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A QUESTION OF ZHOU, SHI AND DUAN ON NONPOWER SUBGROUPS OF FINITE GROUPS

C.S. Anabanti

Institut für Analysis & Zahlentheorie, Technische Universität Graz, Austria, and Department of Mathematics and Applied Mathematics, University of Pretoria, South Africa.
E-Mail anabanti@math.tugraz.at, chimere.anabanti@up.ac.za, chimere.anabanti@unn.edu.ng

A.B. Aroh

Department of Mathematics, University of Nigeria, Nsukka, Nigeria.
E-Mail blaise.aroh.231881@unn.edu.ng

S.B. Hart

Department of Economics, Mathematics and Statistics, Birkbeck, University of London, UK.
E-Mail s.hart@bbk.ac.uk

A.R. Oodo

Department of Mathematics, University of Nigeria, Nsukka, Nigeria.
E-Mail amara.oodo.231880@unn.edu.ng

Abstract. A subgroup H of a group G is called a *power subgroup* of G if there exists a non-negative integer m such that $H = \langle g^m : g \in G\rangle$. Any subgroup of G which is not a power subgroup is called a *nonpower subgroup* of G. Zhou, Shi and Duan, in a 2006 paper, asked whether for every integer k ($k \geq 3$), there exist groups possessing exactly k nonpower subgroups. We answer this question in the affirmative by giving an explicit construction that leads to at least one group with exactly k nonpower subgroups, for all $k \geq 3$, and infinitely many such groups when k is composite and greater than 4. Moreover, we describe the number of nonpower subgroups for the cases of elementary abelian groups, dihedral groups, and 2-groups of maximal class.

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1. Introduction. A subgroup H of a group G is called a *power subgroup* of G if there exists a non-negative integer m such that $H = G^m$, where $G^m := \langle g^m : g \in G\rangle$. The identity subgroup and the whole group are examples of power subgroups of any group G. If H is a power subgroup of G, then H is normal in G; but the converse is not necessarily true. For instance, no subgroup of index 2 in the quaternion group Q_8 of order 8 is a power subgroup of Q_8, even though they are normal subgroups. A subgroup of G which is not a power subgroup is called a *nonpower subgroup* of G.

Let k be the number of nonpower subgroups of a group G. The authors (Zhou, Shi and Duan) of [4] proved the following:

(a) $k \in (0, \infty)$ if and only if G is a finite noncyclic group;

(b) $k = 0$ if and only if G is a cyclic group;

(c) $k = \infty$ if and only if G is an infinite noncyclic group.

They also remarked that neither $k = 1$ nor $k = 2$ is possible in any group. With respect to the case $k \geq 3$, they asked (see [4, Problem]):

Question 1. (Zhou, Shi and Duan) *For any integer k ($k \geq 3$), do there exist groups possessing exactly k nonpower subgroups?*

In this paper, we show that the answer to this question is yes. In fact, we prove that there is at least one group possessing exactly k nonpower subgroups for each $k \geq 3$ (see Theorem 5). Our method of proof also shows that there are infinitely many such groups for each $k > 4$ and k not prime. The constructions we used are given in Section 2; part of it involves the direct product of a dihedral group with a carefully chosen cyclic group.

There are further questions one could ask. For example, given a positive integer n, what is the maximum number of nonpower subgroups in a group of order n? To supply further examples of the possible numbers of nonpower subgroups in a group of a given order, we also explore in Section 3 some special cases: elementary abelian p-groups, dihedral groups, and 2-groups of maximal class. For example, we observe (see Corollary 10) that the elementary abelian p-group $C_p \times C_p$ (p prime) contains exactly $p + 1$ nonpower subgroups, and the generalised quaternion group Q_{2^n} (where $n \geq 3$) contains exactly $2^{n-1} - 1$ nonpower subgroups (see Theorem 16). All the groups studied here are finite.

We end this introductory section by briefly establishing the notation we will use. For a positive integer n, we write C_n for the cyclic group of order n, with D_{2n} being the dihedral group of order $2n$.

Notation. Let G be a group. We write $s(G)$ for the total number of subgroups in G. Also, we write $ps(G)$ for the number of power subgroups, and $nps(G)$ for the number of non-power subgroups. For example, in $C_2 \times C_2$ we have $s(G) = 5$, $ps(G) = 2$ and $nps(G) = 3$.

2. Groups with exactly k nonpower subgroups.

In this section, we give constructions that supply, for each $k \geq 3$, at least one finite group containing exactly k nonpower subgroups. Moreover, for $k \neq 4$ and k not prime, our constructions give infinitely many finite groups containing exactly k nonpower subgroups.

Remark 2. Let G be a finite group. If n is coprime to $|G|$, then $G^n = G$ as the map $g \mapsto g^n$, while not a homomorphism, is certainly a bijection from G to itself in this case. More generally, $G^{mn} = G^m$ for any positive integer m.

LEMMA 3. *Let A and B be finite groups such that $|A|$ and $|B|$ are coprime. Then every subgroup of $A \times B$ is of the form $U \times V$, where $U \leq A$ and $V \leq B$. Moreover, a subgroup of $A \times B$ is a power subgroup if and only if it is of the form $U \times V$, where U is a power subgroup of A and V is a power subgroup of B. In particular,*

$$s(A \times B) = s(A) \times s(B); \tag{1}$$

$$nps(A \times B) = s(A) \times s(B) - ps(A) \times ps(B). \tag{2}$$

Proof. Let $G = A \times B$. The fact that the subgroups of G in this case are the direct products of subgroups of A and B is well-known, but we include the proof for completeness. Suppose $H \leq G$ and let $(a, b) \in H$. Since $|A|$ and $|B|$ are coprime, the orders r and s of a and b respectively are also coprime. Therefore, there exist integers q and t such that $rq + st = 1$. Now $(a, b)^{st} = (a, 1)$ and $(a, b)^{rq} = (1, b)$. Hence, $(a, 1)$ and $(1, b)$ are elements of H. It follows that $H = U \times V$, where $U = \{a \in A : (a, 1) \in H\}$ and $V = \{b \in B : (1, b) \in H\}$. Therefore, $s(G) = s(A) \times s(B)$.

Consider the power subgroup G^m of G, for a positive integer m. We have that $G^m = A^m \times B^m$, because this group is generated by elements $(x, y)^m = (x^m, y^m)$, and we have observed that (x^m, y^m) is contained in a subgroup H if and only if $(x^m, 1) \in H$ and $(1, y^m) \in H$. For the converse, suppose that $U = A^\ell$ and $V = B^m$, for some positive integers m and ℓ. We may assume that ℓ divides $|A|$ and m divides $|B|$, by Remark 2. Now, let $n = \ell m$. Since ℓ and m are therefore coprime, we have that $A^n = A^\ell$, and $B^n = B^m$. Therefore, $U \times V = G^n$. Thus, a subgroup of G is a power subgroup if and only if it is of the form $U \times V$, where U is a power subgroup of A and V is a power subgroup of B. In particular, $ps(G) = ps(A) \times ps(B)$. Hence, $nps(G) = s(G) - ps(G) = s(A) \times s(B) - ps(A) \times ps(B)$. □

Let n be a positive integer. Zhou et al. showed that $nps(C_n) = 0$. We also note that $s(C_n) = ps(C_n) = \tau(n)$, where $\tau(n)$ is the number of divisors of n.

COROLLARY 4. *Suppose $G = A \times C_n$, where n is a positive integer and A is a finite group whose order is coprime to n. Then $nps(G) = \tau(n) \times nps(A)$.*

Proof. We have that $s(C_n) = ps(C_n) = \tau(n)$. Therefore in Equation (2), we have $nps(G) = (s(A) - ps(A))\tau(n) = \tau(n) \times nps(A)$. □

Before the next result we note that if p is an odd prime, then $nps(D_{2p}) = p$. This is because D_{2p} has exactly $p + 3$ subgroups; the p cyclic subgroups of order 2 are the nonpower subgroups. The remaining groups (the trivial subgroup, the cyclic subgroup of index 2, and the whole group) are the power subgroups D_{2p}^{2p}, D_{2p}^2 and D_{2p}^1, respectively. For a full description of nonpower subgroups in arbitrary dihedral groups, see Section 3.

THEOREM 5. *Let k be a positive integer, with $k \geq 3$. Then there exists a finite group G with exactly k nonpower subgroups. If k is composite and $k > 4$, then there are infinitely many such groups.*

Proof. Let k be a positive integer with $k \geq 3$. Then either k is divisible by 4, or k is divisible by an odd prime p (or both). Suppose first that k is divisible by an odd prime p. Let q be any odd prime other than p, and let $r = \frac{k}{p} - 1$. Then $\tau(q^r) = \frac{k}{p}$. We observe that $nps(D_{2p}) = p$. Therefore, by Corollary 4, we get $nps(D_{2p} \times C_{q^r}) = k$. On the other hand, if k is divisible by 4, then let $r = \frac{k}{4} - 1$, and let q be any prime greater than 3. A quick calculation shows that $nps(C_3 \times C_3) = 4$; whence $nps((C_3 \times C_3) \times C_{q^r}) = k$. We note that, in each case, if $k > 4$ and k is composite, then the exponent r is strictly positive. Therefore, since there are infinitely many choices for q, there are infinitely many finite groups G with exactly k nonpower subgroups. □

3. Special cases.

NOTATION. For a prime p and a positive integer n, we write C_p^n for the elementary abelian p-group of finite rank n, and denote the number of subgroups of rank r in C_p^n by $N_p(n, r)$.

THEOREM 6. ([3, Theorem 1]) *Let V be a vector space of dimension n over the finite field $GF(q)$, where q is a prime power. The number of subspaces of V of dimension r is*

$$\Big(\frac{q^n - 1}{q - 1}\Big)\Big(\frac{q^{n-1} - 1}{q^2 - 1}\Big) \cdots \Big(\frac{q^{n-r+1} - 1}{q^r - 1}\Big).$$

REMARK. (a) The group $G = C_p^n$ can be realised as an n-dimensional vector space (say V) over $GF(p)$. Now, the number of subgroups of rank r in C_p^n is equal to the number of subspaces of dimension r in V. In the light of Theorem 6 therefore, given any prime p and positive integers n and r, with $n > r \geq 2$, we have that

$$N_p(n, r) = \Big(\frac{p^n - 1}{p - 1}\Big)\Big(\frac{p^{n-1} - 1}{p^2 - 1}\Big) \cdots \Big(\frac{p^{n-r+1} - 1}{p^r - 1}\Big) = \prod_{k=0}^{r-1} \Big(\frac{p^{n-k} - 1}{p^{k+1} - 1}\Big). \tag{3}$$

(b) $N_p(n, 0) = 1 = N_p(n, n)$ for any prime p and natural number n, and for $n > 1$,

$$N_p(n, 1) = \frac{p^n - 1}{p - 1} = \sum_{k=0}^{n-1} p^k = N_p(n, n - 1).$$

PROPOSITION 7. *For prime p and positive integers n and r (with $n > r \geq 2$), we have:*
(a) $N_p(n - 1, r) = \Big(\frac{p^{n-r}-1}{p^r-1}\Big) N_p(n - 1, r - 1)$;
(b) $N_p(n, r) = p^r N_p(n - 1, r) + N_p(n - 1, r - 1)$.

Proof. Setting $n = n - 1$ and $r = r - 1$ in Equation (3), we have that

$$N_p(n - 1, r - 1) = \Big(\frac{p^{n-1} - 1}{p - 1}\Big) \cdots \Big(\frac{p^{n-r+1} - 1}{p^{r-1} - 1}\Big) = \prod_{k=0}^{r-2} \Big(\frac{p^{n-(k+1)} - 1}{p^{k+1} - 1}\Big). \tag{4}$$

Setting $n = n-1$ in Equation (3), we have that

$$N_p(n-1,r) = \Big(\frac{p^{n-1}-1}{p-1}\Big)\cdots\Big(\frac{p^{n-r+1}-1}{p^{r-1}-1}\Big)\Big(\frac{p^{n-r}-1}{p^r-1}\Big) = \prod_{k=0}^{r-1}\Big(\frac{p^{n-(k+1)}-1}{p^{k+1}-1}\Big)$$

$$= N_p(n-1,r-1)\Big(\frac{p^{n-r}-1}{p^r-1}\Big)\ \big(\text{from Equation (4)}\big), \tag{5}$$

which settles the (a) part. For the (b) part, we multiply Equation (5) by p^r, add the result to Equation (4) and regroup the terms to get the desired result. □

The recurrence relations given in Proposition 7 would be a good source for OEIS `https://oeis.org/`. We now turn to the first main result of this study; see Theorem 8.

THEOREM 8. *For prime, p and a natural number $n > 1$,*

$$nps(C_p^n) = s(C_p^n) - 2 = \sum_{r=1}^{n-1} N_p(n,r).$$

Proof. Let p be a prime and $n > 1$ be an integer. We write $G = C_p^n$. For $m \in \mathbb{N} \cup \{0\}$,

$$G^m = \begin{cases} \{1\}, & \text{if } m \equiv 0 \mod p \\ G, & \text{if } m \not\equiv 0 \mod p. \end{cases}$$

This tells us that the only power subgroups of G are the unique subgroups of ranks 0 and n (viz; the two trivial subgroups). That is, $nps(G) = s(G) - 2$. In particular, the nonpower subgroups of G are the subgroups of ranks $1, 2, \ldots, n-1$. Thus, the number of nonpower subgroups of G is $\sum_{r=1}^{n-1} N_p(n,r)$. □

The following result is an immediate consequence of Theorem 8.

COROLLARY 9. *Let $n > 1$ and p be prime. Then the elementary abelian p-group C_p^n contains exactly $\sum_{r=1}^{n-1} N_p(n,r)$ nonpower subgroups.*

In particular, when $n = 2$, we have the following.

COROLLARY 10. *Let p be prime. The elementary abelian p-group C_p^2 contains exactly $p+1$ nonpower subgroups.*

DEFINITION. A 2-*group of maximal class* is a group of order 2^n and nilpotency class $n-1$ for $n \geq 3$.

REMARK. It is known (for instance, see Theorem 1.2 and Corollary 1.7 of [1]) that any 2-group of maximal class belongs to one of the following three classes:

(i) $\langle x, y \mid x^{2^{n-1}} = y^2 = 1, xy = yx^{-1}\rangle$, $n \geq 3$ (Dihedral);

(ii) $\langle x, y \mid x^{2^{n-1}} = 1, x^{2^{n-2}} = y^2, xy = yx^{-1}\rangle$, $n \geq 3$ (Generalised quaternion);

(iii) $\langle x, y \mid x^{2^{n-1}} = y^2 = 1, xy = yx^{2^{n-2}-1}\rangle$, $n \geq 4$ (Semidihedral).

Definition. For $n \geq 3$, we write

$$D_{2n} := \langle x, y \mid x^n = 1 = y^2, xy = yx^{-1}\rangle$$

for the dihedral group of order $2n$.

Remark. $D_{2n} = \{1, x, \ldots, x^{n-1}, y, xy, \ldots, x^{n-1}y\}$. In D_{2n}, each element of $\{y, xy, \ldots, x^{n-1}y\}$ is an involution. In particular, there are $n+1$ involutions in D_{2n} when n is even.

Theorem 11. ([2]) *For $n > 2$, $s(D_{2n}) = \tau + u$, where τ is the number of positive divisors of n and u is the sum of the positive divisors of n.*

Proposition 12. *Let $G = D_{2n}$, $n > 2$. Writing u for the sum of positive divisors of n and r for the number of even proper divisors of n, we have the following: (i) if n is odd, then $nps(G) = u - 1$; (ii) if n is even, then $nps(G) = s(G) - (r+2)$; (iii) if n is a power of 2, then $nps(G) = u$; (iv) if $n = 2p$ for an odd prime p, then $nps(G) = s(G) - 3 = 3p + 4$.*

Proof. Let τ denote the number of positive divisors of n and u denote the sum of positive divisors of n. By Theorem 11, $s(G) = \tau + u$.

Let $m \in \mathbb{N} \cup \{0\}$ be arbitrary. Then

$$G^{2m+1} = \langle 1, x^{2m+1}, \ldots, x^{-(2m+1)}, y, xy, \ldots, x^{n-1}y\rangle.$$

As $\{1, y, xy, \ldots, x^{n-1}y\} \subseteq G^{2m+1}$, we see immediately that $|G^{2m+1}| > \frac{1}{2}|G|$. The fact that G^{2m+1} is a subgroup of G helps us to conclude that $G^{2m+1} = G$.

On the other hand,

$$G^{2m} = \langle 1, x^{2m}, x^{4m}, \ldots, x^{-4m}, x^{-2m}\rangle = \langle x^{2m}\rangle.$$

(i) Let n be odd. Then $\langle x^{2m}\rangle$ is of the form $\langle x^v\rangle$, where v is a positive divisor of n. Therefore the set of all power subgroups of G is given as

$$\{G\} \cup \{\langle x^v\rangle \mid v \text{ is a positive divisor of } n\}.$$

Thus $ps(G) = \tau + 1$, and we conclude that $nps(G) = (\tau + u) - (\tau + 1) = u - 1$.

(ii) Let n be even. Then $\langle x^{2m}\rangle$ is of the form $\langle x^\mu\rangle$, where μ is an even proper divisor of n. Therefore the set of all power subgroups of G is given as

$$\{\{1\}, G\} \cup \{\langle x^\mu\rangle \mid \mu \text{ is an even proper divisor of } n\}. \tag{6}$$

So $ps(G) = r + 2$, where r is the number of even proper divisors of n. Whence, $nps(G) = s(G) - (r + 2)$.

(iii) Let $n = 2^\ell \geq 4$. In the light of (6), the set of power subgroups of G is

$$\{\{1\}, G, \langle x^2 \rangle, \langle x^4 \rangle, \langle x^8 \rangle, \dots, \langle x^{n/2} \rangle\},$$

where $\langle x^2 \rangle \cong C_{n/2}$, $\langle x^4 \rangle \cong C_{n/4}$, $\langle x^8 \rangle \cong C_{n/8}$, $\dots, \langle x^{n/2} \rangle \cong C_2$. So $ps(G) = \tau$. Therefore, $nps(G) = s(G) - ps(G) = (\tau + u) - \tau = u$.

(iv) Let $n = 2p$ for an odd prime p. In the light of (6), the set of power subgroups of G is

$$\{\{1\}, G\} \cup \{\langle x^\mu \rangle \mid \mu \text{ is an even proper divisor of } 2p\} = \{\{1\}, G, \langle x^2 \rangle\},$$

where $\langle x^2 \rangle \cong C_p$. Hence, $ps(G) = 3$, and we conclude that $nps(G) = s(G) - 3 = \tau + u - 3 = 4 + (1 + 2 + p + 2p) - 3 = 3p + 4$. □

COROLLARY 13. *Given an integer* $n \geq 3$, $s(D_{2^n}) = 2^n + n - 1$ *and* $nps(D_{2^n}) = 2^n - 1$.

Proof. The results follow from a direct application of Theorem 11 and Proposition 12(iii) since the number of positive divisors of 2^{n-1}, which is the same as the number of subgroups of D_{2^n} in $\langle x \rangle$, is n, and the sum of positive divisors of 2^{n-1}, which is the same as the number of subgroups of D_{2^n} not contained in $\langle x \rangle$, is $2^n - 1$. □

DEFINITION. For $n \geq 3$, we write

$$Q_{2^n} := \langle x, y \mid x^{2^{n-1}} = 1, x^{2^{n-2}} = y^2, xy = yx^{-1} \rangle$$

for the generalised quaternion group of order 2^n.

REMARK. $Q_{2^n} = \{1, x, \dots, x^{2^{n-1}-1}, y, xy, \dots, x^{2^{n-1}-1}y\}$. Each element of $\{y, xy, \dots, x^{2^{n-1}-1}y\}$ has order 4 in Q_{2^n}, and the element $x^{2^{n-2}}$ is the unique involution in Q_{2^n}.

DEFINITION. For $n \geq 4$, we write

$$SD_{2^n} := \langle x, y \mid x^{2^{n-1}} = y^2 = 1, xy = yx^{2^{n-2}-1} \rangle$$

for the semidihedral group of order 2^n.

REMARK. $SD_{2^n} = \{1, x, \dots, x^{2^{n-1}-1}, y, xy, \dots, x^{2^{n-1}-1}y\}$. In SD_{2^n}, any element of $\{xy, x^3y, \dots, x^{2^{n-1}-1}y\} \cup \{x^{2^{n-3}}, x^{-(2^{n-3})}\}$ has order 4 while elements of $\{y, x^2y, \dots, x^{2^{n-1}-2}y\} \cup \{x^{2^{n-2}}\}$ are involutions. SD_{2^n} contains $2^{n-2} + 1$ involutions and $2^{n-2} + 2$ elements of order 4.

LEMMA 14. *Let G be any of the three 2-groups of maximal class. If A is a noncyclic proper normal subgroup of G, then* $[G : A] = 2$.

Proof. Let G be any of the three 2-groups of maximal class and of order 2^n, and let A be a noncyclic proper normal subgroup of G. Clearly, $A \not\subset \langle x\rangle$. Let $a \in A$ be such that $a \in \{y, xy, \dots, x^{2^{n-1}-1}y\}$. Now, suppose G is either dihedral or generalised quaternion. We have that $a = x^iy$ for some $i \in \{0, 1, \dots, 2^{n-1} - 1\}$. Using the relation $xy = yx^{-1}$, we obtain that $xax^{-1} = x^2(x^iy) = x^2a$. As A is normal in G and $a \in A$, we deduce that $(xax^{-1})a^{-1} = x^2 \in A$. So $\langle x^2\rangle \subseteq A$. Let G be a semidihedral group. If $a = x^{2i+1}y$ for some $i \in \{0, 1, \dots, 2^{n-2} - 1\}$, then using the relation $xy = yx^{2^{n-2}-1}$, we obtain that $xax^{-1} = yx^{-2i-3}$. Therefore $a(xax^{-1}) = x^{2i+1}yyx^{-2i-3} = x^{-2}$. As A is normal in G and $a \in A$, we conclude that $x^{-2} \in A$; whence $\langle x^{-2}\rangle = \langle x^2\rangle \subseteq A$. If $a = x^{2i}y$ for some $i \in \{0, 1, \dots, 2^{n-2} - 1\}$, then using the relation $xy = yx^{2^{n-2}-1}$, we obtain that $xax^{-1} = yx^{2^{n-2}-2i-2}$. So $a(xax^{-1}) = x^{2i}yyx^{2^{n-2}-2i-2} = x^{2^{n-2}-2} \in A$. But the order of $x^{2^{n-2}-2}$ is the same as the order of x^2; whence $\langle x^{2^{n-2}-2}\rangle = \langle x^2\rangle \subseteq A$. In all the cases, we have these three in common: $[G : \langle x^2\rangle] = 4$, $\langle x^2\rangle \subseteq A \subseteq G$ and $\langle x^2\rangle \neq A \neq G$. Therefore $[G : A] = 2$. □

Proposition 15. *Let G be any of the three 2-groups of maximal class, and of order 2^n for some $n \geq 4$. Given $k \in \{1, 2, \dots, n-2\}$, the number of subgroups of order 2^{n-k} is $2^k + 1$.*

Proof. Let $G = G_{2^n}$ be any of the three 2-groups of maximal class, and of order 2^n for some $n \geq 4$, and let $k \in \{1, 2, \dots, n-2\}$ be arbitrary. We show that there are $2^k + 1$ subgroups of size 2^{n-k}. The first case ($k = 1$) follows from the well-known fact that there are 3 subgroups of index 2 in G; the subgroups of index 2 in G are

$$\langle x\rangle, \langle x^2, y\rangle \text{ and } \langle x^2, xy\rangle,$$

where

$$\langle x\rangle \cong C_{2^{n-1}} \text{ and } \langle x^2, y\rangle \cong G_{2^{n-1}} \cong \langle x^2, xy\rangle.$$

Let H be a non-trivial subgroup of G. Recall that every non-trivial subgroup of a 2-group is contained in an index 2-subgroup of the group. Let $k \in \{1, 2, \dots, n-2\}$, and suppose H is a subgroup of size 2^{n-k} in G. In the light of Lemma 14, H is contained in either $\langle x\rangle$ or one of the noncyclic subgroups of index 2 in any (noncyclic) subgroup of G which is isomorphic to $G_{2^{n-k+1}}$. But there are 2^k noncyclic subgroups of index 2^k in G_{2^n} for any $k \in \{1, 2, \dots, n-2\}$, where $n \geq 4$. Thus, the subgroups of size 2^{n-k} (i.e., subgroups of index 2^k) in G_{2^n} are the unique cyclic subgroup of size 2^{n-k} and the 2^k non-cyclic subgroups of index 2^k. Therefore there are $1 + 2^k$ subgroups of size 2^{n-k} in G_{2^n}. □

Theorem 16. *Given an integer $n \geq 3$, $s(Q_{2^n}) = 2^{n-1} + n - 1$ and $nps(Q_{2^n}) = 2^{n-1} - 1$.*

Proof. In the light of Proposition 15, the number of subgroups of size 2^k in Q_{2^n} and D_{2^n} are equal for each $k \in \{2, 3, \dots, n-1\}$. As the the number of subgroups of index 2 in both D_8 and Q_8 is 3, one sees immediately that the assertion is also true for both D_8 and Q_8. The distinction between the number of subgroups of various sizes in Q_{2^n} and D_{2^n} (where $n \geq 3$) is in the subgroups of size 2. In particular, we have only one subgroup of size 2 in Q_{2^n} as opposed in D_{2^n}, where there are $2^{n-1}+1$ subgroups of size 2. Thus,

$$\begin{aligned} s(Q_{2^n}) =& s(D_{2^n}) - (2^{n-1}+1) + 1 \\ =& 2^{n-1} + n - 1 \text{ (by Corollary 13).} \end{aligned}$$

For the second part, let $m \in \mathbb{N} \cup \{0\}$ be arbitrary, and $G = Q_{2^n}$ for $n \geq 3$. Firstly, $G^{4m+1} = \langle 1, x^{4m+1}, \ldots, x^{-(4m+1)}, y, xy, \ldots, x^{2^{n-1}-1}y\rangle$. But $\{1, y, xy, \ldots, x^{2^{n-1}-1}y\} \subseteq G^{4m+1}$; whence $|G^{4m+1}| > \frac{1}{2}|G|$. As G^{4m+1} is a subgroup of G, we conclude that $G^{4m+1} = G$. Secondly, $G^{4m+3} = \langle 1, x^{4m+3}, \ldots, x^{-(4m+3)}, y^{-1}, (xy)^{-1}, \ldots, (x^{2^{n-1}-1}y)^{-1}\rangle$. As $|\{1, y^{-1}, (xy)^{-1}, \ldots, (x^{2^{n-1}-1}y)^{-1}\}| > \frac{1}{2}|G|$, we deduce that $G^{4m+3} = G$. Thirdly, $G^{4m+2} = \langle 1, x^{4m+2}, \ldots, x^{-(4m+2)}, x^{2^{n-2}}\rangle = \langle x^2\rangle \cong C_{2^{n-2}}$. Finally, $G^{4m} = \langle 1, x^{4m}, x^{8m}, \ldots, x^{-8m}, x^{-4m}\rangle = \langle x^{4m}\rangle$. If $G = Q_8$, then $\langle x^{4m}\rangle \cong \{1\}$. If $G = Q_{16}$, then $\langle x^{4m}\rangle \cong \{1\}$ or $\langle x^4\rangle$, where $\langle x^4\rangle \cong C_2$. Now, let $n \geq 5$, and suppose $\langle x^{4m}\rangle \neq \{1\}$. Then $\langle x^{4m}\rangle$ is exactly one of the following occuring subgroups of Q_{2^n}:

$$\langle x^{2^{n-2}}\rangle, \langle x^{2^{n-3}}\rangle, \ldots, \langle x^4\rangle,$$

where

$$\langle x^{2^{n-2}}\rangle \cong C_2, \langle x^{2^{n-3}}\rangle \cong C_4, \ldots, \langle x^4\rangle \cong C_{2^{n-3}}.$$

Therefore, $ps(Q_{2^n}) = n$; whence $nps(Q_{2^n}) = 2^{n-1} + (n-1) - n = 2^{n-1} - 1$. □

THEOREM 17. *Given an integer $n \geq 4$,*

$$s(SD_{2^n}) = 3(2^{n-2}) + n - 1 \textit{ and } nps(SD_{2^n}) = 3(2^{n-2}) - 1.$$

Proof. In the light of Proposition 15, the number of subgroups of size 2^k in SD_{2^n} and D_{2^n} are equal for each $k \in \{2, 3, \ldots, n-1\}$. The distinction between the number of subgroups of various sizes in SD_{2^n} and D_{2^n} is in the subgroups of size 2. In particular, we have only $2^{n-2}+1$ subgroups of size 2 in SD_{2^n} whilst there are $2^{n-1}+1$ subgroups of size 2 in D_{2^n}. Thus,

$$\begin{aligned} s(SD_{2^n}) =& s(D_{2^n}) - (2^{n-1}+1) + (2^{n-2}+1) \\ =& 3(2^{n-2}) + n - 1 \text{ (by Corollary 13).} \end{aligned}$$

For the second part, let $m \in \mathbb{N} \cup \{0\}$ be arbitrary, and $G = SD_{2^n}$ for $n \geq 4$. Then

$$G^{4m+1} = G = G^{4m+3}$$

follows from similar arguments as in the proof of Theorem 16. On the other hand, the results for G^{4m} and G^{4m+2} are also the same with the results for the generalised quaternion cases. Thus, $ps(SD_{2^n}) = n$; whence $nps(SD_{2^n}) = 3(2^{n-2}) + (n-1) - n = 3(2^{n-2}) - 1$. □

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