VANISHING ELEMENTS AND SYLOW SUBGROUPS

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Abstract. Let G be a finite group and $p > 3$ be a prime. A vanishing conjugacy class of G is a conjugacy class of G which consists of vanishing elements of G. A p-singular vanishing conjugacy class is a vanishing conjugacy class with elements of order divisible by p. We investigate the structure of a Sylow p-subgroup P of a group G with exactly one p-singular vanishing conjugacy class. In particular, we show that P' is a subnormal subgroup of G. We also prove that $|P/\mathbf{O}_p(G)| \leqslant p$ or G has a composition factor isomorphic to $\mathrm{PSL}_2(q)$, where $q = p^f$, $f \geqslant 2$.

1. Introduction

Let G be a finite group, p be a prime and $\mathrm{Irr}(G)$ be the set of irreducible characters of G. A classical theorem of Burnside [11, Theorem 3.15] states that if $\chi \in \mathrm{Irr}(G)$ is such that $\chi(1) > 1$, then $\chi(g) = 0$ for some $g \in G$. It is also well known that if a p-element g of G is such that $\chi(g) = 0$ for some $\chi \in \mathrm{Irr}(G)$, then p divides $\chi(1)$. This provides some relationship between vanishing elements of G whose orders are divisible by p and irreducible characters of G whose degrees are divisible by p. Since character degrees of finite groups have been widely studied, it is interesting to then study corresponding problems on vanishing elements.

The famous Ito-Michler theorem states that p does not divide every irreducible character degree of G if and only if G has a normal abelian Sylow p-subgroup. A variation of the Ito-Michler theorem was studied by Goldstein et al. [9] where they classified groups G with exactly one irreducible character of degree divisible by p. They proved [9, Corollary] that a Sylow p-subgroup of G is almost normal by showing that the normaliser $\mathbf{N}_G(P)$, of a Sylow p-subgroup P of G is either G or a maximal subgroup of G.

Analogous to the Ito-Michler theorem is a corresponding result on vanishing elements. Dolfi et al. [7, Theorem C] proved that if G has no vanishing elements of order divisible by p, then G has a normal Sylow p-subgroup P and either G is abelian or $G/\mathbf{O}_{p'}(G)$ is a Frobenius group with the kernel $P\mathbf{O}_{p'}(G)/\mathbf{O}_{p'}(G)$ and $\mathbf{O}_{p'}(G)$ is nilpotent. Recall that a vanishing conjugacy class of G is a conjugacy class of G which consists of vanishing elements of G. A p-singular vanishing conjugacy class is a vanishing conjugacy class with elements of order divisible by p. In this article we shall study an analogue of a problem studied by Goldstein et al. [9]:

Problem 1. Let G be a finite group and p be a prime. Describe the Sylow p-subgroups of G with exactly one p-singular vanishing conjugacy class.

One wonders if P is also almost normal in the sense of Goldstein et al. [9, Corollary]. A quick look at a Frobenius group G with an abelian kernel of odd composite order

Date: February 1, 2021.

2010 *Mathematics Subject Classification.* Primary 20C15; 20D20.

Key words and phrases. vanishing elements, p-singular elements, Sylow p-subgroups.

and a Frobenius complement shows that this is not true. Indeed, G has exactly one vanishing conjugacy class of order 2 but the normalizer of a Sylow 2-subgroup is not maximal in G. In fact, the normalizer is a minimal subgroup of G. A Sylow p-subgroup P may be almost normal in another sense, which is that the largest normal p-subgroup $\mathbf{O}_p(G)$ of G is either P or a maximal subgroup of P. With this in mind, the result of Goldstein et al. [9, Theorem] implies the following corollary:

Corollary 1.1. *Let p be a prime and P be a Sylow p-subgroup of a finite group G. Let G have exactly one irreducible character whose degree is divisible by a prime p. Then P' is subnormal. Suppose that one of the following holds:*

(a) *G is p-solvable;*
(b) *G has no composition factor isomorphic to $\mathrm{PSL}_2(q)$, $q = p^f$, $f \geqslant 2$ or M_{11}.*

Then $|P/\boldsymbol{O}_p(G)| \leqslant p$.

The case when $G \cong \mathrm{PSL}_2(q)$, $q = p^f$, $f \geqslant 2$ is an exception since G has one irreducible character of degree divisible by p, the Steinberg character, but $|P| = p^f$. The other exception M_{11} is when $p = 3$.

Solvable groups in Problem 1 have Sylow p-subgroups P that are almost normal in the sense of Corollary 1.1, that is, $P/\mathbf{O}_p(G)$ is cyclic of prime order (see [15, Theorem B]). We extend this to general finite groups and obtain our first result below.

Theorem A. *Let p be a prime and P be a Sylow p-subgroup of a finite group G. Let G have exactly one p-singular vanishing conjugacy class. If $p > 3$, then P' is subnormal. Suppose that one of the following holds:*

(a) *$p \neq 3$ and G is p-solvable;*
(b) *$p > 3$ and G has no composition factor isomorphic to $\mathrm{PSL}_2(q)$, with $q = p^f$, $f \geqslant 2$.*

Then $|G/\boldsymbol{O}_p(G)| \leqslant p$.

We remark that since, using the classification of finite simple groups, 2-solvable groups are solvable, the result follows from [15, Theorem B] when $p = 2$. The case when $p = 3$ and G is 3-solvable depends on the following question:

Is it true that if a 3-element x of a 3-solvable group G is non-vanishing, then $x \in \mathbf{F}(G)$?

If this is true, then the condition in Theorem A(a) that $p \neq 3$ is not needed. Note that the question above is true if we replace 3 with any $p \geqslant 5$ by [5, Theorem A], even without the condition that G is p-solvable.

Since $p \in \{2, 3\}$ has been excluded in at least one of the conditions of Theorem A, we have the following result:

Theorem B. *Let $p \in \{2, 3\}$ and P be a Sylow p-subgroup of a finite group G. Suppose that G has exactly one p-singular vanishing conjugacy class. Assume that:*

(a) *G is not p-solvable for $p = 3$, and*
(b) *G has no composition factor isomorphic to $\mathrm{PSL}_2(q)$ with $q = p^f$, $f \geqslant 2$.*

Then the following holds:

(i) *If $p = 2$, then $|P/\boldsymbol{O}_p(G)| \leqslant p^3$.*
(ii) *If $p = 3$, then $|P/\boldsymbol{O}_p(G)| \leqslant p^2$.*

The examples of the extremal bounds come from A_7. Note that in Theorem A, P' is subnormal when $p > 3$. It would be interesting to know that if G has n p-singular vanishing conjugacy classes for some prime $p > 3$, is there a function $f(n)$ such that $P^{f(n)}$ is subnormal in G, where P^a is the ath derived group of P?

Dolfi et al. [6] showed that if all the p-elements of G are non-vanishing, then G has a normal Sylow p-subgroup. This was generalised in [8] when they proved that N is normal subgroup of G whose p-elements are all non-vanishing in G, then N has a normal Sylow p-subgroup. We consider solvable groups which has exactly one vanishing conjugacy class consisting of p-elements.

Theorem C. *Let p be an odd prime and P be a Sylow p-subgroup. If G is a solvable group which has exactly one vanishing conjugacy class consisting of p-elements, then $|P/\boldsymbol{O}_p(G)| \leqslant p$.*

Our notation is standard is mainly taken from [11]. We shall recall some notation when it seems necessary.

2. Preliminaries

In this section, we collect some results that we need for the proofs of the theorems above. We start off with a sufficient condition for a subgroup to be Frobenius group. Recall that for a subset A of G, $k_G(A)$ denotes the number of conjugacy classes of G contained in A.

Lemma 2.1. *Let H and K be normal solvable subgroups of a finite group G such that $1 < K < H$. If $k_G(H\backslash K) = 1$ and $\gcd(|H{:}K|, |K|) = 1$, then H is a Frobenius group with kernel K and a complement of prime order.*

Proof. Since $k_G(H\backslash K) = 1$, we have that H/K is a chief factor of G. This means that H/K is an elementary abelian group of order q^t for some prime q. Since $\gcd(|H{:}K|, |K|) = 1$, it follows that $H\backslash K$ contains elements of the same order and hence are elements of order q. Let Q be a Sylow q-subgroup of H. Then $H = KQ$ and $Q \cong H/K$. Note that for any element $g \in Q\backslash\{1\}$, $\mathbf{C}_H(g) = Q$. Hence, H is a Frobenius group with an abelian complement Q. Therefore Q is a cyclic group of order q. □

The next lemma shows that every element of odd order of an almost simple group outside the simple group is a vanishing element.

Lemma 2.2. *Suppose that $S \trianglelefteq G \leqslant \mathrm{Aut}(S)$, where S is a nonabelian simple group. Let $x \in G$ be an element of odd order. If $x \in G\backslash S$, then is a vanishing element of G.*

Proof. Using [5, Lemma 2.1] and [5, Theorem 2.3], if x is non-vanishing, then $x \in S$. Hence the result follows. □

Theorem 2.3. [5, Theorem A] *Let G be a finite group and x be an element whose order is coprime to 6. If x is a non-vanishing element of G, then $x \in \mathbf{F}(G)$.*

The existence of p-defect zero characters is guaranteed in finite simple groups G for all primes $p \geqslant 5$ dividing $|G|$.

Lemma 2.4. [10, Corollary 2.2] *Let G be a non-abelian finite simple group and p be a prime. If G is a finite group of Lie type, or if $p \geqslant 5$, then there exists $\chi \in \mathrm{Irr}(G)$ of p-defect zero.*

We have the following result which gives a sufficient condition for a finite group to have p-singular vanishing elements when a normal subgroup has an irreducible character of p-defect zero.

Lemma 2.5. [2, Lemma 2.2] *Let G be a finite group, N a normal subgroup of G and p a prime. If N has an irreducible character of p-defect zero, then every p-singular element of N is a vanishing element in G.*

3. p-solvable groups

In this section we prove Theorem C and part of Theorem A.

Proof of Theorem C. Let $\mathcal{C}$ the conjugacy class. We may assume that P is not normal and so $\mathcal{C} \subsetneq \mathbf{O}_p(G)$. Consider $G/\mathbf{O}_p(G) = \overline{G}$. If $\overline{N} = N/\mathbf{O}_p(G) = \mathbf{O}_p(\overline{G}) \neq 1$, then $\mathbf{O}_p(G) < N$, a normal p-subgroup of G, a contradiction.

Let $\mathbf{O}_p(G) \leqslant H, K$ be normal subgroups of G such that H/K is a chief factor of G and $\mathcal{C} \subseteq H \backslash K$. Hence $\overline{H}/\overline{K}$ is a chief factor of $\overline{G}$ and $\mathcal{C} \subseteq \overline{H} \backslash \overline{K}$. Since $k_{\overline{G}}(\overline{H} \backslash \overline{K}) = 1$ and $\gcd(|\overline{H}/\overline{K}|, |\overline{K}|) = 1$, we have that $\overline{H}$ is a Frobenius group with kernel $\overline{K}$ and a Frobenius complement of order p by Lemma 2.1. The result then follows. □

Theorem 3.1. *Let $p \neq 3$ be a prime and P be a Sylow p-subgroup of a finite group G. Suppose that G has exactly one p-singular vanishing conjugacy class. If G is p-solvable, then $|P/\mathbf{O}_p(G)| \leqslant p$.*

Proof. Let $\mathcal{C}$ be the p-singular vanishing conjugacy class of G. If $\mathcal{C}$ does not consists of p-elements, then G is a normal Sylow p-group using [6, Theorem A]. We may assume that $\mathcal{C}$ contains p-elements.

If G is solvable, the result follows by [15, Theorem B]. Suppose that G is non-solvable. If $p = 2$, then G is solvable, so we may assume that $p \geqslant 5$. Since the vanishing prime graph of G is disconnected, by [7, Theorem B], there exist normal subgroup M and N such that N is the solvable radical of G, M/N is simple and G/N is almost simple group. Moreover, since G is p-solvable, either $\mathcal{C} \subseteq N$ or $\mathcal{C} \subseteq G \backslash M$.

Suppose that $\mathcal{C} \subseteq N$. By Lemma 2.2, $p \nmid |G{:}M|$ and so $p \nmid |G{:}N|$. Note that all the p-elements outside $\mathbf{O}_p(G)$ are vanishing elements contained in $\mathcal{C}$ by Theorem 2.3. There exists normal subgroups H, K of G such that $\mathcal{C} = H \backslash K$, H/K is a p-group and $\mathbf{O}_p(G) \leqslant K$. If $K = \mathbf{O}_p(G)$, then H is the Sylow p-subgroup of G and the result follows. We may assume that $\mathbf{O}_p(G) < K$. Then $k_{\overline{G}}(\overline{H} \backslash \overline{K}) = 1$ and $\gcd(|\overline{H}{:}\overline{K}|, |\overline{K}|) = 1$ where $\overline{S} = S/\mathbf{O}_p(G)$. Using Lemma 2.1, $H/\mathbf{O}_p(G)$ is a Frobenius group with kernel $K/\mathbf{O}_p(G)$ and H/K is of order p. The result then follows.

Suppose that $\mathcal{C} \subseteq G \backslash M$. We note that G/M is solvable. Since all the p-elements in N are non-vanishing, we have that N has a normal Sylow p-subgroup by [8, Theorem A]. It is sufficient to show that a Sylow p-subgroup of G/M is cyclic. Let us consider $\mathrm{Out}(M/N)$. Since G is p-solvable, $p \nmid |M/N|$. Let M/N be either a sporadic simple group, an alternating group A_n, $n \geqslant 5$, $n \neq 6$ or the Tits group ${}^2\mathrm{F}_4(2)'$. Then $|\mathrm{Out}(M/N)| = \{1, 2\}$ and the result follows since $p \geqslant 5$. We may assume that M/N is a finite simple group of finite of Lie type. Then $|\mathrm{Out}(M/N)| = dfg$, where d is the order of diagonal automorphisms, f, field automorphisms and g, graph automorphisms with some exceptions (see [4, pages xv-xvi]). Since $p \nmid |M/N|$, $p \nmid d$. Also $p \nmid g$ and so $p \mid f$. Since the group of order f is cyclic, our result follows. □

4. Almost simple groups

In this section we show Theorems A and B hold for almost simple groups. We begin by looking at almost simple groups whose socle is a sporadic simple group.

Theorem 4.1. *Let G be a finite almost simple group whose socle is a sporadic simple group or the Tits' group ${}^2\mathrm{F}_4(2)'$. Let P be a Sylow p-subgroup of G for some prime p. If G has exactly one p-singular vanishing conjugacy class, then P is cyclic of prime order.*

Proof. Using the Atlas [4], we have that our result. □

We consider non-abelian simple groups with some properties that are sufficient for the corresponding almost simple to have exactly one p-singular vanishing conjugacy class but with $|P| \geqslant p^2$. A problem of a similar flavour was considered in [13, Theorem 1.2] and [14, Theorem 1.1].

Theorem 4.2. *Let M be non-abelian simple group which is either an alternating group or a finite simple group of Lie type with the exception of the Tits' group ${}^2\mathrm{F}_4(2)'$. Let P be a Sylow p-subgroup of M for some prime p. Suppose that the following holds:*

(a) *all the p-singular vanishing conjugacy classes of M consists of elements with the same p-power order;*
(b) *the number of p-singular vanishing conjugacy classes of M is at most the size of the outer automorphism group of the group M;*
(c) *P is not cyclic.*

Then one of the following holds:

(i) *$p = 2$ and $M \cong \mathrm{A}_7$;*
(ii) *$p = 3$ and $M \cong \mathrm{A}_7$;*
(iii) *$M \cong \mathrm{PSL}_2(q)$ where $q = p^f$, p a prime and $f \geqslant 2$ an integer.*

Proof. Suppose that M is an alternating group A_n, $7 \leqslant n \leqslant 10$. Then the result follows using the Atlas [4].

Assume that $n \geqslant 11$. First let $p \geqslant 5$. Let $p + 3 \leqslant n$. Then M has elements of order $3p$. Hence M has vanishing elements of order p and $3p$. Note that these elements are also vanishing elements of any almost simple group with socle M by Lemma 2.5. The result follows. We may assume that $M \cong \mathrm{A}_p$, A_{p+1} or A_{p+2}. Then a Sylow p-subgroup of M is cyclic and so M does not satisfy (c). If $p = 2$, then since $n \geqslant 11$, M has elements of 10 and 14. If $p = 3$, then M has elements of order 15 and 21. In all cases, these are vanishing elements of M which are also vanishing elements of any almost simple group with socle M by Lemma 2.5.

Suppose that M is a finite group of Lie type. Note that all the elements of M are vanishing elements of M which are also vanishing elements of any almost simple group with socle M by Lemma 2.5. Assume that $p = 2$. If M has elements of order $2r$, for some odd prime r, then the result follows since M has vanishing elements of orders 2 and $2r$. So we may assume that the centralizer of every involution contained is a 2-group. It follows from [17, III, Theorem 5] that M is isomorphic to one of the following groups: $\mathrm{PSL}_2(p)$, where p is a Fermat or Mersenne prime, $\mathrm{PSL}_3(4)$ or ${}^2\mathrm{B}_2(2^{2m+1})$, $m \geqslant 1$. In the case when $M \cong \mathrm{PSL}_2(p)$, where is Fermat or Mersenne prime, then its Sylow 2-subgroups are cyclic. If $M \cong \mathrm{PSL}_3(4)$ or $M \cong {}^2\mathrm{B}_2(2^{2m+1})$, then M has elements of order 2 and 4.

We may assume that $p \geqslant 3$. We first suppose that $M \cong \mathrm{PSL}_2(q)$, $q = r^f$, f is a positive integer. Then its structure and character table is known. If $p \neq r$, then a Sylow p-subgroup of M is cyclic. If $p = r$, then M has two conjugacy classes with elements of order p. These are the only p-singular elements. Since $|\mathrm{Out}(M)| \geqslant 2$, we have that

M satisfies properties (a) and (b) of the hypothesis. If $f \geqslant 2$, then a Sylow p-subgroup of M is not cyclic and the result follows.

Suppose that $M \cong \mathrm{PSL}_n(q)$ or $\mathrm{PSU}_n(q)$, $q = r^f$, with $n \geqslant 3$ and $q \geqslant 3$ or $n \geqslant 4$ and $q \geqslant 2$. If $p = r$, then the result follows by [18, Theorem 1]. We may assume that $p \neq r$. The orders of the maximal tori for M are well known. If p divides $q^{n_i} \pm 1$, where $n_i < n$, then there exists a torus T of M such that $\dfrac{q^{n_i} \pm 1}{\gcd(n, q \pm 1)}$ divides $|T|$ and $|T|$ is divisible two distinct primes. Hence the result follows since M contains the at least two p-singular elements of distinct orders. If p divides $q^n \pm 1$, then there exists a torus T, the Singer cycle, such that p divides $\dfrac{q^n \pm 1}{(n-1)\gcd(n, q \pm 1)} = |T|$. Since T is cyclic, property (c) is not satisfied.

Suppose that $M \in \{\mathrm{PSp}_4(q) \mid q \geq 4\} \cup \{\mathrm{PSp}_{2n}(q) \mid n \geq 3\} \cup \{\mathrm{PSO}_{2n+1}(q) \mid n \geq 3\} \cup \{\mathrm{PSO}^{\pm}_{2n}(q) \mid n \geq 4\}$. If $p = r$, then the result follows by [18, Theorem 1]. Assume that p divides $|T|$ for some maximal torus T of M. Now either $|T|$ divisible by two distinct primes or T is cyclic. In the first case M is has two p-singular elements of distinct orders and in the second case a Sylow p-subgroup is cyclic. In both cases property (a) or (c) is not satisfied.

Suppose that M is an exceptional finite group of Lie type. Consider $M \cong {}^2\mathrm{B}_2(8)$. The result follows using the Atlas [4]. Assume that $M \cong {}^2\mathrm{B}_2(q^2)$, $q^2 = 2^{2n+1}$ with n, a positive integer. Then all the odd Sylow subgroups of M are cyclic. Hence M does not satisfy property (c).

Suppose that $M \cong {}^2\mathrm{G}_2(q^2)$, $q^2 = 3^{2n+1}$ with n, a positive integer. Then all the Sylow l-subgroups for $l \neq 2, 3$ are cyclic. Hence Sylow p-subgroups of M are cyclic. For $p = 3$, we have that M has elements of order 6 and so does not satisfy (a).

Suppose that $M \cong {}^2\mathrm{F}_4(q^2)$, $q^2 > 2$. Then for every prime divisor l of $|{}^2\mathrm{F}_4(q^2)|$, there exists an l-singular element of ${}^2\mathrm{F}_4(q^2)$ whose order is divisible by two distinct primes with the exception of possibly prime divisors of $q^4 + \sqrt{2}q^3 + q^2 + \sqrt{2}q + 1$ and $q^4 - \sqrt{2}q^3 + q^2 - \sqrt{2}q + 1$. Now, for any prime divisor $l \neq 3$ of the two exceptions, Sylow l-subgroups are cyclic. In all cases, either M has two p-singular elements of distinct orders or Sylow p-subgroups of M are cyclic.

Suppose that $M \cong \mathrm{G}_2(q)$, $q > 2$. Then the maximal tori of M are well known. If p divides $q^2 - 1$, then there exist p-singular element whose order is divisible by two primes. So M has two p-singular elements of distinct orders. If p divides $q^2 \pm q + 1$, then Sylow p-subgroups of M are cyclic.

Suppose that $M \cong {}^3\mathrm{D}_4(q)$, $q \geqslant 2$. Considering the maximal tori of M, for every prime divisor l of $|M|$, either there exists an l-singular element whose order is divisible by two distinct primes or Sylow l-subgroups are cyclic.

Suppose that $M \cong \mathrm{F}_4(q)$, $q \geqslant 2$. Considering the maximal tori of M, we have that for every prime divisor l of $|M|$, there exists an l-singular element of M whose order is divisible by two distinct primes except possibly prime divisors of $q^4 - q^2 + 1$. In this exception case, the Sylow l-subgroups of M are cyclic.

Suppose that $M \cong \mathrm{E}_6(q)$, $q \geqslant 2$ or ${}^2\mathrm{E}_6(q)$, $q \geqslant 2$. The orders of maximal tori of M are known. We have that for every prime divisor l of $|M|$, there exists an l-singular element of M whose order is divisible by two distinct primes except possibly prime divisors of $q^6 \pm q^3 + 1$. In this case, the Sylow l-subgroups of M are cyclic.

Suppose that $M \cong \mathrm{E}_7(q)$, $q \geqslant 2$. If $q = 3$, then every prime divisor l of $|M|$, there exists an l-singular element of M whose order is divisible by two distinct primes except for 757 and 1093 which are primes. If $q \neq 3$, then every prime divisor l of $|M|$, there exists an l-singular element of M whose order is divisible by two distinct primes.

Suppose that $M \cong \mathrm{E}_8(q)$, $q \geqslant 2$. We have that for every prime divisor l of $|M|$, there exists an l-singular element of M whose order is divisible by two distinct primes except possibly prime divisors of $\dfrac{q^{10}-q^5+1}{q^2-q+1}$, $\dfrac{q^{10}+q^5+1}{q^2+q+1}$, q^8-q^4+1 and $\dfrac{q^{10}+1}{q^2+1}$. In these cases, the Sylow l-subgroups of M are cyclic. □

Theorem 4.3. *Let P be a Sylow p-subgroup of a finite almost simple group G for some prime p. Suppose that G has exactly one p-singular vanishing conjugacy class. If the socle of G is not isomorphic to* $\mathrm{PSL}_2(q)$*, where $q = p^f$, $f \geqslant 2$, then one of the following is true:*

(a) *P is cyclic;*
(b) *$G \cong \mathrm{A}_7$, $p = 2$ and $|P| = 2^3$;*
(c) *$G \cong \mathrm{A}_7$, $p = 3$ and $|P| = 3^2$.*

Proof. Let M be the socle of G. If M is a sporadic simple group or the Tits' group ${}^2\mathrm{F}_4(2)'$, then the result follows by Theorem 4.1. We may assume that that M is an alternating group or a finite simple group of Lie type with the exception of the Tits' group ${}^2\mathrm{F}_4(2)'$. Suppose that either the p-singular vanishing conjugacy classes of M consists of elements which do not have the same p-power order. Note that by the proof of Theorem 4.2, these are vanishing elements of G. Then G has p-singular vanishing elements of distinct orders, a contradiction. Suppose that the number of p-singular vanishing conjugacy classes of M is more than the size of the outer automorphism group of the group M. Then G has more than two p-singular vanishing conjugacy classes. Hence using Theorem 4.2 and its proof, if P is not cyclic we only need to consider groups G with socle M such that $M \cong \mathrm{A}_7$. Since S_7 has vanishing elements of order 2, 3 and 6, it follows that S_7 does not satisfy the hypothesis. If $G \cong \mathrm{A}_7$, then using the character table in the Atlas [4], we have the above cases. Hence the result follows. □

5. Proofs of Theorems A and B

Proof of Theorem A. If $p \neq 3$ and G is p-solvable, then $|P/\mathbf{O}_p(G)| \leqslant p$ by Theorem 3.1. We may assume that $p > 3$ and G has no composition factor isomorphic to $\mathrm{PSL}_2(q)$, where $q = p^f$, $f \geqslant 2$. We may also assume that G is not p-solvable. Note that the vanishing prime graph of G is disconnected. Using [7, Theorem B], there exists normal subgroups M and N of G such that M/N is a non-abelian simple group, G/N is an almost simple group and N is a solvable radical of G. Since G is not p-solvable, p divides $|M/N|$ and so M/N has p-singular vanishing elements by Lemma 2.5. This means that N has no vanishing elements which are p-singular. Using [8, Theorem A], we have that N has a normal Sylow p-subgroup. Considering G/N, we have also have that $|PN/N| \leqslant p$ by Theorem 4.3. Hence $|P/\mathbf{O}_p(G)| \leqslant p$ as required. □

Proof of Theorem B. Note that the vanishing prime graph of G is disconnected. Using [7, Theorem B], there exists normal subgroups M and N of G such that M/N is a non-abelian simple group, G/N is an almost simple group and N is a solvable radical of G. Since G is not p-solvable, p divides $|M/N|$. Observe that M/N has a vanishing element of order divisible p. By Theorem 4.3, if P is not cyclic, then $G/N \cong \mathrm{A}_7$. Using [8, Theorem A], we have that N has a normal Sylow p-subgroup since N has no vanishing elements which are p-singular. Then $|PN/N| = p^3$ if $p = 2$ and $|PN/N| = p^2$ if $p = 3$. The result then follows. □

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