FINITE GROUPS WITH FEW CHARACTER VALUES

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Abstract. A classical theorem on character degrees states that if a finite group has less than four character degrees, then the group is solvable. We prove a corresponding result on character values by showing that if a finite group has less than eight character values in its character table, then the group is solvable. This confirms a conjecture of T. Sakurai. We also classify non-solvable groups with exactly eight character values.

1. Introduction

Let G be a finite group and $\mathrm{Irr}(G)$ be the set of complex irreducible characters of G. Recall the definition of a character degree set:

$$\mathrm{cd}(G) := \{\chi(1) \mid \chi \in \mathrm{Irr}(G)\}.$$

The set $\mathrm{cd}(G)$ has received much attention from many authors and has been shown to have strong influence on the structure of G. In particular, we have the famous Ito-Michler theorem which states that G has a normal abelian Sylow p-subgroup if and only if p does not divide any member of $\mathrm{cd}(G) \setminus \{1\}$. On the other extreme is the Thompson's theorem states that if p divides every member of $\mathrm{cd}(G) \setminus \{1\}$, then G has a normal p-complement. In this article we shall study a bigger set found in a character table:

$$\mathrm{cv}(G) = \{\chi(g) \mid \chi \in \mathrm{Irr}(G), g \in G\}.$$

Little has been done in the study of this set. The first article in this direction is from T. Sakurai [9]. The author studied groups with very few character values. We shall compare results in [9] with corresponding character degree results. It is well known that $|\mathrm{cd}(G)| = 1$ if and only if G is abelian. Sakurai also shows that if $|\mathrm{cv}(G)| \leqslant 3$, then G is abelian. The converse is not true since $|\mathrm{cv}(G)| = n$ for a cyclic group G of order n.

If $|\mathrm{cd}(G)| = 2$ with $\mathrm{cd}(G) = \{1, n\}$, then either G has an abelian normal subgroup of index n or $m = p^e$ for a prime p and G is direct product of a p-group and abelian group. It turns out that a non-abelian group G is such that $|\mathrm{cv}(G)| = 4$ if and only if G is the generalized dihedral group of a non-trivial elementary abelian 3-group([9, Theorem]). A generalized dihedral group of a non-trivial elementary abelian p-group is a group $G_n = [\mathrm{C}_p \times \mathrm{C}_p \times \cdots \times \mathrm{C}_p] \rtimes \mathrm{C}_2$, the semidirect product of the direct product of n copies of C_n and C_2, where C_2 acts on $[\mathrm{C}_n \times \mathrm{C}_n \times \cdots \times \mathrm{C}_n]$ without non-trivial fixed points. For groups with more than four character values, Sakurai remarked the difficulty to classify these groups. However, a conjecture was proposed which states that if the group has less than eight character values, then the group is solvable [9, Remark]. Our aim is to settle this conjecture:

Theorem 1.1. *If $|\mathrm{cv}(G)| < 8$, then G is solvable.*

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Isaacs [5, Theorem 12.] proved a result that is now considered classical which states that if $|\mathrm{cd}(G)| < 4$, then G is solvable. Hence Theorem 1.1 is the corresponding result for character values. Our proof uses the classification of finite simple groups.

Malle and Moretó [7] classified non-solvable with $|\mathrm{cd}(G)| = 4$ (see Theorem 2.2). In this article, we classify a corresponding result for character values:

Theorem 1.2. *Let G be a finite non-solvable group. Then $|\mathrm{cv}(G)| = 8$ if and only if $G \cong \mathrm{PSL}_2(5)$ or $\mathrm{PGL}_2(5)$.*

2. Preliminary Results

In this section we list some preliminary results, mostly on character degrees, we need to prove our main results. We shall use freely these well known results in the following lemma. Let C_r denote the cyclic group of order r for some positive integer r.

Lemma 2.1. *Let G be a finite group, N be a normal subgroup of G and n be a positive integer. Then the following holds:*
(a) $\mathrm{cv}(G/N) \leqslant \mathrm{cv}(G)$,
(b) $\mathrm{cv}(C_n) = n$,
(c) *If G is non-abelian, then* $0 \in \mathrm{cv}(G)$.

The following is a result of Malle and Moretó [7] in which they classified non-solvable groups with four character degrees.

Theorem 2.2. [7, Theorem A] *Let G be a non-solvable with $|\mathrm{cd}(G)| = 4$. Then one of the following holds:*
(a) *$G \cong \mathrm{PSL}_2(2^f) \times A$ for some $f \geqslant 2$ and some abelian subgroup A;*
(b) *G has a normal subgroup U such that $U \cong \mathrm{PSL}_2(q)$ or $\mathrm{SL}_2(q)$ for some odd $q \geqslant 5$ and if $C = \boldsymbol{C}_G(U)$, then $C \leqslant \boldsymbol{Z}(G)$ and $G/C \cong \mathrm{PGL}_2(q)$; or*
(c) *the group G has a normal subgroup of index 2 that is a direct product of $\mathrm{PSL}_2(9)$ and central subgroup C. Furthermore, $G/C \cong M_{10}$, the stabilizer of a point in M_{11} in its natural permutation representation.*

Conversely, if one of (a)-(c) holds, then $|\mathrm{cd}(G)| = 4$.

He and Zhu [4] described non-solvable groups with five and six character degrees:

Theorem 2.3. [4, Corollary C] *If G is a non-solvable group and L be the solvable radical of G. If $5 \leqslant |\mathrm{cd}(G)| \leqslant 6$, then G'/L is isomorphic to $\mathrm{PSL}_2(p^f)$, $p^f \geqslant 4$, $\mathrm{PSL}_3(4)$ or ${}^2\mathrm{B}_2(2^f)$, where $2^f = 2^{2m+1}$ for some integer $m \geqslant 1$. Moreover, G/L is an almost simple group.*

We classify almost simple groups with five character degrees.

Theorem 2.4. *Let G be a finite almost simple group, that is $S \trianglelefteq G \leqslant Aut(S)$ for some non-abelian simple group S. Then $|\mathrm{cd}(G)| = 5$, if and only if one of the following holds:*
(a) *$G \cong \mathrm{PSL}_2(q)$, q odd,* $\mathrm{cd}(G) = \{1, (q+\epsilon)/2, q \pm 1, q\}$,
(b) *$G \cong \mathrm{PSL}_2(2^f)\langle\varphi\rangle$, $f > 2$ prime,* $\mathrm{cd}(G) = \{1, 2^f - 1, 2^f, (2^f \pm 1)f\}$,
(c) *$G \cong \mathrm{PSL}_2(2^f)\langle\varphi^{f/2}\rangle$, $f > 2$ even,* $\mathrm{cd}(G) = \{1, 2^f, 2^f + 1, 2(2^f \pm 1)\}$,
(d) *$G \cong \mathrm{PGL}_2(q)\langle\varphi^{f/2}\rangle$, $f > 2$ even, q odd,* $\mathrm{cd}(G) = \{1, q, q+1, 2(q \pm 1)\}$,
(e) *$G \cong \mathrm{PGL}_2(3^f)\langle\varphi\rangle$, $f > 2$ even,* $\mathrm{cd}(G) = \{1, 3^f, 3^f - 1, (3^f \pm 1)f\}$,
(f) *$G \cong \mathrm{PSL}_2(9)\langle\varphi\rangle \cong \mathrm{S}_6$, $f = 2$,* $\mathrm{cd}(G) = \{1, 5, 9, 10, 16\}$,
(g) *$G \cong \mathrm{PSL}_2(5^f)\langle\varphi\rangle$, $f > 2$ prime,* $\mathrm{cd}(G) = \{1, 5^f, (5^f + 1)/2, 5^f - 1, (5^f \pm 1)f\}$,

(h) $G \cong \mathrm{PSL}_2(q)\langle\delta\varphi^{f/2}\rangle$, $f > 2$ *even*, $q \neq 9$, p *odd*, $\mathrm{cd}(G) = \{1, q, q+1, 2(q \pm 1)\}$.

Proof. Note that the every simple group S has an irreducible character χ that is extendible to $\mathrm{Aut}(S)$. Suppose that r is a prime that divides the character degree of χ. Then by Thompson's theorem [5, Corollary 12.2] G has a normal r-complement, a contradiction.

We may assume that no prime divides every character degree of G. Since G has five character degrees, there is no prime dividing four character degrees of G. These groups are the groups in [2, Theorem A]. It is sufficient to consider groups in case (c) since the groups in (a)-(b) and (d)-(f) can be ruled out using the character tables in the Atlas [1] and [11].

By inspecting [2, Table 1], we have our result. □

3. Main Results

Theorem 3.1. *Let G be a finite almost simple group, that is $S \trianglelefteq G \leqslant Aut(S)$ for some non-abelian simple group S. Then either $|\mathrm{cv}(G)| \geqslant 9$ or $|\mathrm{cv}(G)| = 8$ and $G \cong \mathrm{A}_5 \cong \mathrm{PSL}_2(5)$ or $G \cong \mathrm{S}_6 \cong \mathrm{PGL}_2(5)$. In particular, if $|\mathrm{cd}(G)| \geqslant 5$, then $|\mathrm{cv}(G)| \geqslant 9$.*

Proof. Since G is non-solvable, $|\mathrm{cd}(G)| \geqslant 4$ by [5, Theorem 12.15]. Suppose that $|\mathrm{cd}(G)| = 4$. By Theorem 2.2, it is sufficient to consider the following groups $\mathrm{PSL}_2(2^f)$ for some $f \geqslant 2$, $\mathrm{PGL}_2(q)$ for some odd $q \geqslant 5$ and M_{10}. The character tables of these groups can be found in the Atlas [1], [3] and [10]. In particular, $|\mathrm{cv}(G)| \geqslant 9$ except for $\mathrm{PSL}_2(5)$ and $\mathrm{PGL}_2(5)$. In the latter cases $|\mathrm{cv}(G)| = 8$ as required.

We may assume that $|\mathrm{cd}(G)| = 5$. It is sufficient to consider groups in Theorem 2.4. For cases (a) and (f), we have that $|\mathrm{cv}(G)| \geqslant 9$, using character tables in [3] and Atlas [1]. For cases (b), (e) and (g) we have that $C = G/\mathrm{PGL}_2(q)$ is cyclic of order greater than 2. Then by [8, Theorem 1.1], the character table of C has a column with pairwise different values. In particular, $a, b \in \mathrm{cv}(C) \setminus \{-1, 0, 1\}$. Note that the Steinberg character ψ of $\mathrm{PSL}_2(q)$ is extendible to G and also $\psi(s) = -1$ for some $s \in \mathrm{PSL}_2(q)$ by [3, Theorems 4.7 and 4.9]. Hence $-1 \in \mathrm{cv}(G)$ and $|\mathrm{cv}(G)| = |\mathrm{cd}(G) \cup \{a, b, -1, 0\}| \geqslant 9$. Suppose that C is of order 2. Groups with this property are the ones in cases (c), (d) and (h). Consider the characters $\chi_1, \chi_2 \in \mathrm{Irr}(G)$ such that $\chi_1(1) = 2(q+1)$ and $\chi_2(1) = 2(q-1)$. Then since $\mathrm{PGL}_2(q)$ has no irreducible characters of degrees $2(q \pm 1)$, $\chi_i = \theta_{i1} + \theta_{i2}$ by [5, Theorem 6.2], where $\theta_{11}, \theta_{12}, \theta_{21}, \theta_{22} \in \mathrm{Irr}(S)$, $\theta_{11}(1) = \theta_{12}(1) = q + 1$ and $\theta_{21}(1) = \theta_{22}(1) = q - 1$. Using [3, Theorems 4.7 and 4.9], we have that $\theta_{11}(g) = \theta_{12}(g) = 1$ and $\theta_{21}(g) = \theta_{22}(g) = -1$ for some $g \in \mathrm{PGL}_2(q)$. Hence $\chi_1(g) = 2$, $\chi_1(g) = -2$ and $|\mathrm{cd}(G)| \cup \{-2, -1, 0, 2\}| \geqslant 9$.

Going forward we may assume that $|\mathrm{cd}(G)| \geqslant 6$. Suppose that $|\mathrm{cd}(G)| \geqslant 6$ and $S \cong \mathrm{PSL}_2(q)$. We may assume that G is not isomorphic to $\mathrm{PSL}_2(2^f)$, $f \geqslant 2$, $\mathrm{PSL}_2(p^f)$, p odd or $\mathrm{PGL}_2(p^f)$, p odd. Let $H \in \{\mathrm{PSL}_2(2^f), f \geqslant 2\} \cup \{\mathrm{PSL}_2(q), q \text{ odd }\} \cup \{\mathrm{PGL}_2(q), q \text{ odd}\}$. If $G \cong HC_r$, where $r \mid f$, $r > 2$. Then arguing as above, we have that $a, b \in \mathrm{cv}(C_r) \setminus \{-1, 1\}$ and $|\mathrm{cv}(G)| = |\mathrm{cd}(G) \cup \{-1, 0, a, b\}| \geqslant 10$. If $G \cong HC_2$, $2 \mid f$. Then either $2(q-1)$ or $2(q+1)$ is a character degree of G. Arguing as above, we have that either $-2 \in \mathrm{cv}(C)$ or $2 \in \mathrm{cv}(C)$. Since $-1, 0 \in \mathrm{cv}(G)$ and $|\mathrm{cd}(G)| \geqslant 6$ we have that $|\mathrm{cv}(G)| \geqslant 9$.

If $S \cong \mathrm{PSL}_3(4)$, then the result follows using the character table in the Atlas [1]. Suppose that $S \cong {}^2\mathrm{B}_2(2^f)$, where $2^f = 2^{2m+1}$ for some integer $m \geqslant 1$. If $G \cong {}^2\mathrm{B}_2(2^f)$, it follows that $|\mathrm{cv}(G)| \geqslant 9$ by the character table in [11, Theorem 13]. We may assume that $S < G < \mathrm{Aut}(S)$. Then $C = G/S$ is cyclic of odd order. Using the argument as in the case of $\mathrm{PSL}_2(q)$ above, we have that $a, b \in \mathrm{cv}(C) \setminus \{-1, 1\}$. Hence $|\mathrm{cv}(G)| \geqslant |\mathrm{cd}(G) \cup \{a, b, 0\}| = 9$.

If $|\mathrm{cd}(G)| \geqslant 7$, then by the column orthogonality, there is a character value which is a negative number and so $|\mathrm{cv}(G)| \geqslant 9$. This concludes our proof. $\square$

Proof of Theorem 1.1. We prove our result using induction on $|G|$. Since $|\mathrm{cv}(G)| < 8$, every homomorphic image of G has the same property. Let N be a minimal normal subgroup of G. If another such minimal normal subgroup, say M, exists then G is the subdirect product of solvable groups G/N and G/M, and so is solvable. This implies that N is unique and non-abelian. Hence G is almost simple group. The result follows by Theorem 3.1. $\square$

Before we prove Theorem 1.2, we recall a definition. If $N \trianglelefteq G$, then $\mathrm{Irr}(G|N) = \{\chi \in \mathrm{Irr}(G) \mid N \nsubseteq \ker\chi\}$ and $\mathrm{cd}(G|N) = \{\chi(1) \mid \chi \in \mathrm{Irr}(G|N)\}$.

Proof of Theorem 1.2. Since G is non-solvable, $|\mathrm{cd}(G)| \geqslant 4$. If $5 \leqslant |\mathrm{cd}(G)| \leqslant 6$, then by Theorem 2.3, G'/L is isomorphic to $\mathrm{PSL}_2(p^f)$, $p^f \geqslant 4$, $\mathrm{PSL}_3(4)$ or ${}^2\mathrm{B}_2(2^f)$, where $2^f = 2^{2m+1}$ for some integer $m \geqslant 1$, where L is the solvable radical of G. Specifically, G/L is an almost simple group. Using Theorem 3.1, $|\mathrm{cv}(G)| \geqslant |\mathrm{cv}(G/L)| \geqslant 9$.

If $|\mathrm{cd}(G)| = 4$, then using Theorem 2.2 and Theorem 3.1, it is sufficient to consider $G \cong \mathrm{PSL}_2(5) \times A$ for some abelian subgroup A or $G/C \cong \mathrm{PGL}_2(5)$, where $C = \mathbf{C}_G(U)$, $U = \mathrm{PSL}_2(5)$ or $\mathrm{SL}_2(5)$ with $C \leqslant \mathbf{Z}(G)$ and U is a normal subgroup of G. Suppose that $G \cong \mathrm{PSL}_2(5) \times A$ with non-trivial A. Consider $\theta \in \mathrm{Irr}(\mathrm{PSL}_2(5))$ such that $\theta(1) = 5$ and $\phi \in \mathrm{Irr}(A)$ such that $\phi(g) = k < 0$. Then $\chi(1 \times g) = \theta(1) \times \phi(g) = 5k < 0$ and $|\mathrm{cv}(G)| \geqslant 9$. Suppose that G is a group in the latter case. We want to show that $C = 1$. Suppose that $C \neq 1$. Then $\mathbf{Z}(G)U = N$ and $\mathrm{cd}(G|N) \geqslant 3$ by [6, Theorem B]. Then there is a $\chi \in \mathrm{Irr}(G)$ such that $\chi(1) = m \in \mathrm{cd}(G|N) \setminus \{1, 2\}$. For $1 \neq z \in \mathbf{Z}(G)$, we have that $\chi(z) = -m$. This means that $|\mathrm{cv}(G)| \geqslant |\mathrm{PGL}_2(5) \cup \{-m\}| = 9$. Therefore $G = \mathrm{PGL}_2(5)$ and that concludes our argument. $\square$

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