending $C \in H^1_f(K_{v_0}, V_f)$ to $(\log C)(\eta)$; here we adapt notation of [15, section 3.5].

Those maps log, $\log_\eta$ were adapted from the following logarithms of [1, sections 3.1-3.4]. Let F be a field containing K_1; let A be an elliptic curve over $\mathbb{Q}$. There is an isomorphism

$$\log^{1,A}_F : H^1_f(F, J_{BeDaPr}) \overset{\cong}{\to} \mathrm{Hom}_F(\mathrm{Fil}^1 J^{dR}_{BeDaPr}, F)$$

given in [1, p. 1070] as the composition of three vertical maps in that source's diagram: the isomorphism $H^1_f(F, \cdot) \cong \mathrm{Ext}^1_{cris}(\mathbb{Q}_p, \cdot)$, then a comparison isomorphism, then a Poincare duality map. Precomposing $\log^{1,A}_F$ with the canonical map $H^1_f(F, J^{\mathbb{Z}_p}_{BeDaPr}) \to H^1_f(F, J_{BeDaPr})$ yields a map

$$\log^{1,A,\mathbb{Z}_p}_F : H^1_f(F, J^{\mathbb{Z}_p}_{BeDaPr}) \to \mathrm{Hom}_F(\mathrm{Fil}^1 J^{dR}_{BeDaPr}, F).$$

From $\log_F^{1,A}$ and $\log_F^{1,A,\mathbb{Z}_p}$, for each $\eta \in \mathrm{Fil}^1 J^{dR}_{BeDaPr}$ we obtain compatible maps

$$\begin{aligned} \log^{1,A}_{\eta,F} : H^1_f(F, J_{BeDaPr}) &\to F \qquad (6)\\ \log^{1,A,\mathbb{Z}_p}_{\eta,F} : H^1_f(F, J^{\mathbb{Z}_p}_{BeDaPr}) &\to F \qquad (7)\end{aligned}$$

sending a cohomology class C to $(\log_F^{1,A} C)(\eta)$ and $(\log_F^{1,A,\mathbb{Z}_p} C)(\eta)$ respectively. We may take $\eta = \omega_f \wedge \omega_A^{k/2-1}\eta_A^{k/2-1}$ for [1]'s differential forms ω_f, ω_A, η_A.

6. *L*-functions

6.1. The Rankin-Selberg L-function

For classical cusp forms g, h that are eigenforms for the Hecke operators away from their levels, we have the classical Rankin-Selberg L-function $L(s, g \times h)$. We write $L^p(s, g \times h)$ for the L-function $L(s, g \times h)$ without the Euler factor over p.

For classical modular forms $g, h \in S_{k'}(\Gamma_0(M), \chi)$ for a common Dirichlet character χ, the related function $D(s, g, h^c) = \sum_{n=1}^{\infty} a(n, g)a(n, h^c)/n^s$ satisfies

$$\langle g, h\rangle_{\Gamma_0(M)} = \mathrm{vol}(\Gamma_0(M)\backslash\mathfrak{h}) \cdot (\Gamma(k')/(4\pi)^{k'})\mathrm{res}_{s=k'} D(s, g, h^c) \qquad (8)$$

as can be shown using the Rankin-Selberg method [34, p. 35], where

$$\langle g, h\rangle_{\Gamma_0(M)} = \textstyle\int_{\Gamma_0(M)\backslash\mathfrak{h}} g(\tau)\overline{h(\tau)}y^{k'}\mathrm{d}x\mathrm{d}y/y^2$$

is the Petersson inner product (denoting $\tau = x + iy \in \mathbb{C}$).

See, for instance, [1, pp. 1088-1089] and [3, pp. 222-224].

6.2. Local characters and representations

The following expressions will appear in p-adic L-function interpolation formulas. Notation is adapted from that of [11, 12, 37].

For a (not necessarily unitary) character $\eta : \mathbb{Q}_p^\times \to \mathbb{C}^\times$, define

$$g_p(\eta) = \sum_{u \in (\mathbb{Z}_p/C(\eta))^\times} \eta^{-1}(u) \exp\left(-2\pi i \frac{u}{C(\eta)}\right)$$

where the ideal $C(\eta)$ of $\mathbb{Z}_p$ is identified with a generator of the form p^t, $t \in \mathbb{Z}$. Similarly, for a (not necessarily unitary) character $\eta : K_{v_0}^\times \to \mathbb{C}^\times$, define

$$g_{v_0}(\eta) = \frac{\eta(C(\eta))}{N_{K/\mathbb{Q}}(C(\eta))} \sum_{u \in (O_{K,v_0}/C(\eta))^\times} \eta^{-1}(u) \exp\left(-2\pi i \mathrm{Tr}_{K_{v_0}/\mathbb{Q}_p}\left(\frac{u}{2\delta C(\eta)}\right)\right)$$

where the ideal $C(\eta)$ of O_{K,v_0} is identified with a generator of the form p^t, $t \in \mathbb{Z}$, and where the number $\delta \in (\mathbb{R}_{>0} i) \cap K$ is chosen so that the fractional ideal $\{\mathrm{Tr}_{K/\mathbb{Q}}(x\overline{y}/(2\delta)) : x, y \in O_K\}$ of $\mathbb{Q}$ is coprime to pN.

Recall that for (not necessarily unitary) characters $\eta : \mathbb{Q}_p^\times \to \mathbb{C}^\times$ and $\eta' : \mathbb{Q}_p^\times \to \mathbb{C}^\times$, we have the principal series representation $\pi(\eta, \eta')$ or (for $\eta' = \eta|\cdot|_p^{-1}$) special representation $\sigma(\eta, \eta')$, which is an infinite-dimensional irreducible subquotient of the space of locally constant functions $GL_2(\mathbb{Q}_p) \to \mathbb{C}$ on which the character

$$\begin{pmatrix} a & * \\ 0 & d \end{pmatrix} \mapsto \eta(a)\eta'(d)|a/d|_p^{1/2}$$

gives the action of the Borel subgroup of $GL_2(\mathbb{Q}_p)$ by left translation.

For P and λ as in subsection 4.2, let the characters $\eta_{P,p}, \eta'_{P,p} : \mathbb{Q}_p^\times \to \mathbb{C}^\times$ be so that for the automorphic representation $\pi(F(\lambda, P)) \cong \widehat{\otimes}_v \pi_v(F(\lambda, P))$ of $GL_2(\mathbb{A}_\mathbb{Q})$ corresponding to $F(\lambda, P)$ (see for example [10], [11], [14, section 9] and [20, p. 333]), we have that $\pi_p(F(\lambda, P))$ is equivalent to $\pi(\eta_{P,p}, \eta'_{P,p})$ or $\sigma(\eta_{P,p}, \eta'_{P,p})$.

6.3. p-adic L-functions

This subsection gives interpolation formulas for different p-adic L-functions. Fix an even positive integer k. In this subsection, the character $P_{ac} : \Gamma_K^- \to \overline{\mathbb{Q}}_p^\times$ and the anticyclotomic character $\psi : Z_K(\mathfrak{C})_{\text{tor}} \to \overline{\mathbb{Q}}_p^\times$ are allowed to vary so that, for the characters $P = P_{ac} \circ pr_{ac} : \Gamma_K \to \overline{\mathbb{Q}}_p^\times$ and $P\psi : Z_K(\mathfrak{C}) \to \overline{\mathbb{Q}}_p^\times$, there is a positive integer $n \geq k/2$, depending on P_{ac}, such that $(P\psi)^{alg}(z_\infty) = z_\infty^n \overline{z}_\infty^{-n}$. Define the variable $j = (2n-k)/2 \in \mathbb{Z}_{\geq 0}$. Each interpolation formula evaluates $P_{ac} : \widehat{O}_L^{ur}[[\Gamma_K^-]] \to \overline{\mathbb{Q}}_p$ at a p-adic L-function.

6.3.1. The Katz p-adic L-function of Hida and Tilouine

Let the Hecke character $\xi : K^\times \backslash \mathbb{A}_K^\times \to \mathbb{C}^\times$ with infinity type $z_\infty \mapsto z_\infty^{k+j} \overline{z}_\infty^{-j}$ be such that $\xi\xi^{-c}$ has p-adic avatar induced from $(P\psi) \circ sq$.

From a 1993 paper of Hida and Tilouine [12] (see also [37, Definition 7.8]), there is a Katz p-adic L-function $L_{Katz}^-(K) \in \widehat{O}_L^{ur}[[\Gamma_K^-]]$ satisfying the interpolation formula

$$P_{ac}(L_{Katz}^-(K)) = F_{93} C_{1,93} C_{2,93}^n \tag{9}$$

where the function F_{93} and the constants $C_{1,93}$, $C_{2,93}$ are given by

$$\begin{aligned} F_{93} = L(1, \xi\xi^{-c}) g_{v_0}((\xi^{-1}\xi^c)_{v_0}) \Gamma(2n+1) \\ \cdot (1 - p^{-1}\xi\xi^{-c}(\overline{v}_0))(1 - \xi\xi^{-c}(\overline{v}_0)) \prod_{v|\mathfrak{C}} (1 - (\mathrm{N}v)^{-1}\xi\xi^{-c}(v)), \end{aligned}$$

$$C_{1,93} = 2\mathrm{im}(\delta)/\pi, \; C_{2,93} = (\Omega_p/\Omega_\infty)^4 (\pi/\mathrm{im}(\delta))^2$$

for certain periods $\Omega_p \in \widehat{O}_L^{ur}$, $\Omega_\infty \in \mathbb{C}$ defined in [12], using subsection 6.2's δ.

6.3.2. The L-function of Hida

Consider Hida families

$$\lambda : h_{2n+1}(N|D_K|, \widehat{O}_L^{ur}) \to \widehat{O}_L^{ur}[[\Gamma_K]] \text{ and } \lambda' : h_k(N, \widehat{O}_L^{ur}) \to \widehat{O}_L^{ur}[[\Gamma_K]].$$

Fix Q so that $F(\lambda', Q) = f^c$, $\psi'_Q = 1$ and $k(Q) = k$. Let $P_1 = P \circ sq \circ pr_{v_0}$. Assume $\psi'_{P_1} = 1$ (this is an assumption about λ). Assume that each of $p^{\gamma(p)} = C(\eta_{Q,p})$, $p^{\gamma'(p)} = C(\eta'_{Q,p})$, $p^{\delta(p)} = C(\eta'_{P_1,p})$ has at least one factor of p. Assume $F(\lambda, P_1)$ is p-ordinary.

From a 1991 paper of Hida [11], there is a p-adic L-function $D_Q^- \in \widehat{O}_L^{ur}[[\Gamma_K^-]]$ satisfying the interpolation formula

$$P_{ac}(D_Q^-) = F_{91} C_{1,91} C_{2,91}^n \tag{10}$$

with

$$\begin{aligned} F_{91} = & \frac{1}{W'(F(\lambda, P_1))} \cdot \frac{g_p(\eta_{Q,p}) g_p(\eta'_{Q,p})}{g_p(\eta'_{P_1,p})} \\ & \cdot \frac{\Gamma(n + \frac{1}{2}k)\Gamma(n - \frac{1}{2}k + 1) p^{\delta(p)}}{\eta_{P_1,p}(p^{\gamma(p)+\gamma'(p)}) \eta'_{P_1,p} \eta_{P_1,p}^{-1}(p^{\delta(p)})} \cdot \frac{L^p(0, F(\lambda, P_1) \times f)}{\langle F(\lambda, P_1), F(\lambda, P_1) \rangle_{\Gamma_0(N|D_K|)}}, \\ C_{1,91} = & \, \eta_{Q,p}(p^{\gamma(p)+\gamma'(p)}) \cdot \psi_Q \psi'_Q(-1) W'(f^c) \sqrt{N|D_K|}/(2\pi), \\ C_{2,91} = & \, 1/(16\pi^2 |D_K|) \end{aligned}$$

where for primitive cusp forms $f_1 \in S_{k'}(\Gamma_1(M))$ of level M, the number $W'(f_1)$ is described in Hida [11, pp. 344-345] as part of a decomposition $W(f_1) = W'(f_1) W_p(f_1)$ of the W factor $W(f_1) \in \mathbb{C}$ with $|W(f_1)| = 1$ such that

$$M^{k'/2-1} f_1|_{k'} \begin{pmatrix} 0 & -1 \\ M & 0 \end{pmatrix} = W(f_1) f_1^c;$$

see also [5, exercises 1.5.4, 5.5.1 and section 5.10] and [11, pp. 344-345] (where formula (4.10b) should have no minus sign).

6.3.3. The L-function of Wan

For this subsubsection, let $I = \widehat{O}_L^{ur}[[\Gamma_K]]$. Fix an irreducible component of $I \widehat{\otimes}_{O_L} \widehat{O}_L^{ur}$, and let that component's associated ring have normalization $\widehat{I}^{ur}$ (see [37, before Theorem 1.1]).

For a set S of finitely many places of K including v_0 and $\overline{v}_0$, Wan [37, section 7.5] defines two related p-adic L-functions in $\widehat{I}^{ur}[[\Gamma_K]]$ which we call $\widetilde{L}_{Wan}^S$ and L_{Wan}^S (Wan calls them $\mathcal{L}_{\mathbf{f},\xi,\mathcal{K}}^{S,Hida}$ and $\mathcal{L}_{\mathbf{f},\xi,\mathcal{K}}^S$ respectively). For the Hida family $Q : I \to \overline{\mathbb{Q}}_p$ corresponding to f, write $\widetilde{L}_{Wan}^S(f) = Q(\widetilde{L}_{Wan}^S)$ and $L_{Wan}^S(f) = Q(L_{Wan}^S)$; let their images under pr_{ac} be $\widetilde{L}_{Wan}^{-,S}(f)$ and $L_{Wan}^{-,S}(f)$ respectively.

We have $\widetilde{L}_{Wan}^{-,N_p}(f) = D_Q^- L_{Katz}^-(K)$, and $L_{Wan}^{-,N_p}(f) = C_{Wan} \widetilde{L}_{Wan}^{-,N_p}(f)$ for a constant $C_{Wan} \in O_{\overline{\mathbb{Q}}_p}$ which Wan calls $C_{\mathbf{f},K,\xi}$. Starting with $\widetilde{L}_{Wan}^{N_p}$ or $L_{Wan}^{N_p}$ and omitting Euler factors at primes over $S \backslash N_p$ yields $\widetilde{L}_{Wan}^S$ or L_{Wan}^S respectively.

6.3.4. The BDP L-function

Let the Hecke character $\chi : K^\times \backslash \mathbb{A}_K^\times \to \mathbb{C}^\times$ be so that $P\psi$ is the p-adic avatar of $\chi^{-1}\mathrm{N}^{k/2}$. Let $\mathfrak{b}$ be an ideal of O_K coprime to Np, and let b_N be a number in O_K, such that $\mathfrak{b}\mathfrak{C} = b_N O_K$.

From a 2013 paper of Bertolini, Darmon and Prasanna [1], there is a p-adic L-function $L_{BDP}^{-,N_p}(f) \in \widehat{O}_L^{ur}[[\Gamma_K^-]]$ satisfying the interpolation formula

$$P_{ac}(L_{BDP}^{-,N_p}(f)) = F_{13} C_{1,13} C_{2,13}^n \tag{11}$$

with

$$F_{13} = \Gamma\left(n + \frac{1}{2}k\right)\Gamma\left(n - \frac{1}{2}k + 1\right)(P\psi)^{alg}(\mathfrak{b})\psi_{(O_K/\mathfrak{C})^\times}(N_{K/\mathbb{Q}}\mathfrak{b})$$
$$\cdot (1 - \chi^{-1}(\overline{v}_0)a(p,f) + \chi^{-2}(\overline{v}_0)p^{k-1})^2 \cdot L(0, \theta_{\chi^{-1}} \times f),$$

$$C_{1,13} = \frac{2\sqrt{|D_K|}}{4\pi(-1)^{k/2}} W(F(\lambda', Q)),\ C_{2,13} = \frac{\Omega_p^4 \pi^2}{\Omega_\infty^4} \cdot \frac{b_N^2 |\mathfrak{b}|_{\mathbb{A},K}}{-N}$$

where $\psi_{(O_K/\mathfrak{C})^\times}$ is obtained by precomposing ψ with the projection from $Z_K(\mathfrak{C})_{\mathrm{tor}}$ to its $(O_K/\mathfrak{C})^\times/\{\pm 1\}$ part, using the decomposition (2), and the periods Ω_p, Ω_∞ are as before, following the argument of [15, section 5.2].

As in [15, section 5.1], for a set S of finitely many places of K such that $S \supseteq N_p$, define

$$L_{BDP}^{-,S}(f) = L_{BDP}^{-,N_p}(f) \cdot \prod_{v \in S \setminus N_p} L_v(f)$$

where $L_v(f)$ is the Euler factor at v.

6.4. Comparison and missing factor

The line of argument of [15] is followed and adapted. The arithmetic $\widehat{O}_L^{ur}$-algebra map $P_{ac} : \widehat{O}_L^{ur}[[\Gamma_K^-]] \to \widehat{O}_L^{ur}$ and the character χ are as before.

From subsection 6.3's interpolation formulas (9), (10) and (11), we obtain

$$P_{ac}(\widetilde{L}_{Wan}^{-,N_p}(f)) = P_{ac}(D_Q^- L_{Katz}^-(K)) = C(f, P_{ac}) P_{ac}(L_{BDP}^{-,N_p}(f))$$

writing $C(f, P_{ac}) = \widetilde{F}\widetilde{C}_1\widetilde{C}_2^n$ where we define

$$\widetilde{F} := \frac{F_{91}F_{93}}{F_{13}},\ \widetilde{C}_1 := \frac{C_{1,91}C_{1,93}}{C_{1,13}},\ \widetilde{C}_2 := \frac{C_{2,91}C_{2,93}}{C_{2,13}}.$$

As in the argument of [15, section 5.2], $\widetilde{F}$ is a constant times the nth power of a constant. (We have $U\theta_{\chi^{-1}} = F(\lambda, P_1)$ and

$$\frac{L(1, \xi\xi^{-c})\Gamma(2n+1)}{\langle U\theta_{\chi^{-1}}, U\theta_{\chi^{-1}}\rangle_{\Gamma_0(N|D_K|)}} = (\text{constant})(16\pi^2)^n$$

by equation (8) and the fact that

$$\mathrm{res}_{s=2n+1} D(s, U\theta_{\chi^{-1}}, U\theta_{\chi^{-1}}^c) = (\text{constant}) L(1, \xi\xi^{-c})$$

holds.) So we can write $C(f, P_{ac}) = C_1 C_2^n$ for constants C_1, C_2.

6.5. Interpolation

Lemma 6.1. *There is a constant $C_1 \in \widehat{O}_L^{ur}[1/p]^\times$ and a p-adic unit $u \in \widehat{O}_L^{ur}[[\Gamma_K^-]]^\times$ such that for all P_{ac} with $\phi(pN) = (p-1)\phi(N) \mid n$, we have $P_{ac}(C_1 u) = C(f, P_{ac})$.*

Proof. Let C_1 be as in subsection 6.4, and let u be such that $P_{ac}(u) = C_2^n$ identically; this is possible since $\psi^{\phi(pN)} = 1$ and the infinity type exponents of $P\psi$ are $\pm n$. □

The maps P_{ac} as in Lemma 6.1 are dense in Spec $\widehat{O}_L^{ur}[[\Gamma_K^-]]$, so $\widetilde{L}_{Wan}^{-,N_p}(f) = D_Q^- L_{Katz}^-(K) = C_1 u L_{BDP}^{-,N_p}(f)$. So we have shown the following theorem.

Theorem 6.2. *In $\widehat{O}_L^{ur}[[\Gamma_K^-]] \otimes_{O_L} L = \widehat{O}_L^{ur}[[\Gamma_K^-]][1/p]$, we have*

$$(L_{BDP}^{-,N_p}(f)) = (\widetilde{L}_{Wan}^{-,N_p}(f)) \supseteq (L_{Wan}^{-,N_p}(f)).$$

7. From Wan's *L*-function to cohomology: Iwasawa theory

This section collects progress in one direction of an Iwasawa main conjecture and, as a consequence, links Wan's L-function to the cohomology of the $(\mathbb{Q}_p/\mathbb{Z}_p)$-representation W_f of the modular form f.

7.1. Notation

We use and adapt notation of [11, 35, 37].

As in subsection 4.2, take a cuspidal Hida family $\lambda : h_{k,w}(N, O_L) \to I$, with I a finite-rank $\mathbb{Z}_p[[t]]$-module and an integrally closed domain, and let the continuous $\mathbb{Z}_p$-algebra map $Q : I \to \overline{\mathbb{Q}}_p$ correspond to f via λ, with $Q[I] = O_L$.

Choose an irreducible component of $I \widehat{\otimes}_{O_L} \widehat{O}_L^{ur}$, and let the normalization of that component's associated ring be $\widehat{I}^{ur}$ (see [37, before Theorem 1.1]).

Let T_λ be the Galois representation coming from λ. (In [35, section 3.3.10], T_λ is denoted by $\rho_{\mathbf{f}}$; Hecke duality identifies that source's $\mathbf{f}$ with our λ.) We have $T_\lambda \cong I^2$ and $T_f^g \cong O_L^2$; for a sufficiently large $L/\mathbb{Q}_p$, we have $T_\lambda \otimes_I O_L \cong T_f^g$.

Let $\Psi_K : G_K \twoheadrightarrow \Gamma_K \subseteq O_L[[\Gamma_K]]^\times$, $\Psi_- : G_K \twoheadrightarrow \Gamma_K^- \subseteq O_L[[\Gamma_K^-]]^\times$ be the standard projections. Write $(\cdot)^* = \mathrm{Hom}_{O_L}(\cdot, L/O_L)$ for the Pontryagin dual. The module $O_L[[\Gamma_K]]$ acts on $O_L[[\Gamma_K]]^*$ so that $(xF)(y) = F(yx)$ for $x, y \in \Lambda_K$ and $F \in O_L[[\Gamma_K]]^*$. Define the modules

$$\begin{aligned} T_{\lambda,K,\xi} &= T_\lambda \sigma_{\xi^{-c}}(\epsilon^{2-(\kappa/2)}) \otimes_{I[[\Gamma_K]]} I[[\Gamma_K]](\Psi_K^{-c}) \\ T_{f,K,\xi} &= T_f^g \sigma_{\xi^{-c}}(\epsilon^{2-(\kappa/2)}) \otimes_{O_L[[\Gamma_K]]} O_L[[\Gamma_K]](\Psi_K^{-c}). \end{aligned}$$

For any finite set S of finite places of K, such that S includes v_0, $\overline{v}_0$ and all places at which V_f ramifies, define the modules

$$\begin{aligned} Sel^S_{\lambda,K,\xi} &= \{c \in H^1(K, T_{\lambda,K,\xi} \otimes I[[\Gamma_K]]^*) : c \text{ unramified at } \overline{v}_0 \text{ and outside } S\} \\ Sel^S_{f,K,\xi} &= \{c \in H^1(K, T_{f,K,\xi} \otimes O_L[[\Gamma_K]]^*) : c \text{ unramified at } \overline{v}_0 \text{ and outside } S\} \\ X^S_{\lambda,K,\xi} &= (Sel^S_{\lambda,K,\xi})^* \\ X^S_{f,K,\xi} &= (Sel^S_{f,K,\xi})^* \\ \widehat{X}^S_{\lambda,K,\xi} &= X^S_{\lambda,K,\xi} \otimes_{I[[\Gamma_K]]} \widehat{I}^{ur}[[\Gamma_K]] \\ \widehat{X}^S_{f,K,\xi} &= X^S_{f,K,\xi} \otimes_{O_L[[\Gamma_K]]} \widehat{O}^{ur}_L[[\Gamma_K]]. \end{aligned}$$

7.2. Main conjecture for Hida families

Wan proved the following main conjecture (see [37]; the result is in the final proof of that source's Theorem 1.2):

Theorem 7.1 (Main conjecture)**.** *Assume some nebentypus-1 weight-2 specialization f_0 of a Hida family λ satisfies:*

- *(i) f_0 is the ordinary stabilization of a newform of level divisible by some odd prime q not split in K.*
- *(ii) The Galois representation $T^g_{f_0}$ has irreducible residual representation $\overline{T^g_{f_0}}|G_K$, and $\overline{T^g_{f_0}}$ is ramified at q.*

Suppose the Hecke character $\xi : K^\times\backslash\mathbb{A}^\times_K \to \mathbb{C}$ is of infinity type $z_\infty \mapsto z^u_\infty \overline{z}^{-u}_\infty$ for some positive integer u divisible by $p-1$, and is such that the p-adic avatar of $\xi|\cdot|^u_K(\omega_{v_0} \cdot \omega_{\overline{v}_0})$ factors through Γ_K.

Let S be a set of finitely many places of K, including all places dividing pND_K.

Then, letting $P_1, \ldots, P_t$ be the height 1 primes in $\widehat{I}^{ur}[[\Gamma_K]]$ dividing L^S_{Wan} that are pullbacks of height 1 primes in $\widehat{I}^{ur}$, we have

$$L^S_{Wan}\widehat{I}^{ur}[[\Gamma_K]]_{p,P_1,\ldots,P_t} \supseteq \mathrm{Fitt}_{\widehat{I}^{ur}[[\Gamma_K]]_{p,P_1,\ldots,P_t}} \widehat{X}^S_{\lambda,K,\xi}$$

in which the notation $\widehat{I}^{ur}[[\Gamma_K]]_{p,P_1,\ldots,P_t}$ indicates localization with respect to the primes P_i and p as in [37].

7.3. From Hida families to modular forms

We follow the argument in the proof of [37, Theorem 1.2]. In this subsection, S is a set of finitely many places of K including all places over pND_K.

Recall $L^S_{Wan}(f) = Q(L^S_{Wan}) \in \widehat{O}^{ur}_L[[\Gamma_K]] \otimes_{O_L} L$ for Q corresponding to f via λ. Applying Q to Theorem 7.1's result, and noting $\mathrm{Fitt}_{R/I}(M/IM) = (\mathrm{Fitt}_R M)(R/I)$ for an R-module M and ideal I in a noetherian ring R (see [6, Corollary 20.5], [35, section 3.1.5] and [37, section 2.2]), we obtain

$$L^S_{Wan}(f)(\widehat{O}^{ur}_L[[\Gamma_K]] \otimes_{O_L} L) \supseteq \mathrm{Fitt}_{\widehat{O}^{ur}_L[[\Gamma_K]]\otimes_{O_L} L}\left(\frac{\widehat{X}^S_{\lambda,K,\xi} \otimes_{O_L} L}{(\ker Q)(\widehat{X}^S_{\lambda,K,\xi} \otimes_{O_L} L)}\right)$$

and therefore

$$L^S_{Wan}(f)(\widehat{O}^{ur}_L[[\Gamma_K]] \otimes_{O_L} L) \supseteq \mathrm{char}_{\widehat{O}^{ur}_L[[\Gamma_K]]\otimes_{O_L} L}\left(\frac{\widehat{X}^S_{\lambda,K,\xi} \otimes_{O_L} L}{(\ker Q)(\widehat{X}^S_{\lambda,K,\xi} \otimes_{O_L} L)}\right)$$

because the characteristic ideal is the minimum principal ideal containing the Fitting ideal (see the last sentence in the proof of [15, Corollary 3.4.2]).

Now Wan [37, Proposition 2.4] proved an $O_L[[\Gamma_K]]$-module version of the following result for f of weight 2; that argument carries through for higher weight to give:

Theorem 7.2. *There is an $\widehat{O}^{ur}_L[[\Gamma_K]]$-module exact sequence*

$$M \to \widehat{X}^S_{\lambda,K,\xi}/(\ker Q)\widehat{X}^S_{\lambda,K,\xi} \to \widehat{X}^S_{f,K,\xi} \to 0$$

where $M \otimes_{O_L} L$ has annihilator of codimension ≥ 2 in Spec $\widehat{O}^{ur}_L[[\Gamma_K]] \otimes L$*, i.e., is pseudo-null. In $\widehat{O}^{ur}_L[[\Gamma_K]] \otimes_{O_L} L$, this implies*

$$\mathrm{char}_{\widehat{O}^{ur}_L[[\Gamma_K]]\otimes_{O_L} L}\left(\frac{\widehat{X}^S_{\lambda,K,\xi} \otimes_{O_L} L}{(\ker Q)(\widehat{X}^S_{\lambda,K,\xi} \otimes_{O_L} L)}\right) = \mathrm{char}_{\widehat{O}^{ur}_L[[\Gamma_K]]\otimes_{O_L} L}(\widehat{X}^S_{f,K,\xi} \otimes_{O_L} L)$$

so

$$L^S_{Wan}(f)(\widehat{O}^{ur}_L[[\Gamma_K]] \otimes_{O_L} L) \supseteq \mathrm{char}_{\widehat{O}^{ur}_L[[\Gamma_K]]\otimes_{O_L} L}(\widehat{X}^S_{f,K,\xi} \otimes_{O_L} L).$$

7.4. From Greenberg to anticyclotomic: characteristic ideals

In this subsection, the set S is as before. The following arguments are adapted from [15, section 3.4] (that source's Σ is our $S\backslash N_p$).

Define $M = T_{f,K,\xi} \otimes_{O_L} O_L[[\Gamma^-_K]]^*$ and $\mathcal{M} = T_{f,K,\xi} \otimes_{O_L} O_L[[\Gamma_K]]^*$ analogously to [15]. For $\widetilde{M} \in \{M, \mathcal{M}\}$ and $\bullet \in \{\text{ac,Gr}\}$:

(a) Identify $H^1(K_S/K, \widetilde{M})$ with the space of classes in $H^1(K, \widetilde{M})$ unramified at all primes outside S (see [30, Lemma 1.5.3]).

(b) Let $H^1_\bullet(K, \widetilde{M}) \subseteq H^1(K_S/K, \widetilde{M})$ be the space of classes in $H^1(K, \widetilde{M})$ satisfying the following conditions:

  - If $\bullet =$ ac ("ac" is for "anticyclotomic"): no condition at v_0, unramified at finite primes outside N_p splitting in K, 0 at all other primes.
  - If $\bullet =$ Gr ("Gr" is for "Greenberg"): no condition at v_0, unramified at all other primes.

  Let $H^1_{\bullet,S\backslash N_p}(K, \widetilde{M}) \subseteq H^1(K_S/K, \widetilde{M})$ be the space of classes in $H^1(K, \widetilde{M})$ satisfying the above conditions at primes outside $S\backslash N_p$ (but not necessarily at primes in $S\backslash N_p$: the conditions are relaxed at these primes).

(c) Define

$$\begin{aligned} X_{\bullet,S\setminus N_p}(\widetilde{M}) &= (H^1_{\bullet,S\setminus N_p}(K,\widetilde{M}))^* \\ \widehat{X}_{\bullet,S\setminus N_p}(M) &= X_{\bullet,S\setminus N_p}(M)\otimes_{O_L[[\Gamma_K^-]]}\widehat{O}_L^{ur}[[\Gamma_K^-]] \\ \widehat{X}_{\bullet,S\setminus N_p}(\mathcal{M}) &= X_{\bullet,S\setminus N_p}(\mathcal{M})\otimes_{O_L[[\Gamma_K]]}\widehat{O}_L^{ur}[[\Gamma_K]] \end{aligned}$$

Note that $Sel^S_{f,K,\xi} = H^1_{Gr,S\setminus N_p}(K,\mathcal{M})$ and $X^S_{f,K,\xi} = X_{Gr,S\setminus N_p}(\mathcal{M})$.

The argument of [15, section 3.4] goes through, yielding

Theorem 7.3.

$$\frac{(\gamma_+ - 1)\widehat{O}_L^{ur}[[\Gamma_K]] + \mathrm{char}_{\widehat{O}_L^{ur}[[\Gamma_K]]}\widehat{X}^S_{f,K,\xi}}{(\gamma_+ - 1)\widehat{O}_L^{ur}[[\Gamma_K]]} \supseteq \mathrm{char}_{\widehat{O}_L^{ur}[[\Gamma_K^-]]}\widehat{X}_{ac,S\setminus N_p}(M).$$

7.5. Half of an Iwasawa main conjecture

From Theorems 7.2 and 7.3, we have the following over $\widehat{O}_L^{ur}[[\Gamma_K^-]][1/p]$.

Theorem 7.4. *For S including all places of K dividing pND_K, we have*

$$L^{-,S}_{Wan}(f)\cdot\widehat{O}_L^{ur}[[\Gamma_K^-]][1/p] \supseteq \mathrm{char}_{\widehat{O}_L^{ur}[[\Gamma_K^-]][1/p]}(\widehat{X}_{ac,S\setminus N_p}(M)).$$

There is a $\mu = 0$ result for $L^{-,N_p}_{BDP}(f)$ due to Hsieh ([13, Theorem B]; see also the Remark on the previous page in that source). Recall the isomorphism $\widehat{O}_L^{ur}[[\Gamma_K^-]] \cong \widehat{O}_L^{ur}[[t]]$ sending γ_- to $1+t$.

Theorem 7.5 (Hsieh's $\mu = 0$)**.** *As in the Weierstrass preparation theorem, factor the p-adic L-function $L^{-,N_p}_{BDP}(f)$ as $L^{-,N_p}_{BDP}(f) = p^\mu R(t)U(t)$, where $\mu\in\mathbb{Q}$, $U(t)\in\widehat{O}_L^{ur}[[t]]^\times$ and the monic distinguished polynomial $R(t)\in\widehat{O}_L^{ur}[t]$ is chosen so that $\deg R$ is minimized. Then $\mu = 0$.*

Using that theorem, the reasoning of [15, Theorem 6.1.6] goes through to prove half of an Iwasawa main conjecture:

Theorem 7.6. *For any set S of finitely many places of K containing N_p (possibly $S = N_p$), we have*

$$L^{-,S}_{Wan}(f)\cdot\widehat{O}_L^{ur}[[\Gamma_K^-]] \supseteq \mathrm{char}_{\widehat{O}_L^{ur}[[\Gamma_K^-]]}(\widehat{X}_{ac,S\setminus N_p}(M)).$$

7.6. Consequences

Let the continuous $\widehat{O}_L^{ur}$-algebra map $P_1 : \widehat{O}_L^{ur}[[\Gamma_K^-]] \to \overline{\mathbb{Q}}_p$ send each element of Γ_K^- to 1; under the identification $\widehat{O}_L^{ur}[[\Gamma_K^-]] \cong \widehat{O}_L^{ur}[[t]]$ with $\gamma_- \mapsto 1+t$, the map P_1 has the effect of substituting $t = 0$.

Define

$$C(W) := \#H^0(K_{v_0},W)\cdot\#H^0(K_{\overline{v}_0},W)\cdot\textstyle\prod_{v\in S'}\#H^1_{ur}(K_v,W)$$

where S' is the set of finite places v of K such that $v\nmid p$, V_f is ramified at v, and v is above a rational prime that splits in K.

The argument of [15, section 6.2] finally yields

Theorem 7.7. *We have*

$$\mathrm{ind}_p P_1(L^{-,N_p}_{Wan}(f)) \le \mathrm{ind}_p(C(W_f)\#H^1_{ac}(K,W_f)).$$

8. From cohomology to Ш

In this section, we adapt an argument of Jetchev, Skinner and Wan [15, section 3.5] to relate $\#H^1_{ac}(K, W_f)$ and $\#Ш(K, W_f)$.

8.1. Main formula

Theorem 8.1. *Suppose the following hold.*

(i) *Congruence: $k/2$ is not congruent to 0 or 1 modulo $p-1$.*

(ii) *Rank 1: The O_L-module* im AJ_K *has rank 1.*

(iii) *Finiteness of Sha:* $Ш(K, W_f)[p^\infty]$ *has finite cardinality as a set.*

(iv) *Localization: For each place $v \mid p$ of K, the localization map $H^1_f(K, W_f) \to H^1_f(K_v, W_f)$ restricts to a map*

$$(\text{im } AJ_K) \otimes_{O_L} (L/O_L) \to (\text{im } AJ_{K_v}) \otimes_{O_L} (L/O_L)$$

of which the kernel is torsion.

(v) *Local corank 1: For each place $v \mid p$ of K, the O_L-module $H^1_f(K_v, W_f)$ has corank 1.*

Define δ_{v_0} to be the cokernel of the localization map

$$\text{loc}_{v_0}/\text{tor} : H^1_f(K, T_f) \to H^1_f(K_{v_0}, T_f)/H^1_f(K_{v_0}, T_f)_{\text{tor}}.$$

Then

$$\#H^1_{ac}(K, W_f) = \#Ш(K, W_f) \cdot (\#\delta_{v_0})^2 \tag{12}$$

and $H^1_f(K, T_f) \cong O_L$.

Proof. We show that for $(T, V, W) = (T_f, V_f, W_f)$, the hypotheses of [15, Proposition 3.2.1] are true, which yields (12); we also show $Ш(K, W_f) = Ш_f(K, W_f)$ and then prove $H^1_f(K, T_f) \cong O_L$.

As in [15, section 3.5] (noting assumption (i), $V_f^c \cong V_f \cong \text{Hom}_L(V_f, L(\epsilon))$, $p \nmid N$ and that the G_K-representation $T_f/m_L T_f$ is irreducible), to apply [15, Proposition 3.2.1], it is enough to show the following two hypotheses of [15] for $W = W_f$: (corank 1) the O_L-modules $H^1_f(K, W)_{\text{div}}$, $H^1_f(K_{v_0}, W)$, $H^1_f(K_{\overline{v}_0}, W)$ have corank 1, and (sur) the localization maps $H^1_f(K, W)_{\text{div}} \to H^1_f(K_{v_0}, W)$ and $H^1_f(K, W)_{\text{div}} \to H^1_f(K_{\overline{v}_0}, W)$ are surjections.

In the short exact sequence

$$(\text{im } AJ_K) \otimes_{O_L} (L/O_L) \hookrightarrow H^1_f(K, W_f) \twoheadrightarrow Ш(K, W_f)[p^\infty]$$

of O_L-modules, the first term has corank 1 because im AJ_K has rank 1 (assumption (ii)), and the third term has corank 0 (assumption (iii)), so $H^1_f(K, W_f)$ and $H^1_f(K, W_f)_{\text{div}}$ have corank 1. So by assumption (v), (corank 1) holds for $W = W_f$.

Let $v \mid p$ be a place of K. The O_L-module $(\text{im } AJ_K) \otimes_{O_L} (L/O_L)$ has corank 1, and it is isomorphic as an O_L-module to L/O_L. By assumption (iv), $(\text{im } AJ_K) \otimes_{O_L} (L/O_L)$ is sent by the localization map to $(\text{im } AJ_{K_v}) \otimes_{O_L} (L/O_L)$ with torsion kernel, so $(\text{im } AJ_{K_v}) \otimes_{O_L} (L/O_L)$ has corank at least 1. But $(\text{im } AJ_{K_v}) \otimes_{O_L} (L/O_L)$ is an O_L-submodule of $H^1_f(K_v, W_f)$, which has corank 1 (assumption (v)). So as O_L-modules, $(\text{im } AJ_{K_v}) \otimes_{O_L} (L/O_L) = H^1_f(K_v, W_f) \cong L/O_L$, and each class in $H^1_f(K_v, W_f)$ is the image of some class in $(\text{im } AJ_K) \otimes_{O_L} (L/O_L) \subseteq H^1_f(K, W_f)_{\text{div}}$. This implies (sur) for $W = W_f$.

So [15, Proposition 3.2.1] applies, yielding (12).

There are quotient maps

$$\begin{array}{ccc} H^1_f(K, W_f) & & \\ \downarrow & & \\ Ш(K, W_f) & = & H^1_f(K, W_f)/((\text{im } AJ_K) \otimes_{O_L} (L/O_L)) \\ \downarrow & & \\ Ш_f(K, W_f) & = & H^1_f(K, W_f)/H^1_f(K, W_f)_{\text{div}}. \end{array}$$

Since $(\text{im } AJ_K) \otimes_{O_L} (L/O_L)$ is divisible, it is the maximal p-divisible subgroup of $H^1_f(K, W_f)$ (because $H^1_f(K, W_f)$ has corank 1), so $Ш(K, W_f) = Ш_f(K, W_f)$.

For a uniformizer $\varpi_L \in m_L$ of L, taking the long exact G_K-cohomology of the short exact sequence

$$T_f \overset{\varpi_L}{\hookrightarrow} T_f \twoheadrightarrow T_f/\varpi_L T_f$$

implies that the sequence

$$(T_f/\varpi_L T_f)^{G_K} \to H^1(K, T_f) \overset{\varpi_L}{\to} H^1(K, T_f)$$

is exact. The left term is 0: it is an O_L-submodule of the G_K-module $T_f/m_L T_f$, and $T_f/m_L T_f$ is irreducible and not 1-dimensional. So $\varpi_L : H^1(K, T_f) \to H^1(K, T_f)$ is injective and $H^1(K, T_f)$ is torsion free. Now $H^1_f(K, T_f)$ is finitely generated as an O_L-module, and

$$\text{rank}_{O_L} H^1_f(K, T_f) = \dim_L H^1_f(K, V_f) = \text{corank}_{O_L} H^1_f(K, W_f) = 1 \qquad (13)$$

so $H^1_f(K, T_f) \cong O_L$. (Proof of (13): We have $H^1(G_K, T_f) \otimes_{O_L} L \cong H^1(G_K, V_f)$. There is no divisible part in the cokernel of $H^1(K, V_f) \to H^1(K, W_f)$, since that cokernel is the image of the connecting map $H^1(K, W_f) \to H^2(K, T_f)$, which is the torsion subgroup of $H^2(K, T_f)$, and $H^2(K, T_f)$ is a finitely generated O_L-module.) □

8.2. Finding $\#\delta_{v_0}$

We now adapt [15, section 3.5] to find a formula for $\#\delta_{v_0}$.

In this subsection, assume that $H^1_f(K, T_f) \cong O_L$ as O_L-modules and that $H^1_f(K_{v_0}, T_f)/\text{tor} \cong O_L$ is a torsion-free rank-1 O_L-module (both of which are implied by the hypotheses of Proposition 8.1).

Define $C_0 = \mathrm{cor}_{K_1/K} AJ_{K_1}(\Delta_{Ma19}) \in H^1_f(K, T_f)$. In this subsection, assume that the image of C_0 in $H^1_f(K_{v_0}, T_f)$ is not torsion; then $\mathrm{loc}_{v_0}/\mathrm{tor} : O_L \cong H^1_f(K, T_f) \to H^1_f(K_{v_0}, T_f)/\mathrm{tor} \cong O_L$ is injective.

Recall the map $\log_\omega$ for differential forms $\omega \in \tilde{\tilde{V}}$ from subsection 5.8.

Theorem 8.2. *Choose the differential form $\omega \in \tilde{\tilde{V}}$ so that $\log_\omega$ restricts to an isomorphism $\log_\omega : H^1_f(K_{v_0}, T_f)/\mathrm{tor} \xrightarrow{\cong} O_L$ with inverse $\exp_\omega$. (Interpret $H^1_f(K_{v_0}, T_f)/\mathrm{tor}$ as a subgroup of $H^1_f(K_{v_0}, V_f) \cong (H^1_f(K_{v_0}, T_f)/\mathrm{tor}) \otimes_{O_L} L$.) Then*

$$\#\delta_{v_0} = \frac{(O_L : O_L \log_\omega(\mathrm{loc}_{v_0} C_0)) \cdot (H^1_f(K_{v_0}, T_f) : \exp_\omega(pO_L))}{p^{[L:\mathbb{Q}_p]} \cdot \#H^0(K_{v_0}, W_f) \cdot (H^1_f(K, T_f) : O_L C_0)}. \tag{14}$$

Proof. We argue as in [15], replacing that source's A_f, $\#A_f[\mathfrak{p}^\infty](\mathbb{F}_p)$, ω_f, P with $H^1_f(\cdot, T_f)$, $(H^1_f(K_{v_0}, T_f) : \exp_\omega(pO_L))$, ω, C_0 respectively.

The rank-1 O_L-modules

$$H^1_f(K_{v_0}, T_f)/\mathrm{tor} \supseteq (\mathrm{loc}_{v_0}/\mathrm{tor}) H^1_f(K, T_f) \supseteq O_L \mathrm{loc}_{v_0} C_0$$

are of finite index in one another, so we have

$$\begin{aligned}
\#\delta_{v_0} &= (H^1_f(K_{v_0}, T_f)/\mathrm{tor} : (\mathrm{loc}_{v_0}/\mathrm{tor}) H^1_f(K, T_f)) \\
&= (H^1_f(K_{v_0}, T_f)/\mathrm{tor} : O_L \mathrm{loc}_{v_0} C_0) \ / \ ((\mathrm{loc}_{v_0}/\mathrm{tor}) H^1_f(K, T_f) : O_L \mathrm{loc}_{v_0} C_0) \\
&= (H^1_f(K_{v_0}, T_f)/\mathrm{tor} : O_L \mathrm{loc}_{v_0} C_0) \ / \ (H^1_f(K, T_f) : O_L C_0) \qquad (15)
\end{aligned}$$

because $\mathrm{loc}_{v_0}/\mathrm{tor}$ is injective.

Since $\log_\omega : H^1_f(K_{v_0}, T_f)/\mathrm{tor} \to O_L$ is an isomorphism, the numerator in the last fraction in (15) is

$$\begin{aligned}
&(H^1_f(K_{v_0}, T_f)/\mathrm{tor} : O_L \mathrm{loc}_{v_0} C_0) \\
&\quad = (\log_\omega(H^1_f(K_{v_0}, T_f)/\mathrm{tor}) : \log_\omega(O_L \mathrm{loc}_{v_0} C_0)) \\
&\quad = (O_L : O_L \log_\omega(\mathrm{loc}_{v_0} C_0)) \ / \ (O_L : \log_\omega(H^1_f(K_{v_0}, T_f)/\mathrm{tor})). \qquad (16)
\end{aligned}$$

Finally, the denominator in the last fraction in (16) is

$$\begin{aligned}
&(O_L : \log_\omega(H^1_f(K_{v_0}, T_f)/\mathrm{tor})) \\
&\quad = (O_L : pO_L) \ / \ (\log_\omega(H^1_f(K_{v_0}, T_f)/\mathrm{tor}) : pO_L) \\
&\quad = (O_L : pO_L) \ / \ (H^1_f(K_{v_0}, T_f)/\mathrm{tor} : \exp_\omega(pO_L)) \\
&\quad = (O_L : pO_L) \cdot \#H^1_f(K_{v_0}, T_f)_{\mathrm{tor}} \ / \ (H^1_f(K_{v_0}, T_f) : \exp_\omega(pO_L)) \\
&\quad = (O_L : pO_L) \cdot \#H^0(K_{v_0}, W_f) \ / \ (H^1_f(K_{v_0}, T_f) : \exp_\omega(pO_L)). \qquad (17)
\end{aligned}$$

For the last equation, note that $H^1_f(K_{v_0}, T_f)_{\mathrm{tor}} = H^1(K_{v_0}, T_f)_{\mathrm{tor}}$ which is isomorphic to the image of the connecting map $H^0(K_{v_0}, W_f) \to H^1(K_{v_0}, T_f)$; but this image is isomorphic to $H^0(K_{v_0}, W_f)$ since $V_f^{G_{K_{v_0}}} = 0$.

Noting that $(O_L : pO_L) = p^{[L:\mathbb{Q}_p]}$ (because p does not ramify in $L/\mathbb{Q}_p$) and combining (15), (16) and (17) yields (14). □

9. From Ш to Heegner cycles: an Euler system result

9.1. Masoero's theorem

In this subsection, we link the order of Ш to subsection 5.2's Heegner cycles Δ_{Ma19}, Z_{Ma19} by describing and slightly adapting a result of Masoero [21], which that paper proved by adapting arguments of Kolyvagin [18, 19] (as reorganized by McCallum [22]) about Euler systems and Shafarevich-Tate groups.

Theorem 9.1. *([21, Theorem 7.3 and next sentence, Corollary 7.11]; see also [23, Theorem 13.1]). In addition to subsection 2.2's hypotheses, assume that:*

(i) The cohomology class $C_0 = \mathrm{cor}_{K_1/K} AJ_{K_1}(\Delta_{Ma19}) \in H^1(K, T_f)$ *is not torsion.*

(ii) If $g \in GL_2(O_{\mathbb{Q}(f)} \otimes_{\mathbb{Z}} \mathbb{Z}_p)$ *and* $\det g$ *is a* $(k-1)$*th power in* $\mathbb{Z}_p^\times$*, then* g *is in the image of the representation* $\rho_{f,p} : G_{\mathbb{Q}} \to GL_2(O_{\mathbb{Q}(f)} \otimes_{\mathbb{Z}} \mathbb{Z}_p)$ *corresponding to* T_p*.*

Then

$$(\mathrm{im}\ AJ_K) \otimes \mathbb{Q} = L \cdot C_0 \subseteq H^1_f(K, V_f),$$

the group $Ш(K, W_f)$ *has finite cardinality, and*

$$\mathrm{ind}_p \# Ш(K, W_f) \leq 2\mathrm{ind}_p(AJ_{K_1}(Z_{Ma19}), \mathrm{im}\ AJ_{K_1}).$$

Remarks 9.2. (a) Although Masoero's paper assumes that every prime dividing N splits in K, the paper's argument goes through under our more general Heegner hypothesis. The only place where Masoero uses the splitting assumption is to deduce the existence of an ideal $\mathfrak{C}$ of O_K for which $\mathbb{Z}/N\mathbb{Z} \cong O_K/\mathfrak{C}$ [21, section 4.1]; such a $\mathfrak{C}$ still exists if each prime over N may split or ramify in K with the square of the prime not dividing N in the latter case.

(b) Condition (ii) excludes only finitely many p for given f and K; see [21, section 4.2]. Masoero assumes $p \nmid h_K$ to define Kolyvagin classes (see the argument between Remark 4.2 and Proposition 4.3 in [21]). To adapt Masoero's reasoning to the case $p \mid h_K$, one might need to use universal Euler system arguments along the lines of Rubin ([30, sections 4.2-4.4]; see in particular [30, Remark 4.4.3]).

10. Comparing Heegner cycles: Abel-Jacobi maps

Recall subsection 5.1's groups $G(t, r)$ and projections π_B, $\pi_{B,1}$, as well as subsection 5.2's cycles and varieties.

10.1. The section's main result

In this section, we prove the following result.

Theorem 10.1. *In addition to subsection 2.2's hypotheses, assume that:*

(i) We have

$$\mathrm{ind}_p(AJ_{K_1}(Z_{Ma19}), \mathrm{im}\ AJ_{K_1}) = \mathrm{ind}_p(AJ_{K_1}(\Delta_{Ma19}), \mathrm{im}\ AJ_{K_1}). \quad (18)$$

(ii) When the $h_K \cdot \phi(N)$ elements of $\mathrm{Gal}(K_1/K) \times (\Gamma_0(N)/\Gamma_1(N))$ act on the point of $X_1(N)$ associated to E_{x_1}, the corresponding $h_K \cdot \phi(N)$ images of that point are distinct.

(iii) The map (7) with $\eta = \omega_f \wedge \omega_A^{k/2-1} \eta_A^{k/2-1}$, $A = E_{x_1}$ and $F = L$ has image in O_L.

Then

$$\mathrm{ind}_p(AJ_{K_1}(Z_{Ma19}), \mathrm{im}\ AJ_{K_1}) \leq \mathrm{ind}_p AJ_L^{1,E_{x_1}}(Z_{BeDaPr13}) \left(\omega_f \wedge \omega_{E_{x_1}}^{k/2-1} \eta_{E_{x_1}}^{k/2-1}\right). \quad (19)$$

Remarks 10.2. We always have

$$\mathrm{ind}_p(AJ_{K_1}(Z_{Ma19}), \mathrm{im}\ AJ_{K_1}) = \mathrm{ind}_p(\mathrm{cor}_{K_1/K} AJ_{K_1}(\Delta_{Ma19}), \mathrm{im}\ AJ_K) \quad (20)$$

because

$$AJ_{K_1}(Z_{Ma19}) = \mathrm{res}_{K_1/K}(\mathrm{cor}_{K_1/K} AJ_{K_1}(\Delta_{Ma19}))$$

and, noting that $p \nmid h_K$,

$$\begin{aligned} \mathrm{cor}_{K_1/K} AJ_{K_1}(Z_{Ma19}) &= \mathrm{cor}_{K_1/K} \mathrm{res}_{K_1/K}(\mathrm{cor}_{K_1/K} AJ_{K_1}(\Delta_{Ma19})) \\ &= h_K \mathrm{cor}_{K_1/K} AJ_{K_1}(\Delta_{Ma19}). \end{aligned}$$

(To see why these imply (20), use versions of the next subsection's lemmas with pushforwards and pullbacks replaced by the maps $\mathrm{res}_{K_1/K}$, $\mathrm{cor}_{K_1/K}$.) Equation (18) says the $\mathrm{cor}_{K_1/K}$ on (20)'s right hand side can be removed without changing the p-indices, so (18) is stronger than (20).

10.2. Correspondences, Galois actions and p-indices

The following two lemmas are key tools in our argument.

Lemma 10.3. *Let varieties U_1, U_2 be defined over K_1, with associated Abel-Jacobi maps $AJ_{K_1}^{U_1}$, $AJ_{K_1}^{U_2}$. Let Δ_1, Δ_2 be cycles defined over K_1 in U_1, U_2 respectively. Let P be a correspondence from U_1 to U_2, with induced pushforward and pullback maps P_*, P^* between the Chow groups of U_1 and U_2. Write*

$$M_1 := \mathrm{ind}_p(AJ_{K_1}^{U_1}(\Delta_1), \mathrm{im}\ AJ_{K_1}^{U_1})$$

if this number is well defined, and write

$$M_2 := \mathrm{ind}_p(AJ_{K_1}^{U_2}(\Delta_2), \mathrm{im}\ AJ_{K_1}^{U_2})$$

if this number is well defined.

(a) *Assume* $AJ^{U_2}_{K_1}(\Delta_2)$ *is not torsion and* $P_*\Delta_1 = \alpha\Delta_2$ *for some* $\alpha \in \mathbb{Z}_p^\times$. *Then* $AJ^{U_1}_{K_1}(\Delta_1)$ *is not torsion, both of* M_1, M_2 *are well defined, and* $M_1 \leq M_2$.

(b) *Assume* $AJ^{U_1}_{K_1}(\Delta_1)$ *is not torsion and* $\alpha\Delta_1 = P^*\Delta_2$ *for some* $\alpha \in \mathbb{Z}_p^\times$. *Then* $AJ^{U_2}_{K_1}(\Delta_2)$ *is not torsion, both of* M_1, M_2 *are well defined, and* $M_1 \geq M_2$.

Proof. (a) Assume for a contradiction that $AJ^{U_1}_{K_1}(\Delta_1)$ is torsion. Then so is

$$P_* AJ^{U_1}_{K_1}(\Delta_1) = AJ^{U_2}_{K_1}(P_*\Delta_1) = \alpha AJ^{U_2}_{K_1}(\Delta_2),$$

so $AJ^{U_2}_{K_1}(\Delta_2)$ is also torsion, contrary to assumption. So $AJ^{U_1}_{K_1}(\Delta_1)$ is not torsion.

Therefore, there are non-torsion classes $C_1 \in \text{im } AJ^{U_1}_{K_1}$ and $C_2 \in \text{im } AJ^{U_2}_{K_1}$ for which $AJ^{U_1}_{K_1}(\Delta_1) = p^{M_1}C_1$ and $AJ^{U_2}_{K_1}(\Delta_2) = p^{M_2}C_2$, so

$$\begin{aligned}
M_2 &= \text{ind}_p(\alpha AJ^{U_2}_{K_1}(\Delta_2), \text{im } AJ^{U_2}_{K_1}) && \text{(since } \alpha \in \mathbb{Z}_p^\times\text{)}\\
&= \text{ind}_p(AJ^{U_2}_{K_1}(P_*\Delta_1), \text{im } AJ^{U_2}_{K_1}) && \text{(by assumption)}\\
&= \text{ind}_p(P_* AJ^{U_1}_{K_1}(\Delta_1), \text{im } AJ^{U_2}_{K_1}) && \text{(Abel-Jacobi maps commute with correspondences)}\\
&= \text{ind}_p(p^{M_1}P_*(C_1), \text{im } AJ^{U_2}_{K_1})\\
&= M_1 + \text{ind}_p(P_*(C_1), \text{im } AJ^{U_2}_{K_1}) \geq M_1.
\end{aligned}$$

(b) In the argument for part (a), replace P_* with P^* and swap M_1 with M_2, U_1 with U_2, Δ_1 with Δ_2 and C_1 with C_2. We obtain $M_1 \geq M_2$. □

Lemma 10.4. *Let* U *be a variety defined over* K, *with associated Abel-Jacobi map* $AJ^U_{K_1}$. *Let* Δ *be a cycle defined over* K_1 *in* U. *Suppose that*

$$\text{ind}_p(AJ^U_{K_1}(\Delta), \text{im } AJ^U_{K_1})$$

is well defined. Then for any $\sigma \in \text{Gal}(K_1/K)$, *we have*

$$\text{ind}_p(AJ^U_{K_1}(\Delta), \text{im } AJ^U_{K_1}) = \text{ind}_p(AJ^U_{K_1}(\sigma\Delta), \text{im } AJ^U_{K_1}).$$

Proof. If a non-torsion class $C_1 \in \text{im } AJ^U_{K_1}$ satisfies $AJ^U_{K_1}(\Delta) = p^{M_1}C_1$ for some $M_1 \in \mathbb{Z}_{\geq 0}$, then applying $\sigma \in \text{Gal}(K_1/K)$ and noting that Abel-Jacobi maps are Galois equivariant yields $AJ^U_{K_1}(\sigma\Delta) = p^{M_1}(\sigma C_1)$, and σC_1 is non-torsion since C_1 is non-torsion.

Conversely, if a non-torsion class $C_2 \in \text{im } AJ^U_{K_1}$ satisfies $AJ^U_{K_1}(\sigma\Delta) = p^{M_2}C_2$ for some $M_2 \in \mathbb{Z}_{\geq 0}$, then applying σ^{-1} similarly yields $AJ^U_{K_1}(\Delta) = p^{M_2}(\sigma^{-1}C_2)$, and $\sigma^{-1}C_2$ is non-torsion.

The desired result follows. □

For the rest of this section, Theorem 10.1's hypotheses are assumed.

10.3. From Masoero to Castella

First, in this subsection, we link the p-index of Masoero's Heegner cycle Δ_{Ma19} to the p-index of Castella's Heegner cycle Δ_{Ca13}.

The "forget the second $(\mathbb{Z}/N\mathbb{Z})$-basis vector" map $\mathcal{E}(\Gamma(N)) \to \mathcal{E}(\Gamma_1(N))$ gives maps $P_r : \widetilde{\mathcal{E}}^r(\Gamma(N)) \to \widetilde{\mathcal{E}}^r(\Gamma_1(N))$. Passing to Chow groups, we obtain pushforward maps $P_{r,*}$ and pullback maps P_r^*.

To help perform the calculations below, we define a $\mathbb{Q}$-linear map[3]

$$P_{r,*} : \mathbb{Q}[G(2,r) \times (\Gamma_0(N)/\Gamma(N))] \to \mathbb{Q}[G(1,r) \times (\Gamma_0(N)/\Gamma_1(N))]$$

so that the element

$$(((z_{11}, z_{12}, \varepsilon_1), \dots, (z_{r1}, z_{r2}, \varepsilon_r)), s) \in ((\mathbb{Z}/N\mathbb{Z})^2 \rtimes \{\pm 1\})^r \rtimes S_r = G(2,r)$$

(where each z_{ij} is in $\mathbb{Z}/N\mathbb{Z}$, each ε_i is in $\{\pm 1\}$ and $s \in S_r$) is sent by $P_{r,*}$ to

$$(((z_{11}, \varepsilon_1), \dots, (z_{r1}, \varepsilon_r)), s) \in ((\mathbb{Z}/N\mathbb{Z}) \rtimes \{\pm 1\})^r \rtimes S_r = G(1,r),$$

and an element $b \in \Gamma_0(N)/\Gamma(N)$ is sent by $P_{r,*}$ to the image of b under the quotient map $\Gamma_0(N)/\Gamma(N) \to \Gamma_0(N)/\Gamma_1(N)$. Then, for any $\sigma \in \mathbb{Q}[G(2,r) \times (\Gamma_0(N)/\Gamma(N))]$, any cycle Z of $\widetilde{\mathcal{E}}^r(\Gamma(N))$ and any cycle Z_1 of $\widetilde{\mathcal{E}}^r(\Gamma_1(N))$, we have

$$P_{r,*}(\sigma \cdot Z) = P_{r,*}(\sigma) \cdot P_{r,*}(Z) \tag{21}$$

$$P_r^*(P_{r,*}(\sigma) \cdot Z_1) = \sigma \cdot P_r^*(Z_1). \tag{22}$$

Equations (21), (22) are easily shown by first considering the cases $\sigma \in G(2,r)$ and $\sigma \in \Gamma_0(N)/\Gamma(N)$, then extending by $\mathbb{Q}$-linearity.

The maps $P_{k-2,*}$, P_{k-2}^* act on the cycles Δ_{Ma19}, Δ_{Ca13} as follows.

Proposition 10.5. *We have*

$$P_{k-2,*}(\Delta_{Ma19}) = \Delta_{Ca13} \tag{23}$$

$$P_{k-2}^*(\Delta_{Ca13}) = N \cdot \Delta_{Ma19}. \tag{24}$$

Proof. For the pushforward, we have

$$P_{k-2,*}(i_x)_*((\mathrm{Graph}(\sqrt{D_K}))^{k/2-1}) = (i_{x_1})_*((\mathrm{Graph}(\sqrt{D_K}))^{k/2-1}),$$

so by (21), for each $\sigma \in G(2, k-2)$ and each $b \in \Gamma_0(N)/\Gamma(N)$, we have

$$\begin{aligned} &P_{k-2,*}\big(b\sigma(i_x)_*((\mathrm{Graph}(\sqrt{D_K}))^{k/2-1})\big) \\ &\qquad = P_{k-2,*}(b)P_{k-2,*}(\sigma)(i_{x_1})_*((\mathrm{Graph}(\sqrt{D_K}))^{k/2-1}). \end{aligned} \tag{25}$$

[3]Our reason for using the same notation $P_{r,*}$ for different maps is explained by (21) and (22).

Multiplying (25) by

$$\frac{1}{|\Gamma_0(N)/\Gamma(N)|} \cdot \frac{c_2(\sigma)}{|G(2,k-2)|}$$

(recall that the expression $c_2(\sigma)$ was defined in subsection 5.1) and then summing over all $\sigma \in G(2,k-2)$ and all $b \in \Gamma_0(N)/\Gamma(N)$ yields (23).

For the pullback, we have

$$P_{k-2}^*(i_{x_1})_*((\mathrm{Graph}(\sqrt{D_K}))^{k/2-1}) = \sum_{\widetilde{x}} (i_{\widetilde{x}})_*((\mathrm{Graph}(\sqrt{D_K}))^{k/2-1})$$

where $\widetilde{x}$ runs over the N points in the inverse image of x_1 under the map $X(N) \to X_1(N)$. Arguing as before, using (22) instead of (21), yields (24). $\square$

The Abel-Jacobi map commutes with correspondences, so applying the Abel-Jacobi map to (23) and (24) yields

$$P_{k-2,*}AJ_{K_1}(\Delta_{Ma19}) = AJ_{K_1}^1(\Delta_{Ca13}) \tag{26}$$

$$P_{k-2}^* AJ_{K_1}^1(\Delta_{Ca13}) = N \cdot AJ_{K_1}(\Delta_{Ma19}). \tag{27}$$

Since $p \nmid N$ and $AJ_{K_1}(\Delta_{Ma19})$ is not torsion, Lemma 10.3 yields the following.

Proposition 10.6. *We have*

$$\mathrm{ind}_p(AJ_{K_1}(\Delta_{Ma19}), \mathrm{im}\ AJ_{K_1}) = \mathrm{ind}_p(AJ_{K_1}^1(\Delta_{Ca13}), \mathrm{im}\ AJ_{K_1}^1). \tag{28}$$

10.4. Galois action on Castella's Heegner cycle

By the theory of complex multiplication, there is a bijection between elements $\sigma \in \mathrm{Gal}(K_1/K)$ and ideal classes $[\mathfrak{a}]$ of O_K so that the elliptic curve σE_{x_1} corresponds to the same point of $X_1(N)$ as $E_{x_1}/E_{x_1}[\mathfrak{a}]$. It is easily checked that $\sigma\Delta_{Ca13} = \Delta_{Ca13,\mathfrak{a}}$ for σ thus corresponding to $[\mathfrak{a}]$. Applying Lemma 10.4, we obtain:

Proposition 10.7. *For a nonzero ideal* $\mathfrak{a}$ *of* O_K*, we have*

$$\mathrm{ind}_p(AJ_{K_1}^1(\Delta_{Ca13}), \mathrm{im}\ AJ_{K_1}^1) = \mathrm{ind}_p(AJ_{K_1}^1(\Delta_{Ca13,\mathfrak{a}}), \mathrm{im}\ AJ_{K_1}^1).$$

10.5. From Castella to BDP

As in [2, proof of Lemma 3.4], define Π_{k-2} to be the image of

$$\begin{array}{rcl} \widetilde{\mathcal{E}}^{k-2}(\Gamma_1(N)) \times E_{x_1}^{k/2-1} & \hookrightarrow & \widetilde{\mathcal{E}}^{k-2}(\Gamma_1(N)) \times (\widetilde{\mathcal{E}}^{k-2}(\Gamma_1(N)) \times E_{x_1}^{k-2}) \\ (w,a) & \mapsto & (w,(w,(a,\sqrt{D_K}\cdot a))) \end{array}$$

and view Π_{k-2} as a correspondence from $\widetilde{\mathcal{E}}^{k-2}(\Gamma_1(N)) \times E_{x_1}^{k-2}$ to $\widetilde{\mathcal{E}}^{k-2}(\Gamma_1(N))$.

Proposition 10.8. *We have*

$$\Delta_{Ca13,\mathfrak{a}} = \pi_{B,1}\Pi_{k-2}\Delta_{\varphi(E_{x_1},\mathfrak{a})}. \tag{29}$$

Proof. For $\sigma_1 = ((z_i, \varepsilon_{1i})_{i=1}^{k-2}, s_1) \in G(1, k-2)$ and $\sigma_0 = ((\varepsilon_{0i})_{i=1}^{k-2}, s_0) \in G(0, k-2)$, we have

$$\Pi_{k-2}\sigma_1\sigma_0\mathrm{Graph}(\varphi(E_{x_1}, \mathfrak{a}))^{k-2} = \sigma\mathrm{Graph}(\sqrt{D_K})^{k/2-1}_{E_{x_1}/E_{x_1}[\mathfrak{a}]} \tag{30}$$

where

$$\sigma = ((z_i, \varepsilon_{1i}\varepsilon_{0,s_0\circ s_1^{-1}(i)})_{i=1}^{k-2}, s_1 \circ s_0^{-1}) \in G(1, k-2).$$

Multiplying (30) by

$$\frac{c_1(\sigma_1)}{|G(1, k-2)|} \cdot \frac{c_0(\sigma_0)}{|G(0, k-2)|}$$

and then summing over all $(\sigma_1, \sigma_0) \in G(1, k-2) \times G(0, k-2)$, we obtain

$$\Pi_{k-2}\Delta_{\varphi(E_{x_1}, \mathfrak{a})} = \pi_{1,k-2}(\mathrm{Graph}(\sqrt{D_K})^{k/2-1}_{E_{x_1}/E_{x_1}[\mathfrak{a}]}) \tag{31}$$

and applying $\pi_{B,1}$ yields (29). □

Define $Q_\mathfrak{a}$ to be the correspondence from $\widetilde{\mathcal{E}}^{k-2}(\Gamma_1(N))$ to itself that sends a cycle to its intersection with the fiber at $E_{x_1}/E_{x_1}[\mathfrak{a}]$ in $\widetilde{\mathcal{E}}^{k-2}(\Gamma_1(N))$.

Proposition 10.9. *We have*

$$\Pi_{k-2}\Delta_{\varphi(E_{x_1}, \mathfrak{a})} = \phi(N) \cdot Q_\mathfrak{a}\Delta_{Ca13,\mathfrak{a}}. \tag{32}$$

Proof. By definition, Δ_{Ca13} is

$$\sum_{b\in\Gamma_0(N)/\Gamma_1(N)} \frac{1}{\phi(N)} \sum_{\sigma\in G(1,k-2)} \frac{c_1(\sigma)}{|G(1, k-2)|} b\sigma(\mathrm{Graph}(\sqrt{D_K})^{k/2-1}_{E_{x_1}/E_{x_1}[\mathfrak{a}]}). \tag{33}$$

Because of Theorem 10.1's assumption (ii), applying $Q_\mathfrak{a}$ to Δ_{Ca13} eliminates the terms in (33) involving a nontrivial $b \in \Gamma_0(N)/\Gamma_1(N)$ and preserves the terms in (33) with $b = 1$. Therefore $\phi(N) \cdot Q_\mathfrak{a}\Delta_{Ca13,\mathfrak{a}}$ is equal to

$$\sum_{\sigma\in G(1,k-2)} \frac{c_1(\sigma)}{|G(1, k-2)|}\sigma(\mathrm{Graph}(\sqrt{D_K})^{k/2-1}_{E_{x_1}/E_{x_1}[\mathfrak{a}]}) = \pi_{1,k-2}(\mathrm{Graph}(\sqrt{D_K})^{k/2-1}_{E_{x_1}/E_{x_1}[\mathfrak{a}]})$$

which is $\Pi_{k-2}\Delta_{\varphi(E_{x_1}, \mathfrak{a})}$ by (31), so (32) is proved. □

Let $\sigma \in \mathrm{Gal}(K_1/K)$ correspond to the ideal class $[\mathfrak{a}]$ as before. Define $R_\mathfrak{a}$ to be the correspondence from $\widetilde{\mathcal{E}}^{k-2}(\Gamma_1(N))$ to $\widetilde{\mathcal{E}}^{k-2}(\Gamma_1(N)) \times E_{x_1}^{k-2}$ given by the variety in

$$(\widetilde{\mathcal{E}}^{k-2}(\Gamma_1(N)) \times E_{x_1}^{k-2}) \times \widetilde{\mathcal{E}}^{k-2}(\Gamma_1(N))$$

whose points are precisely the points of the form

$$(((w_1 \bmod E_{x_1}[\mathfrak{a}], \ldots, w_{k-2} \bmod E_{x_1}[\mathfrak{a}])_{E_{x_1}/E_{x_1}[\mathfrak{a}]}, (w_1, \ldots, w_{k-2})),$$
$$(w_1 \bmod E_{x_1}[\mathfrak{a}], \ldots, w_{k/2-1} \bmod E_{x_1}[\mathfrak{a}], x_1, \ldots, x_{k/2-1})_{E_{x_1}/E_{x_1}[\mathfrak{a}]})$$

where the w_i are points in E_{x_1} and the x_i are points in $E_{x_1}/E_{x_1}[\mathfrak{a}]$. The subvariety $\mathrm{Graph}(\sqrt{D_K})^{k/2-1}_{E_{x_1}/E_{x_1}[\mathfrak{a}]}$ of $\widetilde{\mathcal{E}}^{k-2}(\Gamma_1(N))$ is interpreted as the variety of points of the form

$$(w_1, \ldots, w_{k/2-1}, \sqrt{D_K} w_1, \ldots, \sqrt{D_K} w_{k/2-1}).$$

Proposition 10.10. *We have*

$$\Delta_{\varphi(E_{x_1},\mathfrak{a})} = \pi_{1,k-2}\pi_{0,k-2}R_{\mathfrak{a},*}\Pi_{k-2}\Delta_{\varphi(E_{x_1},\mathfrak{a})}. \tag{34}$$

Proof. By (31), we have

$$\Pi_{k-2}\Delta_{\varphi(E_{x_1},\mathfrak{a})} = \pi_{1,k-2}(\mathrm{Graph}(\sqrt{D_K})^{k/2-1}_{E_{x_1}/E_{x_1}[\mathfrak{a}]})$$

and applying $\pi_{1,k-2}\pi_{0,k-2}R_{\mathfrak{a},*}$ yields (34). □

Combining Lemma 10.3 with Propositions 10.8, 10.9, 10.10 in that order yields

$$\begin{aligned}\mathrm{ind}_p(AJ^{1,E_{x_1}}_{K_1}(\Delta_{\varphi(E_{x_1},\mathfrak{a})}), \mathrm{im}\ AJ^{1,E_{x_1}}_{K_1}) &\leq \mathrm{ind}_p(AJ^1_{K_1}(\Delta_{Ca13,\mathfrak{a}}), \mathrm{im}\ AJ^1_{K_1}) \\ &\leq \mathrm{ind}_p(AJ^1_{K_1}(\Pi_{k-2}\Delta_{\varphi(E_{x_1},\mathfrak{a})}), \mathrm{im}\ AJ^1_{K_1}) \\ &\leq \mathrm{ind}_p(AJ^{1,E_{x_1}}_{K_1}(\Delta_{\varphi(E_{x_1},\mathfrak{a})}), \mathrm{im}\ AJ^{1,E_{x_1}}_{K_1}),\end{aligned}$$

which means that all of the p-indices are equal. The additional fact that

$$\mathrm{ind}_p(AJ^{1,E_{x_1}}_{K_1}(\Delta_{\varphi(E_{x_1},\mathfrak{a})}), \mathrm{im}\ AJ^{1,E_{x_1}}_{K_1}) \leq \mathrm{ind}_p(AJ^{1,E_{x_1}}_{L}(\Delta_{\varphi(E_{x_1},\mathfrak{a})}), \mathrm{im}\ AJ^{1,E_{x_1}}_{L})$$

now implies:

Proposition 10.11. *We have*

$$\mathrm{ind}_p(AJ^1_{K_1}(\Delta_{Ca13,\mathfrak{a}}), \mathrm{im}\ AJ^1_{K_1}) \leq \mathrm{ind}_p(AJ^{1,E_{x_1}}_{L}(\Delta_{\varphi(E_{x_1},\mathfrak{a})}), \mathrm{im}\ AJ^{1,E_{x_1}}_{L}).$$

10.6. Conclusion

By Theorem 10.1's assumption (i) (that is, (18)) and Propositions 10.6, 10.7 and 10.11, we have

$$\mathrm{ind}_p(AJ_{K_1}(Z_{Ma19}), \mathrm{im}\ AJ_{K_1}) \leq \mathrm{ind}_p(AJ^{1,E_{x_1}}_{L}(\Delta_{\varphi(E_{x_1},\mathfrak{a})}), \mathrm{im}\ AJ^{1,E_{x_1}}_{L})$$

for each nonzero ideal $\mathfrak{a}$ of O_K. Since $Z_{BeDaPr13}$ is defined as a $\mathbb{Z}_p$-linear combination of cycles (note that $(k/2-1)!$ and $\mathrm{N}(\mathfrak{a})$ are coprime to p) whose Abel-Jacobi images' p-indices are at least the p-index of $AJ_{K_1}(Z_{Ma19})$, it follows that

$$\mathrm{ind}_p(AJ_{K_1}(Z_{Ma19}), \mathrm{im}\ AJ_{K_1}) \leq \mathrm{ind}_p(AJ^{1,E_{x_1}}_{L}(Z_{BeDaPr13}), \mathrm{im}\ AJ^{1,E_{x_1}}_{L}). \tag{35}$$

Using Theorem 10.1's assumption (iii), we have

$$(\text{RHS of (35)}) \leq \mathrm{ind}_p AJ^{1,E_{x_1}}_{L}(Z_{BeDaPr13})\left(\omega_f \wedge \omega^{k/2-1}_{E_{x_1}}\eta^{k/2-1}_{E_{x_1}}\right). \tag{36}$$

Equations (35) and (36) imply (19), so Theorem 10.1 is proved.

11. Final argument

We now prove this paper's main theorem:

Theorem 11.1. *Suppose all the assumptions of subsection 2.2 hold, together with the following technical hypotheses.*

(i) For each place $v \mid p$ of K, the O_L-module $H^1_f(K_v, W_f)$ has corank 1, and the localization map $H^1_f(K, W_f) \to H^1_f(K_v, W_f)$ restricts to a map

$$(\text{im } AJ_K) \otimes_{O_L} (L/O_L) \to (\text{im } AJ_{K_v}) \otimes_{O_L} (L/O_L)$$

of which the kernel is torsion.

(ii) The cohomology class $C_0 = \text{cor}_{K_1/K} AJ_{K_1}(\Delta_{Ma19}) \in H^1_f(K, T_f)$ has a non-torsion image in $H^1_f(K_{v_0}, T_f)$ under localization.

(iii) Theorem 9.1's assumption (ii) holds.

(iv) Theorem 10.1's assumptions (i), (ii) and (iii) hold.

(v) The prime p is coprime to the product of the two fractions

$$\frac{\#H^0(K_{\overline{v}_0}, W_f)\Pi_{v\in S'}(\#H^1_{ur}(K_v, W_f))}{\#H^0(K_{v_0}, W_f)}$$

and

$$\frac{(O_L : O_L \log_\omega(\text{loc}_{v_0} C_0))^2 (H^1_f(K_{v_0}, T_f) : \exp_\omega(pO_L))^2}{p^{2[L:\mathbb{Q}_p]-k}(H^1_f(K, T_f) : O_L C_0)^2}$$

where S' is as described just before Theorem 7.7, and the differential form ω and its associated maps $\log_\omega$, $\exp_\omega$ are described in Theorem 8.2 and subsection 5.8.

Then we have

$$2\text{ind}_p AJ_L^{1,E_{x_1}}(Z_{BeDaPr13})\left(\omega_f \wedge \omega_{E_{x_1}}^{k/2-1}\eta_{E_{x_1}}^{k/2-1}\right) = \text{ind}_p \#\text{Ш}(K, W_f).$$

Proof. By Bertolini, Darmon and Prasanna's [1, Theorem 5.13] (with that source's $\chi = \text{N}^{k/2}$, $j = k/2-1$, $r = k-2$, $c = 1$, $\varepsilon_f = 1$), noting the correspondence between section 7's P_1 and subsubsection 6.3.4's P_{ac} corresponding to $\chi = \text{N}^{k/2}$,

$$P_1(L_{BDP}^{-,N_p}(f)) = (1 - p^{-k/2}a(p,f) + p^{-1})^2 \cdot \left(\frac{1}{(k/2-1)!}\sum_{[\mathfrak{a}]}\frac{1}{\text{N}^{k/2-1}(\mathfrak{a})} AJ_L^{1,E_{x_1}}(\Delta_{\varphi_\mathfrak{a}})\left(\omega_f \wedge \omega_{E_{x_1}}^{k/2-1}\eta_{E_{x_1}}^{k/2-1}\right)\right)^2.$$

Since $a(p,f)$ is a p-adic unit and $k/2 \geq 2$, this implies

$$2\mathrm{ind}_p AJ_L^{1,E_{x_1}}(Z_{BeDaPr13})\left(\omega_f \wedge \omega_{E_{x_1}}^{k/2-1}\eta_{E_{x_1}}^{k/2-1}\right) = k + \mathrm{ind}_p P_1(L_{BDP}^{-,N_p}(f)). \quad (37)$$

By Theorem 6.2,

$$\mathrm{ind}_p P_1(L_{BDP}^{-,N_p}(f)) \leq \mathrm{ind}_p P_1(L_{Wan}^{-,N_p}(f)). \quad (38)$$

By Theorem 7.7,

$$\begin{aligned}\mathrm{ind}_p P_1(L_{Wan}^{-,N_p}(f)) \leq{}& \mathrm{ind}_p(\#H^0(K_{v_0}, W_f)) + \mathrm{ind}_p(\#H^0(K_{\overline{v}_0}, W_f)) \\ &+ \mathrm{ind}_p(\#H^1_{ac}(K, W_f)) + \sum_{v\in S'} \mathrm{ind}_p(\#H^1_{ur}(K_v, W_f)). \quad (39)\end{aligned}$$

By Theorem 9.1, the O_L-module im AJ_K has rank 1 and the group $Ш(K, W_f)$ has finite cardinality, so all the hypotheses of Theorems 8.1 and 8.2 hold. By those theorems,

$$\begin{aligned}&\mathrm{ind}_p(\#H^1_{ac}(K, W_f)) \\ &= \mathrm{ind}_p(\#Ш(K, W_f)) - 2[L:\mathbb{Q}_p] + 2\mathrm{ind}_p(O_L : O_L \log_\omega(\mathrm{loc}_{v_0} C_0)) \\ &\quad + 2\mathrm{ind}_p(H^1_f(K_{v_0}, T_f) : \exp_\omega(pO_L)) - 2\mathrm{ind}_p(\#H^0(K_{v_0}, W_f)) \\ &\quad - 2\mathrm{ind}_p(H^1_f(K, T_f) : O_L C_0). \quad (40)\end{aligned}$$

Again by Theorem 9.1,

$$\mathrm{ind}_p(\#Ш(K, W_f)) \leq 2\mathrm{ind}_p(AJ_{K_1}(Z_{Ma19}), \mathrm{im}\ AJ_{K_1}). \quad (41)$$

By Theorem 10.1,

$$2\mathrm{ind}_p(AJ_{K_1}(Z_{Ma19}), \mathrm{im}\ AJ_{K_1}) \leq 2\mathrm{ind}_p AJ_L^{1,E_{x_1}}(Z_{BeDaPr13})\left(\omega_f \wedge \omega_{E_{x_1}}^{k/2-1}\eta_{E_{x_1}}^{k/2-1}\right). \quad (42)$$

Each of (37) to (42) is either an equation or an inequality in the $\leq$ direction. Combining those six statements in that order yields

$$\begin{aligned}&2\mathrm{ind}_p AJ_L^{1,E_{x_1}}(Z_{BeDaPr13})\left(\omega_f \wedge \omega_{E_{x_1}}^{k/2-1}\eta_{E_{x_1}}^{k/2-1}\right) \\ &\leq 2\mathrm{ind}_p AJ_L^{1,E_{x_1}}(Z_{BeDaPr13})\left(\omega_f \wedge \omega_{E_{x_1}}^{k/2-1}\eta_{E_{x_1}}^{k/2-1}\right) - \mathrm{ind}_p(\#H^0(K_{v_0}, W_f)) \\ &\quad + \mathrm{ind}_p(\#H^0(K_{\overline{v}_0}, W_f)) + \sum_{v\in S'} \mathrm{ind}_p(\#H^1_{ur}(K_v, W_f)) \\ &\quad + 2\mathrm{ind}_p(O_L : O_L \log_\omega(\mathrm{loc}_{v_0} C_0)) + 2\mathrm{ind}_p(H^1_f(K_{v_0}, T_f) : \exp_\omega(pO_L)) \\ &\quad + k - 2[L:\mathbb{Q}_p] - 2\mathrm{ind}_p(H^1_f(K, T_f) : O_L C_0). \quad (43)\end{aligned}$$

Theorem 11.1's assumption (v) forces equality in (43), hence equality in each of (37) to (42). In particular, equality occurs in (41) and (42), so

$$2\mathrm{ind}_p AJ_L^{1,E_{x_1}}(Z_{BeDaPr13})\left(\omega_f \wedge \omega_{E_{x_1}}^{k/2-1}\eta_{E_{x_1}}^{k/2-1}\right) = \mathrm{ind}_p \#Ш(K, W_f). \quad \square$$

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