Distribution-free double sampling precedence monitoring scheme to detect unknown shifts in the location parameter

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Abstract

In most applications, parametric monitoring schemes are used to monitor the majority of industrial and non-industrial processes in order to improve the quality of the outputs or services. However, parametric monitoring schemes are known to underperform when the normality assumption is not met or when there is not enough information about the symmetry or asymmetry nature of the process underlying distribution. Hence, in this paper, a new nonparametric Phase II Shewhart-type double sampling (DS) monitoring scheme based on the precedence statistic is proposed in order to efficiently monitor quality processes when the underlying process distribution departs from normality. The performance is investigated using the average run-length (*ARL*), standard deviation of the run-length (*SDRL*), expected *ARL* (*EARL*) and expected average number of observations to signal (*EANOS*) and the average sample sizes (*ASS*) metrics. The latter metrics are computed using Monte Carlo simulation and exact formulae. In general, it is shown that the new DS precedence scheme outperforms the existing basic Shewhart precedence scheme with and without supplementary runs-rules in many situations. A real-life illustrative example based on a filling process of milk bottles is provided to demonstrate the application and implementation of the new DS precedence monitoring scheme.

Keywords: Asymmetric / symmetric distributions; Control chart; Distribution-free; Double sampling; Order statistics; Phase I; Phase II; Precedence scheme; Robustness.

1. Introduction

Statistical process monitoring (SPM) is a field of quality control that uses statistical tools (or methods) to monitor and control industrial and non-industrial processes; in this field, an application of a collection of statistical techniques are implemented to ensure that high quality products are produced, see Montgomery[1]. A control chart / monitoring scheme is the most used tool in SPM application due to its visual appeal and ease of implementation. For some recent contributions to different types of monitoring schemes, see for instance Chen[2], Katebi and Moghadam[3] and, Sunthornwat and Areepong[4]. The basic Shewhart monitoring scheme is a one rule statistical tool used to detect a shift (or change) in the process parameter and it is highly

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recommended because of its simplicity and high efficiency in monitoring large shifts in the process parameter. Despite these qualities, the Shewhart scheme is criticized because of its slowness in detecting small-to-moderate shifts in the process parameter.

To improve the Shewhart scheme's ability towards detection of small-to-moderate process shifts while maintaining its strength towards the detection of large process shifts, Croasdale[5] introduced a double sampling (DS) procedure in a SPM context. Croasdale[5]'s DS scheme is based on two unconnected samples of sizes n_1 and n_2 (where $n_1 \leq n_2$), used in stages 1 and 2, respectively. Later on, Daudin[6] developed a new DS scheme based on a master sample of size n which is divided into two connected samples of sizes n_1 and n_2 (i.e. $n = n_1 + n_2$). The sample of size n_1 is used in stage 1, while in stage 2, the two samples are combined. Since then, many researchers have been developing the DS monitoring scheme; for some recent developments, see for example: [3, 7-12]. Daudin[6] and Costa[13] reported that the DS $\bar{X}$ monitoring scheme perform better than the Shewhart $\bar{X}$ scheme; hence, these two articles and others, suggest that for parametric schemes, the DS-type schemes are superior to the basic Shewhart-type schemes. Note though, there is no research work that addresses the latter for nonparametric schemes; therefore, the objective of this paper is, in part, to investigate whether this is also the case for nonparametric schemes.

Most of the research works in SPM are based on parametric settings (or assumptions). In this case, the control limits and properties of these types of schemes are easy to compute and they are mostly based on exact formulae or closed-form expressions. Monitoring schemes based on these settings are named parametric schemes and they perform better when the underlying process distribution is normally distributed or when the data follow a specific distribution. When these assumptions are violated, the performance of parametric schemes degrades considerably. To solve this problem, distribution-free (or nonparametric) schemes are recommended. A monitoring scheme is said to be a distribution-free scheme if the characteristics of its in-control (IC) run-length distribution remain the same (or approximately closer to each other) across all continuous distributions. These schemes are designed using nonparametric tests such as the sign, signed rank, Wilcoxon rank-sum, Mann-Whitney, median, etc. For a recent overview of nonparametric monitoring schemes, readers are referred to Chakraborti and Graham[14] and, Koutras and Triantafyllou[15]. Distribution-free schemes are preferred because of their IC robustness and attractive properties under non-normal underlying process distributions. However, nonparametric charts are expected to be less sensitive than their parametric counterparts when the underlying process distribution is normal or specified prior to their design (see for example, Chakraborti and Graham[14]).

When underlying process parameters are known (i.e. Case K), monitoring can immediately take place; however, when parameters are unknown (i.e. Case U), monitoring schemes require two phases in their design. The control limits and the process parameters are determined in Phase I when the process is considered to be IC and the actual process monitoring is achieved in Phase II. For more details on Phase II nonparametric monitoring schemes, readers are referred to the review paper by Chakraborti and Graham[14]. While some of the nonparametric schemes are designed under the assumption of Case K (i.e. the sign and signed-rank statistics), most of these schemes are designed under the assumption of Case U (i.e. the precedence, exceedance, Mann-Whitney, Wilcoxon rank-sum).

The precedence and exceedance monitoring schemes are a class of nonparametric Phase II monitoring schemes that can be used to monitor the j^{th} order statistic of a continuous process distribution, where j is a non-zero integer; see Chakraborti et al[16]. For memory-type (i.e. exponentially weighted moving average (EWMA), cumulative sum (CUSUM) and generally weighted moving average (GWMA)) schemes based on the precedence or exceedance statistics, readers need to consult Graham et al[17,18,19], Mukherjee et al[20], Chakraborty et al[21] and Karakani et al[22]. For the one-sided basic and weighted precedence schemes, Balakrishnan et al[23] used the Lehmann alternatives approach to formulate exact expressions to calculate some of the run-length distribution metrics. For the Shewhart-type schemes, Chakraborti et al[24] reported that the basic Shewhart precedence scheme is slow in detecting small shifts in the process parameter. To solve this problem, they proposed the 2-*of*-2 standard runs-rules (SRR) precedence schemes. Thereafter, Malela-Majika et al[25] further investigated the 2-*of*-2 SRR and improved runs-rules (IRR) precedence schemes based on the minimum and median statistics. More recently, Malela-Majika et al[26,27] studied the generalized 2-*of*-(h+1) and w-of-w one- and two-sided SRR and IRR precedence schemes (with integers $h > 0$ and $w > 1$). The focus of this paper is on the Shewhart-type precedence monitoring schemes.

Note that the only two nonparametric DS schemes that exist are for Case K using the EWMA scheme based on the sign statistic. That is, Yang and Wu[28,29] proposed the DS EWMA sign schemes for separately monitoring the location and scale parameters of quality characteristics from symmetric and asymmetric distributions. While the Case U scenario for the parametric DS schemes literature has a number of studies (see for instance: [30-39]); currently, there exists no research work that addresses the Case U scenario for nonparametric DS schemes. Therefore, in this paper, the DS Shewhart-type scheme based on the precedence statistic for Case U scenarios is proposed.

The rest of this paper is organized as follows: in Section 2, the theoretical foundation of the DS precedence schemes and its operational procedure are presented. The sensitivity analysis of the proposed scheme is discussed in Section 3 and its performance is compared to some existing Shewhart precedence schemes. A real-life illustrative example is provided in Section 4 and the concluding remarks are given in Section 5.

2. Design of the basic and double sampling precedence schemes

2.1 Basic precedence monitoring scheme

Assume that a reference (i.e. Phase I) random sample of size m, $X = \{X_i,\ i = 1, 2, \ldots, m\}$, is available from the in-control (IC) process with an unknown c.d.f. (cumulative distribution function) $F_X(x)$. Let $Y = \{Y_{tk},\ t = 1, 2, \ldots; k = 1, 2, \ldots, n\}$ be a t^{th} test (or Phase II) sample of size n with c.d.f. $G_Y(y)$ which is of the same nature as the one of the Phase I sample with a difference in the location parameter, that is, $F_X(x) = G_Y(x + \delta)$ where δ is the shift in the location parameter; see Chakraborti et al[16]. When $\delta = 0$, the process is said to be IC; in this case, $F_X(x) = G_Y(x)$. Otherwise, the process is said to be out-of-control (OOC). For simplicity in the notations, $F_X(x)$ and $G_Y(y)$ are simply denoted by F and G henceforth. Note that the Phase II samples are assumed to be independent and identically distributed (i.i.d.) of one another and of the Phase I sample.

The basic two-sided precedence monitoring is a class of nonparametric monitoring schemes that uses the j^{th} quantiles of the test sample as the charting statistics. The most used quantiles are the minimum, median and maximum statistics where $j = 1, \frac{n+1}{2}$ (assuming that n is odd) and n, respectively. A single charting statistic of the precedence scheme denoted as $Y_{(j:n)}$ is compared to the lower and upper control limits (denoted as LCL and UCL), where $Y_{(j:n)}$ represents the j^{th} order statistic of a test sample of size n. The LCL and UCL of the precedence monitoring scheme are estimated from the IC Phase I sample such that $\widehat{LCL} = X_{(a:m)}$ and $\widehat{UCL} = X_{(b:m)}$ where a and b are two non-zero positive constant known as charting constants such that $1 \leq a < b \leq m$. The basic precedence scheme gives a signal if a single point plots either on or above the $\widehat{UCL}$ or, on or below the $\widehat{LCL}$. Otherwise, the process is considered to be IC. The basic precedence scheme is known to be slow in detecting large shifts in the process. In the effort to solve this problem and at the same time improve the sensitivity of the existing basic precedence scheme towards the detection of small-to-moderate shifts, in this paper, a DS precedence monitoring scheme is introduced in the next subsection.

2.2 Double sampling precedence scheme designs and operation

The proposed DS precedence monitoring scheme is a two-stage precedence scheme which is divided into five charting regions as shown in Figure 1: $A = \left(-\infty, X_{(a_2:m)}\right] \cup \left[X_{(b_2:m)}, +\infty\right)$, $B = \left(X_{(a_2:m)}, X_{(a_1:m)}\right] \cup \left[X_{(b_1:m)}, X_{(b_2:m)}\right)$ and $C = \left(X_{(a_1:m)}, X_{(b_1:m)}\right)$ in Stage 1, and $D = \left(-\infty, X_{(c_1:m)}\right] \cup \left[X_{(c_2:m)}, +\infty\right)$ and $E = \left(X_{(c_1:m)}, X_{(c_2:m)}\right)$ in the Stage 2, where $X_{(\ell:m)}$ represents the ℓ^{th} ($\ell \in \{a_2, a_1, b_1, b_2, c_1, c_2\}$) order statistic of the reference (or Phase I) sample of size m and ℓ represents the position of the order statistic on the reference sample (and it is also referred to as the charting constant).

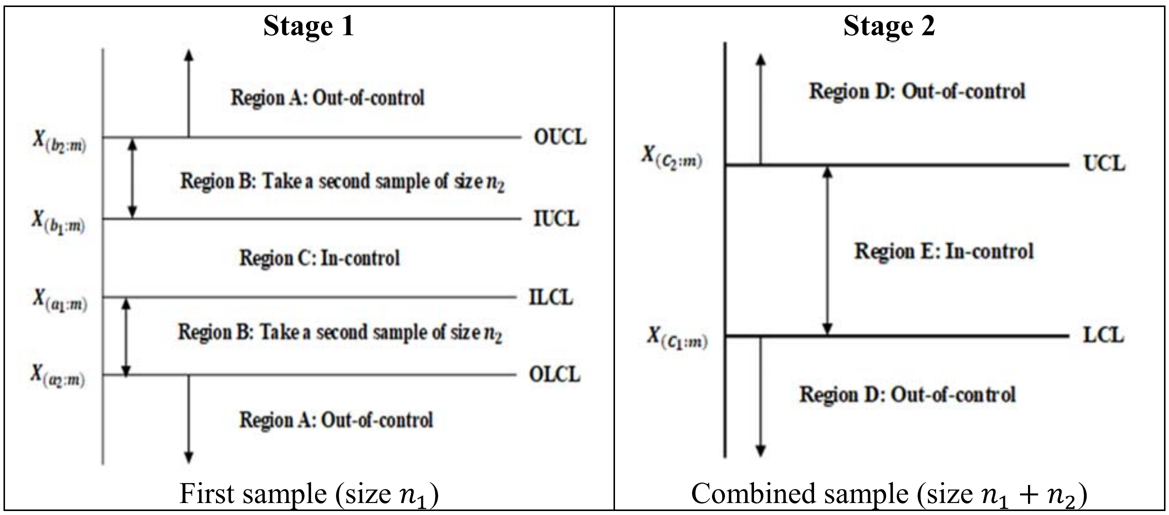


Figure 1. Charting regions of the DS precedence scheme

Chakraborti and Van der Laan[40] showed that the IC probability distribution of the precedence statistics is symmetric. Therefore, the choices of the charting constants a_2, a_1, b_1, b_2, c_1 and c_2 of the DS precedence scheme are such that in stage 1, $a_2 = m - b_2 + 1$, $a_1 = m - b_1 + 1$ and in stage 2, $c_1 = m - c_2 + 1$. Thus, the stage 1's outer lower and upper control limits (denoted as $OLCL$ and $OUCL$) and inner lower and upper control limits (denoted as $ILCL$ and $IUCL$) as well as the stage 2's LCL and UCL of the DS precedence scheme are estimated from the IC reference (i.e. Phase I) sample of size m as follows: $\widehat{OLCL} = X_{(a_2:m)}$, $\widehat{ILCL} = X_{(a_1:m)}$, $\widehat{IUCL} = X_{(b_1:m)}$, $\widehat{OUCL} = X_{(b_2:m)}$, $\widehat{LCL} = X_{(c_1:m)}$ and $\widehat{UCL} = X_{(c_2:m)}$. In Phase II, at each sampling time, a sample of size n is collected (i.e. Y_{tk},) and split into two subsamples Y_{1tp} and Y_{2tq} ($p = 1, 2, \dots, n_1$ and $q = 1, 2, \dots, n_2$) of sizes n_1 and n_2 ($n_2 > n_1$), respectively. Once the two subgroups are formed, then the Phase II operation procedure of the DS precedence scheme is as follows:

1. Take a sample of size n_1 and compute $Y_{(j:n_1)}$ at the t^{th} sampling time of the first sample. For simplicity, in this paper, it is assumed that n_1 is odd i.e. $n_1 = 2r + 1$ (where r is a positive integer) so that $j = r + 1$ corresponds to the unique test sample median of the Y_{1tp} sample in stage 1.
2. If $Y_{(j:n_1)} \in C$, the process is considered to be IC.
3. If $Y_{(j:n_1)} \in A$, the process is said to be OOC.
4. If $Y_{(j:n_1)} \in B$, take a second sample of size n_2.
5. At the t^{th} sampling time, compute the plotting statistic of the combined sample, $Y_{(h:n)}$. We assume that n_2 is even so that $n\ (= n_1 + n_2)$ gives an odd number, i.e. $n = 2s + 1$ (where s is a positive integer) so that $h = s + 1$ corresponds to the unique test sample median.
6. The process is declared OOC at stage 2, if $Y_{(h:n)} \in D$. Otherwise, the process is said to be IC.

The flow chart given in Figure 2 summarizes the operational procedure of the proposed DS precedence scheme.

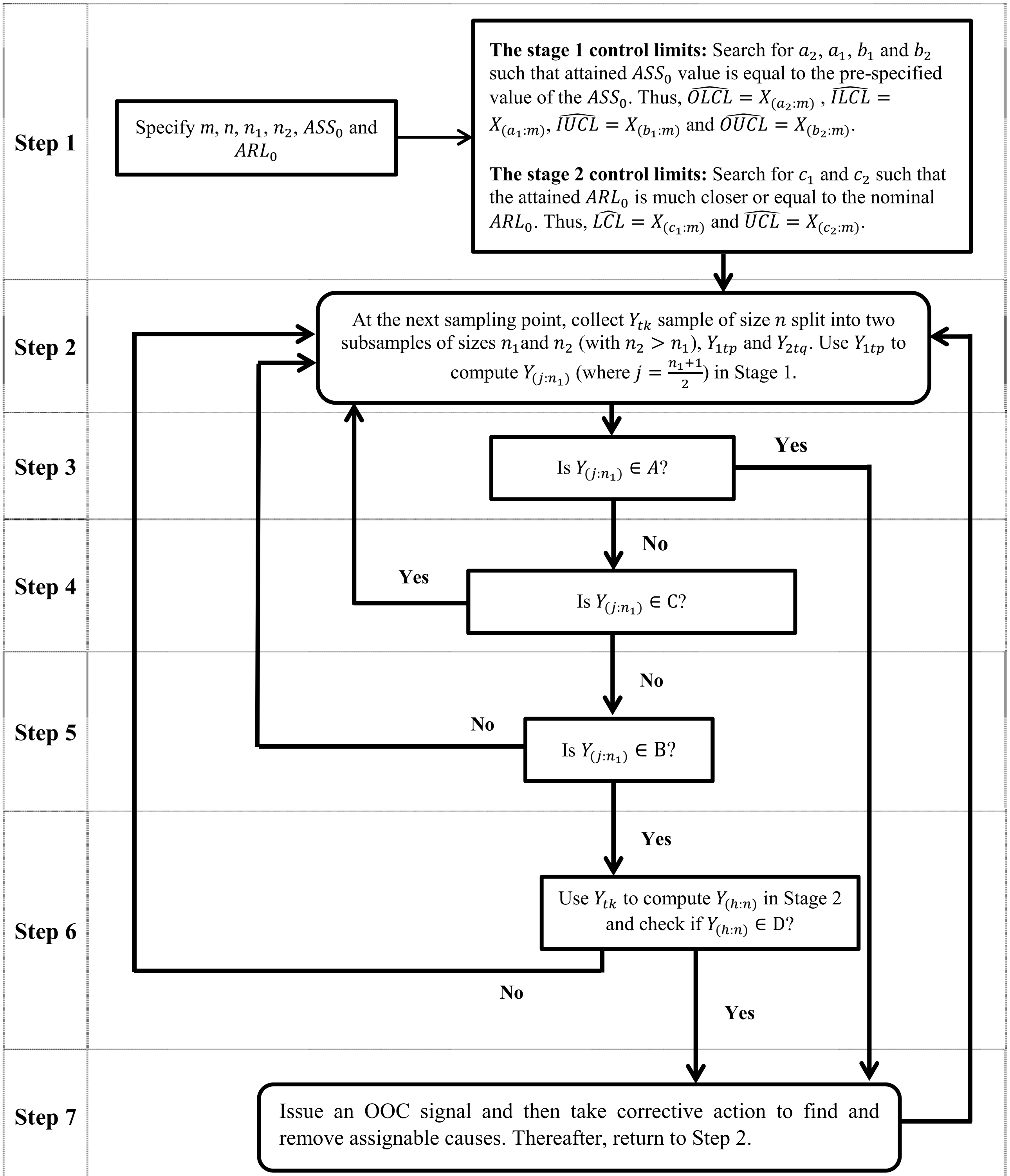


Figure 2: Flow chart illustrating the operation of the DS precedence scheme

3. Sensitivity analysis of the DS precedence scheme

3.1 Run-length properties of the DS precedence scheme

The sensibility of the DS precedence monitoring scheme depends on the combination (m, n_1, n_2, $\bar{n}$, a_2, a_1, b_1, b_2, c_1, c_2). There are three main steps in the search of the optimal design parameters of the proposed scheme. Firstly, the IC average sample size (ASS_0) and nominal IC average run-length (ARL_0) values are set to some pre-specified values, say $\bar{n}$ and $NARL_0$, respectively. Note that the $NARL_0$ must be set to some high recommended values such as 200, 370 and 500. Secondly, search for the stage 1 charting constants a_2, a_1, b_1 and b_2 that provide an attained ASS_0 value much closer or equal to $\bar{n}$. Finally, search for the stage 2 charting constants c_1 and c_2 such that the combination (m, n_1, n_2, $\bar{n}$, a_2, a_1, b_1, b_2, c_1, c_2) yields the attained ARL_0 value much closer or equal to the $NARL_0$ value and yields the smallest OOC (ARL_δ) for some specific shift value. Therefore, the optimization model is given by

$$\underset{m,n_1,n_2,\bar{n},a_2,a_1,b_1,b_2,c_1,c_2}{\text{Min}} ARL_\delta \tag{1}$$

subject to

$$ASS_0 \approx \bar{n} \tag{2}$$

and

$$ARL_0 \approx NARL_0 \tag{3}$$

For the DS precedence scheme with unknown process parameters (hereafter, Case U), in the ideal case, the combination (m, n_1, n_2, $\bar{n}$, a_2, a_1, b_1, b_2, c_1, c_2) is determined such that the attained ARL_0 and ASS_0 values are equal to $NARL_0$ and $\bar{n}$, respectively. However, since the charting constants are integers, we may not get control limits that exactly yield $NARL_0$ and $\bar{n}$ values. In such case, we recommend using control limits that yield attained ARL_0 and ASS_0 which are the closest to $NARL_0$ and $\bar{n}$ values, respectively.

In Phase II, the process is said to be OOC in stage 1 if the charting statistic $Y_{(j:n_1)} \geq \widehat{OUCL}$ or $Y_{(j:n_1)} \leq \widehat{OLCL}$ and in stage 2, the process is considered to be OOC if $Y_{(h:n)} \geq \widehat{UCL}$ or $Y_{(h:n)} \leq \widehat{LCL}$. The Phase II random variable Y follows a beta distribution with parameters j and $n_1 - j + 1$ in stage 1, and j and $n - h + 1$ in stage 2 (see Gibbons and Chakraborti[41] and Malela-Majika et al[26,27]). The conditional probabilities that the charting statistic plots in regions A, B, C, D and E are defined by

$$p_A = P\left(Y_{(j:n_1)} \leq X_{(a_2:m)} \middle| X_{(a_2:m)} = x_{(a_2:m)}\right) + P\left(Y_{(j:n_1)} \geq X_{(b_2:m)} \middle| X_{(b_2:m)} = x_{(b_2:m)}\right) \tag{4}$$

$$p_B = 2\left[P\left(Y_{(j:n_1)} \leq X_{(b_2:m)} \middle| X_{(b_2:m)} = x_{(b_2:m)}\right) - P\left(Y_{(j:n_1)} \leq X_{(b_1:m)} \middle| X_{(b_1:m)} = x_{(b_1:m)}\right)\right] \tag{5}$$

$$p_C = P\left(Y_{(j:n_1)} \leq X_{(b_1:m)} \middle| X_{(b_1:m)} = x_{(b_1:m)}\right) - P\left(Y_{(j:n_1)} \leq X_{(a_1:m)} \middle| X_{(a_1:m)} = x_{(a_1:m)}\right) \tag{6}$$

$$p_D = P\big(Y_{(h:n)} \le X_{(c_1:m)} \big| X_{(c_1:m)} = x_{(c_1:m)}\big) + P\big(Y_{(h:n)} \ge X_{(c_2:m)} \big| X_{(b_2:m)} = x_{(c_2:m)}\big) \tag{7}$$

and

$$p_E = P\big(Y_{(h:n)} \le X_{(c_2:m)} \big| X_{(c_2:m)} = x_{(c_2:m)}\big) - P\big(Y_{(h:n)} \le X_{(c_1:m)} \big| X_{(c_1:m)} = x_{(c_1:m)}\big). \tag{8}$$

Thus, (4)-(8) can also be written as:

$$p_A = I\big(1 - \Psi\big(U_{(a_2:m)}\big), j, n_1 - j + 1\big) + I\big(\Psi\big(U_{(b_2:m)}\big), j, n_1 - j + 1\big), \tag{9}$$

$$p_B = 2\left[I\big(\Psi\big(U_{(b_2:m)}\big), j, n_1 - j + 1\big) - I\big(\Psi\big(U_{(b_1:m)}\big), j, n_1 - j + 1\big)\right], \tag{10}$$

$$p_C = I\big(\Psi\big(U_{(b_1:m)}\big), j, n_1 - j + 1\big) - I\big(\Psi\big(U_{(a_1:m)}\big), j, n_1 - j + 1\big), \tag{11}$$

$$p_D = I\big(1 - \Psi\big(U_{(c_1:m)}\big), h, n - h + 1\big) + I\big(\Psi\big(U_{(c_2:m)}\big), h, n - h + 1\big), \tag{12}$$

and

$$p_E = I\big(\Psi\big(U_{(c_2:m)}\big), h, n - h + 1\big) - I\big(\Psi\big(U_{(c_1:m)}\big), h, n - h + 1\big), \tag{13}$$

respectively, where $I(.,.,.)$ denotes the incomplete beta function and $U_{(\ell:m)}$ represents the ℓ^{th} ($\ell \in \{a_2, a_1, b_1, b_2, c_1, c_2\}$) order statistic of a sample of size m from the Uniform(0,1) distribution and $\Psi = GF^{-1}$ is a conversion (or transformation) function that depends on the c.d.f. F and G. It is important to know that the process is IC if $G \equiv F$. In this case, $\Psi(u) = GF^{-1}(u) = u$ for any $u \in (0,1)$. Note that p_B represents the conditional probability of taking the second sample which is also known as the conditional probability of going to stage 2 (denoted as p_2). Therefore, the unconditional average sample size (ASS) for a specific shift is defined by

$$ASS(\delta) = \int_0^1 \int_0^t (n_1 + n_2 p_2)\, f_{b_1 b_2}(u,t) du\, dt \tag{14}$$

where δ is the shift (or change) in the location parameter, with p_2 defined in (10) and

$$f_{b_1 b_2}(u,t) = \frac{m!}{(b_1 - 1)!\ (b_2 - b_1 - 1)!\ (m - b_2)!} t^{b_1 - 1} (t - u)^{b_2 - b_1 - 1} (1 - t)^{m - b_2}$$

is the joint p.d.f. (probability distribution function) of the b_1^{th} and b_2^{th} order statistics in a sample of size m from the Uniform (0,1) distribution; see Chakraborti et al[24] for more details on the latter joint p.d.f. The attained ASS_0 is computed by setting δ to zero in (14). Again, the stage 1 charting constants a_2, a_1, b_1 and b_2 are computed such that the $ASS_0 \approx \bar{n}$.

In this paper, the Monte Carlo simulation is used to determine the stage 2 charting constants (i.e. c_1 and c_2) and compute the run-length characteristics of the proposed DS precedence scheme. Thus, the Monte Carlo algorithm steps are as follows:

Step 1: Specify the Phase I sample size (m), Phase II sample sizes (n_1 and n_2), number of simulations (ω), the stage 1 charting statistics (a_2, a_1, b_1 and b_2) that yield the $ASS_0 \approx \bar{n}$.

Step 2: Set c_2 (i.e. a positive integer) such that $\frac{m}{2} < c_2 < m$ and compute the corresponding value of c_1 ($c_1 = m - c_2 + 1$). For instance, when $m = 100$, if $c_2 = 84$, then $c_1 = 100 - 84 + 1 = 17$.

Step 3: Generate a Phase I sample X of size m from some specific p.d.f. (such as the $N(0,1)$). The estimated control limits are given by $\widehat{OLCL} = X_{(a_2:m)}$, $\widehat{ILCL} = X_{(a_1:m)}$, $\widehat{IUCL} = X_{(b_1:m)}$, $\widehat{OUCL} = X_{(b_2:m)}$, $\widehat{LCL} = X_{(c_1:m)}$ and $\widehat{UCL} = X_{(c_2:m)}$. For instance, from our previous example in Step 2, the stage 2 control limits will be given by $\widehat{LCL} = X_{(17:100)}$ and $\widehat{UCL} = X_{(84:100)}$.

Step 4: Randomly generate a Phase II test sample, Y_{tk}, of size n (from the same distribution as the one of the Phase I sample), then split into two subsamples, Y_{1tp} and Y_{2tq}, of sizes n_1 and n_2, respectively. Here, the Phase II sample is generated from the $N(\delta,1)$ distribution where δ represents the shift in the process location. Compute the charting statistic $Y_{(j:n_1)}$ in stage 1. Compare $Y_{(j:n_1)}$ to the stage 1 control limits. If $Y_{(j:n_1)}$ plots in region C (i.e. the process is IC), we have to generate the next test sample and compute the next charting statistic which is also compared to the control limits. This process continues until we get an OOC signal and record the number of samples that plotted IC. This number represents one value of the run-length distribution.

Step 5: However, if $Y_{(j:n_1)}$ plots in region B, compute the charting statistic $Y_{(h:n)}$ in stage 2 from the combined sample (i.e. Y_{tk}) and compare it to the stage 2 control limits obtained in Step 2. If $Y_{(h:n)}$ plots in region E (i.e. the process is IC), we have to return to Step 4. Otherwise, if $Y_{(h:n)}$ plots in region D, the process is OOC and record the number of samples that plotted in IC in both stages 1 and 2. The computed value denotes a single value of the run-length distribution.

Step 6: Steps 4 and 5 must be repeated ω times and then go to the next step.

Step 7: Using the unconditional run-length (URL) values obtained in Step 6, we compute the unconditional ARL ($UARL$) as follows:

$$UARL = \frac{1}{\omega}\sum_{i=1}^{\omega} URL_{i}. \qquad (15)$$

Step 8: For $\delta = 0$, if the ARL_0 value is much closer to $NARL_0$, record the control limits $\widehat{OLCL}$, $\widehat{ILCL}$, $\widehat{IUCL}$, $\widehat{OUCL}$, $\widehat{LCL}$ and $\widehat{UCL}$. If not, then repeat the previous Steps 2 to 7.

Step 9: Finally, to calculate the OOC run-length properties, implement Step 3 to 7 using the control limits found in Step 8 by varying the shift parameter (i.e. δ = 0 until 2.5 in increments of 0.25). Note that $\delta = 0$ provides the IC $UARL$ values and $\delta \neq 0$ provides the OOC $UARL$ values and in this paper, these are simply denoted as ARL_0 and ARL_δ, respectively.

Note that other characteristics of the run-length distribution can be computed using PROC UNIVARIATE in SAS®9.4 and in this study, ω=50000.

The ARL metric is much criticized when it comes to the assessment of the performance of a control chart for a range of shifts or overall performance (see for example Ou et al[42]). Therefore, in this paper, the expected ARL value is also used to assess the overall performance of the DS precedence scheme. The $EARL$ is mathematically defined by

$$EARL = \frac{1}{\Delta}\sum_{\delta=\delta_{min}}^{\delta_{max}} ARL(\delta), \quad (16)$$

where $\delta_{\min}$ and $\delta_{\max}$ are the lower and upper bound of the δ in location parameter, respectively, $ARL(\delta)$ is the ARL value for a specific δ and Δ represents the number of increments between $\delta_{\min}$ and $\delta_{\max}$. The smaller the ARL or $EARL$, the better the performance of a monitoring scheme.

The average number of observation to signal (*ANOS*) is also used to investigate the properties of the DS precedence scheme. The *ANOS* and expected *ANOS* (*EANOS*) are mathematically defined by

$$ANOS(\delta) = ASS(\delta) \cdot ARL(\delta) \quad (17)$$

and

$$EANOS = \frac{1}{\Delta}\sum_{\delta=\delta_{min}}^{\delta_{max}} ANOS(\delta), \quad (18)$$

respectively; where the $ASS(\delta)$ and $ARL(\delta)$ are defined in (14) and (15), respectively.

3.2 Determination of the charting constants

The stage 1 charting constants a_2, a_1, b_1 and b_2 are determined by setting (14) to the pre-specified ASS_0 value (i.e. $\bar{n}$). Note that since the charting constants are integers, it might be possible that the charting constant do not yield the exact pre-specified $\bar{n}$ value. In this case, it is recommended to choose the charting constant that yields the attained ASS_0 value as close as possible to $\bar{n}$. In this paper, (14) is used in Mathcad®14 to determine the stage 1 charting constants. From Table 1, it is shown that, when m = 100, $n_1 = 3$ and $n_2 = 6$, then the charting constants (a_2, a_1, b_1, b_2) = (33, 45, 56, 68), (16, 38, 63, 85) and (18, 45, 56, 83) so that the DS precedence scheme yields an attained ASS_0 value of 5, 6 and 6.99 (≈ 7), respectively.

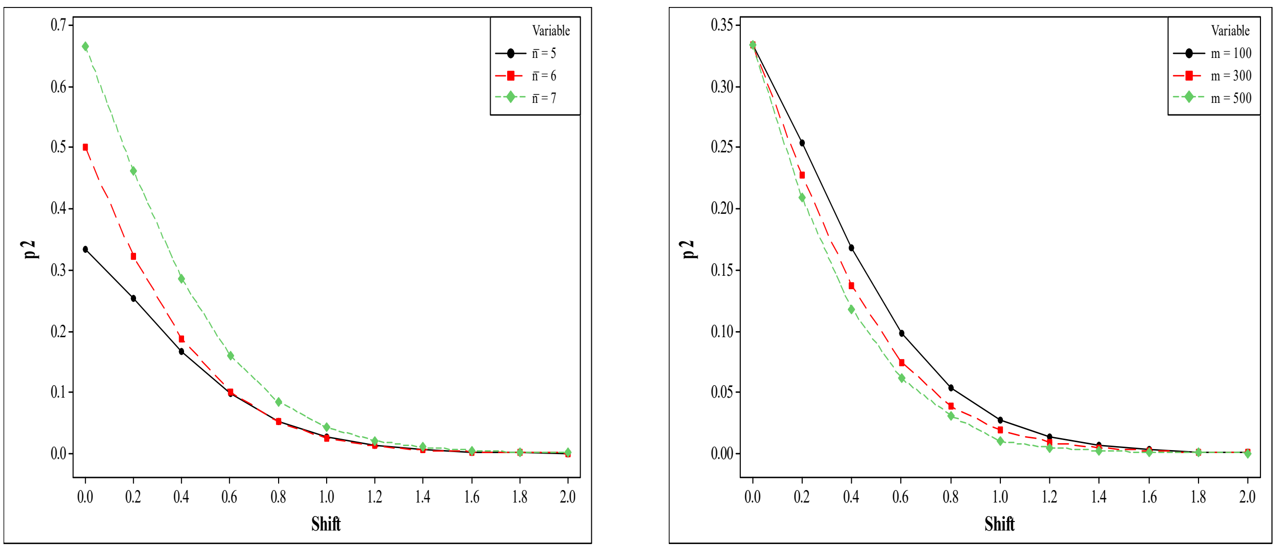
The stage 2 charting constants c_1 and c_2 are determined such that the combination (a_2, a_1, b_1, b_2, c_1, c_2) yields an attained ARL_0 that is closer or equal to $NARL_0$. For instance, for $NARL_0$ value of 500, when m = 100, $n_1 = 3$ and $n_2 = 12$, it is found that (a_2, a_1, b_1, b_2, c_1, c_2) = (12, 22, 79, 89, 18, 83) so that the DS precedence scheme yields attained ASS_0 and ARL_0 values of 5 and 505.73, where the probability of moving to stage 2 (i.e. p_2) is 0.1663. Table 1 displays the charting constants, p_2, attained ASS_0, ARL_0 and IC standard deviation of the run-length ($SDRL_0$) values of the DS precedence scheme when $m \in \{100,500\}$, $n_1 \in \{3,5\}$, $n_2 \in \{6,12\}$, $\bar{n} \in \{5,6,7\}$ and $NARL_0 \in \{200, 370, 500\}$.

From Table 1, it can be seen that the larger the Phase I sample size, the closer the attained ARL_0 value to the $NARL_0$ and the smaller the $SDRL_0$ value, which means there is more stability in the performance of the DS precedence scheme for larger Phase I sample sizes. When m and n_1 are kept fixed, the larger the value of n_2, the higher is the variability in the IC run-length values. The wider (narrower) the distance between the IUCL, OUCL, OLCL and ILCL, the larger (smaller) is the probability of moving to stage 2.

Table 1. Charting constants, p_2 values, attained ASS_0, ARL_0 and $SDRL_0$ values of the DS precedence chart when $m \in \{100,500\}$, $n_1 \in \{3,5\}$, $n_2 \in \{6,12\}$ with $\bar{n} \in \{5,6,7\}$ and $NARL_0 \in \{200,370,500\}$

			Stage 1 charting constants				p_2	ASS_0	Stage 2 charting constants			
m	n_1	n_2	a_2	a_1	b_1	b_2			c_1	c_2	ARL_0	$SDRL_0$
100	3	6	33	45	56	68	0.3334	5.00	13	88	228.78	468.88
									12	89	341.23	683.66
									11	90	535.17	1264.82
			16	38	63	85	0.4997	6.00	14	87	173.10	335.21
									12	89	374.26	766.11
									11	90	585.62	1411.87
			18	45	56	83	0.6654	6.99	13	88	224.94	424.23
									12	89	344.04	798.78
									11	90	535.71	1206.00
		12	12	22	79	89	0.1663	5.00	20	81	239.69	515.65
									19	82	350.05	926.88
									18	83	505.73	1171.83
			26	36	65	75	0.2502	6.00	20	81	187.01	385.16
									18	83	387.77	889.76
									17	84	607.36	1569.00
			33	45	56	83	0.6654	7.00	19	82	231.91	541.10
									18	83	352.34	970.92
									17	84	537.36	1556.78
	5	6	4	6	95	97	0.0035	5.02	50	51	363.12	829.69
									40	61	509.71	1127.10
			21	29	72	80	0.1657	5.99	16	85	212.88	432.90
									15	86	315.98	653.44
									14	87	469.23	1047.63
			14	32	69	87	0.3331	7.00	16	85	207.97	395.58
									15	86	299.53	603.13
									14	87	452.19	1011.14
		12	4	6	95	97	0.0035	5.03	50	51	365.19	726.58
									44	57	504.02	1103.20
			23	27	74	78	0.0829	6.00	22	79	182.26	396.29
									20	81	375.31	966.34
									19	82	565.77	1579.44
			6	23	78	95	0.1671	7.00	22	79	199.64	444.83
									20	81	408.88	944.14
									19	82	616.36	1617.39
500	3	6	167	94	334	407	0.3334	5.00	66	435	197.48	220.52
									57	444	383.79	443.71
									54	447	496.82	583.38
			89	194	307	412	0.5	6.00	65	436	197.15	221.41
									57	444	358.33	411.59
									53	448	502.80	591.09
			32	198	303	469	0.6668	7.00	64	437	208.08	234.14
									57	444	360.60	421.52
									53	448	500.26	586.21
		12	14	90	411	487	0.1667	5.00	103	398	202.44	230.89
									92	409	362.08	618.48
									87	414	500.06	618.48
			62	130	371	439	0.2499	6.00	98	403	201.09	235.91
									89	412	378.17	450.87
									85	416	510.21	609.22
			16	131	370	485	0.3335	7.00	98	403	200.01	234.20
									89	412	371.81	439.51
									85	416	511.37	610.63
	5	6	30	34	467	471	0.0018	5.01	236	265	201.29	224.66
									161	340	372.89	421.60
									138	363	502.19	567.88
			77	199	302	424	0.5684	6.00	76	425	201.52	230.30
									68	433	362.63	424.97
									64	437	504.53	597.62
			123	185	316	378	0.3336	7.00	77	424	193.21	222.57
									68	433	375.53	444.27
									64	437	518.06	623.26
		12	35	38	463	466	0.0017	5.01	212	289	200.89	222.76
									174	327	369.57	419.03
									159	342	503.24	578.24
			84	112	389	417	0.0835	6.00	107	394	194..64	226.56
									98	403	363.87	432.92
									94	407	484.71	585.04
			87	134	367	414	0.1668	7.00	105	396	202.35	234.70
									97	404	358.91	438.90
									93	408	485.00	588.56

Figure 3 displays the IC and OOC probability of taking a second sample (i.e. moving to stage 2) for different sample sizes when $\bar{n} \in \{5,6,7\}$. From Figure 3(a), it can be seen that when the triplet (m, n_1, n_2) is kept fixed, the larger (smaller) the value of $\bar{n}$, the higher (lower) is the IC value of p_2. For small and moderate shifts, the larger (smaller) the value of $\bar{n}$, the higher (smaller) are the OOC values of p_2. Moreover, the OOC p_2 values remain almost the same for large shifts in the process location. Figure 3(b) shows that when $\bar{n}$ is kept fixed, the IC p_2 remains almost the same, regardless of the Phase I sample size. For small and moderate shifts, the smaller the Phase I sample size, the higher are the OOC p_2 values. For large shifts, the OOC p_2 remains almost the same regardless of the Phase I sample size.



(a) $(m, n_1, n_2) = (100, 3, 6)$ for different $\bar{n}$ values (b) $(n_1, n_2, \bar{n}) = (3,6,5)$ for different m values

Figure 3. The probability of moving to stage 2 (i.e. p_2 values)

3.3 IC performance and robustness analysis

Nonparametric schemes are usually preferred over their parametric counterparts because of their IC robustness property. A monitoring scheme is said to be robust when the IC characteristics of the run-length distribution are the same across all continuous probability distributions. Thus, to check the robustness of the DS precedence scheme various probability distributions (symmetric, heavy-tailed and skewed) are considered in this paper. To be more precisely, the following were used: (i) the standard normal distribution (denoted as $N(0,1)$) to study the effect of symmetric distributions, (ii) the Student's t-distribution with degrees of freedom $v = 3$ and 12 (denoted $t(v)$) to study the effect of symmetric distributions with heavier tails, (iii) gamma distribution with parameters $\alpha = 1$ and 3 and $\beta = 1$ (denoted as $GAM(\alpha, \beta)$) to study the effect of skewed distributions, and (iv) double exponential distribution with parameters μ =0 and β =1 (denoted as *DEXP*(μ,β)) to investigate the effect of symmetric distributions with higher peak.

In Table 2, the robustness of the DS precedence scheme is investigated through the IC characteristics of the run-length distribution (ARL_0, $SDRL_0$, 5th, 25th, 50th, 75th and 95th percentiles of the run-length (PRL) denoted as P_5, P_{25}, P_{50}, P_{75}, P_{95}, respectively) when $m \in$ {100, 500}, $n_1 \in \{3,5\}$, $n_2 = 6$ and $\bar{n} \in \{5,7\}$ for a $NARL_0$ value of 500 under different distributions. From Table 2, it can be observed that for $\bar{n}$ = 5, the combination (m, n_1, n_2, a_2, a_1, b_1, b_2, c_1, c_2) = (100, 3, 6, 33, 45, 56, 68, 11, 90) yields an attained ARL_0 values of 535.18, 523.17, 533.94, 539.73, 530.69 and 531.18 under the *N*(0,1), *t*(3), *t*(12), *GAM*(1,1), *GAM*(3,1) and *DEXP*(0,1) distributions, respectively, and the combination (m, n_1, n_2, a_2, a_1, b_1, b_2, c_1, c_2) = (500, 3, 6, 167, 94, 334, 407, 54, 447) yields attained ARL_0 values of 496.82, 496.62, 496.85, 495.38, 494.97 and 495.96 under the *N*(0,1), *t*(3), *t*(12), *GAM*(1,1), *GAM*(3,1) and *DEXP*(0,1) distributions, respectively. In addition, when (m, n_1, n_2, a_2, a_1, b_1, b_2, c_1, c_2) = (500, 3, 6, 167, 94, 334, 407, 54, 447), the P_5 value under the *N*(0,1), *t*(3), *t*(12), *GAM*(1,1), *GAM*(3,1) and *DEXP*(0,1) distributions, is equal to 21, 23, 22, 23, 23 and 22, respectively. This is also true for other characteristics of the run-length distribution (see Table 2). These findings reveal that attained the IC characteristics of the run-length distribution are much closer to each other in each case. Therefore, the proposed DS precedence scheme is IC robust. Moreover, the larger the Phase I sample size the closer the attained ARL_0 values to the $NARL_0$ value of 500 and to each other. This shows that the DS precedence scheme is more robust for large sample sizes.

Table 2. IC robustness of the DS precedence chart when $m \in \{100, 500\}$, $n_1 \in \{3,5\}$, $n_2 = 6$ and $ASS_0 \in \{5,7\}$ for an nominal ARL_0 value of 500 under different distributions

(n_1,n_2)	m	ASS_0	a_2	a_1	b_1	b_2	c_1	c_2	Distribution	ARL_0	$SDRL_0$	P_5	P_{25}	P_{50}	P_{75}	P_{95}
(3,6)	100	5	33	45	56	68	11	90	*N* (0,1)	535.18	1164.82	12	76	213	546	1996
									t (3)	523.17	1100.88	12	75	212	547	1968
									t (12)	533.94	1190.94	12	73	211	543	2011
									GAM (1,1)	539.73	1204.47	12	74	212	459	1974
									GAM (3,1)	530.69	1238.17	12	75	214	539	1958
									DEXP (0,1)	531.18	1267.15	12	75	212	540	1979
		7	18	45	56	83	11	90	*N* (0,1)	535.71	1206.10	13	76	214	546	1974
									t (3)	539.83	1248.75	12	75	210	546	2027
									t (12)	546.14	1308.20	13	77	214	550	2043
									GAM (1,1)	547.65	1386.00	12	75	215	555	2038
									GAM (3,1)	540.40	1350.21	12	74	212	548	2016
									DEXP (0,1)	546.81	1369.89	12	75	213	555	2081
	500	5	167	94	334	407	54	447	*N* (0,1)	496.82	583.38	21	122	306	643	1588
									t (3)	496.62	577.94	23	125	308	656	1589
									t (12)	496.85	583.89	22	124	310	655	1595
									GAM (1,1)	495.38	585.54	23	125	309	651	1595
									GAM (3,1)	494.97	575.03	23	126	311	653	1578
									DEXP (0,1)	495.96	580.57	22	125	311	656	1587
		7	32	198	303	469	53	448	*N* (0,1)	500.26	586.21	22	125	314	662	1626
									t (3)	496.20	580.49	22	124	310	652	1588
									t (12)	501.34	583.52	22	127	315	666	1603
									GAM (1,1)	499.72	585.19	22	126	313	660	1602
									GAM (3,1)	504.44	587.71	23	127	314	664	1619
									DEXP (0,1)	499.95	587.50	22	125	312	659	1597
(5,6)	100	5	4	6	95	97	40	61	*N* (0,1)	506.53	1048.70	13	79	219	541	1856
									t (3)	507.33	1053.37	13	79	218	542	1859
									t (12)	509.67	1024.30	13	79	220	549	1883
									GAM (1,1)	507.15	1020.15	14	79	219	546	1873
									GAM (3,1)	508.37	1060.98	13	80	220	545	1835
									DEXP (0,1)	511.26	1003.00	14	79	220	553	1900
		7	14	32	69	87	14	87	*N* (0,1)	450.89	913.94	11	63	177	454	1657
									t (3)	458.63	1129.48	11	64	180	460	1690
									t (12)	453.97	1094.48	10	63	180	460	1648
									GAM (1,1)	450.30	966.30	10	63	180	459	1660
									GAM (3,1)	455.04	1010.26	10	63	181	461	1647
									DEXP (0,1)	456.81	1010.78	11	65	180	462	1698
	500	5	30	34	467	471	138	363	*N* (0,1)	497.22	568.80	23	127	317	660	1575
									t (3)	498.96	568.39	23	130	319	660	1583
									t (12)	499.47	571.77	23	129	317	664	1576
									GAM (1,1)	501.07	570.83	23	130	318	669	1600
									GAM (3,1)	500.67	581.86	22	129	317	662	1594
									DEXP (0,1)	502.21	567.36	23	127	317	674	1609
		7	123	185	316	378	64	437	*N* (0,1)	519.42	616.32	23	129	322	676	1695
									t (3)	518.48	625.84	23	127	317	673	1687
									t (12)	521.80	626.08	23	128	322	678	1696
									GAM (1,1)	522.39	627.58	23	128	322	684	1697
									GAM (3,1)	520.62	624.62	23	128	319	683	1691
									DEXP (0,1)	522.96	617.96	24	130	325	687	1693

3.4 Out-of-control performance analysis

The first step in the investigation of the performance profile of a nonparametric monitoring scheme is to check whether it is IC robust. Since the DS precedence scheme is IC robust, it is of interest to compare its sensitivity under different probability distributions and with other existing schemes. In this section, Monte Carlo simulations are used to compute the characteristics of the run-length distribution of the DS precedence scheme as explained in Section 3.1. Tables 3 and 4 present the ARL_δ profile of the DS precedence monitoring scheme when $m \in \{100, 500\}$, $n_2 \in \{6,12\}$, $\bar{n} \in \{5, 7\}$ for a $NARL_0$ value of 500 under the $N(0,1)$, $t(3)$,

t(12), *GAM*(1,1), *GAM*(3,1) and *DEXP*(0,1) distributions with $n_1 = 3$ and 5. The findings in Tables 3 and 4 can be summarized as follows:

- The DS precedence scheme performs better for large Phase I sample sizes regardless of the sample size. For instance, under the t(3) distribution, for $\delta = 0.25$ and ($\bar{n}$,n_1,n_2) = (5,3,6) when m = 100 and 500, the DS precedence scheme gives a signal for the first time on the 270th and 143th subgroups, respectively. This shows that the larger the value of m the more sensitive is the monitoring scheme.
- When m, $\bar{n}$ and n_1 are kept fixed, the larger the value of n_2, the more sensitive is the scheme, especially for large $\bar{n}$ and small n_1 values.
- For small stage 1 sample size, say $n_1 = 3$, when m and n_2 are kept fixed, there is no significant difference in the performance of the DS precedence scheme as $\bar{n}$ increases (see Table 3). However, as n_1 increases, say $n_1 = 5$, when m and n_2 are kept fixed, the performance of the DS precedence scheme increases as $\bar{n}$ increases (see Table 4). This is achieved at the expense of the inspection and production costs.
- When m, $\bar{n}$ and n_2 are kept fixed, the larger the stage 1 sample size n_1, the less sensitive is the scheme. In this case, it is recommended to use reasonably large $\bar{n}$ values to improve the performance; but keep in mind high costs of inspection and production.
- Under the t-distribution, the DS precedence scheme performs better for small degrees of freedom. Thus, the larger the degrees of freedom, the less sensitive is the scheme.
- The DS precedence scheme performs worst under skewed distributions especially for small shifts in the location parameter. Under the *GAM*(α,β), the DS precedence scheme is ARL-biased. This can be remedied by increasing the Phase I sample size or the shape parameter. The larger the shape parameter, the better is the performance.
- It can also be noticed that, when m=100, under the *GAM*(α,β) distribution, the proposed DS precedence scheme performs worst for small shifts in the location parameter (especially, when $\delta \leq 0.25$) regardless of the magnitude of the shape parameter (α) and average sample size ($\bar{n}$). However, as α and $\bar{n}$ increase, the sensitivity of the DS precedence scheme under the *GAM*(α,β) distribution increases as well but not to the extent of surpassing its sensitivity under the normal, t and double exponential distributions. Therefore, for small shifts, it is recommended to use high reasonable $\bar{n}$ values to increase its sensitivity under small shifts. For moderate shifts, the DS precedence scheme performs better under the *GAM*(α,β) distribution as compared to its performance under the *DEXP*(0,1) distribution. For large shifts in the

process parameter, the performance is the same regardless of the nature of the underlying process distribution.

- The DS precedence scheme performs better under symmetric distributions with heavier tails. Thus, the DS precedence scheme is more sensitive under the *t*-distribution than the *N*(0,1), *GAM*(1,1), *GAM*(3,1) and *DEXP*(0,1) distributions. However, under symmetric distributions, the performance deteriorates as the peak of the distribution get higher. Therefore, the DS precedence scheme performs better under the *N*(0,1) distribution than the *DEXP*(0,1) distribution.

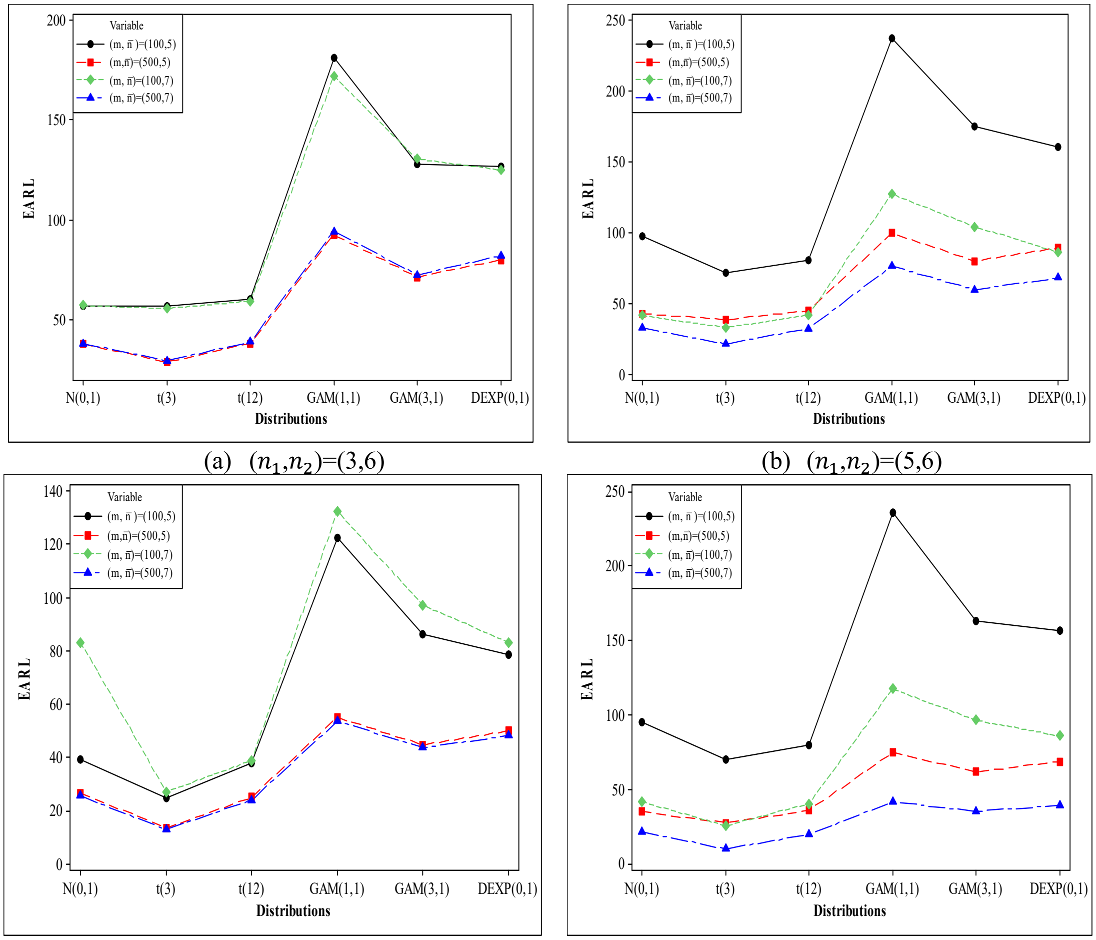
Table 3. OOC *ARL* profile of the DS precedence scheme when $m \in \{100, 500\}$, $n_1 = 3$, $n_2 \in \{6,12\}$, $\bar{n} \in \{5, 7\}$ for a $NARL_0$ value of 500 under different distributions

		m	100						500					
(n_1,n_2)	$\bar{n}$	**Distribution**	0.25	0.50	0.75	1.00	1.50	2.00	0.25	0.50	0.75	1.00	1.50	2.00
(3,6)	5	*N* (0,1)	262.51	56.33	13.54	4.76	1.48	1.05	171.75	37.14	10.46	4.05	1.42	1.04
		t (3)	270.35	55.48	8.71	2.40	1.05	1.00	143.32	20.50	4.29	1.67	1.03	1.00
		t (12)	276.70	62.82	14.05	4.58	1.39	1.04	175.65	36.34	9.93	3.72	1.32	1.03
		GAM (1,1)	660.06	303.76	88.95	30.12	4.25	1.31	380.15	120.15	38.15	13.40	2.43	1.06
		GAM (3,1)	519.05	177.22	50.39	15.70	2.68	1.17	305.78	84.11	25.93	9.05	2.00	1.07
		DEXP (0,1)	421.78	217.66	86.28	29.87	4.32	1.40	310.69	113.24	38.41	13.80	2.51	1.18
	7	*N* (0,1)	264.41	56.90	13.93	4.76	1.47	1.05	174.20	36.99	10.45	4.09	1.41	1.04
		t (3)	266.23	53.94	8.67	2.38	1.05	1.00	146.29	21.33	4.45	1.70	1.03	1.00
		t (12)	272.00	62.15	14.21	4.67	1.39	1.04	179.70	37.44	9.94	3.72	1.32	1.03
		GAM (1,1)	610.87	288.55	92.83	34.35	4.42	1.32	385.77	122.73	39.63	14.13	2.56	1.07
		GAM (3,1)	530.81	182.73	50.33	15.54	2.66	1.17	309.49	86.18	26.73	9.29	2.05	1.08
		DEXP (0,1)	414.89	212.61	87.00	29.96	4.43	1.40	315.30	117.65	40.28	14.50	2.60	1.18
(3,12)	5	*N* (0,1)	194.43	30.47	6.72	2.64	1.22	1.04	127.35	21.63	5.99	2.60	1.27	1.06
		t (3)	133.44	10.87	2.19	1.24	1.03	1.01	68.23	7.21	2.05	1.27	1.04	1.01
		t (12)	189.31	27.86	5.99	2.33	1.17	1.03	121.31	19.23	5.23	2.30	1.21	1.04
		GAM (1,1)	583.41	119.29	24.95	5.96	1.20	1.00	257.07	53.70	13.22	4.11	1.16	1.00
		GAM (3,1)	417.92	77.63	16.09	4.51	1.28	1.01	209.98	41.65	10.66	3.76	1.30	1.02
		DEXP (0,1)	335.09	103.77	23.43	6.12	1.39	1.08	228.31	52.00	13.02	4.10	1.37	1.11
	7	*N* (0,1)	348.23	113.78	29.06	6.38	1.31	1.02	123.50	19.94	5.37	2.29	1.15	1.02
		t (3)	145.56	11.65	2.04	1.13	1.00	1.00	65.94	6.43	1.78	1.15	1.02	1.00
		t (12)	196.61	27.15	5.34	2.00	1.06	1.00	117.49	17.95	4.60	2.01	1.11	1.02
		GAM (1,1)	630.48	129.74	26.73	6.19	1.22	1.00	254.42	50.08	11.69	3.38	1.04	1.00
		GAM (3,1)	474.00	86.93	15.65	4.23	1.19	1.00	207.27	39.68	9.71	3.27	1.13	1.00
		DEXP (0,1)	348.23	113.78	29.06	6.38	1.31	1.02	221.83	49.63	11.58	3.47	1.22	1.05

Table 4. OOC *ARL* profile of the DS precedence scheme when $m \in \{100, 500\}$, $n_1 = 5$, $n_2 \in \{6,12\}$, $\bar{n} \in \{5, 7\}$ for a $NARL_0$ value of 500 under different distributions

		m	100						500					
$(\boldsymbol{n_1},\boldsymbol{n_2})$	$\bar{n}$	**Distribution**	0.25	0.50	0.75	1.00	1.50	2.00	0.25	0.50	0.75	1.00	1.50	2.00
(5,6)	5	*N* (0,1)	346.88	152.37	59.35	20.34	2.84	1.14	187.62	45.15	14.59	6.08	2.03	1.22
		t (3)	299.18	90.97	26.86	10.02	2.64	1.35	177.01	38.13	10.58	3.69	1.25	1.02
		t (12)	319.76	113.78	34.55	12.23	2.74	1.34	197.11	48.15	15.16	6.15	1.94	1.17
		GAM (1,1)	709.13	401.70	197.14	89.25	22.70	6.14	377.86	132.99	53.49	25.01	6.72	2.20
		GAM (3,1)	598.62	287.57	105.00	45.63	9.62	2.82	321.16	99.84	36.81	15.88	4.21	1.67
		DEXP (0,1)	423.01	272.57	156.14	83.03	22.24	5.92	317.24	129.67	53.83	25.68	6.80	2.22
	7	*N* (0,1)	198.02	36.96	8.30	3.20	1.23	1.02	154.06	28.61	7.68	3.05	1.22	1.01
		t (3)	169.45	21.12	3.75	1.41	1.01	1.00	109.52	12.95	2.79	1.31	1.01	1.00
		t (12)	201.61	36.72	8.20	2.92	1.17	1.01	154.27	27.23	7.06	2.71	1.15	1.01
		GAM (1,1)	542.72	164.31	41.11	12.64	1.93	1.04	334.20	87.86	25.21	8.14	1.57	1.00
		GAM (3,1)	487.40	102.54	25.34	7.57	1.59	1.03	269.02	63.16	17.51	5.96	1.48	1.02
		DEXP (0,1)	320.61	138.10	44.37	12.49	2.00	1.10	285.87	86.11	25.28	8.40	1.65	1.06
(5,12)	5	*N* (0,1)	342.65	149.66	54.23	19.74	2.64	1.16	155.53	34.53	11.37	5.11	1.86	1.18
		t (3)	292.96	88.12	26.14	10.10	2.64	1.35	125.45	25.77	7.57	2.9	1.17	1.01
		t (12)	317.29	107.98	34.23	12.2	2.75	1.33	159.89	35.29	11.53	5.03	1.76	1.14
		GAM (1,1)	706.77	396.68	193.01	90.02	22.65	6.05	297.00	89.59	36.45	18.14	5.09	1.78
		GAM (3,1)	547.47	269.42	106.05	45.15	9.53	2.82	255.21	71.28	26.06	11.78	3.45	1.49
		DEXP (0,1)	412.45	259.28	155.02	84.16	21.58	6.07	262.1	87.64	37.09	18.15	5.07	1.82
	7	*N* (0,1)	211.03	27.5	5.75	2.24	1.13	1.01	104.13	15.63	4.18	1.89	1.08	1.01
		t (3)	138.26	9.07	1.86	1.14	1.01	1.00	50.11	4.70	1.48	1.08	1.00	1.00
		t (12)	207.49	25.51	5.01	1.97	1.09	1.01	96.65	13.68	3.6	1.67	1.05	1.00
		GAM (1,1)	553.71	123.58	20.16	4.66	1.11	1.00	201.55	35.83	7.71	2.32	1.01	1.00
		GAM (3,1)	484.12	74.24	13.58	3.65	1.15	1.00	170.27	29.01	6.81	2.39	1.05	1.00
		DEXP (0,1)	381.59	104.91	21.13	4.77	1.23	1.04	187.56	35.29	7.73	2.47	1.11	1.02

Figure 4 displays the $EARL$ profile of the DS precedence profile under different distributions when $m \in$ {100, 500}, $\bar{n} \in${5,7} and $\delta = 0.25$, 0.5, 0.75, 1, 1.5 and 2 for a $NARL_0$ value of 500. In terms of the overall performance, when *m* is kept fixed, for small stage 1 sample size, there is a slight difference in the overall performance of the DS precedence scheme as the $\bar{n}$ value increases. However, as the stage 1 sample size increases, there is a significant difference in the performance of the DS precedence scheme as $\bar{n}$ value increases. In this case, the larger the $\bar{n}$ value, the more sensitive the scheme becomes. Figure 4 also reveals that the larger the Phase I sample size, the better is the overall performance of the proposed DS precedence scheme. Since the minimum *EARL* value is yielded under the *t*-distribution, it can be deduced that the DS precedence scheme is more sensitive under symmetric distribution with heavy tails regardless of the sample sizes.



(a) (n_1,n_2)=(3,6) (b) (n_1,n_2)=(5,6)

(c) (n_1,n_2)=(3,12) (d) (n_1,n_2)=(5,12)

Figure 4. $EARL$ profile of the DS precedence scheme when $\delta \in \{$ 0.25, 0.5, 0.75, 1, 1.5, 2$\}$, $m \in$ {100, 500} and $\bar{n} \in${5,7} for a $NARL_0$ value of 500

The pattern and findings in terms of the $EANOS$ remains the same regardless of the Phase I sample size. Figure 5 displays the $EANOS$ profile of the DS precedence scheme under different distributions when $m = 500$, $(n_1, n_2) = (3,12)$, $\bar{n} \in${5,7} and $\delta = 0.25$ (0.25) 3 for a $NARL_0$ value of 500. Thus, Figure 5 shows that the smaller $\bar{n}$, the better the $EANOS$ profile. In other words, the DS precedence scheme is cost effective for small $\bar{n}$.

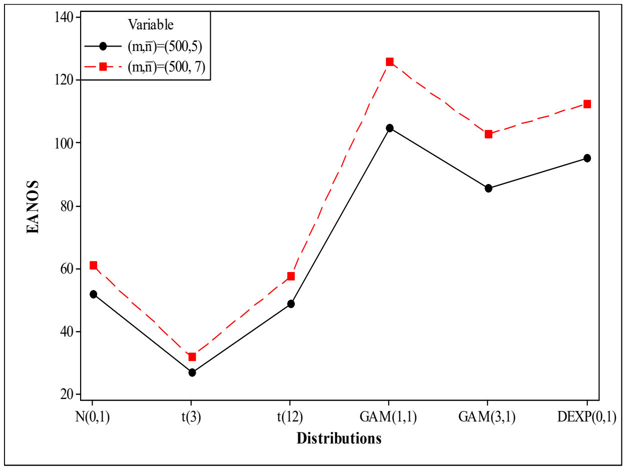
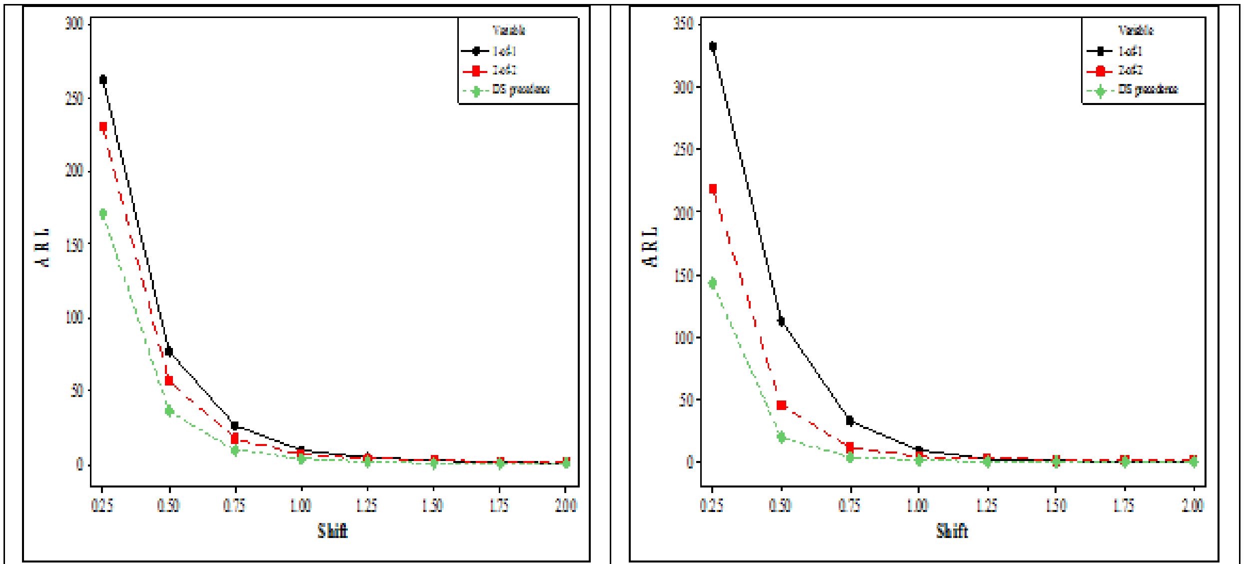


Figure 5. $EANOS$ profile of the DS precedence scheme when δ = 0.25 (0.25) 3, m =500, (n_1, n_2) = (3,12) and $\bar{n}$ ∈{5,7} for a $NARL_0$ value of 500

3.5 Comparison with the existing Shewhart precedence scheme

In this section, the proposed DS precedence scheme is compared to the existing basic Shewhart precedence scheme (denoted as 1-*of*-1) by Chakraborti et al[16] and the 2-*of*-2 SRR precedence scheme (denoted as 2-*of*-2) by Chakraborti et al[24] and Malela-Majika et al[25]. The performances of the competing schemes are computed when m = 500 and n = 5 (which is equivalent to $\bar{n}$ = 5 for the proposed scheme) under normal and non-normal distributions. Assuming that $NARL_0$ = 500, m = 500 and (n_1,n_2) = (3,6) so that $\bar{n}$ = 5, the OOC performance comparison of the competing schemes in terms of the *ARL* and *EARL* values under the *N*(0,1) and *t*(3) distributions are displayed in Figures 6 and 7, respectively.

Figure 6 shows that the proposed DS precedence scheme is superior to both the 1-*of*-1 and 2-*of*-2 precedence schemes regardless of the nature of the distribution. In addition, it is observed that the 2-*of*-2 precedence scheme outperforms the 1-*of*-1 precedence scheme for small shifts in the process parameter and the converse is true for large shifts. Figure 7 reveals that the DS precedence scheme has a better overall performance when compared to both the 1-*of*-1 and 2-*of*-2 precedence schemes regardless of the type of distribution. The 2-*of*-2 outperforms the 1-*of*-1 precedence scheme in terms of the *EARL* values. The 2-*of*-2 and DS precedence schemes perform better under the *t*(3) distribution; however, the 1-*of*-1 precedence scheme performs better under the *N*(0,1) distribution.



(a) $N(0,1)$ distribution	(b) $t(3)$ distribution

Figure 6. ARL comparison of three precedence monitoring scheme under the $N(0,1)$ and $t(3)$ distributions when $\delta = 0.25\ (0.25)\ 2$, $m = 500$ and $\bar{n} = 5$ for a $NARL_0$ value of 500

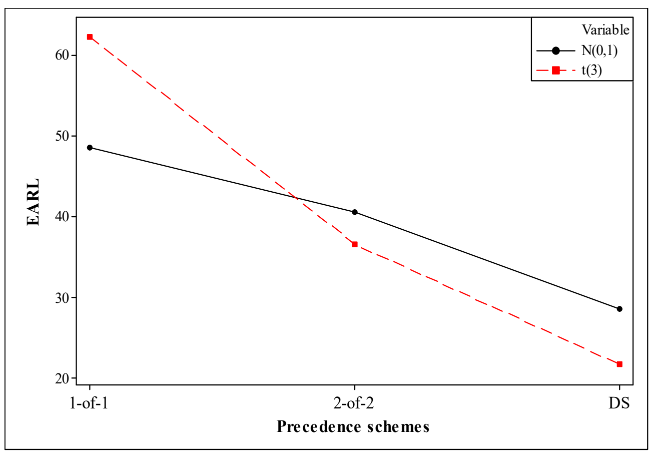


Figure 7. $EARL$ comparison of three precedence monitoring schemes under the $N(0,1)$ and $t(3)$ distributions when $(\delta_{min},\delta_{max})$ = (0.25, 2), $m = 500$ and $\bar{n} = 5$ for a $NARL_0$ value of 500

4. Illustrative example

To illustrate the implementation and application of the DS precedence monitoring scheme, the data on a production process of 500 milliliters (*ml*) milk bottles adopted from Castagliola et al[43], where the quality characteristic of interest is the volume (in *ml*) of milk within each bottle. The data contains two datasets for which the goodness of fit test for normality is not rejected. The first dataset has twenty retrospective or Phase I samples, each of size $n = 5$, collected when the process was considered to be IC, that is, m =100. The second dataset contains twenty test (i.e. Phase II) samples each of size 5. In this example, each test (i.e. Phase II) sample is considered to be a master sample which is then divided into two subgroups of sizes 1 and 4 (i.e. n_1 = 1 and n_2 = 4), in Stages 1 and 2, respectively, such that $n = n_1 + n_2 = 5$.

For a $NARL_0$ value of 370 and a pre-specified $\bar{n}$ value of 3, the stage 1 charting constants a_2, a_1, b_1 and b_2 are equal to 10, 36, 65 and 91, respectively, so that the DS precedence scheme yields an attained ASS_0 value of 3.06 with the following estimated stage 1 control limits: $\widehat{OLCL} = X_{(10:100)}$ = 498.83, $\widehat{ILCL} = X_{(36:100)}$ = 499.71, $\widehat{IUCL} = X_{(65:100)}$ = 500.26 and $\widehat{OUCL} = X_{(91:100)}$ = 501.51. The corresponding stage 2 charting constants c_1 and c_2 are equal to 6 and 95, respectively, so that the estimated stage 2 control limits are given by: $\widehat{LCL} = X_{(6:100)}$ = 498.59 and $\widehat{UCL} = X_{(95:100)}$ = 501.83. A plot of the stages 1 and 2 charting statistics (i.e. $Y_{(j:n_1)}$ and $Y_{(h:n)}$) of the DS precedence scheme are shown in Figure 8. It is seen that at the first, second and third sampling points, the DS precedence scheme plotted in region C (i.e. the IC region) when $t = 1$, 2 and 3. At the fourth and fifth sampling points, the process moved to stage 2. The stage 2 charting statistics at the fourth and fifth sampling times plotted in region E (i.e. the IC region) when $t = 4$ and 5. At the sixth sampling point of stage 1, the charting statistic plotted in region A, indicating that the process is OOC. Thus, the DS precedence scheme was able to detect a shift in the process location on the sixth subgroup.

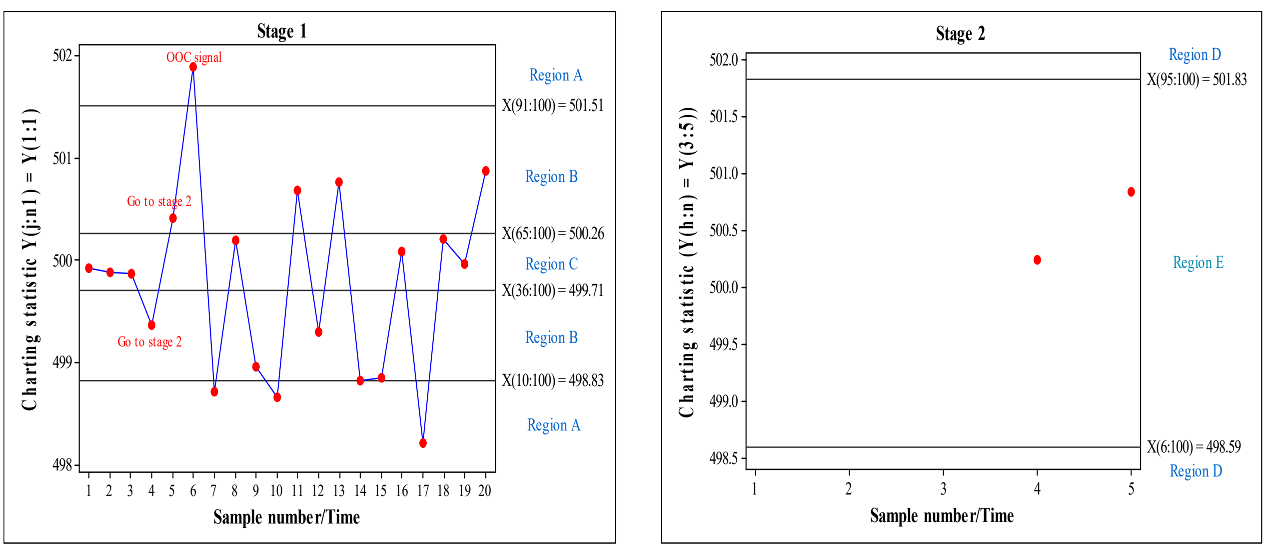


Figure 8. The Phase II DS precedence monitoring scheme for the filling process of 500 *ml* milk bottles

5. Concluding remarks

In this paper, a new nonparametric DS monitoring scheme based on the precedence statistic is introduced. The DS precedence scheme is a two-stage monitoring scheme that uses the j^{th} order statistic in stage 1 and the h^{th} order statistic in stage 2 as the charting statistics. The performance of the DS precedence scheme is investigated using the following run-length properties metrics: ARL, $SDRL$, PRL, $EARL$ and $EANOS$. It is observed that the proposed scheme has attractive run-length properties and better overall performance especially for large Phase I sample sizes. However, for small Phase I sample sizes, the performance of the DS precedence scheme degrades and the attained ARL_0 is too far away from the desired $NARL_0$ value. To improve the sensitivity of the proposed scheme, it is observed that when the stage 1 test sample is small, it is better to use small $\bar{n}$ values since there is no significant increase in the sensitivity of the proposed scheme by using large $\bar{n}$ values. However, for a large stage 1 test sample, reasonably large $\bar{n}$ values can be used. Note though, very large $\bar{n}$ values are not recommended as they lead to high inspection and production costs since the expected total sample size at each sampling point is large. It is observed that the performance of the proposed scheme tend to improve for a large n_2 test sample with a small stage 1 test sample and reasonably large $\bar{n}$ values. Finally, it is shown that the proposed scheme performs better than the existing Shewhart precedence scheme with and without runs-rules.

For future research purpose, the proposed scheme can be studied using the side-sensitive design (see for example Motsepa et al[39]). Moreover, to further improve its performance, the synthetic or group-runs double sampling monitoring scheme based on the precedence statistics can also be studied. Other charting statistics (e.g. sign, signed-rank, exceedance, Wilcoxon rank sum, etc.) can be integrated with the double sampling procedure to form other new efficient nonparametric monitoring schemes. Given the work of Yang and Wu[28,29], other nonparametric memory-type double sampling schemes can also be introduced in the SPM literature.

Data availability statement

The data used in the application example was taken from Castagliola et al[43] and the SAS® v.9.4 programs used here can be requested from the authors.

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