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Semi-order units in vector lattices

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Declaration

I, CHITANGA PAINOS declare that the dissertation, which I hereby submit for the degree Masters of Science in Mathematics at the University of Pretoria, is my own work and has not previously been submitted by me for a degree at this or any other tertiary institution.

Signature:
Date:

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Semi-order units in vector lattices

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Chapter 1

Introduction

1.1 Continuous Functions in Analysis

Spaces of continuous functions come equipped with a variety of structures making it a useful and versatile object of study in mathematics. Indeed, the space $C(X)$ of real-valued continuous functions on a topological space X is a ring [8] and a vector lattice [7]. If X is compact then $C(X)$ is a Banach algebra [9], and a Banach lattice when equipped with the uniform norm. Hence $C(X)$ serves simultaneously as an example of many different mathematical objects.

Spaces of continuous functions are often "typical" objects in certain classes of objects. Every Archimedean vector lattices can be represented as a space of (extended) real-valued continuous functions on some topological space X, see [10]. Also commutative C^*- algebras can be represented as a space of continuous functions, see for instance [5].

In this thesis we study $C(X)$ as a vector lattice and a locally convex topological vector space, and the interaction between these structures. In the case of a compact space X, the norm and the order are intimately related to one another. Indeed, one may define the norm through the order structure.

$$||f|| = \inf\{\lambda > 0 : |f| \leq \lambda \mathbf{1}\}.$$

Furthermore the order bounded and norm bounded sets in $C(X)$ are identical, a linear functional ϕ on $C(X)$ is order bounded if and only if it is norm bounded. Following [4], we aim to generalize these results to a non-compact space X. In particular, we consider $C(X)$ equipped with the compact open topology, and the relationship between this topology and the order structure on $C(X)$.

1.2 Outline

In chapter 2 we give the definition of a semi-order unit and the semi-order unit topology. Many vector lattices have few semi-order units, for example $C(X)$ for a connected non-compact space Tychonoff space X. Therefore given a vector lattice E, we consider the order adherence $\widetilde{E}$, the collection of all limits of order convergent nets in E viewed as a subspace of its order bidual. We will find the semi-order units of $\widetilde{E}$ for certain explicit of vector lattices E.

We dedicate chapter 3 to $C(X)$. In this chapter we study $C_{co}(X)$ from the order theoretic point of view. We discuss closed ideals in $C_{co}(X)$, and we compare the order dual and the topological dual. We also discuss the order theoretic characterization of the compact open topology in terms of semi-order unit in $\widetilde{C(X)}$.

In particular, we end the thesis by showing that X is realcompact if and only if the semi-order unit topology inherited from the order adherence agrees with the compact-open topology on $C(X)$. The Stone-$\check{C}$ech compactification and the realcompactfication of X are constructed as necessary preliminary material for the chapter.

The Appendix contains some results which are used throughout the thesis.

Chapter 2

General Vector Lattices

2.1 Preliminaries

Recall that a vector lattice is an ordered vector space which is a lattice with respect to its partial order. We will assume that the reader is familiar with some basic terminology on ordered vector spaces, in particular vector lattices, see for instance [11], [2]. We follow the notation of [4].

Definition 2.1.1. Let E be a vector lattice. A positive element u in E is called a semi-order unit if for each v in E, there exists a $\lambda > 0$ such that $v \wedge nu \leq \lambda u$ for all $n \in \mathbb{N}$.

Proposition 2.1.2. *Let E be a vector lattice. A positive element $u \in E$ is a semi-order unit if and only if for all $v \in E$, there exists a $\mu > 0$ such that $|v| \wedge nu \leq \mu u$ for all $n \in \mathbb{N}$.*

Proof. Suppose there exists a $\mu > 0$ such that $|v| \wedge nu \leq \mu u$, for all $n \in \mathbb{N}$. Since E is a lattice, $v \leq |v|$. Hence for each $n \in \mathbb{N}$ we have that

$$v \wedge nu \leq |v| \wedge nu \leq \mu u.$$

This holds for each $n \in \mathbb{N}$. By setting $\lambda = \mu$, we have $v \wedge nu \leq \lambda u$, so that u is a semi-order unit.
Now suppose u is a semi-order unit and fix $v \in E$. Since E is a lattice, $|v| \in E$ so by assumption, there exists a $\lambda > 0$ such that $|v| \wedge nu \leq \lambda u$ for all $n \in \mathbb{N}$. By setting $\mu = \lambda$ we get the desired result. $\square$

Proposition 2.1.3. *Let u be a positive element in a vector lattice E and $\lambda > 0$. Then for $v \in E$, we have that $|v| \wedge nu \leq \lambda u$ for all $n \in \mathbb{N}$ if and only if $|v| \wedge tu \leq \lambda u$ for all $t > 0$.*

Proof. Suppose $|v| \wedge nu \leq \lambda u$ for all $n \in \mathbb{N}$. Fix $t > 0$. Since $\mathbb{N}$ is not bounded from above, there exists an $N \in \mathbb{N}$ such that $N > t$. It follows that $Nu > tu$. Thus

$$|v| \wedge tu \leq |v| \wedge Nu \leq \lambda u.$$

Since $t > 0$ was arbitrary, it follows that $|v| \wedge tu \leq \lambda u$ for all $t > 0$. Suppose $|v| \wedge tu \leq \lambda u$ for all $t > 0$. Since $\mathbb{N} \subseteq \mathbb{R}^+$, it follows that if $n \in \mathbb{N}$, then $n > 0$ so that $|v| \wedge nu \leq \lambda u$. Since n was abitrary the result follows. $\square$

Using the above propositions, we will sometimes use $|v|$ rather than v since the definitions will be equivalent.
Note that it also holds for any subset of $\mathbb{R}$ which is not bounded from above.
Remark: If A is a non-empty set in $\mathbb{R}$ and is bounded from below, then $\inf A$ exists. For $\lambda > 0$ define a set

$$B = \{\lambda a : a \in A\}.$$

Then B is bounded below and $\inf B = \lambda \inf A$.

Theorem 2.1.4. *Let u be a semi-order unit in a vector lattice E. Then the function defined by*

$$\rho(v) = \inf\{\lambda > 0 : |v| \wedge nu \leq \lambda u, \;\; n \in \mathbb{N}\}$$

for v in E defines a semi-norm on E.

Proof. Since u is a semi-order-unit, the set

$$\{\lambda > 0 : |v| \wedge nu \leq \lambda u, \;\; n \in \mathbb{N}\}$$

is non-empty and is bounded below by 0, so its infimum exists. By the characterization of an infimum, we have that

$$0 \leq \inf\{\lambda > 0 : |v| \wedge nu \leq \lambda u, \;\; n \in \mathbb{N}\}$$

so that $\rho(v) \geq 0$ for all $v \in E$. Hence

$$\rho : E \longrightarrow [0, \infty).$$

Let $v \in E$ and let $\alpha \in \mathbb{R}$.
If $\alpha = 0$, then

$$\begin{aligned} |\alpha|\rho(v) = 0 \; \rho(v) &= 0 \;\; \inf\{\lambda > 0 : |v| \wedge nu \leq \lambda u, n \in \mathbb{N}\} \\ &= \inf\{\lambda > 0 : |0| \wedge nu \leq \lambda u, n \in \mathbb{N}\} \\ &= \rho(\alpha v). \end{aligned}$$

If $\alpha \neq 0$, then

$$\begin{aligned}
\rho(\alpha v) &= \inf\{\lambda > 0 : |\alpha v| \wedge nu \leq \lambda u, n \in \mathbb{N}\} \\
&= \inf\{\lambda > 0 : |\alpha||v| \wedge nu \leq \lambda u, n \in \mathbb{N}\} \\
&= \inf\{\lambda > 0 : |\alpha|(|v| \wedge \frac{n}{|\alpha|}u) \leq \lambda u, n \in \mathbb{N}\} \\
&= \inf\{\lambda > 0 : |\alpha|(|v| \wedge tu) \leq \lambda u, t > 0\} \\
&= \inf\{\lambda > 0 : |\alpha|(|v| \wedge nu) \leq \lambda u, n \in \mathbb{N}\} \\
&= \inf\{\lambda > 0 : (|v| \wedge nu) \leq \frac{\lambda}{|\alpha|}u, n \in \mathbb{N}\} \\
&= \inf\{|\alpha|\mu > 0 : (|v| \wedge nu) \leq \mu u, n \in \mathbb{N}\} where \mu = \frac{\lambda}{|\alpha|} \\
&= |\alpha| \inf\{\mu > 0 : (|v| \wedge nu) \leq \mu u, n \in \mathbb{N}\} \text{ (using the Remark before the theorem)} \\
&= |\alpha|\rho(v).
\end{aligned}$$

Hence $\rho(\alpha v) = |\alpha|\rho(v)$.
Now it only remains to show the triangle inequality. Let $v_1, v_2 \in E$. Since E is a vector lattice, $|v_1 + v_2| \leq |v_1| + |v_2|$. Hence from Theorem 6.5 of [11] we have that for each $n \in \mathbb{N}$,

$$|v_1 + v_2| \wedge nu \leq (|v_1| + |v_2|) \wedge nu = |v_1| \wedge nu + |v_2| \wedge nu. \tag{2.1}$$

Recall that if A and B are nonempty subsets of $\mathbb{R}$ that are bounded from below, then if $C = \big\{a+b : a \in A, b \in B\big\}$, it follows that $\inf C = \inf A + \inf B$. In addition, if $A \subseteq B$ then $\inf B \leq \inf A$. Using this argument, we have that

$$\begin{aligned}
\rho(v_1) + \rho(v_2) &= \inf\big\{\lambda_1 : |v_1| \wedge nu \leq \lambda_1 u, n \in \mathbb{N}\big\} + \inf\big\{\lambda_2 : |v_2| \wedge nu \leq \lambda_2 u, n \in \mathbb{N}\big\} \\
&= \inf\{\lambda_1 + \lambda_2 : |v_1| \wedge nu \leq \lambda_1 u \text{ and } |v_2| \wedge nu \leq \lambda_2 u, n \in \mathbb{N}\}.
\end{aligned}$$

Let λ_1 and λ_2 be such that $|v_1| \wedge nu \leq \lambda_1 u$ and $|v_2| \wedge nu \leq \lambda_2 u$.
Then for $\lambda = \lambda_1 + \lambda_2$, using (2.1) :

$$\begin{aligned}
|v_1 + v_2| \wedge nu &\leq |v_1| \wedge nu + |v_2| \wedge nu \\
&\leq \lambda_1 u + \lambda_2 u \\
&= \lambda u.
\end{aligned}$$

Hence $\big\{\lambda_1 + \lambda_2 > 0 : |v_1| \wedge nu \leq \lambda_1 u, \text{ and } |v_2| \wedge nu \leq \lambda_2 u \ n \in \mathbb{N}\big\}$ is a subset of the set $\big\{\lambda > 0 : |v_1 + v_2| \wedge nu \leq \lambda u, n \in \mathbb{N}\big\}$. Taking the infimum we obtain the result $\rho(v_1 + v_2) \leq \rho(v_1) + \rho(v_2)$.
Hence ρ defines a semi-norm on a vector lattice E. $\square$

Example 2.1.5. Let X be a discrete space and consider $C(X) = \mathbb{R}^X$. The followings are equivalent.

i. u is a semi-order unit.

ii. The support of u is finite.

Proof of ii $\implies$ i . Suppose spt u is finite. Define a set A as

$$A = \text{spt } u = \{x \in X : u(x) \neq 0\} = \{x_1, ..., x_n\}.$$

Fix $f \in C(X) = \mathbb{R}^X$. Then for any $x \notin A$, $u(x) = 0$ so that $nu(x) = 0$ for all $n \in \mathbb{N}$. Hence for $x \notin A$ we have that

$$\min\{|f(x)|, nu(x)\} = 0. \tag{2.2}$$

Define

$$a = \max\{|f(x_i)| : i = 1, ..., n\}.$$

Clearly $a \neq 0$ because it is a maximum of a finite set of positive elements. Since $u(x) \neq 0$ for $x \in A$, the set

$$\left\{\frac{a}{u(x)} : x \in A\right\}$$

is well defined and non-empty. Define

$$\lambda = \max\left\{\frac{a}{u(x)} : x \in A\right\}.$$

It follows that $\lambda > 0$ and $\lambda \geq \frac{a}{u(x)}$ for all $x \in A$. Hence, if $x \in A$ then $|f(x)| \leq a \leq \lambda u(x)$ so that,

$$\min\{|f(x)|, nu(x)\} \leq \lambda u(x). \tag{2.3}$$

Combining (2.2) and (2.3) we have that for $x \in X$

$$\min\{|f(x)|, nu(x)\} \leq \lambda u(x) \text{ for all } n \in \mathbb{N}$$

so that

$$|f| \wedge nu \leq \lambda u, n \in \mathbb{N}.$$

Since f was arbitrary, it follows that u is a semi-order unit. $\square$

Proof of i $\implies$ ii. Let u be a semi-order unit. Then for any $f \in C(X)$, there exists a $\lambda > 0$ such that $|f| \wedge nu \leq \lambda u$ for all $n \in \mathbb{N}$.
With a view for a contradiction, suppose that spt u is not finite. Let

$$A = \{x_i : i \in \mathbb{N}\} \subseteq \text{spt } u.$$

Define a function as

$$f(x) = \begin{cases} iu(x_i) & if \;\; x = x_i \in A \\ 0 & if \;\; x \notin A. \end{cases}$$

Then $f \in C(X)$. Since u is a semi-order unit, it follows that there exists a $\lambda > 0$ such that

$$|f| \wedge nu \leq \lambda u, \;\text{ for all }\; n \in \mathbb{N}.$$

In particular for $i, n \in \mathbb{N}$, we have that

$$\min\{iu(x_i), nu(x_i)\} \leq \lambda u(x_i).$$

Hence for $i, n \in \mathbb{N}$, we have that $iu(x_i) \leq \lambda u(x_i)$ or $nu(x_i) \leq \lambda u(x_i)$. Since $u(x_i) > 0$, it follows that for $i, n \in \mathbb{N}$, we have that $n \leq \lambda$ or $i \leq \lambda$. Since this holds for any $i, n \in \mathbb{N}$, this means that $\mathbb{N}$ is bounded. However this is a contradiction hence, spt u is finite. □

Definition 2.1.6. Let E be a vector lattice. By the semi-order unit topology on E, we will mean the locally convex topology generated by the collection of semi-norms associated to the family of all semi-order units in E.

Definition 2.1.7. Let E be a vector lattice. A subset D of E is upward directed if for every $x, y \in D$, there exists $z \in D$ such that $x \leq z$ and $y \leq z$.

We note that the downward directed set is defined in a similar way.
Let I and J be directed sets.

Definition 2.1.8. A net $S = (v_\beta)_{\beta \in J}$ in a vector lattice E is said to be decreasing if $\beta_0 < \beta_1$ implies that $v_{\beta_1} \leq v_{\beta_0}$. In this case we write $v_\beta\downarrow$. If, in addition $\inf(v_\beta, \beta \in J) = v$ for some $v \in E$, we say (v_β) decreases to v and we write $v_\beta \downarrow v$.

Definition 2.1.9. A net $S = (u_\alpha)_{\alpha \in I}$ in a vector lattice E is said to be order convergent to $u \in E$ if there exists a net $(v_\beta)_{\beta \in J}$ in E such that $v_\beta \downarrow 0$ and for every $\beta \in J$ there exists $\alpha_\beta \in I$ such that $|u_\alpha - u| \leq v_\beta$ whenever $\alpha \geq \alpha_\beta$. We denote this by

$$u_\alpha \xrightarrow{o} u \tag{2.4}$$

and say $(u_\alpha)_{\alpha \in I}$ is o-convergent to u.

Lemma 2.1.10. *Let E be a vector lattice and $D \subseteq E$, D downward directed and bounded below. If $y = \inf D$ then there exists a net in D that order converges to y.*

Proof. Since D is downward directed, it follows that for $u, v \in D$, there exists a $w \in D$ such that $w \leq v$ and $w \leq u$. Now let $I = (D, \leq^o)$ where for $u, v \in D$, $u \leq^o v$ if and only if $v \leq u$. Define a map

$$S : I \ni u \longmapsto u \in D.$$

Then S is a decreasing net and since $D \downarrow y$ in E, it follows that $S \downarrow y$ hence $S \xrightarrow{o} y$. □

Definition 2.1.11. Let E be a vector lattice and $(x_\alpha)_{\alpha \in I}$ a net in E. We say that $\limsup x_\alpha = z$ if there exists an $\alpha_0 \in I$ such that

$$\inf_{\alpha' \geq \alpha_0} \sup_{\alpha \geq \alpha'} x_\alpha = z.$$

We also say that $\liminf x_\alpha = z$ if there exists an $\alpha_0 \in I$ such that

$$\sup_{\alpha' \geq \alpha_0} \inf_{\alpha \geq \alpha'} x_\alpha = z.$$

We note that, in general, if $\limsup x_\alpha$ and $\liminf x_\alpha$ exist then $\liminf x_\alpha \leq \limsup x_\alpha$.

Theorem 2.1.12. *Let E be a Dedekind complete vector lattice, $(x_\alpha)_{\alpha \in I}$ a net in E and x a point in E. Then the followings are equivalent:*

i. $x_\alpha \xrightarrow{o} x$.

ii. $\liminf x_\alpha = x = \limsup x_\alpha$.

Proof. Suppose that $x_\alpha \xrightarrow{o} x$. It suffices to show that $\liminf x_\alpha \geq x \geq \limsup x_\alpha$. Since $x_\alpha \xrightarrow{o} x$, then there exists nets $(w_\beta)_{\beta \in J} \uparrow x$ and $(v_\beta)_{\beta \in J} \downarrow x$ such that for all $\beta \in J$, there exists $\alpha_\beta \in I$ such that if $\alpha \geq \alpha_\beta$ then $w_\beta \leq x_\alpha \leq v_\beta$. Therefore there exists an $\alpha_0 \in I$ such that $\{x_\alpha : \alpha \geq \alpha_0\}$ is order bounded. Hence $\liminf x_\alpha$ and $\limsup x_\alpha$ exist because E is Dedekind complete. Now fix $\beta_0 \in J$ and let $\alpha_0 = \alpha_{\beta_0}$. Then $w_{\beta_0} \leq x_\alpha \leq v_{\beta_0}$ for all $\alpha \geq \alpha_0$. Now $\left(\sup_{\alpha \geq \alpha'} x_\alpha\right)_{\alpha' \geq \alpha_0}$ is a decreasing net in α_0.
For all $\beta \geq \beta_0$, there exists $\alpha_\beta \geq \alpha_0$ such that if $\alpha \geq \alpha_\beta$ then $w_\beta \leq x_\alpha \leq v_\beta$. Hence $w_\beta \leq \sup_{\alpha \geq \alpha_\beta} x_\alpha \leq v_\beta$. Now

$$\limsup x_\alpha = \inf_{\alpha' \geq \alpha_0} \sup_{\alpha \geq \alpha'} x_\alpha \leq \inf_{\beta \geq \beta_0} \sup_{\alpha \geq \alpha_\beta} x_\alpha \leq \inf_{\beta \geq \beta_0} v_\beta = x.$$

We also have that

$$\liminf x_\alpha = \sup_{\alpha' \geq \alpha_0} \inf_{\alpha \geq \alpha'} x_\alpha \geq \sup_{\beta \geq \beta_0} \inf_{\alpha \geq \alpha_\beta} x_\alpha \geq \inf_{\beta \geq \beta_0} w_\beta = x.$$

Hence $\limsup x_\alpha \leq \liminf x_\alpha$ so that $\limsup x_\alpha = \liminf x_\alpha = x$.
Now suppose that $\limsup x_\alpha = x = \liminf x_\alpha$. Then there exists an $\alpha_0 \in I$ such that

$$\inf_{\alpha' \geq \alpha_0} \sup_{\alpha \geq \alpha'} x_\alpha = x = \sup_{\alpha' \geq \alpha_0} \inf_{\alpha \geq \alpha'} x_\alpha.$$

Let $J = \{\alpha \in I : \alpha \geq \alpha_0\}$ with the ordering inherited from I. Then J is upward directed.
Let $v_\alpha = \sup\{x_{\alpha'} : \alpha' \geq \alpha\}$ for all $\alpha \in J$. Then $v_\alpha \downarrow x$. Also let $w_\alpha = \inf\{x_{\alpha'} : \alpha' \geq \alpha\}$. Then $w_\alpha \uparrow x$. Define $u_\alpha = \sup\{v_\alpha - x, x - w_\alpha\}$. Then $(u_\alpha)_{\alpha \in J}$ is a net in E and $u_\alpha \downarrow 0$. Now if $\alpha' \geq \alpha$ then

$$w_\alpha = \inf_{\alpha' \geq \alpha} x_{\alpha'} \leq x_\alpha \leq \sup_{\alpha' \geq \alpha} x_{\alpha'} = v_\alpha.$$

That is $w_\alpha - x \leq x_{\alpha'} - x \leq v_\alpha - x$ so that $|x_{\alpha'} - x| \leq u_\alpha$.
Hence $x_\alpha \xrightarrow{o} x$. □

Theorem 2.1.13. *Let E and F be vector lattices and let $\tau : E \longrightarrow F$ a linear map. Then the following are equivalent:*

i. *τ is a lattice homomorphism.*

ii. *$\tau[x \vee y] = \tau x \vee \tau y$ for all $x, y \in E$.*

iii. *$\tau[x \wedge y] = \tau x \wedge \tau y$ for all $x, y \in E$.*

iv. *$\tau[x \vee y] = \tau x \vee \tau y$ whenever $x \vee y = 0$ holds in E.*

v. *$\tau x^+ = [\tau x]^+$*

vi. *$\tau |x| = |\tau x|$*

Proof. For the proof of this result, see [1, Theorem 1.31]. □

2.2 Order Bounded Operators and Order Adherence

Definition 2.2.1. Let E and F be be a vector lattices. A linear operator

$$T : E \longrightarrow F$$

is said to be order bounded if it maps order intervals in E into order intervals in F.

Theorem 2.2.2. *Let E and F be vector lattices and denote by $\mathcal{L}_b(E,F)$ the collection of all order bounded operators from E into F. Then $\mathcal{L}_b(E,F)$ is an ordered vector space.*

Proof. Suppose $T, S \in \mathcal{L}(E,F)$.
Fix $[x,y] \subset E$. Since $T \in \mathcal{L}_b(E,F)$, the order bounded interval $[x,y]$ is mapped to some bounded interval say $[a,b]$ in F and similarly S is mapped into an order bounded $[c,d]$.
Claim 1: $(T+S)([x,y]) \subseteq [a+c, b+d]$.
Fix $v \in [x,y]$ then

$$a \leq T(v) \leq b \text{ and } c \leq S(v) \leq d.$$

It then follows that

$$a + c \leq (T+S)(v) = T(v) + S(v) \leq b + d.$$

Hence $(T+S)(v) \in [a+c, b+d]$, proving the claim.
$\lambda > 0$.
Claim 2: λT is mapped into $[\lambda a, \lambda b]$.
Fix $v \in [x,y]$, then $a \leq T(v) \leq b$. Since $\lambda > 0$ it follows that

$$\lambda a \leq \lambda T(v) = T(\lambda v) \leq \lambda b.$$

Hence λT is order bounded.
Suppose $\lambda = -1$. Now if $v \in [x,y]$ then $a \leq T(v) \leq b$ so that

$$-b \leq -T(v) = T(-v) \leq -a.$$

That is $(-T)([x,y]) \subseteq [-b, -a]$. Hence it is a linear vector space. □

When F is Dedekind complete, we present the following result.

Theorem 2.2.3. *Let E,F be vector lattices with F being Dedekind complete and let $D \subseteq \mathcal{L}_b(E,F)$ be upward directed and bounded from above. Then*

$$S(x) = \sup\{Tx : T \in D\}, x \geq 0, x \in E$$

extends to an operator $\widetilde{S} \in \mathcal{L}_b(E,F)$ such that $\widetilde{S} = \sup D$ in $\mathcal{L}_b(E,F)$.

Proof. We prove for the case when $D \subseteq \mathcal{L}_b(E,F)_+$.
Consider T_1 and T_2 in D. Since D is upward directed there exists an operator

$T_3 \in D$ such that $T_1 \leq T_3$ and $T_2 \leq T_3$ in particular if $x_1, x_2 \in E_+$ then $T_1 x_1 \leq T_3 x_1$ and $T_2 x_2 \leq T_3 x_2$. Hence

$$T_1 x_1 + T_2 x_2 \leq T_3 x_1 + T_3 x_2 = T_3(x_1 + x_2) \leq S(x_1 + x_2).$$

Taking the supremum we have $S(x_1) + S(x_2) \leq S(x_1 + x_2)$.
For the reverse inequality, consider $T \in D$ then

$$T(x_1 + x_2) = Tx_1 + Tx_2 \leq S(x_1) + S(x_2).$$

Hence $S(x_1+x_2) \leq S(x_1)+S(x_2)$ so that $S(x_1+x_2) = S(x_1)+S(x_2)$. Hence the map S is additive and it follows from extension Lemma (see [11, Lemma 20.1]) that there exists a uniquely determined positive linear operator $\widetilde{S} : E \longrightarrow F$ such that $\widetilde{S}$ extends S.
Claim: $\widetilde{S} = \sup D$.
For $x \in E_+$ and $T \in D$, we have that

$$Tx \leq \sup\{Tx : T \in D\} = \widetilde{S}(x).$$

Hence $T \leq \widetilde{S}$. Since T was an arbitary element of D we have that $\widetilde{S}$ is an upper bound of D. Now Let R be any other upper bound of D. Then $T \leq R$ for all $T \in D$. For $x \in E_+$ we have that

$$\widetilde{S}(x) = \sup\{Tx : T \in D\} \leq Rx.$$

Thus $\widetilde{S} \leq R$ so that $\widetilde{S}$ is the supremum of D proving the claim. □

If we replace F by $\mathbb{R}$, then the collection of all bounded linear operators from E to $\mathbb{R}$ form a vector space called the order dual of E and we denote it by $E^{\sim}$.
The bidual of E is defined in the similar way and we denote it by $E^{\sim\sim}$.

Theorem 2.2.4. *Let E and F be vector lattices. Assume $T : E \longrightarrow F$ is an interval preserving operator and let $T^* : F^{\sim} \longrightarrow E^{\sim}$ be defined by $T^*\phi = \phi \circ T$ for all $\phi \in F^{\sim}$. Then T^* is a lattice homomorphism.*

Proof. The proof of this result can be found in [2, Theorem 2.19]. □

Corollary 2.2.5. *Let E and F be vector lattices. Assume $T : E \longrightarrow F$ is a vector lattice isomorphism. Then T^* is a vector lattice isomorphism.*

Definition 2.2.6. Let E be a vector lattice. The order adherence of E are all elements $u \in E^{\sim\sim}$ such that there exists a net $S = (u_\alpha)_{\alpha \in I}$ in E with $u_\alpha \xrightarrow{o} u$ in $E^{\sim\sim}$ and we denote this by $\widetilde{E}$. That is

$$\widetilde{E} = \big\{u \in E^{\sim\sim} : \exists (u_\alpha)_{\alpha \in I} \subseteq E,\ \ u_\alpha \xrightarrow{o} u\big\}.$$

Theorem 2.2.7. *The space $\widetilde{E}$ is a sublattice of $E^{\sim\sim}$.*

Proof. Let $u, v \in \widetilde{E}$. Then there exist nets $(u_\alpha)_{\alpha \in I}$ and $(v_\beta)_{\beta \in J}$ in E such that

$$u_\alpha \xrightarrow{o} u \text{ and } v_\beta \xrightarrow{o} v,$$

where I and J are directed sets. Let $w_{(\alpha,\beta)}$ with the component-wise ordering be the net defined by

$$w_{(\alpha,\beta)} = u_\alpha + v_\beta.$$

Using the triangle inequality we have,

$$\begin{aligned} |(u+v) - w_{(\alpha,\beta)}| &= |(u+v) - (u_\alpha + v_\beta)| \\ &= |(u - u_\alpha) + (v - u_\beta)| \\ &\leq |u - u_\alpha| + |v - v_\beta|. \end{aligned}$$

Since $u_\alpha \xrightarrow{o} u$ it follows that there exists a net $(f_\gamma)_{\gamma \in K_A}$ with K_A a directed set so that $f_\gamma \downarrow 0$ and for each $\gamma \in K_A$, there exists $\alpha_\gamma \in I$ such that $|u - u_\alpha| \leq f_\gamma$ whenever $\alpha \geq \alpha_\gamma$.
Since $v_\beta \xrightarrow{o} v$ it follows that there exists a net $(g_\lambda)_{\lambda \in K_B}$ with K_B a directed set so that $g_\lambda \downarrow 0$ and for each $\lambda \in K_B$ there exists $\beta_\lambda \in J$ such that $|v - v_\beta| \leq g_\beta$ whenever $\beta \geq \beta_\lambda$.
Let $h_{(\gamma,\lambda)} = f_\gamma + g_\lambda$, with the component-wise ordering. Now if $(\gamma_0, \lambda_0) \leq (\gamma_1, \lambda_1)$ then $\gamma_0 \leq \gamma_1$ and $\lambda_0 \leq \lambda_1$ so that $f_{\gamma_1} \leq f_{\gamma_0}$ and $g_{\lambda_1} \leq g_{\lambda_0}$. Hence

$$h_{(\gamma_1,\lambda_1)} = f_{\gamma_1} + g_{\lambda_1} \leq f_{\gamma_0} + g_{\lambda_0} = h_{(\gamma_0,\lambda_0)}.$$

Thus $h_{(\gamma,\lambda)} \downarrow$. Since $f_\gamma \downarrow 0$ and $g_\lambda \downarrow 0$ it follows that $h_{(\gamma,\lambda)} \downarrow 0$.
Now for any (γ, λ), $(\alpha, \beta) \geq (\alpha_\gamma, \beta_\lambda)$ implies that $\alpha \geq \alpha_\gamma$ and $\beta \geq \beta_\lambda$ hence

$$|(u+v) - w_{(\alpha,\beta)}| \leq f_\gamma + g_\lambda = h_{(\gamma,\lambda)}$$

so that $w_{(\alpha,\beta)} \xrightarrow{o} u+v$. Hence there is a net $(w_{(\alpha,\beta)})_{(\alpha,\beta) \in I \times J}$ in E that order converges to $u+v$ so that $u+v \in \widetilde{E}$.
Let $\beta \in \mathbb{R}$. Since $u \in \widetilde{E}$, there exists a net $(u_\alpha)_{\alpha \in I}$ in E such that $u_\alpha \xrightarrow{o} u$. Define $v_\alpha = \beta u_\alpha$ for each $\alpha \in I$. Then

$$|\beta u - v_\alpha| = |\beta u - \beta u_\alpha| = |\beta(u - u_\alpha)| = |\beta||u - u_\alpha|.$$

Since $u_\alpha \xrightarrow{o} u$ it follows that there exists a net $(f_\gamma)_{\gamma \in K_A}$ so that $f_\gamma \downarrow 0$ and for each $\gamma \in K_A$, there exists $\alpha_\gamma \in I$ such that $|u - u_\alpha| \leq f_\gamma$ whenever $\alpha \geq \alpha_\gamma$. Let $g_\gamma = |\beta| f_\gamma$. Then $g_\gamma \downarrow 0$ and if $\alpha > \alpha_\gamma$ then

$$|\beta u - v_\alpha| \leq |\beta| f_\gamma = g_\gamma.$$

Hence $v_\alpha \xrightarrow{o} \beta u$ so that $\beta u \in E$. Since $0 \in E \subseteq \widetilde{E}$ we have that E is a linear subspace of $E^{\sim\sim}$. Since $\widetilde{E}$ inherits the ordering from $E^{\sim\sim}$, it is an ordered vector space.
It remains to show that it is a lattice. It is sufficient to prove tat if $u \in \widetilde{E}$, then $|u| \in \widetilde{E}$.
Suppose $u \in \widetilde{E}$ then there exists a net $(u_\alpha)_{\alpha \in I}$ in E such that $u_\alpha \xrightarrow{o} u$. Hence there exist a net $f_\gamma \downarrow 0$ such that for each $\gamma \in J$ we can find an $\alpha_\gamma \in I$ and if $\alpha \geq \alpha_\gamma$, then $|u_\alpha - u| \leq f_\gamma$.
Since E is a vector lattice, $|u_\alpha| \in E$ for each $\alpha \in I$ thus $(|u_\alpha|)_{\alpha \in I}$ is a net in E.
Now, for each $\gamma \in J$, if $\alpha \geq \alpha_\gamma$, by the reverse triangle inequality, we have that

$$||u_\alpha| - |u|| \leq |u_\alpha - u| \leq f_\gamma.$$

Thus $|u_\alpha| \xrightarrow{o} |u|$. Hence $\widetilde{E}$ is a lattice. □

2.3 Lattice Theory

In this section we give more general results on general vector lattices which will be used in the next chapter. The approach is to establish each result in a general setup and apply it to a specific vector lattice. In particular it will be applied to the space of real valued continuous functions on a topological space X.
Let E be a vector lattice and I an ideal in E. We denote by E/I, the quotient space and we use $[x]$ to denote the elements of E/I. In particular,

$$[x] = x + I = \{x + y : y \in I\}.$$

For $x, y \in E$, we will say $[x] \leq [y]$ if there exists $x_1 \in [x]$ and $y_1 \in [y]$ such that $x_1 \leq y_1$. Observe that this defines a partial ordering on E/I.

Theorem 2.3.1. *The space E/I is a vector lattice and we call it the quotient vector lattice of E modulo the ideal I.*

The proof of this result can be found in [11].

Proposition 2.3.2. *Let E and F be vector lattices and let $T : E \longrightarrow F$ be a lattice homomorphism. Then* $\ker T$ *is an ideal in E.*

Proof. The zero vector is in $\ker T$. If $x, y \in \ker T$ and $\lambda \in \mathbb{R}$, then $T(x+y) = Tx + Ty = 0 + 0 = 0$. Also $T(\lambda x) = \lambda T x = \lambda 0 = 0$, so that $\ker T$ is a linear vector space. Finally since T is a lattice homomorphism we have that $T(x \vee y) = Tx \vee Ty = 0 \vee 0 = 0$, hence $\ker T$ is an ideal in E. □

Theorem 2.3.3. *Let E and F be vector lattices and let $T : E \longrightarrow F$ be a lattice homomorphism onto F. Then the map $\widetilde{T} : E/\ker T \longrightarrow F$ defined by*

$$\widetilde{T}(x + \ker T) = Tx$$

is well defined and is a lattice isomorphism.

Proof. Suppose $x_1 + \ker T = x_2 + \ker T$. Then $x_1 - x_2 \in \ker T$ so that $T(x_1 - x_2) = 0$. Hence $Tx_1 = Tx_2$ and $\widetilde{T}(x_1 + \ker T) = \widetilde{T}(x_2 + \ker T)$. Hence $\widetilde{T}$ is well defined. The map $\widetilde{T}$ is linear by abstract results.
Surjectivity: Since T is onto, let $y \in F$, then there exists an $x \in E$ such that $Tx = y$. Hence

$$\widetilde{T}(x + \ker T) = Tx = y$$

so that $\widetilde{T}$ is surjective.
Injectivity: Let $x + \ker T \in \ker \widetilde{T}$. Then $0 = \widetilde{T}(x + \ker T) = Tx$. Hence $x \in \ker T$. Thus $x + \ker T = \ker T$ which is the zero element of the quotient space. Hence $\ker \widetilde{T} = \{\ker T\}$ so that $\widetilde{T}$ is injective.
Finally let $[x_1] = x_1 + \ker T$ and $[x_2] = x_2 + \ker T$ be elements in $E/\ker T$. Then

$$\begin{aligned}
\widetilde{T}([x_1] \vee [x_2]) &= \widetilde{T}([x_1 \vee x_2]) \\
&= T(x_1 \vee x_2) \ \ (\text{definition of } \widetilde{T}) \\
&= Tx_1 \vee Tx_2 \ \ (T \ \text{ is a lattice homomorphism}) \\
&= \widetilde{T}([x_1]) \vee \widetilde{T}([x_2]) \ \ (\text{definition of } \widetilde{T}).
\end{aligned}$$

Hence $\widetilde{T}([x_1] \vee [x_2]) = \widetilde{T}([x_1]) \vee \widetilde{T}([x_2])$ so that $\widetilde{T}$ is a lattice isomorphism. □

Definition 2.3.4. Let E be a vector lattice. $E^\sim$ separates E if for every $0 \neq x \in E$ there exists $\phi \in E^\sim$ such that $\phi(x) \neq 0$.

Note that if $F \subseteq E$, we will say F separates $E^\sim$ if for every $0 \neq \phi \in E^\sim$, there exists an $x \in F$ such that $\phi(x) \neq 0$.

Definition 2.3.5. Let E be a vector lattice and $I \subseteq E$ an ideal in E. We denote by $I^\perp$ the subspace of $E^\sim$ whose members vanishes on I. That is

$$I^\perp = \{\phi \in E^\sim : \phi[I] = \{0\}\},$$

and we say $I^\perp$ is the annihilator of I.

Theorem 2.3.6. *Let E be a vector lattice and $I \subseteq E$ an ideal in E. Suppose $E^\sim$ separates E and $(E/I)^\sim$ separates E/I. Let $Q : E \longrightarrow E/I$ defined by $f \longmapsto f + I$ be the quotient map. Then the map $Q^* : (E/I)^\sim \longrightarrow I^\perp$ defined by*

$$Q^*(\phi) = \phi \circ Q$$

is a lattice isomorphism.

Proof. If $\phi \in (E/I)^\sim$ and $f \in I$ then $Q^*[\phi](f) = \phi \circ Q(f) = \phi(Q(f)) = \phi(0 + I) = 0$ so that $\phi \circ Q \in I^\perp$. Hence Q^* is well defined. The operator Q^* is linear since an adjoint of a linear transformation is linear. Since the map $Q : E \longrightarrow E/I$ is surjective, the adjoint operator Q^* is injective.
Surjectivity: Let $\gamma \in I^\perp$. Then define a map $\phi : E/I \longrightarrow \mathbb{R}$ given by

$$\phi(f + I) = \gamma(f), \;\; f \in E.$$

First suppose that $\gamma \in E^\sim_+$. If $f \in E$ and $f \geq 0$ then $\gamma(f) \geq 0$. Hence if $f + I \in E/I$ such that $f + I$ is a positive element then there exists a positive element $f' \in E$ such that $f' + I = f + I$ so that $\phi(f + I) = \phi(f' + I) = \gamma(f') \geq 0$. Hence ϕ is a positive functional on E/I so that $\phi \in (E/I)^\sim_+$.
Now for any $\gamma \in E^\sim$, let

$$\phi_0(f + I) = \gamma^+(f), \;\; \phi_1(f + I) = \gamma^-(f), \;\; f \in E.$$

Then ϕ_0 and ϕ_1 are positive functionals on E/I so that $\phi_0, \phi_1 \in (E/I)^\sim_+$. Let $\phi = \phi_0 - \phi_1$. Then $\phi \in (E/I)^\sim$. Thus Q^* is surjective.
It remains to show that Q^* and its inverse map are both positive maps so that it is a lattice isomorphism. To this end, let $\phi \geq 0$. Then for any positive element $f + I$ of E/I we have that $\phi(f + I) \geq 0$. So if $f \geq 0$ in E then

$$[Q^*(\phi)](f) = \phi(Q(f)) = \phi(f + I) \geq 0.$$

Now suppose $Q^*\phi \geq 0$. Let $f + I$ be a positive element of E/I then there exists an $f' \in E$ such that $f' \geq 0$ and $f + I = f' + I$. Hence

$$\begin{aligned}
\phi(f + I) &= \phi(f' + I) \\
&= \phi(Q(f')) \\
&= [Q^*(\phi)](f') \\
&\geq 0.
\end{aligned}$$

Hence $\phi \geq 0$ if and only if $Q^*(\phi) \geq 0$ so that Q^* is a linear lattice isomorphism. $\square$

Theorem 2.3.7. *Let E be a vector lattice and I an ideal in E. Then $I^{\perp}$ is a band in $E^{\sim}$.*

Proof. If ϕ_0 is the zero functional on E then $\phi_0(x) = 0$ for all $x \in E$. hence $\phi_0(x) = 0$ for all $x \in I$. We have $\phi_0 \in I^{\perp}$. Now if $\phi_1, \phi_2 \in I^{\perp}$ and $\alpha \in \mathbb{R}$, then for $x \in I$

$$(\phi_1 + \alpha\phi_2)(x) = \phi_1(x) + \alpha\phi_2(x) = 0 + 0 = 0.$$

Hence $\phi_1 + \alpha\phi_2 \in I^{\perp}$ and $I^{\perp}$ is a linear subspace of $E^{\sim}$.
Next we will show that if $\psi \in I^{\perp}$ and $x \in I_+$ then $|\psi|(x) = 0$.
If $\psi \in I^{\perp}$ then $\psi \in E^{\sim}$ and $\psi(x) = 0$ for all $x \in I$. Note that if $x \in I_+$ then $x \in E_+$. Since I is an ideal in E, it follows that if $x \in I_+$ and $y \in E$ such that $|y| \leq x$, then $y \in I$ and $\psi(y) = 0$. Hence

$$|\psi|(x) = \sup\{\psi(y) : |y| \leq x\} = \sup\{0 = \psi(y) : |y| \leq x\} = 0.$$

Now let $\psi_1 \in I^{\perp}$ and $\psi_2 \in E^{\sim}$ be such that $|\psi_2| \leq |\psi_1|$. For $x \in I_+$, we have that $|\psi_2|(x) \leq |\psi_1|(x) = 0$. Hence $|\psi_2|(x) = 0$, thus $\sup\{\psi_2(y) : |y| \leq x\} = 0$. Since $x \in E_+$ and $|x| = x$, we have $\psi_2(x) = 0$. Therefore $\psi_2 \in I^{\perp}$. Thus $I^{\perp}$ is an ideal in $E^{\sim}$.

It remains to show that $I^{\perp}$ is a band.
Let D be an upward directed set in $I^{\perp}$ that is bounded from above. For $x \in E_+$, define

$$\phi_1(x) = \sup\{\phi(x) : \phi \in D\}.$$

Then $\phi_1(x) = 0$ for all $x \in I_+$. It follows from Theorem 2.2.3 that ϕ_1 can be extended to a bounded linear operator $\widetilde{\phi_1}$ such that $\widetilde{\phi_1} = \sup D$ and $\widetilde{\phi_1}(I) = 0$ so that $\sup D \in I^{\perp}$. Hence $I^{\perp}$ is a band in $E^{\sim}$. □

Definition 2.3.8. Let E be vector lattice and $A \subseteq E$. Then

$$A^d = \{y \in E : |x| \wedge |y| = 0 \ \ \forall x \in A\}.$$

We call A^d the disjoint complement of A.

Theorem 2.3.9. *Let E be a vector lattice. Then A^d is a band in E.*

Proof. The proof of this result can be found in [11, Theorem 8.4] □

Theorem 2.3.10. *Let E be a vector lattice and B be a band in E such that*

$$E = B \oplus B^d.$$

Suppose $x \in B$ is an order unit of B. Then x is a semi-order unit of E.

Proof. Fix $z \in E$. Then $z = z_1 + z_2$ where $z_1 \in B$ and $z_2 \in B^d$. We also have that $|z| = |z_1| + |z_2|$. Since $x \in B$ is an order unit, there exists a $\lambda > 0$, $\lambda\mathbb{R}$ such that $|z_1| \leq \lambda x$.
For $n \in \mathbb{N}$, we have the following

$$\begin{aligned} |z| \wedge nx &= (|z_1| + |z_2|) \wedge nx \\ &= (|z_1| \wedge nx) + (|z_2| \wedge nx) \\ &= |z_1| \wedge nx \;\; (\because nx \in B \;\text{ and }\; z_2 \in B^d \;\text{ so }\; |z_2| \wedge nx = 0) \\ &\leq |z_1| \\ &\leq \lambda x. \end{aligned}$$

Hence for every $z \in E$, there exists a λ such that $|z| \wedge nx \leq \lambda x$, $n \in \mathbb{N}$. Thus x is a semi-order unit of E. □

Lemma 2.3.11. *Let E be a vector lattice and B a band in E such that*

$$E = B \oplus B^d.$$

Suppose $P_B : E \longrightarrow B$ is the band projection on the first component. The map

$$P_B^* : B^\sim \longrightarrow E^\sim$$

is a lattice isomorphism onto $(B^d)^\perp$.

Proof. P_B is onto so that the adjoint P_B^* is injective and is linear since the adjoint of a linear transformation is linear.
For surjectivity, let $M = (B^d)^\perp$ and $\psi \in M$. Then $\psi \in E^\sim$ and $\psi(x) = 0$ for all $x \in B^d$. We need to prove that there exists a $\Psi \in B^\sim$ such that $P_B^*(\Psi) = \psi$. First we let $\Psi = \psi|_B$. Then Ψ is well defined and $\Psi \in B^\sim$. Now let $x \in E$. Since $E = B \oplus B^d$, then there exist elements y, z with $y \in B$ and $z \in B^d$ such that $x = y + z$. Hence

$$\begin{aligned} [P_B^*\Psi](x) &= \Psi(P_B(x)) \\ &= \Psi(y) \\ &= \psi(y) \;\; (\because \Psi = \psi|_B \;\text{ and }\; y \in B) \\ &= \psi(y) + \psi(z) \;\; (\because \psi \in M \text{ so } \psi(z) = 0) \\ &= \psi(y + z) = \psi(x). \end{aligned}$$

Hence $P_B^*(\Psi) = \psi$ so that P_B^* is surjective.

Finally we show that P_B^* and its inverse are both positive so that P_B^* is a lattice isomorphism.

Suppose $\Psi \geq 0$ and let $x \in E_+$ then $x = y + z$ where y and z are in B_+ and $(B^d)_+$ respectively. Since $y \in B_+$ and $\Psi \geq 0$, we have that $\Psi(y) \geq 0$. Hence

$$[P_B^*\Psi](x) = \Psi(P_B(x)) = \Psi(y) \geq 0.$$

Now suppose $P_B^*\Psi \geq 0$. Since $y \in B_+$, then $y \in E_+$. Hence

$$\Psi(y) = \Psi(P_B(y)) = [P_B^*\Psi](y) \geq 0.$$

Hence P_B^* is a lattice isomorphism between $B^\sim$ and $(B^d)^\perp$. □

Lemma 2.3.12. *Let E be a vector lattice. If $\phi \in E^\sim$ and $f \in E$ then*

$$|\phi(f)| \leq |\phi|(|f|).$$

In particular, if $\phi \in E_+^\sim$ then $|\phi(f)| \leq \phi(|f|)$.

Proof. First let $\psi \in E^\sim$ and $f \in E_+$, then

$$\psi(f) \leq \psi^+(f) \leq |\psi|(f),$$

and we also have that

$$-\psi(f) \leq \psi^-(f) \leq |\psi|(f).$$

Hence

$$|\psi(f)| = \psi(f) \vee (-\psi(f)) \leq |\psi|(f).$$

Now if $f \in E$ then

$$\begin{aligned} |\psi(f)| &= |\psi(f^+) - \psi(f^-)| \\ &\leq |\psi(f^+)| + |\psi(f^-)| \ \text{(triangle inequality)} \\ &\leq |\psi|(f^+) + |\psi|(f^-) \ (f^+, f^- \in E_+) \\ &= |\psi|(|f|). \end{aligned}$$

The last statement follows from the fact that if $\phi \geq 0$, then $f \in E_+$ so $|\phi|(|f|) = \phi(|f|)$. Hence

$$|\phi(f)| \leq |\phi|(|f|) = \phi(|f|).$$

□

2.4 Semi-order Units

We have given the definition of a semi-order unit and a couple of results based on semi-order units but we have not yet characterised them in any vector lattice. In this section, we will compute the semi-order units of some ℓ^p and L^p spaces for $p \in \mathbb{N}$. We will characterise these semi-order units.
We will make use of the following theorems for which the proof is not given.

Theorem 2.4.1. *Let E be a Banach lattice. Then the norm dual of E and the order dual of E coincide. That is $E^\sim = E^*$.*

Theorem 2.4.2 (Baire Category theorem)**.** *Let X be a non-empty complete metric space. If X can be written as a countable union of closed sets, then at least one of those sets has non-empty interior.*

Theorem 2.4.3. *Let E be a reflexive Banach lattice. Then $\widetilde{E} = E$.*

Proof. Since E is a banach lattice, by Theorem 2.4.1, $E^\sim = E^*$. The space E^* is a banach lattice hence $E^{\sim\sim} = E^{**}$. Now by reflexivity we have that

$$E^{\sim\sim} = E^{**} = E.$$

Hence $E = E^{\sim\sim}$. By definition, $\widetilde{E} \subseteq E^{\sim\sim} = E$. For the reverse inclusion, let $x \in E^{\sim\sim}$. Then $x \in E$ because $E = E^{\sim\sim}$. Now let I be any directed set and let $x_\alpha = x$. Then $(x_\alpha)_{\alpha \in I}$ is a net in E and it order converges to x so that $x \in \widetilde{E}$. Hence $E^{\sim\sim} \subseteq \widetilde{E}$ and $E = \widetilde{E}$. $\square$

Theorem 2.4.4. *Let E be a Banach lattice. Then order intervals in E are norm bounded and closed.*

Proof. Let $x \in [u, v] \subset E$ then $u \leq x \leq v$. It then follows that

$$0 \leq x - u \leq v - u \text{ so that } \quad ||x - u|| \leq ||v - u||.$$

Thus

$$||x|| = ||x - u + u|| \leq ||x - u|| + ||u|| \leq ||v - u|| + ||u||.$$

Let $M = ||v - u|| + ||u||$. Now if $x \in [u, v]$ then $||x|| \leq M$ therefore the order interval is bounded. Next, we will show that it is closed.
Let (x_n) be a sequence such that $x_n \in [u, v]$ for each $n \in \mathbb{N}$. Since $x_n \in [u, v]$, it follows that $u \leq x_n \leq v$ so that $0 \leq x_n - u \leq v - u$. By the closedness property of the positive cone, we have that $0 \leq x - u \leq v - u$ so that $u \leq x \leq v$. Hence the order interval $[u, v]$ is closed. $\square$

For the following vector lattices, we compute the semi-order units of $\widetilde{E}$.

i. c_0

ii. ℓ^p, $1 < p < \infty$

iii. $L^p(0,1), 1 < p < \infty$

iv. ℓ^1

v. c_{00} and $s = \mathbb{R}^{\mathbb{N}}$.

For notation we will use e_k to denote the sequence with the element 1 on the k^{th} component and zero everywhere else. We denote by $x(k)$ the k^{th} component of $x \in \mathbb{R}^{\mathbb{N}}$. If x_n is a sequence in $\mathbb{R}^{\mathbb{N}}$, then $x_n(k)$ denotes the k^{th} component of the n^{th} term of the sequence.

Lemma 2.4.5. *Let E be an ideal in $s = \mathbb{R}^{\mathbb{N}}$ and let $x \in E$. Then $y_n = \sum_{k=1}^{n} x(k)e_k$ order converges to x.*

Proof. Fix $x \in E$ and let (e_k) be the standard basis of E.
Now for each $n \in \mathbb{N}$, define

$$u_n = (u_n(k)) = \begin{cases} 0, & k \leq n \\ |x(k)|, & k > n. \end{cases}$$

For each, $n \in \mathbb{N}$, $u_n \leq |x|$ so that $u_n \in E$ since E is an ideal in s and $x \in E$. Thus (u_n) is a net in E. Furthermore $u_{n+1} \leq u_n$ so that $u_n \downarrow$.
For each $n \in \mathbb{N}$, we have that $0 \leq u_n$. Therefore 0 is a lower bound of the set

$$\{u_n : n \in \mathbb{N}\}.$$

Suppose $v \in E$ such that v is a lower bound of $\{u_n : n \in \mathbb{N}\}$. Then $v \leq u_n$ for all $n \in \mathbb{N}$. In particular for each $k \in \mathbb{N}$ there exists a u_n such that $u_n(k) = 0$. Hence $v(k) \leq 0$ for each $k \in \mathbb{N}$ so that $v \leq 0$. This leads to $\inf\{u_n : n \in \mathbb{N}\} = 0$. Thus $u_n \downarrow 0$, we have

$$|y_n - x| = u_n \downarrow 0.$$

From this we see, $y_n \xrightarrow{o} x$. □

Example 2.4.6. The norm dual of c_0 is ℓ^1. By Theorem 2.4.1, the order dual of c_0 is also ℓ^1. Now the norm dual of ℓ^1 is ℓ^∞. Hence the order dual of ℓ^1 is also ℓ^∞. Therefore the order bidual of c_0 is ℓ^∞.
We need to show that $\widetilde{c_0} = \ell^\infty$.

Let $x \in \ell^\infty$ and define $y_n = \sum_{k=1}^{n} x(k)e_k$. Then $y_n \in c_0$ for each $n \in \mathbb{N}$. It follows from Lemma 2.4.5 that $x \in \widetilde{c_0}$ proving the claim.
Now we characterise the semi-order units of ℓ^∞.
Let B be the collection of all semi-order units of ℓ^∞. Define

$$A = \{u \in \ell^\infty : u \geq 0, \ \inf\{|u(k)| : |u(k)| \neq 0\} > 0\}.$$

Next we need to show that $A = B$.

Let $u \in A$ and fix $v \in \ell^\infty$.
Since $u \in A$, it follows that $\inf\{|u(k)| : |u(k)| \neq 0\} > 0$. Let $= \inf\{|u(k)| : |u(k)| \neq 0\}$. If $u(k) = 0$ then

$$|v(k)| \wedge nu(k) = 0, \ \forall n \in \mathbb{N}.$$

Now if $u(k) \neq 0$, then $a \leq |u(k)|$ we have $1 \leq \frac{1}{a}|u(k)|$. Since $v \in \ell^\infty$, let $d = ||v||_\infty$ then $|v(k)| \leq d$, for each $k \in \mathbb{N}$. Therefore

$$|v(k)| \leq d.1 \leq \frac{d}{a}|u(k)|.$$

Hence

$$|v(k)| \wedge nu(x) \leq \frac{d}{a}|v(k)|.$$

Let $\lambda = \frac{d}{a}$. Then $|v| \wedge nu \leq \lambda u$ for each $n \in \mathbb{N}$, as a consequence, u is a semi-order unit of ℓ^∞. Thus $u \in B$. Since $u \in A$ was arbitrary we have that $A \subseteq B$.

For the reverse inclusion. Suppose $u \in B$ and consider $\mathbf{1} = (1, 1, 1, ...)$. Since u is a semi-order unit of ℓ^∞, it follows that there exists a $\lambda > 0$ such that $\mathbf{1} \wedge nu \leq \lambda u$. In particular, we have that

$$1 \wedge n|u(k)| \leq \lambda|u(k)| \text{ for each } \ n \in \mathbb{N}.$$

If $|u(k)| \neq 0$, then for suffiently large N, $1 < N|u(k)|$. We have

$$1 = 1 \wedge N|u(k)| \leq \lambda|u(k)|.$$

Hence $1 \leq \lambda u(k)$ and $\frac{1}{\lambda} \leq |u(k)|$. Since $|u(k)| \neq 0$ was arbitrary, it follows that $\inf\{|u(k)| : |u(k)| \neq 0\} \geq \frac{1}{\lambda}$, therefore $u \in A$. Thus $B \subseteq A$, proving the claim. So the semi-order units of ℓ^∞ are precisely elements $u \in \ell^\infty$ such that $u \geq 0$ and $\inf\{|u(k)| : |u(k)| \neq 0\} > 0$.

Example 2.4.7. ℓ^p with $1 < p < \infty$ is reflexive see [3]. Hence $\widetilde{\ell^p} = \ell^p$ by Theorem 2.4.3. Now we compute the semi-order units of ℓ^p.
Let $u \in \ell^p$ such that $u \geq 0$.
Statement *i*. The order interval $[-u, u]$ has empty interior.
It is sufficient to show that 0 is not an interior point of the order interval. Fix $r > 0$. It will be shown that $B(0, r) \nsubseteq [-u, u]$.
Since $u \in \ell^p$, there exists a $k \in \mathbb{N}$ such that $0 \leq u(k) < r$. Choose $v(k)$ such that $u(k) < v(k) < r$. Now, let $v = (0, ...0, v(k), 0, ...)$. Clearly $v \in \ell^p$ and $u(k) < v(k)$ which implies $v \notin [-u, u]$. Now

$$||v - 0||_p = v(k) < r, \text{ hence } v \in B(0, r).$$

Hence 0 is not an interior point of the order interval $[-u, u]$. By translation we can show that every point of $[-u, u]$ is not an interior point. Hence the order interval has empty interior.
Statement *ii*. ℓ^p has no order unit.
We will prove by contraction. Suppose $u \in \ell^p$ is an order unit. Then for each $v \in \ell^p$, there exists an $n \in \mathbb{N}$ such that $|v| \leq nu$. Hence $v \in [-nu, nu]$. Since $v \in \ell^p$ was abitrary we have that

$$\ell^p = \bigcup_{n \in \mathbb{N}} [-nu, nu].$$

Hence ℓ^p can be written as a countable union of closed empty interior sets, however this contradicts Theorem 2.4.2. Thus u cannot be an order unit of ℓ^p which leads to ℓ^p having no order units.

We now characterise the semi-order units of ℓ^p.

Now let $u \in \ell^p$ such that $u(k) > 0$ for all $k \in \mathbb{N}$.
Statement *iii*. u is a semi-order unit if and only if u is an order unit.
If u is an order unit then for each $v \in \ell^p$, there exists $\lambda > 0$ such that $|v| \leq \lambda u$. Now if $n \in \mathbb{N}$, then $|v| \wedge nu \leq |v| \leq \lambda u$. Hence

$$|v| \wedge nu \leq \lambda u \text{ for each } n \in \mathbb{N}.$$

Thus u is a semi-order unit of ℓ^p.
Now suppose u is a semi-order unit of ℓ^p. If $v \in \ell^p$, then there exists a $\lambda > 0$ such that $|v| \wedge nu \leq \lambda u$ for each $n \in \mathbb{N}$.
Since the infimum is pointwise, we have that for each $k \in \mathbb{N}$,

$$|v(k)| \wedge n|u(k)| \leq \lambda|u(k)| \text{ for each } n \in \mathbb{N}.$$

Now fix $k \in \mathbb{N}$. Since $\mathbb{N}$ is not bounded, there exists an $N \in \mathbb{N}$ such that $\lambda < N$ hence $\lambda u(k) < Nu(k)$ since $u(k) > 0$. It then follows that

$$|v(k)| = |v(k)| \wedge N|u(k)| \leq \lambda |u(k)|.$$

There $|v(k)| \leq \lambda |u(k)|$. Since $k \in \mathbb{N}$, was arbitrary we have that $|v| \leq \lambda u$ so that u is an order unit of ℓ^p. But ℓ^p has no order units, it follows that u cannot be a semi-order unit of ℓ^p.
Statement iv. Let $u_0 \in \ell^p$. If spt u_0 is infinite then u_0 is not a semi-order unit of ℓ^p.
Let $S =$ spt u_0 and $\phi : S \longrightarrow \mathbb{N}$ be a bijection. Let

$$E = \{u \in \ell^p : u(k) = 0 \ \ \forall k \in S^c\}$$

and define $T : E \longrightarrow \ell^p$ and $T^{-1} : \ell^p \longrightarrow E$ by

$$(Tu)(k) := u(\phi^{-1}(k)) \ \text{ and } \ (T^{-1}u)(k) := u(\phi(k)).$$

Let $u, v \in E$ such that $u \neq v$. Then there exists $k \in \mathbb{N}$ such that $u(\phi^{-1}(k)) \neq v(\phi^{-1}(k))$ and $Tu \neq Tv$. Hence T is one to one. Now let $w \in \ell^p$ and for $k \in S^c$, let $w(k) = 0$.

$$w_0(k) = \begin{cases} 0 & \text{if } \phi^{-1}(k) \in S^c \\ w(k) & \text{if } \phi^{-1}(k) \in S \end{cases}$$

Then $w_0 \in E$ and $Tw_0 = w$. Therefore T is surjective.

It remains to show that T and T^{-1} are positive. Let $u \in E$ such that $u \geq 0$. Since the ordering was pointwise, it follows that $u(k) \geq 0$ for all $k \in \mathbb{N}$ so that $(Tu)(k) = u(\phi^{-1}(k)) \geq 0$. Hence T is positive. Also if $u \in \ell^p$ such that $u \geq 0$, then $u(k) \geq 0$ which leads to $(T^{-1}u)(k) = u(\phi(k)) \geq 0$. Hence $T^{-1}u \geq 0$. We conclude that T and T^{-1} are positive and T is a lattice isomorphism.

Now if u_0 is a semi-order unit of E, and then Tu_0 is also a semi-order unit of ℓ^p with full support. By statement $iii.$, Tu_0 is an order unit of ℓ^p contradicting statement $i.$. Thus u_0 is not a semi-order unit of ℓ^p. Hence no element with infinitely many non-zero elements is a semi-order unit of ℓ^p.
Let B be a collection of all semi-order units of ℓ^p and let A be defined as follows

$$A = \{u \in \ell^p : u \geq 0, u(k) \neq 0 \ \text{ for finite } \ k's\}$$

Statement v. Semi order units of ℓ^p are precisely finitely supported elements, that is $A = B$.

We have shown that if $u \in \ell^p$ is not finitely supported, then u is not a semi-order unit of ℓ^p. Equivalently, if $u \notin A$ then $u \notin B$, That is $B \subseteq A$. For the reverse inclusion, suppose $u \in A$. Then u is finitely supported.
Fix $v \in \ell^p$. Then for the case $u(k) = 0, k \in \mathbb{N}$ we have that $|v(k)| \wedge nu(k) = 0$
For the case of $u(k) \neq 0, k \in \mathbb{N}$, we first observe that the set $\{|u(k)| : u(k) \neq 0\}$ is finite since u is finitely supported.

Now let $a = \inf\{|u(k)| : u(k) \neq 0\}$. Then $a > 0$. Since $\ell^p \subset \ell^\infty$, $||v||_\infty$ is well defined. Let $d = ||v||_\infty$, we have $|v(k)| \leq d \leq \frac{d}{a}|u(k)|$. Let $\lambda = \frac{d}{a}$ then $|v| \wedge nu \leq \lambda u$ for each $n \in \mathbb{N}$. Therefore u is a semi-order unit of ℓ^p, hence $u \in B$ proving the statement.

We will use the following result for which we give the proof first. Let $m(.)$ be the usual Lebesgue measure on $\mathbb{R}$ and $T \subset (0, 1)$ be measurable in the Lebesgue sense.

Lemma 2.4.8. *For any measurable $T \subseteq (0, 1)$, $L^p(T)$ has no order units.*

Proof. With a view for a contradiction, suppose $f \in L^p(T)$ is an order unit. We will first show that the order interval $[-f, f]$ has empty interior.

Since f is positive and well defined on T, we can write

$$T = \bigcup_{n \in \mathbb{N}} f^{-1}([0, n]).$$

Since $f > 0$, for $n \in \mathbb{N}$, the set $f^{-1}([0, n])$ is measurable and one of the sets has positive measure. Let $N \in \mathbb{N}$ be such that $S = f^{-1}([0, N])$ has positive measure.
Now fix $\varepsilon > 0$.
We will first show that there exist an $A \subseteq S$ such that $m(A) < \varepsilon$.
Let $h : T \longmapsto \mathbb{R}$ be defined by

$$h(t) = m([0, t) \cap S), t \in T.$$

Since S is measurable and $[0, t)$ is measurable, it follows that $S \cap [0, t)$ is measurable for any $t \in (0, 1)$.
Now

$$h(0) = m([0, 0) \cap S) = m(\emptyset \cap S) = m(\emptyset) = 0$$

$$\text{and } h(1) = m([0,1) \cap S) = m(S).$$

Next we show that h is continuous by let $a \in (0,1)$ and $\delta = \varepsilon$. Then

$$|h(t) - h(a)| = |m([0,t) \cap S) - m([0,a) \cap S)|.$$

If $|t - a| < \delta$, and $t > a$ then $[0,t) = [0,a) \cup [a,t)$. Hence $[0,t) \cap S = ([0,a) \cap S) \cup ([a,t) \cap S)$ which are disjoint. By the additivity of measure, we have that

$$m([0,t) \cap S) = m([0,a) \cap S) + m([a,t) \cap S).$$

Hence

$$\begin{aligned}
|h(t) - h(a)| &= |m([0,t) \cap S) - m([0,a) \cap S)| \\
&\leq m([a,t) \cap S) \\
&\leq m([a,t)) \\
&= |t - a| \\
&< \delta
\end{aligned}$$

If $t < a$ then the argument still holds with the roles of a and t swaped.
Hence $|h(t)-h(a)| < \varepsilon$ whenever $|t-a| < \delta$. Thus h is uniformly continuous. Since h is continuous on $[0,1]$, $h(0) = 0$ and $h(1) = m(S)$. It then follows from the Intermediate Value Theorem that for any number ε_0 such that $0 < \varepsilon_0 < m(S)$, there exists a $t_0 \in (0,1)$ such that $h(t_0) = \varepsilon_0$. That is $m([0,t_0) \cap S) = \varepsilon_0$. Now let $A = [0,t_0) \cap S$ then $A \subseteq S$ and $0 < m(A) < m(S)$.
Let $r > 0$ be given. It will be shown that 0 is not an interior point of $[-f, f]$.
We know that there exists an $A \subseteq S$ such that

$$0 \leq m(A) = \left(\frac{r}{N+1}\right)^p.$$

Now define

$$k(t) = \begin{cases} N + \frac{1}{2} & \text{if } t \in A \\ 0 & \text{if } t \in T/A. \end{cases}$$

Clearly k is well defined and

$$\begin{aligned}\int_T |k(t)|^p dt &= \int_A \big(N + \frac{1}{2}\big)^p dt \\ &= \big(N + \frac{1}{2}\big)^p \int_A 1 dt \\ &= \big(N + \frac{1}{2}\big)^p \; m(A) \\ &< (N+1)^p \left(\frac{r}{N+1}\right)^p \\ &= r^p.\end{aligned}$$

Thus $||k||_p = \left(\int_T |k(t)|^p dt\right)^{\frac{1}{p}} < r$ so that $k \in B(0, r)$. For $t \in A$

$$f(t) \leq N < N + \frac{1}{2} = k(t),$$

so that $k \notin [-f, f]$. Since $r > 0$ was arbitrary, it follows that 0 is not an interior point of $[-f, f]$.

For any function $g \in L^p(T)$, with f an order unit of $L^p(T)$, there exists an $n \in \mathbb{N}$ such that

$$|g| \leq nf, \text{ hence } g \in [-nf, nf].$$

Therefore every function belong to some order interval. Thus

$$L^p(T) = \bigcup_{n \in \mathbb{N}} [-nf, nf].$$

$L^p(T)$ can be written as a union of closed empty interior sets which contradicts the Baire Category Theorem. Hence $L^p(T)$ has no order units. □

Example 2.4.9. We consider the space $L^p(0,1)$, $1 < p < \infty$. The space $L^p(0,1)$ is reflexive so it follows that the order adherence of $L^p(0,1)$ is $L^p(0,1)$. that is

$$\widetilde{L^p(0,1)} = L^p(0,1).$$

Proof. Let $g \in L^p(0,1)$ be a semi-order unit. If

$$S = \{t \in (0,1) : g(t) > 0\},$$

$g|_S$ is an order unit of $L^p(S)$.
Fix $k \in L^p(0,1)$ Define

$$f(t) = \begin{cases} k(t) & \text{if } t \in S \\ 0 & \text{if } t \notin S. \end{cases}$$

Hence

$$\int_0^1 |f(t)|^p dt = \int_S |k(t)|^p dt < \infty,$$

and $f \in L^p(0,1)$. Since g is a semi-order unit of $L^p(0,1)$, there exists a $\lambda > 0$ such that

$$|f| \wedge ng \leq \lambda g.$$

Since $\mathbb{N}$ is not bounded, there exists $N \in \mathbb{N}$ such that $\lambda < N$ and $\lambda g(t) < Ng(t)$. Hence

$$|f(t)| = |f(t)| \wedge Ng(t) \leq \lambda g(t).$$

Which implies that $|f(t)| \leq \lambda g(t)$. Since $t \in S$ was arbitrary, it follows that $f|_S = k \leq \lambda g|_S$. Hence $g|_S$ is an order unit of $L^p(S)$. But $L^p(S)$ has no order units hence there is no set S such that $m(S) > 0$ and $g|_S$ is an order unit of $L^p(S)$. This implies that the semi-order unit of $L^p(0,1)$ are precisely all positive elements $g \in L^p(0,1)$ such that $m(S) = 0$ where

$$S = \{t \in (0,1) : g(t) > 0\}.$$

In another words, $g = 0$ a.e. □

Definition 2.4.10. Let E, F be vector lattices. Suppose $T : E \longrightarrow F$ is a linear operator. We call T an order continuous operator if for any $D \subseteq E$ such that $D \downarrow 0$, we have that $\inf\{|Tx| : x \in D\} = 0$.

Theorem 2.4.11. *Let E be a vector lattice and denote by $E_n^\sim$ the set of all order continuous operators on E. Then $E_n^\sim$ is a band in $E^\sim$.*

Proof. The proof of this result can be found in [11, Theorem 22.2]. □

Example 2.4.12. Consider the space ℓ^1.
Its norm dual is ℓ^∞. Since ℓ^1 is a Banach lattice, it follows from Theorem 2.4.1 that the order dual of ℓ^1 is ℓ^∞.
Statement i. The order continuous order bounded functionals on ℓ^∞ can be identified by ℓ^1.

Fix $a, b \in \ell^1$ and define a map

$$T : \ell^1 \ni a \longmapsto \phi_a \in (\ell^\infty)_n^\sim \text{ by}$$

$$\phi_a(x) = \sum_{k=1}^{\infty} a(k)x(k), \;\; x \in \ell^{\infty}.$$

Let $\phi_a = \phi_b$. Then

$$a(k) = 1a(k) = \phi_a(e_k) = \phi_b(e_k) = 1b(k) = b(k).$$

Hence $a(k) = b(k)$ for all $k \in \mathbb{N}$ so that $a = b$. Hence T is injective.

Next we will show that ϕ_a is order continuous. We prove for the case where $a > 0$. The general case will follow from $\phi_a = \phi_{a^+} - \phi_{a^-}$.

Fix $D \subseteq \ell^{\infty}$ such that $D \downarrow 0$. For each $k \in \mathbb{N}$, let $D_k = \{x(k) : x \in D\}$
Statement*ii*. $\inf D_k = 0$ for each $k \in \mathbb{N}$.

Since $D \downarrow 0$, we have that $x \geq 0$ for all $x \in D$ so that $x(k) \geq 0$. Hence 0 is a lower bound of D_k. Now with a view for a contradiction, suppose it is not the greatest lower bound, then there exists a $\lambda > 0$ such that $\lambda \leq x(k)$ for each $x \in D$. Now λe_k is a lower bound of D and $\lambda e_k > 0$ contradicting the fact that $\inf D = 0$. Hence $\inf D_k = 0$.

Fix $y \in D$ and let

$$D_y = \{x \in D : x \leq y\}.$$

Statement *iii*. $\inf D = \inf D_y$.

Let $x \in D_y$ then $x \in D$ so that $\inf D \leq x$. Hence $\inf D$ is a lower bound of D_y. Thus $\inf D \leq \inf D_y$. Similarly, if $x_0 \in D$, since $y, x_0 \in D$ and D is downward directed, there exists a $w \in D$ so that $w \leq y$ and $w \leq x_0$, leading to $w \in D_y$. Hence $\inf D_y \leq w \leq x_0$, thus $\inf D_y$ is a lower bound and $\inf D_y \leq \inf D$. Hence $\inf D = \inf D_y$.

So we may assume there exist a y such that $D \leq y$. Given $\varepsilon > 0$. Let $N \in \mathbb{N}$ be such that $\sum_{k=N+1}^{\infty} a(k)y(k) < \frac{\varepsilon}{2}$. Since $D \downarrow 0$, it follows from statement *ii*. that the pointwise infimum is 0. Choose an $x \in D$ such that

$a(k)x(k) < \frac{\varepsilon}{2N}$ for all $1 \leq k \leq N$. Then

$$\begin{aligned}\phi_a(x) = \sum_{k=1}^{\infty} a(k)x(k) &= \sum_{k=1}^{N} a(k)x(k) + \sum_{n=N+1}^{\infty} a(k)x(k) \\ &< \sum_{n=1}^{N} \frac{\varepsilon}{2N} + \sum_{n=N+1}^{\infty} a(k)y(k) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}.\end{aligned}$$

So for every $\varepsilon > 0$, there exists an $x \in D$ so that $\phi_a(x) < \varepsilon$. Hence $\inf \phi_a(D) = 0$. Thus ϕ_a is order continuous so that $\phi_a \in (\ell^\infty)^\sim_n$.
Now let $\phi \in (\ell^\infty)^\sim_n$ such that $\phi \geq 0$. We prove that there exists an $a \in \ell^1$ such that $\phi = \phi_a$.
Let $a \in \mathbb{R}^\mathbb{N}$ be defined by $a(k) = \phi(e_k)$.
Now if $x \in \ell^\infty$ then $x = \sum_{k=1}^{\infty} x(k)e_k$. For $N \in \mathbb{N}$, define

$$x_N = \sum_{n=1}^{N} x_n e_n.$$

Then

$$\phi(x_N) = \phi(\sum_{k=1}^{N} x(k)e_k) = \sum_{n=1}^{N} x(k)\phi(e_k) = \sum_{k=1}^{N} a(k)x(k) = \phi_a(x_N).$$

Using Lemma 2.4.5, $x_N \xrightarrow{o} x$. Since ϕ is order continuous we have that $\phi(x_N) \longrightarrow \phi(x)$ so $\phi(x) = \sum_{k=1}^{\infty} a(k)x(k)$. Hence $\phi = \phi_a$.

It only remains to show that $a \in \ell^1$.
Consider $\mathbf{1} = (1, 1, ...)$. Then for each $n \in \mathbb{N}$, we have that $0 \leq \sum_{k=1}^{n} e_k \leq \mathbf{1}$. Since ϕ is order bounded, the order interval $[0, \mathbf{1}]$ in ℓ^∞ is mapped to some bounded interval $[0, \alpha]$ in $\mathbb{R}$. Also $a(k) = \phi(e_k) \geq 0$ so that $|a(k)| = a(k)$. Hence for $n \in \mathbb{N}$

$$0 \leq \sum_{k=1}^{n} a(k) = \sum_{k=1}^{n} \phi(e_k) = \phi(\sum_{k=1}^{n} e_k) \leq \phi(\mathbf{1}) = \alpha < \infty.$$

Therefore $a = \lim_{n \longrightarrow \infty} \sum_{k=1}^{n} a(k) \leq \alpha < \infty$ and $a \in \ell^1$. Thus $\ell^1 = (\ell^\infty)^\sim_n$. It then follows from Theorem 2.4.11 that $(\ell^\infty)^\sim_n$ is a band in $(\ell^\infty)^\sim$. Now

$$\ell^1 = (\ell^\infty)^\sim_n \subseteq (\ell^\infty)^\sim = (\ell^1)^{\sim\sim}.$$

Hence ℓ^1 is a band in $(\ell^1)^{\sim\sim}$. If $\Phi \in \widetilde{\ell^1}$, then there exists a net $(f_\alpha)_{\alpha\in I}$ in ℓ^1 so that $f_\alpha \xrightarrow{o} \Phi$. Now $\limsup f_\alpha = \Phi$ by Theorem 2.1.12. Since ℓ^1 is a band, for each $\alpha_0 \in I$, $\inf\limits_{\alpha\geq\alpha_0} f_\alpha \in \ell^1$. Hence $\Phi = \limsup f_\alpha \in \ell^1$. Thus $\widetilde{\ell^1} = \ell^1$. Now the semi-order units of ℓ^1 are precisely the elements $u \in \ell^p$ such that the set $\{u(k) : u(k) \neq 0\}$ is finite.

Example 2.4.13. Let $s = \mathbb{R}^{\mathbb{N}}$. The order dual of s is c_{00}.
For $a \in c_{00}$ and $x \in S$ define

$$\phi_a(x) = \langle x, a\rangle = \sum_{k=1}^{\infty} x(k)a(k).$$

Clearly the mapping is linear because

$$\begin{aligned}\phi_{a+b}(x) = \langle x, a+b\rangle &= \sum_{k=1}^{\infty} x(k)(a(k)+b(k)) \\ &= \sum_{k=1}^{\infty} x(k)a(k) + \sum_{k=1}^{\infty} x(k)a(k) \\ &= \phi_a(x) + \phi_b(x).\end{aligned}$$

Since $x \in s$ was arbitrary, we have that $\phi_{a+b} = \phi_a + \phi_b$. It remains to show that it is order bounded.

First consider ϕ_{e_j} for $e_j \in c_{00}$. For $x \in s$, we have that

$$\phi_{e_j}(x) = \langle x, e_j\rangle = \sum_{k=1}^{\infty} e_j(k)x(k) = x(j).$$

Let $[x, y]$ be an order interval in s and let $v \in [x, y]$. Then $x \leq v \leq y$. Since the ordering is point-wise we have that for each $k \in \mathbb{N}$, $x(k) \leq v(k) \leq y_n$. Hence

$$x(j) = \phi_{e_j}(x) \leq v(k) = \phi_{e_j}(v) \leq y(k) = \phi_{e_j}(y).$$

The order interval $[x, y]$ is mapped into the order interval $[\phi_{e_j}(x), \phi_{e_j}(y)]$ so that ϕ_{e_k} is order bounded.

Now fix $a \in c_{00}$. Since a has finite non-zero terms, there is a finite $I \subset \mathbb{N}$ finite such that

$$a = \sum_{k\in I} a(k)e_k.$$

Hence $\phi_a = \phi_{\sum_{k\in I} e_k} = \sum_{k\in I} \phi_{e_k}$. Since ϕ_{e_k} is order bounded for each $k \in I$, the sum of finite order bounded operator is order bounded by Theorem 2.2.2, so that $\phi_a \in \mathcal{L}(s, \mathbb{R})$.

Now fix $\phi \in s^{\sim}_{+}$. Define $a(k) = \phi(e_k)$, $k \in \mathbb{N}$ and let $a = (a(k))$.
We will show that $a \in c_{00}$, by considering the set $A = \{k \in \mathbb{N} : \phi(e_k) \neq 0\}$ and prove that A is finite.
With a view for a contradiction, suppose A is not finite. Then for each $k \in A$, $\phi(e_k) > 0$. Define $x \in s$ by $x(k) = \frac{1}{\phi(e_k)}$ for $k \in A$ and $x(k) = 0$ for $k \notin A$. Let $A = \{k_i : i \in \mathbb{N}\}$ and define $x_N = \sum_{i=1}^{N} x(k_i)e_{k_i} \leq x$. Clearly $x \in s$ and

$$\phi(x) \geq \phi(x_N) = \phi\left(\sum_{k_i=1}^{N} x(k_i)e_{k_i}\right) = \sum_{i=1}^{N} 1 = N.$$

This holds for all $N \in \mathbb{N}$, so $\phi(x)$ is an upper bound for $\mathbb{N}$, which is a contradiction. The set A is finite, so that $a \in c_{00}$. Now for $x \in s$, we have that $x = \sum_{n=1}^{\infty} x_n e_k$. Since A is finite we have the following.

$$\phi_a(x) = \sum_{k=1}^{\infty} a(k)x(k) = \sum_{k=1}^{\infty} x(k)\phi(e_k) = \sum_{k\in A} x(k)\phi(e_k) = \sum_{k\in A} \phi(x(k)e_k) = \phi(x).$$

Hence $\phi_a = \phi$ and $s^{\sim} = c_{00}$. Now the order dual of c_{00} is also s. Therefore $\widetilde{\widetilde{s}} = s$.
The semi-order units of s are the elements which are finitely supported.
It also follows that $\widetilde{\widetilde{c_{00}}} = c_{00}$ and every element in c_{00} is a semi-order unit.

Chapter 3

The vector lattice C(X)

Recall that a topological space X is Tychonoff if X is a completely regular Haursdoff space.

In this chapter, we declare X to be a Tychonoff space. We denote by $C(X)$ the space of real valued continuous function from X into $\mathbb{R}$. We define the point-wise ordering on $C(X)$ as follows. For $f, g \in C(X)$

$$f \leq g \iff f(x) \leq g(x) \text{ for all } \quad x \in X.$$

$C(X)$ equipped with this point-wise ordering is a vector lattice.

For each $K \subseteq X$ compact and $f \in C(X)$, we define

$$||f||_K = \sup\{|f(x)| : x \in K\}.$$

This defines a semi-norm on $C(X)$. The family of all such semi-norms defines a locally convex topology on $C(X)$ called the compact-open topology. We denote $C(X)$ equipped with this topology by $(C(X), \tau_{co})$, but will simply write $C_{co}(X)$. In this chapter, we study $C_{co}(X)$ from the order theoretic point of view. We discuss closed ideals in $C_{co}(X)$. We compare the order and topological duals.

We also discuss the order theoretic characterization of the compact open topology in terms of semi-order unit in suitable subspace of $C(X)^{\sim\sim}$.

The Stone-$\check{C}$ech compactification and the realcompactfication of X are constructed as necessary preliminary material for the chapter.

3.1 Compactifications

Let X and Y be topological spaces. Recall that a map $\tau : X \longrightarrow Y$ is a homeomorphism if τ is a continuous bijection such that $\tau^{-1} : Y \longrightarrow X$ is continuous. If τ is injective, but not onto, then $\tau : X \longrightarrow \tau[X]$ becomes a bijection. Now if τ is continuous and $\tau^{-1} : \tau[X] \longrightarrow X$ is continuous we say τ is a homeomorphic embedding.

Definition 3.1.1. Let X be a topological space. A pair (Y, Δ) where Y is a compact Hausdorff space and

$$\Delta : X \longrightarrow Y$$

is a homeomorphic embedding of X in Y such that $\overline{\Delta[X]} = Y$ is called a compactification of X.

If (Y, Δ) is a compactification of X, we will usually identify X with $\Delta[X]$ and consider X as a subspace of Y. We are interested in a special compactification in which every bounded real-valued continuous function on X can be extended to a continuous function on this compactification. That compactification is called the $Stone - \check{C}ech$ Compactification of X and is denoted by βX. In this section, we will construct this space.

Definition 3.1.2. A subspace S of X is $C-$embedded in X, if every continuous function on S can be extended to a continuous function on X.

Definition 3.1.3. A subspace S of X is C^*-embedded in X, if every bounded continuous function on X can be extended to a bounded continuous function on X.

The following diagram illustrates this, with e, the inclusion map.

$$e : S \ni x \longmapsto x \in X$$

$$\begin{tikzcd} S \arrow[r, "e"] \arrow[dr, "f"'] & X \arrow[d, "g"] \\ & \mathbb{R} \end{tikzcd}$$

This means that for every $f \in C(S)$, there is a $g \in C(X)$ such that $f = g \circ e$.

Theorem 3.1.4. *Let $B \subseteq X$ and let $f : B \to Z$ be a continuous map of B into a Hausdorff space Z. Then there exists at most one extension of f to a continuous function*

$$\tilde{f} : \overline{B} \to Z.$$

Proof. Suppose $\tilde{f}_1$ and $\tilde{f}_2$ are two distinct extensions of f. Since $\tilde{f}_1$ and $\tilde{f}_2$ are distinct, there exists an $x \in \overline{B}$ such that $\tilde{f}_1(x) \neq \tilde{f}_2(x)$. Since Z is Hausdorff, there exists two disjoint open sets U_1 and U_2 in Z such that $\tilde{f}_1(x) \in U_1$ and $\tilde{f}_2(x) \in U_2$. Pick a neighbourhood V of x such that $f_1(V) \subseteq U_1$ and $f_2(V) \subseteq U_2$. Since $x \in \overline{B}$, every open neighbourhood of x intersects B. In particular, $V \cap B \neq \emptyset$. So let $y \in V \cap B$ so that $y \in B$. Since $\tilde{f}_1$ and $\tilde{f}_2$ are extensions of f, it implies that they are equal on B hence $\tilde{f}_1(y) = f(y) = \tilde{f}_2(y)$. But $\tilde{f}_1(y) \in U_1$ and $\tilde{f}_2(y) \in U_2$, which contradicts the fact that U_1 and U_2 are disjoint. $\square$

Corollary 3.1.5. *Let Y and Z be topological spaces with Z Haursdorff. Let $X \subseteq Y$ satisfy $\overline{X} = Y$. Suppose f and g are continuous maps from Y into Z. If $f\big|_X = g\big|_X$, then $f = g$ on Y.*

Theorem 3.1.6. *Let $\{X_\alpha : \alpha \in J\}$ be a collection of topological spaces, Y another topological space and let σ be a map*

$$\sigma : Y \longrightarrow \prod_{\alpha \in J} X_\alpha.$$

Then σ is continuous with respect to the product topology if and only if $\pi_\alpha \circ \sigma$ is continuous for each $\alpha \in J$ where π_α is the projection onto X_α

Proof. Suppose σ is continuous. Since the projection map π_α is continuous for each $\alpha \in J$, it follows that the composition $\pi_\alpha \circ \sigma$ is continuous for each $\alpha \in J$. Now suppose that $\pi_\alpha \circ \sigma$ is continuous for each $\alpha \in J$. Then for U open in X_α the preimage of U under $\pi_\alpha \circ \sigma$ is an open set. That is $(\pi_\alpha \circ \sigma)^{-1}[U]$ is open. But

$$(\pi_\alpha \circ \sigma)^{-1}[U] = \sigma^{-1}[\pi_\alpha^{-1}[U]].$$

It follows that the preimage under σ of each subbasic open set for the product topology is open. Hence σ is continuous. $\square$

Let A and B be sets and τ a mapping from A into B. For each map $g \in \mathbb{R}^B$, the composition $g \circ \tau \in \mathbb{R}^A$. So the map τ induces a map

$$\tau' : \mathbb{R}^B \longrightarrow \mathbb{R}^A \text{ defined by } \tau' g = g \circ \tau, \ \ g \in \mathbb{R}^B.$$

There is a duality between the properties "one-to-one" and "onto." In particular we have the following.

Lemma 3.1.7. *Let τ be a mapping from a set A into a set B and let τ' be the induced mapping. Then*

i. τ' is a linear lattice homomorphism.

ii. τ' *is one-to-one if and only if* τ *is onto.*

iii. τ' *is onto if and only if* τ *is one-to-one.*

Proof of (i). Let $f, g \in \mathbb{R}^B$ and $\alpha, \beta \in \mathbb{R}$. For $x \in A$ we have that

$$\begin{aligned}\tau'(\alpha f + \beta g)(x) &= (\alpha f + \beta g)(\tau x)\\ &= \alpha f(\tau x) + \beta g(\tau x)\\ &= \alpha(\tau' f)(x) + \beta(\tau' g)(x)\\ &= (\alpha\tau' f + \beta\tau' g)(x)\end{aligned}$$

Similarly, $\tau'(f \vee g)(x) = (\tau' f \vee \tau' g)(x)$
Since this is true for all $x \in A$, it follows that $\tau'(\alpha f + \beta g) = \alpha\tau' f + \beta\tau' g$ and $\tau'(g \vee f) = \tau' g \vee \tau' f$ so that τ' is a linear lattice homomorphism. $\square$

Proof of (ii). Suppose τ is onto and let $f, g \in \mathbb{R}^B$ such that $\tau' f = \tau' g$. Let $y \in B$. Since τ is onto, it follows that there exists $x \in A$ such that $\tau(x) = y$. Now

$$\begin{aligned}g(y) &= g(\tau(x))\\ &= [g \circ \tau](x)\\ &= [f \circ \tau](x) \ \ (\because f \circ \tau = g \circ \tau)\\ &= f(\tau(x))\\ &= f(y).\end{aligned}$$

Since this is true for all $y \in B$ we have that $f = g$ so that τ' is injective.
Now assume that τ is not onto. Then there exists a $y \in B$ such that $y \notin \tau[A]$.

Let $\mathbf{1}_{\{y\}}$ be the indicator function on y.
Then if $x \in X$, $\tau'\mathbf{1}_{\{y\}}(x) = \mathbf{1}_{\{y\}}(\tau x) = 0$. Also $\tau'\mathbf{0}(x) = \mathbf{0}(\tau x) = 0$. Hence τ' is not injective. $\square$

Proof of (iii). Assume τ' is onto and let $\tau x_1 = \tau x_2$ for some $x_1, x_2 \in A$.
Fix $f \in \mathbb{R}^A$. Since τ' is onto, there exists a $g \in \mathbb{R}^B$ such that $\tau' g = f$. Now

$$f(x_1) = [\tau' g](x_1) = g(\tau x_1) = g(\tau x_2) = [\tau' g](x_2) = f(x_2).$$

Since this holds for all $f \in \mathbb{R}^A$ we have that $x_1 = x_2$ so that τ is injective.

Now assume that τ is one-to-one and fix $f \in \mathbb{R}^A$.
Since τ is one-to-one, the map $\tau^{-1} : \tau[A] \longrightarrow A$ is well defined and a bijection. Now $f \circ \tau^{-1} \in \mathbb{R}^B$. Let $g = f \circ \tau^{-1}$. Now if $x \in A$, then

$$[\tau' g](x) = [\tau'(f \circ \tau^{-1})](x) = [f \circ \tau^{-1} \circ \tau](x) = f(\tau^{-1}(\tau x)) = f(x).$$

Hence $\tau' g = f$ so that τ' is onto. $\square$

In most application we are interested in topological spaces X and Y and a continuous map $\tau : X \longrightarrow Y$. In this situation, τ' maps $C(Y)$ into $C(X)$ and $C_b(Y)$ into $C_b(X)$.

Theorem 3.1.8. *Let X, Y be Tychonoff space and let $\tau : X \longrightarrow Y$ be a continuous map and let $\tau' : C(Y) \longrightarrow C(X)$ be the induced map defined by $\tau' g = g \circ \tau$. Then*

i. τ' is injective if and only if $\tau[X]$ is dense in Y.

ii. τ' is onto if and only if τ is a homeomorphism onto a $C-$embedded subspace of Y.

iii. $\tau' : C_b(Y) \longrightarrow C_b(X)$ is onto if and only if τ is a homeomorphism onto $\tau[X]$ and $\tau[X]$ is C^*-embedded in Y.

iv. If τ is a homeomorphism then τ' is a lattice isomorphism.

Proof of (i). τ' is injective if and only if $\ker \tau' = \{0\}$. This is equivalent to the following: for all $g \in C(X)$, if $g(\tau x) = 0$ for all $x \in X$ then $g = 0$. The last statements hold if and only if $\tau[X]$ is dense in Y. We prove the last equivalence. Suppose $\tau[X]$ is not dense in Y. Then there is $y \in Y$ such that $y \notin \overline{\tau[X]}$. Since Y is Tychonoff, there exists a function $g \in C(Y)$ such that $g = 0$ on $\overline{\tau[X]}$ and $g(y) = 1$. Now if $x \in X$ then $\tau x \in \tau[X]$, hence $g(\tau x) = 0$. Thus $g(\tau x) = 0$ for all $x \in X$ and $g \neq 0$ on Y proving the contrapositive.

On the other hand, suppose $\tau[X]$ is dense in Y and assume that $g(\tau x) = 0$ for all $x \in X$. Then, if $y \in \tau[X]$ then there exists an $x \in X$ such that $y = \tau x$. Hence
$$g(y) = g(\tau x) = 0.$$
Hence $g|_{\tau[X]} = 0$. It follows from Corollary 3.1.5 that $g = 0$ on Y. □

Proof of (ii). Assume τ' is onto. Fix $f \in C(X)$. Since τ is onto, it follows that there exists a $g \in C(Y)$ such that $\tau' g = f$. Now if $\tau x_1 = \tau x_2$ then $g(\tau x_1) = g(\tau x_2)$. Hence
$$[\tau' g](x_1) = [\tau' g](x_2) \;\; i.e \;\; f(x_1) = f(x_2).$$

Since this holds for each $f \in C(X)$, by the Tychonoff property we have that $x_1 = x_2$ so that τ is injective. A mapping is onto its range so τ^{-1} is well defined as a mapping from $\tau[X]$ onto X.

We will show that τ^{-1} is continuous. Let (x_α) be a net in X so that (τx_α)

converges to some τx in $\tau[X]$. Assume that $x_\alpha \nrightarrow x$. Then there exists $V \ni x$ open and (y_β) a subnet of (x_α) such that $y_\beta \notin V$ for all β. Now there exists $f \in C(X)$ such that $f(x) = 1$ and $f[X \backslash V] = \{0\}$. So $f(y_\beta) = 0$ and $f(x) = 1$. Now τ' is onto. This implies that there exists a $g \in C(Y)$ such that $f = \tau' g$. For all β, $g(\tau y_\beta) = f(y_\beta) = 0$ and $g(\tau x) = f(x) = 1$. Therefore $g(\tau y_\beta) \nrightarrow g(\tau x)$. But $\tau y_\beta \longrightarrow \tau x$ because it is a subnet of (τx_α). Therefore g is not continuous which is a contradiction. Hence τ^{-1} is continuous. Thus τ is a homeomorphism.

Let $h \in C(\tau[X])$. Then $f = h \circ \tau \in C(X)$. Now since τ' is onto, it follows that there exists a $g \in C(Y)$ such that $\tau' g = f$. Now if $y \in \tau[X]$ then there exist an $x \in X$ such that $\tau x = y$. Hence

$$h(y) = h(\tau x) = [h \circ \tau](x) = f(x) = [g \circ \tau](x) = g(\tau x) = g(y).$$

Hence $h = g|_{\tau[X]}$. It follows that $\tau[X]$ is $C-$embedded in Y.

Now suppose that τ is a homeomorphic embedding whose image is $C-$embedded in Y. By assumption, τ^{-1} is continuous from $\tau[X]$ onto X. Consider any $f \in C(X)$. Then the function $f \circ \tau^{-1} \in C(\tau[X])$ and by assumption it has a continuous extension g to all of Y. Clearly $f = g \circ \tau$ i.e $f = \tau' g$. Hence τ' is onto. $\square$

Proof of (iii). This result follows by an identical argument as for (ii). $\square$

Proof of (iv). Since τ is a homeomorphism, it follows from (i) and (ii) that τ' is a bijection. From Lemma 3.1.7, τ' is a linear lattice homomorphism so that τ' is a lattice isomorphism. $\square$

Theorem 3.1.9. *Let S, T and X be Hausdorff spaces. Assume that there exist homeomorphisms*

$$\tau_0 : X \longrightarrow S \text{ and } \tau_1 : X \longrightarrow T$$

onto dense subspaces of S and T respectively. If there exist continuous maps

$$\sigma_0 : S \longrightarrow T \text{ and } \sigma_1 : T \longrightarrow S$$

so that

$$\begin{tikzcd} & S \arrow[dd, "\sigma_0"] \\ X \arrow[ur, "\tau_0"] \arrow[dr, "\tau_1"'] & \\ & T \end{tikzcd} \qquad \begin{tikzcd} X \arrow[dd, "\tau_1"'] \arrow[dr, "\tau_0"] & \\ & S \\ T \arrow[ur, "\sigma_1"'] & \end{tikzcd}$$

commute, then σ_0 is a homeomorphism onto T and $\sigma_1 = \sigma_0^{-1}$.

Proof. First observe that the map $\sigma_0 \circ \sigma_1 : S \longrightarrow S$ is the identity map. This is because its restriction on X is the identity on X. Since X is dense in S we have that it is the identity on S. It also follows that $\sigma_1 \circ \sigma_0$ is also the identity map on T. That is

$$\sigma_0 \circ \sigma_1 = id_S \text{ and } \sigma_1 \circ \sigma_0 = id_T.$$

Hence $\sigma_0^{-1} = \sigma_1$ so that σ_0 is a homeomorphism as required. □

Lemma 3.1.10. *Let X be a Tychonoff space.*

i. Let $\mathcal{F} = \{f_\alpha : \alpha \in I\}$ be the collection of all bounded continuous functions from X to $\mathbb{R}$ and let $I_\alpha = [\inf\limits_{x\in X} f_\alpha(x), \sup\limits_{x\in X} f_\alpha(x)]$ for every $\alpha \in I$. Let $Y = \prod_{\alpha\in I} I_\alpha$. Then

$$e : X \ni x \longmapsto (f_\alpha(x))_{\alpha\in I} \in Y$$

is a homeomorphic embedding onto a C^*-embedded subspace of Y.

ii. Let $\mathcal{S} = \{f_\beta : \beta \in J\}$ be the collection of all continuous functions from X to $\mathbb{R}$. Then

$$e : X \ni x \longmapsto (f_\beta(x))_{\beta\in J} \in \mathbb{R}^J$$

is a homeomorphic embedding onto a C-embedded subspace of $\mathbb{R}^J$.

Proof of (i). By definition of e, $\pi_\alpha \circ e = f_\alpha$ for each $\alpha \in I$ which is continuous where π_α is the projection map onto the α component. Hence by Theorem 3.1.6, e is continuous.
Now suppose $x_1, x_2 \in X$ such that $x_1 \neq x_2$. Since X is Tychonoff, there exists a function $f_{\alpha_0} \in \mathcal{F}$ such that $f_{\alpha_0}(x_1) \neq f_{\alpha_0}(x_2)$. Hence $e(x_1) \neq e(x_2)$ so that e is one-to-one. The induced homomorphism $e' : C_b(Y) \longrightarrow C_b(X)$ is onto. Indeed for $f_\alpha \in \mathcal{F} = C_b(X)$.

$$e'\pi_\alpha = \pi_\alpha \circ e = f_\alpha.$$

By Theorem 3.1.8 (*iii*), e is a homeomorphism onto $e[X]$ and $e[X]$ is C^*-embedded in Y. □

Proof of (ii). The result follows a similar argument as for (i) so we omit the proof. □

Lemma 3.1.11. *Let (Y, Δ) be a compactification of X such that every bounded continuous function on X extends to a continuous function $\tilde{f} : Y \longrightarrow \mathbb{R}$. Let K be a compact Haursdorff space and $f : X \longrightarrow K$ a continuous function. Then there exists a unique continuous extension $\tilde{f} : Y \longrightarrow K$ of f.*

Proof. Since K is Tychonoff, we can view K as a subspace of a product of compact interval say $K \subseteq \prod_{\gamma \in M} I_\gamma$ where $M \in \mathbb{N}$ by Lemma 3.1.10.
Hence for all $\gamma \in M$, $f_\gamma : X \longrightarrow I_\gamma$ with $f_\gamma = \pi_\gamma \circ f$ continuous. Therefore, there exists a unique continuous extension of f_γ, $\tilde{f_\gamma} : Y \longrightarrow \mathbb{R}$.
The function $\tilde{f_\gamma} : Y \ni x \longmapsto (\tilde{f_\gamma}(x))_{\gamma \in M} \in \prod_{\gamma \in M} I_\gamma$ is continuous and $\tilde{f}(x) = f(x)$ for all $x \in X$. Now $\tilde{f}[X] = f[X] \subseteq K$ and K is closed in $\prod_{j \in M} I_\gamma$.
Therefore $\tilde{f}[Y] = \tilde{f}[\overline{X}] \subseteq \overline{\tilde{f}[X]} \subseteq K$ by Theorem A.1.3. So $\tilde{f} : Y \longrightarrow K$ is a continuous extension of f. The uniqueness follows from Theorem 3.1.5 since X is dense Y. $\square$

Theorem 3.1.12. *Let X be a Tychonoff space. Then there exists a unique compactification $(\beta X, \beta)$ of X having the property that every bounded continuous function $f : X \longrightarrow \mathbb{R}$ extends uniquely to a continuous function on βX.*

Proof. Let $\mathcal{F}$, Y and e be as given in Lemma 3.1.10 *i*. By Lemma 3.1.10, e is a homeomorphism onto $e[X]$ and $e[X]$ is C^*−embedded in Y. Now let $\beta X = \overline{e[X]}$ and $\beta(x) = e(x)$, $x \in X$.

If $f \in C_b(X)$ then there exists a continuous extension $\tilde{f} : Y \longrightarrow \mathbb{R}$ of f. The restriction, $\tilde{f}|_{\overline{e[X]}}$ is a continuous extension of f to βX. Uniqueness of the extension follows since X is dense in βX. Hence every bounded continuous function $f : X \longrightarrow \mathbb{R}$ extends uniquely to a continuous function on βX.

It remains to show that the space βX is unique. To this end, let $(\widehat{X}, \Delta)$ be a compactification of X such that every $f \in C_b(X)$ extends continuously to $\widehat{X}$. Then by Lemma 3.1.11, $X \xrightarrow{\beta} \beta X$ extends to $\hat{X} \xrightarrow{\widehat{\beta}} \beta X$ and $X \xrightarrow{\Delta} \widehat{X}$ extends to $\beta X \xrightarrow{\widehat{\Delta}} \widehat{X}$. Now by Theorem 3.1.9, βX and $\widehat{X}$ are homeomorphic. $\square$

Definition 3.1.13. A space X is realcompact if for all $x_0 \in \beta X \backslash X$, there exists a function $f \in C(X)$ such that f does not extend to a continuous function

$$\tilde{f} : X \cup \{x_0\} \longrightarrow \mathbb{R}.$$

Definition 3.1.14. Let X be a Tychonoff space. By a realcompactification of a space X, we mean the largest subspace of βX in which X is C−embedded. We denote this space by υX.

We will use the following lemma without proof.

Lemma 3.1.15. *A space X is realcompact if and only if X is homeomorphic to a closed subspace of $\mathbb{R}^\gamma$ for some cardinal γ.*

Theorem 3.1.16. *Let X be a Tychonoff space then there exists a unique realcompact space υX such that every continuous function f on X extends to a continuous function $\tilde{f}$ on υX.*

Proof. The proof is exactly as Theorem 3.1.12. Let $\mathcal{S}$ and e be given as in Lemma 3.1.10 *ii*.
By Lemma 3.1.10*ii*, e is a homeomorphism onto $e[X]$ and $e[X]$ is $C-$embedded in Y, where Y is a closed subset of $\mathbb{R}^J$. Let $\upsilon X = \overline{e[X]}$. Then υX is realcompact by Lemma 3.1.15. Now if $f \in C(X)$, then there exist $\tilde{f} : Y \longrightarrow \mathbb{R}$ a continuous extension of f. Now the restriction $\tilde{f}|_{\upsilon X}$ is an extension of f to υX. Uniqueness of the extension follows since X is dense in υX.
The proof of the uniqueness of υX is similar to the uniqueness of βX and therefore we omit it, see the last part of the proof of Theorem 3.1.12. □

Theorem 3.1.17. *Let X be a Tychonoff space. Then*

i. The map

$$T : C_b(X) \ni f \longmapsto \tilde{f} \in C(\beta X)$$

is a lattice isomorphism.

ii. The map

$$T : C(X) \ni f \longmapsto \tilde{f} \in C(\upsilon X)$$

is a lattice isomorphism.

Proof of (i). Let $f, g \in C_b(X)$ such that $f \neq g$. Then there exists an $x \in X$ such that $f(x) \neq g(x)$. Hence $\tilde{f}(x) \neq \tilde{g}(x)$ so that $Tf \neq Tg$. Thus T is one-to-one.

If $h \in C(\beta X)$ then $f = h|_X \in C_b(X)$. So there exists an extension $\tilde{f} \in C(\beta X)$ of f. Then $\tilde{f}|_X = f = h|_X$. Now h and $\tilde{f}$ are extension of f. By Corollary 3.1.5 $\tilde{f} = h$. Hence $Tf = h$ so that T is onto.

Let $f \geq 0$. Then $\tilde{f} \vee 0 \geq 0$ is an extension of f and by the uniqueness of the extension we have that $Tf \geq 0$. Now let $\tilde{f} \geq 0$. Then since $X \subseteq \beta X$, $f \geq 0$ on X. Hence $T^{-1}\tilde{f} = f \geq 0$. Hence T is a lattice isomorphism. □

Proof of (ii). The proof is identical to the proof of (i) so we omit it. □

3.2 The Compact-Open Topology

Recall that by the compact open topology on $C(X)$ we mean the locally convex topology generated by the collection of basic open sets

$$\{U_f^\varepsilon = \{g \in C(X) : ||g-f||_K = \rho_K(g-f) < \varepsilon\} : f \in C(X) \ \ K \subseteq X \ \text{ compact}, \varepsilon > 0\}.$$

If $(f_\alpha)_{\alpha \in I}$ is a net in $C(X)$ which converges to f in $C_{co}(X)$ we will write $f_\alpha \overset{co}{\longrightarrow} f$.
Note that in the previous section, we established that $C(X)$ and $C(\upsilon X)$ are isomorphic vector lattices. However this map is not a homeomorphism with respect to the compact open topologies on $C(X)$ and $C(\upsilon X)$ respectively. The proof of this fact relies on the following.

Lemma 3.2.1. *Let X be a Tychonoff space such that $X \neq \upsilon X$ and let $z \in \upsilon X \backslash X$. Define*

$$U = \{f \in C(X) : |\tilde{f}(z)| < 1\}.$$

Then U is not open in $C_{co}(X)$.

Proof. It is sufficient to show that the zero function is not an interior point of U. Fix $\varepsilon > 0$. We show that for any K compact in Y,

$$\{f \in C(X) : \rho_K(f) < \varepsilon\} \not\subseteq U.$$

Since $z \notin X$, then $z \notin K$ and since K is compact in X, it is compact in υX. Now υX is completely regular, there exists a function $\bar{g} \in C(\upsilon X)$ such that $\bar{g}(z) = 1$ and $\bar{g}[K] = \{0\}$. Let $g = \bar{g}|_X$. Then $g \in C(X)$ and $\tilde{g} = \bar{g}$ where $\tilde{g} = Tg$, the extension of g to υX. Now $\rho_K(\bar{g}) = 0 < \varepsilon$, but $\tilde{g}(z) = 1$ so that $g \notin U$. Hence $\{f \in C(X) : \rho_K(f) < \varepsilon\} \not\subseteq U$ so that $0 \in U$ is not an interior point of U. Hence U is not open. $\square$

Theorem 3.2.2. *Let X be a Tychonoff space and let*

$$T : C_{co}(X) \ni f \longmapsto \tilde{f} \in C_{co}(\upsilon X).$$

Then T is a homeomorphism if and only if X is realcompact.

Proof. Suppose X is realcompact. Then $X = \upsilon X$ so that T is the identity map and hence a homeomorphism.

Now suppose that X is not realcompact. Then $X \neq \upsilon X$. Since $X \subseteq \upsilon X$, there exists $z \in \upsilon X$ such that $z \notin X$. Define

$$U = \{\tilde{f} \in C(\upsilon X) : \tilde{f}(z) < 1\}.$$

Let $K = \{z\}$. Then K is compact in υX. and $U = \{\tilde{f} : C(\upsilon X) : \rho_K(f) < 1\}$ is open in $C_{co}(\upsilon X)$. By Theorem 3.2.1, U is not open in $C_{co}(X)$. Thus T is not a homeomorphism. □

Theorem 3.2.3. *Let $(f_\alpha)_{\alpha \in I}$ be a net in $C(X)$ and $f \in C(X)$ such that $f_\alpha(x) \uparrow f(x)$ for every $x \in X$. Then $f_\alpha \xrightarrow{co} f$.*

Proof. Fix $K \subseteq X$ compact. Then $f_\alpha|_K$ is continuous on K and $f|_K$ is also continuous on K. Hence $(f_\alpha|_K)_{\alpha \in I}$ is a monotonic increasing net of functions such that $f_\alpha(x) \uparrow f(x)$ for each $x \in K$. Hence by the Dini's Theorem, $f_\alpha|_K \longrightarrow f|_K$ uniformly. Since K was abitrary, it follows that $f_\alpha \xrightarrow{co} f$. □

Theorem 3.2.4. *Let X be a Tychonoff space. Then $B \subseteq C_{co}(X)$ is topologically bounded if and only if for all $K \subseteq X$ compact, there exists a $\lambda_K > 0$ such that if $f \in B$ then $|f(x)| \leq \lambda_K$ for all $x \in K$.*

Proof. Assume that for all $K \subseteq X$ compact, there exists a λ_K such that if $f \in B$ then $|f(x)| \leq \lambda_K$ for all $x \in K$. Fix $U \in \mathcal{N}_0$ where U is a basic open set contain 0. Without loss of generality we may assume that

$$U = \{f \in C(X) : \rho_K(f) < \varepsilon\},$$

for some compact set $K \subseteq X$ and some $\varepsilon > 0$. So by assumption, there exists a $\lambda_K > 0$ such that if $f \in B$ then $|f(x)| \leq \lambda_K$ for all $x \in K$. Now let $\lambda = \frac{2\lambda_K}{\varepsilon}$. Hence if $f \in B$ then $\rho_K(f) \leq \lambda_K = \frac{\lambda\varepsilon}{2}$. Hence

$$\rho_K(\frac{1}{\lambda}f) = \frac{1}{\lambda}\rho_K(f) \leq \frac{1}{\lambda}\lambda_K < \frac{2}{\lambda}\lambda_K = \varepsilon$$

so that $\frac{1}{\lambda}f \in U$. Hence $f \in \lambda U$. Thus $B \subseteq \lambda U$ so that B is topologically bounded.

Now fix $K \subseteq X$ compact. Then the set $U = \{f \in C(X) : \rho_K(f) < 1\}$ is open in $C_{co}(X)$ and contain 0 so that $U \in \mathcal{N}_0$. By assumption, there exists a $\lambda > 0$ such that $B \subseteq \lambda U$. Let $\lambda_K = \lambda$. If $f \in B$ then $f \in \lambda U$ so that $\rho_K(f) < \lambda = \lambda_K$. Hence

$$\rho_K(f) = \sup\{|f(x)| : x \in K\} \leq \lambda_K.$$

This implies that $|f(x)| \leq \lambda_K$ for all $x \in X$. □

3.3 Closed Ideals

In this section we discuss closed ideals in $C_{co}(X)$. We will establish that given any closed ideal I in $C_{co}(X)$, there exists a closed set A in X such that I consists precisely of those $f \in C(X)$ such that $f[A] = \{0\}$.

Theorem 3.3.1. *Let X be a Tychonoff space, A a closed subset of X and define*

$$I_A = \{f \in C(X) : f|_A = 0\}.$$

i. I_A is a closed ideal in $C_{co}(X)$.

ii. If I is a closed ideal in $C_{co}(X)$, then there exists a closed set $A \subseteq X$ such that $I = I_A$.

Proof of (i). Let $f, g \in I_A$. Now for $x \in A$, we have that

$$[f + g](x) = f(x) + g(x) = 0 + 0 = 0.$$

Hence $(f + g)|_A = 0$ so that $f + g \in I_A$. For $\alpha \in \mathbb{R}$ we have

$$[\alpha f](x) = \alpha f(x) = \alpha 0 = 0.$$

So that $\alpha f \in I_A$. Thus I_A is a linear subspace of $C(X)$.

Now suppose that $f \in I_A$ and $g \in C(X)$ such that $|g| \leq |f|$. Since the ordering is point-wise we have that $|g(x)| \leq |f(x)|$ for each $x \in X$. In particular, if $x \in A$ we have that

$$0 \leq |g(x)| \leq |f(x)| = 0.$$

Hence $g(x) = 0$ for all $x \in A$ so that $g \in I_A$. Thus I_A is an ideal in $C(X)$.

It is remains to show that it is closed in $C_{co}(X)$. Let $(f_\alpha)_{\alpha \in J}$ be a net in I_A such that $f_\alpha \xrightarrow{co} f$ for some $f \in C_{co}(X)$. Fix $\varepsilon > 0$ and let $x \in A$. Since $K = \{x\}$ is compact in X and $f_\alpha \xrightarrow{co} f$, there exists an $\alpha_0 \in J$ such that if $\alpha \geq \alpha_0$ then $||f - f_\alpha||_K < \varepsilon$. Hence

$$\begin{aligned}
|f(x)| &= |f(x) - f_{\alpha_0}(x) + f_{\alpha_0}(x)| \\
&\leq |f(x) - f_{\alpha_0}(x)| + |f_{\alpha_0}(x)| \\
&= |f(x) - f_{\alpha_0}(x)| \ \ (\because f_{\alpha_0} \in I_A \ \text{ and } \ x \in A) \\
&= ||f - f_{\alpha_0}||_{\{x\}} \\
&< \varepsilon.
\end{aligned}$$

Hence for all $\varepsilon > 0$ and $x \in A$ we have that $0 \leq |f(x)| < \varepsilon$ so that $f(x) = 0$. Thus $f \in I_A$ so that I_A is a closed ideal in $C_{co}(X)$. $\square$

Proof of (ii). For $f \in C(X)$, let $Z(f)$ denote the zero-set of f, that is

$$Z(f) = \{x \in X : f(x) = 0\}.$$

Since f is continuous, $Z(f)$ is closed in X because it is an inverse image of the closed set $\{0\}$. Let

$$A = \bigcap_{f \in I_+} Z(f),$$

where $I_+ = \{f \in I : f \geq 0\}$. A is closed because an arbitrary intersection of closed sets is closed.

Let $f \in I$. Since I is an ideal, $|f| \in I_+$ and $A \subseteq Z(|f|) = Z(f)$ so that $|f|[A] = \{0\}$. Hence $f|_A = 0$, thus $f \in I_A$ giving $I \subseteq I_A$.

For the reverse inclusion, let $f \in I_A, f \geq 0$. Fix $K \subseteq X$ compact and $\varepsilon > 0$.
Let $x \in Z(f) \cap K$. Then pick a function $f_x \equiv 0$. Since $f \geq 0$, it follows that $f_x \leq f$ on X. Since f is continuous on X, it is continuous at x, so there exists an open set V_x in X containing x such that if $y \in V_x$ then $f(y) \in (-\frac{\varepsilon}{2}, \frac{\varepsilon}{2})$. Now $f \geq 0$ so $f(y) \in [0, \frac{\varepsilon}{2})$ for all $y \in V_x$. Hence

$$0 \leq f(y) - f_x(y) = f(y) - 0 < \frac{\varepsilon}{2} < \varepsilon \text{ for all } y \in V_x.$$

Now suppose that $x \in K \backslash Z(f)$. Since $A \subseteq Z(f)$, there exists a $0 \leq g \in I$ such that $g(x) \neq 0$. Now Let $\alpha = \min\{\frac{f(x)}{2}, \frac{\varepsilon}{2}\}$ and define a function

$$h = \frac{f(x) - \alpha}{g(x)} g$$

Then

$$h(x) = \frac{f(x) - \alpha}{g(x)} g(x) = f(x) - \alpha.$$

Hence

$$f(x) - \varepsilon < h(x) < f(x).$$

Note that $h \in I$ because I is an ideal and $h \geq 0$. Let $f_x = f \wedge h$. Since $0 \leq f_x \leq h$ and $h \in I$ then $f_x \in I$. Clearly $f_x \leq f$ on X.

Since $f(x) - \varepsilon < h(x) < f(x)$, there exist an open set V_x containing x such that for all $y \in V_x$, $f(y) - \varepsilon < h(y) < f(y)$. If $y \in V_x$ then $h(y) < f(y)$ so that $f_x(y) = h(y)$. Hence $f(y) - \varepsilon < f_x(y) < f(y)$. There, if $y \in V_x$ we have $0 \leq f(y) - f_x(y) < \varepsilon$, for all $y \in V_x$.

Now $\{V_x : x \in K\}$ is an open cover for K and since K is compact in X there exists $x_1, ..., x_n \in K$ such that $K \subseteq V_{x_1} \cup ... \cup V_{x_n}$. Fix $y \in K$. Then $y \in V_{x_i}$ for some $i = 1, ..., n$ so that

$$0 \leq f(y) - f_{x_i}(y) < \varepsilon.$$

Define $g = \sup\{f_{x_i} : i = 1, ..., n\}$. Then g is continuous on X and $0 \leq f_{x_i} \leq g$ and $g \in I$. Now if $y \in K$, then $y \in V_{x_i}$ for some $i = 1, ..., n$. Hence

$$0 \leq f(y) - g(y) \leq f(y) - f_{x_i}(y) < \varepsilon.$$

Hence $||f - g||_K < \varepsilon$. This implies that every open subset of $C_{co}(X)$ containing f intersect I, so that $f \in \overline{I}$. Since I is closed in $C_{co}(X)$ we have that $I = \overline{I}$. Thus $I_A \subseteq I$ as required. $\square$

3.4 Order dual of $C(X)$

In this section we establish results on the order dual of $C(X)$. It will be shown that X is realcompact if and only if $C(X)^{\sim} = C_{co}(X)'$, where $C_{co}(X)'$ denote the continuous dual of $C(X)$ with the compact-open topology. The order dual of $C(X)$, $C(X)^{\sim}$ is described in the following result, see [6].

Theorem 3.4.1. *Let X be a Tychonoff space. Then for all $\phi \in C(X)^{\sim}$, there exists μ_ϕ, a regular borel measure on υX with compact support such that*

$$\int_{\upsilon X} f d\mu_\phi = \phi(f) \text{ for all } f \in C(X).$$

X is realcompact if and only if the support of μ_ϕ is contained in X for all $\phi \in C(X)^{\sim}$.

Theorem 3.4.2. *Let X be a Tychonoff space. Then $C(X)^{\sim}$ coincides with $C_{co}(X)'$ if and only if X is realcompact.*

Proof. Assume that X is not realcompact. Then there is a $z \in \upsilon X \backslash X$. Define

$$\phi : C(X) \ni f \longmapsto \tilde{f}(z) \in \mathbb{R}.$$

Statement i. $\phi \in C(X)^{\sim}$.
Let $[h, g]$ be an order interval in $C_{co}(X)$ and let $f \in [h, g]$. Then $h \leq f \leq g$ in $C(X)$. Since $C(X)$ and $C(\upsilon X)$ are isomorphic vector lattices we have that $\tilde{h} \leq \tilde{f} \leq \tilde{g}$ in $C(\upsilon X)$. Now the ordering is point-wise, so we have that $\tilde{h}(z) \leq \tilde{f}(z) \leq \tilde{g}(z)$. Hence $\phi(f) \in [\tilde{h}(z), \tilde{g}(z)]$. Therefore ϕ is order bounded.

Statement *ii.* $\phi \notin C_{co}(X)'$.
The set $A = (-1, 1)$ is open in $\mathbb{R}$ and

$$\phi^{-1}(A) = \{f \in C(X) : |\tilde{f}(z)| < 1\}.$$

Now this set is not open in $C_{co}(X)$ by Lemma 3.2.1 so that ϕ is not continuous. Thus $C(X)^{\sim} \neq C_{co}(X)'$.

Assume that X is realcompact. Fix $\phi \in C(X)^{\sim}$, then by Theorem 3.4.1, there exists a measure μ_ϕ such that

$$\int_X f d\mu_\phi = \phi(f) \text{ for all } f \in C(X).$$

Let K be the support of μ_ϕ. Then K is compact. Let $(f_\alpha)_{\alpha \in I}$ be a net in $C_{co}(X)$ so that $f_\alpha \longrightarrow 0$. Then $||f_\alpha||_K \longrightarrow 0$.
Fix $\varepsilon > 0$. Then there exists $\alpha_\varepsilon \in I$ such that for all $\alpha \geq \alpha_\varepsilon$ and for all $x \in K$, $|f_\alpha(x)| < \frac{\varepsilon}{\mu_\phi(K)}$. Hence

$$|\phi(f_\alpha)| \leq \int_X |f_\alpha| d\mu_\alpha < \varepsilon.$$

So $\phi(f_\alpha) \longrightarrow 0$. Therefore $\phi \in C_{co}(X)'$.
Now suppose that $\phi \in C_{co}(X)'$. Then by Theorem A.1.6, ϕ is topologically bounded. Then by Theorem 3.2.4, if $B \subseteq C(X)$ is order bounded, it is topologically bounded. so that $\phi(B)$ is bounded by Theorem A.1.6. Thus $\phi \in C(X)^{\sim}$ so that $C_{co}(X)' = C(X)^{\sim}$. $\square$

Theorem 3.4.3. *Let X be a realcompact space and ϕ a positive linear functional on C(X). There exists a compact subset K of X and a positive linear functional ϕ' on C(K) such that $\phi = \phi' \circ r_K$, where $r_K : C(X) \longrightarrow C(K)$ is defined by $r_K(f) = f|_K, f \in C(X)$..*

Proof. Suppose ϕ is a positive linear function on $C(X)$, so $\phi \in C(X)^{\sim}$. Since X is a realcompact space, it follows from Theorem 3.4.2 that $C(X)^{\sim}$ and $C_{co}(X)'$ coincide, that is $C_{co}(X)' = C(X)^{\sim}$. Hence $\phi \in C_{co}(X)'$. Thus ϕ is continuous with respect to the topology of compact convergence on $C(X)$.

It follows from Theorem A.1.7 that there exists a compact set K and an $\alpha > 0$ such that

$$|\phi(f)| \leq \alpha||f||_K \text{ for all } f \in C(X).$$

Fix $g \in C(K)$. Since r_K is onto, there exists an $f \in C(X)$ such that $r_K(f) = g$.
Suppose $f_0, f_1 \in C(X)$ satisfy $r_K(f_0) = r_K(f_1) = g$. Then,

$$|\phi(f_0) - \phi(f_1)| = |\phi(f_0 - f_1)| \leq ||f_0 - f_1||_K = 0.$$

So that $\phi(f_1) = \phi(f_0)$. Therefore the map $\phi' : C(K) \longrightarrow \mathbb{R}$ given by $\phi'(g) = \phi(f)$ if $r_K(f) = g$ is well defined. Now if $f \in C(X)$, then

$$[\phi' \circ r_K](f) = \phi'(r_K(f)) = \phi'(f|_K) = \phi(f)$$

Hence $\phi = \phi' \circ r_K$. Since ϕ is linear, we have

$$\phi'((f+h)|_K) = \phi(f+h) = \phi(f) + \phi(h) = \phi'(f|_K) + \phi'(h|_K),$$

so that ϕ' is also linear. Now let $g \in C(K)_+$, then $g \geq 0$ on K. Since r_K is onto, there exists an $f \in C(X)$ such that $f|_K = g$.The function $f \vee 0$ is in $C(X)_+$ and extends g. Since ϕ is positive it follows that $\phi'(g) = \phi(f) \geq 0$. Hence ϕ' is positive thus proving the result. □

3.5 Bands in $C(X)^\sim$ and $C(X)^{\sim\sim}$

Let X be a Tychonoff space and $K \subseteq X$ compact. Denote by r_K the restriction map from $C(X)$ to $C(K)$. In this section we will establish the relationship between $C(K)$ and $C(X)$ together with their order dual spaces. In particular, we will show an isomorphism between $C(K)^\sim$ and a subspace of $C(X)^\sim$ and also between $C(K)^{\sim\sim}$ and a subspace of $C(X)^{\sim\sim}$.

Lemma 3.5.1. *Let $K \subseteq X$ be a compact subset of X. Define a map $r_K : C(X) \longrightarrow C(K)$ by $r_K(f) = f|_K, f \in C(X)$. Then r_K is an onto linear lattice homomorphism.*

Proof. Let $g \in C(K)$ be given. Consider the embedding $X \hookrightarrow \beta X$. Since $K \subseteq X$ is compact, it is closed in βX. Since βX is a compact Hausdorff space, it is normal. Now K being a closed subset of a normal space, it follows from the Tietze Extension theorem that there is $\tilde{g} \in C(\beta X)$ such that $\tilde{g}|_K = g$. Since $X \subseteq \beta X$, $\tilde{g}|_X$ is continuous on X. Let $f = \tilde{g}|_X$, then $f \in C(X)$ and $f|_K = \tilde{g}|_K = g$. Hence $r_K(f) = f|_K = g$. Since $g \in C(K)$ was arbitrary, it follows that r_K is onto. It is easy to see that r_K is linear. It remains to show that it is a lattice homomorphism. By [1, Theorem 1.30], it suffices to show that

$$(f \wedge 0)|_K = f|_K \wedge 0.$$

For $x \in K$, since the ordering is point-wise we have $[f \wedge 0](x) = f(x) \wedge 0$. This holds for each $x \in K$ so that $(f \wedge 0)|_K = f|_K \wedge 0$. Thus r_K is an onto lattice homomorphism. $\square$

For $r_K : C(X) \longrightarrow C(K)$, define the maps

$$r_K^* : C(K)^\sim \longrightarrow C(X)^\sim \text{ and } r_K^{**} : C(X)^{\sim\sim} \longrightarrow C(K)^{\sim\sim}$$

as

$$r_K^*(\phi) = \phi \circ r_K,\ \ \phi \in C(K)^\sim \text{ and } r_K^{**}(\Psi) = \Psi \circ r_K^*,\ \ \Psi \in C(X)^{\sim\sim}.$$

We give the following results about the maps r_K^* and r_K^{**}.

Theorem 3.5.2. *Let $K \subseteq X$ be compact. The mapping r_K^* is a lattice isomorphism onto a band J in $C(X)^\sim$.*

Proof. The restriction map $r_K : C(X) \longrightarrow C(K)$ is an onto lattice homomorphism by Lemma 3.5.1.
Let $I = \{f \in C(X) : f[K] = \{0\}\} = \ker(r_K)$. Then I is an ideal in $C(X)$.
Let $\widetilde{r_K} : C(X)/I \longrightarrow C(K)$ be defined by

$$f + I \longmapsto r_K(f).$$

By Theorem 2.2.4, $\widetilde{r_K}^*$ is a lattice homomorphism and $(\widetilde{r_K}^{-1})^*$ is a lattice homomorphism so $\widetilde{r_K}^* \circ (\widetilde{r_K}^{-1})^*$ is the identity on $(C(X)/I)^\sim$. Also $(\widetilde{r_K}^{-1})^* \circ \widetilde{r_K}^*$ is the identity on $C(K)^\sim$. Therefore $\widetilde{r_K}^*$ is a lattice isomorphism.
Claim: $(C(X)/I)^\sim$ is separating on $C(X)/I$. Let $f+I$ be a non-zero element in $C(X)/I$. Then $f \notin I$. Since $f \notin I$, there is an $x \in K$ such that $f(x) \neq 0$. Hence $f' = r_K(f) \neq 0$. Since $C(K)^\sim$ is separating on $C(K)$ there exists a $\phi \in C(K)^\sim$ such that $\phi(f') \neq 0$. It then follows that $\widetilde{r_K}^*(\phi)(f) \neq 0$. Hence $(C(X)/I)^\sim$ is separating on $C(X)/I$.
Now let $J = I^\perp$. Then by Theorem 2.3.7, J is a band in $C(X)^\sim$.
Let $Q : C(X) \longrightarrow C(X)/I$ be defined by

$$f \longmapsto f + I.$$

Q is a lattice homomorphism onto $C(X)/I$ by Theorem 2.3.6. The map

$$Q^* : (C(X)/I)^\sim \longrightarrow I^\perp = J$$

is a lattice isomorphism.
Since the composition of lattice isomorphisms is a lattice isomophism we have that the map

$$Q^* \circ \widetilde{r_K}^* : C(K)^\sim \longrightarrow J$$

is a lattice isomorphism.
Claim: $Q^* \circ \widetilde{r_K}^* = r_K^*$
Fix $\phi \in C(K)^\sim$ and let $f \in C(X)$. It follows that

$$\begin{aligned}
[Q^* \circ \widetilde{r_K}^*(\phi)](f) &= [Q^*(\phi \circ \widetilde{r_K})](f) \\
&= [\phi \circ \widetilde{r_K} \circ Q](f) \\
&= \phi \circ \widetilde{r_K}(Q(f)) \\
&= \phi \circ \widetilde{r_K}(f + I) \\
&= \phi((\widetilde{r_K}(f + I)) \\
&= \phi(r_K(f)) \\
&= [\phi \circ r_K](f) = r_K^*(f).
\end{aligned}$$

Since this is true for all all $\phi \in C(K)^\sim$ and $f \in C(X)$ we have that $Q^* \circ \widetilde{r_K}^* = r_K^*$ so that r_K^* is a lattice isomorphism between $C(K)^\sim$ and J. □

Theorem 3.5.3. *Let $K \subseteq X$ be compact. Let*

$$I = \{f \in C(X) : f[K] = \{0\}\},$$

and $J = I^\perp$. Then $M = (J^d)^\perp$ is a band in $C(X)^{\sim\sim}$ and r_K^{**} restricted to M is a lattice isomorphism onto $C(K)^{\sim\sim}$.

Proof. $J \subseteq C(X)^\sim$ is non-empty and it follows from Theorem 2.3.9 that J^d is a band in $C(X)^\sim$. Hence an ideal in $C(X)^\sim$. It then follows from Theorem 2.3.7 that $(J^d)^\perp$ is a band in $C(X)^{\sim\sim}$. Hence M is a band in $C(X)^{\sim\sim}$.
In Theorem 3.5.2, we showed that the map

$$r_K^* : C(K)^\sim \longrightarrow J \subseteq C(X)^\sim$$

is a linear lattice isomorphism onto J. We have the following two adjoints for r_K^*,

$$r_K^{**} : C(X)^{\sim\sim} \longrightarrow C(K)^{\sim\sim}$$

$$r_0^{**} : J^\sim \longrightarrow C(K)^{\sim\sim}.$$

Since r_K^* is a lattice isomorphism, it follows from Corollary 2.2.5 that r_0^{**} is a lattice isomorphism.

Since $C(X)^\sim$ is a Dedekind complete vector lattice, see [Theorem 20.2, [11]], and J being a band, it follows that it is a projection band since any band in a Dedekind complete vector space is a projection band. That is, $C(X)^\sim = J \oplus J^d$. Let $P_J : C(X)^\sim \longrightarrow J$ be the band projection onto J and $P_J^* : J^\sim \longrightarrow C(X)^{\sim\sim}$ be its adjoint. It follows from Lemma 2.3.11 that P_J^*

is a lattice isomorphism onto $M = (J^d)^{\perp}$.
Claim: For all $\Psi \in J^{\sim}$,

$$r_K^{**}(P_J^*\Psi) = r_0^{**}(\Psi)$$

That is, the following diagram commutes.

$$\begin{tikzcd} J^{\sim} \arrow[r, "P_J^*"] \arrow[dr, "r_0^{**}"'] & M \arrow[d, "r_K^{**}|_M"] \\ & C(K)^{\sim\sim} \end{tikzcd}$$

Fix $\Psi \in J^{\sim}$ and let $\phi \in C(K)^{\sim}$.

$$\begin{aligned} [r_K^{**}(P_J^*\Psi)](\phi) &= [r_K^{**}(\Psi \circ P_J)](\phi) \\ &= [(\Psi \circ P_J) \circ r_K^*](\phi) \\ &= \Psi \circ P_J(r_K^*(\phi)) \\ &= \Psi(P_J(r_K^*(\phi))) \\ &= \Psi(r_K^*(\phi)) \ \ (\because \ \ r_K^*(\phi) \in J) \\ &= r_0^{**}(\Psi)(\phi). \end{aligned}$$

Since this holds for all $\Psi \in J^{\sim}$ and $\phi \in C(K)^{\sim}$ it follows that the claim is true. Hence r_K^{**} restricted to M is a lattice isomorphism onto $C(K)^{\sim\sim}$. $\square$

3.6 Embedding $C(X)$ into $C(X)^{\sim\sim}$

Let X be a Tychonoff space. For $x \in X$, we denote by ϕ_x an element of $C(X)^{\sim}$ such that $f \longmapsto f(x)$ for every $f \in C(X)$.
If $f \in C(X)$, we denote by Ψ_f the representation of f in $C(X)^{\sim\sim}$ under the canonical embedding.

Lemma 3.6.1. *Let D be a subset of $C(X)_+$ and $\phi \in C(X)_+^{\sim}$. If D is upward directed and bounded from above in $C(X)^{\sim\sim}$, then in $C(X)^{\sim\sim}$*

$$[\sup D](\phi) = \sup\{\Psi_f(\phi) \ : f \in D\}.$$

Proof. Assume $D \subseteq C(X)_+^{\sim\sim}$ is upward direct and bounded from above. Since $C(X)^{\sim\sim}$ is a Dedekind complete vector lattice , see [Theorem 20.2, [11]], it follows that $\sup D$ exists in $C(X)^{\sim\sim}$.

On the other hand, $C(X)^{\sim}$ is a vector lattice and $\mathbb{R}$ is Dedekind complete and it follows from Theorem 2.2.3 that the map $\Phi : C(X)_+^{\sim} \longrightarrow \mathbb{R}$ given by;

$$\Phi(\phi) = \sup\{\Psi_f(\phi) : f \in D\}, \ \phi \in C(X)_+^{\sim}$$

can be extended to $\tilde{\Phi}$ in $C(X)^{\sim\sim}$ such that $\tilde{\Phi} = \sup D$. If $\phi \geq 0$ we have the following

$$[\sup D](\phi) = \tilde{\Phi}(\phi) = \sup\{\Psi_f(\phi)\ : f \in D\}.$$

$\square$

We also note that the similar result holds for downward directed sets with infimum.

Lemma 3.6.2. *Let $f \in C(X)$, $x \in X$ and $0 \leq \phi \leq \phi_x$ in $C(X)^{\sim}$. Then*

$$|\phi(f)| \leq |f(x)|.$$

Proof. Suppose $0 \leq \phi \leq \phi_x$.
Since $\phi \geq 0$ by Lemma 2.3.12 we have that

$$\begin{aligned} |\phi(f)| &\leq \phi(|f|) \\ &\leq \phi_x(|f|) \ \ (\because |f| \text{is positive}) \\ &= |f|(x) \ \ (\text{definition of } \ \phi_x) \\ &= |f(x)|. \end{aligned}$$

$\square$

Lemma 3.6.3. *Let Ψ and Φ be in $C(X)^{\sim\sim}$ and ϕ_x be the point evaluation functional at $x \in X$. Then*

$$(\Phi \vee \Psi)(\phi_x) = \Phi(\phi_x) \vee \Psi(\phi_x)$$

and

$$(\Phi \wedge \Psi)(\phi_x) = \Phi(\phi_x) \wedge \Psi(\phi_x).$$

Proof. Using Theorem 2.1.13, it is sufficient to show that

$$[\Psi^+](\phi_x) = [\Psi(\phi_x)]^+.$$

By definition

$$[\Psi^+](\phi_x) = \sup\{\Psi(\phi) : 0 \leq \phi \leq \phi_x\}.$$

For $0 \leq \phi \leq \phi_x$, it follows from Lemma 3.6.2 that $|\phi(f)| \leq |f(x)|$ for all $f \in C(X)$.

statement: If $0 \leq \phi \leq \phi_x$ then $\phi = k\phi_x$ for some $0 \leq k \leq 1$. Let $0 \leq \phi \leq \phi_x$. Fix $f \in C(X)$. Let $g = f - f(x)\mathbf{1}$, where $\mathbf{1}$ is the constant 1 function on X.

Then $g \in C(X)$ and $f = f(x)\mathbf{1} + g$. Using Lemma 3.6.2 and the definition of g we have that

$$|\phi(g)| \leq |g(x)| = |[f - f(x)\mathbf{1}](x)| = f(x) - f(x) = 0.$$

Hence $|\phi(g)| = 0$ so that $g \in \ker \phi$. Thus

$$\begin{aligned}\phi(f) &= \phi(f(x)\mathbf{1} + g) \\ &= f(x)\phi(\mathbf{1}) + \phi(g) \\ &= \phi(\mathbf{1})f(x) \ \ (\because \ g \in \ker \phi) \\ &= \phi(\mathbf{1})\phi_x(f).\end{aligned}$$

Since $\phi \geq 0$ and $\mathbf{1} \geq 0$, we have $\phi(\mathbf{1}) \geq 0$. On the other hand, by Lemma 3.6.2 $|\phi(\mathbf{1})| \leq |\mathbf{1}(x)| = 1$ Hence

$$0 \leq \phi(\mathbf{1}) \leq 1.$$

Hence setting $k = \phi(\mathbf{1})$ we have the statement.
Finally,

$$[\Psi^+](\phi_x) = \sup\{k\Psi(\phi_x) : 0 \leq k \leq 1\} = [\Psi(\phi_x)]^+$$

$\square$

Lemma 3.6.4. *Let X be realcompact, $f \in C(X)$ and $D = \{f_\alpha : \alpha \in I\}$ be an upward directed and bounded subset of $C(X)$. Then $\Psi_f = \sup\{\Psi_{f_\alpha} : f_\alpha \in D\}$ in $C(X)^{\sim\sim}$ if and only if $f(x) = \sup\{f_\alpha(x) : f_\alpha \in D\}$ for all $x \in X$.*

Proof. Let X be realcompact. Suppose $\Psi_{f_\alpha} \uparrow \Psi_f$ in $C(X)^{\sim\sim}$. Then for all $\psi \in C(X)^{\sim}_{+}$, we have that $\Psi(\psi) = \sup\{\Psi f_\alpha(\psi) : f_\alpha \in D\}$. Now if $\psi = \phi_x$, then

$$f(x) = \Psi_f(\phi_x) = \sup\{\Psi_{f_\alpha}(\phi_x) : f_\alpha \in D\} = \sup\{f_\alpha(x) : f_\alpha \in D\}.$$

Now suppose $D = \{f_\alpha : \alpha \in I\}$ is directed such that $f_\alpha(x) \uparrow f(x)$ for each $x \in X$. Then by Theorem 3.2.3, $f_\alpha \longrightarrow f$ in the compact open topology.

Since X is realcompact, $C(X)^{\sim}$ coincides with the continuous dual of $C_{co}(X)$. Hence if $\phi \in C(X)^{\sim}$ then $\phi \in C_{co}(X)'$. Hence

$$\phi(f_\alpha) \longrightarrow \phi(f).$$

So if $\phi \in C(X)^{\sim}_{+}$, then $\Psi_f(\phi) = \sup\{\Psi_{f_\alpha}(\phi) : f_\alpha \in D\}$. This implies that $\Psi_f = \sup\{\Psi_{f_\alpha} : f_\alpha \in D\}$ by Lemma 3.6.1. $\square$

Lemma 3.6.5. *Let X be realcompact. If Φ and Ψ belong to $C(X)^{\sim\sim}$ and A and B are subsets of $C(X)$ such that $\Phi = \sup\{\Phi_f : f \in A\}$ and $\Psi = \sup\{\Psi_g : g \in B\}$. Then $\Phi \leq \Psi$ if and only if $\sup\{f(x) : f \in A\} \leq \sup\{g(x) : g \in B\}$ for all $x \in X$.*

Proof. Assume $\sup\{f(x) : f \in A\} \leq \sup\{g(x) : g \in B\}$ for all $x \in X$. For $x \in X$, $f \in A$ and $g \in B$, we have that

$$f(x) \geq f(x) \wedge g(x) = [f \wedge g](x),$$

so that $f(x) \geq \sup_{g \in B}[f \wedge g](x)$.
Since $\sup\{f(x) : f \in A\} \leq \sup\{g(x) : g \in B\}$, for any $f \in A$, we have

$$f(x) \leq \sup\{f(x) : f \in A\} \leq \sup\{g(x) : g \in B\}.$$

Now

$$\begin{aligned} f(x) &= f(x) \wedge f(x) \\ &\leq f(x) \wedge \sup\{g(x) : g \in B\} \\ &= \sup_{g \in B}\{g(x) \wedge f(x) : g \in B\} \\ &= \sup_{g \in B}[g \wedge f](x) \end{aligned}$$

Hence $f(x) = \sup_{g \in B}[g \wedge f](x)$, $x \in X$ for all $x \in X$. Since $f \in C(X)$ and $f(x) = \sup_{g \in B}[g \wedge f](x)$ we have, by Lemma 3.6.4 that

$$\begin{aligned} \Phi_f &= \sup_{g \in B}\{\Phi_f \wedge \Psi_g : g \in B\} \\ &\leq \sup\{\Psi_g : g \in B\} \\ &= \Psi. \end{aligned}$$

Hence $\sup\{\Phi_f : f \in A\} \leq \Psi$. Thus $\Phi \leq \Psi$.

Now suppose that $\Phi \leq \Psi$. It follows that for all $\phi \in (C(X)^{\sim})_{+}$, $\Phi(\phi) \leq \Psi(\phi)$. In particular, $\Phi(\phi_x) \leq \Psi(\phi_x)$ for every $x \in X$. Hence using Lemma 3.6.1 it follows that

$$\sup\{\Phi_f(\phi_x) : f \in A\} = \Phi(\phi_x) \leq \Psi(\phi_x) = \sup\{\Psi(\phi_x) : g \in B\}.$$

Hence $\sup\{f(x) : f \in A\} \leq \sup\{g(x) : g \in g \in B\}$. $\square$

$\widetilde{C(X)}$ is the order adherence of $C(X)$ in $C(X)^{\sim\sim}$. It is the space of all elements $\Phi \in C(X)^{\sim\sim}$ such that there exists a net $(f_\alpha)_{\alpha\in I}$ in $C(X)$ giving $\Phi_{f_\alpha} \xrightarrow{o} \Phi$ in $C(X)^{\sim\sim}$.

Lemma 3.6.6. *Let X be realcompact. If Φ and Ψ belong to $\widetilde{C(X)}$, then $\Phi \leq \Psi$ if and only if $\Phi(\phi_x) \leq \Psi(\phi_x)$ for all $x \in X$.*

Proof. Suppose $\Phi \leq \Psi$. Then for any $\phi \in C(X)^{\sim}_{+}$, it follows that $\Phi(\phi) \leq \Psi(\phi)$. In particular if $\phi = \phi_x$ we get the result.

Now suppose that $\Phi(\phi_x) \leq \Psi(\phi_x)$ for all $x \in X$. Since $\Phi, \Psi \in \widetilde{C(X)}$, it follows that there exists a net $(f_\alpha)_{\alpha\in I}$ and a net $(g_\beta)_{\beta\in J}$ such that $\Phi_{f_\alpha} \xrightarrow{o} \Phi$ and $\Psi_{g_\beta} \xrightarrow{o} \Psi$.

It follows from Theorem 2.1.12 that

$$\Phi = \liminf \Phi_{f_\alpha} \text{ and } \Psi = \limsup \Psi_{g_\beta}.$$

Since $\Phi(\phi_x) \leq \Psi(\phi_x)$, it follows from Definition 2.1.11 that there exists $\alpha_0 \in I$ and $\beta_0 \in J$ such that for all $\alpha' \geq \alpha_0$ and for all $\beta' \geq \beta_0$,

$$\inf\{\Phi_{f_\alpha} : \alpha \geq \alpha'\}(\phi_x) \leq \Phi(\phi_x) \leq \Psi(\phi_x) \leq \sup\{\Psi_{g_\beta} : \beta \geq \beta'\}(\phi_x).$$

It then follows from Lemma 3.6.5 that $\inf\{\Phi_{f_\alpha} : \alpha \geq \alpha'\} \leq \sup\{\Psi_{g_\beta} : \beta \geq \beta'\}$ for all $\alpha' \in I, \beta' \in J$ such that $\alpha' \geq \alpha_0$ and $\beta' \geq \beta_0$, so that $\Phi \leq \Psi$. □

Theorem 3.6.7. *The space $\widetilde{C(X)}$ of order adherence of $C(X)$ is a sublattice of $C(X)^{\sim\sim}$ containing semi-order units*

$$e_K = \inf\{f \in C(X) : f \geq 0 \text{ and } f[K] = \{1\}\}, \quad K \subseteq X \text{ compact.}$$

Proof. It follows from Theorem 2.2.7 that $\widetilde{C(X)}$ is a sublattice of $C(X)^{\sim\sim}$. We show that $e_K \in \widetilde{C(X)}$. Let

$$D = \{f \in C(X) : f \geq 0 \text{ and } f[K] = \{1\}\}.$$

Then the constant $\mathbf{1}$ function is in D so that $D \neq \emptyset$. Now let $f, g \in D$ and define $h = f \wedge g$. Then $h \geq 0$, $h[K] = 1$ so that $h \in D$. But $h \leq f, g$, hence D is downward direct and bounded below by $\mathbf{0}$. Hence by Lemma 2.1.10, there exists a net in D that order converges to e_K. Hence $e_K \in \widetilde{C(X)}$.

Finally we show that e_K is a semi-order unit of $\widetilde{\widetilde{C(X)}}$.
From Lemma 3.5.2, we showed that $r_K^* : C(K)^\sim \longrightarrow J$ is a lattice isomorphism, where J is a band in $C(X)^\sim$. Since $C(X)^\sim$ is Dedekind complete we have that $C(X)^\sim = J \oplus J^d$. We can rewrite it as

$$C(X)^\sim = C(K)^\sim \oplus W,$$

where W is the disjoint complement of $C(K)^\sim$ in $C(X)^\sim$. We will show that e_K vanishes on W.

Let $\phi \geq 0$ be in W. Now $\phi \geq 0$ and $\phi \in C(X)^\sim$, so there exists a compact set K_ϕ and a positive functional $\phi' \in C(K_\phi)^\sim$ such that $\phi = \phi' \circ r_{K_\phi}$, where r_{K_ϕ} is the restriction map. Hence ϕ is non-negative regular borel measure with compact support contained in K_ϕ. Now $K_\phi = (K_\phi \cap K) \cup (K_\phi \backslash K)$, we can express ϕ as a sum of two non-negative measures ϕ_1 and ϕ_2 such that spt$\phi_1 \subseteq K \cap K_\phi$ and spt $\phi_2 \subset \overline{K_\phi \backslash K}$. Since spt $\phi_1 \subseteq K \cap K_\phi \subseteq K$ it follows that $\phi_1 \in C(K)^\sim$. Since $\phi_1 \in C(K)^\sim$ and $\phi \in W$ it follows that $\phi_1 \wedge \phi = 0$. But $\phi_1 \leq \phi$. Thus $\phi = 0$. Hence $\phi = \phi_2$ and $\phi(K) = 0$. Since ϕ is regular, for any $\varepsilon > 0$, there exists a compact set F contained in $K_\phi \backslash K$ such that $\phi(K_\phi \backslash F) < \varepsilon$. Let $g \in C(X)$ satisfy $0 \leq g \leq 1$, $g[K] = \{1\}$ and $g[F] = \{0\}$. It then follows from the definition of e_K that

$$0 \leq e_K(\phi) \leq g(\phi) \leq ||g||_{K_\phi} \phi(K_\phi \backslash F) < \varepsilon.$$

Thus e_K is a member of $C(X)^{\sim\sim}$ which vanishes on the disjoint complement of J in $C(X)^\sim$.

Now we show that e_K is an order unit of $M \subseteq C(X)^{\sim\sim}$, where $M = (J^d)^\perp$. To this end, let

$$\mathcal{A} = \{f \in C(X) : f \geq 0, f[K] = 1\}.$$

For $\phi \in C(K)^\sim$ and the restriction map $r_K^* : C(X)^\sim \longrightarrow C(K)^\sim$ we have the following;

$$\begin{aligned}
(r_K^{**} e_K)(\phi) = e_K(r_K^* \phi) &= \inf\{\Psi_f : f \in \mathcal{A}\}(r_K^* \phi) \\
&= \inf\{\Psi_f(r_K^* \phi) : f \in \mathcal{A}\} \\
&= \inf\{\phi(r_K f) : f \in \mathcal{A}\} \\
&= \phi(\mathbf{1}_K).
\end{aligned}$$

Where $\mathbf{1}_K$ is the constant function $\mathbf{1}$ on $C(K)$. Hence $r_K^{**}(e_K)$ is the constant function $\mathbf{1}$ in $C(K)^{\sim\sim}$.
Let $\Gamma : C(K) \longrightarrow C(K)^{\sim\sim}$ be defined by $f \longmapsto \Phi_f$ for all $f \in C(K)$. Then Γ

is a lattice isomorphism onto $\Gamma[C(K)]$. Since $\mathbf{1}_K$ is an order unit of $C(K)$, it follows that $\Gamma(\mathbf{1}_K) = \Gamma_{\mathbf{1}_K}$ is an order unit of $\Gamma[C(K)]$. Now if $\Phi \in C(K)^{\sim\sim}_+$ and $\psi \in C(K)^{\sim}_+$. Then

$$\begin{aligned}\Phi(\psi) &\leq ||\Phi||||\psi|| \\ &= ||\Phi||\psi(\text{spt}(\psi)) \\ &= ||\Phi||\Gamma_{\mathbf{1}_K}(\psi) \\ &= ||\Phi||\Gamma_{\mathbf{1}_K}(\psi)\end{aligned}$$

Hence $\Phi \leq ||\Phi||\Gamma_{\mathbf{1}_K}$ so that $\Gamma_{\mathbf{1}_K}$ is an order unit of $C(K)^{\sim\sim}$. By Theorem 3.5.3, r_K^{**} is a lattice isomorphism on to M. It follows that e_K is an order unit of M. It then follows from Theorem 2.3.10 that e_K is a semi-order unit of $C(X)^{\sim\sim}$. □

3.7 Semi-order units in $\widetilde{C(X)}$ and the compact open topology

Let u be a semi-order unit of $C(X)$ and ρ_u the associated semi-norm. By semi-order unit topology on $C(X)$, we mean the locally convex space generated by the collection of semi-norms, $\{\rho_u : u \text{ is a semi-order unit on } C(X)\}$. In this section, we establish some results regarding the semi-order unit topology. It will be shown that if u is a semi-order unit of $C(X)$, then the set $S = \{x \in X : u(x) \neq 0\}$ is bounded away from 0. Furthermore, it is a clopen set. We establish the following equivalence. If X is realcompact, then the semi-order unit topology agree with the compact open topology if and only if X is a union of clopen sets.

Proposition 3.7.1. *Let u be a semi-order unit of $C(X)$. The set*

$$S = \{x \in X : u(x) \neq 0\}$$

is both open and closed in X.

Proof. We first show that S is open.
Since u is a continuous function on X and $\mathbb{R}\backslash\{0\} = (-\infty, 0)\cup(0, \infty)$ is open in $\mathbb{R}$, we have that $u^{-1}(\mathbb{R}\backslash\{0\})$ is open in X. That is $S = u^{-1}(\mathbb{R}\backslash\{0\})$ is open.

Now we show that S is closed.
Since u is a semi-order-unit, $u \geq 0$. It follows that $\sqrt{u}$ is well defined function and continuous on X, as u is a semi-order unit, there exist a $\lambda > 0$ such that

$$\sqrt{u} \wedge nu \leq \lambda u, \text{ for all } n \in \mathbb{N}.$$

Let $x \in S$. Then $\sqrt{u(x)} \wedge nu(x) \leq \lambda u(x)$. Since $u(x) > 0$, $\{nu(x) : n \in \mathbb{N}\}$ is not bounded, it follows that $\sqrt{u(x)} \leq \lambda u(x)$, hence $u(x) \leq \lambda^2 u^2(x)$. This implies that

$$\frac{1}{\lambda^2} \leq u(x).$$

Let $a = \frac{1}{\lambda^2}$ then $0 < a \leq u(x),\ x \in S$. Hence $0 < a \leq u(x)$ for all $x \in S$. So if $x \in S$ then $x \in u^{-1}([a, \infty))$, clearly $u^{-1}([a, \infty)) \subseteq S$. Hence $S = u^{-1}([a, \infty))$. The set $[a, \infty)$ is closed in $\mathbb{R}$. Since u is continuous on X, the set $S = u^{-1}([a, \infty))$ is closed in X. Thus S is both closed and open. $\square$

Lemma 3.7.2. *Let X be a realcompact space and $S \subseteq X$. Assume $f|_S$ is bounded for all $f \in C(X)$. Then $\overline{S}$ is compact.*

Proof. Fix $f \in C(X)$. Then by Lemma 3.1.11 there exist a unique extension $\tilde{f}$ of f such that $\tilde{f} : \beta X \longrightarrow \overline{\mathbb{R}}$ and $\tilde{f}$ is continuous on βX.

Suppose there exists $x_0 \in \overline{S}^{\beta X} \backslash X$, where $\overline{S}^{\beta X}$ denotes the closure of S in βX. Since $f|_S$ is bounded, it follows that $\tilde{f}(x_0) \in \mathbb{R}$. Since f is arbitrary, it follows that every continuous function extends to a real-valued continuous function on $X \cup \{x_0\}$ which is a contradiction since X is realcompact.
Hence $\overline{S}^{\beta X} \backslash X = \emptyset$, so it follows that $\overline{S}^{\beta X} \subseteq X$. Now $\overline{S}^{\beta X} \subseteq X$ compact in βX implies that $\overline{S}^{\beta X}$ compact in X. $\overline{S} \subseteq \overline{S}^{\beta X}$ implies that $\overline{S}$ is compact. $\square$

Proposition 3.7.3. *Let X be a realcompact space. If u is a semi-order unit of $C(X)$ then the set*

$$S = \{x \in X : u(x) \neq 0\}$$

is compact.

Proof. By Lemma 3.7.2, it suffices to show that for any $f \in C(X)$, $f|_S$ is bounded.

Fix $f \in C(X)$. Since u is a semi-order unit, u is continuous so that u^2 is also continuous. It follows that there exists a $\delta > 0$ such that

$$u^2 \wedge nu \leq \delta u \text{ for all } n \in \mathbb{N}.$$

Hence we have that u is bounded by δ, that is $u(x) \leq \delta$ for all $x \in X$. Since $f \in C(X)$, u a semi-order unit, there exists a $\lambda > 0$, such that $|f| \wedge nu \leq \lambda u$ for all $n \in \mathbb{N}$. In particular, if $x \in S$ then $u(x) > 0$ so that the sequence $(nu(x))$ is not bounded. It follows that

$$|f(x)| \leq \lambda u(x) \leq \lambda\delta.$$

Hence f is bounded on S. Since f is an arbitrary function in $C(X)$, we have that every function in $C(X)$ is bounded on S. It then follows from Lemma 3.7.2 that $\overline{S}$ is compact. Since S is closed we have that $S = \overline{S}$ so that S is compact as required. $\square$

Proposition 3.7.4. *Let $u \in C(X)$ such that $u > 0$. Suppose $S = \{x \in X : u(x) \neq 0\}$ is compact in X. Then u is a semi-order unit.*

Proof. Fix $f \in C(X)$. Since S is compact and f is continuous, it follows that $f|_S$ is bounded. So there exists $M \in \mathbb{R}$, $M > 0$ such that

$$|f(x)| \leq M \text{ for all } x \in S. \tag{3.1}$$

Since S is compact and u is continuous, we have that $u[S]$ is compact in $\mathbb{R}$. Hence $\inf u[S]$ exists and belong to $u[S]$ so that $\inf u[S] > 0$.
Define $c = \inf u[S]$. It follows that $0 < c \leq u(x)$ for all $x \in S$. Hence

$$0 < \frac{1}{u(x)} \leq \frac{1}{c} \text{ for all } \ x \in S. \tag{3.2}$$

Using (3.1) and (3.2) we have

$$\frac{|f(x)|}{u(x)} \leq \frac{1}{c}M, \text{ so that } |f(x)| \leq \frac{M}{c}u(x) \text{ for all } x \in S.$$

Let $\lambda = \frac{M}{c}$. Then for $x \in S$, $|f(x)| \leq \lambda u(x)$. Hence for $x \in S$, we have that

$$\min\{|f(x)|, nu(x)\} \leq \lambda u(x) \text{ for all } n \in \mathbb{N}.$$

For $x \notin S$, $u(x) = 0$ so that $nu(x) = 0 = \lambda u(x)$ for all $n \in \mathbb{N}$. Hence

$$\min\{|f(x)|, nu(x)\} \leq \lambda u(x), \ \ x \in X \ \ n \in \mathbb{N}.$$

Thus $|f| \wedge nu \leq \lambda u$ for all $n \in \mathbb{N}$. Since f is arbitrary it follows that u is a semi-order unit of $C(X)$. $\square$

Proposition 3.7.5. *Let S be a clopen subset of X and f the indicator function of S. Then f is continuous.*

Proof. Let $B = \{0, 1\}$. We will show that for any V open in $\mathbb{R}$, the set $f^{-1}[V] = f^{-1}[V \cap B]$ is open in X.

Let V be open in $\mathbb{R}$ then $V \cap B$ is one of $\emptyset$, $\{0\}$, $\{1\}$ or B. Now $f^{-1}[\emptyset] = \emptyset$ which is open, $f^{-1}[\{0\}] = X \backslash S$ which is open, and $f^{-1}[\{1\}] = S$ is open in X. Finally $f^{-1}[B] = X$ which is also open. Hence f is continuous on X. $\square$

Theorem 3.7.6. *Let X be a realcompact space. The semi-order unit topology on C(X) coincides with the topology of compact convergence if and only if X is a union of open compact sets.*

Proof. Let u be a semi-order unit on $C(X)$. Then the set $S = \{x \in X : u(x) \neq 0\}$ is open, closed and compact; that is, S is an open compact set. Note that we have shown that there is an $a > 0$ such that if $x \in S$, then $u(x) > a$ and we have also shown that there is a $\delta > 0$ such that if $x \in, S$ then $u(x) \leq \delta$. These together yield

$$0 < a \leq u(x) \leq \delta \text{ for all } x \in S.$$

This implies that u is bounded away from zero on S. Since $u(x) > 0$ for all $x \in S$, it follows that $\{nu(x) : n \in \mathbb{N}\}$ is not bounded on S. Using this fact, we have that

$$\begin{aligned}\rho(f) &= \inf\{\lambda > 0 : |f| \wedge nu \leq \lambda u \text{ for all } n \in \mathbb{N}\} \\ &= \inf\left\{\lambda > 0 : |f(x)| \wedge nu(x) \leq \lambda u(x),\ \ x \in X \text{ for all } n \in \mathbb{N}\right\} \\ &= \inf\left\{\lambda > 0 : |f(x)| \leq \lambda u(x)\ \ x \in S\right\} \\ &= \inf\left\{\lambda > 0 : f|_S \leq \lambda u|_S\right\}.\end{aligned}$$

Since $u \leq \delta$ on S, it follows that if $f|_S \leq \lambda u|_S$ then $f|_S \leq \lambda\delta$. Hence $\frac{1}{\delta}||f||_S \leq \rho(f)$. Also $a \leq u(x), x \in S$ so that $1 \leq \frac{1}{a}u(x)$. Hence

$$|f(x)| \leq ||f||_S \leq \frac{||f||_S}{a}u(x) \text{ for all } x \in S.$$

Thus $|f| \wedge nu \leq \frac{||f||_S}{a}u, n \in \mathbb{N}$. Hence $\frac{||f||_S}{a} \in \{\lambda > 0 : |f| \wedge nu \leq \lambda u\}$ so that $\rho(f) \leq \frac{||f||_S}{a}$. Thus

$$\frac{1}{\delta}||f||_S \leq \rho(f) \leq \frac{1}{a}||f||_S$$

This implies that the semi-norm associated with u is equivalent to $||.||_S$. Hence semi-order unit topology on $C(X)$ is coarser than the topology of compact convergence.

Suppose $X = \bigcup_{\alpha \in I} A_\alpha$ where each A_α is an open, compact set.
We have already proved that the semi-order unit topology on $C(X)$ is generally coarser than the topology of compact convergence. So it remains to show the reverse inclusion and to this end, fix $K \subseteq X$ compact. Since $\{A_\alpha : \alpha \in I\}$

is an open cover for X, it also covers K. Hence there exist $A_1, ...A_n$ such that

$$K \subseteq \bigcup_{i=1}^{n} A_i.$$

Let $S = \bigcup_{i=1}^{n} A_i$. Then S is open compact in X. Let u be an indicator function on S. Since S is clopen, it follows from Proposition 3.7.5 that u is continuous on X. It then follows from Proposition 3.7.4 that u is a semi-order unit. It follows that the semi-norm associated with the function u is equal to $||.||_S$. Indeed

$$\begin{aligned}
\rho_u(f) &= \inf\{\lambda > 0 : |f| \wedge nu \leq \lambda u \text{ for all } n \in \mathbb{N}\} \\
&= \inf\left\{\lambda > 0 : |f(x)| \wedge nu(x) \leq \lambda u(x), \;\; x \in X \text{ for all } n \in \mathbb{N}\right\} \\
&= \inf\left\{\lambda > 0 : |f(x)| \leq \lambda u(x) \;\; x \in S\right\} \\
&= \inf\left\{\lambda > 0 : f|_S \leq \lambda\right\} (\because u|_S = 1) \\
&= ||f||_S.
\end{aligned}$$

Hence $\rho_u(f) = ||f||_S$.

Since $K \subseteq S$, we have that

$$\sup\{|f(x)| : x \in K\} \leq \sup\{|f(x)| : x \in S\}$$

that is $||f||_K \leq ||f||_S$. Hence the topology of compact convergence is coarser that the semi-order unit topology. This implies that the topology of compact convergence is equal to semi-order unit topology.

Now suppose that the semi-order unit topology on $C(X)$ agrees with the topology of compact convergence i.e $C_{co}(X) = C_{sou}(X)$. It will be shown that X is a union of open compact sets. It is sufficient to show that each $x \in X$ belongs to an open compact set A_x so that $X = \bigcup\limits_{x \in X} A_x$.

Fix $x_0 \in X$. Let K be the singleton $\{x_0\}$. Then K is compact. Since the topology of compact convergence and semi-order-unit topology coincide, there exist semi-order units $u_1, ..., u_n$ with associated semi-norms $\rho_1, ..., \rho_n$ such that

$$\sup\{\rho_i : i = 1, ..., n\} \geq ||.||_K.$$

Claim: There exists $1 \leq i \leq n$ such that $x_0 \in \{x \in X : u_i(x) \neq 0\}$.
Assume that it is not true. Since $\{x \in X : u_i(x) \neq 0\}$ is a compact set in X, it follows that

$$B = \bigcup_{i=1}^{n} \{x \in X : u_i(x) \neq 0 \text{ for some } \;\; i = 1, ..., n\}$$

is closed in X and $x_0 \notin B$. Since X is completely regular, we can separate singletons and closed sets with a continuous function. Hence, there exists an $f \in C(X)$ such $f(x_0) = 1$ and $f(x) = 0$ for all $x \in B$. Hence $\rho_i(f) = 0$ for each $i = 1,..,n$ and $\sup\{\rho_i(f) : i = 1,...,n\} = 0$, but $||f||_K = 1$. This implies $\sup\{\rho_i(f) : i = 1,...,n\} < 1$. This is however a contradiction, since $\sup\{\rho_i : i = 1,...,n\} \geq ||.||_K$. Hence $u_i(x_0) \neq 0$ for some $i = 1,...,n$. That is there, is an i_0 such that $u_{i_0}(x_0) \neq 0$. It follows that $x_0 \in \{x \in X : u_{i_0} \neq 0\} = A_{x_0}$ which is an open compact set by Proposition 3.7.3.

Since x_0 was arbitrary, it follows that it is true that for all $x \in X$ and that there exists an A_x open compact such that $x \in A_x$. Hence $X = \bigcup\limits_{x \in X} A_x$ as required. $\square$

Lemma 3.7.7. *Let X be a topological space and let $S \subseteq .X$ Suppose that some function $h \in C(X)$ is not bounded on S. Then S contains a copy of $\mathbb{N}$, C-embedded in X on which h approaches infinity.*

Proof. By replacing h with $|h|$ if necessary, we may suppose that h is positive. S is non-empty for if it is empty then h is bounded on S which is a contradiction.

Now pick arbitrary $x_1 \in S$. Since h is not bounded on S it follows that for any $M \in \mathbb{R}, M > 0$, there exists an $x \in S$ such that $h(x) > M$. In particular there exists an $x_2 \in S$ such that

$$h(x_2) \geq h(x_1) + 1.$$

Inductively, there exists a sequence (x_n) in S such that $h(x_{n+1}) \geq h(x_n) + 1$ for all $n \in \mathbb{N}$. Let $A = \{x_n : n \in \mathbb{N}\}$ and define

$$H : \mathbb{N} \longrightarrow A \text{ by } n \longmapsto x_n.$$

Then H is a bijection by construction. It remains to show that singletons are open in A with the subspace topology inherited from X. To this end let $x_n \in A$, then

$$\{x_n\} = h^{-1}(\{y \in \mathbb{R} : h(x_{n-1}) + 1 < y < h(x_{n+1}) + 1\}) \cap A,$$

which is open by the continuity of h. Hence A carries discrete topology. Thus S contains a copy of $\mathbb{N}$. By the construction of the set A, the function h is unbounded on A.

Claim: $h|_A : A \longrightarrow h[A]$ is a homeomorphism.
Any function is onto its range, hence $h|_A$ is onto $h[A]$. Let $x_n, x_m \in A$ with $x_n \neq x_m$. Then $n \neq m$. So assume $m < n$, then by the construction of the set A, $h(x_m) < h(x_m) + 1 \leq h(x_{m+1}) \leq h(x_n)$. Hence $h(x_m) < h(x_n)$ so that $h(x_m) \neq h(x_n)$. Hence h is injective so that h is a bijection. h is continuous on X, so $h|_A$ is continuous on A

Since A is equipped with a discrete topology, it suffices to show that one point set are open in $h[A]$.

Let $y \in h[A]$. Then there exists $n \in \mathbb{N}$ such that $y = h(x_n)$. Now let $U = (y - \frac{1}{2}, y + \frac{1}{2})$. Then U is open in $\mathbb{R}$ and $\{h(x_n)\} = h[A] \cap U$. Thus one point sets are open in $h[A]$ with the topology inherited from $\mathbb{R}$, so that h is a homeomorphism.

It then follows from [8, Theorem 1.9] that A is C-embedded in X. $\square$

Theorem 3.7.8. *Let X be realcompact and Ψ a semi-order unit on $\widetilde{C(X)}$. Then the closure in X of $\{x \in X : \Psi(\phi_x) \neq 0\}$ is compact, where ϕ_x denotes the point-evaluation functional of $x \in X$.*

Proof. Let $S = \{x \in X : \Psi(\phi_x) \neq 0\}$. With a view towards a contradiction, suppose that $\overline{S}$ is not compact. Since X is realcompact, it follows from Lemma 3.7.2 that there exist a continuous function h which is not bounded on S. Hence by Theorem 3.7.7 there exist a discrete infinite subset A of S such that A is C-embeddeble in X. Since the subspace topology on A is a discrete topology, any function $f : A \longrightarrow \mathbb{R}$ is continuous. Define

$$f(x_n) = n\Psi(\phi_{x_n}).$$

Since A is C-embedded in X, f can be extended to a continous function g on X. Since Ψ is a semi-order unit, $g \wedge n\Psi \leq \lambda\Psi$ for all $n \in \mathbb{N}$. In particular,

$$n\Psi(\phi_{x_n}) = g(x_n) \wedge n\Psi(\phi_{x_n}) \leq \lambda\Psi(\phi_{x_n})$$

for some $\lambda > 0$ and for all $n \in \mathbb{N}$. Since $\Psi(\phi_{x_n}) \neq 0$ and is positive because Ψ is a semi-order unit, we have that $n \leq \lambda$ for all $n \in \mathbb{N}$. This is a contradiction because $\mathbb{N}$ is not bounded from above. Hence $\overline{S}$ is compact. $\square$

Denote by τ_{sou} the topology induced on $C(X)$ as a subspace of $\widetilde{C(X)}$ equipped with the semi-order unit topology. We denote this topological space by a pair $(C(X), \tau_{sou})$ or simply $C_{sou}(X)$.

Theorem 3.7.9. *Let X be realcompact and Ψ a semi-order-unit in $\widetilde{C(X)}$. Then there exists a real number M such that $\Psi(\phi_x) \leq M$ for all x in X.*

Proof. With a view towards a contradiction, suppose Ψ does not satisfy the conclusion. Then for each $M > 0$, there exists an $x \in X$ such that $\Psi(\phi_x) \geq M$. In particular for each $n \in \mathbb{N}$, there exists an $x_n \in X$ such that

$$\Psi(\phi_{x_n}) \geq n^3.$$

For each $n \in \mathbb{N}$, let ψ_n be the point evaluation at x_n, acting as a functional on $C(K)$. Then $\psi_n \in C(K)^\sim$ and $r_K^* \psi_n = \phi_{x_n}$. We have

$$||\psi_n|| = \sup_{||f||_K=1} |\psi_n(f)| = \sup_{||f||_K=1} |f(x_n)| = 1.$$

Therefore $\sum_{n=1}^{\infty} \frac{1}{n^2}\psi_n$ is absolutely convergent in $C(K)^\sim$. Because $C(K)^\sim$ is a Banach lattice, $\sum_{n=1}^{\infty} \frac{1}{n^2}\psi_n$ converges in norm to some ψ in $C(K)^\sim$. Note that $\frac{1}{n^2}\psi_n \leq \psi$ for every $n \in \mathbb{N}$. Hence $\frac{1}{n^2}\phi_{x_n} = r_K^*(\frac{1}{n^2}\psi_n) \leq r_K^*\psi$ so that

$$\Psi(r_K^*\psi) \geq \Psi(\frac{1}{n^2}\phi_{x_n}) > n$$

for all $n \in \mathbb{N}$, a contradiction. □

Proposition 3.7.10. *Let X be realcompact, Ψ a semi-order unit of $\widetilde{C(X)}$ with the associated seminorm ρ_Ψ. Let K be the closure in X of $\{x \in X : \Psi(\phi_x) \neq 0\}$. Then for all $f \in C(X)$*

$$\rho_\Psi(f) \leq ||f||_K \rho_\Psi(\mathbf{1}).$$

Proof. Fix $f \in C(X)$. Consider $x \in K$. We have that $|f(x)| \leq ||f||_K$. By Lemma 3.6.3 we have that, for all $n \in \mathbb{N}$

$$(|f| \wedge n\Psi)(\phi_x) = ((x) \wedge n\Psi(\phi_x) \leq ||f||_K \wedge n\Psi(\phi_x).$$

By definition of ρ_Ψ,

$$\rho_\Psi(||f||_K\mathbf{1}) = \inf\{\lambda > 0 : ||f||_K\mathbf{1} \wedge n\Psi \leq \lambda\Psi, \text{ for all } n \in \mathbb{N}\}.$$

Therefore $(||f||_K\mathbf{1}) \wedge (n\Psi) \leq \rho_\Psi(||f||_K\mathbf{1})\Psi$ in $C(X)^{\sim\sim}$. So, again using Lemma 3.6.3 we have

$$\begin{aligned} ||f||_K \wedge (n\Psi(\phi_x)) &= (||f||_K\mathbf{1}) \wedge (n\Psi)(\phi_x) \\ &\leq \rho_\Psi(||f||_K\mathbf{1})\Psi(\phi_x). \end{aligned}$$

Hence

$$\begin{aligned}(|f| \wedge n\Psi)(\phi_x) &\leq \rho_\Psi(||f||_K \mathbf{1})\Psi(\phi_x) \\ &= ||f||_K \rho_\Psi(\mathbf{1})\Psi(\phi_x).\end{aligned}$$

Since $\Psi(\phi_x) = 0$ for all $x \notin K$ we have that for all $x \in X$,

$$(|f| \wedge n\Psi)(\phi_x) \leq ||f||_K \rho_\Psi(\mathbf{1})\Psi(\phi_x).$$

Hence $f \wedge n\Psi \leq ||f||_K \rho_\Psi(\mathbf{1})\Psi$ for all $n \in \mathbb{N}$ so that

$$||f||_K \rho_\Psi(\mathbf{1}) \in \{\lambda > 0 : f \wedge n\Psi \leq \lambda\Psi\}.$$

Taking the infimum of the set $\{\lambda > 0 : f \wedge n\Psi \leq \lambda\Psi\}$ we obtain $\rho_\Psi(f) \leq ||f||_K \rho_\Psi(\mathbf{1})$. □

Proposition 3.7.11. *The topology τ_{sou} is finer than the topology of compact convergence.*

Proof. Let K be a compact set in X and denote by e_K the semi-order unit associated with K and ρ_{e_K} be the associated semi-norm. We show that $||f||_K \leq \rho_{e_K}(f)$ for all $f \in C(X)$. Fix $f \in C(X)$. By definition of ρ_{e_K} we have that $|f| \wedge ne_K \leq \rho_{e_K}(f)e_K$ for all $n \in \mathbb{N}$. It follows from Lemma 3.6.3 that

$$|f(x)| \wedge ne_K(\phi_x) \leq \rho_{e_K}(f)e_K(\phi_x),\ \ x \in X.$$

Since $e_K(\phi_x) = 1$ for all $x \in K$, we have that $|f(x)| \leq \rho_{e_K}(f)$ for all $x \in K$. Hence $||f||_K \leq \rho_{e_K}(f)$. Let U be an open neighbourhood of 0 with respect to the compact-open topology. For every compact set $K \subseteq X$, there exists a $\varepsilon_K > 0$ so that $\{f \in C(X) : ||f||_K < \varepsilon_K\} \subseteq U$. Since $||.||_K \leq \rho_{e_K}$, $\{f \in C(X) : \rho_{e_K} < \varepsilon_K\} \subseteq \{f \in C(X) : ||f||_K < \varepsilon_K\} \subseteq U$. Hence U is an open neighbourhood of 0 with respect to $\tau_{sou,}$ the semi-order unit topology. This shows that τ_{sou} is finer than the compact-open topology. □

Theorem 3.7.12. *Let X be realcompact.*

i. If K is a compact subset of X with associated semi-order unit e_K in $\widetilde{C(X)}$ then $\rho_{e_K}(f) = ||f||_K$ for all $f \in C(X)$.

ii. Let Ψ be semi-order unit in $\widetilde{C(X)}$ and $A = \{x \in X : \Psi(\phi_x) \neq 0\}$. Let $K = \overline{A}$. Then $\rho_\Psi(.)$ and $||.||_K$ are equivalent on $C(X)$.

Proof of (i). We showed in the proof of Proposition 3.7.11 and in Proposition 3.7.10 that

$$||f||_K \le \rho_{e_K}(f) \le \rho_{e_K}(\mathbf{1})||f||_K, \;\; f \in C(X).$$

We will show that $\rho_{e_K}(\mathbf{1}) = 1$.

$$\rho_{e_K}(\mathbf{1}) = \inf\{\lambda > 0 : \mathbf{1} \wedge ne_K \le \lambda e_K\}$$

$$\begin{aligned}
\mathbf{1} \wedge ne_K \le \lambda e_K &\iff (\mathbf{1} \wedge ne_K)(\phi_x) \le \lambda e_K(\phi_x) \;\; \forall x \in X \;\; \text{(By Lemma 3.6.6)} \\
&\iff 1 \wedge ne_K(\phi_x) \le \lambda e_K(\phi_x) \;\; \forall x \in K \;\; (e_K(\phi_x) = 0 \;\; \forall x \notin K) \\
&\iff 1 \wedge n \le \lambda \;\; (x \in K \implies e_K(\phi_x) = 1) \\
&\iff \lambda \ge 1.
\end{aligned}$$

Hence $\rho_{e_K}(\mathbf{1}) = 1$ so that

$$||f||_K \le \rho_e(f) \le ||f||_K.$$

Hence $\rho_{e_K}(f) = ||f||_K$ thus proving (i). □

Proof of (ii). Let $f \in C(X)$. We note that

$$|f(x)| \wedge n\Psi(\phi_x) \le \rho_\Psi(f)\Psi(\phi_x)$$

Because $\Psi(\phi_x) > 0$ for all $x \in A$, we have $|f(x)| \le \rho_\Psi(f)$, $x \in A$. By Theorem 3.7.9, there exists an $M > 0$ such that $\Psi(\phi_x) \le M$ and $\Psi(\phi_x) \ne 0$ for $x \in A$. Hence

$$\sup\{|f(x)| : x \in A\} \le \rho_\Psi(f)M.$$

Thus

$$\frac{1}{M}||f||_K \le \rho_\Psi(f) \le \rho_\Psi(\mathbf{1})||f||_K.$$

Therefore $||.||_K$ and $\rho_\Psi(.)$ are equivalent on $C(X)$. □

We showed in Theorem 3.1.17 that $T : C(X) \longrightarrow C(\upsilon X)$ is a lattice isomorphism so the adjoint $T^* : C(\upsilon X)^\sim \longrightarrow C(X)^\sim$ is a lattice isomorphism. Similarly the adjoint $C(X)^{\sim\sim} \longrightarrow C(\upsilon X)^{\sim\sim}$ is a lattice isomorphism.

Now if $\Psi \in \widetilde{C(X)}$, then there exists a net $(f_\alpha)_{\alpha\in I}$ such that $\Psi_{f_\alpha} \xrightarrow{o} \Psi$. The net $(Tf_\alpha)_{\alpha\in I} = (\tilde{f}_\alpha)_{\alpha\in I}$ is in $C(\upsilon X)$ and $\Psi_{\tilde{f}_\alpha} \xrightarrow{o} \Psi_{\tilde{f}}$. We deduce that $\widetilde{C(X)}$ and $\widetilde{C(\upsilon X)}$ are isomorphic vector lattices. The semi-order units in these two spaces are in 1 to 1 correspondence.
Hence $C_{sou}(X)$ and $C_{sou}(\upsilon X)$ are homeomorphic. In particular, T is a homeomorphism.

Lemma 3.7.13. Ψ *is a semi-order unit of* $\widetilde{C(X)}$ *if and only if* $T^{**}\Psi$ *is a semi-order unit of* $\widetilde{C(\upsilon X)}$.

Proof. We will show that

$$\rho_{\Psi}(f) = \rho_{T^{**}\Psi}(Tf) \text{ for all } f \in C(X).$$

For $\lambda > 0$, $|f| \leq n\Psi \leq \lambda\Psi$ in $\widetilde{C(X)}$ if and only if $|T^{**}f| \wedge nT^{**}\Psi \leq \lambda T^{**}\Psi$ in $\widetilde{C(\upsilon X)}$. But $Tf = T^{**}f$ in $\widetilde{C(\upsilon X)}$.
So for $\lambda > 0$,

$$|f| \leq n\Psi \leq \lambda\Psi \text{ in } \widetilde{C(X)} \iff |Tf| \wedge nT^{**}\Psi \leq \lambda T^{**}\Psi \text{ in } \widetilde{C(\upsilon X)}.$$

Therefore $\rho_{\Psi}(f) = \rho_{T^{**}\Psi}(Tf)$, for all $f \in C(X)$. □

Using Theorem 3.7.12, and the above Lemma together with the previous discussion we obtain the following theorem.

Theorem 3.7.14. *Let* X *be Tychonoff. Then the following statements are equivalent.*

i. X *is realcompact*

ii. $T : C_{co}(X) \ni f \longmapsto \tilde{f} \in C_{co}(\upsilon X)$ *is a homeomorphism.*

iii. $C_{sou}(X) = C_{co}(X)$.

proof of (i) $\implies$ *(iii).* This follows immediately from Theorem 3.7.12. □

proof of (iii) $\implies$ *(ii).* Suppose $C_{sou}(X) = C_{co}(X)$. Using the previous discussion we have that $C_{sou}(X)$ and $C_{sou}(\upsilon X)$ are homeomorphic. Also using Theorem 3.7.12, υX is realcompact so $C_{sou}(\upsilon X) = C_{co}(\upsilon X)$. Together with the assumption we have that

$$C_{co}(X) = C_{sou}(X) \cong C_{sou}(\upsilon X), = C_{co}(\upsilon X).$$

Hence $C_{co}(X)$ and $C_{co}(\upsilon X)$ are homeomorphic. □

Proof of (i) $\iff$ *(ii).* This is Theorem 3.2.2. This completes the proof. □

Appendix A

Here we collect together some useful results which are used in the thesis or helps us to recap some concepts but do not belong to any scope of the essay.

A.1 Topology

Definition A.1.1. A topological space X is completely regular if for any closed set $B \subseteq X$ and every $x_0 \notin B$ then there exist a continuous function $f : X \longrightarrow \mathbb{R}$ such that $f(x_0) = 1$ and $f[B] = \{0\}$.

Definition A.1.2. A topological space X is a Tychonoff space if X is a completely regular Hausdorff space.

Theorem A.1.3. *Let V be a topological space, $B \subseteq V$ and $f : V \longrightarrow \mathbb{R}$ be a continuous function. Then*

$$f[\overline{B}] \subseteq \overline{f[B]}.$$

Proof. Let $y \in f[\overline{B}]$. Then there exists $x \in \overline{B}$ such that $f(x) = y$. Since $x \in \overline{B}$, there exists a net $(x_\alpha)_{\alpha \in I}$ such that $x_\alpha \longrightarrow x$. Now $(f(x_\alpha))_{\alpha \in I}$ is a net in $f[B]$. Since f is continuous, $f(x_\alpha) \longrightarrow f(x)$. Hence $y \in \overline{f[B]}$. Thus $f[\overline{B}] \subseteq \overline{f[B]}$. □

Definition A.1.4. Let V be a topological vector space. A set B is bounded in B if for every neighbourhood U of 0, there exists a $\lambda > 0$ such that $B \subset \lambda U$. Equivalently, B is topologically bounded if for all $U \in \mathcal{N}_0$, there exists a $\lambda > 0$ such that

$$\frac{1}{\lambda} B \subseteq U.$$

Definition A.1.5. Let V be a topological space and $\phi : V \longrightarrow \mathbb{R}$ a linear functional. Then ϕ is topologically bounded if for every $B \subseteq V$ bounded with respect to a topology on V, $\phi(B)$ is bounded in $\mathbb{R}$.

We denote by V' a space of continuous functionals on the topological space V

Theorem A.1.6. *Let $\phi \in V'$. Then ϕ is topologically bounded.*

Proof. Fix $B \subseteq V$ bounded. Since ϕ is continuous, with $\varepsilon = 1$, there exists a $U \in \mathcal{N}_0$, such that if $f \in U$ then $|\phi(f)| = |\phi(f) - \phi(\mathbf{0})| < 1$. Since $U \in \mathcal{N}_0$, there exists a $\lambda > 0$ such that $B \subseteq \lambda U$. Now if $f \in B$, then $|\phi(f)| \leq \lambda$. Hence ϕ is topologically bounded. $\square$

Theorem A.1.7. *Let X be realcompact and $\phi \in C_{co}(X)'$. Then there exists a compact set $K \subseteq X$ and $\alpha > 0$ such that $|\phi(f)| \leq \alpha||f||_K$, for all $f \in C(X)$*

Proof. Fix $\phi \in C(X)^{\sim}_{+}$. Then there exist a regular borel measure μ_ϕ such that

$$\int_X f d\mu_\phi = \phi(f).$$

Let $K = \text{spt}\mu_\phi$. Then $K \subseteq X$ since X is realcompact. Now if $f \geq 0$ Then

$$|\phi(f)| = \phi(f) = \int_X f d\mu_\phi \leq \int_X ||f||_K d\mu_\phi = ||f||_K \mu_\phi(K).$$

Let $\alpha = \mu_\phi(K)$ thus we get the result. $\square$

Definition A.1.8. Two semi-norms $||.||$ and $||.||_0$ on E are equivalent if there exists positive real numbers α and β such that

$$\alpha||f|| \leq ||f||_0 \leq \beta||f||, \;\; f \in E.$$

Lemma A.1.9. *Let V be a vector space and $\mathcal{P}$ be a collection of all semi-norms on V. If the subbase elements of the topology of X are of the form $\{x \in V : \rho(x - x_0) < \varepsilon\}$ for ε, $x_0 \in X$ and $\rho \in \mathcal{P}$. Then a set U is open if and only if for each $x \in U$, there exists $\rho_1, ..., \rho_n$ and $\varepsilon_1, ..., \varepsilon_n$ such that*

$$\bigcap_{k=1}^{n} \{x \in V : \rho_k(x - x_0) < \varepsilon_k\} \subseteq U$$

Theorem A.1.10 (Tietze Extension Theorem)**.** *Let V be a normal space and $B \subseteq V$. Suppose $f : B \longrightarrow \mathbb{R}$ is continuous. Then there exists $\tilde{f} : V \longrightarrow \mathbb{R}$ a continuous extension of f.*

Theorem A.1.11 (Dini's Theorem)**.** *Let V be a compact topological space and $(f_n)_{n\in\mathbb{N}}$ be a monotonic increasing sequence of continuous function on V which convergences pointwise to a continuous function $f : V \longrightarrow \mathbb{R}$. Then $f_n \longrightarrow f$ uniformly.*

A.2 Ordered vector spaces

Definition A.2.1. Let E be a set. A relation $\leq$ on E is called a partial order if:

i. $x \leq x$ for every $x \in E$ ($\leq$ is reflexive).

ii. $x \leq y$ and $y \leq z$ implies $x \leq z$ ($\leq$ is transitive).

iii. $x \leq y$ and $y \leq x$ implies $x = y$ ($\leq$ is anti-symmetric).

The set E equipped with partial ordering is called a partially ordered set.

Definition A.2.2. Let E be a non-empty partially ordered set. If $x \vee y$ and $x \wedge y$ exist for every $x, y \in E$ we say E is a lattice.

Definition A.2.3. Let $(E,\leq)$ be a vector space equipped with partial ordering such that the following properties hold for all x, y, $z \in E$ and $\lambda \geq 0$.

i. $x \leq y$ implies $x + z \leq y + z$.

ii. $x \leq y$ implies $\lambda x \leq \lambda y$.

Then $(E, \leq)$ is called an ordered vector space.

Definition A.2.4. Let $(E, \leq)$ be an ordered vector space. Then

$$E^+ = \{x \in E : 0 \leq x\}$$

is called the positive cone of E and the elements of E^+ are called the positive elements of E.

Definition A.2.5. A Riesz space is an ordered vector space which is lattice with respect to its partial order and we will denote it by E.

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