Some new inequalities for generalized convex functions pertaining generalized fractional integral operators and their applications

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Abstract

In this paper, authors establish a new identity for a differentiable function using generic integral operators. By applying it, some new integral inequalities of trapezium, Ostrowski and Simpson type are obtained. Moreover, several special cases have been studied in detail. Finally, many useful applications have been found.

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1. INTRODUCTION AND PRELIMINARIES

DEFINITION 1.1. [49] $\Lambda : \mathrm{I} \subseteq \mathbb{R} \to \mathbb{R}$ is called convex function on I, if

$$\Lambda((1-\zeta)\flat_1 + \zeta\flat_2) \leq (1-\zeta)\Lambda(\flat_1) + \zeta\Lambda(\flat_2), \quad \forall \flat_1, \flat_2 \in \mathrm{I},\ \zeta \in [0,1].$$

THEOREM 1.2. (Trapezium inequality) Suppose that $\Lambda : \mathrm{I} \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function, $\flat_1, \flat_2 \in \mathrm{I}$ with $\flat_1 < \flat_2$, then

$$\Lambda\left(\frac{\flat_1+\flat_2}{2}\right) \leq \frac{1}{\flat_2-\flat_1}\int_{\flat_1}^{\flat_2} \Lambda(\ell)d\ell \leq \frac{\Lambda(\flat_1)+\Lambda(\flat_2)}{2}. \tag{1}$$

Interested readers are referred to [4]-[6],[15; 19; 20; 22; 25; 26; 28],[33]-[38],[45; 47; 52; 53; 55; 56].

THEOREM 1.3. (Ostrowski inequality) Assume that $\Lambda : \mathrm{I} \to \mathbb{R}$ be a differentiable function on I°, $\flat_1, \flat_2 \in \mathrm{I}^\circ$ with $\flat_1 < \flat_2$. If $|\Lambda'(\ell)| \leq \mathscr{K}$, then

$$\left|\Lambda(\ell) - \frac{1}{\flat_2-\flat_1}\int_{\flat_1}^{\flat_2} \Lambda(\xi)\,d\xi\right| \leq \mathscr{K}(\flat_2-\flat_1)\left[\frac{1}{4} + \frac{\left(\ell - \frac{\flat_1+\flat_2}{2}\right)^2}{(\flat_2-\flat_1)^2}\right], \quad \forall \ell \in [\flat_1, \flat_2]. \tag{2}$$

For other recent published papers about Ostrowski type inequalities, see [1]-[3],[7]-[14],[17],[29]-[32],[39]-[41],[43; 44; 50; 54; 57].

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THEOREM 1.4. (Simpson inequality) Let $\Lambda : [\flat_1,\flat_2] \to \mathbb{R}$ be four times differentiable on $(\flat_1,\flat_2)$ and suppose that

$$\|\Lambda^{(4)}\|_\infty := \sup_{\ell\in(\flat_1,\flat_2)} |\Lambda^{(4)}| < +\infty.$$

Then

$$\left|\int_{\flat_1}^{\flat_2}\Lambda(\ell)d\ell - \frac{\flat_2-\flat_1}{3}\left[\frac{\Lambda(\flat_1)+\Lambda(\flat_2)}{2}+2\Lambda\left(\frac{\flat_1+\flat_2}{2}\right)\right]\right| \le \frac{(\flat_2-\flat_1)^5}{2880}\|\Lambda^{(4)}\|_\infty. \quad (3)$$

About Simpson type inequalities, see [27; 42; 51; 57].

In our paper we will establish some new trapezium, Ostrowski and Simpson type inequalities pertaining generalized convex functions with respect to another function. Moreover, many useful applications will be given. Hence, it is important to recall some essential facts relevant to fractional integrals.

DEFINITION 1.5. [34] Assume that $\Lambda \in \mathscr{L}[\flat_1,\flat_2]$, then κ–fractional integrals, $\eta,\kappa > 0$ with $\flat_1 \ge 0$ are

$$I_{\flat_1^+}^{\eta,\kappa}\Lambda(\xi_1) = \frac{1}{\kappa\Gamma_\kappa(\eta)}\int_{\flat_1}^{\xi_1}(\xi_1-\xi)^{\frac{\eta}{\kappa}-1}\Lambda(\xi)\,d\xi, \quad \flat_1 < \xi_1,$$

and

$$I_{\flat_2^-}^{\eta,\kappa}\Lambda(\xi_1) = \frac{1}{\kappa\Gamma_\kappa(\eta)}\int_{\xi_1}^{\flat_2}(\xi-\xi_1)^{\frac{\eta}{\kappa}-1}\Lambda(\xi)\,d\xi, \quad \flat_2 > \xi_1, \quad (4)$$

respectively.

DEFINITION 1.6. [35; 36] S is called ϖ–convex set, if

$$\varpi(\jmath)\flat_2 + (1-\varpi(\jmath))\flat_1 \in S, \;\; \forall \flat_1,\flat_2 \in S, \; \jmath\in[0,1].$$

DEFINITION 1.7. $\Lambda : S\to\mathbb{R}$ on ϖ–convex set S is called ϖ–convex function, if

$$\Lambda(\flat_1 + \jmath e^{i\varpi}(\flat_2-\flat_1)) \le (1-\jmath)\Lambda(\flat_1) + \jmath\Lambda(\flat_2), \;\; \forall \flat_1,\flat_2\in S, \; \jmath\in[0,1].$$

Raina, in [48], defined for $\rho,\delta > 0$ and $|z| < R$, the following class of functions:

$$\mathscr{F}_{\rho,\delta}^{\sigma}(z) = \mathscr{F}_{\rho,\delta}^{\sigma(0),\sigma(1),\dots}(z) = \sum_{k=0}^{+\infty}\frac{\sigma(k)}{\Gamma(\rho k+\delta)}z^k. \quad (5)$$

Choosing $|z| \leq 1$, we take hypergeometric function. For $\sigma = (1,1,\dots)$ with $\rho = \eta, (\Re(\eta) > 0), \delta = 1$ and $z \in \mathbb{C}$ in (5), we obtain Mittag–Leffler function

$$\mathscr{E}_\eta(z) = \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(1+\eta k)}.$$

DEFINITION 1.8. $S \neq \emptyset$ is called generalized convex set, if

$$m\flat_1 + \zeta \mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2 - m\flat_1) \in S, \qquad \forall\, \flat_1, \flat_2 \in S, \ \ m \in (0,1], \ \ \zeta \in [0,1]. \tag{6}$$

DEFINITION 1.9. Λ is called generalized convex function, if

$$\Lambda(m\flat_1 + \zeta \mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2 - m\flat_1)) \leq (1-\zeta)\Lambda(m\flat_1) + \zeta\Lambda(\flat_2), \quad \forall\, \flat_1, \flat_2 \in S, \ m \in (0,1], \ \ \zeta \in [0,1]. \tag{7}$$

REMARK 1.10. Taking $m = 1$ and $\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2 - \flat_1) = \flat_2 - \flat_1 > 0$, we get definition 1.1. For suitable choice of $\mathscr{F}^{\sigma}_{\rho,\delta}(\cdot)$, we obtain definition 1.7. This describes the reasons and motivations of newly defined notions and the relation with these known definitions.

DEFINITION 1.11. [23; 24] Assume that $\Lambda_2 : [\flat_1, \flat_2] \to \mathbb{R}$ is an increasing and positive monotone function on $[\flat_1, \flat_2]$, with continuous derivative on $(\flat_1, \flat_2)$. The left-right- fractional integrals of Λ_1 with respect to Λ_2 on $[\flat_1, \flat_2]$, $\eta > 0$ are

$$I^{\eta,\Lambda_2}_{\flat_1+}\Lambda_1(\xi_1) = \frac{1}{\Gamma(\eta)} \int_{\flat_1}^{\xi_1} \frac{\Lambda_2'(\xi)\Lambda_1(\xi)}{[\Lambda_2(\xi_1) - \Lambda_2(\xi)]^{1-\eta}}\, d\xi, \ \flat_1 < \xi_1, \tag{8}$$

and

$$I^{\eta,\Lambda_2}_{\flat_2-}\Lambda_1(\xi_1) = \frac{1}{\Gamma(\eta)} \int_{\xi_1}^{\flat_2} \frac{\Lambda_2'(\xi)\Lambda_1(\xi)}{[\Lambda_2(\xi) - \Lambda_2(\xi_1)]^{1-\eta}}\, d\xi, \ \flat_2 > \xi_1, \tag{9}$$

respectively.

Function $\varpi : [0,+\infty) \to [0,+\infty)$ constructed from Sarikaya et al. in [45; 46], has the following properties:

$$\int_0^1 \frac{\varpi(\xi)}{\xi}\, d\xi < +\infty, \tag{10}$$

$$\frac{1}{A_1} \leq \frac{\varpi(\varepsilon)}{\varpi(\xi_1)} \leq A_1 \text{ for } \frac{1}{2} \leq \frac{\varepsilon}{\xi_1} \leq 2, \tag{11}$$

$$\frac{\varpi(\xi_1)}{\xi_1^2} \leq A_2 \frac{\varpi(\varepsilon)}{\varepsilon^2} \text{ for } \varepsilon \leq \xi_1, \tag{12}$$

$$\left| \frac{\varpi(\xi_1)}{\xi_1^2} - \frac{\varpi(\varepsilon)}{\varepsilon^2} \right| \leq A_3 |\xi_1 - \varepsilon| \frac{\varpi(\xi_1)}{\xi_1^2} \text{ for } \frac{1}{2} \leq \frac{\varepsilon}{\xi_1} \leq 2, \tag{13}$$

where $A_1, A_2, A_3 > 0$ are independent of $\varepsilon, \xi_1 > 0$. Moreover, Sarikaya et al. defined the following useful operators:

$$_{\flat_1^+}I_{\varpi}\Lambda(\xi_1) = \int_{\flat_1}^{\xi_1} \frac{\varpi(\xi_1 - \xi)}{\xi_1 - \xi}\Lambda(\xi)\,d\xi, \quad \flat_1 < \xi_1, \tag{14}$$

$$_{\flat_2^-}I_{\varpi}\Lambda(\xi_1) = \int_{\xi_1}^{\flat_2} \frac{\varpi(\xi - \xi_1)}{\xi - \xi_1}\Lambda(\xi)\,d\xi, \quad \flat_2 > \xi_1. \tag{15}$$

About their efficiency, see [18; 21; 45]. Finally, Farid in [16], defined the following generic operators:

$$G_{\flat_1+}^{\varpi,\Lambda_2}\Lambda_1(\xi_1) = \int_{\flat_1}^{\xi_1} \frac{\varpi(\Lambda_2(\xi_1) - \Lambda_2(\xi))}{\Lambda_2(\xi_1) - \Lambda_2(\xi)}\Lambda_2'(\xi)\Lambda_1(\xi)\,d\xi, \; \flat_1 < \xi_1, \tag{16}$$

and

$$G_{\flat_2-}^{\varpi,\Lambda_2}\Lambda_1(\xi_1) = \int_{\xi_1}^{\flat_2} \frac{\varpi(\Lambda_2(\xi) - \Lambda_2(\xi_1))}{\Lambda_2(\xi) - \Lambda_2(\xi_1)}\Lambda_2'(\xi)\Lambda_1(\xi)\,d\xi, \; \flat_2 > \xi_1, \tag{17}$$

respectively.

The paper is constructed in this way: In Section 2, we will find an interesting identity with parameter λ and using generic integral operators form auxiliary equality, some new integral inequalities of trapezium, Ostrowski and Simpson type will be obtain. Section 3 is devoted to useful applications.

2. MAIN RESULTS

Let $\mathscr{P} = [m\flat_1, \flat_2]$, where $\flat_1 < \flat_2$ for some fixed $m \in (0,1]$ with $\zeta \in [0,1]$.

$$\Pi_m^{\varpi,\Upsilon}(\ell,\zeta) := \int_0^{\zeta} \frac{\varpi\left(\Upsilon\left(m\flat_1 + \xi\mathscr{F}_{\rho,\delta}^{\sigma}(\ell - m\flat_1)\right) - \Upsilon(m\flat_1)\right)}{\Upsilon\left(m\flat_1 + \xi\mathscr{F}_{\rho,\delta}^{\sigma}(\ell - m\flat_1)\right) - \Upsilon(m\flat_1)} \tag{18}$$

$$\times\Upsilon'\left(m\flat_1 + \xi\mathscr{F}_{\rho,\delta}^{\sigma}(\ell - m\flat_1)\right)d\xi < +\infty,$$

and

$$\Xi_m^{\varpi,\Upsilon}(\ell,\zeta) := \int_{\zeta}^{1} \frac{\varpi\left(\Upsilon\left(m\ell + \mathscr{F}_{\rho,\delta}^{\sigma}(\flat_2 - m\ell)\right) - \Upsilon\left(m\ell + \xi\mathscr{F}_{\rho,\delta}^{\sigma}(\flat_2 - m\ell)\right)\right)}{\Upsilon\left(m\ell + \mathscr{F}_{\rho,\delta}^{\sigma}(\flat_2 - m\ell)\right) - \Upsilon\left(m\ell + \xi\mathscr{F}_{\rho,\delta}^{\sigma}(\flat_2 - m\ell)\right)} \tag{19}$$

$$\times \Upsilon'\left(m\ell+\xi \mathscr{F}^{\sigma}_{\rho,\lambda}(\flat_2-m\ell)\right)d\xi<+\infty.$$

The following lemma will help us to find new results.

LEMMA 2.1. Let $\Lambda:\mathscr{P}\to\mathbb{R}$ be a differentiable function on $\mathscr{P}^{\circ}$ and $\lambda\in\mathbb{R}$. Assume that $\Lambda'\in\mathscr{L}(\mathscr{P})$ and $\mathscr{F}^{\sigma}_{\rho,\lambda}(\flat_2-m\flat_1)>0$, then

$$\frac{\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)\Lambda\left(m\flat_1+\zeta\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)\right)+\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)\Lambda(m\ell)}{\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\flat_1)}$$

$$-\frac{\lambda}{\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\flat_1)}\times\left[\frac{\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)\Lambda\left(m\flat_1+\zeta\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)\right)}{\Pi^{\varpi,\Upsilon}_m(\ell,1)}+\frac{\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)\Lambda(m\ell)}{\Xi^{\varpi,\Upsilon}_m(\ell,0)}\right]$$

$$+\frac{\lambda}{\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\flat_1)}\times\left[\frac{\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)\Lambda(m\flat_1)}{\Pi^{\varpi,\Upsilon}_m(\ell,1)}+\frac{\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)\Lambda(m\ell+\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell))}{\Xi^{\varpi,\Upsilon}_m(\ell,0)}\right]$$

$$-\frac{1}{\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\flat_1)}\times\left[\frac{G^{\varpi,\Upsilon}_{\left(m\flat_1+\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)\right)^-}\Lambda(m\flat_1)}{\Pi^{\varpi,\Upsilon}_m(\ell,1)}+\frac{G^{\varpi,\Upsilon}_{(m\ell)^+}\Lambda(m\ell+\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell))}{\Xi^{\varpi,\Upsilon}_m(\ell,0)}\right]$$

$$=\frac{\left[\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)\right]^2}{\Pi^{\varpi,\Upsilon}_m(\ell,1)\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\flat_1)}\times\int_0^1\left[\Pi^{\varpi,\Upsilon}_m(\ell,\zeta)-\lambda\right]\Lambda'\left(m\flat_1+\zeta\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)\right)d\zeta \quad (20)$$

$$-\frac{\left[\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)\right]^2}{\Xi^{\varpi,\Upsilon}_m(\ell,0)\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\flat_1)}\times\int_0^1\left[\Xi^{\varpi,\Upsilon}_m(\ell,\zeta)-\lambda\right]\Lambda'\left(m\ell+\zeta\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)\right)d\zeta.$$

We denote

$$T_{\Lambda,\Pi^{\varpi,\Upsilon}_m,\Xi^{\varpi,\Upsilon}_m}(\lambda;\ell,\flat_1,\flat_2):=\frac{\left[\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)\right]^2}{\Pi^{\varpi,\Upsilon}_m(\ell,1)\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\flat_1)} \quad (21)$$

$$\times\int_0^1\left[\Pi^{\varpi,\Upsilon}_m(\ell,\zeta)-\lambda\right]\Lambda'\left(m\flat_1+\zeta\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)\right)d\zeta$$

$$-\frac{\left[\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)\right]^2}{\Xi^{\varpi,\Upsilon}_m(\ell,0)\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\flat_1)}\times\int_0^1\left[\Xi^{\varpi,\Upsilon}_m(\ell,\zeta)-\lambda\right]\Lambda'\left(m\ell+\zeta\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)\right)d\zeta.$$

PROOF. By integrating by parts (21), we derive

$$T_{\Lambda,\Pi_m^{\varpi,\Upsilon},\Xi_m^{\varpi,\Upsilon}}(\lambda;\ell,\flat_1,\flat_2)=\frac{\left[\mathscr{F}_{\rho,\delta}^{\sigma}(\ell-m\flat_1)\right]^2}{\Pi_m^{\varpi,\Upsilon}(\ell,1)\mathscr{F}_{\rho,\delta}^{\sigma}(\flat_2-m\flat_1)}$$

$$\times\left\{\int_0^1\Pi_m^{\varpi,\Upsilon}(\ell,\zeta)\Lambda'\left(m\flat_1+\zeta\mathscr{F}_{\rho,\delta}^{\sigma}(\ell-m\flat_1)\right)d\zeta-\lambda\int_0^1\Lambda'\left(m\flat_1+\zeta\mathscr{F}_{\rho,\delta}^{\sigma}(\ell-m\flat_1)\right)d\zeta\right\}$$

$$-\frac{\left[\mathscr{F}_{\rho,\delta}^{\sigma}(\flat_2-m\ell)\right]^2}{\Xi_m^{\varpi,\Upsilon}(\ell,0)\mathscr{F}_{\rho,\delta}^{\sigma}(\flat_2-m\flat_1)}$$

$$\times\left\{\int_0^1\Xi_m^{\varpi,\Upsilon}(\ell,\zeta)\Lambda'\left(m\ell+\zeta\mathscr{F}_{\rho,\delta}^{\sigma}(\flat_2-m\ell)\right)d\zeta-\lambda\int_0^1\Lambda'\left(m\ell+\zeta\mathscr{F}_{\rho,\delta}^{\sigma}(\flat_2-m\ell)\right)d\zeta\right\}$$

$$=\frac{\left[\mathscr{F}_{\rho,\delta}^{\sigma}(\ell-m\flat_1)\right]^2}{\Pi_m^{\varpi,\Upsilon}(\ell,1)\mathscr{F}_{\rho,\delta}^{\sigma}(\flat_2-m\flat_1)}\times\left\{\frac{\Pi_m^{\varpi,\Upsilon}(\ell,\zeta)\Lambda\left(m\flat_1+\zeta\mathscr{F}_{\rho,\delta}^{\sigma}(\ell-m\flat_1)\right)}{\mathscr{F}_{\rho,\delta}^{\sigma}(\ell-m\flat_1)}\right|_0^1-\frac{1}{\mathscr{F}_{\rho,\delta}^{\sigma}(\ell-m\flat_1)}$$

$$\times\int_0^1\frac{\varpi\left(\Upsilon\left(m\flat_1+\zeta\mathscr{F}_{\rho,\delta}^{\sigma}(\ell-m\flat_1)\right)-\Upsilon(m\flat_1)\right)}{\Upsilon\left(m\flat_1+\zeta\mathscr{F}_{\rho,\delta}^{\sigma}(\ell-m\flat_1)\right)-\Upsilon(m\flat_1)}$$

$$\times\Upsilon'\left(m\flat_1+\zeta\mathscr{F}_{\rho,\delta}^{\sigma}(\ell-m\flat_1)\right)\Lambda\left(m\flat_1+\zeta\mathscr{F}_{\rho,\delta}^{\sigma}(\ell-m\flat_1)\right)d\zeta$$

$$\left.-\frac{\lambda}{\mathscr{F}_{\rho,\delta}^{\sigma}(\ell-m\flat_1)}\Lambda\left(m\flat_1+\zeta\mathscr{F}_{\rho,\delta}^{\sigma}(\ell-m\flat_1)\right)\right|_0^1\Bigg\}-\frac{\left[\mathscr{F}_{\rho,\delta}^{\sigma}(\flat_2-m\ell)\right]^2}{\Xi_m^{\varpi,\Upsilon}(\ell,0)\mathscr{F}_{\rho,\delta}^{\sigma}(\flat_2-m\flat_1)}$$

$$\times\left\{\frac{\Xi_m^{\varpi,\Upsilon}(\ell,\zeta)\Lambda\left(m\ell+\zeta\mathscr{F}_{\rho,\delta}^{\sigma}(\flat_2-m\ell)\right)}{\mathscr{F}_{\rho,\delta}^{\sigma}(\flat_2-m\ell)}\right|_0^1$$

$$-\frac{1}{\mathscr{F}_{\rho,\delta}^{\sigma}(\flat_2-m\ell)}\times\int_0^1\frac{\varpi\left(\Upsilon\left(m\ell+\mathscr{F}_{\rho,\delta}^{\sigma}(\flat_2-m\ell)\right)-\Upsilon\left(m\ell+\zeta\mathscr{F}_{\rho,\delta}^{\sigma}(\flat_2-m\ell)\right)\right)}{\Upsilon\left(m\ell+\mathscr{F}_{\rho,\delta}^{\sigma}(\flat_2-m\ell)\right)-\Upsilon\left(m\ell+\zeta\mathscr{F}_{\rho,\delta}^{\sigma}(\flat_2-m\ell)\right)}$$

$$\times\Upsilon'\left(m\ell+\zeta\mathscr{F}_{\rho,\delta}^{\sigma}(\flat_2-m\ell)\right)\Lambda\left(m\ell+\zeta\mathscr{F}_{\rho,\delta}^{\sigma}(\flat_2-m\ell)\right)d\zeta$$

$$\left.-\frac{\lambda}{\mathscr{F}_{\rho,\delta}^{\sigma}(\flat_2-m\ell)}\Lambda\left(m\ell+\zeta\mathscr{F}_{\rho,\delta}^{\sigma}(\flat_2-m\ell)\right)\right|_0^1\Bigg\}$$

$$= \frac{\mathscr{F}^{\sigma}_{\rho,\delta}(\ell - m\flat_1)\Lambda\left(m\flat_1 + \zeta \mathscr{F}^{\sigma}_{\rho,\delta}(\ell - m\flat_1)\right) + \mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2 - m\ell)\Lambda(m\ell)}{\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2 - m\flat_1)}$$

$$-\frac{\lambda}{\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2 - m\flat_1)} \times \left[\frac{\mathscr{F}^{\sigma}_{\rho,\delta}(\ell - m\flat_1)\Lambda\left(m\flat_1 + \zeta \mathscr{F}^{\sigma}_{\rho,\delta}(\ell - m\flat_1)\right)}{\Pi^{\varpi,\Upsilon}_{m}(\ell,1)} + \frac{\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2 - m\ell)\Lambda(m\ell)}{\Xi^{\varpi,\Upsilon}_{m}(\ell,0)}\right]$$

$$+\frac{\lambda}{\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2 - m\flat_1)} \times \left[\frac{\mathscr{F}^{\sigma}_{\rho,\delta}(\ell - m\flat_1)\Lambda(m\flat_1)}{\Pi^{\varpi,\Upsilon}_{m}(\ell,1)} + \frac{\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2 - m\ell)\Lambda(m\ell + \mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2 - m\ell))}{\Xi^{\varpi,\Upsilon}_{m}(\ell,0)}\right]$$

$$-\frac{1}{\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2 - m\flat_1)} \times \left[\frac{G^{\varpi,\Upsilon}_{\left(m\flat_1 + \mathscr{F}^{\sigma}_{\rho,\delta}(\ell - m\flat_1)\right)^-}\Lambda(m\flat_1)}{\Pi^{\varpi,\Upsilon}_{m}(\ell,1)} + \frac{G^{\varpi,\Upsilon}_{(m\ell)^+}\Lambda(m\ell + \mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2 - m\ell))}{\Xi^{\varpi,\Upsilon}_{m}(\ell,0)}\right].$$

REMARK 2.2. *a.* Taking $m = 1$, $\lambda = 0$, $\mathscr{F}^{\sigma}_{\rho,\delta}(\ell - m\flat_1) = \ell - m\flat_1$, $\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2 - m\ell) = \flat_2 - m\ell$, $\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2 - m\flat_1) = \flat_2 - m\flat_1$ and $\Upsilon(\zeta) = \zeta = \varpi(\zeta)$ in Lemma 2.1, then

$$T_{\Lambda}(\ell,\flat_1,\flat_2) := \Lambda(\ell) - \frac{1}{\flat_2 - \flat_1}\int_{\flat_1}^{\flat_2} \Lambda(\zeta)\, d\zeta.$$

b. Choosing $m = 1$, $\lambda = 1$, $\mathscr{F}^{\sigma}_{\rho,\delta}(\ell - m\flat_1) = \ell - m\flat_1$, $\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2 - m\ell) = \flat_2 - m\ell$, $\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2 - m\flat_1) = \flat_2 - m\flat_1$ and $\Upsilon(\zeta) = \zeta = \varpi(\zeta)$ in Lemma 2.1, then

$$\overline{T_{\Lambda}}(\ell,\flat_1,\flat_2) := \frac{(\ell - \flat_1)\Lambda(\flat_1) + (\flat_2 - \ell)\Lambda(\flat_2)}{\flat_2 - \flat_1} - \frac{1}{\flat_2 - \flat_1}\int_{\flat_1}^{\flat_2} \Lambda(\zeta)\, d\zeta.$$

c. Taking $m = 1$, $\ell = \frac{\flat_1 + \flat_2}{2}$, $\mathscr{F}^{\sigma}_{\rho,\delta}(\ell - m\flat_1) = \ell - m\flat_1$, $\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2 - m\ell) = \flat_2 - m\ell$, $\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2 - m\flat_1) = \flat_2 - m\flat_1$ and $\Upsilon(\zeta) = \zeta = \varpi(\zeta)$ in Lemma 2.1, then

$$T_{\Lambda}(\lambda;\flat_1,\flat_2) := \lambda\left[\frac{\Lambda(\flat_1) + \Lambda(\flat_2)}{2}\right] + (1 - \lambda)\Lambda\left(\frac{\flat_1 + \flat_2}{2}\right) - \frac{1}{\flat_2 - \flat_1}\int_{\flat_1}^{\flat_2} \Lambda(\zeta)\, d\zeta.$$

THEOREM 2.3. Let $\Lambda : \mathscr{P} \to \mathbb{R}$ be a differentiable function on $\mathscr{P}^{\circ}$ and $\lambda \in [0,1]$. If $|\Lambda'|^q$ is generalized convex function on $\mathscr{P}$ and $\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2 - m\flat_1) > 0$, then for $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$\left|T_{\Lambda,\Pi^{\varpi,\Upsilon}_{m},\Xi^{\varpi,\Upsilon}_{m}}(\lambda;\ell,\flat_1,\flat_2)\right|$$

$$\leq \frac{\left[\mathscr{F}^{\sigma}_{\rho,\delta}(\ell - m\flat_1)\right]^2}{\sqrt[q]{2}\Pi^{\varpi,\Upsilon}_{m}(\ell,1)\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2 - m\flat_1)}\sqrt[p]{B^{\varpi,\Upsilon}_{\Pi_m}(\ell;\lambda,p)} \times \sqrt[q]{|\Lambda'(m\flat_1)|^q + |\Lambda'(\ell)|^q} \qquad (22)$$

$$+\frac{\left[\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)\right]^2}{\sqrt[q]{2}\,\Xi^{\varpi,\Upsilon}_m(\ell,0)\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\flat_1)}\sqrt[p]{B^{\varpi,\Upsilon}_{\Xi_m}(\ell;\lambda,p)}\times\sqrt[q]{|\Lambda'(m\ell)|^q+|\Lambda'(\flat_2)|^q},$$

where

$$B^{\varpi,\Upsilon}_{\Pi_m}(\ell;\lambda,p):=\int_0^1\left|\Pi^{\varpi,\Upsilon}_m(\ell,\zeta)-\lambda\right|^p d\zeta,\ \ B^{\varpi,\Upsilon}_{\Xi_m}(\ell;\lambda,p):=\int_0^1\left|\Xi^{\varpi,\Upsilon}_m(\ell,\zeta)-\lambda\right|^p d\zeta. \quad (23)$$

PROOF. Applying Lemma 2.1, generalized convexity of $|\Lambda'|^q$, Hölder's inequality, we get

$$\left|T_{\Lambda,\Pi^{\varpi,\Upsilon}_m,\Xi^{\varpi,\Upsilon}_m}(\lambda;\ell,\flat_1,\flat_2)\right|$$

$$\leq\frac{\left[\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)\right]^2}{\Pi^{\varpi,\Upsilon}_m(\ell,1)\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\flat_1)}\times\int_0^1\left|\Pi^{\varpi,\Upsilon}_m(\ell,\zeta)-\lambda\right|\left|\Lambda'\left(m\flat_1+\zeta\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)\right)\right|d\zeta$$

$$+\frac{\left[\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)\right]^2}{\Xi^{\varpi,\Upsilon}_m(\ell,0)\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\flat_1)}\times\int_0^1\left|\Xi^{\varpi,\Upsilon}_m(\ell,\zeta)-\lambda\right|\left|\Lambda'\left(m\ell+\zeta\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)\right)\right|d\zeta$$

$$\leq\frac{\left[\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)\right]^2}{\Pi^{\varpi,\Upsilon}_m(\ell,1)\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\flat_1)}$$

$$\times\left(\int_0^1\left|\Pi^{\varpi,\Upsilon}_m(\ell,\zeta)-\lambda\right|^p d\zeta\right)^{\frac{1}{p}}\left(\int_0^1\left|\Lambda'\left(m\flat_1+\zeta\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)\right)\right|^q d\zeta\right)^{\frac{1}{q}}$$

$$+\frac{\left[\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)\right]^2}{\Xi^{\varpi,\Upsilon}_m(\ell,0)\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\flat_1)}$$

$$\times\left(\int_0^1\left|\Xi^{\varpi,\Upsilon}_m(\ell,\zeta)-\lambda\right|^p d\zeta\right)^{\frac{1}{p}}\left(\int_0^1\left|\Lambda'\left(m\ell+\zeta\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)\right)\right|^q d\zeta\right)^{\frac{1}{q}}$$

$$\leq\frac{\left[\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)\right]^2}{\Pi^{\varpi,\Upsilon}_m(\ell,1)\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\flat_1)}\sqrt[p]{B^{\varpi,\Upsilon}_{\Pi_m}(\ell;\lambda,p)}\left(\int_0^1\left[(1-\zeta)|\Lambda'(m\flat_1)|^q+\zeta|\Lambda'(\ell)|^q\right]d\zeta\right)^{\frac{1}{q}}$$

$$+\frac{\left[\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)\right]^2}{\Xi^{\varpi,\Upsilon}_m(\ell,0)\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\flat_1)}\sqrt[p]{B^{\varpi,\Upsilon}_{\Xi_m}(\ell;\lambda,p)}\left(\int_0^1\left[(1-\zeta)|\Lambda'(m\ell)|^q+\zeta|\Lambda'(\flat_2)|^q\right]d\zeta\right)^{\frac{1}{q}}$$

$$= \frac{\left[\mathscr{F}^{\sigma}_{\rho,\delta}(\ell - m\flat_1)\right]^2}{\sqrt[q]{2}\Pi^{\varpi,\Upsilon}_m(\ell,1)\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2 - m\flat_1)} \sqrt[p]{B^{\varpi,\Upsilon}_{\Pi_m}(\ell;\lambda,p)} \times \sqrt[q]{|\Lambda'(m\flat_1)|^q + |\Lambda'(\ell)|^q}$$

$$+ \frac{\left[\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2 - m\ell)\right]^2}{\sqrt[q]{2}\Xi^{\varpi,\Upsilon}_m(\ell,0)\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2 - m\flat_1)} \sqrt[p]{B^{\varpi,\Upsilon}_{\Xi_m}(\ell;\lambda,p)} \times \sqrt[q]{|\Lambda'(m\ell)|^q + |\Lambda'(\flat_2)|^q}.$$

COROLLARY 2.4. Taking $p = 2 = q$ in Theorem 2.3, we have

$$\left|T_{\Lambda,\Pi^{\varpi,\Upsilon}_m,\Xi^{\varpi,\Upsilon}_m}(\lambda;\ell,\flat_1,\flat_2)\right|$$

$$\leq \frac{\left[\mathscr{F}^{\sigma}_{\rho,\delta}(\ell - m\flat_1)\right]^2}{\sqrt{2}\Pi^{\varpi,\Upsilon}_m(\ell,1)\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2 - m\flat_1)} \sqrt{B^{\varpi,\Upsilon}_{\Pi_m}(\ell;\lambda,2)} \times \sqrt{|\Lambda'(m\flat_1)|^2 + |\Lambda'(\ell)|^2} \tag{24}$$

$$+ \frac{\left[\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2 - m\ell)\right]^2}{\sqrt{2}\Xi^{\varpi,\Upsilon}_m(\ell,0)\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2 - m\flat_1)} \sqrt{B^{\varpi,\Upsilon}_{\Xi_m}(\ell;\lambda,2)} \times \sqrt{|\Lambda'(m\ell)|^2 + |\Lambda'(\flat_2)|^2}.$$

COROLLARY 2.5. Choosing $|\Lambda'| \leq \mathscr{K}$ in Theorem 2.3, we get

$$\left|T_{\Lambda,\Pi^{\varpi,\Upsilon}_m,\Xi^{\varpi,\Upsilon}_m}(\lambda;\ell,\flat_1,\flat_2)\right| \leq \frac{\mathscr{K}}{\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2 - m\flat_1)} \tag{25}$$

$$\times \left[\frac{\left[\mathscr{F}^{\sigma}_{\rho,\delta}(\ell - m\flat_1)\right]^2}{\Pi^{\varpi,\Upsilon}_m(\ell,1)} \sqrt[p]{B^{\varpi,\Upsilon}_{\Pi_m}(\ell;\lambda,p)} + \frac{\left[\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2 - m\ell)\right]^2}{\Xi^{\varpi,\Upsilon}_m(\ell,0)} \sqrt[p]{B^{\varpi,\Upsilon}_{\Xi_m}(\ell;\lambda,p)} \right].$$

COROLLARY 2.6. Taking $m = 1$, $\lambda = 0$, $\mathscr{F}^{\sigma}_{\rho,\delta}(\ell - m\flat_1) = \ell - m\flat_1$, $\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2 - m\ell) = \flat_2 - m\ell$, $\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2 - m\flat_1) = \flat_2 - m\flat_1$ and $\Upsilon(\zeta) = \zeta = \varpi(\zeta)$ in Theorem 2.3, we obtain

$$\left|T_{\Lambda}(\ell,\flat_1,\flat_2)\right| \leq \frac{1}{\sqrt[q]{2}\sqrt[p]{p+1}(\flat_2 - \flat_1)} \tag{26}$$

$$\times \left\{ (\ell - \flat_1)^2 \sqrt[q]{|\Lambda'(\flat_1)|^q + |\Lambda'(\ell)|^q} + (\flat_2 - \ell)^2 \sqrt[q]{|\Lambda'(\ell)|^q + |\Lambda'(\flat_2)|^q} \right\}.$$

COROLLARY 2.7. Choosing $\ell = \frac{\flat_1 + \flat_2}{2}$ in Corollary 2.6, we have

$$\left|T_{\Lambda}(\flat_1,\flat_2)\right| \leq \frac{(\flat_2 - \flat_1)}{4\sqrt[q]{2}\sqrt[p]{p+1}} \tag{27}$$

$$\times \left\{ \sqrt[q]{|\Lambda'(\flat_1)|^q + \left|\Lambda'\left(\frac{\flat_1 + \flat_2}{2}\right)\right|^q} + \sqrt[q]{\left|\Lambda'\left(\frac{\flat_1 + \flat_2}{2}\right)\right|^q + |\Lambda'(\flat_2)|^q} \right\}.$$

COROLLARY 2.8. Taking $m = 1$, $\lambda = 1$, $\mathscr{F}^{\sigma}_{\rho,\delta}(\ell - m\flat_1) = \ell - m\flat_1$, $\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2 - m\ell) =$

$\flat_2 - m\ell$, $\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2 - m\flat_1) = \flat_2 - m\flat_1$ and $\Upsilon(\zeta) = \zeta = \varpi(\zeta)$ in Theorem 2.3, we get

$$\left|\overline{T_{\Lambda}}(\ell, \flat_1, \flat_2)\right| \le \frac{1}{\sqrt[q]{2}\sqrt[p]{p+1}(\flat_2 - \flat_1)} \tag{28}$$

$$\times\left\{(\ell - \flat_1)^2 \sqrt[q]{|\Lambda'(\flat_1)|^q + |\Lambda'(\ell)|^q} + (\flat_2 - \ell)^2 \sqrt[q]{|\Lambda'(\ell)|^q + |\Lambda'(\flat_2)|^q}\right\}.$$

COROLLARY 2.9. Choosing $m = 1$, $\lambda = \frac{1}{3}$, $\mathscr{F}^{\sigma}_{\rho,\delta}(\ell - m\flat_1) = \ell - m\flat_1$, $\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2 - m\ell) = \flat_2 - m\ell$, $\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2 - m\flat_1) = \flat_2 - m\flat_1$ and $\Upsilon(\zeta) = \zeta = \varpi(\zeta)$ in Theorem 2.3, we obtain

$$\left|T_{\Lambda}\left(\frac{1}{3}; \flat_1, \flat_2\right)\right| \le \frac{1}{\sqrt[q]{2}(\flat_2 - \flat_1)} \sqrt[p]{\frac{2^{p+1}+1}{3^{p+1}(p+1)}} \tag{29}$$

$$\times\left\{(\ell - \flat_1)^2 \sqrt[q]{|\Lambda'(\flat_1)|^q + |\Lambda'(\ell)|^q} + (\flat_2 - \ell)^2 \sqrt[q]{|\Lambda'(\ell)|^q + |\Lambda'(\flat_2)|^q}\right\}.$$

COROLLARY 2.10. Substituting $\lambda = 0$ and $\varpi(\zeta) = \zeta$ in Theorem 2.3, we have

$$\left|T_{\Lambda,\Pi^{\Upsilon}_{m},\Xi^{\Upsilon}_{m}}(0; \ell, \flat_1, \flat_2)\right| \le \frac{1}{\sqrt[q]{2}\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2 - m\flat_1)} \tag{30}$$

$$\times\left\{\sqrt[q]{\mathscr{F}^{\sigma}_{\rho,\delta}(\ell - m\flat_1)}\sqrt[p]{B^{\Upsilon}_1(\ell; p)} \times \sqrt[q]{|\Lambda'(m\flat_1)|^q + |\Lambda'(\ell)|^q}\right.$$

$$\left. + \sqrt[q]{\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2 - m\ell)}\sqrt[p]{B^{\Upsilon}_2(\ell; p)} \times \sqrt[q]{|\Lambda'(m\ell)|^q + |\Lambda'(\flat_2)|^q}\right\},$$

where

$$B^{\Upsilon}_1(\ell; p) := \int_{m\flat_1}^{m\flat_1 + \mathscr{F}^{\sigma}_{\rho,\delta}(\ell - m\flat_1)} \left[\Upsilon(\zeta) - \Upsilon(m\flat_1)\right]^p d\zeta \tag{31}$$

and

$$B^{\Upsilon}_2(\ell; p) := \int_{m\ell}^{m\ell + \mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2 - m\ell)} \left[\Upsilon(m\ell + \mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2 - m\ell)) - \Upsilon(\zeta)\right]^p d\zeta. \tag{32}$$

COROLLARY 2.11. For $\lambda = 0$ and $\varpi(\zeta) = \frac{\zeta^{\alpha}}{\Gamma(\alpha)}$ in Theorem 2.3, we get

$$\left|T_{\Lambda,\Pi^{\Upsilon}_{m},\Xi^{\Upsilon}_{m}}(0; \ell, \flat_1, \flat_2)\right| \le \frac{1}{\sqrt[q]{2}\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2 - m\flat_1)} \tag{33}$$

$$\times\left\{\sqrt[q]{\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)}\sqrt[p]{B^{\Upsilon}_3(\ell;p,\alpha)}\times\sqrt[q]{|\Lambda'(m\flat_1)|^q+|\Lambda'(\ell)|^q}\right.$$

$$\left.+\sqrt[q]{\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)}\sqrt[p]{B^{\Upsilon}_4(\ell;p,\alpha)}\times\sqrt[q]{|\Lambda'(m\ell)|^q+|\Lambda'(\flat_2)|^q}\right\},$$

where

$$B^{\Upsilon}_3(\ell;p,\alpha):=\int_{m\flat_1}^{m\flat_1+\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)}\left[\Upsilon(\zeta)-\Upsilon(m\flat_1)\right]^{p\alpha}d\zeta \tag{34}$$

and

$$B^{\Upsilon}_4(\ell;p,\alpha):=\int_{m\ell}^{m\ell+\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)}\left[\Upsilon(m\ell+\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell))-\Upsilon(\zeta)\right]^{p\alpha}d\zeta. \tag{35}$$

COROLLARY 2.12. Substituting $\lambda=0$ and $\varpi(\zeta)=\frac{\zeta^{\frac{\alpha}{\kappa}}}{\kappa\Gamma_\kappa(\alpha)}$ in Theorem 2.3, we obtain

$$\left|T_{\Lambda,\Pi^{\Upsilon}_m,\Xi^{\Upsilon}_m}(0;\ell,\flat_1,\flat_2)\right|\le\frac{1}{\sqrt[q]{2}\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\flat_1)} \tag{36}$$

$$\times\left\{\sqrt[q]{\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)}\sqrt[p]{B^{\Upsilon}_5(\ell;p,\alpha,\kappa)}\times\sqrt[q]{|\Lambda'(m\flat_1)|^q+|\Lambda'(\ell)|^q}\right.$$

$$\left.+\sqrt[q]{\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)}\sqrt[p]{B^{\Upsilon}_6(\ell;p,\alpha,\kappa)}\times\sqrt[q]{|\Lambda'(m\ell)|^q+|\Lambda'(\flat_2)|^q}\right\},$$

where

$$B^{\Upsilon}_5(\ell;p,\alpha,\kappa):=\int_{m\flat_1}^{m\flat_1+\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)}\left[\Upsilon(\zeta)-\Upsilon(m\flat_1)\right]^{\frac{p\alpha}{\kappa}}d\zeta \tag{37}$$

and

$$B^{\Upsilon}_6(\ell;p,\alpha,\kappa):=\int_{m\ell}^{m\ell+\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)}\left[\Upsilon(m\ell+\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell))-\Upsilon(\zeta)\right]^{\frac{p\alpha}{\kappa}}d\zeta. \tag{38}$$

COROLLARY 2.13. For $\lambda=0$, $\forall\,\xi\in[0,\zeta]$, $\varpi_\Upsilon(\ell,\zeta)=\zeta(\Upsilon(m\flat_1+\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1))-\zeta)^{\alpha-1}$ and $\forall\,\xi\in[\zeta,1]$, $\varpi_\Upsilon(\ell,\zeta)=\zeta(\Upsilon(m\ell+\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell))-\zeta)^{\alpha-1}$ in Theorem 2.3, we have

$$\left|T_{\Lambda,\Pi^{\Upsilon}_m,\Xi^{\Upsilon}_m}(0;\ell,\flat_1,\flat_2)\right|\le\frac{\left[\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)\right]^{\frac{q+1}{q}}}{\sqrt[q]{2}\left[\Upsilon(m\flat_1+\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1))-\Upsilon(m\flat_1)\right]\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\flat_1)} \tag{39}$$

$$\times\sqrt[p]{B^{\Upsilon}_7(\ell;p)}\times\sqrt[q]{|\Lambda'(m\flat_1)|^q+|\Lambda'(\ell)|^q}$$

$$+\frac{\left[\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)\right]^{\frac{q+1}{q}}}{\sqrt[q]{2}\left[\Upsilon^{\alpha}(m\ell+\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell))-\Upsilon^{\alpha}(m\ell)\right]\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\flat_1)}$$

$$\times\sqrt[p]{B^{\Upsilon}_{8}(\ell;p,\alpha)}\times\sqrt[q]{|\Lambda'(m\ell)|^{q}+|\Lambda'(\flat_2)|^{q}},$$

where

$$B^{\Upsilon}_{7}(\ell;p):=\int_{m\flat_1}^{m\flat_1+\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)}\left[\Upsilon(\zeta)-\Upsilon(m\flat_1)\right]^{p}d\zeta \tag{40}$$

and

$$B^{\Upsilon}_{8}(\ell;p,\alpha):=\int_{m\ell}^{m\ell+\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)}\Big[\Upsilon^{\alpha}(m\ell+\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell))-\Upsilon^{\alpha}(\zeta)\Big]^{p}d\zeta. \tag{41}$$

COROLLARY 2.14. Substituting $\lambda=0$ and $\varpi(\zeta)=\frac{\zeta}{\alpha}\exp(-A\zeta)$, where $A=\frac{1-\alpha}{\alpha}$, in Theorem 2.3, we get

$$\left|T_{\Lambda,\Pi^{\Upsilon}_{m},\Xi^{\Upsilon}_{m}}(0;\ell,\flat_1,\flat_2)\right|\le\frac{\left[\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)\right]^{\frac{q+1}{q}}}{\sqrt[q]{2}\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\flat_1)} \tag{42}$$

$$\times\sqrt[p]{B^{\Upsilon}_{9}(\ell;p,A)}\times\sqrt[q]{|\Lambda'(m\flat_1)|^{q}+|\Lambda'(\ell)|^{q}}$$

$$+\frac{\left[\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)\right]^{\frac{q+1}{q}}}{\sqrt[q]{2}\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\flat_1)}\sqrt[p]{B^{\Upsilon}_{10}(\ell;p,A)}\times\sqrt[q]{|\Lambda'(m\ell)|^{q}+|\Lambda'(\flat_2)|^{q}},$$

where

$$B^{\Upsilon}_{9}(\ell;p,A):=\int_{m\flat_1}^{m\flat_1+\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)}\Big\{1-\exp\left[A\left(\Upsilon(m\flat_1)-\Upsilon(\zeta)\right)\right]\Big\}^{p}d\zeta \tag{43}$$

and

$$B^{\Upsilon}_{10}(\ell;p,A):=\int_{m\ell}^{m\ell+\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)}\Big\{1-\exp\left[A\left(\Upsilon(\zeta)-\Upsilon(m\ell+\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell))\right)\right]\Big\}^{p}d\zeta. \tag{44}$$

THEOREM 2.15. Let $\Lambda:\mathscr{P}\to\mathbb{R}$ be a differentiable function on $\mathscr{P}^{\circ}$ and $\lambda\in[0,1]$. If $|\Lambda'|^{q}$ is generalized convex on $\mathscr{P}$ and $\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\flat_1)>0$, then for $q\ge1$, we have

$$\left|T_{\Lambda,\Pi^{\varpi,\Upsilon}_{m},\Xi^{\varpi,\Upsilon}_{m}}(\lambda;\ell,\flat_1,\flat_2)\right|\le\frac{\left[\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)\right]^{2}}{\Pi^{\varpi,\Upsilon}_{m}(\ell,1)\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\flat_1)}\left[B^{\varpi,\Upsilon}_{\Pi_m}(\ell;\lambda,1)\right]^{1-\frac{1}{q}} \tag{45}$$

$$\times\sqrt[q]{\Big[B^{\varpi,\Upsilon}_{\Pi_m}(\ell;\lambda,1)-E^{\varpi,\Upsilon}_{\Pi_m}(\ell;\lambda)\Big]|\Lambda'(m\flat_1)|^{q}+E^{\varpi,\Upsilon}_{\Pi_m}(\ell;\lambda)|\Lambda'(\ell)|^{q}}$$

$$+\frac{\left[\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)\right]^2}{\Xi^{\varpi,\Upsilon}_m(\ell,0)\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\flat_1)}\left[B^{\varpi,\Upsilon}_{\Xi_m}(\ell;\lambda,1)\right]^{1-\frac{1}{q}}$$

$$\times\sqrt[q]{\left[B^{\varpi,\Upsilon}_{\Xi_m}(\ell;\lambda,1)-G^{\varpi,\Upsilon}_{\Xi_m}(\ell;\lambda)\right]|\Lambda'(m\ell)|^q+G^{\varpi,\Upsilon}_{\Xi_m}(\ell;\lambda)|\Lambda'(\flat_2)|^q},$$

where

$$E^{\varpi,\Upsilon}_{\Pi_m}(\ell;\lambda):=\int_0^1\zeta\left|\Pi^{\varpi,\Upsilon}_m(\ell,\zeta)-\lambda\right|d\zeta,\quad G^{\varpi,\Upsilon}_{\Xi_m}(\ell;\lambda):=\int_0^1\zeta\left|\Xi^{\varpi,\Upsilon}_m(\ell,\zeta)-\lambda\right|d\zeta.\quad(46)$$

PROOF. By Lemma 2.1, generalized convexity of $|\Lambda'|^q$ and power mean inequality, we get

$$\left|T_{\Lambda,\Pi^{\varpi,\Upsilon}_m,\Xi^{\varpi,\Upsilon}_m}(\lambda;\ell,\flat_1,\flat_2)\right|$$

$$\leq\frac{\left[\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)\right]^2}{\Pi^{\varpi,\Upsilon}_m(\ell,1)\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\flat_1)}\times\int_0^1\left|\Pi^{\varpi,\Upsilon}_m(\ell,\zeta)-\lambda\right|\left|\Lambda'\left(m\flat_1+\zeta\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)\right)\right|d\zeta$$

$$+\frac{\left[\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)\right]^2}{\Xi^{\varpi,\Upsilon}_m(\ell,0)\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\flat_1)}\times\int_0^1\left|\Xi^{\varpi,\Upsilon}_m(\ell,\zeta)-\lambda\right|\left|\Lambda'\left(m\ell+\zeta\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)\right)\right|d\zeta$$

$$\leq\frac{\left[\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)\right]^2}{\Pi^{\varpi,\Upsilon}_m(\ell,1)\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\flat_1)}$$

$$\times\left(\int_0^1\left|\Pi^{\varpi,\Upsilon}_m(\ell,\zeta)-\lambda\right|d\zeta\right)^{1-\frac{1}{q}}\left(\int_0^1\left|\Pi^{\varpi,\Upsilon}_m(\ell,\zeta)-\lambda\right|\left|\Lambda'\left(m\flat_1+\zeta\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)\right)\right|^q d\zeta\right)^{\frac{1}{q}}$$

$$+\frac{\left[\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)\right]^2}{\Xi^{\varpi,\Upsilon}_m(\ell,0)\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\flat_1)}$$

$$\times\left(\int_0^1\left|\Xi^{\varpi,\Upsilon}_m(\ell,\zeta)-\lambda\right|d\zeta\right)^{1-\frac{1}{q}}\left(\int_0^1\left|\Xi^{\varpi,\Upsilon}_m(\ell,\zeta)-\lambda\right|\left|\Lambda'\left(m\ell+\zeta\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)\right)\right|^q d\zeta\right)^{\frac{1}{q}}$$

$$\leq\frac{\left[\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)\right]^2}{\Pi^{\varpi,\Upsilon}_m(\ell,1)\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\flat_1)}\sqrt[p]{B^{\varpi,\Upsilon}_{\Pi_m}(\ell;\lambda,p)}$$

$$\times\left(\int_0^1\left|\Pi^{\varpi,\Upsilon}_m(\ell,\zeta)-\lambda\right|\left[(1-\zeta)|\Lambda'(m\flat_1)|^q+\zeta|\Lambda'(\ell)|^q\right]d\zeta\right)^{\frac{1}{q}}$$

$$+\frac{\left[\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)\right]^2}{\Xi^{\varpi,\Upsilon}_m(\ell,0)\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\flat_1)}\sqrt[p]{B^{\varpi,\Upsilon}_{\Xi_m}(\ell;\lambda,p)}$$

$$\times\left(\int_0^1\left|\Xi^{\varpi,\Upsilon}_m(\ell,\zeta)-\lambda\right|\left[(1-\zeta)|\Lambda'(m\ell)|^q+\zeta|\Lambda'(\flat_2)|^q\right]d\zeta\right)^{\frac{1}{q}}$$

$$=\frac{\left[\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)\right]^2}{\Pi^{\varpi,\Upsilon}_m(\ell,1)\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\flat_1)}\left[B^{\varpi,\Upsilon}_{\Pi_m}(\ell;\lambda,1)\right]^{1-\frac{1}{q}}$$

$$\times\sqrt[q]{\left[B^{\varpi,\Upsilon}_{\Pi_m}(\ell;\lambda,1)-E^{\varpi,\Upsilon}_{\Pi_m}(\ell;\lambda)\right]|\Lambda'(m\flat_1)|^q+E^{\varpi,\Upsilon}_{\Pi_m}(\ell;\lambda)|\Lambda'(\ell)|^q}$$

$$+\frac{\left[\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)\right]^2}{\Xi^{\varpi,\Upsilon}_m(\ell,0)\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\flat_1)}\left[B^{\varpi,\Upsilon}_{\Xi_m}(\ell;\lambda,1)\right]^{1-\frac{1}{q}}$$

$$\times\sqrt[q]{\left[B^{\varpi,\Upsilon}_{\Xi_m}(\ell;\lambda,1)-G^{\varpi,\Upsilon}_{\Xi_m}(\ell;\lambda)\right]|\Lambda'(m\ell)|^q+G^{\varpi,\Upsilon}_{\Xi_m}(\ell;\lambda)|\Lambda'(\flat_2)|^q}.$$

COROLLARY 2.16. For $q=1$ in Theorem 2.15, we get

$$\left|T_{\Lambda,\Pi^{\varpi,\Upsilon}_m,\Xi^{\varpi,\Upsilon}_m}(\lambda;\ell,\flat_1,\flat_2)\right|\leq\frac{\left[\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)\right]^2}{\Pi^{\varpi,\Upsilon}_m(\ell,1)\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\flat_1)} \tag{47}$$

$$\times\left[\left(B^{\varpi,\Upsilon}_{\Pi_m}(\ell;\lambda,1)-E^{\varpi,\Upsilon}_{\Pi_m}(\ell;\lambda)\right)|\Lambda'(m\flat_1)|+E^{\varpi,\Upsilon}_{\Pi_m}(\ell;\lambda)|\Lambda'(\ell)|\right]$$

$$+\frac{\left[\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)\right]^2}{\Xi^{\varpi,\Upsilon}_m(\ell,0)\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\flat_1)}$$

$$\times\left[\left(B^{\varpi,\Upsilon}_{\Xi_m}(\ell;\lambda,1)-G^{\varpi,\Upsilon}_{\Xi_m}(\ell;\lambda)\right)|\Lambda'(m\ell)|+G^{\varpi,\Upsilon}_{\Xi_m}(\ell;\lambda)|\Lambda'(\flat_2)|\right].$$

COROLLARY 2.17. Taking $|\Lambda'|\leq\mathscr{K}$ in Theorem 2.15, we have

$$\left|T_{\Lambda,\Pi^{\varpi,\Upsilon}_m,\Xi^{\varpi,\Upsilon}_m}(\lambda;\ell,\flat_1,\flat_2)\right|\leq\frac{\mathscr{K}}{\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\flat_1)} \tag{48}$$

$$\times\left[\frac{\left[\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)\right]^2}{\Pi^{\varpi,\Upsilon}_m(\ell,1)}B^{\varpi,\Upsilon}_{\Pi_m}(\ell;\lambda,1)+\frac{\left[\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)\right]^2}{\Xi^{\varpi,\Upsilon}_m(\ell,0)}B^{\varpi,\Upsilon}_{\Xi_m}(\ell;\lambda,1)\right].$$

COROLLARY 2.18. Choosing $m=1,\lambda=0,\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)=\ell-m\flat_1,\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)=\flat_2-m\ell,\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\flat_1)=\flat_2-m\flat_1$ and $\Upsilon(\zeta)=\zeta=\varpi(\zeta)$ in Theorem 2.15, we get

$$\left|T_{\Lambda}(\ell,\flat_1,\flat_2)\right|\leq\frac{1}{2\sqrt[q]{3}(\flat_2-\flat_1)} \tag{49}$$

$$\times\left\{(\ell-\flat_1)^2\sqrt[q]{|\Lambda'(\flat_1)|^q+2|\Lambda'(\ell)|^q}+(\flat_2-\ell)^2\sqrt[q]{2|\Lambda'(\ell)|^q+|\Lambda'(\flat_2)|^q}\right\}.$$

COROLLARY 2.19. Taking $\ell=\frac{\flat_1+\flat_2}{2}$ in Corollary 2.18, we obtain

$$\left|T_{\Lambda}(\flat_1,\flat_2)\right|\leq\frac{(\flat_2-\flat_1)}{8\sqrt[q]{3}} \tag{50}$$

$$\times\left\{\sqrt[q]{|\Lambda'(\flat_1)|^q+2\left|\Lambda'\left(\frac{\flat_1+\flat_2}{2}\right)\right|^q}+\sqrt[q]{2\left|\Lambda'\left(\frac{\flat_1+\flat_2}{2}\right)\right|^q+|\Lambda'(\flat_2)|^q}\right\}.$$

COROLLARY 2.20. Choosing $m=1,\lambda=1,\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)=\ell-m\flat_1,\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)=\flat_2-m\ell,\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\flat_1)=\flat_2-m\flat_1$ and $\Upsilon(\zeta)=\zeta=\varpi(\zeta)$ in Theorem 2.15, we have

$$\left|\overline{T_{\Lambda}}(\ell,\flat_1,\flat_2)\right|\leq\frac{1}{2\sqrt[q]{3}(\flat_2-\flat_1)} \tag{51}$$

$$\times\left\{(\ell-\flat_1)^2\sqrt[q]{2|\Lambda'(\flat_1)|^q+|\Lambda'(\ell)|^q}+(\flat_2-\ell)^2\sqrt[q]{|\Lambda'(\ell)|^q+2|\Lambda'(\flat_2)|^q}\right\}.$$

COROLLARY 2.21. Taking $m=1,\lambda=\frac{1}{3},\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)=\ell-m\flat_1,\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)=\flat_2-m\ell,\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\flat_1)=\flat_2-m\flat_1$ and $\Upsilon(\zeta)=\zeta=\varpi(\zeta)$ in Theorem 2.15, we get

$$\left|T_{\Lambda}\left(\frac{1}{3};\flat_1,\flat_2\right)\right|\leq\frac{1}{2\sqrt[q]{243}(\flat_2-\flat_1)} \tag{52}$$

$$\times\left\{(\ell-\flat_1)^2\sqrt[q]{185|\Lambda'(\flat_1)|^q+58|\Lambda'(\ell)|^q}+(\flat_2-\ell)^2\sqrt[q]{195|\Lambda'(\ell)|^q+48|\Lambda'(\flat_2)|^q}\right\}.$$

COROLLARY 2.22. Substituting $\lambda=0$ and $\varpi(\zeta)=\zeta$ in Theorem 2.15, we obtain

$$\left|T_{\Lambda,\Pi^{\Upsilon}_{m},\Xi^{\Upsilon}_{m}}(0;\ell,\flat_1,\flat_2)\right|\leq\frac{1}{\left[\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)\right]^{\frac{q+1}{q}}\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\flat_1)} \tag{53}$$

$$\times\Big[B_1^{\Upsilon}(\ell;1)\Big]^{1-\frac{1}{q}}\sqrt[q]{\Big[B_1^{\Upsilon}(\ell;1)\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)-C_1^{\Upsilon}(\ell)\Big]|\Lambda'(m\flat_1)|^q+C_1^{\Upsilon}(\ell)|\Lambda'(\ell)|^q}$$

$$+\frac{1}{\big[\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)\big]^{\frac{q+1}{q}}\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\flat_1)}\Big[B_2^{\Upsilon}(\ell;1)\Big]^{1-\frac{1}{q}}$$

$$\times\sqrt[q]{\Big[B_2^{\Upsilon}(\ell;1)\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)-E_1^{\Upsilon}(\ell)\Big]|\Lambda'(m\ell)|^q+E_1^{\Upsilon}(\ell)|\Lambda'(\flat_2)|^q},$$

where

$$C_1^{\Upsilon}(\ell):=\int_{m\flat_1}^{m\flat_1+\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)}(\zeta-m\flat_1)(\Upsilon(\zeta)-\Upsilon(m\flat_1))\,d\zeta, \tag{54}$$

$$E_1^{\Upsilon}(\ell):=\int_{m\ell}^{m\ell+\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)}(\zeta-m\ell)(\Upsilon(m\ell+\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell))-\Upsilon(\zeta))\,d\zeta. \tag{55}$$

COROLLARY 2.23. For $\lambda=0$ and $\varpi(\zeta)=\frac{\zeta^{\alpha}}{\Gamma(\alpha)}$ in Theorem 2.15, we have

$$\big|T_{\Lambda,\Pi_m^{\Upsilon},\Xi_m^{\Upsilon}}(0;\ell,\flat_1,\flat_2)\big|\le\frac{1}{\big[\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)\big]^{\frac{q+1}{q}}\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\flat_1)} \tag{56}$$

$$\times\Big[B_3^{\Upsilon}(\ell;1,\alpha)\Big]^{1-\frac{1}{q}}$$

$$\times\sqrt[q]{\Big[B_3^{\Upsilon}(\ell;1,\alpha)\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)-C_1^{\Upsilon}(\ell,\alpha)\Big]|\Lambda'(m\flat_1)|^q+C_1^{\Upsilon}(\ell,\alpha)|\Lambda'(\ell)|^q}$$

$$+\frac{1}{\big[\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)\big]^{\frac{q+1}{q}}\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\flat_1)}\Big[B_4^{\Upsilon}(\ell;1,\alpha)\Big]^{1-\frac{1}{q}}$$

$$\times\sqrt[q]{\Big[B_4^{\Upsilon}(\ell;1,\alpha)\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)-E_1^{\Upsilon}(\ell,\alpha)\Big]|\Lambda'(m\ell)|^q+E_1^{\Upsilon}(\ell,\alpha)|\Lambda'(\flat_2)|^q},$$

where

$$C_1^{\Upsilon}(\ell,\alpha):=\int_{m\flat_1}^{m\flat_1+\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)}(\zeta-m\flat_1)\big[\Upsilon(\zeta)-\Upsilon(m\flat_1)\big]^{\alpha}\,d\zeta, \tag{57}$$

$$E_1^{\Upsilon}(\ell,\alpha):=\int_{m\ell}^{m\ell+\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)}(\zeta-m\ell)\big[\Upsilon(m\ell+\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell))-\Upsilon(\zeta)\big]^{\alpha}\,d\zeta. \tag{58}$$

COROLLARY 2.24. Substituting $\lambda=0$ and $\varpi(\zeta)=\frac{\zeta^{\frac{\alpha}{\kappa}}}{\kappa\Gamma_{\kappa}(\alpha)}$ in Theorem 2.15, we get

$$\left|T_{\Lambda,\Pi_m^{\Upsilon},\Xi_m^{\Upsilon}}(0;\ell,\flat_1,\flat_2)\right| \leq \frac{1}{\left[\mathscr{F}_{\rho,\delta}^{\sigma}(\ell-m\flat_1)\right]^{\frac{q+1}{q}}\mathscr{F}_{\rho,\delta}^{\sigma}(\flat_2-m\flat_1)}\left[B_5^{\Upsilon}(\ell;1,\alpha,\kappa)\right]^{1-\frac{1}{q}} \tag{59}$$

$$\times\sqrt[q]{\left[B_5^{\Upsilon}(\ell;1,\alpha,\kappa)\mathscr{F}_{\rho,\delta}^{\sigma}(\ell-m\flat_1)-C_1^{\Upsilon}(\ell,\alpha,\kappa)\right]|\Lambda'(m\flat_1)|^q+C_1^{\Upsilon}(\ell,\alpha,\kappa)|\Lambda'(\ell)|^q}$$

$$+\frac{1}{\left[\mathscr{F}_{\rho,\delta}^{\sigma}(\flat_2-m\ell)\right]^{\frac{q+1}{q}}\mathscr{F}_{\rho,\delta}^{\sigma}(\flat_2-m\flat_1)}\left[B_6^{\Upsilon}(\ell;1,\alpha,\kappa)\right]^{1-\frac{1}{q}}$$

$$\times\sqrt[q]{\left[B_6^{\Upsilon}(\ell;1,\alpha,\kappa)\mathscr{F}_{\rho,\delta}^{\sigma}(\flat_2-m\ell)-E_1^{\Upsilon}(\ell,\alpha,\kappa)\right]|\Lambda'(m\ell)|^q+E_1^{\Upsilon}(\ell,\alpha,\kappa)|\Lambda'(\flat_2)|^q},$$

where

$$C_1^{\Upsilon}(\ell,\alpha,\kappa) := \int_{m\flat_1}^{m\flat_1+\mathscr{F}_{\rho,\delta}^{\sigma}(\ell-m\flat_1)}(\zeta-m\flat_1)\left[\Upsilon(\zeta)-\Upsilon(m\flat_1)\right]^{\frac{\alpha}{\kappa}}d\zeta, \tag{60}$$

$$E_1^{\Upsilon}(\ell,\alpha,\kappa) := \int_{m\ell}^{m\ell+\mathscr{F}_{\rho,\delta}^{\sigma}(\flat_2-m\ell)}(\zeta-m\ell)\left[\Upsilon(m\ell+\mathscr{F}_{\rho,\delta}^{\sigma}(\flat_2-m\ell))-\Upsilon(\zeta)\right]^{\frac{\alpha}{\kappa}}d\zeta. \tag{61}$$

COROLLARY 2.25. For $\lambda=0$, $\forall\,\xi\in[0,\zeta]$, $\varpi_{\Upsilon}(\ell,\zeta)=\zeta(\Upsilon(m\flat_1+\mathscr{F}_{\rho,\delta}^{\sigma}(\ell-m\flat_1))-\zeta)^{\alpha-1}$ and $\forall\,\xi\in[\zeta,1]$, $\varpi_{\Upsilon}(\ell,\zeta)=\zeta(\Upsilon(m\ell+\mathscr{F}_{\rho,\delta}^{\sigma}(\flat_2-m\ell))-\zeta)^{\alpha-1}$ in Theorem 2.15, we obtain

$$\left|T_{\Lambda,\Pi_m^{\Upsilon},\Xi_m^{\Upsilon}}(0;\ell,\flat_1,\flat_2)\right| \leq \frac{1}{\left[\mathscr{F}_{\rho,\delta}^{\sigma}(\ell-m\flat_1)\right]^{\frac{q+1}{q}}\mathscr{F}_{\rho,\delta}^{\sigma}(\flat_2-m\flat_1)} \tag{62}$$

$$\times\left[B_7^{\Upsilon}(\ell;1,\alpha)\right]^{1-\frac{1}{q}}\sqrt[q]{\left[B_7^{\Upsilon}(\ell;1,\alpha)\mathscr{F}_{\rho,\delta}^{\sigma}(\ell-m\flat_1)-C_1^{\Upsilon}(\ell)\right]|\Lambda'(m\flat_1)|^q+C_1^{\Upsilon}(\ell)|\Lambda'(\ell)|^q}$$

$$+\frac{1}{\left[\mathscr{F}_{\rho,\delta}^{\sigma}(\flat_2-m\ell)\right]^{\frac{q+1}{q}}\mathscr{F}_{\rho,\delta}^{\sigma}(\flat_2-m\flat_1)}\left[B_8^{\Upsilon}(\ell;1,\alpha)\right]^{1-\frac{1}{q}}$$

$$\times\sqrt[q]{\left[B_8^{\Upsilon}(\ell;1,\alpha)\mathscr{F}_{\rho,\delta}^{\sigma}(\flat_2-m\ell)-L_2^{\Upsilon}(\ell,\alpha)\right]|\Lambda'(m\ell)|^q+L_2^{\Upsilon}(\ell,\alpha)|\Lambda'(\flat_2)|^q},$$

where

$$L_2^{\Upsilon}(\ell,\alpha) := \int_{m\ell}^{m\ell+\mathscr{F}_{\rho,\delta}^{\sigma}(\flat_2-m\ell)}(\zeta-m\ell)\left[\Upsilon^{\alpha}(m\ell+\mathscr{F}_{\rho,\delta}^{\sigma}(\flat_2-m\ell))-\Upsilon^{\alpha}(\zeta)\right]d\zeta. \tag{63}$$

COROLLARY 2.26. Substituting $\lambda=0$ and $\varpi(\zeta)=\frac{\zeta}{\alpha}\exp(-A\zeta)$, where $A=\frac{1-\alpha}{\alpha}$

in Theorem 2.15, we have

$$\left|T_{\Lambda,\Pi_m^{\Upsilon},\Theta_m^{\Upsilon}}(0;\ell,\flat_1,\flat_2)\right| \leq \frac{1}{(1-\alpha)\left[\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)\right]^{\frac{q+1}{q}}\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\flat_1)} \tag{64}$$

$$\times\left\{\left[B_9^{\Upsilon}(\ell;1,A)\right]^{1-\frac{1}{q}}\sqrt[q]{L_3^{\Upsilon}(\ell,A)|\Lambda'(m\flat_1)|^q+L_4^{\Upsilon}(\ell,A)|\Lambda'(\ell)|^q}\right.$$

$$+\frac{1}{(1-\alpha)\left[\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)\right]^{\frac{q+1}{q}}\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\flat_1)}$$

$$\left.\times\left[B_{10}^{\Upsilon}(\ell;1,A)\right]^{1-\frac{1}{q}}\sqrt[q]{L_5^{\Upsilon}(\ell,A)|\Lambda'(m\ell)|^q+L_6^{\Upsilon}(\ell,A)|\Lambda'(\flat_2)|^q}\right\},$$

where

$$L_3^{\Upsilon}(\ell,A):=\int_{m\flat_1}^{m\flat_1+\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)}\left(m\flat_1+\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)-\zeta\right) \tag{65}$$

$$\times\left\{1-\exp\left[A\left(\Upsilon(m\flat_1)-\Upsilon(\zeta)\right)\right]\right\}d\zeta,$$

$$L_4^{\Upsilon}(\ell,A):=\int_{m\flat_1}^{m\flat_1+\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)}(\zeta-m\flat_1)\left\{1-\exp\left[A\left(\Upsilon(m\flat_1)-\Upsilon(\zeta)\right)\right]\right\}d\zeta, \tag{66}$$

$$L_5^{\Upsilon}(\ell,A):=\int_{m\ell}^{m\ell+\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)}\left(m\ell+\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)-\zeta\right) \tag{67}$$

$$\times\left\{1-\exp\left[A\left(\Upsilon(\zeta)-\Upsilon(m\ell+\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell))\right)\right]\right\}d\zeta,$$

$$L_6^{\Upsilon}(\ell,A):=\int_{m\ell}^{m\ell+\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)}(\zeta-m\ell)\left\{1-\exp\left[A\left(\Upsilon(\zeta)-\Upsilon(m\ell+\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell))\right)\right]\right\}d\zeta. \tag{68}$$

3. APPLICATIONS

For $\flat_1, \flat_2 \in \mathbb{R}$ and $0 < \flat_1 < \flat_2$ we recall:

(1) arithmetic mean:

$$\mathrm{A}(\flat_1, \flat_2) = \frac{\flat_1 + \flat_2}{2};$$

(2) harmonic mean:

$$\mathrm{H}(\flat_1, \flat_2) = \frac{2}{\frac{1}{\flat_1} + \frac{1}{\flat_2}};$$

(3) logarithmic mean:

$$\mathrm{L}(\flat_1, \flat_2) = \frac{\flat_2 - \flat_1}{\ln|\flat_2| - \ln|\flat_1|};$$

(4) generalized log-mean:

$$\mathrm{L}_r(\flat_1, \flat_2) = \left[\frac{\flat_2^{r+1} - \flat_1^{r+1}}{(r+1)(\flat_2 - \flat_1)}\right]^{\frac{1}{r}}; \; r \in \mathbb{R} \setminus \{-1, 0\}.$$

From the main results in Section 2, we get

PROPOSITION 3.1. Let $r, \flat_1, \flat_2 \in \mathbb{R}$ with $0 < \flat_1 < \flat_2$, then for $r, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$\left|\mathrm{A}^r(\flat_1, \flat_2) - \mathrm{L}_r^r(\flat_1, \flat_2)\right| \leq \frac{r(\flat_2 - \flat_1)}{4\sqrt[p]{p+1}} \tag{69}$$

$$\times \left\{ \sqrt[q]{\mathrm{A}\left(\flat_1^{q(r-1)}, \left(\frac{\flat_1 + \flat_2}{2}\right)^{q(r-1)}\right)} + \sqrt[q]{\mathrm{A}\left(\left(\frac{\flat_1 + \flat_2}{2}\right)^{q(r-1)}, \flat_2^{q(r-1)}\right)} \right\}.$$

PROOF. Taking $m = 1$, $\lambda = 0$, $\ell = \frac{\flat_1 + \flat_2}{2}$, $\mathscr{F}^{\sigma}_{\rho,\delta}(\ell - m\flat_1) = \ell - m\flat_1$, $\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2 - m\ell) = \flat_2 - m\ell$, $\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2 - m\flat_1) = \flat_2 - m\flat_1$, $\Lambda(\zeta) = \zeta^r$ and $\Upsilon(\zeta) = \zeta = \varpi(\zeta)$ in Theorem 2.3, we get (69).

PROPOSITION 3.2. Let $r, \flat_1, \flat_2 \in \mathbb{R}$ with $0 < \flat_1 < \flat_2$, then for $r, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, we obtain

$$\left|\mathrm{A}(\flat_1^r, \flat_2^r) - \mathrm{L}_r^r(\flat_1, \flat_2)\right| \leq \frac{r(\flat_2 - \flat_1)}{4\sqrt[p]{p+1}} \tag{70}$$

$$\times\left\{\sqrt[q]{A\left(\flat_1^{q(r-1)},\left(\frac{\flat_1+\flat_2}{2}\right)^{q(r-1)}\right)}+\sqrt[q]{A\left(\left(\frac{\flat_1+\flat_2}{2}\right)^{q(r-1)},\flat_2^{q(r-1)}\right)}\right\}.$$

PROOF. Choosing $m=1$, $\lambda=1$, $\ell=\frac{\flat_1+\flat_2}{2}$, $\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)=\ell-m\flat_1$, $\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)=\flat_2-m\ell$, $\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\flat_1)=\flat_2-m\flat_1$, $\Lambda(\zeta)=\zeta^r$ and $\Upsilon(\zeta)=\zeta=\varpi(\zeta)$ in Theorem 2.3, we have (70).

PROPOSITION 3.3. Let $\flat_1,\flat_2\in\mathbb{R}$ with $0<\flat_1<\flat_2$, then for $q>1$ and $\frac{1}{p}+\frac{1}{q}=1$, we get

$$\left|\frac{1}{A(\flat_1,\flat_2)}-\frac{1}{L(\flat_1,\flat_2)}\right|\le\frac{(\flat_2-\flat_1)}{4\sqrt[p]{p+1}} \tag{71}$$

$$\times\left\{\frac{1}{\sqrt[q]{H\left(\flat_1^{2q},\left(\frac{\flat_1+\flat_2}{2}\right)^{2q}\right)}}+\frac{1}{\sqrt[q]{H\left(\left(\frac{\flat_1+\flat_2}{2}\right)^{2q},\flat_2^{2q}\right)}}\right\}.$$

PROOF. Taking $m=1$, $\lambda=0$, $\ell=\frac{\flat_1+\flat_2}{2}$, $\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)=\ell-m\flat_1$, $\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)=\flat_2-m\ell$, $\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\flat_1)=\flat_2-m\flat_1$, $\Lambda(\zeta)=\dfrac{1}{\zeta}$ and $\Upsilon(\zeta)=\zeta=\varpi(\zeta)$ in Theorem 2.3, we obtain (71).

PROPOSITION 3.4. Let $\flat_1,\flat_2\in\mathbb{R}$ with $0<\flat_1<\flat_2$, then for $q>1$ and $\frac{1}{p}+\frac{1}{q}=1$, we have

$$\left|\frac{1}{H(\flat_1,\flat_2)}-\frac{1}{L(\flat_1,\flat_2)}\right|\le\frac{(\flat_2-\flat_1)}{4\sqrt[p]{p+1}} \tag{72}$$

$$\times\left\{\frac{1}{\sqrt[q]{H\left(\flat_1^{2q},\left(\frac{\flat_1+\flat_2}{2}\right)^{2q}\right)}}+\frac{1}{\sqrt[q]{H\left(\left(\frac{\flat_1+\flat_2}{2}\right)^{2q},\flat_2^{2q}\right)}}\right\}.$$

PROOF. Choosing $m=1$, $\lambda=1$, $\ell=\frac{\flat_1+\flat_2}{2}$, $\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)=\ell-m\flat_1$, $\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)=\flat_2-m\ell$, $\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\flat_1)=\flat_2-m\flat_1$, $\Lambda(\zeta)=\dfrac{1}{\zeta}$ and $\Upsilon(\zeta)=\zeta=\varpi(\zeta)$ in Theorem 2.3, we get (72).

PROPOSITION 3.5. Let $r,\flat_1,\flat_2\in\mathbb{R}$ with $0<\flat_1<\flat_2$ and $r>1$, then for $q\ge1$, we obtain

$$\left|A^r(\flat_1,\flat_2)-L_r^r(\flat_1,\flat_2)\right|\le\sqrt[q]{\frac{2}{3}}\,\frac{r(\flat_2-\flat_1)}{8} \tag{73}$$

$$\times\left\{\sqrt[q]{\mathrm{A}\left(\flat_1^{q(r-1)},2\left(\frac{\flat_1+\flat_2}{2}\right)^{q(r-1)}\right)}+\sqrt[q]{\mathrm{A}\left(2\left(\frac{\flat_1+\flat_2}{2}\right)^{q(r-1)},\flat_2^{q(r-1)}\right)}\right\}.$$

PROOF. Taking $m=1$, $\lambda=0$, $\ell=\frac{\flat_1+\flat_2}{2}$, $\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)=\ell-m\flat_1$, $\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)=\flat_2-m\ell$, $\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\flat_1)=\flat_2-m\flat_1$, $\Lambda(\zeta)=\zeta^r$ and $\Upsilon(\zeta)=\zeta=\varpi(\zeta)$ in Theorem 2.15, we have (73).

PROPOSITION 3.6. Let $r,\flat_1,\flat_2\in\mathbb{R}$ with $0<\flat_1<\flat_2$ and $r>1$, then for $q\geq 1$, we get

$$\left|\mathrm{A}(\flat_1^r,\flat_2^r)-\mathrm{L}_r^r(\flat_1,\flat_2)\right|\leq\sqrt[q]{\frac{2}{3}}\frac{r(\flat_2-\flat_1)}{8} \tag{74}$$

$$\times\left\{\sqrt[q]{\mathrm{A}\left(2\flat_1^{q(r-1)},\left(\frac{\flat_1+\flat_2}{2}\right)^{q(r-1)}\right)}+\sqrt[q]{\mathrm{A}\left(\left(\frac{\flat_1+\flat_2}{2}\right)^{q(r-1)},2\flat_2^{q(r-1)}\right)}\right\}.$$

PROOF. Choosing $m=1$, $\lambda=1$, $\ell=\frac{\flat_1+\flat_2}{2}$, $\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)=\ell-m\flat_1$, $\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)=\flat_2-m\ell$, $\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\flat_1)=\flat_2-m\flat_1$, $\Lambda(\zeta)=\zeta^r$ and $\Upsilon(\zeta)=\zeta=\varpi(\zeta)$ in Theorem 2.15, we obtain (74).

PROPOSITION 3.7. Let $\flat_1,\flat_2\in\mathbb{R}$ with $0<\flat_1<\flat_2$, then for $q\geq 1$, we have

$$\left|\frac{1}{\mathrm{A}(\flat_1,\flat_2)}-\frac{1}{\mathrm{L}(\flat_1,\flat_2)}\right|\leq\sqrt[q]{\frac{4}{3}}\frac{(\flat_2-\flat_1)}{8} \tag{75}$$

$$\times\left\{\frac{1}{\sqrt[q]{\mathrm{H}\left(2\flat_1^{2q},\left(\frac{\flat_1+\flat_2}{2}\right)^{2q}\right)}}+\frac{1}{\sqrt[q]{\mathrm{H}\left(\left(\frac{\flat_1+\flat_2}{2}\right)^{2q},2\flat_2^{2q}\right)}}\right\}.$$

PROOF. Taking $m=1$, $\lambda=0$, $\ell=\frac{\flat_1+\flat_2}{2}$, $\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)=\ell-m\flat_1$, $\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)=\flat_2-m\ell$, $\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\flat_1)=\flat_2-m\flat_1$, $\Lambda(\zeta)=\dfrac{1}{\zeta}$ and $\Upsilon(\zeta)=\zeta=\varpi(\zeta)$ in Theorem 2.15, we get (75).

PROPOSITION 3.8. Let $\flat_1,\flat_2\in\mathbb{R}$ with $0<\flat_1<\flat_2$, then for $q\geq 1$, we obtain

$$\left|\frac{1}{\mathrm{H}(\flat_1,\flat_2)}-\frac{1}{\mathrm{L}(\flat_1,\flat_2)}\right|\leq\sqrt[q]{\frac{4}{3}}\frac{(\flat_2-\flat_1)}{8} \tag{76}$$

$$\times\left\{\frac{1}{\sqrt[q]{\mathrm{H}\left(\flat_1^{2q},2\left(\frac{\flat_1+\flat_2}{2}\right)^{2q}\right)}}+\frac{1}{\sqrt[q]{\mathrm{H}\left(2\left(\frac{\flat_1+\flat_2}{2}\right)^{2q},\flat_2^{2q}\right)}}\right\}.$$

PROOF. Choosing $m=1, \lambda=1, \ell=\frac{\flat_1+\flat_2}{2}, \mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)=\ell-m\flat_1, \mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)=\flat_2-m\ell, \mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\flat_1)=\flat_2-m\flat_1, \Lambda(\zeta)=\dfrac{1}{\zeta}$ and $\Upsilon(\zeta)=\zeta=\varpi(\zeta)$ in Theorem 2.15, we have (76).

Finally, we will find some new error estimations pertaining quadrature formula. Let denote $\mathscr{Q}:\flat_1=\varsigma_0<\varsigma_1<\ldots<\varsigma_k=\flat_2$. The following quadrature formulas are very useful in the sequel.

$$\int_{\flat_1}^{\flat_2}\Lambda(\ell)d\ell=\mathrm{M}(\Lambda,\mathscr{Q})+\mathrm{E}(\Lambda,\mathscr{Q}),\quad \int_{\flat_1}^{\flat_2}\Lambda(\ell)d\ell=\mathrm{T}(\Lambda,\mathscr{Q})+\mathrm{E}^*(\Lambda,\mathscr{Q})$$

where

$$\mathrm{M}(\Lambda,\mathscr{Q}):=\sum_{j=0}^{k-1}\Lambda\left(\frac{\varsigma_j+\varsigma_{j+1}}{2}\right)(\varsigma_{j+1}-\varsigma_j),\quad \mathrm{T}(\Lambda,\mathscr{Q}):=\sum_{j=0}^{k-1}\frac{\Lambda(\varsigma_j)+\Lambda(\varsigma_{j+1})}{2}(\varsigma_{j+1}-\varsigma_j),$$

and $\mathrm{E}(\Lambda,\mathscr{Q}), \mathrm{E}^*(\Lambda,\mathscr{Q})$ are denoted their corresponding errors.

PROPOSITION 3.9. Let $\Lambda:[\flat_1,\flat_2]\to\mathbb{R}$ be a differentiable function on $(\flat_1,\flat_2)$, where $\flat_1<\flat_2$. If $|\Lambda'|^q$ is convex on $[\flat_1,\flat_2]$, then for $q>1$ and $\frac{1}{p}+\frac{1}{q}=1$, we have

$$\left|\mathrm{E}(\Lambda,\mathscr{Q})\right|\le\frac{1}{4\sqrt[q]{2}\sqrt[p]{p+1}}\times\sum_{j=0}^{k-1}(\varsigma_{j+1}-\varsigma_j)^2 \tag{77}$$

$$\times\left\{\sqrt[q]{|\Lambda'(\varsigma_j)|^q+\left|\Lambda'\left(\frac{\varsigma_j+\varsigma_{j+1}}{2}\right)\right|^q}+\sqrt[q]{\left|\Lambda'\left(\frac{\varsigma_j+\varsigma_{j+1}}{2}\right)\right|^q+|\Lambda'(\varsigma_{j+1})|^q}\right\}.$$

PROOF. From Theorem 2.3 for $m=1, \lambda=0, \ell=\frac{\flat_1+\flat_2}{2}, \mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)=\ell-m\flat_1, \mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)=\flat_2-m\ell, \mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\flat_1)=\flat_2-m\flat_1$ and $\Upsilon(\zeta)=\zeta=\varpi(\zeta)$ on $[\varsigma_j,\varsigma_{j+1}]$ $(j=0,\ldots,k-1)$ of $\mathscr{Q}$, we get

$$\left|\Lambda\left(\frac{\varsigma_j+\varsigma_{j+1}}{2}\right)-\frac{1}{\varsigma_{j+1}-\varsigma_j}\int_{\varsigma_j}^{\varsigma_{j+1}}\Lambda(\ell)d\ell\right|\le\frac{(\varsigma_{j+1}-\varsigma_j)}{4\sqrt[q]{2}\sqrt[p]{p+1}} \tag{78}$$

$$\times\left\{\sqrt[q]{|\Lambda'(\varsigma_j)|^q+\left|\Lambda'\left(\frac{\varsigma_j+\varsigma_{j+1}}{2}\right)\right|^q}+\sqrt[q]{\left|\Lambda'\left(\frac{\varsigma_j+\varsigma_{j+1}}{2}\right)\right|^q+|\Lambda'(\varsigma_{j+1})|^q}\right\}.$$

From (78), we have

$$\left|\mathrm{E}(\Lambda,\mathscr{Q})\right| = \left|\int_{\flat_1}^{\flat_2}\Lambda(\ell)d\ell - \mathrm{M}(\Lambda,\mathscr{Q})\right|$$

$$\leq \left|\sum_{j=0}^{k-1}\left\{\int_{\varsigma_j}^{\varsigma_{j+1}}\Lambda(\ell)d\ell - \Lambda\left(\frac{\varsigma_j+\varsigma_{j+1}}{2}\right)(\varsigma_{j+1}-\varsigma_j)\right\}\right|$$

$$\leq \sum_{j=0}^{k-1}\left|\left\{\int_{\varsigma_j}^{\varsigma_{j+1}}\Lambda(\ell)d\ell - \Lambda\left(\frac{\varsigma_j+\varsigma_{j+1}}{2}\right)(\varsigma_{j+1}-\varsigma_j)\right\}\right|$$

$$\leq \frac{1}{4\sqrt[q]{2}\sqrt[p]{p+1}} \times \sum_{j=0}^{k-1}(\varsigma_{j+1}-\varsigma_j)^2$$

$$\times\left\{\sqrt[q]{|\Lambda'(\varsigma_j)|^q + \left|\Lambda'\left(\frac{\varsigma_j+\varsigma_{j+1}}{2}\right)\right|^q} + \sqrt[q]{\left|\Lambda'\left(\frac{\varsigma_j+\varsigma_{j+1}}{2}\right)\right|^q + |\Lambda'(\varsigma_{j+1})|^q}\right\}.$$

PROPOSITION 3.10. Let $\Lambda : [\flat_1,\flat_2] \to \mathbb{R}$ be a differentiable function on $(\flat_1,\flat_2)$, where $\flat_1 < \flat_2$. If $|\Lambda'|^q$ is convex on $[\flat_1,\flat_2]$, then for $q \geq 1$, we obtain

$$\left|\mathrm{E}(\Lambda,\mathscr{Q})\right| \leq \frac{1}{8\sqrt[q]{3}} \times \sum_{j=0}^{k-1}(\varsigma_{j+1}-\varsigma_j)^2 \tag{79}$$

$$\times\left\{\sqrt[q]{|\Lambda'(\varsigma_j)|^q + 2\left|\Lambda'\left(\frac{\varsigma_j+\varsigma_{j+1}}{2}\right)\right|^q} + \sqrt[q]{2\left|\Lambda'\left(\frac{\varsigma_j+\varsigma_{j+1}}{2}\right)\right|^q + |\Lambda'(\varsigma_{j+1})|^q}\right\}.$$

PROOF. The proof is analogous as to that of Proposition 3.9 taking $m = 1$, $\lambda = 0$, $\ell = \frac{\flat_1+\flat_2}{2}$, $\mathscr{F}^{\sigma}_{\rho,\delta}(\ell - m\flat_1) = \ell - m\flat_1$, $\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2 - m\ell) = \flat_2 - m\ell$, $\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2 - m\flat_1) = \flat_2 - m\flat_1$ and $\Upsilon(\zeta) = \zeta = \varpi(\zeta)$ using Theorem 2.15.

PROPOSITION 3.11. Let $\Lambda : [\flat_1,\flat_2] \to \mathbb{R}$ be a differentiable function on $(\flat_1,\flat_2)$, where $\flat_1 < \flat_2$. If $|\Lambda'|^q$ is convex on $[\flat_1,\flat_2]$, then for $q > 1$ and $\frac{1}{p}+\frac{1}{q} = 1$, we have

$$\left|\mathrm{E}^*(\Lambda,\mathscr{Q})\right| \leq \frac{1}{4\sqrt[q]{2}\sqrt[p]{p+1}} \times \sum_{j=0}^{k-1}(\varsigma_{j+1}-\varsigma_j)^2 \tag{80}$$

$$\times\left\{\sqrt[q]{|\Lambda'(\varsigma_j)|^q + \left|\Lambda'\left(\frac{\varsigma_j+\varsigma_{j+1}}{2}\right)\right|^q} + \sqrt[q]{\left|\Lambda'\left(\frac{\varsigma_j+\varsigma_{j+1}}{2}\right)\right|^q + |\Lambda'(\varsigma_{j+1})|^q}\right\}.$$

PROOF. By Theorem 2.3 for $m = 1, \lambda = 1, \ell = \frac{\flat_1+\flat_2}{2}, \mathscr{F}^{\sigma}_{\rho,\delta}(\ell - m\flat_1) = \ell - m\flat_1, \mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2 - m\ell) = \flat_2 - m\ell, \mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2 - m\flat_1) = \flat_2 - m\flat_1$ and $\Upsilon(\zeta) = \zeta = \varpi(\zeta)$ on $[\varsigma_j, \varsigma_{j+1}]$ $(j = 0,\dots,k-1)$ of $\mathscr{Q}$, we get

$$\left|\frac{\Lambda(\varsigma_j)+\Lambda(\varsigma_{j+1})}{2} - \frac{1}{\varsigma_{j+1}-\varsigma_j}\int_{\varsigma_j}^{\varsigma_{j+1}} \Lambda(\ell)d\ell\right| \leq \frac{(\varsigma_{j+1}-\varsigma_j)}{4\sqrt[q]{2}\sqrt[p]{p+1}} \tag{81}$$

$$\times\left\{\sqrt[q]{|\Lambda'(\varsigma_j)|^q + \left|\Lambda'\left(\frac{\varsigma_j+\varsigma_{j+1}}{2}\right)\right|^q} + \sqrt[q]{\left|\Lambda'\left(\frac{\varsigma_j+\varsigma_{j+1}}{2}\right)\right|^q + |\Lambda'(\varsigma_{j+1})|^q}\right\}.$$

From (81), we get

$$\left|\mathrm{E}^*(\Lambda,\mathscr{Q})\right| = \left|\int_{\flat_1}^{\flat_2}\Lambda(\ell)d\ell - \mathrm{T}(\Lambda,\mathscr{Q})\right|$$

$$\leq \left|\sum_{j=0}^{k-1}\left\{\int_{\varsigma_j}^{\varsigma_{j+1}}\Lambda(\ell)d\ell - \frac{\Lambda(\varsigma_j)+\Lambda(\varsigma_{j+1})}{2}(\varsigma_{j+1}-\varsigma_j)\right\}\right|$$

$$\leq \sum_{j=0}^{k-1}\left|\left\{\int_{\varsigma_j}^{\varsigma_{j+1}}\Lambda(\ell)d\ell - \frac{\Lambda(\varsigma_j)+\Lambda(\varsigma_{j+1})}{2}(\varsigma_{j+1}-\varsigma_j)\right\}\right|$$

$$\leq \frac{1}{4\sqrt[q]{2}\sqrt[p]{p+1}} \times \sum_{j=0}^{k-1}(\varsigma_{j+1}-\varsigma_j)^2$$

$$\times\left\{\sqrt[q]{|\Lambda'(\varsigma_j)|^q + \left|\Lambda'\left(\frac{\varsigma_j+\varsigma_{j+1}}{2}\right)\right|^q} + \sqrt[q]{\left|\Lambda'\left(\frac{\varsigma_j+\varsigma_{j+1}}{2}\right)\right|^q + |\Lambda'(\varsigma_{j+1})|^q}\right\}.$$

PROPOSITION 3.12. Let $\Lambda : [\flat_1,\flat_2] \to \mathbb{R}$ be a differentiable function on $(\flat_1,\flat_2)$, where $\flat_1 < \flat_2$. If $|\Lambda'|^q$ is convex on $[\flat_1,\flat_2]$, then for $q \geq 1$, we obtain

$$\left|\mathrm{E}^*(\Lambda,Q)\right| \leq \frac{1}{8\sqrt[q]{3}} \times \sum_{j=0}^{k-1}(\varsigma_{j+1}-\varsigma_j)^2 \tag{82}$$

$$\times\left\{\sqrt[q]{2|\Lambda'(\varsigma_j)|^q + \left|\Lambda'\left(\frac{\varsigma_j+\varsigma_{j+1}}{2}\right)\right|^q} + \sqrt[q]{\left|\Lambda'\left(\frac{\varsigma_j+\varsigma_{j+1}}{2}\right)\right|^q + 2|\Lambda'(\varsigma_{j+1})|^q}\right\}.$$

PROOF. The proof is analogous as to that of Proposition 3.11 taking $m = 1, \lambda = 1, \ell = \frac{\flat_1+\flat_2}{2}, \mathscr{F}^{\sigma}_{\rho,\delta}(\ell - m\flat_1) = \ell - m\flat_1, \mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2 - m\ell) = \flat_2 - m\ell, \mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2 - m\flat_1) = \flat_2 - m\flat_1$ and $\Upsilon(\zeta) = \zeta = \varpi(\zeta)$ using Theorem 2.15.

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