A hybrid homogeneously weighted moving average control chart for process monitoring: Discussion

Short running head: HHWMA scheme

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Abstract

The sensitivity of a monitoring scheme depends on many factors including the variance of the charting statistic which is very important in the computation of the control limits. This paper discusses the computation of the variance of the recently proposed hybrid homogeneously weighted moving average (HHWMA) $\bar{X}$ scheme which was based on an incorrect assumption. The correct variance is used to evaluate the run-length characteristics of the HHWMA $\bar{X}$ scheme. It is observed that the incorrect variance has a significant impact on the sensitivity (or performance) of the HHWMA $\bar{X}$ scheme.

Keywords: Average run-length; HWMA; Hybrid HWMA; Standard deviation of the run-length; Median run-length

1. Introduction

Recently, Adeoti and Koleoso[1] proposed a new hybrid homogeneously weighted moving average (HHWMA) monitoring scheme to efficiently monitor changes in the process mean. Alevizakos et al[2] reported that the mathematical expression of the variance of the HHWMA $\bar{X}$ statistic proposed by Adeoti and Koleoso[1] is incorrect because they assumed that the covariance between two consecutive charting statistics is zero. Note though, Alevizakos et al[2] did not provide or derive the correct variance expression of the HHWMA $\bar{X}$ statistic. In this paper, we provide the correct mathematical expressions of the variance and control limits of the HHWMA $\bar{X}$ scheme. Moreover, few results from Adeoti and Koleoso[1] are compared to the ones using the correct control limits.

2. Hybrid homogeneously weighted moving average monitoring scheme

The HHWMA monitoring scheme is a combination of two HWMA schemes where two different smoothing parameters (say, λ_1 and λ_2) are considered. In this section, we show how the mean, variance and control limits of the HHWMA scheme are computed.

2.1 Properties of the HHWMA $\bar{X}$ monitoring scheme

Let X_{ij} ($i = 1, 2, \ldots; j = 1, 2, \ldots, n$) denote the j^{th} observation in the i^{th} sample of size $n \geq 1$ and assume that the X_{ij} are independent and identically distributed (i.i.d.) normal random variables, i.e. $X_{ij} \sim N(\mu_0 + \delta\sigma_0, \sigma_0)$, where μ_0 is the in-control (IC) mean value, σ_0 is the IC standard deviation and δ is the magnitude of the shift in standard deviation units. When $\delta = 0$, it implies that $X_{ij} \sim N(\mu_0, \sigma_0)$ and hence, the process is considered to be IC. However, when $\delta \neq 0$, the process is out-of-control

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(OOC). Let $\bar{X}_i = \sum_{j=1}^{n} X_{ij}/n$ be the sample mean of the i^{th} subgroup; then, the plotting statistic of the HHWMA $\bar{X}$ scheme (denoted as HH_i) is defined as

$$HH_i = \lambda_2 H_i + (1-\lambda_2)\bar{H}_{i-1}, \qquad \bar{H}_0 = \mu_0, \tag{1}$$

where

$$H_i = \lambda_1 \bar{X}_i + (1-\lambda_1)\bar{\bar{X}}_{i-1}, \qquad \bar{\bar{X}}_0 = \mu_0 \tag{2}$$

and

$$\bar{\bar{X}}_{i-1} = \frac{1}{i-1}\sum_{u=1}^{i-1} \bar{X}_u \ \text{ and } \bar{H}_{i-1} = \frac{1}{i-1}\sum_{u=1}^{i-1} H_u, \tag{3}$$

where λ_q ($0 < \lambda_q \leq 1$; $q = 1$ and 2) are the smoothing constants, $\bar{\bar{X}}_{i-1}$ ($\bar{H}_{i-1}$) is the mean of the previous $i-1$ subgroup sample means (charting statistics), respectively; with $\bar{\bar{X}}_0$ and $\bar{H}_0$ set equal to the target mean μ_0. Equation (2) can be written as

$$H_i = \lambda_1 \bar{X}_i + \frac{1-\lambda_1}{i-1}\sum_{u=1}^{i-1} \bar{X}_u. \tag{4}$$

The correct computation of the variance of HH_i is as follows:

- For $i = 1$,

$$\begin{aligned} HH_1 &= \lambda_2 H_1 + (1-\lambda_2)\bar{H}_0 \\ &= \lambda_2\left(\lambda_1 \bar{X}_1 + (1-\lambda_1)\bar{\bar{X}}_0\right) + (1-\lambda_2)\mu_0 \\ &= \lambda_1\lambda_2 \bar{X}_1 + [\lambda_2\,(1-\lambda_1) + (1-\lambda_2)]\mu_0 \end{aligned}$$

and hence,

$$HH_1 = \lambda_1\lambda_2 \bar{X}_1 + (1-\lambda_1\lambda_2)\mu_0. \tag{5}$$

The expected value and variance of HH_1 are given by

$$E(HH_1) = [\lambda_1\lambda_2 + (1-\lambda_1\lambda_2)]\mu_0 = \mu_0 \tag{6}$$

and

$$Var(HH_1) = Var[\lambda_1\lambda_2 \bar{X}_1 + (1-\lambda_1\lambda_2)\mu_0] = \lambda_1^2\lambda_2^2\, Var(\bar{X}_1) = \lambda_1^2\lambda_2^2 \frac{\sigma_0^2}{n}. \tag{7}$$

- For $i = 2$,

$$\begin{aligned} HH_2 &= \lambda_2 H_2 + (1-\lambda_2)\bar{H}_1 \\ &= \lambda_2\left(\lambda_1 \bar{X}_2 + (1-\lambda_1)\bar{\bar{X}}_1\right) + (1-\lambda_2)\bar{H}_1 \\ &= \lambda_1\lambda_2 \bar{X}_2 + \lambda_2\,(1-\lambda_1)\bar{X}_1 + (1-\lambda_2\,)\left[\lambda_1\,\bar{X}_1 + (1-\lambda_1)\bar{\bar{X}}_0\right] \\ &= \lambda_1\lambda_2 \bar{X}_2 + (\lambda_1+\lambda_2-2\lambda_1\lambda_2)\bar{X}_1 + (1-\lambda_1)(1-\lambda_2)\mu_0 \end{aligned}$$

and thus,

$$E(HH_2) = [\lambda_1\lambda_2 + (\lambda_1+\lambda_2-2\lambda_1\lambda_2) + (1-\lambda_1)(1-\lambda_2)]\,\mu_0 = \mu_0 \tag{8}$$

and

$$\begin{aligned} Var(HH_2) &= Var[\lambda_1\lambda_2 \bar{X}_2 + (\lambda_1+\lambda_2-2\lambda_1\lambda_2)\bar{X}_1 + (1-\lambda_1)(1-\lambda_2)\mu_0] \\ &= [\lambda_1^2\lambda_2^2 + (\lambda_1+\lambda_2-2\lambda_1\lambda_2)^2]\frac{\sigma_0^2}{n}. \end{aligned} \tag{9}$$

- For $i > 2$,

$HH_i = \lambda_2 H_i + (1-\lambda_2)\bar{H}_{i-1}$

$= \lambda_2\left(\lambda_1\bar{X}_i + (1-\lambda_1)\bar{\bar{X}}_{i-1}\right) + \frac{(1-\lambda_2)}{i-1}\sum_{k=1}^{i-1} H_k$

$= \lambda_1\lambda_2\bar{X}_i + \lambda_2\,(1-\lambda_1)\bar{\bar{X}}_{i-1} + \lambda_1\,(1-\lambda_2)\bar{\bar{X}}_{i-1} + \frac{(1-\lambda_1)\,(1-\lambda_2)}{i-1}\sum_{k=1}^{i-1}\bar{\bar{X}}_{k-1}$

$= \lambda_1\lambda_2\bar{X}_i + (\lambda_1+\lambda_2-2\lambda_1\lambda_2)\bar{\bar{X}}_{i-1} + \frac{(1-\lambda_1)\,(1-\lambda_2)}{i-1}\sum_{k=0}^{i-2}\bar{\bar{X}}_k$

$= \lambda_1\lambda_2\bar{X}_i + (\lambda_1+\lambda_2-2\lambda_1\lambda_2)\bar{\bar{X}}_{i-1} + \frac{(1-\lambda_1)\,(1-\lambda_2)}{i-1}\sum_{k=1}^{i-2}\bar{\bar{X}}_k + \frac{(1-\lambda_1)\,(1-\lambda_2)}{i-1}\mu_0$

$= \lambda_1\lambda_2\bar{X}_i + \frac{(\lambda_1+\lambda_2-2\lambda_1\lambda_2)}{i-1}\bar{X}_{i-1} + \frac{(\lambda_1+\lambda_2-2\lambda_1\lambda_2)}{i-1}\sum_{u=1}^{i-2}\bar{X}_u + \frac{(1-\lambda_1)\,(1-\lambda_2)}{i-1}\sum_{u=1}^{i-2}\left(\sum_{k=u}^{i-2}\frac{1}{k}\right)\bar{X}_u + \frac{(1-\lambda_1)\,(1-\lambda_2)}{i-1}\mu_0$

and finally,

$$HH_i = \lambda_1\lambda_2\bar{X}_i + \frac{(\lambda_1+\lambda_2-2\lambda_1\lambda_2)}{i-1}\bar{X}_{i-1} + \frac{1}{i-1}\sum_{u=1}^{i-2}\left[\lambda_1+\lambda_2-2\lambda_1\lambda_2 + (1-\lambda_1)\,(1-\lambda_2)\sum_{k=u}^{i-2}\frac{1}{k}\right]\bar{X}_u + \frac{(1-\lambda_1)\,(1-\lambda_2)}{i-1}\mu_0. \quad (10)$$

Next, the mean of the HHWMA statistic is given by

$$E(HH_i) = \lambda_1\lambda_2 E(\bar{X}_i) + \frac{(\lambda_1+\lambda_2-2\lambda_1\lambda_2)}{i-1}E(\bar{X}_{i-1}) + \frac{1}{i-1}\sum_{u=1}^{i-2}\left[\lambda_1+\lambda_2-2\lambda_1\lambda_2 + (1-\lambda_1)(1-\lambda_2)\sum_{u=1}^{i-2}\frac{1}{k}\right]E(\bar{X}_u) + \frac{(1-\lambda_1)\,(1-\lambda_2)}{i-1}\mu_0$$

$$= \left[\lambda_1\lambda_2 + \frac{(\lambda_1+\lambda_2-2\lambda_1\lambda_2)}{i-1} + \frac{1}{i-1}\sum_{u=1}^{i-2}(\lambda_1+\lambda_2-2\lambda_1\lambda_2) + \frac{(1-\lambda_1)(1-\lambda_2)}{i-1}\sum_{u=1}^{i-2}\left(\sum_{k=u}^{i-2}\frac{1}{k}\right) + \frac{(1-\lambda_1)\,(1-\lambda_2)}{i-1}\right]\mu_0,$$

which reduces to

$$E(HH_i) = [\lambda_1\lambda_2 + (\lambda_1+\lambda_2-2\lambda_1\lambda_2) + (1-\lambda_1)(1-\lambda_2)]\,\mu_0 = \mu_0. \quad (11)$$

Finally, the corresponding variance is given by

$$Var(HH_i) = \left[\lambda_1^2\lambda_2^2 + \frac{(\lambda_1+\lambda_2-2\lambda_1\lambda_2)^2}{(i-1)^2} + \frac{1}{(i-1)^2}\sum_{u=1}^{i-2}\left(\lambda_1+\lambda_2-2\lambda_1\lambda_2 + (1-\lambda_1)(1-\lambda_2)\sum_{k=u}^{i-2}\frac{1}{k}\right)^2\right]\frac{\sigma_0^2}{n}. \quad (12)$$

Therefore,

$$E(HH_i) = \mu_0 \quad (13a)$$

and

$$Var(HH_i) = \begin{cases} \lambda_1^2\lambda_2^2\dfrac{\sigma_0^2}{n}, & \text{for } i = 1 \\ [\lambda_1^2\lambda_2^2 + (\lambda_1 + \lambda_2 - 2\lambda_1\lambda_2)^2]\dfrac{\sigma_0^2}{n}, & \text{for } i = 2 \\ \left[\lambda_1^2\lambda_2^2 + \dfrac{(\lambda_1 + \lambda_2 - 2\lambda_1\lambda_2)^2}{(i-1)^2} + \dfrac{1}{(i-1)^2}\sum_{u=1}^{i-2}\left(\lambda_1 + \lambda_2 - 2\lambda_1\lambda_2 + (1-\lambda_1)(1-\lambda_2)\sum_{k=u}^{i-2}\dfrac{1}{k}\right)^2\right]\dfrac{\sigma_0^2}{n}, & \text{for } i > 2. \end{cases} \quad (13b)$$

The correct upper and lower control limits and centerline of the HHWMA $\bar{X}$ scheme are

$$UCL_i/LCL_i = \mu_0 \pm L\sqrt{Var(HH_i)} \text{ and } CL_i = \mu_0, \quad (14)$$

where L is the coefficient of the control limits which is selected such that the attained IC average run-length (*ARL*) of the HHWMA scheme is equal to some prespecified value. The HHWMA $\bar{X}$ scheme gives a signal at the sampling time i if HH_i plots beyond the control limits defined in Equation (14). It is worth mentioning that Adeoti and Koleoso[1] incorrectly derived the variance of HH_i as

$$Var(HH_i) = \begin{cases} \lambda_1^2\lambda_2^2\dfrac{\sigma_0^2}{n}, & \text{for } i = 1 \\ \left[\lambda_1^2\lambda_2^2 + \dfrac{(\lambda_1(1-\lambda_2))^2}{(i-1)} + \dfrac{(\lambda_2(1-\lambda_1))^2}{(i-1)} + \dfrac{(1-\lambda_1)^2(1-\lambda_2)^2}{(i-1)^2}\right]\dfrac{\sigma_0^2}{n}, & \text{for } i > 1. \end{cases} \quad (15)$$

2.2 Features of the HHWMA monitoring scheme

From Equations (10), (13a), (13b) and (14) the following important features of the HHWMA scheme can be deduced:

(i) If $\lambda_1 = \lambda_2 = 1$, the HHWMA monitoring scheme reduces to the Shewhart scheme.

(ii) If $\lambda_1 = 1$ and $\lambda_2 \neq 1$, or $\lambda_1 \neq 1$ and $\lambda_2 = 1$, the HHWMA scheme reduces to the basic HWMA scheme by Abbas[3].

(iii) If $\lambda_1 = \lambda_2 = \lambda$ with $0 < \lambda \leq 1$, the HHWMA scheme reduces to the DHWMA scheme discussed in Alevizakos et al[2].

The above features explain the flexibility of the HHWMA scheme which makes it more efficient than the Shewhart, HWMA and DHWMA monitoring schemes.

3. Discussion of the results

Table 1 displays the *ARL*, standard deviation of the run-length (*SDRL*) and median run-length (*MRL*) results of the HHWMA $\bar{X}$ scheme adapted from Adeoti and Koleoso[1] using the variance in Equation (15) and the ones found using the correct variance (i.e. Equation (13b)) which are provided in parentheses. These results are computed using Monte Carlo simulations with 20000 replications when $\lambda_1 = 0.1$, $\lambda_2 \in \{0.5, 0.75\}$ and $\delta \in \{0, 0.25, 0.5, 0.75, 1, 1.5, 2\}$ for a nominal IC *ARL* of 500. It can be noticed that the results are significantly different in terms of the *ARL*, *SDRL* and *MRL*. Note that the run-length profiles of the HHWMA $\bar{X}$ scheme reported by Adeoti and Koleoso[1] are much smaller than the ones found using the correct variance, for most shift values. It can also be noticed that the widths of the control limits (i.e., L values) are significantly different; for instance, when $\lambda_1 = 0.1$ and $\lambda_2 = 0.5$, then L is equal to 3.570 (see Adeoti and Koleoso[1]); however, using the correct variance, L is equal to

2.459 (see Table 1). Moreover, it is observed that using the correct expression of the variance, if λ_1 is kept fixed, as λ_2 increases, L increases as well - this contradicts the pattern of L reported in Adeoti and Koleoso[1].

Table 1. Run-length characteristics of the HHWMA $\bar{X}$ scheme when $\lambda_1 = 0.1$ and $\lambda_1 \in \{0.5, 0.75\}$ for a nominal ARL_0 of 500

	L = 3.570 (2.459) λ_2 **= 0.5**			**L = 3.170 (2.796)** λ_2 **= 0.75**		
Shift	***ARL***	***SDRL***	***MRL***	***ARL***	***SDRL***	***MRL***
0.00	501.28 (499.20)	426.95 (348.24)	405 (454)	508.79 (501.61)	372.27 (372.05)	452 (429)
0.25	56.51 (83.57)	55.3 (60.54)	40 (72)	70.74 (83.30)	57.31 (56.86)	57 (72)
0.50	18.12 (28.40)	14.65 (19.54)	14 (25)	22.84 (29.05)	16.19 (18.27)	19 (26)
0.75	9.88 (14.33)	6.56 (9.55)	8 (13)	11.97 (14.97)	7.59 (8.97)	10 (14)
1.00	6.73 (8.77)	3.62 (5.63)	6 (8)	7.71 (9.30)	4.33 (5.34)	7 (8)
1.50	4.35 (4.51)	1.58 (2.67)	4 (4)	4.54 (4.89)	1.98 (2.52)	4 (5)
2.00	3.44 (2.88)	0.99 (1.68)	3 (3)	3.28 (3.21)	1.23 (1.59)	3 (3)

ARL, average run-length; *SDRL*, standard deviation of the run-length; *MRL*, median run-length.

In the Table 2, the HHWMA $\bar{X}$ scheme (i.e. when $\lambda_1 \neq \lambda_2$) is compared to the DHWMA $\bar{X}$ scheme (i.e. when $\lambda_1 = \lambda_2$) in terms of the *ARL*, *SDRL* and *MRL* profiles when *n* = 1 for a nominal ARL_0 value of 500. Table 2 also compares the performances of the HHWMA and DHWMA $\bar{X}$ schemes in terms of the expected *ARL* (*EARL*), expected *SDRL* (*ESDRL*) and expected *MRL* (*EMRL*) profiles. The *EARL*, *ESDRL* and *EMRL* are mathematically defined by

$$EARL = \frac{1}{\Delta}\sum_{\delta=\delta_{\min}}^{\delta_{\max}} ARL(\delta),\ ESDRL = \frac{1}{\Delta}\sum_{\delta=\delta_{\min}}^{\delta_{\max}} SDRL(\delta) \quad \text{and} \quad EMRL = \frac{1}{\Delta}\sum_{\delta=\delta_{\min}}^{\delta_{\max}} MRL(\delta), \qquad (16)$$

respectively, where the $\delta \in [\delta_{\min}, \delta_{\max}]$, Δ is the number of increments from $\delta_{\min}$ to $\delta_{\max}$ of Riemann sum, $ARL(\delta)$, $SDRL(\delta)$ and $MRL(\delta)$ are the *ARL*, *SDRL* and *MRL* for a specific shift δ in the process parameter. In this paper, we use increments of 0.25 in the summations in Equation (16), with $\delta_{\min}$= 0.25 and $\delta_{\max}$= 2.

The results in Table 2 can be summarized as follows:

- When the process is IC, for a fixed value of λ_1, the IC *SDRL* profile of the HHWMA $\bar{X}$ scheme decreases (increases) as λ_2 increases (decreases) and the width of the control limits gets wider (narrower) as λ_2 increases (decreases).
- In terms of the *ARL* and *MRL* profiles, for a fixed value of λ_1, the sensitivity of the HHWMA $\bar{X}$ scheme decreases as λ_2 increases. However, the *SDRL* profile remains almost at the same level.
- If $\lambda_1 < \lambda_2$, the DHWMA $\bar{X}$ scheme with λ_1 performs better than the HHWMA $\bar{X}$ scheme in terms of the *ARL* and *MRL* profiles. The converse is true if $\lambda_1 > \lambda_2$.

Table 2. The *ARL* (first row), *SDRL* (second row) and *MRL* (third row) of the HHWMA and DHWMA $\bar{X}$ schemes along with their corresponding *L* values when $\lambda_1 \in \{0.1, 0.15, 0.2, 0.25, 0.5\}$, $\lambda_2 \in \{0.1, 0.15, 0.2, 0.25, 0.5\}$ and *n* = 1 for a nominal ARL_0 value of 500

λ_1 =	0.1				
λ_2 =	0.1	0.15	0.2	0.25	0.5
L =	1.201	1.416	1.629	1.801	2.459
Shift					
	499.88	500.51	502.41	499.45	499.20
0.00	776.84	609.45	528.34	449.51	348.20
	54.00	194.00	360.00	400.00	454.00
	40.93	51.96	62.12	68.31	83.57
0.25	56.10	61.75	65.09	63.78	60.54
	16.00	29.00	42.00	51.00	72.00
	13.32	16.35	19.82	21.78	28.40
0.50	15.96	17.56	19.09	19.64	19.54
	7.00	10.00	14.00	17.00	25.00
	6.91	8.40	9.93	10.99	14.33
0.75	7.42	8.33	9.09	9.54	9.55
	5.00	6.00	7.00	9.00	13.00
	4.42	5.28	5.94	6.74	8.77
1.00	4.37	4.85	5.13	5.43	5.63
	3.00	4.00	5.00	5.00	8.00
	3.14	3.69	4.28	4.59	5.99
1.25	2.95	3.23	3.45	3.54	3.71
	1.00	3.00	4.00	4.00	5.00
	2.40	2.71	3.03	3.44	4.51
1.50	2.16	2.32	2.41	2.58	2.67
	1.00	1.00	3.00	3.00	4.00
	1.92	2.17	2.43	2.67	3.56
1.75	1.65	1.83	1.90	1.98	2.08
	1.00	1.00	1.00	2.00	3.00
	1.59	1.76	1.96	2.17	2.88
2.00	1.27	1.39	1.49	1.60	1.68
	1.00	1.00	1.00	1.00	3.00
EARL	9.33	11.54	13.69	15.09	19.00
ESDRL	11.49	12.66	13.46	13.51	13.18
EMRL	4.38	6.88	9.63	11.50	16.63

λ_1 =	0.15				
λ_2 =	0.1	0.15	0.2	0.25	0.5
L =	1.416	1.692	1.934	2.152	2.739
Shift					
	500.51	500.26	499.96	501.79	499.74
0.00	609.45	479.55	408.96	378.48	359.39
	194.00	377.00	441.00	463.00	430.00
	51.96	65.69	75.34	83.42	91.43
0.25	61.75	65.29	65.53	67.32	63.10
	29.00	48.00	60.00	69.00	80.00
	16.35	20.98	24.22	27.42	31.71
0.50	17.56	19.69	20.36	21.37	20.13
	10.00	16.00	19.00	23.00	28.00
	8.40	10.39	12.14	13.78	16.04
0.75	8.33	9.23	10.01	10.39	9.90
	6.00	8.00	10.00	12.00	15.00
	5.28	6.36	7.45	8.22	9.93
1.00	4.85	5.38	5.81	6.11	5.97
	4.00	5.00	6.00	7.00	9.00
	3.69	4.38	4.94	5.64	6.77
1.25	3.23	3.57	3.67	3.95	3.88
	3.00	4.00	4.00	5.00	6.00
	2.71	3.26	3.75	4.20	5.00
1.50	2.32	2.59	2.78	2.89	2.74
	1.00	3.00	3.00	4.00	5.00
	2.17	2.54	2.90	3.24	3.94
1.75	1.83	1.99	2.10	2.22	2.16
	1.00	1.00	3.00	3.00	4.00
	1.76	2.05	2.30	2.60	3.22
2.00	1.39	1.58	1.67	1.78	1.76
	1.00	1.00	1.00	2.00	3.00
EARL	11.54	14.46	16.63	18.57	21.01
ESDRL	12.66	13.67	13.99	14.50	13.71
EMRL	6.88	10.75	13.25	15.63	18.75

λ_1 =	0.2				
λ_2 =	0.1	0.15	0.2	0.25	0.5
L =	1.629	1.934	2.189	2.394	2.883
Shift					
	502.41	499.96	500.05	501.71	502.01
0.00	528.34	408.96	346.33	351.12	395.63
	360.00	441.00	459.00	471.00	422.00
	62.12	75.34	84.15	89.42	92.36
0.25	65.09	65.53	66.43	65.55	65.01
	42.00	60.00	71.00	76.00	78.00
	19.82	24.22	28.17	30.53	32.33
0.50	19.09	20.36	21.44	21.23	20.52
	14.00	19.00	24.00	26.00	29.00
	9.93	12.14	14.01	15.29	16.68
0.75	9.09	10.01	10.36	10.54	10.04
	7.00	10.00	12.00	13.00	15.00
	5.94	7.45	8.49	9.28	10.27
1.00	5.13	5.81	6.08	6.11	5.95
	5.00	6.00	7.00	8.00	9.00
	4.28	4.94	5.80	6.37	7.08
1.25	3.45	3.67	4.03	4.20	3.94
	4.00	4.00	5.03	6.00	6.00
	3.03	3.75	4.27	4.64	5.29
1.50	2.41	2.78	2.90	2.96	2.88
	3.00	3.00	4.00	4.00	5.00
	2.43	2.90	3.28	3.60	4.12
1.75	1.90	2.10	2.23	2.28	2.15
	1.00	3.00	3.00	3.00	4.00
	1.96	2.30	2.64	2.91	3.33
2.00	1.49	1.67	1.80	1.84	1.73
	1.00	1.00	2.00	3.00	3.00
EARL	13.69	16.63	18.85	20.26	21.43
ESDRL	13.46	13.99	14.41	14.34	14.03
EMRL	9.63	13.25	16.00	17.38	18.63

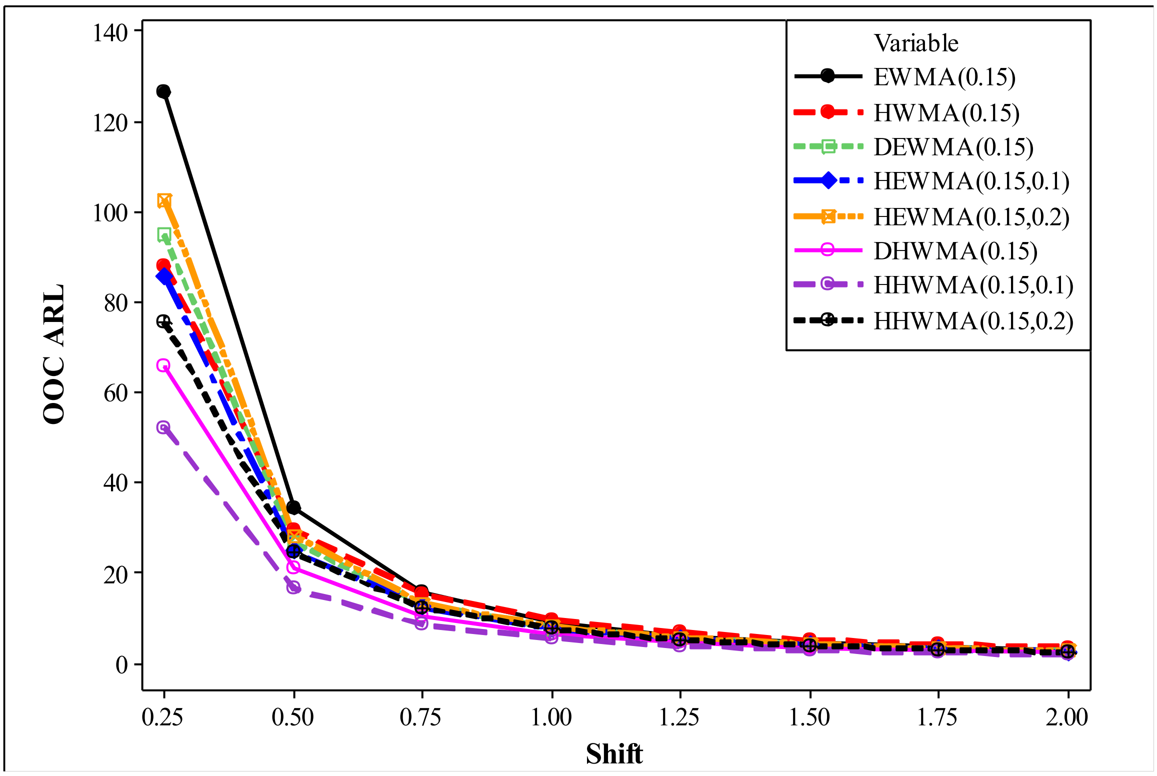
λ_1 =	0.25				
λ_2 =	0.1	0.15	0.2	0.25	0.5
L =	1.801	2.152	2.394	2.577	2.962
Shift					
	499.45	501.79	501.71	499.69	499.20
0.00	449.51	378.48	351.12	346.33	417.02
	400.00	463.00	471.00	449.00	396.00
	68.31	83.42	89.42	93.01	94.17
0.25	63.78	67.32	65.55	65.48	69.42
	51.00	69.00	76.00	80.00	77.00
	21.78	27.42	30.53	32.34	32.52
0.50	19.64	21.37	21.23	21.59	20.71
	17.00	23.00	26.00	29.00	29.00
	10.99	13.78	15.29	16.42	16.86
0.75	9.54	10.39	10.54	10.64	10.29
	9.00	12.00	13.00	15.00	15.00
	6.74	8.22	9.28	10.03	10.33
1.00	5.43	6.11	6.11	6.34	5.82
	5.00	7.00	8.00	9.00	9.00
	4.59	5.64	6.37	6.84	7.28
1.25	3.54	3.95	4.20	4.20	3.94
	4.00	5.00	6.00	6.00	7.00
	3.44	4.20	4.64	5.02	5.39
1.50	2.58	2.89	2.96	3.02	2.80
	3.00	4.00	4.00	5.00	5.00
	2.67	3.24	3.60	3.94	4.13
1.75	1.98	2.22	2.28	2.28	2.09
	2.00	3.00	3.00	4.00	4.00
	2.17	2.60	2.91	3.12	3.35
2.00	1.60	1.78	1.84	1.86	1.69
	1.00	2.00	3.00	3.00	3.00
EARL	15.09	18.57	20.26	21.34	21.75
ESDRL	13.51	14.50	14.33	14.43	14.60
EMRL	11.50	15.63	17.38	18.88	18.63

λ_1 =	0.5				
λ_2 =	0.1	0.15	0.2	0.25	0.5
L =	2.459	2.739	2.883	2.962	3.071
Shift					
	499.20	499.74	502.01	499.20	500.40
0.00	348.20	359.39	395.63	417.02	477.48
	454.00	430.00	422.00	396.00	358.00
	83.57	91.43	92.36	94.17	119.42
0.25	60.54	63.10	65.01	69.42	104.83
	72.00	80.00	78.00	77.00	89.00
	28.40	31.71	32.33	32.52	36.12
0.50	19.54	20.13	20.52	20.71	27.24
	25.00	28.00	29.00	29.00	29.00
	14.33	16.04	16.68	16.86	17.15
0.75	9.55	9.90	10.04	10.29	11.45
	13.00	15.00	15.00	15.00	15.00
	8.77	9.93	10.27	10.33	10.19
1.00	5.63	5.97	5.95	5.82	6.22
	8.00	9.00	9.00	9.00	9.00
	5.99	6.77	7.08	7.28	6.91
1.25	3.71	3.88	3.94	3.94	4.94
	5.00	6.00	6.00	7.00	6.00
	4.51	5.00	5.29	5.39	5.07
1.50	2.67	2.74	2.88	2.80	2.73
	4.00	5.00	5.00	5.00	5.00
	3.56	3.94	4.12	4.13	3.92
1.75	2.08	2.16	2.15	2.09	2.03
	3.00	4.00	4.00	4.00	4.00
	2.88	3.22	3.33	3.35	3.18
2.00	1.68	1.76	1.73	1.69	1.58
	3.00	3.00	3.00	3.00	3.00
EARL	19.00	21.01	21.43	21.75	25.25
ESDRL	13.18	13.71	14.03	14.60	20.13
EMRL	16.63	18.75	18.63	18.63	20.00

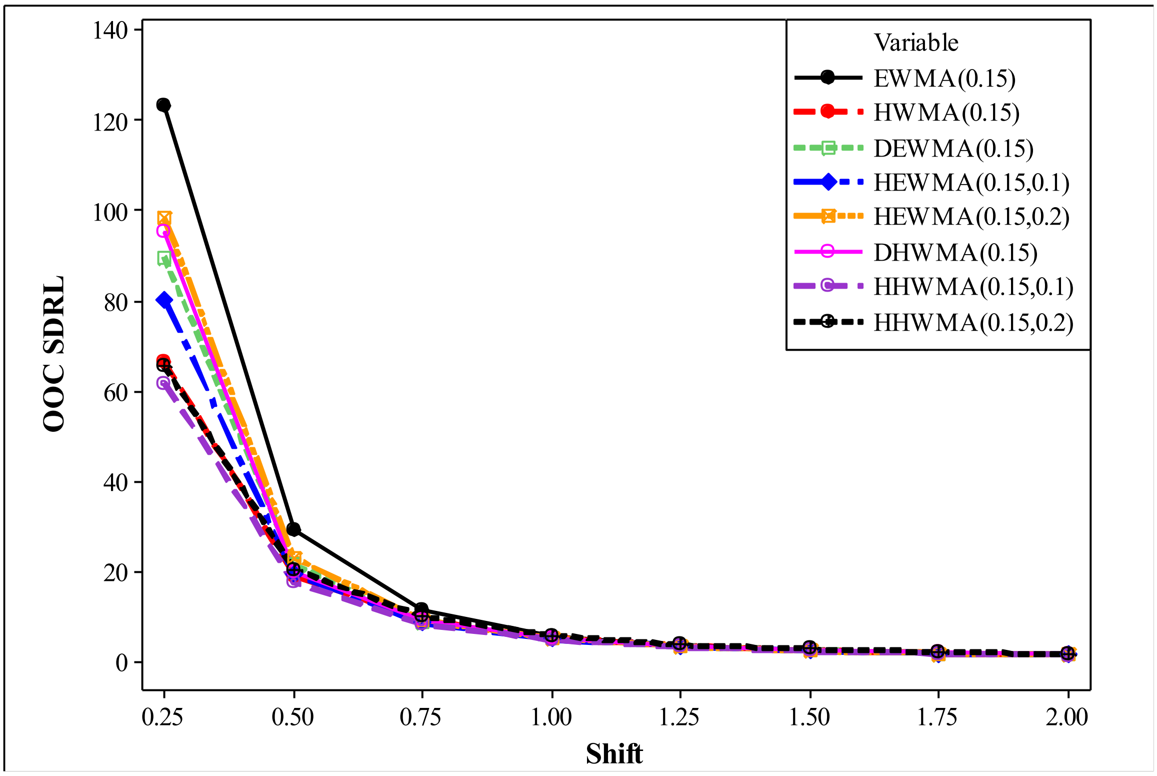
EARL, Expected Average Run-Length; *ESDRL*, Expected Standard Deviation of the Run-Length; *EMRL*, Expected Median Run-Length.

- If a and b are two positive numbers in the interval $(0, 1]$, the performance of the HHWMA $\bar{X}$ schemes with $\lambda_1 = a$ and $\lambda_2 = b$ is the same as the one of the HHWMA $\bar{X}$ schemes with $\lambda_1 = b$ and $\lambda_2 = a$.
- In terms of the overall performance (*EARL* and *EMRL* profiles), the DHWMA $\bar{X}$ scheme performs better for small smoothing parameters.
- The performance of the HHWMA $\bar{X}$ scheme in terms of the *ARL*, *MRL*, *EARL* and *EMRL* deteriorates when λ_1 or/and λ_2 increase(s).

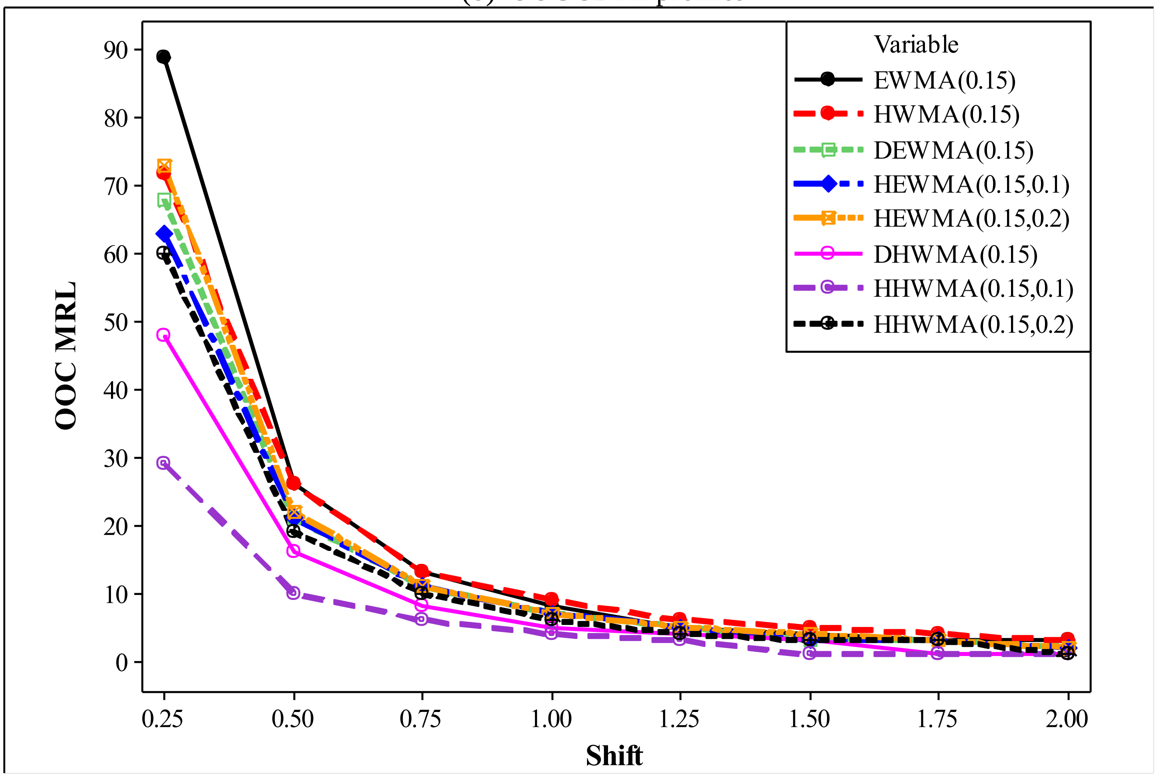
Given the recommendation outlined in Alevizakos et al[2] that the DHWMA $\bar{X}$ scheme should be evaluated for $\lambda \geq 0.14$ due to large IC *SDRL* values when λ is small; hence, Figure 1 compares the OOC performance of the HHWMA $\bar{X}$ scheme with the ones of the EWMA, HWMA, DEWMA, HEWMA and DHWMA $\bar{X}$ schemes when $n = 1$, $\lambda = 0.15$, $\lambda_1 = 0.15$ and $\lambda_2 = 0.1$ and 0.2 for a nominal ARL_0 of 500. It can be noticed that in terms of the *ARL* and *MRL* profiles, the HHWMA $\bar{X}$ scheme with $\lambda_1 = 0.15$ and $\lambda_2 = 0.1$ performs better than all the competing schemes from small to large shifts followed by the DHWMA $\bar{X}$ scheme. These findings also hold in terms of the *SDRL* profile for small and moderate shifts. However, for large shifts, the *SDRL* values are almost the same. The DHWMA and HHWMA $\bar{X}$ schemes are superior to the HEWMA and DEWMA $\bar{X}$ schemes for small to large shifts in terms of the *ARL* and *MRL* profiles. Moreover, both the DHWMA and HHWMA $\bar{X}$ schemes perform much better than the EWMA and HWMA $\bar{X}$ schemes.



(a) OOC *ARL* profiles



(b) OOC *SDRL* profiles



(c) OOC *MRL* profiles

Figure 1. Comparison of the OOC performance of the HHWMA $\bar{X}$ scheme with the ones of the EWMA, HWMA, DEWMA, HEWMA and DHWMA $\bar{X}$ schemes when $n = 1$, $\lambda = 0.15$, $\lambda_1 = 0.15$ and $\lambda_2 = 0.1$ and 0.2 for a nominal ARL_0 of 500

4. Conclusion and recommendations

The HHWMA scheme is more flexible and has very interesting properties compared to the Shewhart, EWMA and HWMA schemes. A comparative analysis of the performance of the HHWMA $\bar{X}$ scheme with the ones of the existing schemes reveals that the HHWMA $\bar{X}$ scheme is superior to the competing monitoring schemes considered in this paper. Researchers are recommended to use the correct variance of the HHWMA statistic to expand and improve the existing HHWMA scheme under both the violation and non-violation of the normality assumption.

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References

1. Adeoti OA, Koleoso SO. A hybrid homogeneously weighted moving average control chart for process monitoring. *Qual Reliab Eng Int*. 2020;36(6):2170-2186.
2. Alevizakos V, Chatterjee K, Koukouvinos C. The extended homogeneously weighted moving average control chart. *Qual Reliab Eng Int*. 2021; https://doi.org/10.1002/qre.2849.
3. Abbas N. Homogeneously weighted moving average control chart with an application in substrate manufacturing process. *Comput Ind Eng*. 2018;120:460-470.