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Constructing some discrete 4-D hyperchaotic systems

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Abstract: Modeling real life phenomena often leads to complex nonlinear dynamics such as bifurcation and chaos. The study of such problems has attracted interest of many scientists over the past decades. In this paper, we present a method for constructing some discrete four dimensional (4-D) hyperchaotic systems. A nonclassical procedure for discretising autonomous 4-D continuous hyperchaotic systems is applied; a parameter is introduced in this process. By adjusting this parameter, until we obtain exactly two equal-positive Lyapunov exponents, a new discrete 4-D hyperchaotic system is realised. We prove that these discrete systems are bounded-input bounded-output (BIBO) stable. Our illustrative results show that the constructed discrete systems and their continuous counterparts have similar phase portraits.

Keywords: Chaos, Difference equations, Hyperchaos, Lyapunov exponents.
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1 Introduction

Ever since the introduction of supercomputers, mathematical modeling of complex physical phenomena has surged to unimaginable proportions. As a result of these developments, the science realm has seen the appearance of many complicated models which come very close to fully describing the dynamics of complex systems; these models often arise from research areas such as biomathematics, engineering, physics and numerous other fields. The analyses of these systems gets much more difficult when the dynamics exhibit bifurcation, chaos and hyperchaos phenomena; see for instance [19]. When such challenges are realised, the dynamic behaviour of the system may become sensitive to initial values; a slight change in the initial values might result in a completely different dynamic behaviour, which makes the system hard to control.

The most common category of mathematical models is based on differential equations; it is known as continuous-time models. Another category is known as the discrete-time models; it is governed by difference equations.
Our main objective is to construct some discrete 4-D hyperchaotic models. These are systems that have at least two positive Lyapunov exponents; see for instance [9, 18, 20]. Discrete models are very important because they often exhibit more complex dynamic behaviours than continuous models; see for instance [10, 11, 12, 14, 22]. In recent years, we have seen an increase in the use of numerical methods, such as the Euler and the non-standard finite difference (NSFD) methods, to construct discrete models from continuous ones; see for instance [2, 3, 4, 6, 7]. In this paper, we do not explicitly use any known numerical method; instead, we apply a nonclassical discretisation technique to a discrete Euler type discrete model; this is explained in the next section.
The rest of this paper is organised as follows, Section 2 is dedicated to the construction of discrete models, numerical experiments are in Section 3 and the conclusion is in Section 4.

2 The method

In order to do justice to the nonstandard finite difference (NSFD) method and to emphasise the fact that we are not applying NSFD in this paper, readers are given a brief reminder of the main distinguishing features of the NSFD method. This is done because our construction technique uses a trick that is almost similar to the NSFD approach.

2.1 NSFD method in brief

Consider the following initial-value problem, which is given by the system of first-order ordinary differential equations:

$$\frac{dX(t)}{dt} = F(X(t)), \;\; X(t_0) = X_0, \;\; t \in [t_0, T] \tag{1}$$

where

$$X = X(t) = (x_1(t), x_2(t), \cdots, x_n(t)) \;\text{ and }\; F(X) = (f_1(X), f_2(X), \cdots, f_n(X)) \tag{2}$$

Definition 1 *Let*

$$t_i = t_0 + ih \; (i = 0, 1, 2, \cdots, N), \;\; h = \frac{T - t_0}{N}, \;\; X^i \approx X(t_i). \tag{3}$$

A finite difference method for approximating the system in (1) is called an NSFD scheme if at least one of the following implementations is applied; see for instance [1, 2, 15].

1. The first order derivative is discretized as follows:

$$\frac{dX}{dt} \longrightarrow \frac{X^{i+1} - X^i}{\phi(q, h)} \tag{4}$$

where

$$q > 0, \;\; \phi(q, h) = h + \mathcal{O}(h^2), \;\; h \to 0. \tag{5}$$

2. The nonlinear terms and some linear terms in $F(X)$ are approximated in a nonlocal manner; for instance:

$$x_k^2(t_i) \longrightarrow x_k^{i+1}x_k^i \text{ and } x_k(t_i) \longrightarrow 2x_k^i - x_k^{i+1}, \text{ etc.} \tag{6}$$

2.2 The method

An Euler based system of difference equations corresponding to the system in (1) would be in the form:

$$X^{i+1} - X^i = F(X^i) \tag{7}$$

and in explicit form system (7) may be expressed as follows:

$$X^{i+1} = X^i + F(X^i). \tag{8}$$

We discretise the first term X^i that is on the right hand side of Eq. (8) in a nonlocal way as follows:

$$X^i \longrightarrow \omega X^i - \left[(\varphi - 1)X^{i+1}\right], \ \varphi >> 1; \tag{9}$$

see Definition 1 (ii) above. Upon substitution of Eq. (9) into (8), we obtain

$$X^{i+1} = \varphi X^i - \left[(\varphi - 1)X^{i+1}\right] + F(X^i). \tag{10}$$

After rearrangement of the system in (10), the following system is realised

$$X^{i+1} = \frac{\varphi X^i + F(X^i)}{\varphi}. \tag{11}$$

Remark 2 *The value of φ is not unique; however, it has to be much greater than one in order to generate hyperchaos. It can always be adjusted, tuned up accordingly, to suit parameters of different systems. This is addressed in the next subsection.*

2.3 Finding appropriate value of φ

For the purpose of this study, an appropriate value of φ is defined as the value corresponding to two equal-positive Lyapunov exponents. We introduce the method in [18] that will be used to calculate Lyapunov exponents of the discrete system in (11). Lyapunov exponents are numerical values that are used to determine chaotic behaviour of attractors; see for instance [9] and [18]. Non-chaotic attractors have only non-positive Lyapunov exponents; whereas chaotic attractors have at least one positive Lyapunov exponent. Let

$$x^i = (x_1^i, x_2^i, x_3^i, x_4^i) \text{ and } G = J(x^0)J(x^1)\cdots J(x^M), \tag{12}$$

H - be the right hand side of Eq. (11) at time t given by

$$H(X) = \frac{\varphi X + F(X)}{\varphi}, \tag{13}$$

and J - be defined by the Jacobian matrix

$$J_{kj} = \left(\frac{\partial H_k}{\partial x_j}\right) \; (j = 1, 2, 3, 4) \text{ and } (k = 1, 2, 3, 4). \tag{14}$$

If $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are the eigenvalues of the matrix G in Eq. (12), then the Lyapunov exponents (LE) of the system in (11) are as follows:

$$LE_j = \frac{1}{M} \ln |\lambda_j| \,, \; (j = 1, 2, 3, 4). \tag{15}$$

2.4 Stability analysis

Definition 3 *[8, pp 138]*
A system is external or BIBO-stable if for any bounded input $u(t)$ it responds with a bounded output $y(t)$, meaning that

$$\{\|u(t)\| \leq M_1 < \infty | 0 < t < \infty\} \Rightarrow \{\|y(t)\| \leq M_2 < \infty | 0 < t < \infty\}$$

where $\|\cdot\|$ is the Euclidean norm.

Theorem 4 *If φ is large enough, then the system in (11) is BIBO stable.*

Proof. Let

$$\epsilon_i = \lim_{\varphi \to \infty} \frac{F(X^i)}{\varphi} \quad \text{and} \quad \delta = \max_{\forall \, i} |\epsilon_i|.$$

Taking the limit of Eq. (11) as φ approaches infinity, we get

$$\begin{cases} \lim\limits_{\varphi \to \infty} \sup X^{i+1} = X^i + \epsilon_i \\ \qquad\qquad\qquad \leq X^i + \delta. \end{cases} \tag{16}$$

Thus, the proof is complete. ■

3 Numerical experiments

In order to validate discrete system (11), we devote this section to numerical simulations of various hyperchaotic systems. All simulations are done in MATLAB R2014a; $N = 40000$ and $M = 600$ as defined in equations (3) and (15), respectively. In all examples, the state variables are x, y, z and w while a, b, c, d and k are the parameters. [17, 23] The following hyperchaotic system is considered

$$\begin{cases} x' = -y - z - aw, \\ y' = x, \\ z' = b(1 - y^2) - cz, \\ w' = dx. \end{cases} \tag{17}$$

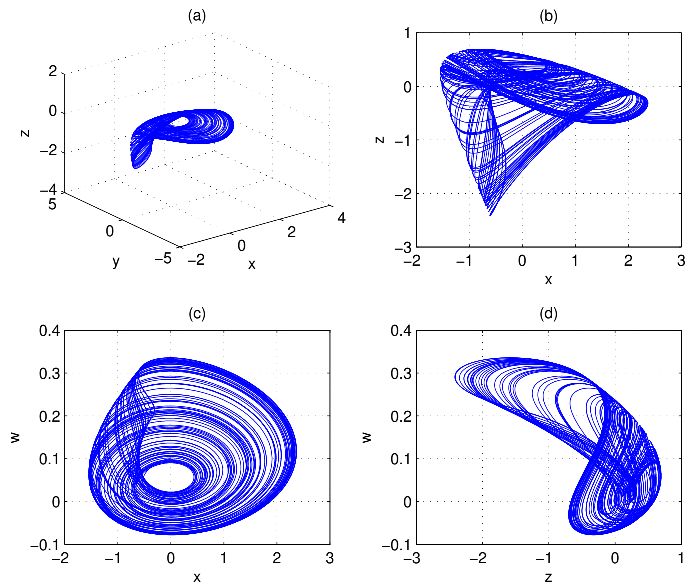

Figure 1: This is the phase portrait obtained from Experiment 3 with parameters $(a, b, c, d) = (1.378, 0.5, 0.6, 0.097)$, $\varphi = 65$ and initial values $X(0) = (0, 0, 0, 0.1)$. (a) x-y-z plane (b) x-z plane (c) x-w plane (d) z-w plane.

Applying the discretisation in Eq. (10) to the system in (17) and rearranging, we obtain

$$\begin{cases} x^{i+1} = \dfrac{\varphi x^i - y^i - z^i - a w^i}{\varphi}, \\ y^{i+1} = \dfrac{\varphi y^i + x^i}{\varphi}, \\ z^{i+1} = \dfrac{\varphi z^i + b(1 - (y^i)^2) - c z^i}{\varphi}, \\ w^{i+1} = \dfrac{\varphi w^i + d x^i}{\varphi}. \end{cases} \tag{18}$$

Using the initial values $X(0) = (0, 0, 0, 0.1)$, the system in (18) produces $\varphi = 65$ and the following Lyapunov exponents:

$$LE_1 = 0.00209,\ LE_2 = 0.00209,\ LE_3 = -0.01319,\ LE_4 = 0. \tag{19}$$

For parameter values $(a, b, c, d) = (1.378, 0.5, 0.6, 0.097)$ and $X(0) = (0, 0, 0, 0.1)$, we obtain the hyperchaotic attractors shown in Fig. 1. Different plane combinations are plotted in sub-figures 1:(a)-(d).

The following hyperchaotic system is considered [13, 23]

$$\begin{cases} x' = a(y - x) + yz, \\ y' = cx - y - xz + w, \\ z' = xy - bz, \\ w' = -xz + dw. \end{cases} \tag{20}$$

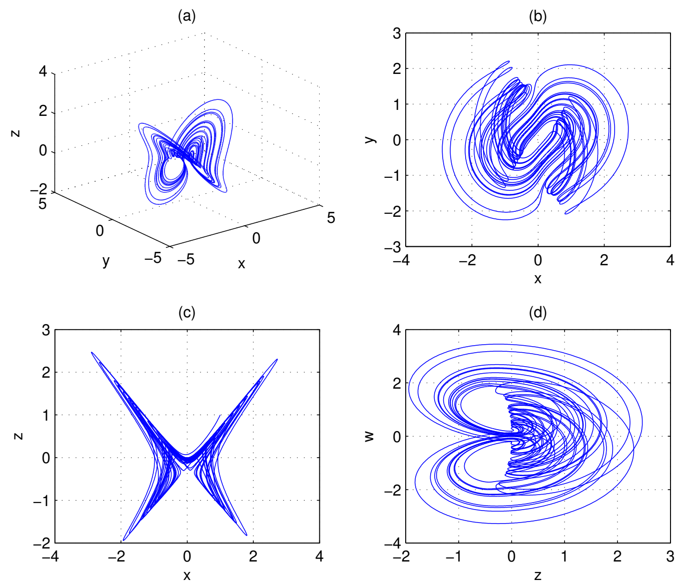

Figure 2: This phase portrait is obtained from Experiment 3 with parameters $(a, b, c, d) = (0.3, 0.5, 0.8, 0.063)$, $\varphi = 85$ and initial values $X(0) = (1, 0, 1, 0)$. (a) x-y-z plane (b) x-y plane (c) x-z plane (d) z-w plane.

Similarly, discretising the system in (20) and rearranging, we obtain

$$\begin{cases} x^{i+1} = \dfrac{\varphi x^i + a(y^i - x^i) + y^i z^i}{\varphi}, \\ y^{i+1} = \dfrac{\varphi y^i + cx^i - y^i - x^i z^i + w^i}{\varphi}, \\ z^{i+1} = \dfrac{\varphi z^i + x^i y^i - bz^i}{\varphi}, \\ w^{i+1} = \dfrac{\varphi w^i - x^i z^i + dw^i}{\varphi}. \end{cases} \tag{21}$$

Using the initial values $X(0) = (1, 0, 1, 0)$, the system in (21) produces $\varphi = 85$ and the following Lyapunov exponents:

$$LE_1 = 0.00392,\ LE_2 = 0.00392,\ LE_3 = -0.01063,\ LE_4 = -0.01063. \tag{22}$$

In this Experiment, we use parameter values $(a, b, c, d) = (0.3, 0.5, 0.8, 0.063)$, $\varphi = 85$ and $X(0) = (1, 0, 1, 0)$; the phase portraits shown in Fig. 2 are obtained. Different plane combinations are plotted in sub-figures 2:(a)-(d).

The following hyperchaotic system is considered [21, 23]

$$\begin{cases} x' = ax + dz - yz, \\ y' = xz - by, \\ z' = c(x - z) + xy, \\ w' = c(y - w) + xz. \end{cases} \tag{23}$$

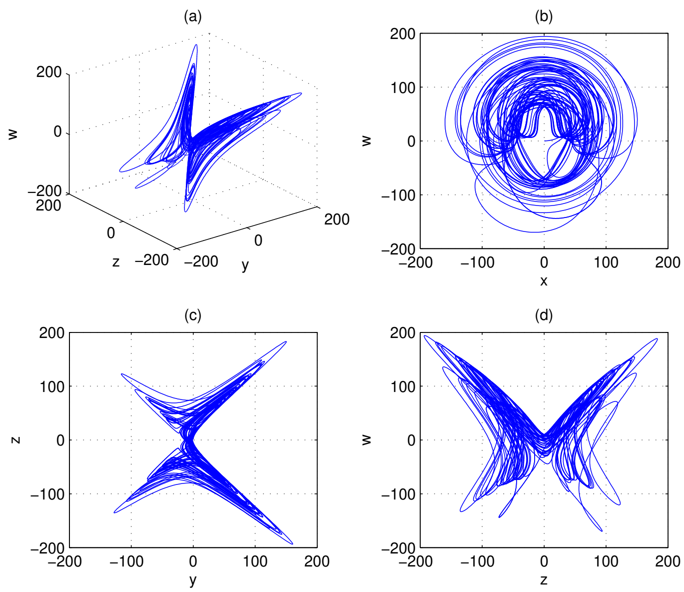

Figure 3: This phase portrait is obtained from Experiment 3 with parameters $(a, b, c, d) = (16, 40, 20, 8)$, $\varphi = 2980$ and initial values $X(0) = (1, 0, 1, 0)$. (a) y-z-w plane (b) x-w plane (c) y-z plane (d) z-w plane.

Similarly, discretising the system in (23) and rearranging, we obtain

$$\begin{cases} x^{i+1} = \dfrac{\varphi x^i + a x^i + d z^i - y^i z^i}{\varphi}, \\ y^{i+1} = \dfrac{\varphi y^i + x^i z^i - b y^i}{\varphi}, \\ z^{i+1} = \dfrac{\varphi z^i + c(x^i - z^i) + x^i y^i}{\varphi}, \\ w^{i+1} = \dfrac{\varphi w^i + c(y^i - w^i) + x^i z^i}{\varphi}. \end{cases} \tag{24}$$

For this Experiment, our parameters are much bigger compared to the rest and work better with big initial values when calculating Lyapunov exponents. Using the initial values $X(0) = (100, 110, 100, 100)$, the system in (24) produces $\varphi = 2980$ and the following Lyapunov exponents:

$$LE_1 = 0.00220,\ LE_2 = 0.00220,\ LE_3 = -0.00673,\ LE_4 = -0.01888. \tag{25}$$

In this case, parameters used are $(a, b, c, d) = (16, 40, 20, 8)$, $\varphi = 2980$ and the initial values are $X(0) = (1, 0, 1, 0)$. We obtained the phase portraits shown in Fig. 3. Different plane combinations are plotted in sub-figures 3:(a)-(d). The following hyperchaotic system is considered [16, 23]

$$\begin{cases} x' = a(y - x), \\ y' = (d - z)x + cy - w, \\ z' = xy - bz, \\ w' = x + k. \end{cases} \tag{26}$$

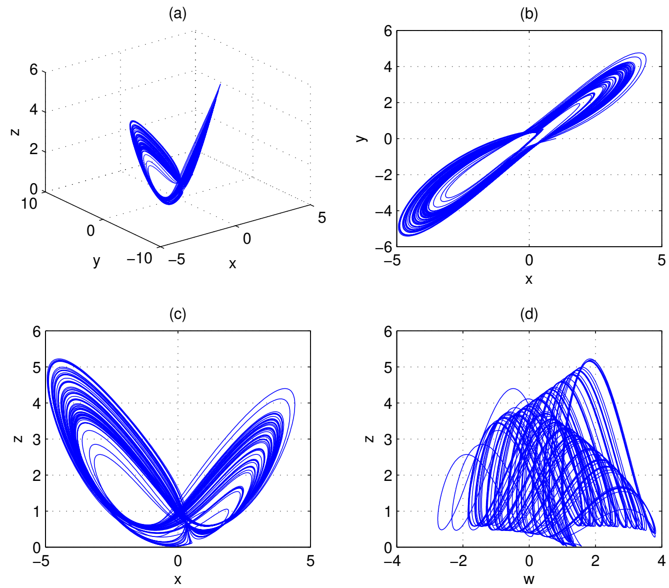

Figure 4: This phase portrait is obtained from Experiment 3 with parameters $(a, b, c, d, k) = (8, 3, 4, -2, 0.2)$, $\varphi = 145$ and the initial values are $X(0) = (1, 0, 1, 0)$. (a) x-y-z plane (b) x-y plane (c) x-z plane (d) w-z plane.

Finally, discretising the system in (26) and rearranging, we obtain

$$\begin{cases} x^{i+1} = \dfrac{\varphi x^i + a(y^i - x^i)}{\varphi}, \\ y^{i+1} = \dfrac{\varphi y^i + (d - z^i)x^i + cy^i - w^i}{\varphi}, \\ z^{i+1} = \dfrac{\varphi z^i + x^i y^i - bz^i}{\varphi}, \\ w^{i+1} = \dfrac{\varphi w^i + x^i + k}{\varphi}. \end{cases} \tag{27}$$

Using the initial values $X(0) = (1, 0, 1, 0)$, the system in (27) produces $\varphi = 145$ and the following Lyapunov exponents:

$$LE_1 = 0.01287,\ LE_2 = 0.01287,\ LE_3 = -0.00211,\ LE_4 = -0.02276. \tag{28}$$

The parameters used are $(a, b, c, d, k) = (8, 3, 4, -2, 0.2)$, $\varphi = 145$ and the initial values are $X(0) = (1, 0, 1, 0)$. We obtained the phase portraits shown in Fig. 4. Again, different plane combinations are plotted in sub-figures 4:(a)-(d).

4 Conclusion

In this paper, a simple method for constructing discrete 4-D hyperchaotic systems is developed. Four hyperchaotic systems have been constructed. Numerical simulation results that are obtained from the constructed discrete hyperchaotic systems are consistent with theoretical results and practical circuit implementation of the considered continuous hyperchaotic systems.

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