Analysis of linear and nonlinear mathematical models for beams, cables, columns and rods

by

Kirstin Hohls

Submitted in fulfilment of the requirements for the degree

Magister Scientiae

in the Department of Mathematics and Applied Mathematics
in the Faculty of Natural and Agricultural Sciences

University of Pretoria
Pretoria

(December 2021)

Declaration of Authorship

I, the undersigned, hereby declare that the dissertation submitted herewith for the degree Magister Scientiae to the University of Pretoria contains my own, independent work and has not previously been submitted by me for any degree at any other university.

Name: Kirstin Hohls

Date: December 2021

Summary

The aim of the dissertation was to conduct a literature study on mathematical modelling, numerical computation and mathematical analysis of a linear and nonlinear Timoshenko model for the vibration of a rod. In this dissertation the word “rod” is used as a collective name for beams, cables and columns.

The general linear existence theory was stated and applied to a linear rod model with axial force (the adapted Timoshenko rod model). In order to do this, the model considered was written in weak variational form and the required properties shown to hold. It was discovered that, assuming the axial force S is constant, a critical value for S exists such that if S is less than its critical value, then the required properties of the theory do not hold.

The spectral theory for a linear rod model was extended to include an axial force for any combination of three boundary conditions, where S is greater than its critical value. This was done while improving on the rigour of the exposition in Van Rensburg and Van der Merwe (2006). The eigenfunctions for a pinned-pinned rod were then calculated and used to generate a series solution. In the case where S was not greater than this critical value, a formal series solution was investigated.

The finite element method (FEM) was then applied to the adapted Timoshenko rod model with pinned-pinned boundary conditions and the convergence investigated. The first five eigenvalues in the increasing sequence were calculated using the spectral theory and a critical value for S was approximated. For illustrative purposes, the approximations found using FEM were then compared to the results of the series solution.

The semi-linear Timoshenko rod model of Sapir and Reiss (1979) with pinned-pinned boundary conditions was then studied. The problem was written in variational form to apply FEM to the semi-linear model and an original algorithm was derived. The results were compared to those of the linear model for small initial displacement where the axial force neared its critical value. Approximations where the axial force surpassed the critical value of the linear model were also investigated. An interval for a critical value of the nonlinear model (less than that of the linear model) was found. This discovery is contrary to the popular belief that the critical values for linear and nonlinear models are equal.

Peradze and Kalichava (2020) also constructed an algorithm for the semi-linear problem. Three separate estimates were found for the algorithm and

then added to form a “total error estimate”. The structure and readability of the article was improved upon and inconsistencies identified.

The existence theory for the Sapir-Reiss semi-linear Timoshenko rod model presented by Ammari (2002) was shown to be incomplete and contain crucial errors. Some of the identified errors and exclusions were rectified.

In the articles studied, improvements were made regarding the presentation of the work, connections established, and the integrated result written up.

Contents

Acknowledgements

I wish to offer my gratitude to the National Research Foundation for providing the funding for my studies.

I would also like to express my deep gratitude to Professor NF Janse van Rensburg for his invaluable guidance, time and endless patience as my supervisor.

Chapter 1

Introduction

1.1 Mathematical models for the motion of a rod

Beams, cables, columns and rods are mentioned in the title of the dissertation. The terminology is well understood in the engineering literature and Applied Mathematics where the focus is on real world applications and indicates that application is important. The expression "analysis of" is to make it clear that the approach is mathematical.

In this dissertation, "rod" is used to refer to a one-dimensional continuum following, for example, [Ant96] and [LA12]. This implies that "rod" is a collective name for beams, cables, columns etc. In [LA12], Lang and Arnold highlight the importance of rod models in industrial and engineering applications.

The Cosserat model is the most general geometrically exact rod model. An important special case is the local linear Timoshenko model in [VDL21]. In this dissertation our attention is restricted to linear and semi-linear Timoshenko models for small oscillations. Simplified theories such as the Euler-Bernoulli theory are possible, but are not considered. An important objective of the study is the comparison of different models for the same application by theoretical analysis as well as simulation using the finite element method.

To model the motion of a rod, one must first describe the rod in mathematical terms. Consider, for example, a cylinder with length ℓ and radius a. Using coordinates, it is possible to identify each point in the beam and its axis is well defined. Cross-sections may also be rectangular or any other shape as long as the axis passes through the centroids. Such a rod is called prismatic. When the diameter of the cross-sections are small compared to its length, the rod is modelled as a one-dimensional continuum.

The interval $[0, \ell]$ is the reference configuration for the motion of a rod in space. The current configuration (real position at any time) is then determined by six scalar functions or two vector functions: $\bar{r}(x,t)$ is the position of $x \in [0, \ell]$ at time t and $\bar{n}(x,t)$ is the orientation of a cross-section at time t.

So called equations of motion can be derived from conservation laws of momentum and angular momentum, but this is beyond the scope of this study. These equations of motion can be found in [Ant96], [LA12] and [VDL21], for example.

This work is restricted to small planar oscillations of a rod. Let $\{\bar{e}_1, \bar{e}_2, \bar{e}_3\}$ denote an orthonormal set "fixed in space" forming a right-handed triad. To describe the motion, a reference position is also needed. For this the line segment joining the zero vector and $\ell\bar{e}_1$ is used. The position of x in the reference configuration at time t is given by

$$\bar{r}(x,t) = u(x,t)\bar{e}_1 + w(x,t)\bar{e}_2.$$

The rotation of a cross section is described by $\phi(x,t)$.

In Sections 1.2 and 1.3, the mathematical models are introduced.

1.2 The Timoshenko theory

1.2.1 The original Timoshenko model

Consider the original Timoshenko model for a beam. Details on the derivation of the model can be found in [Tim37, pp.337-338], [Fun65, pp.323-324] and [Inm94, pp.337-338].

In the equation below, $w(x,t)$ denotes the transverse displacement or deflection of the rod's axis and $\phi(x,t)$ the angle of rotation of a cross-section (assuming plane cross-sections remain plane). In textbooks, sketches are used to define $w(x,t)$ and $\phi(x,t)$.

Equations of motion

$$\rho A \partial_t^2 w = \partial_x V + P, \tag{1.2.1}$$
$$\rho I \partial_t^2 \phi = V + \partial_x M. \tag{1.2.2}$$

In these equations ρ denotes the density, A the area of a cross section, I the area moment of inertia, M the moment, V the shear force and P the transverse load.

Constitutive equations

$$M = EI\partial_x \phi, \tag{1.2.3}$$
$$V = AG\kappa^2(\partial_x w - \phi), \tag{1.2.4}$$

where E and G are elastic constants and κ^2 the shear correction factor.

Dimensionless form

The length of the beam is denoted by ℓ. Set

$$\tau = \frac{t}{T}, \quad \xi = \frac{x}{\ell}, \quad w^*(\xi,\tau) = \frac{w(x,t)}{\ell} \quad \text{and} \quad \phi^*(\xi,\tau) = \phi(x,t),$$

where T is to be specified. Following [VV06], the forces, force densities and moments are then scaled by $AG\kappa^2$, $AG\kappa^2\ell^{-1}$ and $AG\kappa^2\ell$ respectively. That is,

$$P^*(\xi,\tau) = \frac{\ell P(x,t)}{AG\kappa^2}, \quad V^*(\xi,\tau) = \frac{V(x,t)}{AG\kappa^2} \quad \text{and} \quad M^*(\xi,\tau) = \frac{M(x,t)}{AG\kappa^2\ell}.$$

The following dimensionless constants are introduced:

$$\alpha = \frac{A\ell^2}{I} \quad \text{and} \quad \beta = \frac{AG\kappa^2\ell^2}{EI}.$$

Lastly, choose (as in [VV06])

$$T = \ell\sqrt{\frac{\rho}{G\kappa^2}}.$$

From this point onward, the notation V, M, P, w and ϕ is used instead of V^*, M^*, P^*, w^* and ϕ^* respectively. The equations of motion and constitutive equations then form the Timoshenko model. This is referred to as Model T, which can be applied to "beams", "cables" and even "wires".

Model T Find $w(x,t)$ and $\phi(x,t)$ such that

$$\partial_t^2 w = \partial_x V + P, \tag{1.2.5}$$

$$\frac{1}{\alpha}\partial_t^2 \phi = V + \partial_x M, \tag{1.2.6}$$

$$M = \frac{1}{\beta}\partial_x \phi, \tag{1.2.7}$$

$$V = \partial_x w - \phi. \tag{1.2.8}$$

Three sets of boundary conditions are considered for the model.
Pinned-pinned rod:

$$w(0,t) = \partial_x\phi(0,t) = w(1,t) = \partial_x\phi(1,t) = 0, \tag{1.2.9}$$

Cantilever rod:

$$w(0,t) = \phi(0,t) = \partial_x\phi(1,t) = \partial_x w(1,t) - \phi(1,t) = 0, \tag{1.2.10}$$

Clamped-clamped rod:

$$w(0,t) = \phi(0,t) = \phi(1,t) = w(1,t) = 0. \tag{1.2.11}$$

If the load $P = 0$, then $w = \phi = 0$ is an equilibrium solution for the model.

It should be noted that $\frac{\beta}{\alpha} = \frac{G\kappa^2}{E}$. Also, $\kappa^2 \in [\frac{1}{2}, 1]$ and therefore $\gamma = \frac{\beta}{\alpha} \in [\frac{1}{6}, \frac{1}{2}]$ (see for example [LVV09] where references are provided). Significant variation is possible for the constant α and thus also for β.

For theoretical purposes it is necessary to assume that $\alpha > \beta > 1$ (see Section 2.3). This also happens to be realistic from a physical perspective. Elementary calculations show that, even for $\beta = 2$, no one will consider the relevant object to be a beam (or cable). On the other hand, it will be shown that β cannot be too large since a beam is supposed to resist bending effectively. For a cable, β (and therefore α) can be large and the simpler Euler-Bernoulli model may be used.

1.2.2 Axial force

The equation of motion for longitudinal vibration is given by

$$\rho A \partial_t^2 u = \partial_x S + Q, \tag{1.2.12}$$

where u is the displacement, S is the axial force and Q the axial load (see for example [Inm94]).

Using **Hooke's Law** in its simplest form yields the constitutive equation

$$S = AE\partial_x u. \tag{1.2.13}$$

For the dimensionless form, the force S is scaled in exactly the same way as the force V in the previous section. Hooke's Law then becomes

$$S = \frac{1}{\gamma}\partial_x u. \tag{1.2.14}$$

The other constitutive equations are the same as for the standard Timoshenko model.

In Equation (1.2.1) for the standard Timoshenko model, it is tacitly assumed that the shear force V is equal to the transverse force. However, due to the bending of the rod, the axis becomes curved (and is referred to as the deflection curve). It is hence more realistic to assume that the transverse force F is given by

$$F = S\sin\theta + V\cos\theta,$$

where θ is the rotation of the tangent vector. For a linear model and some nonlinear models, it is assumed that θ is sufficiently small to justify the assumption $\sin\theta = \partial_x w$ and $\cos\theta = 1$. (See, for example, [SR79] and [LL91].) Consequently,

$$F = S\partial_x w + V.$$

The dimensionless equations of motion for a Timoshenko rod with axial force are given below.

Model T-AF

$$\begin{aligned} \partial_t^2 u &= \partial_x S + Q, & (1.2.15)\\ \partial_t^2 w &= \partial_x (S\partial_x w) + \partial_x V + P, & (1.2.16)\\ \frac{1}{\alpha}\partial_t^2 \phi &= V + \partial_x M. & (1.2.17) \end{aligned}$$

The constitutive equations are (1.2.7), (1.2.8) and (1.2.14).

The boundary conditions for the standard Timoshenko rod (1.2.9), (1.2.10) and (1.2.11) remain. Boundary conditions for the displacement u are required. For the pinned-pinned rod as well as the clamped-clamped rod they are

$$u(0,t) = 0, \quad u(1,t) = D, \tag{1.2.18}$$

with D given. For the cantilever rod the boundary conditions are

$$u(0,t) = \partial_x u(1,t) = 0. \tag{1.2.19}$$

1.2.3 Adapted Timoshenko model

Equation (1.2.15) decouples from the rest of the system since it is the one-dimensional wave equation and together with the standard boundary conditions the system is "well formulated". Once this problem is solved, S and u are known. Consequently, the transverse vibration may be considered, but note that S is now a "time dependent" parameter. This variant is not considered in the present study. However, in some realistic applications, $\partial_t Q = 0$ and then $\partial_t S = 0$ if there is no dynamic boundary forcing. Equation (1.2.15) then reduces to

$$0 = \frac{dS}{dx} + Q.$$

This equation, together with the boundary conditions above, determine S and u uniquely. The model is referred to as the adapted Timoshenko theory. If $Q = 0$ for a rod with boundary conditions (1.2.18), then $S = \dfrac{D}{\gamma}$, a constant. This constant is denoted by S_0 and can be positive or negative. (This case is often referred to as a pre-stressed rod.) In [CVV18], the case where S_0 is positive and "large" is studied.

Model AT

Equations of motion

$$\begin{aligned} \partial_t^2 w &= \partial_x (S\partial_x w) + \partial_x V + P, & (1.2.20)\\ \frac{1}{\alpha}\partial_t^2 \phi &= V + \partial_x M. & (1.2.21) \end{aligned}$$

with constitutive equations

$$\begin{aligned} M &= \frac{1}{\beta}\partial_x \phi, & (1.2.22)\\ V &= \partial_x w - \phi. & (1.2.23) \end{aligned}$$

The three sets of boundary conditions are the same as for the Timoshenko model.

Pinned-pinned rod:

$$w(0,t) = \partial_x\phi(0,t) = w(1,t) = \partial_x\phi(1,t) = 0, \tag{1.2.24}$$

Cantilever rod:

$$w(0,t) = \phi(0,t) = V(1,t) = M(1,t) = 0, \tag{1.2.25}$$

Clamped-clamped rod:

$$w(0,t) = \phi(0,t) = w(1,t) = \phi(1,t) = 0. \tag{1.2.26}$$

It is important to note the fact that Model T is a special case of Model AT, where the force $S = 0$.

1.2.4 Damping

Although damping terms are included in some of the relevant publications, damping is not the main concern in this study. Different constitutive equations to model damping are found in the literature, where [Inm94] is a useful reference. Viscous damping is a possibility: The “load” P in (1.2.20) is then due to damping and is replaced by $-\mu_1\partial_t w$ for some $\mu_1 > 0$. Another possibility is strain rate damping, where (1.2.22) changes to $M = \frac{1}{\beta}\partial_x\phi + \mu_2\partial_t\partial_x\phi$ with $\mu_2 > 0$. Strain rate damping will not be considered in this dissertation.

In [Amm02] a damping term of the form $-\mu_3\partial_t\phi$ is introduced in (1.2.21). No explanation is given, which makes one sceptical. However, in the mathematical analysis the type of damping does not matter provided that it dissipates energy (see Section 2.5). For the sake of comparison, the same damping terms as [Amm02] are included. It is assumed that μ_1 and μ_3 may be zero.

1.3 Nonlinear Timoshenko rod models

1.3.1 Semi-linear Timoshenko rod model

Recall the Timoshenko model for a rod with an axial force (Model T-AF). For the nonlinear theory, the axial force S is given by

$$S = EA\left(\partial_x s - 1\right), \tag{1.3.1}$$

where s is the arc length function. In dimensionless form, S is rewritten as

$$S = \frac{1}{\gamma}\left(\partial_x s - 1\right). \tag{1.3.2}$$

The constitutive equation for S is then found by approximating s. The approximation

$$\partial_x s - 1 \approx \partial_x u + \frac{1}{2}\left(\partial_x w\right)^2$$

is often used (see for example [SR79], [LL91] or [VDL21]).

Model SLT

Equations of motion

$$\partial_t^2 u = \partial_x S + Q, \tag{1.3.3}$$
$$\partial_t^2 w = \partial_x(S\partial_x w) + \partial_x V + P_2, \tag{1.3.4}$$
$$\frac{1}{\alpha}\partial_t^2 \phi = V + \partial_x M, \tag{1.3.5}$$

with constitutive equations

$$M = \frac{1}{\beta}\partial_x \phi, \tag{1.3.6}$$
$$V = \partial_x w - \phi, \tag{1.3.7}$$
$$S = \frac{1}{\gamma}\partial_x u + \frac{1}{2\gamma}(\partial_x w)^2. \tag{1.3.8}$$

The three sets of boundary conditions considered are the same as for Model T-AF. It is important to note that in [SR79] only pinned-pinned boundary conditions are considered.

1.3.2 Semi-linear Timoshenko beam of Sapir and Reiss

In [SR79] the authors consider the case where the axial load density Q is zero. Two additional assumptions are made:

$$\partial_t^2 u = 0 \quad \text{and} \quad \partial_x S = 0. \tag{1.3.9}$$

It follows that $\int_0^1 S(t) = S(t)$, and hence

$$S(t) = \frac{1}{\gamma}u(1,t) + \frac{1}{2\gamma}\int_0^1 (\partial_x w(\cdot,t))^2. \tag{1.3.10}$$

The boundary conditions (1.2.18) for u are now redundant, but $D = u(1)$ now becomes a parameter in the constitutive Equation (1.3.10). That is,

$$S(t) = \frac{D}{\gamma} + \frac{1}{2\gamma}\int_0^1 (\partial_x w(\cdot, t))^2. \tag{1.3.11}$$

Alternatively, one may use the axial force $S_0 = D/\gamma$ as a given parameter. Model SLT is thereby simplified.

Model SLT-SR Equations of motion

$$\partial_t^2 w = S\partial_x^2 w + \partial_x V, \tag{1.3.12}$$
$$\frac{1}{\alpha}\partial_t^2 \phi = V + \partial_x M, \tag{1.3.13}$$

with constitutive equations

$$M = \frac{1}{\beta}\partial_x \phi, \tag{1.3.14}$$
$$V = \partial_x w - \phi, \tag{1.3.15}$$
$$S = S_0 + \frac{1}{2\gamma}\int_0^1 (\partial_x w(\cdot, t))^2. \tag{1.3.16}$$

The boundary conditions for the rod are

$$w(0,t) = \partial_x\phi(0,t) = w(1,t) = \partial_x\phi(1,t) = 0 \quad \text{(pinned-pinned)}, \tag{1.3.17}$$
$$w(0,t) = \phi(0,t) = w(1,t) = \phi(1,t) = 0 \quad \text{(clamped-clamped)}. \tag{1.3.18}$$

The model was first derived from two-dimensional elasticity in [SR79], but in this dissertation it is considered as one of the special cases in [VDL21]. In [SR79] only pinned-pinned boundary conditions are considered and only the case where $S_0 < 0$. In the article, the motion of a buckled rod is analysed.

Remark. *The report [VDB16] was consulted to write Sections 1.2 and 1.3. An improved version of this report is the article [VDL21], which appeared recently.*

1.4 Mathematical analysis and computation

The aim of this research is to study a number of model problems for partial differential equations with the same real world application. The study includes existence theory, spectral theory, finite element (FEM) analysis and implementation.

The articles [VV06], [CST16], [CVV18], [PK20] and [Amm02] were initially chosen to be the main sources for the study. As expected, other sources were needed as the research progressed.

The existence of solutions for the linear models T and AT was to be proved using the general theory in [VV02]. The proofs in the article itself were not to be studied in depth, but rather the applicability of the theory.

Modal analysis for linear problems was considered next. In the paper [CVV18] a rigorous basis for modal analysis is provided using results from functional analysis. It was thought that this should be applied to Models T and AT. In the articles [VV06] and [CST16], natural frequencies and modes of vibration for the standard Timoshenko rod (Model T) were derived.The purpose of this study was to compare the articles, evaluate the level of rigour and investigate the extension of the results to Model AT.

Application of FEM to the linear models was planned and hence it was natural to investigate convergence. For this purpose the paper [BV13] was identified.

The model presented in [SR79] was of great importance for the study because although it is not linear, the application is the same as for Model AT. It is significant that the semi-linear model SLT-SR differs “slightly” from Model AT. The main difference is the term containing the integral of $(\partial_x w)^2$, which should be negligible for small vibrations. The authors of [SR79] were interested in properties of solutions, especially buckling and post-buckling behaviour. For this research the main concern was existence, FEM analysis and application as well as comparison to the linear model, Model AT.

I order to carry out FEM analysis of the nonlinear model, the article [PK20] was chosen. In it the authors present an algorithm to simulate the oscillation of solutions of Model SLT-SR. They prove that the FEM approximation converges to the exact solution and derive error estimates.

The biggest challenge was to study the existence theory for Model SLT-SR in [Amm02]. In the nonlinear theory one is confronted by mathematical methods not encountered in the linear theory.

Chapter 2

Analysis of linear vibration models

In this chapter a general second order hyperbolic problem in variational form is considered (linear vibration models are special cases). General existence results from the literature ([VV02] and [VS19]) are discussed as well as the theoretical foundation for modal analysis in [CVV18]. The standard Timoshenko theory is used as an example.

2.1 Variational approach to the Timoshenko model

Much has been written in books and articles on the Timoshenko rod model. However, the variational approach to the theory attracted less attention despite certain advantages. The first being that the variational form is effectively the same for all possible homogeneous boundary conditions. Secondly, there is a single point of departure for the existence theory, FEM theory and FEM application. Consequently, a detailed treatment of the variational approach is desirable.

The variational form and weak variational form for Model T in this section and Section 2.3 can easily be adapted for the Adapted Timoshenko model and the semi-linear Timoshenko model of Sapir and Reiss ([SR79]) in later chapters. The weak variational form also serves as a non-trivial motivation for the general theory in Sections 2.4 to 2.7.

Properties of function spaces are investigated in Sections 2.2 and 2.3 to prepare for the general case in the rest of this chapter as well as other models in Chapters 3 and 5 to 7.

2.1.1 Timoshenko model problem

The boundary conditions considered in Models T, AT and SLT-SR are

$$w(0,t) = \partial_x\phi(0,t) = w(1,t) = \partial_x\phi(1,t) = 0 \quad \text{for } t > 0, \tag{2.1.1}$$
$$w(0,t) = \phi(0,t) = \partial_x\phi(1,t) = \partial_x w(1,t) - \phi(1,t) = 0 \ \text{ for } t > 0, \tag{2.1.2}$$
$$w(0,t) = \phi(0,t) = w(1,t) = \phi(1,t) = 0 \quad \text{for } t > 0. \tag{2.1.3}$$

Equations (2.1.1), (2.1.2) and (2.1.3) are the boundary conditions for a pinned-pinned rod, cantilever rod and clamped-clamped rod respectively.

Problem T Given α and β, find w and ϕ such that

$$\partial_t^2 w = \partial_x V + P - \mu_1\partial_t w \quad \text{in } (0,1) \text{ for } t > 0, \tag{2.1.4}$$
$$\frac{1}{\alpha}\partial_t^2\phi = V + \partial_x M - \mu_3\partial_t\phi \quad \text{in } (0,1) \text{ for } t > 0, \tag{2.1.5}$$

with boundary conditions (2.1.1), (2.1.2) or (2.1.3). The accompanying constitutive equations are

$$M = \frac{1}{\beta}\partial_x\phi \quad \text{in } (0,1) \text{ for } t > 0, \tag{2.1.6}$$
$$V = \partial_x w - \phi \quad \text{in } (0,1), \text{ for } t > 0 \tag{2.1.7}$$

and the initial conditions considered are

$$w(x,0) = w_0(x) \quad \text{for } x \in (0,1), \tag{2.1.8}$$
$$\phi(x,0) = \phi_0(x) \quad \text{for } x \in (0,1), \tag{2.1.9}$$
$$\partial_t w(x,0) = w_d(x) \quad \text{for } x \in (0,1), \tag{2.1.10}$$
$$\partial_t \phi(x,0) = \phi_d(x) \quad \text{for } x \in (0,1). \tag{2.1.11}$$

For the classical Timoshenko theory, $\mu_1 = \mu_3 = 0$. In the rest of the section this will be assumed. Damping will only be mentioned where relevant.

Remark. *The parameters α and β are bounded, with a positive infimum and do not to vary greatly. In order to simplify the theory in this dissertation, α and β are assumed to be constant. The theory still holds if this assumption is not made.*

2.1.2 Variational form

In order to carry out analysis and generate an approximation for a solution of the model problem, it is necessary to derive the variational form. To do this, (2.1.4) and (2.1.5) are multiplied by arbitrary functions v_1 and v_2 respectively, where v_1 and v_2 are in $C^1[0,1]$. The two equations are then integrated over $[0,1]$. This results in

$$\int_0^1 \partial_t^2 w v_1 = \int_0^1 \partial_x V v_1 + \int_0^1 P v_1 \quad \text{in } (0,1) \text{ for } t > 0, \tag{2.1.12}$$

$$\int_0^1 \frac{1}{\alpha} v_2 \partial_t^2 \phi = \int_0^1 V v_2 + \int_0^1 \partial_x M v_2 \quad \text{in } (0,1) \text{ for } t > 0. \tag{2.1.13}$$

The following notation is convenient to define the test functions.

$$T_1[0,1] = \{v \in C^1[0,1] \mid v(0) = v(1) = 0\}, \tag{2.1.14}$$

$$T_2[0,1] = \{v \in C^1[0,1] \mid v(0) = 0\}. \tag{2.1.15}$$

Also, the set of infinitely differentiable functions with support contained in $(0,1)$ is notated $C_0^\infty(0,1)$. More detail and the definition of "support" can be found in Appendix A.

Recall that for f and g in $\mathcal{L}^2(0,1)$, $\int_0^1 fg$ is an inner product on $\mathcal{L}^2(0,1)$.

Definition. The notation $(\cdot,\cdot)$ is used for the **inner product** on $\mathcal{L}^2(0,1)$, where

$$(f,g) = \int_0^1 fg.$$

The **norm** for $\mathcal{L}^2(0,1)$ is induced by $(\cdot,\cdot)$ and notated

$$\|f\|^2 = (f,f).$$

Note that if w_1 and w_2 are in $\mathcal{L}^2(0,1)$, then the notation $w_1 = w_2$ implies equality almost everywhere. A crucial result is that $C_0^\infty(0,1)$ is dense in $\mathcal{L}^2(0,1)$ [OR76, Theorem 2.7].

The generic notation T_P is used throughout the dissertation to denote the set of test functions which is specific to given boundary conditions. This allows for the variational form of different problems to be defined using one notation. For the boundary conditions mentioned in (2.1.1)-(2.1.3), there are three variations for T_P:

Case 1 *(Pinned-pinned rod)* $T_P = T_1[0,1] \times C^1[0,1]$;

Case 2 *(Cantilever rod)* $T_P = T_2[0,1] \times T_2[0,1]$;

Case 3 *(Clamped-clamped rod)* $T_P = T_1[0,1] \times T_1[0,1]$.

In each case, the variational form only differs in the definition of the set of test functions T_P. Using integration by parts in (2.1.12) and (2.1.13), it follows that for every $\langle v_1, v_2 \rangle \in T_P$,

$$\int_0^1 \partial_t^2 w(\cdot,t)v_1 = -\int_0^1 V(\cdot,t)v_1' + \int_0^1 P(\cdot,t)v_1, \tag{2.1.16}$$

$$\int_0^1 \frac{1}{\alpha}\partial_t^2 \phi(\cdot,t)v_2 = \int_0^1 V(\cdot,t)v_2 - \int_0^1 M(\cdot,t)v_2'. \tag{2.1.17}$$

The variational form of Problem T may now be presented.

Problem TV Given the load P, find $\langle w, \phi \rangle$ such that for each $t > 0$, $\langle w(\cdot,t), \phi(\cdot,t) \rangle \in T_P$ and for every $\langle v_1, v_2 \rangle \in T_P$,

$$\int_0^1 \partial_t^2 w(\cdot,t)v_1 = -\int_0^1 V(\cdot,t)v_1' + \int_0^1 P(\cdot,t)v_1, \tag{2.1.18}$$

$$\int_0^1 \frac{1}{\alpha}\partial_t^2 \phi(\cdot,t)v_2 = \int_0^1 V(\cdot,t)v_2 - \int_0^1 M(\cdot,t)v_2'. \tag{2.1.19}$$

The accompanying constitutive equations are

$$M = \frac{1}{\beta}\partial_x \phi,$$
$$V = \partial_x w - \phi.$$

Clearly a solution of Problem T is a solution of Problem TV. It is shown below that if a solution $\langle w, \phi \rangle$ of Problem TV satisfies certain conditions then it is a solution of Problem T.

Proposition 2.1.1. *If $\langle w, \phi \rangle$ is a solution of Problem TV and $\langle w(\cdot,t), \phi(\cdot,t) \rangle$ is in $C^2[0,1] \times C^2[0,1]$ for $t > 0$, then $\langle w, \phi \rangle$ is a solution of Problem T.*

Proof. Suppose that, for each $t > 0$, $\langle w(\cdot,t), \phi(\cdot,t) \rangle \in T_P$ and

$$\int_0^1 \partial_t^2 w(\cdot,t)v_1 = -\int_0^1 V(\cdot,t)v_1' + \int_0^1 P(\cdot,t)v_1, \tag{2.1.20}$$

$$\int_0^1 \frac{1}{\alpha}\partial_t^2 \phi(\cdot,t)v_2 = \int_0^1 V(\cdot,t)v_2 - \int_0^1 M(\cdot,t)v_2' \tag{2.1.21}$$

for every $\langle v_1, v_2\rangle \in T_P$.

Since $C_0^\infty(0,1)$ is contained in $T_1[0,1]$, $T_2[0,1]$ and $C^1[0,1]$, Equations (2.1.20) and (2.1.21) hold for every $v \in C_0^\infty(0,1) \times C_0^\infty(0,1)$.

After performing integration by parts on (2.1.20) it follows that

$$\int_0^1 \left(\partial_t^2 w(\cdot,t) - \partial_x V(\cdot,t) - P(\cdot,t)\right) v_1 = 0.$$

Since $C_0^\infty(0,1)$ is dense in $\mathcal{L}^2(0,1)$, for each $v_1 \in \mathcal{L}^2(0,1)$,

$$\int_0^1 \left(\partial_t^2 w(\cdot,t) - \partial_x V(\cdot,t) - P(\cdot,t)\right) v_1 = 0.$$

That is,

$$\partial_t^2 w(\cdot,t) - \partial_x V(\cdot,t) - P(\cdot,t) = 0. \tag{2.1.22}$$

Also, after performing integration by parts on (2.1.21) it follows that

$$\int_0^1 \left(\frac{1}{\alpha}\partial_t^2 \phi(\cdot,t) - V(\cdot,t) - \partial_x M(\cdot,t)\right) v_2 \;=\; 0.$$

Again by the density of $C_0^\infty(0,1)$ in $\mathcal{L}^2(0,1)$, for each $v_2 \in \mathcal{L}^2(0,1)$,

$$\int_0^1 \left(\frac{1}{\alpha}\partial_t^2 \phi(\cdot,t) - V(\cdot,t) - \partial_x M(\cdot,t)\right) v_2 \;=\; 0$$

This implies that

$$\frac{1}{\alpha}\partial_t^2 \phi(\cdot,t) - V(\cdot,t) - \partial_x M(\cdot,t) = 0. \tag{2.1.23}$$

To show that a solution of the system (2.1.22), (2.1.23) satisfies the boundary conditions, consider Case 2. Note that $w(\cdot,t)$ and $\phi(\cdot,t)$ are in $T_2[0,1]$. That is

$$w(0,t) = \phi(0,t) = 0. \tag{2.1.24}$$

It remains to show that $\partial_x \phi(1,t) = \partial_x w(1,t) - \phi(1,t) = 0$.

If (2.1.22) is multiplied by $v_1 \in T_2[0,1]$ and integrated over $[0,1]$, then, using integration by parts,

$$\int_0^1 \left(\partial_t^2 w(\cdot,t) v_1 + V(\cdot,t) v_1' - P(\cdot,t) v_1\right) - V(1,t) v_1(1) = 0.$$

But $w(\cdot,t)$ satisfies (2.1.20) for $t>0$. Therefore, for each $v_1 \in T_2[0,1]$,

$$V(1,t)v_1(1) = 0.$$

From the constitutive equations, $V = \partial_x w - \phi$. Also, for $v_1 \in T_2[0,1]$, $v_1(1)$ is arbitrary. Therefore

$$\partial_x w(1,t) - \phi(1,t) = 0. \tag{2.1.25}$$

Similarly, if (2.1.23) is multiplied by $v_2 \in T_2[0,1]$ and integrated over $[0,1]$, then, using integration by parts,

$$\int_0^1 \left(\frac{1}{\alpha}\partial_t^2\phi(\cdot,t)v_2 - V(\cdot,t)v_2 + M(\cdot,t)v_2'\right) - M(1,t)v_2(1) = 0.$$

But $\phi(\cdot,t)$ satisfies (2.1.21) for $t>0$. Therefore, for each $v_2 \in T_2[0,1]$,

$$M(1,t)v_2(1) = 0.$$

From the constitutive equations, $M = \frac{1}{\beta}\partial_x\phi$. Also, for $v_2 \in T_2[0,1]$, $v_2(1)$ is arbitrary. Therefore

$$\partial_x\phi(1,t) = 0. \tag{2.1.26}$$

That is, by (2.1.24), (2.1.25) and (2.1.26), the boundary conditions in Case 2 are satisfied.

The proof that the boundary conditions are satisfied for Case 1 or Case 3 is similar.

It can therefore be concluded that $\langle w, \phi\rangle$ is a solution of Problem T. □

For theoretical purposes, the sum of Equations (2.1.16) and (2.1.17) is used. (To implement the finite element method this is not done.) Define the following notation.

Bilinear forms For f and g in $\mathcal{L}^2(0,1)\times\mathcal{L}^2(0,1)$ and u and v in T_P, define the following symmetric bilinear forms:

$$\begin{aligned}
c_T(f,g) &= (f_1,g_1) + \frac{1}{\alpha}(f_2,g_2), && (2.1.27)\\
a(f,g) &= \mu_1(f_1,g_1) + \mu_3(f_2,g_2), && (2.1.28)\\
b_T(u,v) &= \frac{1}{\beta}(u_2',v_2') + (u_1'-u_2, v_1'-v_2), && (2.1.29)\\
(f,g)^{(2)} &= (f_1,g_1) + (f_2,g_2). && (2.1.30)
\end{aligned}$$

Let $f(\cdot,t) = \langle P(\cdot,t), 0\rangle$. Then **Problem TV** may be written as follows:

Find $y = \langle w, \phi\rangle$ such that for each $t > 0$, $y(\cdot,t) \in T_P$ and

$$c_T\left(\partial_t^2 y(\cdot,t), v\right) + a\left(\partial_t y(\cdot,t), v\right) + b_T\left(y(\cdot,t), v\right) = \left(f(\cdot,t), v\right)^{(2)} \quad (2.1.31)$$

for each $v \in T_P$, where

$$\langle w(x,0), \phi(x,0)\rangle = \langle w_0(x), \phi_0(x)\rangle\,, \quad \langle \partial_t w(x,0), \partial_t \phi(x,0)\rangle = \langle w_d(x), \phi_d(x)\rangle\,.$$

The product space $\mathcal{L}^2(0,1)^2 = \mathcal{L}^2(0,1) \times \mathcal{L}^2(0,1)$ features prominently in the theory.

Proposition 2.1.2. *The bilinear form* $(\cdot,\cdot)^{(2)}$ *is an inner product for the space* $\mathcal{L}^2(0,1)^2$*, which is complete.*

Proof. To prove the symmetry, linearity and non-negativity of the bilinear form is trivial. Suppose $f \in \mathcal{L}^2(0,1)^2$ is such that

$$(f,f)^{(2)} = 0.$$

Since $\|f_1\|^2 + \|f_2\|^2 = 0$, it follows that $f_1 = f_2 = 0$ *a.e.* and hence $f = 0$ *a.e.*. Therefore the bilinear form satisfies the conditions in the definition of an inner product. Finally, the cartesian product of two Hilbert spaces is a Hilbert space. □

The variational problem (Problem TV) is suitable for FEM calculations, but not for any of the theory. The spaces of test functions are not complete. In Section 2.3, after discussing Sobolev's embedding theorem in one dimension in Section 2.2, further properties of the relevant function spaces will be derived.

2.2 Sobolev's embedding theorem

Sobolev space theory is required for the theory in this dissertation (existence theory, modal analysis and convergence for the finite element method). In the literature consulted ([OR76], [Sho77] and [Eva98]), weak derivatives and other properties of functions are presented for functions defined on a subset of $\mathbb{R}^n$. It is usually assumed that functions defined on an interval of the real line can be treated as a special case. In Appendix A, results of Sobolev space theory are given for functions defined on a subset of $\mathbb{R}^n$, where those defined on one-dimensional intervals are special cases.

In this section, Sobolev's embedding theorem is stated and proved because the one-dimensional case differs from the multi-dimensional case and is important for the theory in this dissertation. It is shown that for the one-dimensional case, if a function has weak derivatives up to order m, then it may be considered to have continuous derivatives up to order $m-1$. That is, if $u \in H^1(0,1)$, then there exists a unique continuous function v such that $u = v$ *a.e.* and u may be considered to be continuous.

Some preliminary results, used in the proofs, are presented first.

Proposition 2.2.1. *If $f \in C^1[0,1]$, then*

$$\|f\|_{\sup} \leq \sqrt{2}\|f\|_1. \tag{2.2.1}$$

Proof. Let f be an arbitrary function in $C^1[0,1]$. If f has a zero, then using the Fundamental Theorem of Calculus and the Cauchy-Schwartz inequality,

$$|f(x)| \leq \int_0^1 |f'| \leq \|f'\|.$$

Therefore the Poincaré type inequality follows:

$$\|f\|_{\sup} \leq \|f'\|. \tag{2.2.2}$$

If f is positive, then it has a minimum $m > 0$. Let $g = f - m$. Then the function g has a zero in $[0,1]$ and hence $\|g\|_{\sup} \leq \|g'\| = \|f'\|$. Therefore

$$\|f\|_{\sup} \leq m + \|f'\| \leq \|f\| + \|f'\|.$$

Therefore, using the inequality $2|ab| \leq a^2 + b^2$ for real numbers a and b,

$$(\|f\|_{\sup})^2 \quad \leq \quad 2\|f\|^2 + 2\|f'\|^2 = 2\|f\|_1^2,$$

If f is negative, $-f$ may be substituted into (2.2.1). Consequently, (2.2.1) holds for any $f \in C^1[0,1]$. □

Proposition 2.2.2. *If $u \in H^1(0,1)$ then there exists a unique $u^* \in C[0,1] \cap H^1(0,1)$ such that $u = u^*$ a.e. and $\|u^*\|_{\sup} \leq \sqrt{2}\|u\|_1$.*

Proof. Let $u \in H^1(0,1)$. Since $H^1(0,1)$ is the closure of $C^1[0,1]$ using the norm $\|\cdot\|_1$ (see Appendix A), there exists a sequence $(g_n) \subset C^1[0,1]$ such that

$$\|g_n - u\|_1 \to 0.$$

This sequence is Cauchy in $H^1(0,1)$ and hence Cauchy in $(C[0,1], \|\cdot\|_{\text{sup}})$ from Proposition 2.2.1. But $(C[0,1], \|\cdot\|_{\text{sup}})$ is complete. Denote the unique limit of (g_n) in $C[0,1]$ by u^*. Then, since $\|g_n - u^*\| \le \|g_n - u^*\|_{\text{sup}}$, it follows that

$$\|g_n - u^*\| \to 0.$$

Finally, $\|g_n - u\| \to 0$ since $\|g_n - u\| \le \|g_n - u\|_1$ and hence $u = u^*$ a.e. by the uniqueness of limits. Since $\|g_n\|_{\text{sup}} \le \sqrt{2}\|g_n\|_1$, it follows that

$$\|u^*\|_{\text{sup}} \le \sqrt{2}\|u\|_1.$$

Since any two continuous functions equal almost everywhere are identical, it follows that u^* is unique in $C[0,1] \cap H^1(0,1)$ such that $u = u^*$ a.e. and $\|u^*\|_{\text{sup}} \le \sqrt{2}\|u\|_1$. □

Definition. For $u \in C^m[0,1]$, let

$$\|u\|_{\text{sup}}^m = \sum_{k=0}^{m} \|u^{(k)}\|_{\text{sup}}.$$

It is now possible to present the proof of Sobolev's Lemma and Embedding Theorem.

Theorem 2.2.1 (Sobolev's Lemma in one dimension)**.**
If $u \in C^m[0,1]$ *then*

$$\|u\|_{\text{sup}}^{m-1} \le 2\sqrt{m}\|u\|_m. \tag{2.2.3}$$

Proof. By Proposition 2.2.1, the estimate

$$\left(\|u^{(j)}\|_{\text{sup}}\right)^2 \le 2\|u^{(j)}\|^2 + 2\|u^{(j+1)}\|^2 \tag{2.2.4}$$

holds for the jth derivative of u, where $j < m$. Summing over j yields

$$\sum_{j=0}^{m-1} \left(\|u^{(j)}\|_{\text{sup}}\right)^2 \le 4\|u\|_m^2. \tag{2.2.5}$$

The inequality $\left(\sum_{j=0}^{m-1} \|u^{(j)}\|_{\text{sup}}\right)^2 \le m \sum_{j=0}^{m-1} \left(\|u^{(j)}\|_{\text{sup}}\right)^2$ yields (2.2.3). □

Theorem 2.2.2 (Sobolev's Embedding Theorem in one dimension)**.**
If $u \in H^m(0,1)$ then there exists a unique $u^* \in C^{m-1}[0,1]$ such that $u = u^*$ a.e. and $\|u^*\|_{\sup}^{m-1} \leq m\sqrt{2}\|u\|_m$.

Proof. Suppose $u \in H^2(0,1)$.

Since $H^2(0,1)$ is the closure of $C^2[0,1]$ using the norm $\|\cdot\|_2$ (see Appendix A), there exists a sequence $(u_n) \subset C^2[0,1]$ such that

$$\lim_{n\to\infty} \|u_n - u\|_2 = 0. \tag{2.2.6}$$

This implies that

$$\lim_{n\to\infty} \|u_n - u\| = 0, \tag{2.2.7}$$
$$\lim_{n\to\infty} \|u_n' - Du\| = 0, \tag{2.2.8}$$
$$\lim_{n\to\infty} \|u_n'' - D^2u\| = 0. \tag{2.2.9}$$

By Proposition 2.2.2 there exists a unique $u^* \in C[0,1]$ such that $u = u^*$ a.e. and a unique $\hat{u} \in C[0,1]$ such that $Du = \hat{u}$ a.e..

Since (u_n) and (u_n') are contained in $C^1[0,1]$, it follows by Proposition 2.2.1 that they are Cauchy sequences in $(C[0,1], \|\cdot\|_{\sup})$. Thus,

$$\lim_{n\to\infty} \|u_n - v\|_{\sup} = 0,, \tag{2.2.10}$$
$$\lim_{n\to\infty} \|u_n' - w\|_{\sup} = 0. \tag{2.2.11}$$

Hence, by the uniqueness of a limit, $v = u^*$ and, by the implied uniform convergence, $w = (u^*)'$. Also, Equation (2.2.11) implies that

$$\lim_{n\to\infty} \|u_n' - w\| = 0. \tag{2.2.12}$$

Equations (2.2.8) and (2.2.12) imply that $Du = \hat{u} = (u^*)'$ a.e. and hence $u^* \in C^1[0,1]$. Also, from Proposition 2.2.1, $\|u_n\|_{\sup}^1 \leq 2\sqrt{2}\|u_n\|_2$. Hence $\|u\|_{\sup}^1 \leq 2\sqrt{2}\|u\|_2$. This proves that the theorem holds for the case $m = 2$.

Assume that, for $m = 1, 2, \ldots, n$, if $u \in H^m(0,1)$ then there exists a unique $u^* \in C^{m-1}[0,1]$ such that $u = u^*$ a.e. and $\|u^*\|_{\sup}^{m-1} \leq m\sqrt{2}\|u\|_m$.

Suppose $u \in H^{n+1}(0,1)$. Then $u \in H^n(0,1)$ and there exists a unique $u^* \in C^{n-1}[0,1]$ such that $u^* = u$ a.e. But $Du \in H^n(0,1)$ as well. Therefore there exists a unique $\hat{u} \in C^{n-1}[0,1]$ such that $\hat{u} = Du$ a.e.. That is, $\hat{u} = u^{*\prime}$ (as shown in the case $m = 2$). It follows that $u^* \in C^n[0,1]$.

Also,

$$\|u\|_{\sup}^{n} = \|u\|_{\sup}^{n-1} + \|u^{(n)}\|_{\sup} \leq (n+1)\sqrt{2}\|u\|_{n+1}.$$

This proves the theorem for $m = n+1$. Therefore, by mathematical induction, the theorem holds for any $m \in \mathbb{N}$. □

For $u \in H^1(0,1)$, let $u(x)$ be the value $v(x)$ of the unique continuous function v which is equal to u almost everywhere. Then x_0 is a zero of u if $v(x_0) = 0$.

For f and g in $H^1(0,1) \times H^1(0,1) = H^1(0,1)^2$, let

$$(f,g)_1^{(2)} = (f_1, g_1) + (Df_1, Dg_1) + (f_2, g_2) + (Df_2, Dg_2)\,, \qquad (2.2.13)$$

where D denotes the weak derivative.

Proposition 2.2.3. *The bilinear form* $(\cdot,\cdot)_1^{(2)}$ *is an inner product for the Sobolev space* $H^1(0,1)^2$*, which is complete.*

Proof. By the definition of $(\cdot,\cdot)_m$, which is an inner product for $H^m(0,1)$ (See Appendix A), it follows that $(\cdot,\cdot)_1^{(2)}$ is an inner product for $H^1(0,1)^2$. Also, the cartesian product of two Hilbert spaces is a Hilbert space. □

Definition (Value of a function at the boundary)**.** For $u \in H^k(0,1)^2$,

$$u(0) = u^*(0) \quad \text{and} \quad u(1) = u^*(1),$$

where u^* is the unique function in $C^{k-1}(0,1)^2$ such that $u = u^*$ a.e.

2.3 Weak formulation of the standard Timoshenko problem

The weak variational form is formulated at the end of this section. Recall that for a realistic model, $\alpha > \beta > 1$. This fact is discussed in Chapter 1 in the context of rod mechanics. In this chapter it is an assumption for the theory, where it is included in the estimates.

In Section 2.2 it is shown that for the one-dimensional case, if a function has weak derivatives up to order m, then it may be considered to have continuous derivatives up to order $m-1$. That is, if $u \in H^1(0,1)$, then there exists a unique continuous function v such that $u = v$ *a.e.* and u may be considered to be continuous.

In this section, the weak variational form of Problem T is considered. At the end of Section 2.1, the space $\mathcal{L}^2(0,1)^2$ is shown to be a Hilbert space with inner product $(\cdot,\cdot)^{(2)}$. Let $\|\cdot\|_{\mathcal{L}^2}$ denote the norm induced by $(\cdot,\cdot)^{(2)}$ and $\|\cdot\|_{H^1}$ the norm induced by $(\cdot,\cdot)_1^{(2)}$.

Definition (Energy Space)**.** The closure of T_P in $H^1(0,1)^2$ with respect to the norm $\|\cdot\|_{H^1}$ is referred to as the energy space $\mathcal{V}$.

The energy space for each of the three cases of boundary conditions is now characterised. To do this, the notation $\overline{A}$ is used to denote the closure of the set A. Also note that for two sets A and B, $\overline{A \times B} = \overline{A} \times \overline{B}$. For Case 1, $\mathcal{V} = \overline{T_1[0,1]} \times H^1(0,1)$, for Case 2, $\mathcal{V} = \overline{T_2[0,1]} \times \overline{T_2[0,1]}$ and for Case 3, $\mathcal{V} = \overline{T_1[0,1]} \times \overline{T_1[0,1]}$.

The bilinear form b_T is extended to $\mathcal{V}$: for u and v in $\mathcal{V}$,

$$b_T(u,v) = \frac{1}{\beta}\left(Du_2, Dv_2\right) + \left(Du_1 - u_2, Dv_1 - v_2\right). \tag{2.3.1}$$

Proposition 2.3.1. *The bilinear form c_T is an inner product for the space $\mathcal{L}^2(0,1)^2$ and the norm induced by c_T is equivalent to $\|\cdot\|_{\mathcal{L}^2}$.*

Proof. The fact that c_T is an inner product for $\mathcal{L}^2(0,1)^2$ follows easily from the fact that $\alpha > 0$ and c_T is composed of inner products on $\mathcal{L}^2(0,1)$. Let $u \in \mathcal{L}^2(0,1)^2$. Then

$$\min\left\{1, \frac{1}{\alpha}\right\} \|u\|_{\mathcal{L}^2}^2 \le c_T(u,u) \le \max\left\{1, \frac{1}{\alpha}\right\} \|u\|_{\mathcal{L}^2}^2. \tag{2.3.2}$$

□

The inner product c_T for $\mathcal{L}^2(0,1)^2$ will prove to be useful.

Definition (Inertia Space)**.** The space $\mathcal{L}^2(0,1)^2$ with norm induced by c_T, denoted $\|\cdot\|_{\mathcal{W}}$, is referred to as the inertia space $\mathcal{W}$.

Proposition 2.3.2. *For any $u \in \mathcal{V}$,*

$$\|u_1\|^2 \le \|Du_1\|^2 \le 2\beta\left(\|Du_1 - u_2\|^2 + \frac{1}{\beta}\|u_2\|^2\right). \tag{2.3.3}$$

Proof. Since $u_1(0) = 0$, $\|u_1\| \leq \|Du_1\|$ (See Appendix A). Using the triangle inequality and the fact that for a and b in $\mathbb{R}$, $2|ab| \leq a^2 + b^2$, it follows that

$$\begin{aligned}\|Du_1\|^2 &\leq 2\|Du_1 - u_2\|^2 + 2\|u_2\|^2 \\ &\leq 2\beta\left(\|Du_1 - u_2\|^2 + \frac{1}{\beta}\|u_2\|^2\right).\end{aligned}$$

□

Proposition 2.3.3. *For Cases 2 and 3, if $u \in \mathcal{V}$, then*

$$\|u_1\|^2 \leq \|Du_1\|^2 \leq 2\beta b_T(u,u). \tag{2.3.4}$$

Proof. Let $u \in \mathcal{V}$. Then, since $u_2(0) = 0$, (see Appendix A)

$$\|u_2\| \leq \|Du_2\|.$$

The result therefore follows from Proposition 2.3.2. □

The result above does not hold for Case 1.

Theorem 2.3.1. *For any $u \in \mathcal{V}$, there exists a non-zero real number c^2 such that*

$$c^2\|u\|^2_{\mathcal{L}^2} \leq b_T(u,u). \tag{2.3.5}$$

Proof. In Cases 2 and 3, if $u = \langle u_1, u_2\rangle \in \mathcal{V}$, then Propositions 2.3.2 and 2.3.3 hold. It follows that for $u \in \mathcal{V}$,

$$\begin{aligned}\|u\|^2_{\mathcal{L}^2} &= \|u_1\|^2 + \|u_2\|^2 \\ &\leq 2\beta b_T(u,u) + \|Du_2\|^2 \\ &\leq 3\beta b_T(u,u). \end{aligned} \tag{2.3.6}$$

That is, for $c^2 = \frac{1}{3\beta}$, (2.3.5) holds.

In Case 1, if $u \in \mathcal{V}$, a direct proof appears to be impossible. For this reason a proof is given by contradiction. The proof is based on the idea given in the appendix of [VZV09]. Note that for any nonzero $u \in \mathcal{V}$ and $w = \|u\|^{-1}_{\mathcal{L}^2}u$,

$$\frac{b_T(u,u)}{(u,u)^{(2)}} = b_T(w,w). \tag{2.3.7}$$

Let $\mathcal{S}$ denote the unit sphere with centre 0 in $\mathcal{L}^2(0,1)^2$. In order to prove (2.3.5), it is required to prove that for $u \in \mathcal{V} \cap \mathcal{S}$, there exists a real number c^2 such that

$$b_T(u,u) \geq c^2. \tag{2.3.8}$$

Suppose not. Then there exists a sequence (u^n) contained in $T_P \cap \mathcal{S}$ such that $b(u^n, u^n) \to 0$ as $n \to \infty$. This implies that for a given $c^2 \in \mathbb{R}$, there exists $n_0 \in \mathbb{N}$ such that for $n > n_0$,

$$b_T(u^n, u^n) < c^2. \tag{2.3.9}$$

By the contrapositive of the result shown for Cases 2 and 3, it follows that u_2^n does not have a zero in $[0,1]$. Without loss of generality, assume $u_2^n > 0$.

For any $\varepsilon > 0$, there exists $n_1 > n_0$ such that for $n > n_1$,

$$b_T(u^n, u^n) < \varepsilon^2. \tag{2.3.10}$$

For convenience, the notation $w_n = u_1^n$ and $\phi_n = u_2^n$ is used. From the fact that $u^n \in \mathcal{S}$ for each $n \in \mathbb{N}$, it follows that for each $n \in \mathbb{N}$,

$$\|\phi_n\|^2 + \|w_n\|^2 = 1 \tag{2.3.11}$$

and hence

$$\|\phi_n\|^2 \leq 1. \tag{2.3.12}$$

For each $n \in \mathbb{N}$, w_n and ϕ_n are in $C^1[0,1]$ since (u^n) is contained in $T_P \cap \mathcal{S}$. Then, by (2.3.10), for $n > n_1$

$$\|w_n' - \phi_n\| < \varepsilon.$$

Since w_n has a zero in $[0,1]$, it follows from a poincaré type inequality (see Appendix A) and the triangle inequality that for $n > n_1$,

$$\|w_n\| \leq \|w_n'\| \leq \|w_n' - \phi_n\| + \|\phi_n\| < \varepsilon + \|\phi_n\|.$$

That is,

$$\|w_n\|^2 < 2\varepsilon^2 + 2\|\phi_n\|^2. \tag{2.3.13}$$

By (2.3.11) and (2.3.13),

$$1 < \|\phi_n\|^2 + 2\varepsilon^2 + 2\|\phi_n\|^2. \tag{2.3.14}$$

Choose $\varepsilon = \frac{1}{10}$. Then (2.3.12) and (2.3.14) imply that for $n > n_1$

$$\frac{49}{150} < \|\phi_n\|^2 \leq 1. \tag{2.3.15}$$

For each $n \in \mathbb{N}$, the maximum and minimum of ϕ_n (denoted $\phi_{n(\max)}$ and $\phi_{n(\min)}$ respectively) are reached since ϕ_n may be considered to be an element of $C[0,1]$. Therefore, for some $a, b \in [0,1]$, using the Cauchy-Schwartz inequality and the fact that $\phi_n > 0$, it follows that,

$$\phi^2_{n(\max)} - \phi^2_{n(\min)} = \int_a^b (\phi_n^2)' = \int_a^b 2\phi_n \phi_n' \leq 2\|\phi_n\|\|\phi_n'\|. \tag{2.3.16}$$

Thus, (2.3.15) and (2.3.16) imply that

$$\begin{aligned}\phi^2_{n(\min)} &= \int_0^1 \phi^2_{n(\min)} = \int_0^1 \phi_n^2 - \int_0^1 \left(\phi_n^2 - \phi^2_{n(\min)}\right) \\ &\geq \int_0^1 \phi_n^2 - \int_0^1 \left(\phi^2_{n(\max)} - \phi^2_{n(\min)}\right) \\ &> \frac{49}{150} - 2\|\phi_n'\| \end{aligned} \tag{2.3.17}$$

That is,

$$\phi^2_{n(\min)} > \frac{49}{150} - 2\varepsilon = \frac{19}{150} \tag{2.3.18}$$

Again, using the Cauchy-Schwartz inequality,

$$\left| \int_0^1 w_n' - \int_0^1 \phi_n \right| \leq \int_0^1 |w_n' - \phi_n| \leq \|w_n' - \phi_n\| < \frac{1}{10} \tag{2.3.19}$$

Hence,

$$\frac{\sqrt{114} - 3}{30} < \phi_{n(\min)} - \frac{1}{10} \leq \int_0^1 \phi_n - \frac{1}{10} < \int_0^1 w_n'. \tag{2.3.20}$$

But this implies that $w_n(1) > w_n(0)$, which contradicts the fact that $u^n \in T_P$. Therefore, for $u \in \mathcal{V} \cap \mathcal{S}$, there exists $c^2 > 0$ such that

$$b_T(u,u) \geq c^2. \tag{2.3.21}$$

That is, for $u \in \mathcal{V}$, there exists $c^2 > 0$ such that (2.3.5) holds. □

Corollary. *The bilinear form b_T is an inner product for the space $\mathcal{V}$.*

Proof. By Theorem 2.3.1, b_T is a positive definite symmetric bilinear form. □

Let $\|\cdot\|_{\mathcal{V}}$ denote the norm induced by the inner product b_T. The space $\mathcal{V}$ equipped with the norm $\|\cdot\|_{\mathcal{V}}$ is referred to as the energy space.

Proposition 2.3.4. *For any* $u \in \mathcal{V}$,

$$\|Du_1\|^2 \leq 2K^*\|u\|_{\mathcal{V}}^2. \tag{2.3.22}$$

with $K^* = \max\{\beta, c^{-2}\}$.

Proof. From Proposition 2.3.2, for any $u \in \mathcal{V}$,

$$\begin{aligned} \|Du_1\|^2 &\leq 2\beta\left(\|Du_1 - u_2\|^2 + \frac{1}{\beta}\|u_2\|^2\right) \\ &\leq 2\beta\left(\|Du_1 - u_2\|^2 + \frac{1}{\beta}\|u\|_{\mathcal{L}^2}^2\right). \end{aligned} \tag{2.3.23}$$

That is, by Theorem 2.3.1,

$$\begin{aligned} \|Du_1\|^2 &\leq 2\beta\|u\|_{\mathcal{V}}^2 + \frac{2}{c^2}\|u\|_{\mathcal{V}}^2 \\ &\leq 2K^*\|u\|_{\mathcal{V}}^2. \end{aligned} \tag{2.3.24}$$

□

Proposition 2.3.5. *The norms* $\|\cdot\|_{H^1}$ *and* $\|\cdot\|_{\mathcal{V}}$ *are equivalent on* $\mathcal{V}$.

Proof. Let $u \in \mathcal{V}$. Then, for $K_1 = \max\{\beta^{-1}, 2\}$,

$$\begin{aligned} \|u\|_{\mathcal{V}}^2 &\leq K_1\left(\int_0^1 (Du_2)^2 + \int_0^1 (Du_1)^2 + \int_0^1 u_2^2\right) \\ &\leq K_1\|u\|_{H^1}^2. \end{aligned} \tag{2.3.25}$$

Also, using the inequality (2.3.5) and Proposition 2.3.4, it follows that

$$\begin{aligned} \|u\|_{H^1}^2 &= \|u\|_{\mathcal{L}^2}^2 + \|Du_1\|^2 + \|Du_2\|^2 \\ &\leq \frac{2}{c^2}\|u\|_{\mathcal{V}}^2 + 2K^*\|u\|_{\mathcal{V}}^2 + \beta\|u\|_{\mathcal{V}}^2 \\ &\leq 5K^*\|u\|_{\mathcal{V}}^2. \end{aligned} \tag{2.3.26}$$

□

Remark. *From Proposition 2.3.5 it follows that the energy space $\mathcal{V}$ is complete.*

Let J denote the interval $[-\tau, \tau]$ or $[0, \tau]$. As a first step to proceed in formulating the weak variational form, the partial time derivatives of Equation (2.1.31) are replaced by partial weak derivatives. This suggests that a solution $y = \langle w, \phi \rangle$ should be sought for in $H^2([0,1] \times J)^2$. That is, y should be found such that for each $t > 0$, $y(\cdot, t) \in \mathcal{V}$ and, for each $v \in \mathcal{V}$,

$$c_T\left(D_t^2 y(\cdot,t), v\right) + a\left(D_t y(\cdot,t), v\right) + b_T\left(y(\cdot,t), v\right) = \left(f(\cdot,t), v\right)^{(2)}.$$

In order to use the abstract existence theory in Section 2.4, an even weaker form of the problem is considered. It is shown that Problem TV can be written in the form

$$c_T(u'', v) + a(u'v) + b_T(u, v) = (f, v).$$

For any function $g \in \mathcal{L}^2([0,1] \times J)$, let

$$\hat{g}(t) = g(\cdot, t) \in \mathcal{L}^2[0,1].$$

Then the ordinary derivative of $\hat{g}$ is the partial time derivative of g. Let $u_1(t) = w(\cdot, t)$ and $u_2(t) = \phi(\cdot, t)$.

Definition (Derivative)**.** Suppose f is a function defined on some interval (a, b) with values in a Banach space $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$. Let t be any point in (a, b). If $\lim_{h\to 0} h^{-1}(f(t+h) - f(t))$ exists, then it is called the derivative of f at t and is denoted by $f'(t)$. That is, if it exists, the derivative f' at t is such that

$$\lim_{h\to 0} \|h^{-1}(f(t+h) - f(t)) - f'(t)\|_{\mathcal{B}} = 0 \quad \text{and} \quad f'(t) \in \mathcal{B}.$$

Remark. *In this dissertation the Banach spaces considered are the Hilbert spaces $(\mathcal{L}^2(0,1)^2, \|\cdot\|_{\mathcal{L}^2})$, $(\mathcal{W}, \|\cdot\|_{\mathcal{W}})$ and $(\mathcal{V}, \|\cdot\|_{\mathcal{V}})$.*

The weak variational form of Problem T follows.

Problem TW
Given $u_0 \in \mathcal{V}$, $u_d \in \mathcal{W}$ and $f \in C([0,\tau); \mathcal{L}^2(0,1)^2)$, find $u \in C^2((0,\tau); \mathcal{W})$ such that for each $t > 0$, $u(t) \in \mathcal{V}$, $u'(t) \in \mathcal{V}$ and

$$\begin{aligned} c_T\left(u''(t), v\right) + a(u'(t), v) + b_T(u(t), v) &= (f(t), v)^{(2)} \quad \text{for each } v \in \mathcal{V}, \\ \lim_{t\to 0^+} \|u(t) - u_0\|_{\mathcal{V}} &= 0, \\ \lim_{t\to 0^+} \|u'(t) - u_d\|_{\mathcal{L}^2} &= 0. \end{aligned}$$

Remark. *In the weak variational form of a problem, the information regarding the system of partial differential equations and boundary conditions is contained in the bilinear forms and function spaces.*

2.4 Existence and uniqueness of solutions

The results of the general theory in [VV02] and [VS19] are presented in this section. This theory deals with existence and uniqueness of solutions of linear vibration models. The problem formulation in this section is a generalisation of the formulation given in Section 2.3.

The following assumptions are made about the spaces being considered.

S1 The spaces $\mathcal{V}$, $\mathcal{W}$ and $\mathcal{X}$ are real Hilbert spaces, where $\mathcal{V} \subset \mathcal{W} \subset \mathcal{X}$.

S2 The notation for the inner products and induced norms are as follows (following [VS19]).

Table 2.1: Notation for spaces with inner products and induced norms.

Space		Inner product	Norm
Global space	$\mathcal{X}$	$(\cdot,\cdot)_{\mathcal{X}}$	$\|\cdot\|_{\mathcal{X}}$
Inertia space	$\mathcal{W}$	$(\cdot,\cdot)_{\mathcal{W}}$	$\|\cdot\|_{\mathcal{W}}$
Energy space	$\mathcal{V}$	$(\cdot,\cdot)_{\mathcal{V}}$	$\|\cdot\|_{\mathcal{V}}$

The spaces $\mathcal{V}$ and $\mathcal{W}$ are to be constructed for every application. This was done for Problem TW in Section 2.3.

Three bilinear forms, a, b and c are considered, where a and b are defined on $\mathcal{V}$ and c is defined on $\mathcal{W}$. It is assumed that $b = b_1 + b_2$, where b_1 is symmetric and b_2 not. It is possible that b_2 is zero.

The following assumptions are made in [VS19] for the existence results.

A1 $\mathcal{V}$ is dense in $\mathcal{W}$ and $\mathcal{W}$ is dense in $\mathcal{X}$.

A2 There exists a positive constant $C_{\mathcal{W}}$ such that $\|w\|_{\mathcal{X}} \leq C_{\mathcal{W}}\|w\|_{\mathcal{W}}$ for each $w \in \mathcal{W}$ and $c(\cdot,\cdot) = (\cdot,\cdot)_{\mathcal{W}}$.

A3 There exists a positive constant $C_{\mathcal{V}}$ such that $\|v\|_{\mathcal{W}} \leq C_{\mathcal{V}}\|v\|_{\mathcal{V}}$ for each $v \in \mathcal{V}$ and $b_1(\cdot,\cdot) = (\cdot,\cdot)_{\mathcal{V}}$.

A4 The bilinear form a is non-negative, symmetric and bounded on $\mathcal{V}$, i.e. there exists a positive constant K_a such that for u and v in $\mathcal{V}$, $|a(u,v)| \leq K_a\|u\|_{\mathcal{V}}\|v\|_{\mathcal{V}}$.

A4W (Weak damping) The bilinear form a is non-negative, symmetric and bounded on $\mathcal{W}$, i.e. there exists a positive constant C_a such that for v and w in $\mathcal{W}$, $|a(u,v)| \leq C_a\|u\|_{\mathcal{W}}\|v\|_{\mathcal{W}}$.

A5 The bilinear form b_2 is bounded in the following sense: there exists a positive constant $K_1 < C_{\mathcal{V}}^{-1}$ such that for u and v in $\mathcal{V}$,

$$|b_2(u,v)| \leq K_1\|u\|_{\mathcal{V}}\|v\|_{\mathcal{W}}.$$

The following general problem in weak variational form is from [VS19].

Problem GVar
Given a function $f : J \to \mathcal{X}$, find a function $u \in C(J,\mathcal{V})$ such that u' is continuous at 0 with respect to $\|\cdot\|_{\mathcal{W}}$ and for each $t \in J$, $u(t) \in \mathcal{V}$, $u'(t) \in \mathcal{V}$, $u''(t) \in \mathcal{W}$ and

$$c\left(u''(t),v\right) + a\left(u'(t),v\right) + b\left(u(t),v\right) = \left(f(t),v\right)_{\mathcal{X}} \quad \text{for each } v \in \mathcal{V}, \tag{2.4.1}$$

$$\text{while} \quad u(0) = u_0\,, \quad u'(0) = u_d\,.$$

Semigroup theory is used in [VV02] and [VS19] to obtain the following results.

The theorems listed below are the main results of [VS19].

Theorem 2.4.1. *Suppose Assumptions **A1**, **A2**, **A3**, **A4** and **A5** hold. If, for $u_0 \in \mathcal{V}$ and $u_d \in \mathcal{V}$, there exists some $y \in \mathcal{W}$ such that*

$$b\left(u_0,v\right) + a\left(u_d,v\right) = c\left(y,v\right) \quad \textit{for each } v \in \mathcal{V}, \tag{2.4.2}$$

then for each $f \in C^1\left([0,T),\mathcal{X}\right)$ there exists a unique solution

$$u \in C\left([0,T),\mathcal{V}\right) \cap C^1\left([0,T),\mathcal{W}\right) \cap C^1\left((0,T),\mathcal{V}\right) \cap C^2\left((0,T),\mathcal{W}\right)$$

for Problem GVar. If $f = 0$ then $u \in C^1\left([0,\infty),\mathcal{V}\right) \cap C^2\left([0,\infty),\mathcal{W}\right)$.

Definition (The space E_b)**.**

$E_b = \{x \in \mathcal{V} \big|$ there exists a $y \in \mathcal{W}$ such that $c(y,v) = b(x,v)$ for all $v \in \mathcal{V}\}$.

Theorem 2.4.2 (Weak damping).
*Suppose Assumptions **A1**, **A2**, **A3**, **A4W** and **A5** hold. Let J be an interval containing zero, then there exists a unique solution*

$$u \in C^1(J, \mathcal{V}) \cap C^2(J, \mathcal{W})$$

of Problem GVar for each $u_0 \in E_b$, $u_d \in \mathcal{V}$ and each $f \in C^1(J, \mathcal{X})$. If $f = 0$ then $u \in C^1((-\infty, \infty), \mathcal{V}) \cap C^2((-\infty, \infty), \mathcal{W})$.

The symmetric case

If the bilinear form b is symmetric, it is important to note that Theorems 2.4.1 and 2.4.2 reduce to the existence theorems in [VV02] (copied here for convenience).

Theorem 2.4.3. *Suppose Assumptions **A1**, **A2**, **A3** and **A4** hold. If, for $u_0 \in \mathcal{V}$ and $u_d \in \mathcal{V}$, there exists some $y \in \mathcal{W}$ such that*

$$b(u_0, v) + a(u_1, v) = c(y, v) \quad \text{for each } v \in \mathcal{V}, \tag{2.4.3}$$

then for each $f \in C^1([0, \infty), \mathcal{X})$, there exists a unique solution

$$u \in C^1([0, \infty), \mathcal{V}) \cap C^2([0, \infty), \mathcal{W})$$

for Problem GVar.

Theorem 2.4.4 (Weak damping).
*Suppose Assumptions **A1**, **A2**, **A3** and **A4W** hold. Then there exists a unique solution*

$$u \in C^1((-\infty, \infty), \mathcal{V}) \cap C^2((-\infty, \infty), \mathcal{W})$$

of Problem GVar for each $u_0 \in E_b$, each $u_d \in \mathcal{V}$ and each $f \in C^1((-\infty, \infty), \mathcal{X})$.

Remark. *The theory in [VV02] is sufficient for Models T and AT.*

In Section 2.7, this theory is applied to the standard Timoshenko problem.

To interpret the existence theory, the proofs of the theorems in [VV02] and [VS19] were investigated. In both articles, Problem GVar is written as an initial value problem for a first-order differential equation using a linear operator A determined by a, b and c. Semigroup theory is then used to obtain existence results. A necessary condition for the solvability of Problem GVar is that A is the infinitesimal generator of a semigroup and that the initial conditions are contained in the domain of A.

Of special interest is the case of weak damping or no damping. For this case the domain of A is $E_b \times \mathcal{V}$. Therefore, in the case of weak or no damping, by Theorem 2.4.4, if Assumptions A1, A2, A3 and A4W are met, Then the unique solution u for Problem GVar is an element of $C^1((-\infty,\infty),\mathcal{V}) \cap C^2((-\infty,\infty),\mathcal{W})$ for each $\langle u_0, u_1\rangle \in E_b \times \mathcal{V}$, and each $f \in C^1((-\infty,\infty),\mathcal{X})$. Theorem 2.4.4 also implies that since the initial condition is contained in $E_b \times \mathcal{V}$, the solution remains in $E_b \times \mathcal{V}$ for each $t > 0$. The regularity of a solution of Problem T is discussed in more detail in Section 2.7.

2.5 Modal analysis

Consider Problem GVar with $f \equiv 0$, the bilinear form $a \equiv 0$, and a possible solution $u(t) = T(t)\tilde{u}$, where T is a real-valued function. That is, for each $t \in J$, it follows that $u(t) \in \mathcal{V}$, $u''(t) \in \mathcal{W}$ and

$$T''(t)c(\tilde{u}, v) + T(t)b(\tilde{u}, v) = 0 \quad \text{for each } v \in \mathcal{V}. \tag{2.5.1}$$

This results in the following eigenvalue problem and differential equation.

$$b(\tilde{u}, v) = \lambda c(\tilde{u}, v) \quad \text{for each } v \in \mathcal{V}, \tag{2.5.2}$$

$$T''(t) + \lambda T(t) = 0. \tag{2.5.3}$$

If eigenvalues and eigenvectors exist, then for some real numbers B_1 and B_2, the general solution of Equation (2.5.3) is

$$T(t) = B_1 \cos(\sqrt{\lambda}\, t) + B_2 \sin(\sqrt{\lambda}\, t). \tag{2.5.4}$$

In Section 2.6 it is shown that there exists a complete sequence of eigenvectors $(\tilde{u}_m)$ for the eigenvalue problem (2.5.2) with a corresponding sequence (λ_m) of eigenvalues.

In this section the application is discussed. Note that if $u_m(t) = T_m(t)\tilde{u}_m$, where $\tilde{u}_m$ and λ_m are a solution of (2.5.2) and T_m is given by (2.5.4), then $u_m(t)$ is a solution of Problem GVar (with $f \equiv 0$ and $a \equiv 0$), known as a modal solution. The formal series solution then follows.

$$u(t) = \sum_{m=1}^{\infty} T_m(t)\tilde{u}_m. \tag{2.5.5}$$

Energy

It is now shown, using the same method as [CVV18], that the series representation (2.5.5) is valid. That is, the partial sums of the formal series solution can be used to approximate the exact solution and the error of this approximation can be made arbitrarily small in energy. The definition of energy associated with a function is given here for reference.

Definition. The energy E associated with a function u is given by

$$E(t) = \frac{1}{2}b(u(t), u(t)) + \frac{1}{2}c(u'(t), u'(t)).$$

Note that for any solution u of Problem GVar (with $f \equiv 0$),

$$E'(t) = b(u(t), u'(t)) + c(u''(t), u'(t)) = -a(u'(t), u'(t)) \leq 0. \tag{2.5.6}$$

That is,

$$E(t) \leq E(0) \text{ for all } t > 0. \tag{2.5.7}$$

The following partial sum is considered

$$u^M(t) = \sum_{m=1}^{M} T_m(t)\tilde{u}_m. \tag{2.5.8}$$

The notation below is used for convenience. Let

$$u_0^M = \sum_{m=1}^{M} B_m\tilde{u}_m \text{ and } u_d^M = \sum_{m=1}^{M} C_m\tilde{u}_m, \tag{2.5.9}$$

where $B_m = T_m(0)$ and $C_m = T'_m(0)$. The partial sum (2.5.8) satisfies the differential equation (2.4.1), with $f \equiv 0$ and $a \equiv 0$, with the initial conditions $u^M(0) = u_0^M$ and $(u^M)'(0) = u_d^M$. Therefore Equation (2.4.1) – where $f \equiv 0$ and $a \equiv 0$ – is also satisfied by the error function $u^{E_M} = u - u^M$, where $u^{E_M}(0) = u_0 - u_0^M$ and $(u^{E_M})'(0) = u_d - u_d^M$. Let the energy associated with u^{E_M} be denoted by $E^M(t)$. It follows from the inequality (2.5.7) that for all $t > 0$,

$$E^M(t) \leq E^M(0). \tag{2.5.10}$$

It can be concluded that if *both* the partial sum approximation of the initial displacement *and* the partial sum approximation of the initial velocity converge to the initial displacement and initial velocity respectively in the energy norm ($\|\cdot\|_{\mathcal{V}}$) and the inertia norm ($\|\cdot\|_{\mathcal{W}}$) respectively, then *both* the partial sum (2.5.8) converges to the solution in the energy norm and the derivative of the partial sum converges to the derivative of the solution in the inertia norm.

As a result of a theorem in [CVV18], if $C_m = c(u_1, \tilde{u}_m)$, then $\|u_d - u_d^M\|_{\mathcal{W}} \to 0$ as $M \to \infty$. Also, by another theorem in [CVV18], if $B_m = \|\tilde{u}_m\|_{\mathcal{V}}^{-2} b(u_0, \tilde{u}_m)$, then $\|u_0 - u_0^M\|_{\mathcal{V}} \to 0$ as $M \to \infty$. That is, there exist coefficients B_m and C_m such that $E^M(0) \to 0$ as $M \to \infty$ and hence $E^M(t) \to 0$ as $M \to \infty$.

Therefore, for all $t > 0$, the partial sum (2.5.8) converges to the solution in the energy norm and the derivative of the partial sum converges to the derivative of the solution in the inertia norm. Also, the accuracy of the approximations – in terms of partial sums – depends only on the accuracy of the partial sum approximations of u_0 and u_d at time $t = 0$. The accuracy can therefore be guaranteed.

Hence, the limit of the partial sums – the formal solution – satisfies Problem GVar, provided the assumptions A1, A2, A3, A4W, A5 are met and $u_0 \in E_b$, $u_d \in \mathcal{V}$ and $f \in C^1(J, \mathcal{X})$.

2.6 Existence of a complete sequence of eigenvectors

In [CVV18] it is assumed that the bilinear form b is symmetric. In order to prove the existence of a complete sequence of eigenvectors, the following assumptions are made. The first four assumptions (B1-B4) are a more concise version of Assumptions S1-S3 and A1-A3 in Section 2.4 and the last assumption (B5) is additional.

B1 $\mathcal{W}$ is a Hilbert space with inner product c and induced norm $\|\cdot\|_{\mathcal{W}}$.

B2 $\mathcal{V}$ is a Hilbert space with inner product b and induced norm $\|\cdot\|_{\mathcal{V}}$.

B3 There exists a positive constant C_b such that $\|v\|_{\mathcal{W}} \leq C_b\|v\|_{\mathcal{V}}$ for each $v \in \mathcal{V}$.

B4 $\mathcal{V}$ is a dense subset of $\mathcal{W}$.

B5 The embedding of $\mathcal{V}$ into $\mathcal{W}$ is compact.

Alternative of B5: Every bounded set in $\mathcal{V}$ contains a sequence that converges in $\mathcal{W}$ (is pre-compact in $\mathcal{W}$).

The following theorem from [CVV18] is as a result of the Riesz Representation Theorem.

Theorem 2.6.1. *For each $x \in \mathcal{W}$ there exists a unique element $w \in V$ such that*

$$b(w, u) = c(x, u) \quad \textit{for each} \quad u \in V \quad \textit{and} \quad \|w\|_V \leq C_b\|x\|_{\mathcal{W}}.$$

Definition. For each $x \in \mathcal{W}$ define the operator T as $Tx = w$, where w is the unique element described in Theorem 2.6.1.

The operator T is a bounded linear operator from $\mathcal{W}$ to $\mathcal{V}$, i.e.

$$\|Tx\|_{\mathcal{V}} \leq C_b\|x\|_{\mathcal{W}} \quad \text{for each} \quad x \in \mathcal{W}.$$

In addition, by assumption B5, T is compact. Also, since b and c are symmetric, it can be shown that T is symmetric.

Theorem 2.6.2. *The null space $\mathcal{N}(T)$ of the operator T is trivial.*

Proof. If $x \in \mathcal{N}(T)$, then $Tx = 0$. Therefore,

$$c(x, u) = b(Tx, u) = 0 \quad \text{for each } u \in \mathcal{V}.$$

That is, $c(x, u) = 0$ for each $u \in \mathcal{V}$. Since $\mathcal{V}$ is dense in $\mathcal{W}$, it is simple to show that $x = 0$. □

A consequence of Theorem 2.6.2 is that the operator T is invertible.

Proposition 2.6.1. *For each $x \in \mathcal{W}$, if $x \neq 0$ then $c(Tx, x) > 0$.*

Proof. For each $x \in \mathcal{W}$ where $x \neq 0$, by Theorem 2.6.2 it follows that

$$0 < b(Tx, Tx) = c(Tx, x).$$

□

Theorem 2.6.3.

$$b(x, u) = \lambda c(x, u) \quad \textit{for each} \quad u \in \mathcal{V}$$

if and only if

$$Tx = \frac{1}{\lambda}x.$$

Proof. The result follows easily from the fact that $b(Tx, u) = c(x, u)$ for each $u \in \mathcal{V}$. □

For $\mu = \frac{1}{\lambda}$, the eigenvalue problem (2.5.2) can thus be rewritten as

$$T\tilde{u} = \mu\tilde{u}. \tag{2.6.1}$$

Definition (Completeness)**.** If (σ_k) is an orthonormal sequence in a Banach space $\mathcal{B}$ and $v = \sum_{k=1}^{\infty} \langle v, \sigma_k \rangle \sigma_k$ for each $v \in \mathcal{B}$, then (σ_k) is said to be complete in $\mathcal{B}$.

By the theory of symmetric linear compact operators (see [Zei95]) the following properties are true for eigenvalues and eigenvectors of T.

(a) T has an orthonormal sequence of eigenvectors $(\tilde{u}_n)$ with corresponding positive eigenvalues (μ_n);

(b) The orthonormal sequence of eigenvectors $(\tilde{u}_n)$ is complete in $\mathcal{W}$;

(c) Each eigenspace is finite dimensional;

(d) The sequence of eigenvalues (μ_n) is decreasing and $\mu_n \to 0$ as $n \to \infty$.

Therefore, the theorem below follows.

Theorem 2.6.4. *The following statements are true.*

(a) The eigenvalue problem (2.5.2) has an orthonormal sequence of eigenvectors $(\tilde{u}_n)$ with corresponding positive eigenvalues (λ_n);

(b) The orthonormal sequence of eigenvectors $(\tilde{u}_n)$ is complete in $\mathcal{W}$;

(c) Each eigenspace is finite dimensional;

(d) The sequence of eigenvalues (λ_n) is increasing and $\lambda_n \to \infty$ as $n \to \infty$.

The authors in [CVV18] then proceed to find the following result.

Theorem 2.6.5. *The sequence of eigenvectors $(\tilde{u}_n)$, which is complete in $\mathcal{W}$, is also complete in $\mathcal{V}$.*

The well known definition of the Rayleigh quotient follows for reference.

Definition. The Rayleigh quotient $\mathbf{R}$ is defined as

$$\mathbf{R}(v) = \frac{b(v,v)}{c(v,v)} \text{ for each } v \in \mathcal{V}.$$

It follows from the fact that b and c are inner products that the Rayleigh quotient is positive for all elements of $\mathcal{V}$.

2.7 Application of existence theory to a Timoshenko rod

The existence theory of Section 2.4 is now applied to Problem TW from Section 2.1 for completeness.

Theorem 2.7.1. *For $u_0 \in \mathcal{V}$ and $u_d \in \mathcal{V}$, Problem TW has a unique solution*

$$u \in C^1([0,\infty),\mathcal{V}) \cap C^2([0,\infty),\mathcal{W}).$$

Proof. The assumptions A1-A4 are verified in order to use Theorem 2.4.3.

A1 Since $C_0^\infty(0,1)$ is dense in $\mathcal{L}^2(0,1)$ and by the equivalence of the norms $\|\cdot\|_{\mathcal{L}^2}$ and $\|\cdot\|_{\mathcal{W}}$ (Proposition 2.3.1), it follows that $C_0^\infty(0,1)^2$ is dense in $\mathcal{W}$ and $\mathcal{W}$ is dense in $\mathcal{L}^2(0,1)^2$. That is, by the fact that $C_0^\infty(0,1)^2 \subset \mathcal{V} \subset \mathcal{W}$, $\mathcal{V}$ is dense in $\mathcal{W}$ and $\mathcal{W}$ is dense in $\mathcal{L}^2(0,1)^2$.

A2 By the equivalence of the norms $\|\cdot\|_{\mathcal{L}^2}$ and $\|\cdot\|_{\mathcal{W}}$ (Proposition 2.3.1), there exists a positive constant $C_{\mathcal{W}}$ such that $\|w\|_{\mathcal{L}^2} \leq C_{\mathcal{W}}\|w\|_{\mathcal{W}}$ for each $w \in \mathcal{W}$.

A3 By the inequality (2.3.5) and the equivalence of the norms $\|\cdot\|_{\mathcal{L}^2}$ and $\|\cdot\|_{\mathcal{W}}$ (Proposition 2.3.1), it follows that there exists a positive constant $C_{\mathcal{V}}$ such that $\|v\|_{\mathcal{W}} \leq C_{\mathcal{V}}\|v\|_{\mathcal{V}}$ for each $v \in \mathcal{V}$.

A4 In this case, $a \equiv 0$. Therefore the bilinear form a is non-negative, symmetric and bounded on $\mathcal{V}$.

Therefore, by Theorem 2.4.3, there exists a unique solution u for Problem TW, where

$$u \in C^1([0,\infty),\mathcal{V}) \cap C^2([0,\infty),\mathcal{W}).$$

□

The existence of a complete sequence of eigenfunctions of the associated eigenvalue problem is not discussed here since Problem T is a special case of Problem AT presented in Chapter 3.

Regularity

Recall from Section 2.4 that

$$E_b = \{x \in \mathcal{V} \big| \text{ there exists a } y \in \mathcal{W} \text{ such that } c(y,v) = b(x,v) \text{ for all } v \in \mathcal{V}\}.$$

Note that, by definition, E_b is a subset of $\mathcal{V}$. From Section 2.2 this implies that E_b is contained in $H^1(0,1)^2$. It is, however, possible for elements of E_b to have higher order derivatives than just one. The following result characterizes E_b.

Proposition 2.7.1. *Consider the different cases for boundary conditions. In each case,* $E_b \subset H^2(0,1)^2$.

Proof. Recall that $\mathcal{V} \subset H^1(0,1)^2$. Let $u \in E_b$. That is, $u \in \mathcal{V}$ is such that for each $v \in \mathcal{V}$, there exists a $y \in \mathcal{W}$ such that

$$b_T(u,v) = c_T(y,v). \tag{2.7.1}$$

Let $x = \langle x_1, 0\rangle$ and $z = \langle 0, z_2\rangle$, where $x_1, z_2 \in C_0^\infty(0,1)$. Then, x and z are in $\mathcal{V}$ and, by the definitions of b_T and c_T,

$$(Du_1 - u_2, x_1') = (y_1, x_1) \quad \text{and} \tag{2.7.2}$$

$$\frac{1}{\beta}(Du_2, z_2') - (Du_1 - u_2, z_2) = \frac{1}{\alpha}(y_2, z_2). \tag{2.7.3}$$

That is, for any x and z in $C_0^\infty(0,1)$,

$$(Du_1, x') = -(Du_2 - y_1, x) \quad \text{and} \tag{2.7.4}$$

$$\frac{1}{\beta}(Du_2, z') = -\left(u_2 - Du_1 - \frac{1}{\alpha}y_2, z\right). \tag{2.7.5}$$

Hence $(Du_2 - y_1)$ is the weak derivative of Du_1 and $\left(u_2 - Du_1 - \frac{1}{\alpha}y_2\right)$ is the weak derivative of $\frac{1}{\beta}Du_2$. Therefore $\bar{u} \in H^2(0,1)^2$. □

Proposition 2.7.1 and Theorem 2.2.2 imply that for u_0 in E_b, u_0 is considered an element of $C^1[0,1]$. That is, if u_0 is in E_b, then u is in $C^1[0,1]$ for all $t > 0$.

Chapter 3

Adapted Timoshenko rod

In this chapter the adapted Timoshenko model is investigated. The conditions for existence of a weak solution and the completeness of a sequence of eigenfunctions are established. The treatment is similar to that of the Timoshenko theory. As a consequence, the results from Chapter 2 could often be used again with or without some modification. In this chapter the theory will be applied to the adapted Timoshenko model with constant parameters.

3.1 Model: Adapted Timoshenko rod

Considered here is the Adapted Timoshenko rod model where the term representing the axial load, S, is such that $\partial_t S = 0$. The model is the same as Problem T except for one extra term:

$$\partial_t^2 w = \partial_x \left(S \partial_x w\right) + \partial_x V + P. \tag{3.1.1}$$

In the case where $S = 0$, (3.1.1) becomes (2.1.4). A constitutive equation for S is also included:

$$S = \frac{1}{\gamma} \partial_x u. \tag{3.1.2}$$

The boundary conditions considered are the same as in Chapter 1, except for Case 2 (cantilever rod), where an extra term is included:

$$w(0,t) = \phi(0,t) = S\partial_x w(1,t) + V(1,t) = M(1,t) = 0. \tag{3.1.3}$$

The additional term in the boundary conditions, however, does not influence the test functions. Recall that

$$T_1[0,1] = \{v \in C^1[0,1] \mid v(0) = v(1) = 0\}, \tag{3.1.4}$$
$$T_2[0,1] = \{v \in C^1[0,1] \mid v(0) = 0\}, \tag{3.1.5}$$

so that the sets of test functions T_P are defined as follows:

Case 1 (Pinned-pinned rod)

$$T_P = T_1[0,1] \times C^1[0,1], \tag{3.1.6}$$

Case 2 (Cantilever rod)

$$T_P = T_2[0,1] \times T_2[0,1], \tag{3.1.7}$$

Case 3 (Clamped-clamped rod)

$$T_P = T_1[0,1] \times T_1[0,1]. \tag{3.1.8}$$

As before, the inner product notation (u, v) will be used, where

$$(u, v) = \int_0^1 uv.$$

The variational form of the adapted Timoshenko rod model is given below.

Problem ATV Given positive constants α and β and load P, find $\langle w, \phi \rangle$ such that for each $t > 0$, $\langle w(\cdot,t), \phi(\cdot,t) \rangle \in T_P$ and for each $\langle v_1, v_2 \rangle \in T_P$,

$$\left(\partial_t^2 w(\cdot,t), v_1\right) = -\left(S\partial_x w(\cdot,t), v_1'\right) - \left(V(\cdot,t), v_1'\right) + \left(P(\cdot,t), v_1\right), \tag{3.1.9}$$
$$\left(\frac{1}{\alpha}\partial_t^2 \phi(\cdot,t), v_2\right) = -\left(M(\cdot,t), v_2'\right) + \left(V(\cdot,t), v_2\right). \tag{3.1.10}$$

The accompanying constitutive equations are

$$M = \frac{1}{\beta}\partial_x \phi, \tag{3.1.11}$$
$$V = \partial_x w - \phi, \tag{3.1.12}$$
$$S = \frac{1}{\gamma}\partial_x u. \tag{3.1.13}$$

Remark. *As with Problem TV in Chapter 2, if alternative boundary conditions are used, the variational form will remain the same except for the definition of the test functions T_P.*

3.2 Existence

In order to show the existence of a solution, the weak variational form of the problem is considered and the theory of Chapter 2 is applied. To find the weak variational form, Equations (3.1.9) and (3.1.10) are added and bilinear forms are used. This was done in detail in Chapter 2. The variational form of the problem differs from Problem TV only in the definition of the bilinear form b.

Define the symmetric bilinear form b_{AT} for vectors u and v as

$$\begin{aligned} b_{AT}(u,v) &= \frac{1}{\beta}(Du_2, Dv_2) + (Du_1 - u_2, Dv_1 - v_2) + (SDu_1, Dv_1) \\ &= b_T(u,v) + (SDu_1, Dv_1). \end{aligned} \tag{3.2.1}$$

Assumption The axial force S is in $C[0,1]$ and is differentiable.

An important special case of the model is where S is a constant. Additionally, the case where S changes sign is not considered because that is unrealistic in the model under consideration here.

After careful consideration it is found that the proofs of the propositions in Sections 2.1 and 2.3 – up to Proposition 2.3.3 – still hold with the necessary minor adaptations. It is only the properties of the bilinear form b_{AT} which are to be investigated in order to ensure the validity of the adaptation of Theorem 2.3.1 and hence Proposition 2.3.5.

The formulation of the weak variational form is the same as in Section 2.3. Recall the Hilbert space $\mathcal{L}^2(0,1)^2$ with the norm $\|\cdot\|_{\mathcal{L}^2}$ (induced by $(\cdot,\cdot)^{(2)}$). In addition, the space $\mathcal{V}$ denotes the closure of the test functions T_P in $H^1(0,1)^2$ with respect to the norm $\|\cdot\|_{H^1}$ (induced by $(\cdot,\cdot)_1^{(2)}$). Also, the space $\mathcal{W}$ denotes the space $\mathcal{L}^2(0,1)^2$ with norm $\|\cdot\|_{\mathcal{W}}$ (induced by c_T).

The energy space $\mathcal{V}$ for each of the three cases of boundary conditions is characterised as before. For Case 1, $\mathcal{V} = \overline{T_1[0,1]} \times H^1(0,1)$, for Case 2, $\mathcal{V} = \overline{T_2[0,1]} \times \overline{T_2[0,1]}$ and for Case 3, $\mathcal{V} = \overline{T_1[0,1]} \times \overline{T_1[0,1]}$.

By Theorem 2.3.1, for any $u \in \mathcal{V}$ there exists a non-zero real number c^2 such that

$$c^2\|u\|_{\mathcal{L}^2}^2 \le b_T(u,u).$$

The number $K^* = \max\{\beta, c^{-2}\}$ is used in the proposition below, which is an adaptation of the corollary to Theorem 2.3.1.

Proposition 3.2.1.
If

(a) $S \geq 0$, *or*

(b) $S < 0$ *and*

$$\sup |S| \leq \frac{1}{2K^* + 1}, \tag{3.2.2}$$

then b_{AT} *is an inner product for* $\mathcal{V}$.

Proof. (a) If $S \geq 0$, then using the fact that for $u \in \mathcal{V}$, $b_T(u,u) \leq b_{AT}(u,u)$, Theorem 2.3.1 implies the desired result.

(b) Suppose $S < 0$ and (3.2.2) holds. The symmetry and linearity of b_{AT} is trivial. Let $u \in \mathcal{V}$ such that $b_{AT}(u,u) = 0$. By (3.2.2) and Proposition 2.3.4,

$$|(SDu_1, Du_1)| \leq \sup |S| \|Du_1\|^2 < \frac{2K^*}{2K^* + 1} b_T(u,u). \tag{3.2.3}$$

Now

$$b_{AT}(u,u) = b_T(u,u) + (SDu_1, Du_1) \geq b_T(u,u) - \frac{2K^*}{2K^* + 1} b_T(u,u). \tag{3.2.4}$$

Therefore

$$b_{AT}(u,u) \geq \frac{1}{2K^* + 1} b_T(u,u). \tag{3.2.5}$$

That is, $b_T(u,u) = 0$ and hence $u = 0$.

□

Assumption In the remainder of Chapter 3, it is assumed that (3.2.2) holds when $S < 0$.

Let $\|\cdot\|_{\mathcal{V}_A}$ denote the norm induced by the inner product b_{AT}.

Proposition 3.2.2 (Adaptation of Proposition 2.3.5)**.**
The norms $\|\cdot\|_{H^1}$ *and* $\|\cdot\|_{\mathcal{V}_A}$ *are equivalent.*

Proof. The proof is the same as that of Proposition 2.3.5, where (3.2.5) is used instead of the inequality (2.3.5). □

As in Chapter 2, let

$$f(\cdot,t) = \langle P(\cdot,t), 0\rangle \,.$$

The weak form of Problem ATV follows.

Problem ATW

Given $u_0 \in \mathcal{V}$, $u_d \in \mathcal{W}$, and $f \in C([0,\tau);\mathcal{L}^2(0,1)^2)$, find $u \in C^2((0,\tau);\mathcal{W})$ such that for each $t > 0$, $u(t) \in \mathcal{V}$, $u'(t) \in \mathcal{V}$ and

$$\begin{aligned} c_T\left(u''(t), v\right) + b_{AT}(u(t), v) &= (f(t), v)^{(2)} \quad \text{for each} \quad v \in \mathcal{V}, \\ \lim_{t\to 0^+} \|u(t) - u_0\|_{\mathcal{V}_\mathcal{A}} &= 0, \\ \lim_{t\to 0^+} \|u'(t) - u_d\|_{\mathcal{L}^2} &= 0. \end{aligned}$$

Note that if $u(t)$ is continuous with respect to the norm $\|\cdot\|_{\mathcal{V}_\mathcal{A}}$ and $u'(t)$ is continuous with respect to the norm $\|\cdot\|_{\mathcal{L}^2}$, then the initial conditions become

$$u(0) = u_0 \quad \text{and} \quad u'(0) = u_d.$$

Problem ATW, with $u(0) = u_0$ and $u'(0) = u_d$ resembles Problem GVar defined in Chapter 2.

By the equivalence of the norms $\|\cdot\|_{\mathcal{V}_\mathcal{A}}$ and $\|\cdot\|_{H^1}$ (Proposition 3.2.2) and the norms $\|\cdot\|_{\mathcal{L}^2}$ and $\|\cdot\|_{\mathcal{W}}$ (Proposition 2.3.1), it follows that $(\mathcal{V}, \|\cdot\|_{\mathcal{V}_\mathcal{A}})$ and $(\mathcal{W}, \|\cdot\|_{\mathcal{W}})$ are Hilbert spaces. That is, the assumptions S1-S3 are satisfied. Thus, in order to show existence of a unique solution, only the assumptions A1-A4 are required to hold.

Theorem 3.2.1. *For $u_0 \in E_b$ and $u_d \in \mathcal{V}$, Problem ATW has a unique solution*

$$u \in C^1\left([0,\infty), \mathcal{V}\right) \cap C^2\left([0,\infty), \mathcal{W}\right).$$

Proof. The assumptions A1-A4 are verified in order to use Theorem 2.4.3 from Chapter 2.

A1 Since $C_0^\infty(0,1)$ is dense in $\mathcal{L}^2(0,1)$ and by the equivalence of the norms $\|\cdot\|_{\mathcal{L}^2}$ and $\|\cdot\|_{\mathcal{W}}$, it follows that $C_0^\infty(0,1)^2$ is dense in $\mathcal{W}$ and $\mathcal{W}$ is dense in $\mathcal{L}^2(0,1)^2$. That is, by the fact that $C_0^\infty(0,1)^2 \subset \mathcal{V} \subset \mathcal{W}$, $\mathcal{V}$ is dense in $\mathcal{W}$ and $\mathcal{W}$ is dense in $\mathcal{L}^2(0,1)^2$.

A2 By the equivalence of the norms $\|\cdot\|_{\mathcal{L}^2}$ and $\|\cdot\|_{\mathcal{W}}$, there exists a positive constant $C_\mathcal{W}$ such that $\|w\|_{\mathcal{L}^2} \le C_\mathcal{W}\|w\|_{\mathcal{W}}$ for each $w \in \mathcal{W}$.

A3 By the inequality (3.2.5) and the equivalence of the norms $\|\cdot\|_{\mathcal{L}^2}$ and $\|\cdot\|_{\mathcal{W}}$, it follows that there exists a positive constant $C_{\mathcal{V}}$ such that $\|v\|_{\mathcal{W}} \leq C_{\mathcal{V}}\|v\|_{\mathcal{V}_{\mathcal{A}}}$ for each $v \in \mathcal{V}$.

A4 In this case, $a \equiv 0$. Therefore the bilinear form a is non-negative, symmetric and bounded on $\mathcal{V}$.

Therefore, by Theorem 2.4.3, there exists a unique solution u for Problem ATW, where

$$u \in C^1([0,\infty),\mathcal{V}) \cap C^2([0,\infty),\mathcal{W}).$$

□

Theorem 3.2.1 implies that Problem ATW may be written with initial conditions

$$u(0) = u_0, \quad u'(0) = u_d. \tag{3.2.6}$$

3.3 Eigenvalues and eigenfunctions

To apply the results in Section 2.6, consider **Problem AT Eig** below.

$$-((1+S)u_1')' + u_2' - \lambda u_1 = 0, \tag{3.3.1}$$

$$-\frac{1}{\beta}u_2'' + u_2 - u_1' - \frac{\lambda}{\alpha}u_2 = 0. \tag{3.3.2}$$

The boundary conditions are

Case 1 (Pinned-pinned rod): $u_1(0) = u_2'(0) = u_1(1) = u_2'(1) = 0$;

Case 2 (Cantilever rod): $u_1(0) = u_2(0) = u_1'(1) - u_2(1) = u_2'(1) = 0$;

Case 3 (Clamped-clamped rod): $u_1(0) = u_2(0) = u_1(1) = u_2(1) = 0$.

To prove the existence of a complete sequence of eigenfunctions, the weak variational form of the eigenvalue problem is needed. This is presented using the notation of bilinear forms given in Sections 2.1 and 3.2.

Problem AT EigW Find $z \in \mathcal{V}$ such that $z \neq 0$ and, for each $v \in \mathcal{V}$,

$$b_{AT}(z,v) = \lambda c_T(z,v). \tag{3.3.3}$$

The existence of eigenvalues and eigenfunctions is not assumed, but if any do exist, some properties are simple to prove.

Suppose z_m and z_n are solutions of Problem AT EigW. It follows by the symmetry of b_{AT} and c_T that

$$b_{AT}(z_m, z_n) = b_{AT}(z_n, z_m)$$

and hence

$$(\lambda_m - \lambda_n)c_T(z_m, z_n) = 0.$$

That is, any eigenfunctions of Problem AT EigW are orthogonal with respect to the inner product c_T. Also, for a solution z of Problem AT EigW,

$$b_{AT}(z, z) = \lambda c_T(z, z). \tag{3.3.4}$$

By Propositions 2.3.1 and 3.2.1 , since c_T and b_{AT} are inner products and z is a solution of Problem AT EigW (i.e. $z \neq 0$), it follows that $c_T(z, z) > 0$ and $b_{AT}(z, z) > 0$. That is, all eigenvalues λ of Problem AT EigW are positive. Above, inequality (3.2.2) was used but it is also necessary to investigate the existence of zero and negative eigenvalues and the implication for applications.

Existence of a complete sequence of eigenfunctions

In order to use the theory of Section 2.6, it is only required to prove the assumption B5 since B1-B4 have been proved already in Section 3.2. Proofs of compact embedding in the literature require a smooth boundary for the relevant domain (see, for example [Eva98]). This does not make sense for the one-dimensional case. It is therefore proved in this subsection.

The following results are used in the proof of the existence of a complete sequence of eigenfunctions.

Proposition 3.3.1. *If $u \in C[0, 1] \cap H^1(0, 1)$, then for any ξ and x in $[0, 1]$,*

$$|u(\xi) - u(x)| \leq \sqrt{|\xi - x|}\ \|u\|_1.$$

Proof. For any ξ and x in $[0, 1]$, using the Cauchy-Schwartz inequality, the fact that for $u =\in C[0, 1] \cap H^1(0, 1)$ and $\|Du\|^2 \leq \|u\|_1^2$, it follows that

$$|u(\xi) - u(x)| = \left|\int_x^\xi Du\right| \leq \sqrt{|x - \xi|}\|Du\| \leq \sqrt{|x - \xi|}\|u\|_1.$$

□

Proposition 3.3.2. *If a sequence* $(u_n) \subset C[0,1] \cap H^1(0,1)$ *is bounded with respect to* $\|\cdot\|_1$*, then the sequence is equicontinuous.*

Proof. Any function in $C[0,1]$ is uniformly continuous [Rud64, Theorem 4.19]. Equicontinuity follows from Proposition 3.3.1. □

It is now possible to prove that Problem AT EigW has a complete sequence of eigenfunctions by showing that assumption B5 holds.

Theorem 3.3.1. *A bounded sequence in* $H^1(0,1)$ *has a uniformly convergent subsequence in* $C[0,1]$*.*

Proof. Let (u_n) be a bounded sequence in $H^1(0,1)$. Proposition 2.2.2 ensures that there exists a sequence $(u_n^*) \subset C[0,1] \cap H^1(0,1)$ such that for each $n \in \mathbb{N}$, $u_n = u_n^*$ a.e. and $\|u_n^*\|_{\sup} \leq \sqrt{2}\|u_n\|_1$. Hence (u_n^*) is uniformly bounded and, by Proposition 3.3.2, equicontinuous. That is, (u_n^*) is pointwise bounded and equicontinuous on $[0,1]$. Therefore (u_n^*) contains a uniformly convergent subsequence (see [Rud64, Theorem 7.23]). □

Corollary. *Any bounded sequence in* $H^1(0,1)$ *has a uniformly convergent subsequence in* $\mathcal{L}^2(0,1)$*.*

Remark. *Proposition 3.3.1 also holds for vector valued functions, using a norm on* $\mathbb{R}^n$ *instead of an absolute value.*

Theorem 3.3.2. *A bounded sequence in* $\mathcal{V}$ *has a convergent subsequence in* $\mathcal{W}$*.*

Proof. The proof follows from Theorem 3.3.1, the equivalence of the norms $\|\cdot\|_{H^1}$ and $\|\cdot\|_{\mathcal{V}_\mathcal{A}}$ (Proposition 3.2.2) and the fact that for each $v \in \mathcal{V}$, $\|v\|_\mathcal{W} \leq C_\mathcal{V}\|v\|_{\mathcal{V}_\mathcal{A}}$ (Assumption B3). □

By Theorem 2.6.4, since Assumption B5 holds, Problem AT EigW has an orthonormal sequence of eigenvectors $(\tilde{u}_n)$ – which is complete in $\mathcal{V}$ – with corresponding positive eigenvalues (λ_n) forming an increasing sequence where $\lambda_n \to \infty$ as $n \to \infty$.

Regularity

Proposition 3.3.3.

$$C^1[0,1]^2 \cap \mathcal{V} = T_P.$$

Proof. Since $T_P \subset C^1[0,1]^2$ and $T_P \subset \mathcal{V}$, it follows that $T_P \subset C^1[0,1]^2 \cap \mathcal{V}$.

Let $v \in C^1[0,1]^2 \cap \mathcal{V}$. By the definition of $\mathcal{V}$, there exists a sequence (v_n) contained in T_P such that $\|v_n - v\|_{H^1} \to 0$. That is,

$$\|v_1^{(n)} - v_1\|_1 \to 0 \text{ and } \|v_2^{(n)} - v_2\|_1 \to 0.$$

The fact that $C^1[0,1]^2 \cap \mathcal{V} \subset T_P$ follows from Proposition 2.2.1 and that (v_n) is contained in T_P. □

Remark. *It is possible to prove that a solution of Problem AT EigW is actually a classical solution of Problem AT Eig in all three cases of boundary conditions. This is done by showing regularity using induction.*

Proposition 3.3.4. *If the pair $\{u, \lambda\}$ is a solution of Problem AT EigW, then, for all three cases of boundary conditions, $u \in H^2(0,1)^2$ and $\{u, \lambda\}$ satisfies the eigenvalue problem*

$$-D((1+S)u_1') + u_2' - \lambda u_1 = 0, \quad (3.3.5)$$

$$-\frac{1}{\beta}D^2 u_2 + u_2 - u_1' - \frac{\lambda}{\alpha}u_2 = 0. \quad (3.3.6)$$

Proof. Let $\{u, \lambda\}$ be a solution of Problem AT EigW. That is, $u \in \mathcal{V}$ is such that $u \neq 0$ and, for each $v \in \mathcal{V}$,

$$b_{AT}(u, v) = \lambda c_T(u, v) \quad (3.3.7)$$

Let $x = \langle x_1, 0\rangle$ and $z = \langle 0, z_2\rangle$, where x_1 and z_2 are in $C_0^\infty(0,1)$. Then, x and z are in $\mathcal{V}$ and by the definitions of b_{AT} and c_T,

$$(Du_1 - u_2, x_1') + (SDu_1, x_1') = \lambda\,(u_1, x_1) \text{ and} \quad (3.3.8)$$

$$\left(\frac{1}{\beta}Du_2, z_2'\right) - (Du_1 - u_2, z_2) = \lambda\left(\frac{1}{\alpha}u_2, z_2\right). \quad (3.3.9)$$

That is, for any x and z in $C_0^\infty(0,1)$,

$$((1+S)Du_1, x') = -\,(Du_2 - \lambda u_1, x) \text{ and} \quad (3.3.10)$$

$$\left(\frac{1}{\beta}Du_2, z'\right) = -\left(u_2 - Du_1 - \frac{\lambda}{\alpha}u_2, z\right). \quad (3.3.11)$$

Hence $(Du_2 - \lambda u_1)$ is the weak derivative of $(1+S)Du_1$ and $\big(u_2 - Du_1 - \frac{\lambda}{\alpha}u_2\big)$ is the weak derivative of $\frac{1}{\beta}Du_2$. Therefore $u \in H^2(0,1)^2$. Also, by Theorem 2.2.2 (Sobolev's embedding theorem), u can be considered an element of $C^1(0,1)^2 \cap \mathcal{V}$. Hence

$$-D\left((1+S)u_1'\right) + u_2' - \lambda u_1 = 0 \tag{3.3.12}$$

$$-\frac{1}{\beta}D^2u_2 + u_2 - u_1' - \frac{\lambda}{\alpha}u_2 = 0. \tag{3.3.13}$$

By (3.3.12) and (3.3.13), it follows that for any $v = \langle v_1, v_2\rangle \in C^1[0,1]^2$,

$$\begin{aligned}&\big(-(1+S)D^2u_1 - \partial_x S u_1' + u_2' - \lambda u_1, v_1\big)\\ &= -\left(-\frac{1}{\beta}D^2u_2 + u_2 - u_1' - \frac{\lambda}{\alpha}u_2, v_2\right).\end{aligned} \tag{3.3.14}$$

The fact that $u \in C^1(0,1)^2 \cap \mathcal{V}$ implies that $u \in T_P$ (Proposition 3.3.3) and

$$-(1+S)u_1'v_1]_0^1 - \frac{1}{\beta}u_2'v_2]_0^1 + u_2v_1]_0^1 + b_{AT}(u,v) - \lambda c_T(u,v) = 0.$$

Since u is a solution of Problem AT EigW, it follows that for any $v \in C^1[0,1]^2$,

$$-(1+S)u_1'v_1]_0^1 - \frac{1}{\beta}u_2'v_2]_0^1 + u_2v_1]_0^1 = 0. \tag{3.3.15}$$

Case 1: For any $v = \langle v_1, v_2\rangle \in T_P$, Equation (3.3.15) implies that

$$-\frac{1}{\beta}u_2'(1)v_2(1) + \frac{1}{\beta}u_2'(0)v_2(0) = 0. \tag{3.3.16}$$

For $x \in [0,1]$, let $w(x) = x$ and $z(x) = 1-x$. Then, for arbitrary $v_1 \in T[0,1]$, $\langle v_1, w\rangle$ and $\langle v_1, z\rangle$ are in T_P. Therefore, by Equation (3.3.16) it follows that

$$u_2'(1) = 0 \quad \text{and} \quad u_2'(0) = 0.$$

Case 2: Since $S(1) = 0$, (3.3.15) implies that for any $v = \langle v_1, v_2\rangle \in T_P$,

$$-\left(u_1'(1) - u_2(1)\right)v_1(1) - \frac{1}{\beta}u_2'(1)v_2(1) = 0. \tag{3.3.17}$$

For $x \in [0,1]$, let $w(x) = x(1-x)$. Then, for v_1 and v_2 in $T[0,1]$ such that $v_1(1) \neq 0$ and $v_2(1) \neq 0$, $\langle v_1, w\rangle$ and $\langle w, v_2\rangle$ are in T_P. Therefore, by (3.3.17),

$$u_1'(1) - u_2(1) = 0 \quad \text{and} \quad u_2'(1) = 0.$$

Case 3: No further calculations are necessary.

Thus, $u \in H^2(0,1)^2$ and $\{u, \lambda\}$ satisfies the given eigenvalue problem for all three cases of boundary conditions. □

Proposition 3.3.5. *If the pair* $\{u, \lambda\}$ *is a solution of Problem AT EigW and* $S \in C^2[0,1]$*, where* $S > -1$*, then* $u \in H^m(0,1)^2$*, where* $m \geq 3$*.*

Proof. Note that u_1, u_1', u_2 and u_2' are in $H^1(0,1)$. Therefore $D(D^2u_1)$ and $D(D^2u_2)$ exist and hence $u \in H^3(0,1)^2$ with

$$D^3u_1 = D\left(\frac{1}{1+S}\left(u_2' - \lambda u_1 - \partial_x S u_1'\right)\right), \quad (3.3.18)$$

$$D^3u_2 = D\left(\beta u_2 - \beta u_1' - \lambda\gamma u_2\right). \quad (3.3.19)$$

Similarly, it follows that $u \in H^m(0,1)^2$ for $m \geq 3$. □

By Sobolev's Embedding Theorem in one dimension (Theorem 2.2.2), u_1 and u_2 can be considered to be in $C^2[0,1]$. That is, $u \in C^2[0,1]^2 \cap \mathcal{V}$. Therefore the orthonormal sequence of eigenvectors $(\tilde{u}_n)$ of Problem AT EigW – which is complete in $\mathcal{V}$ – is an orthonormal sequence of eigenvectors for Problem AT Eig.

Improved regularity

The following result is used to show the regularity of an eigenfunction of Problem AT EigW.

Theorem 3.3.3. *Problem AT Eig has a classical solution.*

Proof. From Section 3.3, Problem AT Eig has an orthonormal sequence of twice differentiable eigenvectors $(\tilde{u}_n)$ – which is complete in $\mathcal{V}$ – with corresponding positive eigenvalues (λ_n) forming an increasing sequence where $\lambda_n \to \infty$ as $n \to \infty$. It remains to show that the boundary conditions of the eigenvalue problem are satisfied.

By Equations (3.3.12) and (3.3.13) in the proof of Proposition 3.3.4 and by Theorem 2.2.2, it follows that for any solution u to Problem AT EigW, $u \in C^2[0,1]^2 \cap \mathcal{V}$ and

$$-((1+S)u_1')' + u_2' - \lambda u_1 = 0, \quad (3.3.20)$$

$$-\frac{1}{\beta}u_2'' + u_2 - u_1' - \frac{\lambda}{\alpha}u_2 = 0. \quad (3.3.21)$$

It follows from Proposition 3.3.3 that since $u \in (C^2[0,1]^2 \cap \mathcal{V})$, $u \in T_P$. Thus, for both the pinned-pinned and the cantilever case, since $\bar{u} \in T_P$, $\bar{u}$ satisfies the boundary conditions of Problem AT Eig. □

If inequality (3.2.2) does not hold, then the results obtained thus far may not be true. However, when the parameters in Problem AT Eig are constant, results can be achieved using the theory of ordinary differential equations, as is shown in the next section.

3.4 Properties of eigenfunctions

For the results in Section 3.3 it is not necessary that the axial force S be constant. However, by assuming that S is constant, more can be achieved. Assume S is a constant, say S_0, where $S_0 > -1$. The approach in [VV06] for the classical Timoshenko theory (i.e. $S_0 = 0$) is taken in this section. The results presented are more comprehensive than those in either [VV06] or [CST16].

Suppose the function $e^{mx}\bar{w}$ is a solution of Problem AT Eig. This is possible if and only if

$$\begin{bmatrix} -(S_0+1)m^2-\lambda & m \\ -\alpha m & -\frac{1}{\gamma}m^2+(\alpha-\lambda) \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{3.4.1}$$

In order to find a nontrivial solution, it is necessary that the determinant

$$(S_0+1)m^4 + (-S_0\beta + \lambda(1+(S_0+1)\gamma))\, m^2 + \gamma\lambda(\lambda-\alpha) = 0. \tag{3.4.2}$$

The roots m^2 of Equation (3.4.2) are

$$m^2 = -\frac{1}{2(S_0+1)}\left(-S_0\beta + \lambda(1+(1+S_0)\gamma)\right)(1 \pm \Delta^{\frac{1}{2}}), \tag{3.4.3}$$

where

$$\begin{aligned} \Delta &= 1 - \frac{4(S_0+1)(\gamma\lambda(\lambda-\alpha))}{(-S_0\beta+\lambda(1+(1+S_0)\gamma))^2} \\ &= \frac{4\beta\lambda + (S_0\beta+\lambda(1-(S_0+1)\gamma))^2}{(S_0\beta+\lambda(-1-(S_0+1)\gamma))^2}. \end{aligned} \tag{3.4.4}$$

Note that Δ is a continuous function of λ. Therefore, since $\Delta > 0$ for $\lambda > 0$ and $\Delta = 1$ if $\lambda = 0$, there exists a real number $C^* \in (-1, 0)$ such that if $\lambda \in (C^*, 0)$, then $\Delta > 0$. Therefore, considering $\lambda > C^*$, the roots m^2 of Equation (3.4.2) are real numbers and are not equal.

If $S_0 \geq 0$, then $\lambda > 0$ (see Section 3.3). Therefore if $\lambda \leq 0$, then $-1 < S_0 < 0$.

Special cases

If $\lambda = 0$ or $\lambda = \alpha$, then one of the roots m^2 of Equation (3.4.2) is zero and the other root is negative. That is, the four roots m of Equation (3.4.2) are 0 (with multiplicity 2) and two others denoted by $\pm\omega i$.

In the case where $m^2 = 0$ (which only occurs if $\lambda = \alpha$ or $\lambda = 0$), two linearly independent solutions of Problem AT Eig can be found by inspection of Equations (3.3.1) and (3.3.2):

$$\lambda = \alpha : \quad \begin{bmatrix} w(x) \\ \phi(x) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ or } \begin{bmatrix} 1 \\ \alpha x \end{bmatrix}, \qquad \lambda = 0 : \quad \begin{bmatrix} w(x) \\ \phi(x) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ or } \begin{bmatrix} x \\ 1 \end{bmatrix}.$$

Other cases

If $C^* < \lambda < 0$ or $\lambda > \alpha$, then both roots m^2 of Equation (3.4.2) are negative. That is, the four roots m of Equation (3.4.2) can be denoted as $\pm\theta i$ and $\pm\omega i$.

If $0 < \lambda < \alpha$, then one of the roots m^2 of Equation (3.4.2) is negative and the other root is positive. That is, the four roots m of Equation (3.4.2) can be denoted as $\pm\mu$ and $\pm\omega i$.

If $m^2 \neq 0$, then the solutions are of the form $e^{mx}\langle w_1, w_2\rangle$, where $w_1 = m$ and $w_2 = (S_0 + 1)m^2 + \lambda$.

The general solutions of each of the five cases ($C^* < \lambda < 0$, $\lambda = 0$, $0 < \lambda < \alpha$, $\lambda = \alpha$ and $\lambda > \alpha$) are now considered. The forms of the general solution of the system in Problem AT Eig are given below. For the three cases where $\lambda > 0$, the results are very similar to [VV06].

General solution for the cases $C^* < \lambda < 0$ and $\lambda > \alpha$

$$\begin{aligned} \begin{bmatrix} w(x) \\ \phi(x) \end{bmatrix} &= A \begin{bmatrix} \sin\theta x \\ -\frac{\lambda-(S_0+1)\theta^2}{\theta}\cos\theta x \end{bmatrix} + B \begin{bmatrix} \cos\theta x \\ \frac{\lambda-(S_0+1)\theta^2}{\theta}\sin\theta x \end{bmatrix} \\ &\quad + C \begin{bmatrix} \sin\omega x \\ -\frac{\lambda-(S_0+1)\omega^2}{\omega}\cos\omega x \end{bmatrix} + D \begin{bmatrix} \cos\omega x \\ \frac{\lambda-(S_0+1)\omega^2}{\omega}\sin\omega x \end{bmatrix} \end{aligned} \tag{3.4.5}$$

General solution for the case $\lambda = 0$

$$\begin{aligned} \begin{bmatrix} w(x) \\ \phi(x) \end{bmatrix} &= A \begin{bmatrix} x \\ 1 \end{bmatrix} + B \begin{bmatrix} 1 \\ 0 \end{bmatrix} + C \begin{bmatrix} \sin\omega x \\ -\frac{\lambda-(S_0+1)\omega^2}{\omega}\cos\omega x \end{bmatrix} \\ &\quad + D \begin{bmatrix} \cos\omega x \\ \frac{\lambda-(S_0+1)\omega^2}{\omega}\sin\omega x \end{bmatrix} \end{aligned} \tag{3.4.6}$$

General solution for the case $0 < \lambda < \alpha$

$$\begin{aligned}\begin{bmatrix} w(x) \\ \phi(x) \end{bmatrix} &= A \begin{bmatrix} \sinh \mu x \\ \frac{\lambda + (S_0+1)\mu^2}{\mu} \cosh \mu x \end{bmatrix} + B \begin{bmatrix} \cosh \mu x \\ \frac{\lambda + (S_0+1)\mu^2}{\mu} \sinh \mu x \end{bmatrix} \\ &+ C \begin{bmatrix} \sin \omega x \\ -\frac{\lambda - (S_0+1)\omega^2}{\omega} \cos \omega x \end{bmatrix} + D \begin{bmatrix} \cos \omega x \\ \frac{\lambda - (S_0+1)\omega^2}{\omega} \sin \omega x \end{bmatrix} \end{aligned} \tag{3.4.7}$$

General solution for the case $\lambda = \alpha$

$$\begin{aligned}\begin{bmatrix} w(x) \\ \phi(x) \end{bmatrix} &= A \begin{bmatrix} 0 \\ 1 \end{bmatrix} + B \begin{bmatrix} 1 \\ \alpha x \end{bmatrix} + C \begin{bmatrix} \sin \omega x \\ -\frac{\lambda - (S_0+1)\omega^2}{\omega} \cos \omega x \end{bmatrix} \\ &+ D \begin{bmatrix} \cos \omega x \\ \frac{\lambda - (S_0+1)\omega^2}{\omega} \sin \omega x \end{bmatrix} \end{aligned} \tag{3.4.8}$$

In [CST16], the value α is referred to as the "transition frequency". This is because, for the case where all eigenvalues are positive, there is a shift in the "nature" of the vibration modes when $\lambda = \alpha$ is crossed. This also appears true for the case $\lambda > C^*$, with an additional transition frequency at 0.

As in [VV06], it should be noted that ω, μ and θ are uniquely determined by λ:

$$\omega^2 = \frac{1}{2(S_0+1)} \left|\lambda(1 + (1+S_0)\gamma) - S_0\beta\right| \left(\Delta^{\frac{1}{2}} + 1\right) \text{ for } \lambda > 0, \tag{3.4.9}$$

$$\mu^2 = \frac{1}{2(S_0+1)} \left|\lambda(1 + (1+S_0)\gamma) - S_0\beta\right| \left(\Delta^{\frac{1}{2}} - 1\right) \text{ for } \lambda < \alpha, \tag{3.4.10}$$

$$\theta^2 = \frac{1}{2(S_0+1)} \left|\lambda(1 + (1+S_0)\gamma) - S_0\beta\right| \left(1 - \Delta^{\frac{1}{2}}\right) \text{ for } \lambda > \alpha. \tag{3.4.11}$$

Five possibilities of the value of λ have been considered. In each case four linearly independent solutions have been found. The solution space of Problem AT Eig, however, is four-dimensional since it can be written as a system of four linear first-order equations. The following theorem therefore follows.

Theorem 3.4.1. *In each of the five cases* $C^* < \lambda < 0$, $\lambda = 0$, $\lambda < \alpha$, $\lambda = \alpha$ *and* $\lambda > \alpha$ *respectively, a general solution for the system of differential equations in Problem AT exists and has the forms given by* (3.4.5)*,* (3.4.6)*,* (3.4.7)*,*(3.4.8) *and* (3.4.5) *respectively.*

In order to determine the eigenvalues of Problem AT Eig, the following strategy is used. Applying the boundary conditions at zero causes the dimension of the solution space of the system of differential equations to reduce to two. Any eigenspace, therefore, can have a dimension of at most two. Three cases of boundary conditions at zero are considered. These include clamped, free and pinned. The reduction of the solution space will be demonstrated for the first two zero boundary conditions in this subsection and the third zero boundary condition in the next subsection. The boundary conditions considered are

Clamped at $x = 0$: $\quad w(0) = \phi(0) = 0,$

Free at $x = 0$: $\quad w'(0) - \phi(0) = \phi'(0) = 0,$

Pinned at $x = 0$: $\quad w(0) = \phi'(0) = 0.$

Substitution of these zero boundary conditions yields solutions where two of the four coefficients can be written in terms of the other two. Then, any of the three boundary conditions can be applied at $x = 1$ in order to solve for the remaining coefficients. These boundary conditions are

Clamped at $x = 1$: $\quad w(1) = \phi(1) = 0,$

Free at $x = 1$: $\quad w'(1) - \phi(1) = \phi'(1) = 0,$

Pinned at $x = 1$: $\quad w(1) = \phi'(1) = 0.$

Clamped at $x = 0$

The cases $C^* < \lambda < 0$ and $\lambda > \alpha$

$$\begin{aligned}\begin{bmatrix} w(x) \\ \phi(x) \end{bmatrix} &= A \begin{bmatrix} \sin\theta x \\ -\frac{\lambda-(S_0+1)\theta^2}{\theta}\cos\theta x \end{bmatrix} + B \begin{bmatrix} \cos\theta x \\ \frac{\lambda-(S_0+1)\theta^2}{\theta}\sin\theta x \end{bmatrix} \\ &\quad -A\frac{\omega(\lambda-(S_0+1)\theta^2)}{\theta(\lambda-(S_0+1)\omega^2)} \begin{bmatrix} \sin\omega x \\ -\frac{\lambda-(S_0+1)\omega^2}{\omega}\cos\omega x \end{bmatrix} \\ &\quad -B \begin{bmatrix} \cos\omega x \\ \frac{\lambda-(S_0+1)\omega^2}{\omega}\sin\omega x \end{bmatrix}\end{aligned}$$

The case $\lambda = 0$

$$\begin{aligned}\begin{bmatrix} w(x) \\ \phi(x) \end{bmatrix} &= A \begin{bmatrix} x \\ 1 \end{bmatrix} + B \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \frac{\omega A}{\lambda-(S_0+1)\omega^2} \begin{bmatrix} \sin\omega x \\ -\frac{\lambda-(S_0+1)\omega^2}{\omega}\cos\omega x \end{bmatrix} \\ &\quad -B \begin{bmatrix} \cos\omega x \\ \frac{\lambda-(S_0+1)\omega^2}{\omega}\sin\omega x \end{bmatrix}\end{aligned}$$

The case $0 < \lambda < \alpha$

$$\begin{aligned}\begin{bmatrix} w(x) \\ \phi(x) \end{bmatrix} &= A \begin{bmatrix} \sinh \mu x \\ \frac{\lambda + (S_0+1)\mu^2}{\mu} \cosh \mu x \end{bmatrix} + B \begin{bmatrix} \cosh \mu x \\ \frac{\lambda + (S_0+1)\mu^2}{\mu} \sinh \mu x \end{bmatrix} \\ &+ A \frac{\omega(\lambda + (S_0+1)\mu^2)}{\mu(\lambda - (S_0+1)\omega^2)} \begin{bmatrix} \sin \omega x \\ -\frac{\lambda - (S_0+1)\omega^2}{\omega} \cos \omega x \end{bmatrix} \\ &- B \begin{bmatrix} \cos \omega x \\ \frac{\lambda - (S_0+1)\omega^2}{\omega} \sin \omega x \end{bmatrix}\end{aligned}$$

The case $\lambda = \alpha$

$$\begin{aligned}\begin{bmatrix} w(x) \\ \phi(x) \end{bmatrix} &= A \begin{bmatrix} 0 \\ 1 \end{bmatrix} + B \begin{bmatrix} 1 \\ \alpha x \end{bmatrix} + \frac{\omega A}{\lambda - (S_0+1)\omega^2} \begin{bmatrix} \sin \omega x \\ -\frac{\lambda - (S_0+1)\omega^2}{\omega} \cos \omega x \end{bmatrix} \\ &- B \begin{bmatrix} \cos \omega x \\ \frac{\lambda - (S_0+1)\omega^2}{\omega} \sin \omega x \end{bmatrix}\end{aligned}$$

If the boundary conditions at $x = 1$ are applied, the following system is obtained and the other two coefficients can be solved for:

$$\begin{bmatrix} d_1(\mu(\lambda), \omega(\lambda), \theta(\lambda)) & d_2(\mu(\lambda), \omega(\lambda), \theta(\lambda)) \\ d_3(\mu(\lambda), \omega(\lambda), \theta(\lambda)) & d_4(\mu(\lambda), \omega(\lambda), \theta(\lambda)) \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{3.4.12}$$

More detail of the cantilever (clamped-free) and clamped-clamped rods is given in Subsection 3.5.2. In general, for all cases of boundary conditions at $x = 1$ and all possibilities of λ, the coefficient matrix is non-zero.

Free at $x = 0$

The cases $C^* < \lambda < 0$ and $\lambda > \alpha$

$$\begin{aligned}\begin{bmatrix} w(x) \\ \phi(x) \end{bmatrix} &= A \begin{bmatrix} \sin \theta x \\ -\frac{\lambda - (S_0+1)\theta^2}{\theta} \cos \theta x \end{bmatrix} + B \begin{bmatrix} \cos \theta x \\ \frac{\lambda - (S_0+1)\theta^2}{\theta} \sin \theta x \end{bmatrix} \\ &- A \frac{\omega\,(\lambda - S_0\theta^2)}{\theta\,(\lambda - S_0\omega^2)} \begin{bmatrix} \sin \omega x \\ -\frac{\lambda - (S_0+1)\omega^2}{\omega} \cos \omega x \end{bmatrix} \\ &- B \frac{\lambda - (S_0+1)\theta^2}{\lambda - (S_0+1)\omega^2} \begin{bmatrix} \cos \omega x \\ \frac{\lambda - (S_0+1)\omega^2}{\omega} \sin \omega x \end{bmatrix}\end{aligned}$$

The case $\lambda = 0$

$$\begin{bmatrix} w(x) \\ \phi(x) \end{bmatrix} = A \begin{bmatrix} x \\ 1 \end{bmatrix} + B \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

The case $0 < \lambda < \alpha$

$$\begin{aligned}\begin{bmatrix} w(x) \\ \phi(x) \end{bmatrix} &= A \begin{bmatrix} \sinh \mu x \\ \frac{\lambda + (S_0+1)\mu^2}{\mu} \cosh \mu x \end{bmatrix} + B \begin{bmatrix} \cosh \mu x \\ \frac{\lambda + (S_0+1)\mu^2}{\mu} \sinh \mu x \end{bmatrix} \\ &+ A \frac{\omega\,(\lambda + S_0\mu^2)}{\mu\,(\lambda - S_0\omega^2)} \begin{bmatrix} \sin \omega x \\ -\frac{\lambda - (S_0+1)\omega^2}{\omega} \cos \omega x \end{bmatrix} \\ &- B \frac{\lambda + (S_0+1)\mu^2}{\lambda - (S_0+1)\omega^2} \begin{bmatrix} \cos \omega x \\ \frac{\lambda - (S_0+1)\omega^2}{\omega} \sin \omega x \end{bmatrix}\end{aligned}$$

The case $\lambda = \alpha$

$$\begin{aligned}\begin{bmatrix} w(x) \\ \phi(x) \end{bmatrix} &= A \begin{bmatrix} 0 \\ 1 \end{bmatrix} + B \begin{bmatrix} 1 \\ \alpha x \end{bmatrix} - A \frac{\omega}{\lambda - S_0\omega^2} \begin{bmatrix} \sin \omega x \\ -\frac{\lambda - (S_0+1)\omega^2}{\omega} \cos \omega x \end{bmatrix} \\ &- B \frac{\alpha}{\lambda - (S_0+1)\omega^2} \begin{bmatrix} \cos \omega x \\ \frac{\lambda - (S_0+1)\omega^2}{\omega} \sin \omega x \end{bmatrix}\end{aligned}$$

Again, if the boundary conditions at $x = 1$ are applied, the following system is obtained and the other two coefficients can be solved for:

$$\begin{bmatrix} d_1(\mu(\lambda), \omega(\lambda), \theta(\lambda)) & d_2(\mu(\lambda), \omega(\lambda), \theta(\lambda)) \\ d_3(\mu(\lambda), \omega(\lambda), \theta(\lambda)) & d_4(\mu(\lambda), \omega(\lambda), \theta(\lambda)) \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{3.4.13}$$

For all cases of boundary conditions at $x = 1$ and all possibilities of λ, the coefficient matrix is non-zero.

Note that the solution space, and therefore the eigenspace, of each case will have dimension two only if the coefficient matrix of (3.4.12) and (3.4.13) respectively is a zero matrix. If this occurs, λ is referred to as a double eigenvalue. If the coefficient matrix of (3.4.12) and (3.4.13) respectively is nonzero, then λ is an eigenvalue if and only if the determinant

$$d_1 d_4 - d_2 d_3 = 0.$$

For both the clamped and the free case and all of the boundary conditions at $x = 1$, there is at least one nonzero entry in the coefficient matrix of (3.4.12) and (3.4.13) respectively. Therefore all eigenvalues in these cases are simple.

3.5 Computation of eigenvalues and eigenfunctions

3.5.1 Rod with pinned-pinned boundary conditions

The same approach as in the previous subsection is now taken with a rod that has pinned boundary conditions at $x = 0$. Detail is given for a rod that is pinned on both ends since the pinned-clamped and pinned-free rods could be viewed as the clamped-pinned and free-pinned rods, which have already been considered. If the zero boundary values of the pinned-pinned problem are substituted into the general equations, since ω, μ and θ are distinct, nonzero values, it results in the following.

The cases $C^* < \lambda < 0$ and $\lambda > \alpha$

$$\begin{bmatrix} w(x) \\ \phi(x) \end{bmatrix} = A \begin{bmatrix} \sin\theta x \\ -\frac{\lambda-(S_0+1)\theta^2}{\theta}\cos\theta x \end{bmatrix} + C \begin{bmatrix} \sin\omega x \\ -\frac{\lambda-(S_0+1)\omega^2}{\omega}\cos\omega x \end{bmatrix} \tag{3.5.1}$$

Applying the two boundary conditions of the pinned-pinned problem at $x = 1$, the following equation is found

$$\begin{bmatrix} \sin\theta & \sin\omega \\ (\lambda-(S_0+1)\theta^2)\sin\theta & (\lambda-(S_0+1)\omega^2)\sin\omega \end{bmatrix} \begin{bmatrix} A \\ C \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{3.5.2}$$

The case $\lambda = 0$

$$\begin{bmatrix} w(x) \\ \phi(x) \end{bmatrix} = A \begin{bmatrix} x \\ 1 \end{bmatrix} + C \begin{bmatrix} \sin\omega x \\ -\frac{\lambda-(S_0+1)\omega^2}{\omega}\cos\omega x \end{bmatrix} \tag{3.5.3}$$

Applying the two boundary conditions of the pinned-pinned problem at $x = 1$, the following equation is found

$$\begin{bmatrix} 1 & \sin\omega \\ 0 & (\lambda-(S_0+1)\omega^2)\sin\omega \end{bmatrix} \begin{bmatrix} A \\ C \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{3.5.4}$$

The case $0 < \lambda < \alpha$

$$\begin{bmatrix} w(x) \\ \phi(x) \end{bmatrix} = A \begin{bmatrix} \sinh\mu x \\ \frac{\lambda+(S_0+1)\mu^2}{\mu}\cosh\mu x \end{bmatrix} + C \begin{bmatrix} \sin\omega x \\ -\frac{\lambda-(S_0+1)\omega^2}{\omega}\cos\omega x \end{bmatrix}. \tag{3.5.5}$$

Applying the two boundary conditions of the pinned-pinned problem at $x = 1$, the following equation is found

$$\begin{bmatrix} \sinh\mu & \sin\omega \\ (\lambda+(S_0+1)\mu^2)\sinh\mu & (\lambda-(S_0+1)\omega^2)\sin\omega \end{bmatrix} \begin{bmatrix} A \\ C \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{3.5.6}$$

The case $\lambda = \alpha$

$$\begin{bmatrix} w(x) \\ \phi(x) \end{bmatrix} = A \begin{bmatrix} 0 \\ 1 \end{bmatrix} + C \begin{bmatrix} \sin \omega x \\ -\frac{\lambda-(S_0+1)\omega^2}{\omega} \cos \omega x \end{bmatrix} \tag{3.5.7}$$

Applying the two boundary conditions of the pinned-pinned problem at $x = 1$, the following equation is found

$$\begin{bmatrix} 0 & \sin \omega \\ 0 & (\lambda - (S_0+1)\omega^2) \sin \omega \end{bmatrix} \begin{bmatrix} A \\ C \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{3.5.8}$$

Using the results above, the following theorem follows.

Theorem 3.5.1. *The following statements are true for a rod with pinned-pinned boundary conditions:*

(a) The value α is an eigenvalue of the pinned-pinned problem with the corresponding eigenfunction $[0\ 1]^T$.

(b) If $[w\ \phi]^T$ is a non-constant eigenfunction of the pinned-pinned problem, then given $S_0 > -1$, it follows that the sequence of eigenfunctions is given by

$$\begin{bmatrix} w_k(x) \\ \phi_k(x) \end{bmatrix} = \begin{bmatrix} \sin(k\pi x) \\ A_k \cos(k\pi x) \end{bmatrix}, \tag{3.5.9}$$

with k an integer and A_k a constant depending on k and λ_k.

(c) All eigenvalues λ, where $0 \leq < \lambda < \alpha$, are simple.

Proof. (a) Simple substitution shows that for $\lambda = \alpha$, $[0\ 1]^T$ is a solution of the pinned-pinned problem.

(b) When Equation (3.5.2) is solved, it follows that $C \sin \omega = A \sin \theta = 0$. Therefore three options arise that result in a non-trivial solution. Either $A = 0$ and ω is a multiple of π; $C = 0$ and θ is a multiple of π; or A and C are real numbers and both ω and θ are multiples of π, but are not equal. In all three cases, it still follows that for $C^* < \lambda < 0$ and $\lambda > \alpha$, the sequence of eigenfunctions has the given form.

When Equations (3.5.4) and (3.5.6) are solved, it follows that $A = 0$ and therefore, for a non-trivial solution it is required that ω is a multiple of π. That is, for $0 \leq \lambda < \alpha$, the sequence of eigenfunctions has the given form.

When Equation (3.5.8) is solved, it follows that $A \in \mathbb{R}$ and $C \sin \omega = 0$. Therefore, for a non-constant solution it is required that ω is a multiple of π. That is, for $\lambda = \alpha$, the sequence of eigenfunctions has the given form.

(c) From the proof of (b) it can easily be seen that the solution space is one-dimensional when $0 \leq \lambda < \alpha$.

□

Suppose $[w_k \; \phi_k]^T$ is a non-constant eigenfunction of the pinned-pinned problem.

Substituting (3.5.9) into Problem AT Eig, it follows that $[w_k \; \phi_k]^T$ is a solution of the system if and only if the pair (λ_k, A_k) is a solution of

$$
\begin{aligned}
(S_0+1)k^2\pi^2 - A_k k\pi &= \lambda_k, && (3.5.10)\\
\frac{k^2\pi^2 A_k}{\gamma} - \alpha k\pi + \alpha A_k &= \lambda_k A_k. && (3.5.11)
\end{aligned}
$$

From Equation (3.5.11) it is clear that $A_k \neq 0$.

If $\lambda_k = \alpha$ then it follows from Equations (3.5.10) and (3.5.11) that a necessary condition for α to be a double eigenvalue is

$$
\alpha = \frac{k^2\pi^2(S_0+1)}{(1+\gamma)} \quad \text{or} \quad \beta = (S_0+1)k^2\pi^2 - \alpha. \tag{3.5.12}
$$

If there exists an integer k such that (3.5.12) holds, then the eigenvalue $\lambda_k = \alpha$ depends on the elastic constants and the dimensions of the rod.

A necessary condition for 0 to be an eigenvalue is

$$
\frac{k^2\pi^2(S_0+1)}{-S_0} = \beta. \tag{3.5.13}
$$

To determine the distribution of eigenvalues, a different approach is taken compared to [VV06]. Substituting (3.5.10) into (3.5.11), it follows that

$$
\begin{aligned}
A_k^2 + \left(k\pi\left(\frac{1}{\gamma} - S_0 - 1\right) + \frac{\alpha}{k\pi}\right)A_k - \alpha &= 0 && (3.5.14)\\
(S_0+1)k^2\pi^2 - A_k k\pi &= \lambda_k && (3.5.15)
\end{aligned}
$$

That is, two different real values for A_k are obtained, resulting in two different values for λ_k.

$$A_k = \frac{k^2\pi^2\left(S_0 + 1 - \gamma^{-1}\right) - \alpha}{2k\pi}\left(1 \pm \sqrt{1 + \frac{4\alpha}{\left(k\pi\left(\gamma^{-1} - S_0 - 1\right) + \frac{\alpha}{k\pi}\right)^2}}\right) \quad (3.5.16)$$

$$\lambda_k = (S_0 + 1)k^2\pi^2 - A_k k\pi. \quad (3.5.17)$$

It is therefore clear that the system has two solutions (λ_k, A_k) and (λ_k^*, A_k^*). Let λ_k^* denote the larger value. The authors in [VV06] refer to the value λ_k as an eigenvalue of Type 1 and to λ_k^* as one of Type 2.

For some value of S_0 in the interval $(-1, 0)$, it is found that $\lambda_1 = 0$ and it is clear that, for smaller values of S_0, one or more of the eigenvalues may even be negative.

From (3.5.10) and (3.5.11) it can be seen that there exists a value for S_0 such that $\lambda_1 = 0$. Denoting this value by S_{crit}, it follows that

$$\begin{aligned} (S_{crit} + 1)\pi - A_1 &= 0, \\ \pi^2 A_1 - \beta\pi + \beta A_1 &= 0. \end{aligned}$$

That is,

$$(S_{crit} + 1) = \frac{\beta}{\pi^2 + \beta}.$$

Note that $S_{crit} = \frac{-\pi^2}{\pi^2+\beta}$ and hence $-1 < S_{crit} < 0$.

Proposition 3.5.1. $\lambda_1 < 0$ *if and only if* $S_0 \in (-1, S_{crit})$.

Proof. Suppose $\lambda_1 < 0$. By Equation (3.5.10), this is true if and only if

$$A_1 > (S_0 + 1)\pi. \quad (3.5.18)$$

Also, by Equation (3.5.11), $\lambda_1 < 0$ if and only if

$$\pi^2 - \frac{\beta\pi}{A_1} + \beta < 0. \quad (3.5.19)$$

That is, $\lambda_1 < 0$ if and only if

$$\pi^2 + \beta < \frac{\beta}{(S_0 + 1)}. \quad (3.5.20)$$

It therefore follows that $\lambda_1 < 0$ if and only if $S_0 < S_{crit}$. That is, if and only if $S_0 \in (-1, S_{crit})$. □

3.5.2 Rod with clamped boundary conditions

The rod with cantilever (clamped-free) or clamped-clamped boundary conditions from Section 3.4 is now considered in more detail. Recall that when the boundary values at $x = 1$ are applied, an equation of the following form is found (note that μ, ω and θ are distinct and nonzero):

$$\begin{bmatrix} d_1(\mu(\lambda), \omega(\lambda), \theta(\lambda)) & d_2(\mu(\lambda), \omega(\lambda), \theta(\lambda)) \\ d_3(\mu(\lambda), \omega(\lambda), \theta(\lambda)) & d_4(\mu(\lambda), \omega(\lambda), \theta(\lambda)) \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{3.5.21}$$

Eigenvalues of the cantilever or clamped-clamped problem are double if and only if the coefficient matrix is a zero matrix.

Cantilever rod

For the cases where $C^* < \lambda < 0$ and $\lambda > \alpha$,

$$d_2 = \frac{-\lambda + S_0\theta^2}{\theta} \sin\theta + \frac{\lambda - S_0\omega^2}{\omega} \sin\omega$$

and

$$d_4 = \left(\lambda - (S_0 + 1)\theta^2\right) \cos\theta - \left(\lambda - (S_0 + 1)\omega^2\right) \cos\omega.$$

It is not possible for d_2 and d_4 to be zero simultaneously since $\omega \neq \theta$. Hence the coefficient matrix is nonzero.

When $\lambda = 0$ and $\lambda = \alpha$,

$$d_3 = \omega \sin\omega \quad \text{and} \quad d_4 = (S_0 + 1)\omega^2 \cos\omega.$$

Since it is not possible for both $\sin\omega$ and $\cos\omega$ to be zero at the same time, it follows that the coefficient matrix is nonzero.

For the case where $0 < \lambda < \alpha$,

$$d_2 = -\frac{\lambda + S_0\mu^2}{\mu} \sinh\mu - \frac{\lambda - (S_0 + 2)\omega^2}{\omega} \sin\omega \neq 0.$$

That is, the coefficient matrix is nonzero.

In all five cases, there are no double eigenvalues.

Clamped-clamped rod

It can be shown that in each case ($C^* < \lambda < 0$, $\lambda = 0$, $0 < \lambda < \alpha$, $\lambda = \alpha$ and $\lambda > \alpha$) there is at least one nonzero entry in the coefficient matrix. That is, there are no double eigenvalues.

To find frequency equations as has been done in [VV06] is quite a long and tedious process. However, eigenvalues λ can be searched for using the bisection method, where λ is such that the determinant of the coefficient matrix is zero.

The authors of [CST16] state that the entire spectrum, "regardless of the boundary conditions, must be constructed by taking into account that it consists of... portions, neither of which can be disregarded." They also claim to have completed the theory. However, although the "complete" (or whole) spectrum is considered, they do not mention the completeness of the sequence of eigenvectors, which is an important part of the theory. The authors of [VV06] reference [Zei95] with regards to the completeness of the sequence of eigenfunctions.

3.6 Modal analysis

Due to the nature of the eigenfunction expressions, it is convenient to conduct an analysis for the pinned-pinned rod. The analysis done can also be applied to the other cases, where the process is similar.

As a result of Proposition 3.5.1, it is necessary to distinguish between the cases $S_0 > S_{crit}$ and $S_0 \leq S_{crit}$. The case where $S_0 > S_{crit}$ is discussed in Subsection 3.6.1, while the case where $S_0 \leq S_{crit}$ is investigated in Subsection 3.6.2.

3.6.1 Approximation of solutions by partial sums

In this subsection, modal analysis is performed on the Adapted Timoshenko rod model with pinned-pinned boundary conditions and a constant axial force S_0. To be specific, in each case the formal series solution is calculated knowing from Section 2.5 that this can be justified whenever the weak solution actually exists. It is assumed that $S_0 > S_{crit}$, which results in positive eigenvalues (see Proposition 3.5.1).

The form of the formal series solution is given in Section 2.5, which was shown to be valid using convergence in energy as in [CVV18]. The sequence of eigenfunctions with corresponding eigenvalues for the pinned-pinned rod is calculated in Section 3.5.1. The formal series solution obtained is

$$\begin{bmatrix} w(x,t) \\ \phi(x,t) \end{bmatrix} = \sum_{k=1}^{\infty} \left(c_k \cos(\sqrt{\lambda_k}\ t) + d_k \sin(\sqrt{\lambda_k}\ t) \right) \begin{bmatrix} w_k(x) \\ \phi_k(x) \end{bmatrix}, \tag{3.6.1}$$

where c_k is such that

$$\begin{bmatrix} w(x,0) \\ \phi(x,0) \end{bmatrix} = \begin{bmatrix} w_0(x) \\ \phi_0(x) \end{bmatrix} = \sum_{k=1}^{\infty} c_k \begin{bmatrix} w_k(x) \\ \phi_k(x) \end{bmatrix} \tag{3.6.2}$$

and d_k is such that

$$\begin{bmatrix} \partial_t w(x,0) \\ \partial_t \phi(x,0) \end{bmatrix} = \begin{bmatrix} w_d(x) \\ \phi_d(x) \end{bmatrix} = \sum_{k=1}^{\infty} \sqrt{\lambda_k}\ d_k \begin{bmatrix} w_k(x) \\ \phi_k(x) \end{bmatrix}. \tag{3.6.3}$$

That is,

$$c_T\left(\begin{bmatrix} w_0 \\ \phi_0 \end{bmatrix}, \begin{bmatrix} w_n \\ \phi_n \end{bmatrix} \right) = c_T\left(\sum_{k=1}^{\infty} c_k \begin{bmatrix} w_k \\ \phi_k \end{bmatrix}, \begin{bmatrix} w_n \\ \phi_n \end{bmatrix} \right) \tag{3.6.4}$$

and

$$c_T\left(\begin{bmatrix} w_d \\ \phi_d \end{bmatrix}, \begin{bmatrix} w_n \\ \phi_n \end{bmatrix} \right) = c_T\left(\sum_{k=1}^{\infty} \sqrt{\lambda_k} d_k \begin{bmatrix} w_k \\ \phi_k \end{bmatrix}, \begin{bmatrix} w_n \\ \phi_n \end{bmatrix} \right). \tag{3.6.5}$$

Using partial sums it follows by the orthogonality of eigenfunctions (Theorem 2.6.4) that for each natural number n,

$$c_n \left(\|w_n\|^2 + \frac{1}{\alpha}\|\phi_n\|^2 \right) = (w_0, w_n) + \frac{1}{\alpha}(\phi_0, \phi_n) \tag{3.6.6}$$

and

$$\sqrt{\lambda_n} d_n \left(\|w_n\|^2 + \frac{1}{\alpha}\|\phi_n\|^2 \right) = (w_d, w_n) + \frac{1}{\alpha}(\phi_d, \phi_n). \tag{3.6.7}$$

Remark. *If α is an eigenvalue (that is, if there exists an integer m such that* (3.5.12) *holds), then the term*

$$\left(c_0 \cos(\sqrt{\alpha}\ t) + d_0 \sin(\sqrt{\alpha}\ t) \right) \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

is added to the series in (3.6.1)*, where $c_0 = (\phi_0, 1)$ and $d_0 = \alpha^{-\frac{1}{2}}(\phi_d, 1)$.*

3.6.2 Degenerate cases

If $S_0 \leq S_{crit}$, it is easy to see that a formal series solution for the vibration problem can be calculated, but it is unclear if the procedure can be justified. A formal series solution for Problem AT with pinned-pinned boundary conditions and $S_0 \in (-1, S_{crit}]$ is now calculated.

Example 1 Formal series solution for the case $\lambda_1 = 0$

If $\lambda_1 = 0$, then $T_1(t) = at + b$ by (2.5.4). Consequently, the formal series solution is

$$\begin{aligned} \begin{bmatrix} w(x,t) \\ \phi(x,t) \end{bmatrix} &= (at+b) \begin{bmatrix} w_1(x) \\ \phi_1(x) \end{bmatrix} \\ &\quad + \sum_{k=2}^{\infty} \left(c_k \cos(\sqrt{\lambda_k}\ t) + d_k \sin(\sqrt{\lambda_k}\ t) \right) \begin{bmatrix} w_k(x) \\ \phi_k(x) \end{bmatrix}. \end{aligned}$$

If the initial velocity is zero, it follows that $a = 0$. Therefore

$$\begin{bmatrix} w(x,t) \\ \phi(x,t) \end{bmatrix} = b \begin{bmatrix} w_1(x) \\ \phi_1(x) \end{bmatrix} + \sum_{k=2}^{\infty} \left(c_k \cos(\sqrt{\lambda_k} t) + d_k \sin(\sqrt{\lambda_k} t) \right) \begin{bmatrix} w_k(x) \\ \phi_k(x) \end{bmatrix}.$$

Example 2 Formal series solution for the case $\lambda_1 < 0$

If $\lambda_1 < 0$ and $\lambda_k > 0$ for $k > 1$, then $T_1(t) = a_1 e^{\sqrt{-\lambda_1}t} + a_2 e^{-\sqrt{-\lambda_1}t}$ by (2.5.4). Hence, the formal series solution is

$$\begin{aligned} \begin{bmatrix} w(x,t) \\ \phi(x,t) \end{bmatrix} &= \left(a_1 e^{\sqrt{-\lambda_1}t} + a_2 e^{-\sqrt{-\lambda_1}t} \right) \begin{bmatrix} w_1(x) \\ \phi_1(x) \end{bmatrix} \\ &\quad + \sum_{k=2}^{\infty} \left(c_k \cos(\sqrt{\lambda_k} t) + d_k \sin(\sqrt{\lambda_k} t) \right) \begin{bmatrix} w_k(x) \\ \phi_k(x) \end{bmatrix}. \end{aligned}$$

If the initial velocity is zero, it follows that $a_1 = a_2$. Therefore

$$\begin{aligned} \begin{bmatrix} w(x,t) \\ \phi(x,t) \end{bmatrix} &= a_1 \left(e^{\sqrt{-\lambda_1}t} + e^{-\sqrt{-\lambda_1}t} \right) \begin{bmatrix} w_1(x) \\ \phi_1(x) \end{bmatrix} \\ &\quad + \sum_{k=2}^{\infty} \left(c_k \cos(\sqrt{\lambda_k} t) + d_k \sin(\sqrt{\lambda_k} t) \right) \begin{bmatrix} w_k(x) \\ \phi_k(x) \end{bmatrix}. \end{aligned}$$

Furthermore, the energy becomes unbounded which does not describe oscillations and the physical application is therefore questionable.

The examples above do not exhaust all the possibilities of degenerate cases. For example, more than one negative eigenvalue may exist.

Chapter 4

Application of the finite element method to the linear models

Solutions of the linear problems may be calculated using partial sums if a rod is prismatic. Otherwise, the eigenvalues and eigenfunctions must be calculated numerically. This can be done using the finite element method. Only the case of pinned-pinned boundary conditions is considered here, since the cases of other boundary conditions differ mainly in the definitions of the space $\mathcal{V}$.

The main source for the convergence theory of FEM applied to the dynamic problem is [BV13]. The idea in this chapter is to show the application of the theory of [BV13] to Problem AT. A detailed study of the proofs in the article is beyond the scope of this dissertation.

4.1 Variational forms

Recall the weak variational form of Model AT, Problem ATW, from Chapter 3.

Problem ATW

Given $u_0 \in \mathcal{V}$, $u_d \in \mathcal{W}$, and $f \in C([0,\tau);\mathcal{L}^2(0,1)^2)$ find $u \in C^2((0,\tau);\mathcal{W})$

such that for each $t > 0$, $u(t) \in \mathcal{V}$, $u'(t) \in \mathcal{V}$ and, for each $v \in \mathcal{V}$,

$$\begin{aligned} c_T\left(u''(t), v\right) + b_{AT}(u(t), v) &= (f(t), v)^{(2)}, \\ u(0) = u_0, \quad Du(0) &= u_d. \end{aligned}$$

The derivation of the mixed variational form is done in the same way as the standard variational form (Problem ATV). The only difference is that the constitutive equation for V is not substituted in.

Mixed Variational Form Given the load P, find $\langle w, \phi\rangle$ and V such that for each $t > 0$, $\langle w(\cdot,t), \phi(\cdot,t)\rangle \in T_P$, $V(\cdot,t) \in C[0,1]$ and for each $\langle v_1, v_2\rangle \in T_P$ and each $g \in C[0,1]$,

$$\begin{aligned} \left(\partial_t^2 w(\cdot,t), v_1\right) &= -\left(S\partial_x w(\cdot,t), v_1'\right) - \left(V(\cdot,t), v_1'\right) + \left(P(\cdot,t), v_1\right), \\ \left(\partial_t^2 \phi(\cdot,t), v_2\right) &= \alpha\left(V(\cdot,t), v_2\right) - \frac{1}{\gamma}\left(\partial_x \phi(\cdot,t), v_2'\right), \\ \left(V(\cdot,t), g\right) &= \left(\partial_x w(\cdot,t) - \phi(\cdot,t), g\right), \end{aligned}$$

while

$$\langle w(\cdot,0), \phi(\cdot,0)\rangle = \langle w_0, \phi_0\rangle \quad \text{and} \quad \langle \partial_t w(\cdot,0), \partial_t \phi(\cdot,0)\rangle = \langle w_d, \phi_d\rangle.$$

In order to apply the convergence theory to the mixed finite element method, the weak mixed variational form is considered.

Weak Mixed Variational Form Given the load P as well as $u_0 \in \mathcal{V}$ and $u_d \in \mathcal{W}$, find $u = \langle u_1, u_2\rangle \in C^2((0,\tau); \mathcal{W})$ and $V \in C([0,\tau); \mathcal{L}^2(0,1))$ such that for each $t > 0$, $u(t) \in \mathcal{V}$, $u'(t) \in \mathcal{V}$ and, for each $\langle v_1, v_2\rangle \in \mathcal{V}$ and $g \in \mathcal{L}^2(0,1)$,

$$\begin{aligned} \left(u_1''(t), v_1\right) &= -\left(SDu_1(t), Dv_1\right) - \left(V(t), Dv_1\right) + \left(P(t), v_1\right), \\ \frac{1}{\alpha}\left(u_2''(t), v_2\right) &= \left(V(t), v_2\right) - \frac{1}{\beta}\left(Du_2(t), Dv_2\right), \\ \left(V(t), g\right) &= \left(Du_1(t) - u_2(t), g\right), \end{aligned}$$

while

$$u(0) = u_0, \qquad Du(0) = u_d.$$

If S is not constant, numerical integration is required. For the remainder of this chapter, S is assumed to be constant ($S = S_0$), which is true for a pre-stressed rod.

Proposition 4.1.1. *The functions u and V are a solution of the weak mixed variational form if and only if u is a solution of Problem ATW.*

Proof. Let $\langle u_1, u_2\rangle$ and V be a solution of the weak mixed variational form. Then for each $t > 0$, $u(t) \in \mathcal{V}$, $u'(t) \in \mathcal{V}$ and, for each $\langle v_1, v_2\rangle \in \mathcal{V}$ and each $g \in \mathcal{L}^2(0,1)$,

$$\begin{aligned}
(u_1''(t), v_1) &= -S_0\,(Du_1(t), Dv_1) - (V(t), Dv_1) + (P(t), v_1)\,,\\
\frac{1}{\alpha}\,(u_2''(t), v_2) &= (V(t), v_2) - \frac{1}{\beta}\,(Du_2(t), Dv_2)\,,\\
(V(t), g) &= (Du_1(t) - u_2(t), g)\,.
\end{aligned}$$

Since $Dv_1 \in \mathcal{L}^2(0,1)$ and $v_2 \in \mathcal{L}^2(0,1)$, it follows that for each $\langle v_1, v_2\rangle \in \mathcal{V}$,

$$\begin{aligned}
(u_1''(t), v_1) &= -(1+S_0)\,(Du_1(t), Dv_1) + (u_2(t), Dv_1) + (P(t), v_1)\,,\\
\frac{1}{\alpha}\,(u_2''(t), v_2) &= (Du_1(t) - u_2(t), v_2) - \frac{1}{\beta}\,(Du_2(t), Dv_2)\,.
\end{aligned}$$

This is true if, for $f(t) = \langle P(t), 0\rangle$ and, for each pair $\langle v_1, v_2\rangle \in \mathcal{V}$,

$$c_T\,(u''(t), v) + b_{AT}(u(t), v) = (f(t), v)^{(2)}.$$

That is, u is a solution of Problem ATW.

Now suppose u is a solution of Problem ATW. Then for each $t \geq 0$, $u(t) \in \mathcal{V}$, $u'(t) \in \mathcal{V}$ and, for each $v \in \mathcal{V}$,

$$c_T\,(u''(t), v) + b_{AT}(u(t), v) = (f(t), v)^{(2)}. \tag{4.1.1}$$

This is true if $f(t) = \langle P(t), 0\rangle$ and, for each $\langle v_1, v_2\rangle \in \mathcal{V}$,

$$\begin{aligned}
(u_1''(t), v_1) &= -(1+S_0)\,(Du_1(t), Dv_1) + (u_2(t), Dv_1) + (P(t), v_1)\,,\\
\frac{1}{\alpha}\,(u_2''(t), v_2) &= (Du_1(t) - u_2(t), v_2) - \frac{1}{\beta}\,(Du_2(t), Dv_2)\,.
\end{aligned}$$

That is, for each $\langle v_1, v_2\rangle \in \mathcal{V}$ and $g \in H^1(0,1)$,

$$\begin{aligned}
(u_1''(t), v_1) &= -S_0\,(Du_1(t), Dv_1) - (V(t), Dv_1) + (P(t), v_1)\,,\\
\frac{1}{\alpha}\,(u_2''(t), v_2) &= (V(t), v_2) - \frac{1}{\beta}\,(Du_2(t), Dv_2)\,,\\
(V(t), g) &= (Du_1(t) - u_2(t), g)\,.
\end{aligned}$$

Fix an arbitrary $g \in \mathcal{L}^2(0,1)$. Then, since $H^1(0,1)$ is dense in $\mathcal{L}^2(0,1)$, it follows that there exists a sequence $\{g_n\}$ contained in $H^1(0,1)$ such that $g_n \to g$ as $n \to \infty$. Therefore, since for any $n \in \mathbb{N}$,

$$(V(t), g_n) = (Du_1(t) - u_2(t), g_n)\,, \tag{4.1.2}$$

by the uniqueness of a limit

$$(V(t), g) = (Du_1(t) - u_2(t), g) . \tag{4.1.3}$$

Since $g \in \mathcal{L}^2(0,1)$ was arbitrary, it follows that for each $\langle v_1, v_2 \rangle \in \mathcal{V}$ and $g \in \mathcal{L}^2(0,1)$,

$$\begin{aligned}
(u_1''(t), v_1) &= -S_0 (Du_1(t), Dv_1) - (V(t), Dv_1) + (P(t), v_1) , \\
\frac{1}{\alpha} (u_2''(t), v_2) &= (V(t), v_2) - \frac{1}{\beta} (Du_2(t), Dv_2) , \\
(V(t), g) &= (Du_1(t) - u_2(t), g) .
\end{aligned}$$

Therefore $\langle u_1, u_2 \rangle$ and V are a solution of the weak mixed variational form.

Since the initial conditions are the same, they are also satisfied. □

4.2 Dynamic problem

Suppose V_S^h and V_M^h are finite dimensional subspaces of T_P and Y^h is a finite dimensional subspace of $C[0,1]$.

Standard Galerkin Approximation

Find $\left\langle w^h, \phi^h \right\rangle$ such that, for each $t > 0$, $\left\langle w^h(\cdot,t), \phi^h(\cdot,t) \right\rangle \in V_S^h$, and

$$\begin{aligned}
\left(\partial_t^2 w^h(\cdot,t), v_1\right) &= -(1+S_0) \left(\partial_x w^h(\cdot,t), v_1'\right) + \left(\phi^h(\cdot,t), v_1'\right) + (P(\cdot,t), v_1) , \\
\left(\partial_t^2 \phi^h(\cdot,t), v_2\right) &= \alpha \left(\partial_x w^h(\cdot,t) - \phi^h(\cdot,t), v_2\right) - \frac{1}{\gamma} \left(\partial_x \phi^h(\cdot,t), v_2'\right)
\end{aligned}$$

hold for each $\langle v_1, v_2 \rangle \in V_S^h$, while

$$\langle w^h(\cdot,0), \phi^h(\cdot,0) \rangle = \langle w_0^h, \phi_0^h \rangle \quad \text{and} \quad \langle \partial_t w^h(\cdot,0), \partial_t \phi^h(\cdot,0) \rangle = \langle w_d^h, \phi_d^h \rangle .$$

The choice of the finite dimensional subspace V_S^h must now be made. If piecewise linear basis functions are chosen, then although the standard Galerkin approximation above converges in theory, in practice locking occurs. (The term "locking" is used to describe a situation where the discretisation error fails to decrease at the theoretically predicted rate.) One possibility to avoid this, if it is known that the exact solution is sufficiently regular, is to use Hermite piecewise cubic basis functions.

Another alternative is to use piecewise linear basis functions and apply the mixed finite element method.

In order to obtain the mixed Galerkin approximation, some notation is required. Divide the interval $[0, 1]$ into n elements of equal length. Continuous piecewise linear basis functions are used (also known as C^0 piecewise linear basis functions), notated as $\delta_0, \delta_1, \ldots, \delta_n$. Let S^h be the span of the set $\{\delta_0, \ldots, \delta_n\}$ and S_0^h be the span of the set $\{\delta_1, \ldots, \delta_{n-1}\}$. The choices $V_M^h = S_0^h \times S^h$ and $Y^h = S^h$ are made.

Define the following matrices

$$\begin{aligned} K_{ij} &= (\delta_j', \delta_i'), \\ M_{ij} &= (\delta_j, \delta_i), \\ L_{ij} &= (\delta_j, \delta_i'). \end{aligned}$$

More detail on the definition of the basis functions and matrices can be found in Appendix B. The following sub-matrices are also defined. If X is a square matrix, then

- $X_{\{a,b\}}$ is matrix X with rows a and b and columns a and b deleted;
- $X_{R\{a,b\}}$ is matrix X with rows a and b deleted.

Definition. Let x_j denote the value of x at node j. Then

$$\begin{aligned} w_0^h(x) &= \sum_{j=1}^{n-1} w_0(x_j)\delta_j(x), \qquad \phi_0^h(x) = \sum_{j=0}^{n} \phi_0(x_j)\delta_j(x), \\ w_d^h(x) &= \sum_{j=1}^{n-1} w_d(x_j)\delta_j(x), \qquad \phi_d^h(x) = \sum_{j=0}^{n} \phi_d(x_j)\delta_j(x). \end{aligned}$$

The mixed Galerkin approximation is therefore given below.

Mixed Galerkin approximation

Find $\langle w^h, \phi^h \rangle$ and V^h such that, for each $t > 0$, $\langle w^h(\cdot, t), \phi^h(\cdot, t) \rangle \in S_0^h, \times S^h$, $V^h(\cdot, t) \in S^h$ and

$$\begin{aligned} \left(\partial_t^2 w^h(\cdot, t), v_1\right) &= -S_0 \left(\partial_x w^h(\cdot, t), v_1'\right) - \left(V^h(\cdot, t), v_1'\right) + \left(P(\cdot, t), v_1\right), \\ \left(\partial_t^2 \phi^h(\cdot, t), v_2\right) &= \alpha \left(V^h(\cdot, t), v_2\right) - \frac{1}{\gamma} \left(\partial_x \phi^h(\cdot, t), v_2'\right), \\ \left(V^h(\cdot, t), g\right) &= \left(\partial_x w^h(\cdot, t) - \phi^h(\cdot, t), g\right) \end{aligned}$$

hold for each $\langle v_1, v_2\rangle \in S_0^h \times S^h$ and $g \in S^h$, while

$$\langle w^h(\cdot,0), \phi^h(\cdot,0)\rangle = \langle w_0^h, \phi_0^h\rangle \quad \text{and} \quad \langle \partial_t w^h(\cdot,0), \partial_t \phi^h(\cdot,0)\rangle = \langle w_d^h, \phi_d^h\rangle.$$

Since $w^h(\cdot,t) \in S_0^h$, $\phi^h(\cdot,t) \in S^h$ and $V^h(\cdot,t) \in S^h$ for each $t > 0$, it follows that

$$\begin{aligned} w^h(x,t) &= \sum_{j=1}^{n-1} w_j(t)\delta_j(x), && (4.2.1)\\ \phi^h(x,t) &= \sum_{j=0}^{n} \phi_j(t)\delta_j(x), && (4.2.2)\\ d^h(x,t) &= \sum_{j=0}^{n} d_j(t)\delta_j(x), && (4.2.3)\\ V^h(x,t) &= \sum_{j=0}^{n} \left(d_j(t) - \phi_j(t)\right)\delta_j(x), && (4.2.4) \end{aligned}$$

where $(d^h, g) = (\partial_x w^h, g)$ for $g \in C[0,1]$. Note that there exists a one-to-one correspondence between S^h and $\mathbb{R}^{n+1}$. Similarly, there exists a one-to-one correspondence between S_0^h and $\mathbb{R}^{n-1}$.

Definition. For $u \in S^h$, $v \in S_0^h$, let $\bar{\pi}u = \bar{u}$ if $u = \sum_{i=0}^{n} u_i\delta_i$ and $\bar{\pi}v = \bar{v}$ if $v = \sum_{i=1}^{n-1} v_i\delta_i$.

Note that $\bar{\pi}$ is clearly one-to-one. Take $V^h(\cdot,t)$ as an example. Then, since $V^h(\cdot,t) \in S^h$,

$$\bar{\pi}V^h(\cdot,t) = \overline{V^h}(t).$$

These one-to-one correspondences may also be applied to pairs $[w,\phi]^T$.

Using Equations (4.2.1) - (4.2.4), the Galerkin approximation can be rewritten: Find $\bar{w}$, $\bar{\phi}$ and $\bar{d}$ such that, for $i = 1,\dots,n-1$; $k = 0,\dots,n$ and $m = 0,\dots,n$,

$$\begin{aligned} \sum_{j=1}^{n-1} w_j''(\delta_j,\delta_i) &= -S_0 \sum_{j=1}^{n-1} w_j(\delta_j',\delta_i') - \sum_{j=0}^{n} V_j(\delta_j,\delta_i') && (4.2.5)\\ \sum_{j=0}^{n} \phi_j''(\delta_j,\delta_k) &= \alpha \sum_{j=0}^{n} V_j(\delta_j,\delta_k) - \frac{1}{\gamma}\sum_{j=0}^{n} \phi_j\left(\delta_j',\delta_k'\right), && (4.2.6)\\ \sum_{j=0}^{n} d_j(\delta_j,\delta_m) &= \sum_{j=1}^{n-1} w_j(\delta_j',\delta_m), && (4.2.7)\\ \bar{V}(t) &= \bar{d}(t) - \bar{\phi}(t), && (4.2.8) \end{aligned}$$

with

$$\bar{w}(0) = \bar{\pi} w_0^h, \quad \bar{\phi}(0) = \bar{\pi}\phi_0^h, \quad \bar{w}'(0) = \bar{\pi} w_d^h, \quad \bar{\phi}'(0) = \bar{\pi}\phi_d^h.$$

That is, find $\begin{bmatrix} \bar{w} \\ \bar{\phi} \end{bmatrix}$ and $\bar{d}$ such that

$$\begin{bmatrix} M_{\{0,n\}} & [0] \\ [0] & M \end{bmatrix} \begin{bmatrix} \bar{w} \\ \bar{\phi} \end{bmatrix}'' = \begin{bmatrix} -L_{R\{0,n\}} \\ \alpha M \end{bmatrix} \bar{V} + \begin{bmatrix} -S_0 K_{\{0,n\}} & [0] \\ [0] & -\frac{1}{\gamma} K \end{bmatrix} \begin{bmatrix} \bar{w} \\ \bar{\phi} \end{bmatrix},$$

$$M\bar{d} = \begin{bmatrix} \left(L_{R\{0,n\}}\right)^T & [0] \end{bmatrix} \begin{bmatrix} \bar{w} \\ \bar{\phi} \end{bmatrix} \quad \text{and} \quad \bar{V} = \bar{d} - \bar{\phi}$$

with

$$\begin{bmatrix} \bar{w}(0) \\ \bar{\phi}(0) \end{bmatrix} = \begin{bmatrix} \bar{\pi} w_0^h \\ \bar{\pi}\phi_0^h \end{bmatrix}, \qquad \begin{bmatrix} \bar{w}'(0) \\ \bar{\phi}'(0) \end{bmatrix} = \begin{bmatrix} \bar{\pi} w_d^h \\ \bar{\pi}\phi_d^h \end{bmatrix}.$$

Remark. *For rods with cantilever or clamped-clamped boundary conditions, the Galerkin approximation is the same except for the obvious modifications to the matrices M, K and L.*

Using central differences, the following algorithm is obtained to find approximations for $\begin{bmatrix} \bar{w} \\ \bar{\phi} \end{bmatrix}$ and $\bar{d}$.

Algorithm
For each time step k,

$$\begin{aligned} \begin{bmatrix} M_{\{0,n\}} & [0] \\ [0] & M \end{bmatrix} \begin{bmatrix} \bar{w}_{k+1} \\ \bar{\phi}_{k+1} \end{bmatrix} &= (\delta t)^2 \left(\begin{bmatrix} -L_{R\{0,n\}} \\ \alpha M \end{bmatrix} (\bar{d}_k - \bar{\phi}_k) \right. \\ &\quad \left. + \begin{bmatrix} -S_0 K_{\{0,n\}} & [0] \\ [0] & -\frac{1}{\gamma} K \end{bmatrix} \begin{bmatrix} \bar{w}_k \\ \bar{\phi}_k \end{bmatrix} \right) \\ &\quad + \begin{bmatrix} M_{\{0,n\}} & [0] \\ [0] & M \end{bmatrix} \left(2 \begin{bmatrix} \bar{w}_k \\ \bar{\phi}_k \end{bmatrix} - \begin{bmatrix} \bar{w}_{k-1} \\ \bar{\phi}_{k-1} \end{bmatrix} \right), \end{aligned}$$

where

$$M\bar{d}_k = \begin{bmatrix} \left(L_{R\{0,n\}}\right)^T & \bar{0} \end{bmatrix} \begin{bmatrix} \bar{w}_k \\ \bar{\phi}_k \end{bmatrix}.$$

To prepare for the initial time step,

$$\begin{bmatrix} \bar{w}_0 \\ \bar{\phi}_0 \end{bmatrix} = \begin{bmatrix} \bar{\pi} w_0^h \\ \bar{\pi}\phi_0^h \end{bmatrix} \quad \text{and} \quad \frac{1}{2\delta t} \left(\begin{bmatrix} \bar{w}_1 \\ \bar{\phi}_1 \end{bmatrix} - \begin{bmatrix} \bar{w}_{-1} \\ \bar{\phi}_{-1} \end{bmatrix} \right) = \begin{bmatrix} \bar{\pi} w_d^h \\ \bar{\pi}\phi_d^h \end{bmatrix}.$$

The algorithm for the standard Galerkin approximation can be found by substituting (4.2.8) into (4.2.5) and (4.2.6). It is not discussed in this dissertation as it can be found in many textbooks.

Provided that the initial condition u_0 is in E_b, the standard finite element method can be used effectively with Hermite cubic basis functions and yields extremely accurate numerical results.

4.3 Convergence

Convergence for the dynamic problem is studied in this section. The theory presented in [BV13] is applied to the standard finite element method and other sources are quoted for the mixed finite element method.

4.3.1 General results

Recall the general spaces $\mathcal{V}, \mathcal{W}$ and $\mathcal{X}$ and Problem GVar defined in Chapter 2. The standard FEM results in [BV13] are written for a general problem such as Problem GVar. Let V^h be a finite dimensional subspace of $\mathcal{V}$. Note that h is a parameter related to the dimension m of V^h and $h \to 0$ as $m \to \infty$. Let $t_n = n(\delta t)$ and denote the approximation of $u^h(t_n)$ by u_n^h.

Assumption C1 The solution $u \in C(J, \mathcal{V})$ of Problem GVar is such that $(Pu) \in C^2(J)$, where P is the projection operator defined by

$$b(u - Pu, v) = 0 \text{ for each } v \in V^h.$$

Assumption C2 There exists a subspace $\mathcal{G}$ of $\mathcal{V}$ and a positive integer η such that if $u \in \mathcal{G}$, then for some positive real number G

$$\inf_{v \in V^h} \|u - v\|_{\mathcal{V}} \leq G h^\eta \|u\|_{\mathcal{G}},$$

where $\|\cdot\|_{\mathcal{G}}$ is a norm or semi-norm for $\mathcal{G}$.

Definition. Let Π denote the interpolation operator for the relevant space. Define $\Pi^{(2)}$ as follows. For $f \in \mathcal{X}$,

$$\Pi^{(2)} f = \langle \Pi f_1, \Pi f_2 \rangle .$$

Theorem 4.3.1. *Suppose Assumption C2 holds,* $u_0^h = \Pi^{(2)}u_0$ *and* $u_1^h = \Pi^{(2)}u_1$*. If the solution* u *of Problem GVar satisfies Assumption C1,* $u(t) \in \mathcal{G}$ *and* $u'(t) \in \mathcal{G}$*, then*

$$\begin{aligned}\|u(t) - u^h(t)\|_{\mathcal{W}} \;\leq\;& C_{\mathcal{V}}Gh^{\eta}\|u(t)\|_{\mathcal{G}} + \sqrt{2}C_{\mathcal{V}}Gh^{\eta}(3T\max\|u'(t)\|_{\mathcal{G}} \\ &+3C_aT\max\|u(t)\|_{\mathcal{G}} + (2+3C_aT)\|u_0\|_{\mathcal{G}} \\ &+3T\|u_1\|_{\mathcal{G}}) \qquad (4.3.1)\end{aligned}$$

for each $t \in [0,T]$*, where* $C_{\mathcal{V}}$ *and* C_a *are from assumptions A3 and A4W respectively.*

Proof. See [BV13] Theorem 5.2. □

Theorem 4.3.2. *If* $f \in C^2([0,T],\mathcal{X})$ *then*

$$\begin{aligned}\|u^h(t_n) - u_n^h\|_{\mathcal{W}} \;\leq\;& 7T^2\delta t^2\max\|(u^h)^{(4)}\|_{\mathcal{W}} + 7T\delta t\max\|(u^h)'''\|_{\mathcal{W}} \\ &+\sqrt{2C_a}\delta t^4\max\|(u^h)'''\|_{\mathcal{W}} \qquad (4.3.2)\end{aligned}$$

for each $t \in (0,T)$*, where* C_a *is from assumption A4W.*

Proof. See [BV13] Theorem 6.1. □

The following theorem follows from Theorems 4.3.1 and 4.3.2 above.

Theorem 4.3.3 (Error estimate)**.**
Suppose the solution u *of Problem GVar satisfies Assumption C1 and Assumption C2 holds. Also, suppose* $u_0^h = \Pi^{(2)}u_0$*,* $u_d^h = \Pi^{(2)}u_d$*,* $u(t) \in \mathcal{G}$ *and* $u'(t) \in \mathcal{G}$*. If* $f \in C^2([0,T],\mathcal{X}^2)$ *then*

$$\begin{aligned}\|u(t_n) - u_n^h\|_{\mathcal{W}} \;\leq\;& \|u(t_n) - u^h(t_n)\|_{\mathcal{W}} + \|u^h(t_n) - u_n^h\|_{\mathcal{W}} \\ \leq\;& C_{\mathcal{V}}Gh^{\eta}\|u(t_n)\|_{\mathcal{G}} + \sqrt{2}C_{\mathcal{V}}Gh^{\eta}(3T\max\|u'(t_n)\|_{\mathcal{G}} \\ &+3C_aT\max\|u(t_n)\|_{\mathcal{G}} + (2+3C_aT)\|u_0\|_{\mathcal{G}} + 3T\|u_1\|_{\mathcal{G}}) \\ &+7T^2\delta t^2\max\|(u^h)^{(4)}\|_{\mathcal{W}} + 7T\delta t\max\|(u^h)'''\|_{\mathcal{W}} \\ &+\sqrt{2C_a}\delta t^4\max\|(u^h)'''\|_{\mathcal{W}} \qquad (4.3.3)\end{aligned}$$

for each $t \in (0,T)$*.*

4.3.2 Application to Problem AT

In order to apply the general theory, the space V^h and operator P must be identified and Assumptions C1 and C2 shown to hold.

Using piecewise linear basis functions (as in Section 4.2), let $V^h = S_0^h \times S^h$ and P be such that

$$b_{AT}(u - Pu, v) = 0 \text{ for each } v \in S_0^h \times S^h.$$

Then the assumptions become

C1 The solution $u \in C(J, \mathcal{V})$ of Problem ATW is such that $(Pu) \in C^2(J)$.
C2 There exists a subspace $\mathcal{G}$ of $\mathcal{V}$ and a positive integer η such that if $u \in \mathcal{G}$, then for some positive real number G

$$\inf_{v \in S_0^h \times S^h} \|u - v\|_{\mathcal{V}_\mathcal{A}} \leq G h^\eta \|u\|_\mathcal{G},$$

where $\|\cdot\|_\mathcal{G}$ is a norm or semi-norm for $\mathcal{G}$.

Theorem 4.3.4. *Using piecewise linear basis functions, the standard FEM converges for Problem ATW if* $f \in C^2([0,T], \mathcal{L}^2(0,1)^2)$.

Proof. By Theorem 3.2.1 Problem ATW has a unique solution

$$u \in C^1([0,\infty), \mathcal{V}) \cap C^2([0,\infty), \mathcal{W}). \tag{4.3.4}$$

Therefore $u \in C^2([0,\infty), \mathcal{V})$ since $\mathcal{V} \subset \mathcal{W}$. Hence, as in the remark in Section 3 of [BV13], it can be proved that $P(u(\cdot,t)) \in C^2([0,\infty))$ for each $t > 0$. That is, Assumption C1 holds.

Let $\mathcal{G} = H^2(0,1)^2 \cap \mathcal{V}$ and $\|\cdot\|_{H^2}$ denote the natural norm for $H^2(0,1)^2$. Since $u \in C^2([0,\infty), \mathcal{V})$, it follows that, for $i = 1, 2$, (see [SF73])

$$\|u_i - \Pi u_i\| \leq h^2 \|\partial_x^2 u_i\| \quad \text{and} \quad \|\partial_x u_i - \partial_x(\Pi u_i)\| \leq h^2 \|\partial_x^2 u_i\|.$$

Therefore, by the equivalence of the norms $\|\cdot\|_{\mathcal{V}_\mathcal{A}}$ and $\|\cdot\|_{H^1}$ (Proposition 3.2.2),

$$\begin{aligned}
\|u - \Pi u\|_{\mathcal{V}_\mathcal{A}}^2 &\leq K^2 \left(\|u_1 - \Pi u_1\|^2 + \|\partial_x u_1 - \partial_x(\Pi u_1)\|^2 \right. \\
&\quad \left. + \|u_2 - \Pi u_2\|^2 + \|\partial_x u_2 - \partial_x(\Pi u_2)\|^2 \right) \\
&\leq 2K^2 h^2 \left(\|\partial_x^2 u_1\|^2 + \|\partial_x^2 u_2\|^2 \right)
\end{aligned}$$

That is

$$\|u - \Pi u\|^2_{\mathcal{V}_\mathcal{A}} \leq K\sqrt{2}h\|u\|_{H^2}. \tag{4.3.5}$$

Therefore, for $G = K\sqrt{2}$,

$$\inf_{v \in V^h} \|u - v\|^2_{\mathcal{V}_\mathcal{A}} \leq Gh\|u\|_{H^2}. \tag{4.3.6}$$

Hence Assumption C2 holds and Theorem 4.3.3 may be used to find the error estimate for the problem. Since $m \to \infty$ implies that $h \to 0$, the FEM solution converges for Problem ATW. □

Now consider using piecewise Hermite cubic basis functions. The spaces V^h and P are defined in a similar fashion and assumptions C1 and C2 are the same as in the case where piecewise linear basis functions are used.

Theorem 4.3.5. *Using piecewise Hermite cubic basis functions, the standard FEM converges for Problem ATW if $f \in C^2([0,T], \mathcal{L}^2(0,1)^2)$.*

Proof. The proofs that $u \in C^2([0,\infty), \mathcal{V})$ and Assumption C1 holds are similar to those for piecewise linear basis functions.

Since $u \in C^2([0,\infty), \mathcal{V})$, there exists a constant C such that for $i = 1, 2$, (see [SF73])

$$\||\partial_x^m u_i - \partial_x^m(\Pi u_i)\| \leq Ch^{2-m}\|\partial_x^2 u_i\| \quad \text{for} \quad m = 0, 1, 2. \tag{4.3.7}$$

Therefore, following similar arguments to the proof of Theorem 4.3.4, there exists a positive real number G such that

$$\inf_{v \in V^h} \|u - v\|^2_{\mathcal{V}_\mathcal{A}} \leq Gh\|u\|_{H^2}. \tag{4.3.8}$$

That is, Assumption C2 holds and hence Theorem 4.3.3 may be used to find the error estimate for the problem. Since $m \to \infty$ implies that $h \to 0$, the FEM solution converges for Problem ATW. □

Convergence of the mixed finite element method

The convergence of the mixed finite element method will not be discussed here as it is beyond the scope of this dissertation. The interested reader may consult [Sem94] and [LMR16] for more information.

4.4 Numerical experiments

If a rod is not prismatic, then the eigenvalues and eigenfunctions must be calculated numerically. A comparison of numerical and exact results is therefore done to determine the accuracy of the numerical algorithm. In this section the values $\gamma = \dfrac{\beta}{\alpha} = 0.25$ and $\alpha = 1200$ were used throughout.

Eigenvalues and eigenfunctions

Using the system of Equations (3.5.16)-(3.5.17) in Chapter 3, the first five terms of the sequence of eigenvalues are given in Table 4.1, where the influence of the value of S_0 near 0 is investigated. These values are accurate as they are calculated directly. Note that S_0 can be positive or negative. The eigenvalues where $S_0 = -10^{-5}$ and $S_0 = 10^{-5}$ do not differ significantly from those where $S_0 = 0$. They are therefore not included in the table below.

Table 4.1: The first five eigenvalues corresponding to values of S_0 near 0.

E.value	$S_0 = -10^{-1}$	$S_0 = -3.185 \times 10^{-2}$	$S_0 = -10^{-2}$	$S_0 = -10^{-3}$
λ_1	-0.6675	8.14×10^{-6}	0.2140	0.3022
λ_2	0.6270	3.250	4.091	4.437
λ_3	10.92	16.72	18.58	19.34
λ_4	36.56	46.72	49.98	51.32
λ_5	81.44	97.20	102.2	104.3

E.value	$S_0 = -10^{-4}$	$S_0 = 0$	$S_0 = 10^{-4}$	$S_0 = 10^{-3}$	$S_0 = 10^{-2}$
λ_1	0.3110	0.3119	0.3129	0.3217	0.4099
λ_2	4.472	4.476	4.480	4.514	4.861
λ_3	19.42	19.43	19.43	19.51	20.28
λ_4	51.45	51.47	51.48	51.62	52.96
λ_5	104.5	104.6	104.6	104.8	106.9

For the case where $S_0 = -10^{-2}$, the eigenvalues differ significantly from the case where $S_0 = -10^{-3}$. This can be interpreted as the case where the rod may buckle if the compressive force S is increased further. For the case where $S_0 = 10^{-2}$, the eigenvalues also differ significantly from the case where $S_0 = 10^{-3}$. This can be interpreted as the case where the rod stiffens significantly as a result of the tensile force stretching it.

Dynamic problem

In this section the numerical results of Model AT, with $S = S_0$, are calculated after half of a period of the solution and compared to the exact results. As this comparison is for illustrative purposes, for simplicity the initial conditions used are chosen to be the first mode. That is,

$$w_0(x) = \sin(\pi x), \quad \phi_0(x) = \pi\cos(\pi x), \quad \text{and} \quad w_d(x) = \phi_d(x) = 0. \tag{4.4.1}$$

The formal series solution of the model (see Section 3.5.1) is given by

$$\begin{bmatrix} w(x,t) \\ \phi(x,t) \end{bmatrix} = \cos(\sqrt{\lambda_1}\; t) \begin{bmatrix} \sin(\pi x) \\ A_1\cos(\pi x) \end{bmatrix}. \tag{4.4.2}$$

Also, substituting in the initial conditions, the algorithm to find the numerical approximation of the solution becomes

Algorithm
For each time step k,

$$\begin{aligned} \begin{bmatrix} M_{\{0,n\}} & [0] \\ [0] & M \end{bmatrix} \begin{bmatrix} \bar{w}_{k+1} \\ \bar{\phi}_{k+1} \end{bmatrix} &= (\delta t)^2 \begin{bmatrix} -L_{R\{0,n\}} \\ \alpha M \end{bmatrix} (\bar{d}_k - \bar{\phi}_k) \\ &\quad + (\delta t)^2 \begin{bmatrix} -S_0 K_{\{0,n\}} & [0] \\ [0] & -\frac{1}{\gamma}K \end{bmatrix} \begin{bmatrix} \bar{w}_k \\ \bar{\phi}_k \end{bmatrix} \\ &\quad + \begin{bmatrix} M_{\{0,n\}} & [0] \\ [0] & M \end{bmatrix} \left(2\begin{bmatrix} \bar{w}_k \\ \bar{\phi}_k \end{bmatrix} - \begin{bmatrix} \bar{w}_{k-1} \\ \bar{\phi}_{k-1} \end{bmatrix} \right), \end{aligned}$$

where

$$M\bar{d}_k = \begin{bmatrix} \left(L_{R\{0,n\}}\right)^T & \bar{0} \end{bmatrix} \begin{bmatrix} \bar{w}_k \\ \bar{\phi}_k \end{bmatrix}.$$

To prepare for the initial step, $\begin{bmatrix} \bar{w}_0 \\ \bar{\phi}_0 \end{bmatrix} = \begin{bmatrix} \bar{\pi} w_0^h \\ \bar{\pi}\phi_0^h \end{bmatrix}$ and $\begin{bmatrix} \bar{w}_{-1} \\ \bar{\phi}_{-1} \end{bmatrix} = \begin{bmatrix} \bar{w}_1 \\ \bar{\phi}_1 \end{bmatrix}$.

The following tables show, using the given initial conditions (4.4.1) and different values for the force S_0, the numerical (mixed finite element method) and exact (Equation (4.4.2)) results obtained for the deflection and angle of rotation after half of a period of the solution. The first table also includes, for reference, the values of the eigenvalue λ_1 and the value of the dimensionless time t. Since the greatest deflection occurs at the centre of the rod for this set of initial conditions, results for the deflection at $x = \frac{1}{2}$ will be compared and because the greatest deviation occurs at the two endpoints ($x = 0$ and $x = 1$) for this set of initial conditions, results for the deviation at $x = 1$ will be compared.

It should be noted that $S_0 = -10^{-1}$ results in a negative eigenvalue λ_1 and was therefore not considered.

Table 4.2: Deflection at $x = 0.5$ after Half a Period

S_0	λ_1	t	Numer. Result	Exact Result	Relative Error
-10^{-2}	0.2140	6.7910	-1.0004	-1	4.0×10^{-4}
-10^{-3}	0.3022	5.7152	-1.0004	-1	4.0×10^{-4}
-10^{-4}	0.3110	5.6337	-1.0002	-1	2.0×10^{-4}
0	0.3119	5.6248	-1.0001	-1	1.0×10^{-4}
10^{-4}	0.3129	5.6160	-1.0000	-1	0.0×10^{-4}
10^{-3}	0.3217	5.5385	-1.0004	-1	4.0×10^{-4}
10^{-2}	0.4099	4.9070	-1.0001	-1	1.0×10^{-4}

Table 4.3: Angle of rotation at $x = 1$ after Half a Period

S_0	Numerical Result	Exact result	Relative Error
-10^{-2}	3.0174	3.0421	8.119×10^{-3}
-10^{-3}	2.9868	3.0423	1.824×10^{-2}
-10^{-4}	3.0766	3.0423	1.127×10^{-2}
0	3.1077	3.0423	2.012×10^{-2}
10^{-4}	3.1035	3.0423	2.702×10^{-2}
10^{-3}	2.9897	3.0423	1.729×10^{-2}
10^{-2}	3.1208	3.0425	2.574×10^{-2}

From Tables 4.2 and 4.3 it is clear that the numerical approximation of the deflection in the linear model is accurate to at least three significant digits, while the approximation of the angle of rotation is accurate to one significant digit. In the approximations, 28 elements and 3000 time steps were used. A better result is expected if more elements or time steps are used.

Chapter 5

The Sapir-Reiss semi-linear model

The focus of this chapter is on the SLT-SR model described in Section 1.3. Recall that in [SR79] only pinned-pinned boundary conditions are used. For this reason the same is done in this chapter. As mentioned, the model was of great importance for the study since it is not linear but the application is the same as for Model AT (which is linear). It is significant that the semi-linear model SLT-SR differs "slightly" from Model AT as explained below.

More recently, in [ADMPS12] a study similar to [SR79] is done, but shear is not considered in the model. Due to the shear, the model in [SR79] has greater complexity and is, in our view, more interesting.

5.1 Model problem and scope of numerical experiments

Model SLT-SR is almost the same as the pre-stressed Model AT introduced in Chapter 1. It is convenient to refer to the pre-stressed model as Model ATC since $S = S_0 = \frac{D}{\gamma}$ (a constant). Recall that for Model SLT-SR,

$$S(t) \;=\; S_0 + \frac{1}{2\gamma}\int_0^1 (\partial_x w(\cdot,t))^2. \tag{5.1.1}$$

The only difference between the models is the term containing the integral of $(\partial_x w)^2$.

Originally, the main concern regarding [SR79] was existence, FEM analysis and application as well as comparison to the linear model. However, to do a comparison it was necessary to consider buckling. Of importance in this chapter is the discussion in Sections 3.5 and 3.6 regarding the critical load. It was proved that the existence of a zero eigenvalue implies that $S_0 = \frac{-\pi^2}{\pi^2+\beta}$. (This critical value of S_0 is denoted by S_{crit}.) By Proposition 3.5.1, negative eigenvalues exist if and only if $S_0 \in (-1, S_{crit})$. It follows that S_{crit} is important for comparison of the models and that one should distinguish between the cases $S_0 < S_{crit}$ and $S_0 > S_{crit}$.

An investigation into what sense one should consider Model ATC as a linear approximation of Model SLT-SR is conducted. To guide this investigation, the situation is compared to a mechanical system modelled by a system of ordinary differential equations. The nonlinear system $u'' + f(u) = 0$, with $f(0) = 0$ has a well defined linear approximation $w'' = -Aw$, where A is the Jacobian matrix of f at 0.

Suppose $-A$ has a positive eigenvalue λ with corresponding eigenvector w_λ. Then

$$w(t) = \left(c_1 \cosh \sqrt{\lambda}t + c_2 \sinh \sqrt{\lambda}t\right) w_\lambda$$

is a solution of the linear approximation for which $|w(t)|_{\mathbb{R}^n} \to \infty$ as $t \to \infty$.

The nonlinear system can be written as $u''+Au = g(u)$, where $|u|_{\mathbb{R}^n}^{-1}|g(u)|_{\mathbb{R}^n} \to 0$ as $|u|_{\mathbb{R}^n}^{-1} \to 0$. Consider the corresponding first order system for the pair $\langle u, u' \rangle$, that is $u' = v$ with $v' = -Au + g(u)$. In matrix form, this is written as

$$\begin{bmatrix} u \\ v \end{bmatrix}' = \begin{bmatrix} 0 & I \\ -A & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ g(u) \end{bmatrix}.$$

It is well known that the equilibrium 0 is stable if all the eigenvalues of $\begin{bmatrix} 0 & I \\ -A & 0 \end{bmatrix}$ have negative real parts and is unstable if one (or more) eigenvalues are positive or have positive real parts.

By assumption, $-Aw_\lambda = \lambda w_\lambda$ and hence

$$\begin{bmatrix} 0 & I \\ -A & 0 \end{bmatrix} \begin{bmatrix} w_\lambda \\ \sqrt{\lambda} w_\lambda \end{bmatrix} = \sqrt{\lambda} \begin{bmatrix} w_\lambda \\ \sqrt{\lambda} w_\lambda \end{bmatrix}.$$

It follows that the equilibrium 0 of the nonlinear system is unstable and there exists a ball with centre 0 where solutions close to the line determined by $\left\langle w_\lambda, \sqrt{\lambda} w_\lambda \right\rangle$ will leave the ball.

Note that $w = \phi = 0$ is an equilibrium for Model ATC as well as Model SLT-SR. If $S_0 > S_{crit}$, the solution of Problem AT is periodic and the rod oscillates around the zero equilibrium. Sapir and Reiss [SR79] studied post-buckling oscillations of an axially loaded rod. The authors first considered the system where the nonlinear term is omitted. They claim that the system was "linearised" to obtain the eigenvalue problem, but no reference is given. Using the critical load for the linear system, they calculated approximate solutions for the nonlinear system when $S_0 < S_{crit}$.

Analysis of nonlinear stability for partial differential equations is beyond the scope of this dissertation. It was decided to explore the properties of solutions using numerical methods. The model, with pinned-pinned boundary conditions, is solved numerically and compared to a solution of Model ATC. The case where $S_0 > S_{crit}$ is discussed in Section 5.4 and the case where $S_0 \leq S_{crit}$ in Section 5.5.

5.2 Variational forms

Recall the variational form of Model ATC found in Chapter 3. It is natural to use the mixed form for this model as well. The variational form for the semi-linear case is identical except for the definition of S.

Problem SLT-SRV Given the load P, find $\langle w, \phi \rangle$ and V such that for each $t > 0$, $\langle w(\cdot,t), \phi(\cdot,t) \rangle \in T_P$, $V(\cdot,t) \in C(0,1)$, the equations

$$\begin{aligned} \left(\partial_t^2 w(\cdot,t), v_1\right) &= -\left(S(t)\partial_x w(\cdot,t), v_1'\right) - \left(V(\cdot,t), v_1'\right) + \left(P(\cdot,t), v_1\right), \\ \left(\frac{1}{\alpha}\partial_t^2 \phi(\cdot,t), v_2\right) &= \left(V(\cdot,t), v_2\right) - \left(\frac{1}{\beta}\partial_x \phi(\cdot,t), v_2'\right) \end{aligned}$$

hold for each $\langle v_1, v_2 \rangle \in T_P$ with constitutive equations

$$\begin{aligned} V(\cdot,t) &= \partial_x w(\cdot,t) - \phi(\cdot,t), \\ S(t) &= S_0 + \frac{1}{2\gamma}\int_0^1 (\partial_x w(\cdot,t))^2. \end{aligned}$$

Also, for $t = 0$,

$$w(x,0) = w_0(x), \qquad \partial_t w(x,0) = w_d(x), \tag{5.2.1}$$

$$\phi(x,0) = \phi_0(x), \qquad \partial_t \phi(x,0) = \phi_d(x). \tag{5.2.2}$$

As in [PK20], damping is not considered in this chapter. (That is, $a = 0$).

Problem SLT-SRV is considered for FEM implementation. The minimum requirement for the initial states are $\langle w_0, \phi_0 \rangle \in E_b$ and $\langle w_d, \phi_d \rangle \in \mathcal{V}$ (see the remark below).

For completeness, the weak variational form is considered below. In order to find the weak variational form, the bilinear form b_{AT} from Chapter 3 must be rewritten. For u and v in $\mathcal{V}$, let

$$b_0(u, v) = b_T(u, v) + (S_0 Du_1, Dv_1) \tag{5.2.3}$$

and

$$\sigma(u, v) = b_0(u, v) + \frac{1}{2\gamma} \left((Du_1, Du_1)Du_1, Dv_1\right). \tag{5.2.4}$$

Note that $b_0 = b_{AT}$ when $S = S_0$. Interestingly, σ is a semi-linear form. Using the notation $f(t) = \langle P(\cdot, t), 0 \rangle$, the weak variational form of the problem follows.

Problem SLT-SRW

Given $u_0 \in \mathcal{V}$, $u_d \in \mathcal{L}^2(0,1)^2$, and $f \in C([0,\tau); \mathcal{L}^2(0,1)^2)$ find $u \in C^2((0,\tau); \mathcal{W})$ such that for each $t > 0$, $u(t) \in \mathcal{V}$, $u'(t) \in \mathcal{V}$ and

$$\begin{aligned}
c_T\left(u''(t), v\right) + a(u'(t), v) + \sigma(u(t), v) &= (f(t), v) \quad \text{for each} \quad v \in \mathcal{V}, \\
\lim_{t \to 0^+} \|u(t) - u_0\|_{\mathcal{V}} &= 0, \\
\lim_{t \to 0^+} \|u'(t) - u_d\|_{\mathcal{L}^2} &= 0.
\end{aligned}$$

Another weak variational form (based on [PK20]) is considered in Chapter 6. The existence of a solution of the problem above is considered in Chapter 7. The theory is based on [Amm02] and $a \neq 0$.

Remark. *In the linear theory, the initial conditions must be such that* $u_0 \in E_b$ *and* $u_d \in \mathcal{V}$ *in order for existence of a solution to be considered. It is therefore safe to assume that at least the same must be true for the nonlinear problem above.*

5.3 Development of an algorithm

To simulate oscillation of a rod, an algorithm is developed for Problem SLT-SR using a variation of the mixed finite element method.

The notation defined in Chapter 4 is used to find the Galerkin approximation of the model and algorithm to compute the approximation.

Galerkin approximation

Find w^h, ϕ^h and d^h such that, for each $t > 0$, $w^h(\cdot,t) \in S_1^h$, $\phi^h(\cdot,t) \in S^h$, $d^h(\cdot,t) \in S^h$ and the equations

$$\begin{aligned}
\left(\partial_t^2 w^h(\cdot,t), v_1\right) &= -\left(S_0 + \frac{1}{2\gamma}(d^h(\cdot,t), d^h(\cdot,t))\right)\left(\partial_x w^h(\cdot,t), v_1'\right) \\
&\quad - \left(V^h(\cdot,t), v_1'\right), \\
\left(\partial_t^2 \phi^h(\cdot,t), v_2\right) &= \alpha\left(V^h(\cdot,t), v_2\right) - \frac{1}{\gamma}\left(\partial_x \phi^h(\cdot,t), v_2'\right), \\
\left(d^h(\cdot,t), g\right) &= \left(\partial_x w^h(\cdot,t), g\right), \\
V^h(\cdot,t) &= d^h(\cdot,t) - \phi^h(\cdot,t)
\end{aligned}$$

hold for each $v_1 \in S_1^h$, $v_2 \in S^h$ and $g \in S^h$, while

$$\langle w^h(\cdot,0), \phi^h(\cdot,0)\rangle = \langle w_0^h, \phi_0^h\rangle \quad \text{and} \quad \langle \partial_t w^h(\cdot,0), \partial_t \phi^h(\cdot,0)\rangle = \langle w_d^h, \phi_d^h\rangle.$$

Since $w^h(\cdot,t) \in S_1^h$, $\phi^h(\cdot,t) \in S^h$ and $d^h(\cdot,t) \in S^h$ for each $t > 0$, it follows that

$$\begin{aligned}
w^h(x,t) &= \sum_{j=1}^{n-1} w_j(t)\delta_j(x), \\
\phi^h(x,t) &= \sum_{j=0}^{n} \phi_j(t)\delta_j(x), \\
d^h(x,t) &= \sum_{j=0}^{n} d_j(t)\delta_j(x) \\
V^h(x,t) &= \sum_{j=0}^{n} \left(d_j(t) - \phi_j(t)\right)\delta_j(x).
\end{aligned}$$

Also, $\int_0^1 (d^h(\cdot,t))^2$ can be approximated, using the trapezoidal rule, by

$$\int_0^1 (d^h(\cdot,t))^2 \approx \frac{1}{n}\left(\frac{1}{2}(d_0(t))^2 + \sum_{j=1}^{n-1}(d_j(t))^2 + \frac{1}{2}(d_n(t))^2\right). \tag{5.3.1}$$

Therefore the Galerkin approximation can be rewritten: Find $\bar{w}$, $\bar{\phi}$ and $\bar{d}$

such that

$$
\begin{aligned}
S &= S_0 + \frac{1}{2\gamma n}\left(\frac{1}{2}(d_0(t))^2 + \sum_{j=1}^{n-1}(d_j(t))^2 + \frac{1}{2}(d_n(t))^2\right), \\
\sum_{j=1}^{n-1} w_j''\left(\delta_j, \delta_i\right) &= -S\sum_{j=1}^{n-1} w_j(\delta_j', \delta_i') - \sum_{j=0}^{n} V_j\left(\delta_j, \delta_i'\right) \quad \text{for } i = 1, \ldots, n-1, \\
\sum_{j=0}^{n} \phi_j''\left(\delta_j, \delta_k\right) &= \alpha\sum_{j=0}^{n} V_j\left(\delta_j, \delta_k\right) - \frac{1}{\gamma}\sum_{j=0}^{n} \phi_j\left(\delta_j', \delta_k'\right) \quad \text{for } k = 0, \ldots, n, \\
\sum_{j=0}^{n} d_j(\delta_j, \delta_m) &= \sum_{j=1}^{n-1} w_j(\delta_j', \delta_m) \quad \text{for } m = 0, \ldots, n, \\
\bar{V}(t) &= \bar{d}(t) - \bar{\phi}(t)
\end{aligned}
$$

with

$$\bar{w}(0) = \bar{\pi}w_0^h, \quad \bar{\phi}(0) = \bar{\pi}\phi_0^h, \quad \bar{w}'(0) = \bar{\pi}w_d^h, \quad \bar{\phi}'(0) = \bar{\pi}\phi_d^h.$$

This may be constructed as a system of ordinary differential equations.

That is, find $\begin{bmatrix}\bar{w} \\ \bar{\phi}\end{bmatrix}$ and $\bar{d}$ such that

$$M\bar{d} = \begin{bmatrix}\left(L_{R\{0,n\}}\right)^T & \bar{0}\end{bmatrix}\begin{bmatrix}\bar{w} \\ \bar{\phi}\end{bmatrix},$$

$$S(t) = S_0 + \frac{1}{2\gamma n}\left(\frac{1}{2}(d_0(t))^2 + \sum_{j=1}^{n-1}(d_j(t))^2 + \frac{1}{2}(d_n(t))^2\right),$$

$$\bar{V} = \bar{d} - \bar{\phi}$$

and

$$\begin{bmatrix}M_{\{0,n\}} & [0] \\ [0] & M\end{bmatrix}\begin{bmatrix}\bar{w} \\ \bar{\phi}\end{bmatrix}'' = \begin{bmatrix}-L_{R\{0,n\}} \\ \alpha M\end{bmatrix}\bar{V} + \begin{bmatrix}-SK_{\{0,n\}} & [0] \\ [0] & -\frac{1}{\gamma}K\end{bmatrix}\begin{bmatrix}\bar{w} \\ \bar{\phi}\end{bmatrix},$$

with

$$\begin{bmatrix}\bar{w}(0) \\ \bar{\phi}(0)\end{bmatrix} = \begin{bmatrix}\bar{\pi}w_0^h \\ \bar{\pi}\phi_0^h\end{bmatrix}, \quad \begin{bmatrix}\bar{w}'(0) \\ \bar{\phi}'(0)\end{bmatrix} = \begin{bmatrix}\bar{\pi}w_d^h \\ \bar{\pi}\phi_d^h\end{bmatrix}.$$

Using central differences, the following algorithm is obtained to find approximations for $\begin{bmatrix}\bar{w} \\ \bar{\phi}\end{bmatrix}$ and $\bar{d}$.

Algorithm For each time step k,

$$\begin{bmatrix} M_{\{0,n\}} & [0] \\ [0] & M \end{bmatrix} \begin{bmatrix} \bar{w}_{k+1} \\ \bar{\phi}_{k+1} \end{bmatrix} = (\delta t)^2 \begin{bmatrix} -L_{R\{0,n\}} \\ \alpha M \end{bmatrix} (\bar{d}_k - \bar{\phi}_k)$$
$$+ (\delta t)^2 \begin{bmatrix} -S_k K_{\{0,n\}} & [0] \\ [0] & -\frac{1}{\gamma} K \end{bmatrix} \begin{bmatrix} \bar{w}_k \\ \bar{\phi}_k \end{bmatrix} + \begin{bmatrix} M_{\{0,n\}} & [0] \\ [0] & M \end{bmatrix} \left(2 \begin{bmatrix} \bar{w}_k \\ \bar{\phi}_k \end{bmatrix} - \begin{bmatrix} \bar{w}_{k-1} \\ \bar{\phi}_{k-1} \end{bmatrix} \right),$$

where

$$M\bar{d}_k = \begin{bmatrix} \left(L_{R\{0,n\}}\right)^T & \bar{0} \end{bmatrix} \begin{bmatrix} \bar{w}_k \\ \bar{\phi}_k \end{bmatrix},$$
$$S_k = S_0 + \frac{1}{2\gamma n} \left(\frac{1}{2}(d_0^{(k)})^2 + \sum_{j=1}^{n-1} (d_j^{(k)})^2 + \frac{1}{2}(d_n^{(k)})^2 \right).$$

To prepare for the initial step,

$$\begin{bmatrix} \bar{w}_0 \\ \bar{\phi}_0 \end{bmatrix} = \begin{bmatrix} \bar{\pi} w_0^h \\ \bar{\pi} \phi_0^h \end{bmatrix} \quad \text{and} \quad \frac{1}{2\delta t} \left(\begin{bmatrix} \bar{w}_1 \\ \bar{\phi}_1 \end{bmatrix} - \begin{bmatrix} \bar{w}_{-1} \\ \bar{\phi}_{-1} \end{bmatrix} \right) = \begin{bmatrix} \bar{\pi} w_d^h \\ \bar{\pi} \phi_d^h \end{bmatrix}.$$

Remark. *To the best of our knowledge, this algorithm is new.*

5.4 Implementation of FEM

In this section, simulation of the motion for a subcritical load ($S_0 < S_{crit}$) is carried out. This is done to compare the results of linear and semi-linear models and investigate the outcome. The values $\gamma = \frac{\beta}{\alpha} = 0.25$, $\alpha = 1200$ and $\beta = 300$ are used. For $\beta = 300$, the critical value $S_{crit} = -0.03185$ (approximately).

The numerical results of Model SLT-SR with a small initial displacement were calculated and compared to the results of Model ATC after half a period of the linear case. For a small initial displacement, the linear and nonlinear models should behave similarly. Thus, the comparison of the deflection and angle of rotation for the two models is justified.

For the initial condition, a multiple of the first mode of the linear model is used. That is, for $a = 10^{-3}$,

$$w_0(x) = a \sin(\pi x), \quad \phi_0(x) = a\pi \cos(\pi x), \quad w_d(x) = \phi_d(x) = 0. \tag{5.4.1}$$

The exact solution for the linear model is given by

$$\begin{bmatrix} w(x,t) \\ \phi(x,t) \end{bmatrix} = a\cos(\sqrt{\lambda_1}\ t)\begin{bmatrix} \sin(\pi x) \\ A_1\cos(\pi x) \end{bmatrix}. \tag{5.4.2}$$

Substituting in the initial conditions, the algorithm to find the numerical approximation of the solution in Section 5.3 becomes

Algorithm

$$\begin{aligned} M\bar{d}_k &= \begin{bmatrix} \left(L_{R\{0,n\}}\right)^T & \bar{0} \end{bmatrix}\begin{bmatrix} \bar{w}_k \\ \bar{\phi}_k \end{bmatrix}, \\ S_k &= S_0 + \frac{1}{2\gamma n}\left(\frac{1}{2}(d_0^{(k)})^2 + \sum_{j=1}^{n-1}(d_j^{(k)})^2 + \frac{1}{2}(d_n^{(k)})^2\right) \end{aligned}$$

and

$$\begin{aligned} \begin{bmatrix} M_{\{0,n\}} & [0] \\ [0] & M \end{bmatrix}\begin{bmatrix} \bar{w}_{k+1} \\ \bar{\phi}_{k+1} \end{bmatrix} &= (\delta t)^2\begin{bmatrix} -L_{R\{0,n\}} \\ \alpha M \end{bmatrix}(\bar{d}_k - \bar{\phi}_k) \\ &+ (\delta t)^2\begin{bmatrix} -S_k K_{\{0,n\}} & [0] \\ [0] & -\frac{1}{\gamma}K \end{bmatrix}\begin{bmatrix} \bar{w}_k \\ \bar{\phi}_k \end{bmatrix} + \begin{bmatrix} M_{\{0,n\}} & [0] \\ [0] & M \end{bmatrix}\left(2\begin{bmatrix} \bar{w}_k \\ \bar{\phi}_k \end{bmatrix} - \begin{bmatrix} \bar{w}_{k-1} \\ \bar{\phi}_{k-1} \end{bmatrix}\right) \end{aligned}$$

with $\begin{bmatrix} \bar{w}_0 \\ \bar{\phi}_0 \end{bmatrix} = \begin{bmatrix} \bar{\pi} w_0^h \\ \bar{\pi}\phi_0^h \end{bmatrix}$ and $\begin{bmatrix} \bar{w}_{-1} \\ \bar{\phi}_{-1} \end{bmatrix} = \begin{bmatrix} \bar{w}_1 \\ \bar{\phi}_1 \end{bmatrix}$.

Experiment 1

In this experiment, various values of S_0, with $S_0 \geq -0.01 > S_{crit}$ were chosen to compare the solutions of Models ATC and SLT-SR.

The following tables show, for the given initial conditions (5.4.1) and different values for the force S_0, the exact results obtained for Model ATC (Equation (5.4.2)) and the numerical results obtained for Model SLT-SR where the deflection and angle of rotation are calculated after 2 dimensionless units of time. This is done because $t = 2$ is small compared to the period of all linear solutions considered.

For this set of initial conditions, the greatest deflection occurs at the centre of the rod. Results for the deflection at $x = 0.5$ are therefore compared. Also, the greatest deviation occurs at the two endpoints ($x = 0$ and $x = 1$) for this set of initial conditions and hence results for the deviation at $x = 1$ are compared.

Table 5.1: Deflection at $x = 0.5$ and $t = 2$, where $a = 10^{-3}$

Value of S_0	Model ATC	Model SLT-SR	Relative Difference
-10^{-2}	6.017×10^{-4}	6.015×10^{-4}	2.327×10^{-4}
-10^{-3}	4.542×10^{-4}	4.539×10^{-4}	5.725×10^{-4}
0	4.383×10^{-4}	4.381×10^{-4}	6.160×10^{-4}
10^{-3}	4.226×10^{-4}	4.226×10^{-4}	1.420×10^{-4}
10^{-2}	2.863×10^{-4}	2.862×10^{-4}	3.144×10^{-4}
1	9.972×10^{-4}	9.974×10^{-4}	1.705×10^{-4}
1.5	1.073×10^{-4}	1.071×10^{-4}	2.330×10^{-3}
2	-8.759×10^{-4}	-8.762×10^{-4}	3.425×10^{-4}

Table 5.2: Angle of rotation at $x = 1$ and $t = 2$, where $a = 10^{-3}$

Value of S_0	Model ATC	Model SLT-SR	Relative Difference
-10^{-2}	-1.830×10^{-3}	-1.830×10^{-3}	4.371×10^{-4}
-10^{-3}	-1.382×10^{-3}	-1.381×10^{-3}	7.237×10^{-4}
0	-1.334×10^{-3}	-1.333×10^{-3}	7.499×10^{-4}
10^{-3}	-1.286×10^{-3}	-1.285×10^{-3}	3.111×10^{-4}
10^{-2}	-8.710×10^{-4}	-8.706×10^{-4}	4.822×10^{-4}
1	-3.058×10^{-3}	-3.058×10^{-3}	6.540×10^{-5}
1.5	-3.306×10^{-4}	-3.293×10^{-4}	3.872×10^{-3}
2	2.708×10^{-3}	2.709×10^{-3}	4.432×10^{-4}

It was found that the approximate solution of the semi-linear model suggests that the solutions are "period-like". (The term almost periodic is well defined.)

The Tables 5.1 and 5.2 show that for $S_0 > S_{crit}$, the approximations of Model ATC are equal to the approximations of Model SLT-SR to at least 3 significant digits.

Solutions of linear and semi-linear models for S_0 close to the critical value $S_{crit} \approx -0.03185$ are investigated in the experiments that follow.

Experiment 2

In this experiment, the case where $S_0 > S_{crit}$, but with $|S_0 - S_{crit}|$ small is examined. To be specific, the value $S_0 = -3.1 \times 10^{-2}$ is used. In this case, $\lambda_1 = 8.33 \times 10^{-3}$.

Only the deflections are given, since they represent the shape of the rod. As in Experiment 1, the greatest deflection occurs at the centre of the rod and hence the results for the deflection at $x = 0.5$ are compared.

Table 5.3: Deflection at $x = 0.5$, where $a = 10^{-2}$

Value of t	Model SLT-SR	Model ATC	Relative Difference
0	1.00×10^{-2}	1.00×10^{-2}	0.00
1	9.91×10^{-3}	9.96×10^{-3}	4.42×10^{-3}
2	9.65×10^{-3}	9.84×10^{-3}	1.91×10^{-2}
3	9.21×10^{-3}	9.63×10^{-3}	4.32×10^{-2}
4	8.63×10^{-3}	9.34×10^{-3}	7.61×10^{-2}
5	7.91×10^{-3}	8.98×10^{-3}	1.19×10^{-1}
6	7.06×10^{-3}	8.54×10^{-3}	1.74×10^{-1}
7	6.15×10^{-3}	8.03×10^{-3}	2.34×10^{-1}
8	5.15×10^{-3}	7.45×10^{-3}	3.09×10^{-1}
9	4.09×10^{-3}	6.81×10^{-3}	3.99×10^{-1}
10	3.00×10^{-3}	6.12×10^{-3}	5.10×10^{-1}
11	1.87×10^{-3}	5.37×10^{-3}	6.52×10^{-1}
12	7.29×10^{-4}	4.58×10^{-3}	8.41×10^{-1}
13	-4.20×10^{-4}	3.75×10^{-3}	1.11

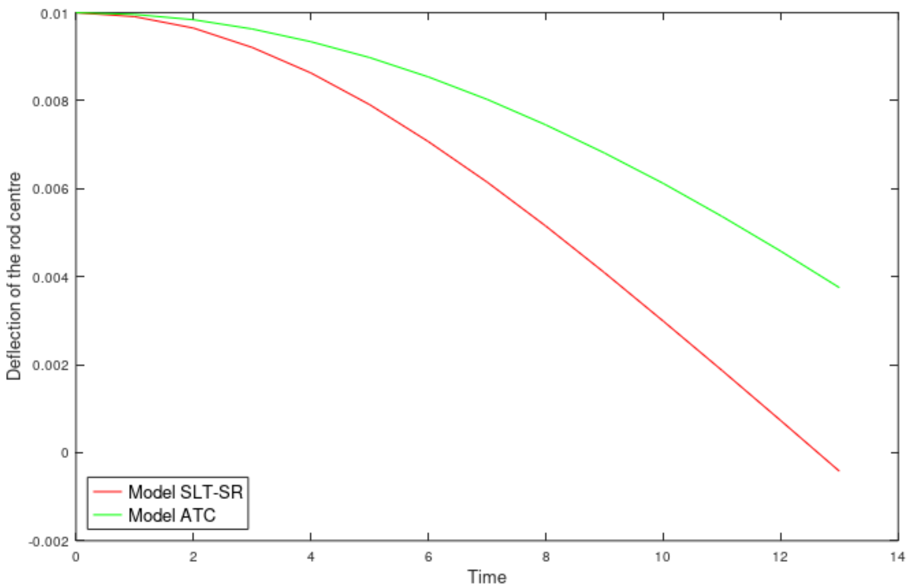

Figure 5.1: Comparison of the results from Table 5.3, $S_0 > S_{crit}$, $a = 10^{-2}$.

It is clear from Table 5.3 that the solutions differ significantly. The solution of Model ATC is periodic and the solution of Model SLT-SR appears to be "period-like". As the graphs in Figure 5.1 indicate, the "period" of the nonlinear model is less than the period of the linear model.

5.5 Buckling

Recall that S_{crit} is the critical load that corresponds to λ_1 being zero. In Model ATC, the solution where $S_0 \leq S_{crit}$ yields unrealistic results (see Subsection 3.6.2) which are interpreted as an indication that the rod could be buckling. However, the results of Experiment 2 lead to the belief that S_{crit} is not "critical" for Model SLT-SR.

Experiment 3

In this experiment, the case $S_0 = S_{crit}$ is examined. Assuming $\lambda_1 = 0$, the value of S_0 was calculated and approximated by $S_0 = -3.185 \times 10^{-2}$. In this case, the solution of Model ATC is a non-zero constant.

Table 5.4: Deflection at $x = 0.5$, where $a = 10^{-2}$

Value of t	Model SLT-SR	Relative Difference
0	1.00×10^{-2}	0.00
1	9.956×10^{-3}	4.37×10^{-3}
2	9.811×10^{-3}	1.89×10^{-2}
3	9.575×10^{-3}	4.25×10^{-2}
4	9.260×10^{-3}	7.40×10^{-2}
5	8.857×10^{-3}	1.14×10^{-1}
6	8.399×10^{-3}	1.60×10^{-1}
7	7.881×10^{-3}	2.12×10^{-1}
8	7.310×10^{-3}	2.69×10^{-1}
9	6.708×10^{-3}	3.29×10^{-1}
10	6.075×10^{-3}	3.93×10^{-1}
11	5.416×10^{-3}	4.58×10^{-1}
12	4.749×10^{-3}	5.25×10^{-1}
13	4.067×10^{-3}	5.93×10^{-1}

As the graph in Figure 5.2 indicates, the solution of Model SLT-SR decreases, while the solution of Model ATC stays constant, which is not physically realistic. The graph for the nonlinear case can be interpreted that the solution may reach zero and oscillate around it. This was confirmed in a later experiment (see Figure 5.3).

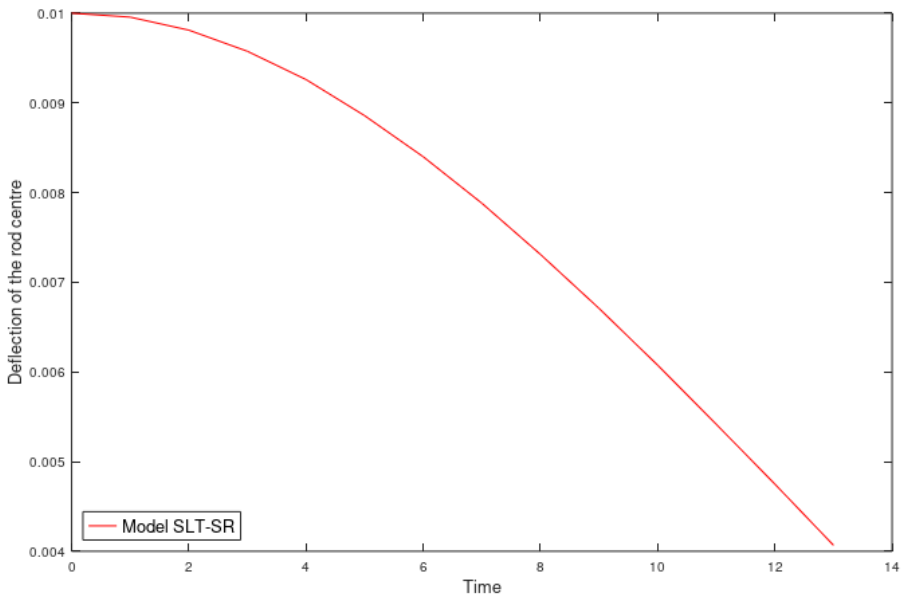

Figure 5.2: Comparison of the linear and nonlinear results from Table 5.4, $S_0 = S_{crit}$, $a = 10^{-2}$.

The following experiment is similar to Experiment 3, but with $\lambda_1 < 0$.

Experiment 4

In this set of results, the case where $S_0 < S_{crit}$, but with $|S_0 - S_{crit}|$ small, is examined. The value $S_0 = -3.2 \times 10^{-2}$ is used and in this case, $\lambda_1 = -1.4610 \times 10^{-3}$.

It is clear from Table 5.5 that the linear solution in this example makes no physical sense since there is no additional force and yet the deflection of the centre of the rod increases. It is possible, however, that the nonlinear approximation reflects the actual deflection. Although the solution of Model SLT-SR decreases, it is slower than for the case where $\lambda_1 = 0$. This can be seen by comparing the values of the deflection at the centre of the rod in each case at $t = 13$. It may be interpreted from the graph for the nonlinear case that it is possible for the solution to reach zero and oscillate around it.

In search of a nonlinear critical value for S_0, Experiment 5 is conducted.

Experiment 5

Choose $S_0 = -3.25 \times 10^{-2}$. The graph below compares the results of Model SLT-SR found in Experiments 2, 3 and 4, as well as for the additional case, where $t \in [0, 30]$.

Table 5.5: Deflection at $x = 0.5$, where $a = 10^{-2}$

Value of t	Model SLT-SR	Model ATC	Relative Difference
0	1.00×10^{-2}	1.00×10^{-2}	0.00
1	9.96×10^{-3}	1.00×10^{-2}	4.36×10^{-3}
2	9.84×10^{-3}	1.00×10^{-2}	1.88×10^{-2}
3	9.64×10^{-3}	1.01×10^{-2}	4.24×10^{-2}
4	9.37×10^{-3}	1.01×10^{-2}	7.36×10^{-2}
5	9.03×10^{-3}	1.02×10^{-2}	1.13×10^{-1}
6	8.64×10^{-3}	1.03×10^{-2}	1.58×10^{-1}
7	8.20×10^{-3}	1.04×10^{-2}	2.09×10^{-1}
8	7.71×10^{-3}	1.05×10^{-2}	2.64×10^{-1}
9	7.19×10^{-3}	1.06×10^{-2}	3.21×10^{-1}
10	6.65×10^{-3}	1.07×10^{-2}	3.81×10^{-1}
11	6.09×10^{-3}	1.09×10^{-2}	4.42×10^{-1}
12	5.52×10^{-3}	1.11×10^{-2}	5.02×10^{-1}
13	4.93×10^{-3}	1.13×10^{-2}	5.62×10^{-1}

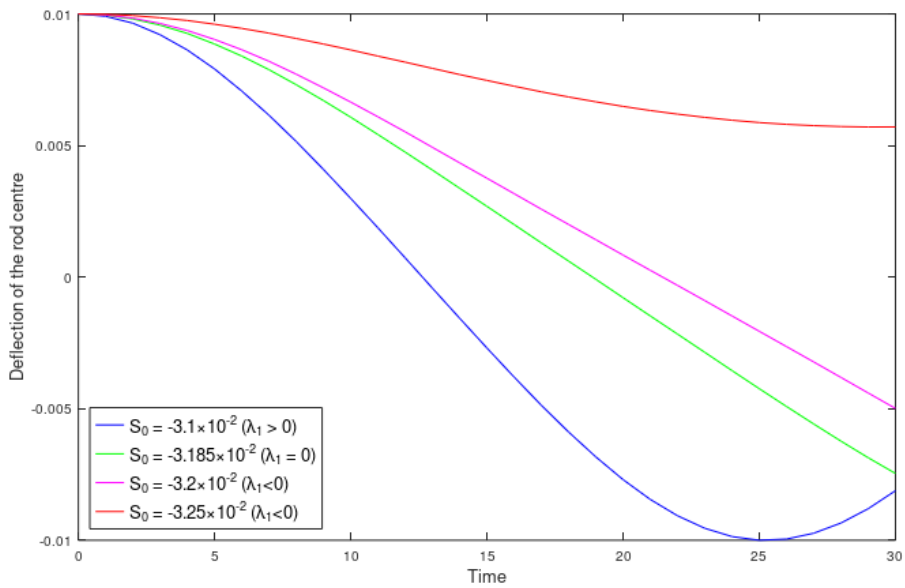

Figure 5.3: Comparison of the deflection of the rod centre for the cases $S_0 = -3.1 \times 10^{-2}$, $S_0 = -3.185 \times 10^{-2}$, $S_0 = -3.2 \times 10^{-2}$ and $S_0 = -3.25 \times 10^{-2}$, where $a = 10^{-2}$.

For $S_0 = -3.2 \times 10^{-2}$, the solution still oscillates around zero even though λ_1 is negative. However, for $S_0 = -3.25 \times 10^{-2}$, the solution no longer oscillates around zero, but possibly some other positive value. This leads to the conjecture that the critical value for S_0 in Model SLT-SR is between $S_0 = -3.2 \times 10^{-2}$ and $S_0 = -3.25 \times 10^{-2}$.

Conclusion

The results in Chapter 5 almost confirm the findings in [SR79]. One difference is that they claim the critical value is the same for the linear and nonlinear models. In this chapter, calculations show that the critical values differ but are close. It is important to note that neither Sapir and Reiss nor we proved any of the conclusions. However, completely different computational methods were used, which render the findings significant. Clearly, more reading and investigations should be done in future research.

Chapter 6

Convergence for the Sapir-Reiss semi-linear model

In Chapter 5 the same approach was taken in solving the nonlinear model as with the linear model. A different approach is considered in this chapter. The system is reduced to one of first-order equations before the finite element method is applied to it. This approach has the advantage that error estimates are calculated using weak derivatives in terms of space and time. That is, convergence for the shear strain and bending strain can also be found. The methods used in this chapter follow closely to those used in [PK20].

Instead of deriving the error estimates using the solution itself, error estimates using a vector valued image of the solution are calculated. This vector valued image, however, does not include the deflection and neither do the actual iterations. The use of the deflection is avoided by using the equality of mixed derivatives. The approach taken in [PK20] is similar to approaches used in Finite Differences. Estimates are first calculated on matrices, vectors and the so-called truncation error before the approximation error is found. This is done using a norm that depends on the element length h.

6.1 Comparison of models

The system considered by the authors of [PK20] is

$$\partial_t^2 w = \left(cd - a + b\int_0^1 (\partial_x w)^2 dx\right)\partial_x^2 w - cd\partial_x\phi, \tag{6.1.1}$$

$$\partial_t^2 \phi = c\partial_x^2\phi - c^2 d\left(\phi - \partial_x w\right), \tag{6.1.2}$$

where $0 \leq x \leq 1$, $0 \leq t \leq T$ and with boundary and initial conditions

$$\partial_t w(0,t) = \partial_t w(1,t) = 0, \quad \partial_t\phi(0,t) = \partial_t\phi(1,t) = 0, \tag{6.1.3}$$

$$\begin{aligned} \partial_t w(x,0) = w_d(x), \quad w(x,0) = w_0(x), \\ \partial_t \phi(x,0) = \phi_d(x), \quad \phi(x,0) = \phi_0(x). \end{aligned} \tag{6.1.4}$$

As part of the justification for this model, the articles [SR79] and [Amm02] (amongst others) are quoted, which consider boundary conditions for a pinned-pinned rod. The authors of [PK20] give no explanation as to why they consider clamped-clamped boundary conditions and no other boundary conditions are mentioned. Also, there is no mention that $w(0,t) = \phi(0,t) = 0$. Fortunately, by choosing the test functions in the variational form according to the boundary conditions (as has been done throughout this dissertation), this does not influence the theory.

The system considered in Chapter 5 is

$$\partial_t^2 w = \left(1 + S_0 + \frac{1}{2\gamma}\int_0^1 (\partial_x w(\cdot,t))^2\right)\partial_x^2 w - \partial_x\phi, \tag{6.1.5}$$

$$\partial_t^2 \phi = \frac{1}{\gamma}\partial_x^2\phi - \alpha\left(\phi - \partial_x w\right), \tag{6.1.6}$$

where $0 \leq x \leq 1$, $0 \leq t \leq T$ and with initial conditions

$$\begin{aligned} \partial_t w(x,0) = w_d(x), \quad w(x,0) = w_0(x), \\ \partial_t \phi(x,0) = \phi_d(x), \quad \phi(x,0) = \phi_0(x). \end{aligned} \tag{6.1.7}$$

Comparing the two models, it was found that no scaling was done for the model in [PK20].

In [PK20], the authors use the notation I_1 to denote the area moment of inertia where I was used in Chapter 1. In addition, I_2 denotes the polar moment of the cross-section. The authors of [SR79] use I_2, the polar moment of a cross-section, instead of I, which is clearly an error. In other articles, including [PK20], this error has been carried over. Fortunately, it is not relevant to the theory. The value I_2 is irrelevant for the model but is of the same order as I_1. The authors of [PK20] also use Δ in their model to denote the "end shortening" of the rod. This value is not dimensionless, whereas all other parameters in their model are.

Rewriting the system (6.1.1) - (6.1.2) using the dimensionless constants defined in Chapter 1, it is found that

$$cd = \frac{A\ell^2 G}{EI} = \frac{\beta}{\kappa^2} \tag{6.1.8}$$

$$a = \frac{A\ell\Delta}{I} = \frac{A\ell^2\Delta}{I\ell} = \alpha D \tag{6.1.9}$$

$$b = \frac{A\ell^2}{2I} = \frac{\alpha}{2} \tag{6.1.10}$$

$$c = \frac{A\ell^2}{I_2} = \chi\alpha, \quad \text{where} \quad \frac{1}{2} \leq \chi \leq 1. \tag{6.1.11}$$

That is, (noting that $\beta = \gamma\alpha$)

$$\partial_t^2 w = \left(\alpha\left(\frac{\gamma}{\kappa^2} - D\right) + \frac{\alpha}{2}\int_0^1 (\partial_x w)^2 dx\right)\partial_x^2 w - \frac{\gamma\alpha}{\kappa^2}\partial_x\phi, \tag{6.1.12}$$

$$\partial_t^2\phi = \chi\alpha\partial_x^2\phi - \frac{\chi\alpha^2\gamma}{\kappa^2}\left(\phi - \partial_x w\right), \tag{6.1.13}$$

where $0 \leq x \leq 1$, $0 \leq t \leq T$.

The form of the problem considered in [PK20] is therefore similar to Problem SLT-SR.

The requirement in [PK20] that $cd - a > 0$ is equivalent to the condition $D < \frac{\gamma}{\kappa^2}$. Typically, $\gamma \in [\frac{1}{6}, \frac{1}{2}]$ and $\kappa^2 \in [\frac{1}{2}, 1]$ (see Chapter 1) and hence the requirement is that $D < 1$. Alternatively, for $S_0 = \frac{D}{\gamma}$ (see Chapter 1), the requirement is equivalent to $S_0 < 2$.

As mentioned in Chapter 5, the initial conditions must satisfy the forced boundary conditions in order for existence of a solution to be considered. It is assumed that

$$w_0, \phi_0 \in H^3(0,1) \quad \text{and} \quad w_d, \phi_d \in H^2(0,1).$$

Although not explicitly stated, the authors of [PK20] consider weak solutions on $(0,1)\times(0,T)$. That is, w and ϕ are assumed to be in $H^2\left((0,1)\times(0,T)\right)$.

For ease of reference, most of the notation used in the remainder of this chapter is the same as that used in [PK20].

6.2 Variational forms and existence theorems

Following the sequence used in the rest of this dissertation, the weak variational form is found before the Galerkin approximation is given. This is omitted in [PK20], although weak derivatives seem to be implied. Once the finite element method has been applied, the authors of [PK20] apply a Crank–Nicolson type of symmetric difference scheme and use a Picard type iteration process to find the algorithm.

Instead of solving the problem as it is, the problem is first reduced to a system of first-order differential equations. Let

$$w_t = \partial_t w, \quad w_x = \partial_x w, \quad \phi_t = \partial_t \phi, \quad \phi_x = \partial_x \phi. \tag{6.2.1}$$

Then (6.1.1), (6.1.2), (6.1.3) and (6.1.4) become

$$\begin{aligned}
\partial_t w_t &= \left(cd - a + b\int_0^1 w_x^2(\cdot,t)\right)\partial_x w_x - cd\phi_x, && (6.2.2)\\
\partial_t w_x &= \partial_x w_t, && (6.2.3)\\
\partial_t \phi_t &= c\partial_x \phi_x - c^2 d\left(\phi - w_x\right), && (6.2.4)\\
\partial_t \phi_x &= \partial_x \phi_t, && (6.2.5)\\
\partial_t \phi &= \phi_t, && (6.2.6)
\end{aligned}$$

where $0 < x < 1$, $0 < t \leq T$ and

$$w_t(0,t) = w_t(1,t) = 0, \quad \phi_t(0,t) = \phi_t(1,t) = 0, \tag{6.2.7}$$

with initial conditions

$$\begin{aligned}
&w_t(x,0) = w_d(x), \quad w_x(x,0) = \partial_x w_0(x),\\
&\phi_t(x,0) = \phi_d(x), \quad \phi_x(x,0) = \partial_x \phi_0(x), \quad \phi(x,0) = \phi_0(x).
\end{aligned} \tag{6.2.8}$$

Note that the solution for w will still need to be found since the solution of this set of equations will give approximations for $\partial_t w$, $\partial_x w$, $\partial_t \phi$, $\partial_x \phi$ and ϕ. This could be done by integration or, as the authors of [PK20] suggest, Taylor expansions.

Although it seems to be implied throughout [PK20] that weak derivatives of both time and space exist, only the initial conditions are assumed to have weak derivatives. For this nonlinear problem it is not necessarily true that the solution will have the required regularity.

In order to find the variational form of this problem, Equations (6.2.2) - (6.2.6) are multiplied by functions $v_i \in C[0,1]$, $i = 1, 2, ..., 5$. and each equation is integrated over the length. Let

$$T_P = \left\{ g = (g_1, g_2, g_3, g_4, g_5) \in (C[0,1])^5 : g_1(0) = g_1(1) = 0, \right. \\ \left. g_3(0) = g_3(1) = 0 \right\}. \qquad (6.2.9)$$

The variational form therefore follows.

Variational Form
Given positive constants a, b, c, d, find $\langle w_t,\, w_x,\, \phi_t,\, \phi_x,\, \phi \rangle$ such that for each $t > 0$, $\langle w_t(\cdot,t)$,$w_x(\cdot,t)$,$\phi_t(\cdot,t)$,$\phi_x(\cdot,t)$, $\phi(\cdot,t)\rangle \in T_P$ and for each $v \in T_P$,

$$\begin{aligned} (\partial_t w_t, v_1) &= \left(\left(cd - a + b \int_0^1 (w_x(\cdot,t))^2 \right) \partial_x w_x - cd\phi_x, v_1 \right), && (6.2.10) \\ (\partial_t w_x, v_2) &= -\left(w_t, v_2' \right), && (6.2.11) \\ (\partial_t \phi_t, v_3) &= \left(c\partial_x \phi_x - c^2 d \left(\phi - w_x \right), v_3 \right), && (6.2.12) \\ (\partial_t \phi_x, v_4) &= -\left(\phi_t, v_4' \right), && (6.2.13) \\ (\partial_t \phi, v_5) &= \left(\phi_t, v_5 \right), && (6.2.14) \end{aligned}$$

with initial conditions

$$\begin{aligned} &w_t(x,0) = w_d(x), \quad w_x(x,0) = \partial_x w_0(x), \\ &\phi_t(x,0) = \phi_d(x), \quad \phi_x(x,0) = \partial_x \phi_0(x), \quad \phi(x,0) = \phi_0(x). \quad (6.2.15) \end{aligned}$$

In order to find the weak variational form, the closure of the test functions T_P (that is, $\overline{T_P}$) must be defined. This is not the space $\mathcal{V}$ as in previous chapters. Now

$$\overline{T_P} = H_0^1(0,1) \times H^1(0,1) \times H_0^1(0,1) \times H^1(0,1)^2, \qquad (6.2.16)$$

where

$$H_0^1(0,1) = \{ w \in H^1(0,1) : \lim_{x \to 0} w(x) = \lim_{x \to 1} w(x) = 0 \}.$$

In this chapter the weak partial derivatives with respect to time and space are denoted D_t and D_x respectively, whereas the weak derivative with respect to space is denoted D as before. The fact that mixed weak partial derivatives are interchangeable comes from the definition of a weak derivative, since mixed partial derivatives of C^∞ functions are equal.

Weak Variational Form

Given positive constants a, b, c, d, find $y = \langle w_t,\ w_x,\ \phi_t,\ \phi_x,\ \phi\rangle$ such that for each $t > 0$, $y(\cdot, t) = \langle w_t(\cdot, t), w_x(\cdot, t), \phi_t(\cdot, t), \phi_x(\cdot, t),\ \phi(\cdot, t)\rangle \in \overline{T_P}$ and for each $v \in \overline{T_P}$,

$$\begin{aligned}
(D_t w_t, v_1) &= \left(\left(cd - a + b\int_0^1 (w_x(\cdot, t))^2\right) D_x w_x - cd\phi_x, v_1\right), & (6.2.17)\\
(D_t w_x, v_2) &= -(w_t, Dv_2), & (6.2.18)\\
(D_t \phi_t, v_3) &= \left(cD_x\phi_x - c^2 d(\phi - w_x), v_3\right), & (6.2.19)\\
(D_t \phi_x, v_4) &= -(\phi_t, Dv_4), & (6.2.20)\\
(D_t \phi, v_5) &= (\phi_t, v_5), & (6.2.21)
\end{aligned}$$

with initial conditions

$$\begin{aligned}
&w_t(x, 0) = w_d(x), \quad w_x(x, 0) = D_x w_0(x),\\
&\phi_t(x, 0) = \phi_d(x), \quad \phi_x(x, 0) = D_x\phi_0(x), \quad \phi(x, 0) = \phi_0(x). \quad (6.2.22)
\end{aligned}$$

Recall the notation $\bar{\pi} f = \bar{f}$ used in Chapter 4: for $u \in S^h$ and $v \in S_0^h$, $\bar{\pi} u = \bar{u}$ if $u = \sum_{i=0}^{n} u_i\delta_i$ and $\bar{\pi} v = \bar{v}$ if $v = \sum_{i=1}^{n-1} v_i\delta_i$. Since piecewise linear basis functions are being used, the i^{th} entry of $\bar{\pi} f$ is the function f at node i. The definition of $\bar{\pi}$ is therefore extended to $H^1(0, 1)$: for $u \in H^1(0, 1)$, let $\bar{\pi} u = \bar{u}$, where $\bar{u}$ denotes the nodal values of the unique continuous function that is equal to u almost everywhere (see Theorem 2.2.2).

Define the following vector valued function:

$$\bar{y}(t) = \left(\overline{w_t}(\cdot, t), \overline{w_x}(\cdot, t), \overline{\phi_t}(\cdot, t), \overline{\phi_x}(\cdot, t), \overline{\phi}(\cdot, t)\right). \quad (6.2.23)$$

Existence theory for the nonlinear problem is discussed in Chapter 7. The fact that the model used there is similar to the one in this chapter suggests that the existence results will hold for both models.

6.3 Algorithm in [PK20]

The first-order system is written as a Galerkin approximation. Each equation of the Galerkin approximation is then written in matrix form. The entire system is written in one matrix differential equation depending only on time, which is discretised twice further to form an algorithm used to approximate the solution.

Application of the finite element method

As in Chapter 4, let the interval $[0,1]$ be divided into n subintervals and δ_j denote the $j'th$ C^0 piecewise linear basis function on $[0,1]$. (More information is given in Appendix B.) Then the approximate solution of the system (6.2.2)-(6.2.6) is found in the form

$$
\begin{aligned}
{w_t}^h(x,t) &= \sum_{j=1}^{n-1} w_{t_j}(t)\delta_j(x), \qquad {w_x}^h(x,t) = \sum_{k=0}^{n} w_{x_k}(t)\delta_k(x),\\
{\phi_t}^h(x,t) &= \sum_{j=1}^{n-1} \phi_{t_j}(t)\delta_j(x), \qquad {\phi_x}^h(x,t) = \sum_{k=0}^{n} \phi_{x_k}(t)\delta_k(x),\\
&\qquad \phi^h(x,t) = \sum_{k=0}^{n} \phi_{_k}(t)\delta_k(x). \qquad (6.3.1)
\end{aligned}
$$

The functions $w_{t_j}(t)$, $w_{x_k}(t)$, $\phi_{t_j}(t)$, $\phi_{x_k}(t)$ and $\phi_{_k}(t)$ above are defined using the system of ordinary differential equations below:

$$
\begin{aligned}
\left(D_t {w_t}^h, \delta_j\right) &= \left(\left(cd - a + b\int_0^1 \left({w_x}^h(\cdot,t)\right)^2\right) D_x {w_x}^h - cd{\phi_x}^h, \delta_j\right), && (6.3.2)\\
\left(D_t {w_x}^h, \delta_k\right) &= -\left({w_t}^h, {\delta_k}'\right), && (6.3.3)\\
\left(D_t {\phi_t}^h, \delta_j\right) &= \left(cD_x{\phi_x}^h - c^2 d\left(\phi^h - {w_x}^h\right), \delta_j\right), && (6.3.4)\\
\left(D_t {\phi_x}^h, \delta_k\right) &= -\left({\phi_t}^h, {\delta_k}'\right), && (6.3.5)\\
\left(D_t \phi^h, \delta_k\right) &= \left({\phi_t}^h, \delta_k\right), && (6.3.6)
\end{aligned}
$$

where $0 < t \leq T$, with the initial conditions

$$
\begin{aligned}
w_{t_j}(0) &= w_d(x_j), \qquad w_{x_k}(0) = \partial_x w_0(x_k), && (6.3.7)\\
\phi_{t_j}(0) &= \phi_d(x_j), \qquad \phi_{x_k}(0) = \partial_x \phi_0(x_k), \quad \phi_{_k}(0) = \phi_0(x_k), && (6.3.8)
\end{aligned}
$$

where $j = 1, ..., n-1$ and $k = 0, 1, ..., n$.

As mentioned above,

$$\bar{\pi}{w_t}^h(\cdot,t) = \langle w_{t1}(t), \ldots w_{tn-1}(t)\rangle \quad \text{and} \quad \bar{\pi}{w_x}^h(\cdot,t) = \langle w_{x0}(t), \ldots w_{xn}(t)\rangle.$$

The notation $\bar{\pi}u = \bar{u}$ could become confusing at this point. To avoid this, the notation, for example

$$w_x^h(t) = \bar{\pi}{w_x}^h(\cdot,t),$$

is used. Define the vector function y^h as follows:

$$y^h(t) = \left(w_t^h(t), w_x^h(t), \phi_t^h(t), \phi_x^h(t), \phi^h(t)\right). \tag{6.3.9}$$

Matrices Recall the matrices M and L defined in Chapter 4. The matrices are also written out in more detail in Appendix B. For a square matrix X, the following notation is used.

- $X_{\{0,n\}}$ is matrix X with rows 0 and n and columns 0 and n deleted;
- X_R is matrix X with rows 0 and n deleted.

The system in Equations (6.3.2)-(6.3.8) can therefore be rewritten as

$$\begin{aligned}
M_{\{0,n\}} D_t(w_t^h(t)) &= \left(cd - a + b w_x^h(t)^T M w_x^h(t)\right) L_R w_x^h(t) \\
&\quad - cd M_R \phi_x^h(t), && (6.3.10)\\
M D_t(w_x^h(t)) &= -\left(L_R\right)^T w_t^h(t), && (6.3.11)\\
M_{\{0,n\}} D_t(\phi_t^h(t)) &= c L_R \phi_x^h(t) - c^2 d M_R \phi^h(t) + c^2 d M_R w_x^h(t), && (6.3.12)\\
M D_t(\phi_x^h(t)) &= -\left(L_R\right)^T \phi_t^h(t), && (6.3.13)\\
M D_t(\phi^h(t)) &= \left(M_R\right)^T \phi_t^h(t), && (6.3.14)
\end{aligned}$$

where $0 < t \leq T$ and with initial conditions

$$\begin{aligned}
w_t^h(0) &= \overline{\pi} w_d, \qquad w_x^h(0) = \overline{\pi}\partial_x w_0, && (6.3.15)\\
\phi_t^h(0) &= \overline{\pi}\phi_d, \qquad \phi_x^h(0) = \overline{\pi}\partial_x \phi_0, \qquad \phi^h(0) = \overline{\pi}\phi_0. && (6.3.16)
\end{aligned}$$

For theoretical purposes, Equations (6.3.10)-(6.3.16) are represented as

$$\mathbf{M} D_t(y^h(t)) = \left(\mathbf{L} + \mathbf{N}(w_x^h(t))\right) y^h(t), \tag{6.3.17}$$

with $0 < t \leq T$ and initial condition

$$y^h(0) = \left(\overline{\pi} w_d, \overline{\pi}\partial_x w_0, \overline{\pi}\phi_d, \overline{\pi}\partial_x \phi_0, \overline{\pi}\phi_0\right), \tag{6.3.18}$$

where (following the notation of [PK20])

$$\mathbf{M} = \begin{bmatrix} \frac{1}{h}M_{\{0,n\}} & & & & \\ & \frac{2}{h}M & & 0 & \\ & & \frac{1}{h}M_{\{0,n\}} & & \\ & 0 & & \frac{2}{h}M & \\ & & & & \frac{2}{h}M \end{bmatrix}, \tag{6.3.19}$$

$$\mathbf{L} = \begin{bmatrix} 0 & \frac{(cd-a)}{h}L_R & 0 & \frac{-cd}{h}M_R & 0 \\ \frac{-2}{h}(L_R)^T & 0 & 0 & 0 & 0 \\ 0 & \frac{c^2d}{h}M_R & 0 & \frac{c}{h}L_R & \frac{-c^2d}{h}M_R \\ 0 & 0 & \frac{-2}{h}(L_R)^T & 0 & 0 \\ 0 & 0 & \frac{2}{h}(M_R)^T & 0 & 0 \end{bmatrix}, \tag{6.3.20}$$

$$\mathbf{N}(v(t)) = bv(t)^T M v(t) \begin{bmatrix} 0 & \frac{1}{h}L_R & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, v(t) \in \mathbb{R}^{n+1}. \tag{6.3.21}$$

If alternative boundary conditions are used, (6.3.17) remains the same except for the definitions of $\mathbf{M}$, $\mathbf{L}$ and $\mathbf{N}$.

Algorithm

Before the algorithm from [PK20] is given, more notation is required.

Let τ denote the grid step on the time interval $[0, T]$, where $0 < \tau < 1$, $\tau = \frac{T}{R}$ and $t_m = m\tau$, $m = 0, 1, ..., R$. Denote the values of vectors of the form $v^h(t_m)$ by v^h_m, $m = 0, 1, ..., R$. Using this notation, let

$$y^h_m = \left(w_{tm}^h, w_{xm}^h, \phi_{tm}^h, \phi_{xm}^h, \phi_m^h\right). \tag{6.3.22}$$

Using a Crank-Nicholson type of symmetric difference scheme (see [PK20]), it follows that

$$\mathbf{M}\frac{y^h_m - y^h_{m-1}}{\tau} = \frac{1}{2}\left(\mathbf{L} + \frac{1}{2}\left(\mathbf{N}(w_{xm}^h) + \mathbf{N}(w_{xm-1}^h)\right)\right)\left(y^h_m + y^h_{m-1}\right), \quad \text{with } y^h_0 = y^h(0). \tag{6.3.23}$$

The authors of [PK20] further discretise the system to perform an iteration process. Let

$$y^h_{m,p} = \left(w_{tm,p}^h, w_{xm,p}^h, \phi_{tm,p}^h, \phi_{xm,p}^h, \phi_{m,p}^h\right), \tag{6.3.24}$$

where, for chosen τ and h, $y^h_{m,p}$ is the p^{th} iteration of y^h_m. It is assumed that y^h_{m-1} is known and has a negligible corresponding error. Let

$$y^h_{m,0} = y^h_{m-1}. \tag{6.3.25}$$

Using a Picard type iteration process (see [PK20]), the following algorithm for $y^h_{m,p}$ is found:

$$\mathbf{M}y^h_{m,p} = \mathbf{M}y^h_{m-1} + \frac{\tau}{2}\left(\mathbf{L} + \frac{1}{2}\left(\mathbf{N}({w_x}^h_{m,p-1}) + \mathbf{N}({w_x}^h_{m-1})\right)\right)\left(y^h_{m,p-1} + y^h_{m-1}\right). \tag{6.3.26}$$

6.4 Truncation error

Once the error estimate for the system has been found, the error of the iteration process (which was assumed to be negligible) is estimated.

The vector $\bar{y}$ defined in (6.2.23) does not necessarily satisfy (6.3.17). That is, there is an error term when it is substituted in. To be precise,

$$\mathbf{M}D_t(\bar{y}(t)) = \left(\mathbf{L} + \mathbf{N}(\overline{w_x}(\cdot, t))\right)\bar{y}(t) + \theta^h. \tag{6.4.1}$$

The term θ^h is referred to as the truncation error. For $z^h(t) = \bar{y}(t) - y^h(t)$,

$$\mathbf{M}D_t(z^h(t)) = \mathbf{L}z^h(t) + \mathbf{N}(\overline{w_x}(\cdot, t))\bar{y}(t) - \mathbf{N}(w^h_x(t))y^h(t) + \theta^h(t), \tag{6.4.2}$$

where

$$\theta^h(t) = \mathbf{M}D_t(\bar{y}(t)) - \left(\mathbf{L} + \mathbf{N}(\overline{w_x}(\cdot, t))\right)\bar{y}(t), \quad \theta^h(0) = \bar{0}. \tag{6.4.3}$$

Equation (6.4.2) resembles a linear differential equation, where the second, third, and fourth terms act as forcing functions. The strategy followed in this chapter is to find bounds for these "forcing functions", then Gronwall's inequality may be used to find the error estimate for the algorithm.

Proposition 6.4.1 (Gronwall's inequality)**.**
Let $f(\cdot)$ be a non-negative, absolutely continuous function on $[0, T]$ and $g(t)$ and $h(t)$ be non-negative, summable functions on $[0, T]$. Suppose

$$f'(t) \leq g(t)f(t) + h(t) \quad \text{a.e. for } t \in [0, T]. \tag{6.4.4}$$

Then, for each $t \in [0, T]$,

$$f(t) \leq e^{\int_0^t g(s)ds}\left(f(0) + \int_0^t h(s)ds\right). \tag{6.4.5}$$

Proof. See [Eva98, Appendix B] □

Vector and matrix norms

Some norms are defined in order to find bounds on the "forcing functions" in Equation (6.4.2). Recall that $h = \frac{1}{n}$. The h-scalar product for two vectors u and v with the same dimension k is defined as

$$(u, v)_h = h \sum_{i=1}^{k} u_i v_i,$$

with associated norm

$$\|u\|_h = (u, u)_h^{\frac{1}{2}}.$$

Let

$$S^h = span\{\delta_0, ..., \delta_n\} \text{ and } S_0^h = span\{\delta_1, ..., \delta_{n-1}\}. \tag{6.4.6}$$

The relation between the $\| \cdot \|_h$ norm and the $\mathcal{L}^2(0,1)$ norm is shown below.

Proposition 6.4.2. *The norm $\| \cdot \|_h$ on S^h and the norm $\| \cdot \|_h$ on S_0^h both converge to the norm in $\mathcal{L}^2(0,1)$.*

Proof. For $u \in S^h$ and $v \in S_0^h$, by the definition of the Riemann integral,

$$\lim_{h \to 0} \|u\|_h^2 = \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n} (u_i)^2 = \int_0^1 (u)^2 = \|u\|^2, \tag{6.4.7}$$

$$\lim_{h \to 0} \|v\|_h^2 = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n-1} (v_i)^2 = \int_0^1 (v)^2 = \|v\|^2. \tag{6.4.8}$$

□

The proposition above implies that any bounds found in terms of the $\| \cdot \|_h$ norm also hold in $\mathcal{L}^2(0,1)$.

Other related norms are now defined. For a symmetric positive definite matrix V and vector u with dimension equal to the order of V, define the norm

$$\|u\|_{V,h} = \left(\frac{1}{h} V u, u \right)_h^{\frac{1}{2}}. \tag{6.4.9}$$

The norm $\| \cdot \|_h$ on a matrix V is represented by

$$\|V\|_h = \max_{u \neq 0} \frac{\|Vu\|_h}{\|u\|_h}. \tag{6.4.10}$$

The following estimates are useful.

Proposition 6.4.3. *For a vector $v \in \mathbb{R}^{n+1}$ and $v_0 \in \mathbb{R}^{n-1}$,*

$$\frac{1}{\sqrt{6}}\|v\|_h \leq \|v\|_{M,h} \leq \|v\|_h, \tag{6.4.11}$$

$$\frac{1}{\sqrt{3}}\|v_0\|_h \leq \|v_0\|_{M_{\{0,n\}},h} \leq \|v_0\|_h. \tag{6.4.12}$$

Also,

$$\|L_R\|_h \leq 1. \tag{6.4.13}$$

Proof. Let $v \in \mathbb{R}^{n+1}$ and $v_0 \in \mathbb{R}^{n-1}$. Then (see Appendix B for the definition of M)

$$\|v\|_{M,h}^2 = \sum_{i=0}^{n}(Mv)_i v_i = \frac{1}{6}h\sum_{i=0}^{N-1}\left(2v_i^2 + 2v_i v_{i+1} + 2v_{i+1}^2\right).$$

Therefore, using the property $|2ab| \leq a^2 + b^2$ for any real numbers a and b, it follows that

$$\frac{1}{6}\|v\|_h^2 \leq \|v\|_{M,h}^2 \leq \|v\|_h^2.$$

The proof that $\frac{1}{\sqrt{3}}\|v_0\|_h \leq \|v_0\|_{M_{\{0,n\}},h} \leq \|v_0\|_h$ is similar.

For the proof of (6.4.13), recall that

$$\|L_R\|_h = \max_{v\neq 0}\frac{\|L_R v\|_h}{\|v\|_h}.$$

Now, for any $v \in \mathbb{R}^{n+1}$, using the property $-2ab \leq a^2 + b^2$ for any real numbers a and b, (see Appendix B for the definition of L)

$$\|L_R v\|_h^2 = \frac{h}{4}\sum_{i=1}^{n-1}(v_{i-1} - v_{i+1})^2 \leq h\sum_{i=0}^{n}(v_i)^2 = \|v\|_h^2.$$

Therefore

$$\|L_R\|_h \leq 1.$$

□

Bounds for the growth of the function $\bar{y}$

The notation defined above is now used to find bounds in order to estimate the "forcing" terms in (6.4.2).

Due to the fact that some expressions are used frequently, shortened notation is defined for convenience. This notation is similar to that used in [PK20]. Recall that the notation $C^{i,j}([0,1]\times[0,T])$ is used for the space of functions, defined on $[0,1]\times[0,T]$, which have i continuous partial space derivatives and j continuous partial time derivatives. Let

$$\omega_0 = \frac{a}{b} + \sqrt{\frac{2}{b}} e_1(0), \tag{6.4.14}$$

where

$$e_1(t) = \Big(\|\partial_t w(x,t)\|^2 + cd\|\partial_x w(x,t) - \phi(x,t)\|^2 + \frac{1}{2b}\big(a - b\|\partial_x w(x,t)\|^2\big)^2 + \frac{1}{c}\|\partial_t \phi(x,t)\|^2 + \|\partial_x \phi(x,t)\|^2 \Big)^{\frac{1}{2}}. \tag{6.4.15}$$

In addition, if $w_x(x,t) \in C^{\rho,0}([0,1]\times[0,T])$, where $\rho = 1$ or $\rho = 2$, let

$$\omega_\rho = \begin{cases} \frac{5}{6} m_0 m_1 & \text{for } \rho = 1, \\ \frac{1}{3}\left(m_0(m_1+m_2) + \frac{1}{2}m_1^2\right) & \text{for } \rho = 2, \end{cases} \tag{6.4.16}$$

where $m_i = \max\limits_{0\le x\le 1}\max\limits_{0\le t\le T}\left\|\dfrac{\partial^i w_x(x,t)}{\partial x^i}\right\|_h$, $i = 0,1,2$.

Let

$$e_2(\bar{y}) = \frac{1}{\sqrt{3cd}}\Big(\|\overline{w_t}\|^2_{M_{\{0,n\}},h} + cd\|\overline{w_x} - \overline{\phi}\|^2_{M,h} + \frac{1}{2b}\big(a - b\|\overline{w_x}\|^2_{M,h}\big)^2 + \frac{1}{c}\|\overline{\phi_t}\|^2_{M_{\{0,n\}},h} + \|\overline{\phi_x}\|^2_{M,h}\Big)^{\frac{1}{2}}. \tag{6.4.17}$$

The values s_0, s_1 and s_2 are given by

$$s_0 = \omega_0 + h^\rho \omega_\rho, \tag{6.4.18}$$

$$s_1 = \frac{a}{b} + \sqrt{\frac{6}{b}cd}\left(e_2(y^h(0)) + \frac{T}{h}\left\|M_R\phi_x^2 - L_R\phi^2\right\|_h\right), \tag{6.4.19}$$

$$s_2 = 3b\left(s_1 + \left(s_1 - \frac{a}{b}\right)^2\right)\max\left\{1, \frac{8}{b}, \frac{c}{2}, \frac{2}{cd}\right\} \tag{6.4.20}$$

and the values α_1, α_2 and α_3 are given by

$$\alpha_1 = 3\left(\eta + \left(s_0 + \sqrt{6}\,(\sqrt{s_0}+\sqrt{s_1})^2 + s_1\right)\frac{b}{2h}\right), \tag{6.4.21}$$

$$\alpha_2 = \frac{3}{2}\left(\eta + s_1\left(1+2\sqrt{6}\right)\frac{b}{h}\right), \tag{6.4.22}$$

$$\alpha_3 = \frac{3}{2}\left(\eta + s_1\frac{b}{h}\right), \tag{6.4.23}$$

where, for p and q positive real numbers depending on a, c and d,

$$\eta^2 = \frac{1}{h^2}p + q. \tag{6.4.24}$$

The propositions below are lemmas which are proved in [PK20]. References are given for convenience.

Proposition 6.4.4 (Lemma 1)**.**
The following inequalities hold:

$$\|\mathbf{M}\|_h \leq 2, \tag{6.4.25}$$

$$\|\mathbf{M}^{-1}\|_h \leq 3, \tag{6.4.26}$$

$$\|\mathbf{L}\|_h \leq \eta. \tag{6.4.27}$$

Proof. See [PK20, Section 3.1.2]. □

Proposition 6.4.5 (Lemma 2)**.**
For $\bar{u}, \bar{v}, \bar{u}^*$ and $\bar{v}^*$ in $S_0^h \times S^h \times S_0^h \times (S^h)^2$,

$$\begin{aligned}\|\mathbf{N}(u_2)\bar{v}^* - \mathbf{N}(v_2)\bar{u}^*\|_h \leq \frac{b}{2h}\Big(\left(\|u_2\|_{M,h}^2 + \|v_2\|_{M,h}^2\right)\|\bar{v}^* - \bar{u}^*\|_h + (\|u_2\|_{M,h} \\ +\|v_2\|_{M,h})\left(\|v_2^*\|_h + \|u_2^*\|_h\right)\|\bar{u} - \bar{v}\|_h\Big).\end{aligned}$$

Proof. See [PK20, Section 3.1.2]. □

Proposition 6.4.6 (Lemmas 3 and 4)**.**
There exists a solution of the problem given by Equations (6.3.10)-(6.3.16), *and the following inequalities are valid for* $0 \leq t \leq T$*:*

$$\|\overline{w_x}(\cdot,t)\|_{M,h}^2 \leq s_0, \tag{6.4.28}$$

$$\|w_x^h(t)\|_{M,h}^2 \leq s_1. \tag{6.4.29}$$

Proof. See [PK20, Section 3.2.2]. □

Proposition 6.4.7 (Lemma 5).
There exists a solution of the problem in (6.3.23) *and the following estimate holds:*

$$\|w_{xm}^{h}\|_{M,h}^{2} \quad \leq \quad s_1, \tag{6.4.30}$$

where $m = 1, 2, ..., R$.

Proof. See [PK20, Section 3.3.2]. □

Proposition 6.4.8 (Lemma 6).
If the grid step τ *satisfies the condition*

$$0 < \tau \leq \frac{1-\epsilon_2}{\alpha_3}, \tag{6.4.31}$$

where ϵ_2 *is an arbitrary number from the interval* $(0, 1)$*, then*

$$\|y_m^h\|_h \quad \leq \quad s_2, \tag{6.4.32}$$

where $m = 1, 2, ..., R$.

Proof. See [PK20, Section 3.4.2]. □

If a function has a continuous partial time derivative of order ℓ, then the partial derivative is bounded.

Definition. For $y(x,t) \in \left(C^{0,\ell}([0,1] \times [0,T])\right)^5$, $\ell = 0, 1, 2$, let

$$m_{\bar{y},\ell} \quad = \quad m_{\overline{w_t},\ell} + m_{\overline{w_x},\ell} + m_{\overline{\phi_t},\ell} + m_{\overline{\phi_x},\ell} + m_{\overline{\phi},\ell}, \tag{6.4.33}$$

$$m_{\overline{v_0},\ell} \quad = \quad \max_{\substack{0\leq t\leq T\\ 1\leq j\leq n-1}} \left\| \frac{d^\ell v_0(x_j,t)}{dt^\ell} \right\|_h, \ \overline{v_0} = \overline{w_t}, \overline{\phi_t}, \tag{6.4.34}$$

$$m_{\overline{v},\ell} \quad = \quad \max_{\substack{0\leq t\leq T\\ 0\leq k\leq n}} \left\| \frac{d^\ell v(x_k,t)}{dt^\ell} \right\|_h, \ \overline{v} = \overline{w_x}, \overline{\phi_x}, \overline{\phi}. \tag{6.4.35}$$

Also, for $y^h(t) \in \left(C^2[0,T]\right)^5$, $\ell = 0, 1, 2$, let

$$m_{y^h,\ell} \quad = \quad m_{w_t^h,\ell} + m_{w_x^h,\ell} + m_{\phi_t^h,\ell} + m_{\phi_x^h,\ell} + m_{\phi^h,\ell}, \tag{6.4.36}$$

$$m_{v_0,\ell} \quad = \quad \max_{\substack{0\leq t\leq T\\ 1\leq j\leq n-1}} \left\| \frac{d^\ell v_{0j}(t)}{dt^\ell} \right\|_h, \ v_0 = w_t^h, \phi_t^h, \tag{6.4.37}$$

$$m_{v,\ell} \quad = \quad \max_{\substack{0\leq t\leq T\\ 0\leq k\leq n}} \left\| \frac{d^\ell v_k(t)}{dt^\ell} \right\|_h, \ v = w_x^h, \phi_x^h, \phi^h. \tag{6.4.38}$$

The authors of [PK20] do not explicitly mention an estimate for $\mathbf{N}(w_x^h(t))$, however it is similar to Proposition 6.4.5. It is given below for convenient reference.

Proposition 6.4.9. *For* $t \in [0,T]$ *and* $\bar{v}(t) \in \mathbb{R}^{n-1} \times \mathbb{R}^{n+1} \times \mathbb{R}^{n-1} \times (\mathbb{R}^{n+1})^2$,

$$\|\mathbf{N}(v_2(t))\bar{v}(t)\|_h \leq \frac{b}{h}\|v_2(t)\|_{M,h}^2 \|\bar{v}\|_h. \tag{6.4.39}$$

Proof. Let $t \in [0,T]$. Then, for $\bar{v}(t) \in \mathbb{R}^{n-1} \times \mathbb{R}^{n+1} \times \mathbb{R}^{n-1} \times (\mathbb{R}^{n+1})^2$, using the estimate given in (6.4.13) ,

$$\begin{aligned}
\|\mathbf{N}(v_2(t))\bar{v}(t)\|_h &= \left\| b v_2(t)^T M v_2(t) \frac{1}{h} L_R v_2(t) \right\|_h \\
&= \frac{b}{h} \|v_2(t)\|_{M,h}^2 \|L_R v_2(t)\|_h \\
&\leq \frac{b}{h} \|v_2(t)\|_{M,h}^2 \|L_R\|_h \|v_2(t)\|_h \\
&\leq \frac{b}{h} \|v_2(t)\|_{M,h}^2 \|\bar{v}(t)\|_h.
\end{aligned} \tag{6.4.40}$$

□

Using the theory above, an estimate may be found for the truncation error defined in (6.4.3).

Proposition 6.4.10. *The following inequality holds for the truncation error in* (6.4.3)*: if* $w_x(x,t) \in C^{\rho,1}([0,1] \times [0,T])$*, where* $\rho = 1$ *or* $\rho = 2$*, and* $w_t(x,t)$*,* $\phi_t(x,t)$*,* $\phi_x(x,t)$*,* $\phi(x,t) \in C^{0,1}([0,1] \times [0,T])$*, then*

$$\|\theta^h(t)\|_h \leq 2m_{\bar{y},1} + \eta m_{\bar{y},0} + \frac{b}{h} s_0 m_{\bar{y},0}.$$

Proof. For any $t \in [0,T]$, using propositions 6.4.4, 6.4.6 and 6.4.9 and the fact that $w_x(x,t) \in C^{\rho,1}([0,1] \times [0,T])$, where $\rho = 1$ or $\rho = 2$, and $w_t(x,t)$, $\phi_t(x,t)$, $\phi_x(x,t)$, $\phi(x,t) \in C^{0,1}([0,1] \times [0,T])$,

$$\begin{aligned}
\|\theta^h(t)\|_h &= \left\| \mathbf{M}\frac{d\bar{y}(t)}{dt} - (\mathbf{L} + \mathbf{N}(\overline{w_x}(\cdot,t)))\,\bar{y}(t) \right\|_h \\
&\leq \left\| \mathbf{M}\frac{d\bar{y}(t)}{dt} \right\|_h + \|\mathbf{L}\bar{y}(t)\|_h + \|\mathbf{N}(\overline{w_x}(\cdot,t))\bar{y}(t)\|_h \\
&\leq \|\mathbf{M}\|_h\, m_{\bar{y},1} + \|\mathbf{L}\|_h\, m_{\bar{y},0} + \frac{b}{h}\|\overline{w_x}(\cdot,t)\|_{M,h}^2 m_{\bar{y},0} \\
&\leq 2m_{\bar{y},1} + \eta m_{\bar{y},0} + \frac{b}{h} s_0 m_{\bar{y},0}.
\end{aligned} \tag{6.4.41}$$

□

6.5 Error estimates

Using bounds found for the so-called truncation errors, Gronwall's inequality is used to find error estimates for the algorithm given in Section 6.3. This is done using the triangle inequality to add three different error estimates in order to find the total estimate.

Semi-discrete error estimate

The first of the error estimates calculated is the semi-discrete error estimate. That is, the error of the finite element method.

Theorem 6.5.1. *The error of the finite element method is estimated by the inequality below, using the notation of* (6.2.23), (6.3.9) *and* (6.4.3)*:*

$$\|\bar{y}(t) - y^h(t)\|_h \leq \left(3\alpha_1 e^{t\alpha_1} t^2 + 3t\right) \max_{0\leq t\leq T} \|\theta^h(t)\|_h. \qquad (6.5.1)$$

Proof. Let $z^h(t) = \bar{y}(t) - y^h(t)$. By (6.4.2), since $z^h(0) = 0$, it follows that

$$\mathbf{M}z^h(t) = \int_0^t \left(\mathbf{L}z^h(u) + \mathbf{N}(\overline{w_x}(\cdot,u))\bar{y}(u) - \mathbf{N}(w_x^h(u))y^h(u) + \theta^h(u)\right) du. \qquad (6.5.2)$$

Also, by Proposition 6.4.5, (6.4.28) and (6.4.29),

$$\begin{aligned}
&\|\mathbf{N}(\overline{w_x}(\cdot,t))\bar{y}(t) - \mathbf{N}(w_x^h(t))y^h(t)\|_h \\
&\leq \frac{b}{2h}\left(\|\overline{w_x}(\cdot,t)\|_{M,h}^2 + \|w_x^h(t)\|_{M,h}^2 + (\|\overline{w_x}(\cdot,t)\|_{M,h}\right. \\
&\quad \left. +\|w_x^h(t)\|_{M,h}\right)\left(\|\overline{w_x}(\cdot,t)\|_h + \|w_x^h(t)\|_h\right)\right) \|z^h(t)\|_h \\
&\leq \frac{b}{2h}\left(s_0 + \sqrt{6}\left(\sqrt{s_0}+\sqrt{s_1}\right)^2 + s_1\right)\|z^h(t)\|_h. \qquad (6.5.3)
\end{aligned}$$

Therefore, from (6.5.2), (6.4.26) and (6.4.27), it follows that

$$\begin{aligned}
\|z^h(t)\|_h &\leq \alpha_1 \int_0^t \|z^h(u)\|_h du + 3\int_0^t \|\theta^h(u)\|_h du \\
&\leq \alpha_1 \int_0^t \|z^h(u)\|_h du + 3t \max_{0\leq t\leq T} \|\theta^h(t)\|_h. \qquad (6.5.4)
\end{aligned}$$

Using Gronwall's inequality (Proposition 6.4.1), with $f(t) = \int_0^t \|z^h(u)\|_h du$, it follows that

$$\int_0^t \|z^h(u)\|_h du \leq 3e^{t\alpha_1}t^2 \max_{0\leq t\leq T} \|\theta^h(t)\|_h. \tag{6.5.5}$$

Therefore (6.5.4) becomes

$$\begin{aligned}\|z^h(t)\|_h &\leq 3\alpha_1 e^{t\alpha_1}t^2 \max_{0\leq t\leq T} \|\theta^h(t)\|_h + 3t \max_{0\leq t\leq T} \|\theta^h(t)\|_h \\ &= \left(3\alpha_1 e^{t\alpha_1}t^2 + 3t\right) \max_{0\leq t\leq T} \|\theta^h(t)\|_h.\end{aligned}$$

□

Corollary. *If*

$$w_x(x,t) \in C^{\rho,1}([0,1]\times[0,T]), \quad \textit{where} \quad \rho = 1 \ \textit{or}\ \rho = 2, \tag{6.5.6}$$

$$w_t(x,t), \phi_t(x,t), \phi_x(x,t), \phi(x,t) \in C^{0,1}([0,1]\times[0,T]), \tag{6.5.7}$$

then the error of the finite element method is estimated by the following inequality.

$$\|\bar{y}(t) - y^h(t)\|_h \leq \left(3\alpha_1 e^{t\alpha_1}t^2 + 3t\right)\left(2m_{\bar{y},1} + \eta m_{\bar{y},0} + \frac{b}{h}s_0 m_{\bar{y},0}\right), \tag{6.5.8}$$

Fully discrete error estimate

Recall that τ denotes the grid step on the time interval $[0,T]$, where $0 < \tau < 1$, $\tau = \frac{T}{R}$ and $t_m = m\tau$, $m = 0,1,...,R$. Also, the approximate values of vectors of the form $v^h(t_m)$ are denoted by v_m^h, $m = 0,1,...,R$. Let

$$y_m^h = \left(w_{tm}^h, w_{xm}^h, \phi_{tm}^h, \phi_{xm}^h, \phi_m^h\right). \tag{6.5.9}$$

Recall from (6.3.23) that along with $y_0^h = y^h(0)$,

$$\mathbf{M}\frac{y_m^h - y_{m-1}^h}{\tau} = \frac{1}{2}\left(\mathbf{L} + \frac{1}{2}\left(\mathbf{N}(w_{xm}^h) + \mathbf{N}(w_{xm-1}^h)\right)\right)\left(y_m^h + y_{m-1}^h\right).$$

Let

$$z_m^h = y^h(t_m) - y_m^h.$$

Note that by letting $y_m^h = y^h(t_m) - z_m^h$, it follows that for $m = 0,1,...R$

$$\begin{aligned}\mathbf{M}\frac{z_m^h - z_{m-1}^h}{\tau} &= \frac{1}{2}\mathbf{L}\left(z_m^h + z_{m-1}^h\right) - \frac{1}{4}\left(\mathbf{N}(w_{xm}^h)\right. \\ &\quad \left.+\mathbf{N}(w_{xm-1}^h)\right)\left(y_m^h + y_{m-1}^h\right) + \frac{1}{4}\left(\mathbf{N}(w_x^h(t_m))\right. \\ &\quad \left.+\mathbf{N}(w_x^h(t_{m-1}))\right)\left(y^h(t_m) + y^h(t_{m-1})\right) + \theta_m^h, \end{aligned} \tag{6.5.10}$$

where $z_0^h = 0$ and

$$\begin{aligned}\theta_m^h &= \mathbf{M}\frac{y^h(t_m) - y^h(t_{m-1})}{\tau} - \frac{1}{2}\mathbf{L}\left(y^h(t_m) + y^h(t_{m-1})\right) \\ &\quad -\frac{1}{4}\left(\mathbf{N}({w_x}^h(t_m)) + \mathbf{N}({w_x}^h(t_{m-1}))\right)\left(y^h(t_m) + y^h(t_{m-1})\right). \quad (6.5.11)\end{aligned}$$

The following proposition is a part of the proof of Theorem 2 in [PK20], but is not explicitly stated.

Proposition 6.5.1. *If, for the functions in* (6.3.1),

$$w_{t_j}(t), w_{x_k}(t), \phi_{t_j}(t), \phi_{x_k}(t), \phi_k(t) \in C^2[0, T],$$

where $j = 1, ..., n-1$ and $k = 0, ..., n$, then the truncation error in (6.5.11) *is estimated by*

$$\max_{0\le\ell\le m} \|\theta_\ell^h\|_h \le \tau\left(m_{y^h,2} + \frac{1}{2}\eta m_{y^h,1} + s_1\frac{b}{h}\left(\frac{1}{2} + \sqrt{6}\right)m_{w_x^h,1}\right). \quad (6.5.12)$$

The error estimate of y_m^h is found below.

Theorem 6.5.2. *Suppose that for the grid step τ, the inequality below holds:*

$$0 < \tau \le \frac{1-\epsilon_1}{\alpha_2}, \quad (6.5.13)$$

where ϵ_1 is an arbitrary number from the interval $(0, 1)$*. Then the following inequality estimates the error of the difference scheme in* (6.3.23)*:*

$$\|y^h(t_m) - y_m^h\|_h \le \frac{3}{2\alpha_2}e^{2t_m(\alpha_2/\epsilon_1)} \max_{0\le\ell\le m} \|\theta_\ell^h\|_h. \quad (6.5.14)$$

Proof. It follows from (6.5.10) that

$$\begin{aligned}z_m^h &= z_{m-1}^h + \frac{\tau\mathbf{M}^{-1}}{2}\Bigg(\mathbf{L}(z_m^h + z_{m-1}^h) + \frac{1}{2}\sum_{i,j=0}^{1}\big(\mathbf{N}(w_x^h(t_{m-i}))y^h(t_{m-j}) \\ &\quad -\mathbf{N}(w_{x\,m-i}^{\ h})y_{m-j}^h\big)\Bigg) + \tau\mathbf{M}^{-1}\theta_m^h. \quad (6.5.15)\end{aligned}$$

Also, by Propositions 6.4.3, 6.4.5 and 6.4.6, for $i, j \in \{0, 1\}$

$$\begin{aligned}&\left\|\mathbf{N}(w_x^h(t_{m-i}))y^h(t_{m-j}) - \mathbf{N}(w_{x\,m-i}^{\ h})y_{m-j}^h\right\|_h \\ \le\ & \frac{b}{2h}\Big[\left(\|w_x^h(t_{m-i})\|_{M,h}^2 + \|w_{x\,m-i}^{\ h}\|_{M,h}^2\right)\|z_{m-j}^h\|_h \\ &+ \left(\|w_x^h(t_{m-i})\|_{M,h} + \|w_{x\,m-i}^{\ h}\|_{M,h}\right)\left(\|w_x^h(t_{m-j})\|_h + \|w_{x\,m-j}^{\ h}\|_h\right)\|z_{m-i}^h\|_h\Big] \\ \le\ & s_1\frac{b}{h}\left(\|z_{m-j}^h\|_h + 2\sqrt{6}\|z_{m-i}^h\|_h\right). \quad (6.5.16)\end{aligned}$$

Therefore, by Proposition 6.4.4 and the definition of α_2,

$$\|z_m^h\|_h \leq \|z_{m-1}^h\|_h + \tau\alpha_2 \left(\|z_m^h\|_h + \|z_{m-1}^h\|_h\right) + 3\tau\|\theta_m^h\|_h. \tag{6.5.17}$$

That is,

$$\begin{aligned}\|z_m^h\|_h &\leq \frac{1+\tau\alpha_2}{1-\tau\alpha_2}\|z_{m-1}^h\|_h + \frac{3\tau}{1-\tau\alpha_2}\|\theta_m^h\|_h \\ &\leq \left(\frac{1+\tau\alpha_2}{1-\tau\alpha_2}\right)^m \|z_0^h\|_h + \tau\frac{3}{1-\tau\alpha_2}\sum_{\ell=0}^{m-1}\left(\frac{1+\tau\alpha_2}{1-\tau\alpha_2}\right)^\ell \|\theta_{m-\ell}^h\|_h \\ &= \left(\frac{1-\tau\alpha_2+2\tau\alpha_2}{1-\tau\alpha_2}\right)^m \|z_0^h\|_h \\ &\quad +\tau\frac{3}{1-\tau\alpha_2}\sum_{\ell=0}^{m-1}\left(\frac{1-\tau\alpha_2+2\tau\alpha_2}{1-\tau\alpha_2}\right)^\ell \|\theta_{m-\ell}^h\|_h. \end{aligned} \tag{6.5.18}$$

Therefore, since $z_0^h = 0$ and by (6.5.13), it follows from the limit of a geometric series that

$$\begin{aligned}\|z_m^h\|_h &\leq \tau\frac{3}{\epsilon_1}\sum_{\ell=0}^{m-1}\left(1+2\tau\frac{\alpha_2}{\epsilon_1}\right)^\ell \|\theta_{m-\ell}^h\|_h \\ &\leq \tau\frac{3}{\epsilon_1}\max_{0\leq\ell\leq m}\|\theta_\ell^h\|_h \sum_{\ell=0}^{m-1}\left(1+2\tau\frac{\alpha_2}{\epsilon_1}\right)^\ell \\ &= \tau\frac{3}{\epsilon_1}\left(\left(1+2\tau\frac{\alpha_2}{\epsilon_1}\right)^m - 1\right)\frac{\epsilon_1}{2\tau\alpha_2}\max_{0\leq\ell\leq m}\|\theta_\ell^h\|_h \\ &\leq \frac{3}{2\alpha_2}\left(1+2t_m\frac{\alpha_2}{m\epsilon_1}\right)^m \max_{0\leq\ell\leq m}\|\theta_\ell^h\|_h \\ &\leq \frac{3}{2\alpha_2}e^{2t_m(\alpha_2/\epsilon_1)}\max_{0\leq\ell\leq m}\|\theta_\ell^h\|_h. \end{aligned} \tag{6.5.19}$$

□

Corollary. *If, for the functions in* (6.3.1),

$$w_{t_j}(t), w_{x_k}(t), \phi_{t_j}(t), \phi_{x_k}(t), \phi_k(t) \in C^2[0,T], \tag{6.5.20}$$

$j = 1, ..., n-1,\ k = 0, ..., n,$ *then*

$$\begin{aligned}\|y^h(t_m) - y_m^h\|_h &\leq \frac{3\tau e^{2t_m(\alpha_2/\epsilon_1)}}{2\alpha_2}\left(m_{y^h,2} + \frac{1}{2}\eta m_{y^h,1}\right. \\ &\quad \left. + s_1\frac{b}{h}\left(\frac{1}{2}+\sqrt{6}\right)m_{w_x^h,1}\right). \end{aligned} \tag{6.5.21}$$

There is now enough information to calculate the error estimate on the difference scheme, which is not mentioned in [PK20]. Instead, an error estimate for the iteration process is also given and then the total error estimate is found.

Error estimate for the iteration process

Recall that it is assumed that y^h_{m-1} is known and has a negligible corresponding error. Also, the algorithm for $y^h_{m,p}$ is given by (6.3.26) and (6.3.25), repeated here for convenience.

$$\mathbf{M}y^h_{m,p} = \mathbf{M}y^h_{m-1} + \tfrac{\tau}{2}\left(\mathbf{L} + \tfrac{1}{2}\left(\mathbf{N}(w^h_{x m,p-1}) + \mathbf{N}(w^h_{x m-1})\right)\right)\left(y^h_{m,p-1} + y^h_{m-1}\right),$$
$$y^h_{m,0} = y^h_{m-1}.$$

Theorem 6.5.3. *Suppose the grid step τ satisfies the inequalities below.*

$$0 < \tau \leq \tfrac{1-\epsilon_2}{\alpha_3}, \tag{6.5.22}$$

$$0 < \tau\left(\eta + \tfrac{3b}{h}\left(1 + \tfrac{8}{3}\tau\tfrac{\alpha_3}{1-q} + 2\left(\tau\tfrac{\alpha_3}{1-q}\right)^2\right)s_2^2\right) \leq \tfrac{2}{3}q, \tag{6.5.23}$$

where ϵ_2 and q are arbitrary numbers from the interval $(0,1)$. *Then*

$$\|y^h_m - y^h_{m,p}\|_h \leq \tau s_2\alpha_3\frac{2}{1-q}q^p, \qquad p = 1, 2, \ldots. \tag{6.5.24}$$

Proof. It follows from (6.3.26) that for $\Delta y^h_{m,p} = y^h_{m,p} - y^h_{m,p-1}$,

$$\begin{aligned}\Delta y^h_{m,p+1} &= \frac{\tau}{2}\mathbf{M}^{-1}\left(\mathbf{L} + \frac{1}{2}\mathbf{N}(w^h_{x m-1})\right)\Delta y^h_{m,p} \\ &\quad + \frac{\tau}{4}\mathbf{M}^{-1}\sum_{\ell=0}^{1}(-1)^\ell\mathbf{N}(w^h_{x m,p-\ell})\left(y^h_{m,p-\ell} + y^h_{n-1}\right).\end{aligned} \tag{6.5.25}$$

Therefore, by Proposition 6.4.5 and the definition of $\mathbf{N}$,

$$\left\|\mathbf{N}(w^h_{x m-1})\Delta y^h_{m,p}\right\|_h \leq \frac{b}{h}\|w^h_{x m-1}\|^2_{M,h}\|\Delta y^h_{m,p}\|_h, \tag{6.5.26}$$

$$\begin{aligned}&\left\|\mathbf{N}(w^h_{x m,p})y^h_{m,p} - \mathbf{N}(w^h_{x m,p-1})y^h_{m,p-1}\right\|_h \\ &\quad\leq \frac{b}{2h}\|\Delta y^h_{m,p}\|_h\left(\sum_{i=0}^{1}\|w^h_{x m,p-i}\|^2_{M,h}\right. \\ &\quad\left. + \sum_{i=0}^{1}\|w^h_{x m,p-i}\|_{M,h}\left(\|w^h_{x m,p}\|_h + \|w^h_{x m,p-1}\|_h\right)\right),\end{aligned} \tag{6.5.27}$$

$$\left\| \left(\mathbf{N}(w_{x\,m,p}^{\,h}) - \mathbf{N}(w_{x\,m,p-1}^{\,h})\right) y_{m-1}^h \right\|_h \\ \leq \frac{b}{h} \sum_{\ell=0}^{1} \|w_{x\,m,p-\ell}^{\,h}\|_{M,h} \|w_{x\,m-1}^{\,h}\|_h \|\Delta y_{m,p}^h\|_h. \tag{6.5.28}$$

Also, by Proposition 6.4.3, it follows that

$$\|w_{x\,m-1}^{\,h}\|_{M,h} \leq \|w_{x\,m-1}^{\,h}\|_h \leq \|y_{m-1}^h\|_h, \tag{6.5.29}$$
$$\|w_{x\,m,p-\ell}^{\,h}\|_{M,h} \leq \|w_{x\,m,p-\ell}^{\,h}\|_h \leq \|y_{m,p-\ell}^h\|_h, \tag{6.5.30}$$
$$\|w_{x\,m,p}^{\,h}\|_{M,h} \leq \|w_{x\,m,p}^{\,h}\|_h \leq \|y_{m,p}^h\|_h. \tag{6.5.31}$$

Therefore

$$\begin{aligned} \|\Delta y_{m,p+1}^h\|_h \;\leq\; & \frac{3\tau}{2}\left(\eta + \frac{b}{2h}\left(\|y_{m-1}^h\|_h^2 + \|y_{m,p}^h\|_h^2 + \|y_{m,p-1}^h\|_h^2\right)\right)\|\Delta y_{m,p}^h\|_h \\ & + \frac{3\tau}{2}\|y_{m-1}^h\|_h\left(\|y_{m,p}^h\|_h + \|y_{m,p-1}^h\|_h\right)\|\Delta y_{m,p}^h\|_h. \\ & + \frac{3\tau}{2}\|y_{m-1}^h\|_h\|y_{m,p}^h\|_h\|y_{m,p-1}^h\|_h\|\Delta y_{m,p}^h\|_h. \end{aligned} \tag{6.5.32}$$

From (6.3.26) and (6.3.25) it follows that

$$\mathbf{M}y_{m,1}^h = \mathbf{M}y_{m-1}^h + \tau\left(\mathbf{L} + \mathbf{N}(w_{x\,m-1}^{\,h})\right) y_{m-1}^h. \tag{6.5.33}$$

Therefore, by Propositions 6.4.7, 6.4.8 and the definition of α_3,

$$\|\Delta y_{m,1}^h\|_h \leq 2\tau s_2 \alpha_3 \quad \text{and} \quad \|y_{m,1}^h\|_h \leq s_2 + 2\tau s_2 \alpha_3. \tag{6.5.34}$$

Let

$$k = 2\tau s_2 \alpha_3. \tag{6.5.35}$$

Then (6.5.23) can be rewritten as

$$0 < \frac{3\tau}{2}\left(\eta + \frac{b}{2h}\sum_{\ell=0}^{2}(\ell+1)s_2^{2-\ell}\left(s_2 + \frac{k}{1-q}\right)^{\ell}\right) \leq q. \tag{6.5.36}$$

Thus, (6.5.32), (6.5.34) and (6.5.36) together with the fact that $\frac{1}{1-q} > 1$ imply that

$$\|\Delta y_{m,2}^h\|_h \leq q\|\Delta y_{m,1}^h\|_h \leq kq. \tag{6.5.37}$$

Therefore, using the triangle inequality, it follows from (6.5.34) and (6.5.37) that

$$\|y_{m,2}^h\|_h \leq \|y_{m,1}^h\|_h + \|\Delta y_{m,2}^h\|_h \leq s_2 + k(1+q). \tag{6.5.38}$$

Using mathematical induction (see [PK20]), (6.5.39) follows for $p \geq 1$:

$$\|\Delta y^h_{m,p}\|_h \leq k q^{p-1}. \tag{6.5.39}$$

This implies that

$$\|y^h_{m,p+\ell} - y^h_{m,p}\|_h \leq \sum_{i=1}^{\ell} \|\Delta y^h_{m,p+i}\|_h \leq k \sum_{i=1}^{\ell} q^{p+i-1} \leq k \frac{q^p}{1-q}. \tag{6.5.40}$$

Therefore, for $p = 0, 1, ...,$ $\ell = 1, 2, ...,$

$$\|\Delta y^h_{m,p+\ell}\|_h \leq 2k \frac{q^{p-1}}{1-q}. \tag{6.5.41}$$

Note that the right hand side of (6.5.41) tends to zero as $p \to \infty$. Hence the sequence $(y^h_{m,p})$, $p = 0, 1, ...,$ is convergent. Thus, letting $p \to \infty$ in Equation (6.3.26) and using Proposition 6.4.5 and the continuity of the matrices $\mathbf{M}$ and $\mathbf{L}$, it follows that

$$\lim_{p\to\infty} y^h_{m,p} = y^h_m.$$

Therefore, by (6.5.40), it follows that for $p = 1, 2, ...,$

$$\|y^h_m - y^h_{m,p}\|_h = \lim_{\ell\to\infty} \|y^h_{m,p+\ell} - y^h_{m,p}\|_h \leq k \frac{q^p}{1-q} = 2\tau s_2 \alpha_3 \frac{q^p}{1-q}. \tag{6.5.42}$$

□

Total error estimate

The error estimates of the previous sections are now used together with the triangle inequality in order to find the total error estimate of the approximation.

In [Amm02] it is mentioned, but not proved, that the regularity of the solution of the linear problem may be increased to $H^3(0,1)$. This is not necessarily true for the nonlinear case. Due to this remark in [Amm02], the authors of [PK20] assume that $w_0(x)$ and $\phi_0(x)$ are in $H^3(0,1)$ and $w_d(x)$ and $\phi_d(x)$ are in $H^2(0,1)$.

Theorem 6.5.4. *Suppose that y is a solution of the weak variational form of the problem given in Section 6.2. In addition, suppose*

$$w_0(x), \phi_0(x) \in H^3(0,1), \quad w_d(x), \phi_d(x) \in H^2(0,1) \tag{6.5.43}$$

and that for the grid step τ *the following inequalities hold:*

$$0 < \tau \leq \tfrac{1-\epsilon_1}{\alpha_2}, \tag{6.5.44}$$

$$0 < \tau \leq \tfrac{1-\epsilon_2}{\alpha_3}, \tag{6.5.45}$$

$$0 < \tau \left(\eta + \tfrac{3b}{h} \left(1 + \tfrac{8}{3} \tau \tfrac{\alpha_3}{1-q} + 2 \left(\tau \tfrac{\alpha_3}{1-q} \right)^2 \right) s_2^2 \right) \leq \tfrac{2}{3} q, \tag{6.5.46}$$

where ϵ_1*,* ϵ_2 *and* q *are arbitrary numbers from the interval* $(0, 1)$*.*

If

$$w_x(x,t) \in C^{\rho,1}([0,1] \times [0,T]), \quad \textit{where} \quad \rho = 1 \ \textit{or} \ \rho = 2, \tag{6.5.47}$$

$$w_t(x,t), \ \phi_t(x,t), \ \phi_x(x,t), \phi(x,t) \in C^{0,1}([0,1] \times [0,T]), \tag{6.5.48}$$

$$w_{t_j}(t), w_{x_k}(t), \phi_{t_j}(t), \phi_{x_k}(t), \phi_k(t) \in C^2[0,T], \tag{6.5.49}$$

for $j = 1, ..., n-1$ *and* $k = 0, ..., n$*. Then, for chosen* τ *and* h *at* $t = t_m$*, the estimate for the total error at the* p^{th} *iteration step is*

$$\begin{aligned} \|\bar{y}(t_m) - y^h_{m,p}\|_h \ \leq \ & \left(3\alpha_1 e^{t_m \alpha_1} t_m^2 + 3t_m\right) \left(2m_{y,1} + \eta m_{y,0} + \frac{b}{h} s_0 m_{y,0} \right) \\ & + \frac{\tau 3 e^{2t_m(\alpha_2/\epsilon_1)}}{2\alpha_2} \Bigg(m_{y^h,2} + \frac{1}{2} \eta m_{y^h,1} \\ & + s_1 \frac{b}{2h} \left(1 + 2\sqrt{6} \right) m_{w^h_x,1} \Bigg) + \tau s_2 \alpha_3 \frac{2}{1-q} q^p. \end{aligned} \tag{6.5.50}$$

If the right hand side of (6.5.50) vanishes as $h \to 0$, then by Proposition 6.4.2 it follows that the approximation converges in the $\mathcal{L}^2(0,1)$ norm.

In [PK20], the result (6.5.50) is given in the form

$$\|\bar{y}(t_m) - y^h_{m,p}\|_h \leq c_1 + c_2 \tau + c_3 q^p. \tag{6.5.51}$$

This is very misleading as the coefficients c_1, c_2 and c_3 all depend on h and do not all vanish as $h \to 0$. In fact, some increase to infinity.

After examining the theorems several times, it was concluded that the error bound calculated in [PK20] is not of any use since it is itself unbounded. To remedy this by finding a bounded error estimate is beyond the scope of this dissertation.

Chapter 7

Existence of a solution for the Sapir-Reiss problem

The article [Amm02] is studied in this chapter. The author investigated the existence of solutions of the nonlinear Timoshenko model of Sapir and Reiss ([SR79]). There is, however, only brief mention of the article [SR79]. Instead, Ammari refers to a few 1991 publications. A standard Timoshenko rod is considered with small vibrations, where it is implicitly assumed that $S_0 = 0$ (that is, the rod is not pre-stressed as in [SR79]). This diminishes the value of the article significantly. A difficult, but exciting undertaking to study this article was anticipated, but instead we were disappointed.

The fact that the definition of the angle ϕ is incorrect in [Amm02] brings into question the author's understanding of the application. Added to this is the absence of explanation as to why natural boundary conditions are treated as forced. Also, the author seems unaware that for a standard as well as an adapted Timoshenko rod, there is a result on the existence of a complete, orthonormal sequence of eigenvectors. This omitted fact is relevant to one of the existence proofs.

The system of partial differential equations and boundary conditions for the Sapir-Reiss model are discussed in Chapter 5 of the dissertation, where an algorithm to implement FEM is derived. Although the weak variational form of the problem is found, no existence theory is discussed. Existence is assumed and the proof postponed to this chapter.

7.1 Local existence result and associated linear problems

Existence results for Problem SLT-SRW (formulated in Section 5.2) are derived in this chapter. The approach is based on the article [Amm02], where the problem is somewhat simplified. It is assumed that $S_0 = 0$ and $f = 0$. Instead of (5.2.3) and (5.2.4), it is assumed in [Amm02] that

$$\sigma(u,v) = b_T(u,v) + \frac{1}{2\gamma}\|Du_1\|^2(Du_1, Dv_1).$$

Problem SLT-SRW Given $u_0 \in \mathcal{V}$ and $u_d \in \mathcal{W}$, find $u \in C^2((0,\tau);\mathcal{W})$ such that for $t > 0$, $u(t) \in \mathcal{V}$, $u'(t) \in \mathcal{V}$ and

$$c_T(u''(t), v) + a(u'(t), v) + \sigma(u(t), v) = 0 \quad \text{for each } v \in \mathcal{V}, \qquad (7.1.1)$$

$$\lim \|(u(t) - u_0)\|_{\mathcal{V}} \to 0,$$
$$\lim \|(u'(t) - u_d)\|_{\mathcal{W}} \to 0.$$

The strategy in [Amm02] is to prove a local existence result. That is, there exists some $\tau > 0$ and a function u such that Equation (7.1.1) is satisfied. Then it must be proved that the solution can be extended. However, to prove the local result, an associated linear problem is solved first.

For greater clarity, two linear problems are considered, Problems 7.1 and 7.2. The local nonlinear problem is then Problem 7.3. It is clearly not necessary to duplicate the complete formulation of the different problems.

Notation J_2: $C^1([0,t];\mathcal{V}) \to C^1([0,t];\mathbb{R})$

$$J_2(t, g(t)) = \int_0^1 [D_x g(t)]^2.$$

Instead of Equation (7.1.1), consider

$$c_T(u''(t), v) + a(u'(t), v) + b_T(u(t), v) + J_2(t, u_1(t))(Du_1(t), Dv_1) = 0 \qquad (7.1.2)$$

for each $v \in \mathcal{V}$.

Problem 7.1 For any $\tau > 0$ and any given $f \in C^1[0,\tau]$, consider, for each $t \in [0,\tau)$,

$$c(u''(t), v) + a(u'(t), v) + b_T(u(t), v) + f(t)(Du_1(t), Dv_1) = 0 \qquad (7.1.3)$$

for each $v \in \mathcal{V}$.

Problem 7.2 For any $\tau > 0$ and any $g \in C^1([0,\tau];\ \overline{T_1(0,1)})$, consider, for each $t \in [0,\tau)$,

$$c(u''(t), v) + a(u'(t), v) + b_T(u(t), v) + J_2(t, g(t))(Du_1(t), Dv_1) = 0 \quad (7.1.4)$$

for each $v \in \mathcal{V}$.

Problem 7.3 There exists some $\tau > 0$ and a function u that solves Equation (7.1.2) on $[0,\tau)$ and satisfies the initial conditions.

To obtain the local existence result the following strategy is followed. First it is proved that Problem 7.1 has a solution. Then, the function $f(t) = J_2(t, g(t))$ is used to prove that Problem 7.2 has a unique solution. This result is then used to prove existence of a solution for Problem 7.3 using fixed point iteration.

Assume the following result for the time being.

Proposition 7.1.1 (Theorem 2.1)**.**
If $u_0 \in E_b$ and $u_d \in \mathcal{V}$, then there exists a unique solution

$$u \in C([0,\tau);\mathcal{V}) \cap C^1((0,\tau);\mathcal{V}) \cap C^2((0,\tau);\mathcal{W})$$

for Problem 7.1 with $u(0) = u_0$ and $u'(0) = u_d$.

Corollary. *If $u_0 \in E_b$ and $u_d \in \mathcal{V}$, then there exists a unique solution*

$$u \in C([0,\tau);\mathcal{V}) \cap C^1((0,\tau);V) \cap C^2((0,\tau);W)$$

for Problem 7.2 with $u(0) = u_0$ and $u'(0) = u_d$.

Define the mapping $\mathcal{A}$: $u_1 = \mathcal{A}g$ where u is the solution in Proposition 7.1.1.

Suppose that B is a subset of $C^1\left([0,\tau];\ \overline{T_1(0,1)}\right)$ such that $\mathcal{A}(B) \subset B$ and $\mathcal{A}$ is a contraction on B. This would imply that a fixed point exists and hence a solution of Problem 7.3.

7.2 The associated linear problems

In this section, the proof of Proposition 7.1.1 (from the previous section) is considered.

In [Amm02], a classical boundary initial value problem is formulated in Section 2, and in Theorem 2.1 it is claimed that a solution exists. The only way to interpret this is that the partial derivatives in the formulation of the problem are weak partial derivatives. Alternatively the author should have stated that "a **weak** derivative exists".

The author of [Amm02] writes the problem in variational form and proves the existence of a weak solution. The approach is standard and can be interpreted as a special case of a more general result in [Eva98, Section 7.2]. The steps in the proofs are as follows.

1. The existence of a basis (e_k) for $\mathcal{V}$ is assumed and functions ξ_k considered such that $u_n = \sum_{k=1}^{n} \xi_k e_k$ is a classical solution of the variational problem.

2. It is proved that the sequences (u_n') and (u_n'') are bounded in $C([0,\tau];\mathcal{V})$ and $C([0,\tau];\mathcal{W})$. Hence, they converge weak* respectively and satisfy the weak variational form in the sense of distributions.

3. The existence part of Proposition 7.1.1 is proved.

4. Gronwall's inequality is used to prove uniqueness.

The "proof" outlined above leaves much to be desired. In Step 2 it should be made clear that the weak* limits are not even functions in $\mathcal{L}^2([0,\tau];\mathcal{L}^2(0,1)^2)$ and in what sense the limit of the sequence (u_n) "satisfies the problem". In Step 3, Ammari claims without any reason that $\lim_{n\to\infty} u_n \in C([0,\tau];H^2(0,1))$ and "... following [Str66] ...", claim to complete the proof in one line. This part of the proof cannot be completed in one sentence as explained below.

To present a proper proof and appreciate the shortcomings in [Amm02], one may consider [Eva98, Section 7.2]. Evans proves existence for the multidimensional wave equation with time-dependent coefficients. First the existence of a weak solution is proved and this part is what is relevant here. Evans defines a bilinear form $B[u,v,t]$ and then the weak problem (without calling it a "weak problem") in the definition of the **weak solution** of the original problem. Problem 7.1 is similar to the problem in [Eva98] and is in some sense a special case. The proof in the book may be used with the relevant minor changes. It is presented on eleven pages and consists of five theorems and their proofs.

7.3 Local existence

The local existence result, Theorem 2.2 in [Amm02, Section 2], is under consideration. In short, the solvability of Problem 7.3 is established.

Theorem 7.3.1 (Theorem 2.2 Part 1).
If $u_0 \in E_b$ and $u_d \in \mathcal{V}$, then there exists some $\tau > 0$ and a unique function

$$u \in C([0,\tau);\mathcal{V}) \cap C^1((0,\tau);\mathcal{V}) \cap C^2((0,\tau);\mathcal{W})$$

that solves Equation (7.1.2)

$$c_T(u''(t), v) + a(u'(t), v) + b_T(u(t), v) + J_2(t, u_1(t))(Du_1(t), Dv_1) = 0$$

on $[0,\tau)$ *with* $u(0) = u_0$ *and* $u'(0) = u_d$.

The proof in [Amm02] is not complete. First, it is necessary to consider two ways to construct a contraction mapping. The viability of both depend on the corollary in the previous section. One possibility is the mapping $\mathcal{A}$ defined after the corollary. In [Amm02], three mappings α_1, α_2 and S are defined as follows. For any functions f and g, let $\alpha_2 f$ be the pair $\langle f, f\rangle$ and $\alpha_2 \langle g, g\rangle = g$. The mapping S is defined by letting $S(\langle g, g\rangle)$ be the solution of Problem 7.2. Finally, $\widetilde{S} = \alpha_1 \circ S \circ \alpha_2$. According to [Amm02], α_1 and α_2 are contractions and “It is easy to see for τ small enough that S is a contraction ... (see for instance [1]).” The problem here is that not only is the existence of a solution of Problem 7.2 required, but also an estimate relating the solution to g.

Ammari ([Amm02]) concludes that $\widetilde{S}$ is a contraction which has a unique fixed point and thus the problem has a unique solution. There is a second part to Theorem 2.2.

Theorem 7.3.2 (Theorem 2.2 Part 2).
Furthermore, at least one of these two affirmations is true:

(a) $\tau = +\infty$,

(b) $\lim_{t\to\tau^-} (\|u(t)\|_{H^2} + \|u'(t)\|_{\mathcal{V}}) = +\infty$.

This statement is not proved at all. The proof of Theorem 7.3.1 is followed by “The proof of the local existence result is now complete.”

If Theorem 7.3.2 is true, then since τ is assumed to be "small enough that S is a contraction," it follows that the only possibility is for part (b) to hold. This must be a serious error, as [Amm02] uses the fact that these norms are uniformly bounded on a maximal interval of existence in the proof of the global existence theorem (Theorem 3.1). The proof described above is clearly not complete.

7.4 Global existence

In Theorem 3.1 of [Amm02, Section 3] it is established that the local solution can be extended to $[0, \infty)$. This is subject to u_0 and u_d satisfying "Condition (3.1)" (see the remark).

Theorem 7.4.1. *If $u_0 \in E_b$, $u_d \in \mathcal{V}$ and u_0 and u_d satisfy "Condition (3.1)", then there exists a unique function*

$$u \in C([0, \infty); \mathcal{V}) \cap C^1((0, \infty); \mathcal{V}) \cap C^2((0, \infty); \mathcal{W})$$

that solves Equation (7.1.2)

$$c_T(u''(t), v) + a(u'(t), v) + b_T(u(t), v) + J_2(t, u_1(t))(Du_1(t), Dv_1) = 0$$

on $[0, \infty)$ *with* $u(0) = u_0$ *and* $u'(0) = u_d$.

Remark. *In essence, "Condition (3.1)" is a bound involving u_0 and u_d and their derivatives.*

Recall from Section 7.3 that [Amm02, Theorem 2] cannot be used to prove [Amm02, Theorem 3.1]. This, however, is not the only shortcoming. It appears that the following result is true:
Suppose $[0, t_m)$ is the maximal interval of existence of the solution and $t_m < \infty$. If $\|u(t)\|_H^2 + \|u(t)\|_V$ is bounded on $[0, t_m)$, then it is a contradiction and the solution can be extended.

This result is used but not stated and, of course, not cited. As in Section 5.1, the theory of ordinary differential equations is considered to motivate why such a general result is probably not true.

Let H be an arbitrary Hilbert space, b a bilinear form on H and f an arbitrary function on H. Consider for each $t > 0$, $u(t) \in H$ and

$$(u''(t), v) + b(u(t), v) + (f(u(t)), v) = 0 \quad \text{for each } v \in H. \tag{7.4.1}$$

Rewritten in a familiar form,

$$u'' + Au + f(u) = 0, \tag{7.4.2}$$

where the linear operator A is defined in such a way that the two problems are equivalent. Suppose a solution is defined on a bounded interval $[0, T)$ and the possibility of an extension is considered. If H is finite dimensional and the range of the pair $\langle u, u' \rangle$ is contained in a bounded subset D of $H \times H$, then $\lim_{t \to T^-} u(t)$ and $\lim_{t \to T^-} u'(t)$ exist and belong to the compact set $\overline{D}$. (The proof is not trivial.) The initial value problem (7.4.2) with

$$u(T) = \lim_{t \to T^-} u(t), \qquad u'(T) = \lim_{t \to T^-} u'(t)$$

can be solved locally to extend the solution.

Model SLT-SR is of the form (7.4.1) or (7.4.2), but the spaces $\mathcal{V}$ and $\mathcal{W}$ are not finite dimensional and hence a closed and bounded set need not be compact. It is concluded that A and f must possess special properties for an extension of a solution to exist. The multi-dimensional wave equation with a nonlinearity is also of the form 7.4.2 (see for example [Eva98]). However, the function f for the examples in the textbook are of a different type to the function in this chapter.

Chapter 8

Conclusion

8.1 Overview

Chapter 1 is an introductory chapter in which the models used throughout the dissertation are introduced and discussed. The original Timoshenko model for a beam is given and written in dimensionless form, forming the linear Model T. Three sets of boundary conditions used throughout the dissertation – pinned-pinned, cantilever and clamped-clamped – are given and the parameters of the model are discussed. The use of the term "rod" as a collective name for beams, cables, wires, etc. is then stated. Next, the possibility of an axial force is considered in the Timoshenko model and the dimensionless equations of motion are given as Model T-AF. Boundary conditions for the additional displacement variable are then stated. In the case of the pinned-pinned or clamped-clamped rod, the rod could be pre-stressed. This is considered in the adapted Timoshenko model, Model AT. Damping is discussed for completion, but is not the main concern in the dissertation. An alternative formulation of the Timoshenko model with axial force is considered using nonlinear theory. This results in the semi-linear Model SLT. The authors of [SR79] make additional assumptions to form the semi-linear Model SLT-SR. In this dissertation Model SLT-SR is considered as one of the special cases in [VDL21].

In Chapter 2, general existence and uniqueness of a solution for a linear model are presented. The standard Timoshenko model problem (Problem T) is used to illustrate the use of the theory. This is done by finding the variational form of the problem, which is shown to be equivalent to the original problem under certain conditions. Sobolev's embedding theorem for a function defined on an interval is also discussed. The theorem is required for application of the theory and is not a special case of the theory for a function defined on $\mathbb{R}^n$. The weak variational form of Problem T (Problem TW) is then found in order for the function spaces in consideration to be complete – a requirement of the theory. The existence and uniqueness theory from [VV02] and [VS19] is then presented for a general Problem GVar. Modal analysis of Problem GVar is also presented and a formal series solution found, which is justified using the energy method given in [CVV18]. The theory of existence of a complete sequence of eigenvectors presented by [CVV18], is also discussed. The general theory is then applied to Problem TW, which has the same form as Problem GVar. Finally, the regularity of the solution guaranteed by the theory is investigated.

The general existence theory and modal analysis discussed in Chapter 2 is applied to the adapted Timoshenko problem (Problem AT) in Chapter 3. In order for this to be done, the weak variational form of Problem AT (Problem ATW) is found. The application of the general theory to Problem ATW is very similar to that of Problem AW. The necessary adaptations are made and it is found that if the axial force S satisfies certain conditions, the general existence theory holds. The existence of a complete sequence of eigenfunctions for the weak variational form of the eigenvalue problem associated with Problem AT (Problem AT EigW) is then shown. This sequence of eigenfunctions has corresponding positive eigenvalues which form an unbounded increasing sequence. Using Sobolev's embedding theorem it is found that the complete sequence of eigenfunctions for Problem AT EigW also satisfies the eigenvalue problem associated with Problem AT. Assuming that the axial force S is constant, it is found that a critical value for S exists which results in the first eigenvalue in the increasing sequence of eigenvalues to be zero. Following [VV06], properties of the sequence of eigenfunctions are then investigated for any combination of pinned, clamped and free boundary conditions, where S is greater than the critical value. The eigenfunctions for a pinned-pinned rod are then calculated and used to generate a series solution. In the case where S is less than or equal to the critical value, a formal series solution can be found, but the procedure cannot necessarily be justified.

In Chapter 4, the solution of Model AT with pinned-pinned boundary conditions is approximated using the finite element method (FEM) and central differences. The approximation is then compared to the series solution found in Chapter 3. In order to implement FEM, the Galerkin approximation is defined using piecewise linear basis functions. However, although the standard Galerkin approximation converges in theory, in practice locking occurs. To avoid this, the mixed finite element method is used in the calculations. The Galerkin approximation is written in matrix notation as a system of ordinary differential equations after which central differences are used to derive an algorithm to approximate the solution. Convergence of the standard FEM for the dynamic problem is studied using [BV13] and [SF73]. References are given for the convergence of the mixed FEM. The first five eigenvalues in the sequence of eigenvalues are found using the theory of Chapter 3 and used to approximate the critical value for S. This is done in order to investigate the behaviour of the model when S nears its critical value. The mixed FEM approximation for the dynamic problem is also found and compared to the series solution to illustrate accuracy.

The semi-linear model from [SR79] (Problem SLT-SR) with pinned-pinned boundary conditions is investigated in Chapter 5. The variational and weak variational forms are derived, but existence of a solution is not discussed as it is dealt with in Chapter 7. Once the Galerkin approximation is found in matrix form, central differences are used to derive an algorithm to approximate the resulting system of ordinary differential equations. These results are compared to the series solution of the linear Model AT with constant axial load (that is, Model ATC) found in Chapter 4. The results where S approaches the critical value such that the first eigenvalue is zero are investigated. This is where possible buckling occurs. It is confirmed that Model ATC does not describe the physical situation in this case, but that Model SLT-SR may be accurate. It is also found that the critical value for Model ATC is not the critical value for Model SLT-SR. An interval where the critical value of Model SLT-SR lies is found, but further investigation is left for future research.

In Chapter 6 the convergence of FEM applied to a model similar to Problem SLT-SR is investigated using [PK20] as a guide. The system is reduced to one of first-order equations before FEM is applied to it. It is found that convergence for FEM applied to this model implies convergence for FEM applied to Problem SLT-SR. An algorithm is derived using piecewise linear basis functions, a Crank-Nicholson type of symmetric difference scheme and a Picard type iteration process. The so-called truncation error is then found and used to calculate error estimates which are added to find a total error estimate. The error estimate found, however, does not imply convergence as [PK20] claims.

In Chapter 7, the existence of a solution for Problem SLT-SR is discussed.

It is presented by [Amm02], but the presentation is shown to be incomplete and contain crucial errors. One of these errors involves the use of the local existence theorem to prove the global existence theorem. The proof in the article follows a pattern used by other researchers. The errors and exclusions are identified and some are rectified.

8.2 Achievements

The aim of the dissertation was to conduct a literature study on modelling, numerical computation and mathematical analysis of a linear and nonlinear Timoshenko model for the vibration of a rod with axial force. This included existence theory and spectral theory.

After introducing and discussing the models used throughout the dissertation, the general linear existence theory was stated and applied to a linear rod model with axial force (the adapted Timoshenko rod model). In order for this to be done, the model considered was written in weak variational form and the required properties shown to hold. Sobolev's embedding theorem – which was also used to apply the theory – was proved for a function defined on an interval. It was discovered that, assuming the axial force S is constant, a critical value for S exists such that if S is less than its critical value, then the required properties of the theory do not hold. The spectral theory for a linear rod model was extended to include an axial force for any combination of pinned, clamped and free boundary conditions, where S is greater than its critical value. This was done while improving on the rigour of the exposition in Van Rensburg and Van der Merwe (2006). The eigenfunctions for a pinned-pinned rod were then calculated and used to generate a series solution. In the case where S was less than or equal to this critical value, a formal series solution could be found, but the procedure could not necessarily be justified.

The finite element method was then applied to the adapted Timoshenko rod model with pinned-pinned boundary conditions and the convergence investigated. The first five eigenvalues in the increasing sequence of eigenvalues were calculated using the spectral theory and a critical value for S was approximated. For illustrative purposes, the series solution was found using the first mode as the initial condition. The approximations found using FEM were then compared to the results of the series solution. An investigation of the solution when S neared its critical value led to the study of a nonlinear model.

The nonlinear model studied was the semi-linear Timoshenko rod model of Sapir and Reiss (1979) with pinned-pinned boundary conditions. The problem was written in weak variational form, but existence of a solution postponed. FEM was applied to the semi-linear model and an original algorithm derived. The results were compared to those of the linear model for small initial displacement where the axial force neared its critical value. Approximations where the axial force surpassed the critical value of the linear model were also investigated and an interval for a critical value of the nonlinear model (less than the critical value of the linear model) was found. This discovery is contrary to the popular belief that the critical values for linear and nonlinear models are the same.

The convergence of the FEM algorithm derived for the semi-linear problem was not studied. An alternative algorithm of Peradze and Kalichava (2020) was considered for convergence instead. This algorithm was more involved than the one derived in the dissertation as it approximated five functions instead of two. Three separate estimates were found for the algorithm and then added to form a "total error estimate". The structure and readability of the article was improved upon and inconsistencies identified.

The existence theory for the Sapir-Reiss semi-linear Timoshenko rod model presented by Ammari (2002) was shown to be incomplete and contain crucial errors. The proof in the article was shown to follow a pattern used by other researchers. The errors and exclusions were identified and some were rectified.

In the articles studied, improvements were made regarding the presentation of the work, connections established and the integrated result written up. In some cases it was necessary to correct and complete work.

8.3 Future research

The investigations for the Sapir-Reiss semi-linear rod show that the critical values for the linear and nonlinear models differ, but are close. This is contrary to the popular belief that they are equal. More reading and investigations are required in this regard.

The convergence of the finite element method for the Sapir-Reiss semi-linear rod investigated by Peradze and Kalichava (2020) considers a rather involved algorithm. Convergence for the algorithm derived in this dissertation, which is simpler, should be investigated.

The existence theory for the semi-linear Timoshenko model of Sapir and Reiss presented in this dissertation is incomplete. Much more research is required.

Finally, the spectral theory of the adapted Timoshenko rod is to be consolidated and written up as a journal article.

Appendix A

Sobolev Space Theory

Unless otherwise stated, the proofs of the results given in this Appendix may be found either in [Eva98] or [OR76]. Both textbooks prove these results for a multi-dimensional space, where the proof for a one-dimensional space is a special case. The exceptional case is where a better result can be obtained in a one-dimensional space than a multi-dimensional space.

Definition (Support of a function)**.**
The closure of the set $\{x \in (0,1) \mid g(x) \neq 0\}$ is referred to as the support of the function g on $(0,1)$.

Let $C_0(0,1)$ denote the functions in $C(0,1)$ with support contained in $(0,1)$. That is, the functions with compact support. Then

$$C_0^\infty(0,1) = C^\infty(0,1) \cap C_0(0,1).$$

Definition (Weak derivatives)**.**
Let m be any positive integer. If $u \in L^2(0,1)$ and there exists a $v \in L^2(0,1)$ such that

$$(u, \phi^{(m)}) = (-1)^m (v, \phi) \quad \text{for each} \quad \phi \in C_0^\infty(0,1)$$

then v is called the m-th order weak derivative of u, denoted $D^m u$.

Note that the m-th order weak derivative $D^m u$ is uniquely determined.

Definition. The set of functions with weak derivatives in $\mathcal{L}^2(0,1)$ up to order m is denoted by $H^m(0,1)$.

Proposition A.0.1. *$C^m[0,1] \subset H^m(0,1)$ and if $u \in C^m[0,1]$, then*

$$D^m u = u^{(m)}.$$

Proposition A.0.2. *$H^m(0,1)$ is a vector space.*

Proposition A.0.3. *The following statements are true.*

(a) If $u \in H^m(0,1)$ and i and j are positive integers such that $i + j \leq m$, then $D^i(D^j u) = D^{i+j}u$.

(b) If $D^k u \in H^m(0,1)$, then $u \in H^{k+m}(0,1)$.

Definition. The bilinear form $(\cdot,\cdot)_m$ is defined by

$$(u,v)_m = (u,v) + (Du,Dv) + \cdots + (D^m u, D^m v).$$

Proposition A.0.4. *The bilinear form $(\cdot,\cdot)_m$ is an inner product for $H^m(0,1)$.*

Definition. Denote the norm for the vector space $H^m(0,1)$ by

$$\|u\|_m = \sqrt{(u,u)_m}.$$

Definition. The Sobolev space $H^m(0,1)$ is the vector space $H^m(0,1)$ with inner product $(\cdot,\cdot)_m$.

Proposition A.0.5. *The space $H^m(0,1)$ is complete.*

Proposition A.0.6. *If $u \in H^m(0,1)$, then there exists a sequence (u_n) contained in $C^m[0,1]$ such that*

$$\|u - u_n\|_m \to 0 \quad as \quad n \to \infty.$$

Proposition A.0.7. *$H^m(0,1)$ is equal to the closure of $C^m[0,1]$ with respect to the $\|\cdot\|_m$ norm.*

Proof. See [OR76, Theorem 4.1] □

The following results are also useful.

Proposition A.0.8 (Poincaré type inequalities)**.** *If $u \in C^1[a,b]$ and u has a zero in $[0,1]$, then*

(a) $\|u\|_{\sup} \leq \sqrt{b-a}\|u'\|$,

(b) $\|u\| \leq (b-a)\|u'\|$.

Proof. By the Fundamental Theorem of Calculus and the Cauchy-Schwartz inequality,

$$|u(x)| \leq \int_a^b |u'| \leq \sqrt{b-a}\|u'\|.$$

Therefore the Poincaré type inequality (a) follows. The inequality in (b) follows from (a) and the fact that $\|u\| \leq \|u\|_{\sup}$. □

Proposition A.0.9. *Suppose* $w \in H^1(0,1)$ *has a zero in* $[0,1]$*. Then*

$$\|w\| \leq \|Dw\|.$$

Proof. By Proposition A.0.7, there exists a sequence (w_n) contained in $C^1[0,1]$ such that $\lim_{n\to\infty} \|w_n\|_1 = \|w\|_1$. That is, such that

$$\lim_{n\to\infty} \|w_n\| = \|w\| \quad \text{and} \quad \lim_{n\to\infty} \|w_n'\| = \|Dw\|.$$

If $Dw = 0$, then $w = 0$ since it is constant and has a zero in $[0,1]$ and the result holds. Suppose $Dw \neq 0$. Then, by Proposition A.0.8,

$$\frac{\|w\|}{\|Dw\|} = \lim_{n\to\infty} \frac{\|w_n\|}{\|w_n'\|} \leq 1.$$

□

Appendix B

Finite Element Method

Suppose the interval $[a, b]$ is divided into N elements with length $h = \frac{b-a}{N}$. For each node $x_i = a + ih$, $i = 0, \dots, N$, the C^0 piecewise linear basis functions are defined as

$$\delta_0(x) = \begin{cases} \frac{x_1 - x}{h} & \text{if } x \in (x_0, x_1) \\ 0 & \text{if } x \notin (x_0, x_1) \end{cases}; \quad \text{(B.0.1)}$$

$$\delta_i(x) = \begin{cases} \frac{x - x_{i-1}}{h} & \text{if } x \in (x_{i-1}, x_i) \\ \frac{x_{i+1} - x}{h} & \text{if } x \in (x_i, x_{i+1}) \\ 0 & \text{if } x \notin (x_{i-1}, x_{i+1}) \end{cases}, i = 1, 2, \dots N-1; \quad \text{(B.0.2)}$$

$$\delta_N(x) = \begin{cases} \frac{x - x_{N-1}}{h} & \text{if } x \in (x_{N-1}, x_N) \\ 0 & \text{if } x \notin (x_{N-1}, x_N) \end{cases}. \quad \text{(B.0.3)}$$

The following matrices are defined.

$$M = (\delta_j, \delta_i)_{0 \le i,j \le N} = \frac{h}{6} \begin{bmatrix} 2 & 1 & & & & & \\ 1 & 4 & 1 & & & & \\ & & \cdot & & & 0 & \\ & & & \cdot & & & \\ & 0 & & & \cdot & & \\ & & & & 1 & 4 & 1 \\ & & & & & 1 & 2 \end{bmatrix}, \quad \text{(B.0.4)}$$

$$K = (\delta_j', \delta_i')_{0\le i,j\le N} = \frac{1}{h}\begin{bmatrix} 1 & -1 & & & & & \\ -1 & 2 & -1 & & & & \\ & & \cdot & & & 0 & \\ & & & \cdot & & & \\ & 0 & & & \cdot & & \\ & & & & -1 & 2 & -1 \\ & & & & & -1 & 1 \end{bmatrix} \tag{B.0.5}$$

and

$$L = (\delta_j, \delta_i')_{0\le i,j\le N} = \frac{1}{2}\begin{bmatrix} -1 & -1 & & & & & \\ 1 & 0 & -1 & & & & \\ & & \cdot & & & 0 & \\ & & & \cdot & & & \\ & 0 & & & \cdot & & \\ & & & & 1 & 0 & -1 \\ & & & & & 1 & 1 \end{bmatrix}. \tag{B.0.6}$$

Depending on the admissible basis functions, corresponding rows and/or columns are deleted.

Bibliography

[ADMPS12] K.T. Andrews, Y. Dumont, M.F. M'Bengue, J. Purcell, M. Shillor. Analysis and simulations of a nonlinear elastic dynamic beam. *Zeitschrift für angewandte Mathematik und Physik ZAMP*, 63(6):1005-1019, 2012.

[Amm02] K. Ammari. Global Existence and Uniform Stabilization of a Nonlinear Timoshenko Beam. *Portugaliae Mathematica*, 59(2): 125-140, 2002.

[Ant96] S. Antman, Dynamical problems for geometrically exact theories of nonlinearly viscoelastic rods, *Journal of Nonlinear Science* 6:1-18, 1996.

[BV13] M. Basson, N. F. J. van Rensburg. Galerkin Finite Element Approximation of General Linear Second Order Hyperbolic Equations. *Numerical Functional Analysis and Optimization*, 34(9): 976-1000, 2013.

[CST16] A. Cazzani, F. Stochino and E. Turco. On the whole spectrum of Timoshenko beams. Part I: a theoretical revisitation. *Zeitschrift für angewandte Mathematik und Physik ZAMP*, 67: 24, 2016.

[CVV18] D. Civin, N.F.J. van Rensburg, A.J. van der Merwe. Using energy methods to compare linear vibration models. *Applied Mathematics and Computation*, 321:602–613, 2018.

[Eva98] L.C. Evans. *Partial Differential Equations.* American Mathematical Society, USA, 1998.

[Fun65] Y.C. Fung. Foundations of Solid Mechanics. Prentice-Hall Inc., Englewood Cliffs, New Jersey, 1965.

[Inm94] D.J. Inman. Engineering Vibration. Prentice-Hall Inc., Englewood Cliffs, New Jersey, 1994.

[LA12] H. Lang, M. Arnold. Numerical aspects in the dynamic simulation of geometrically exact rods, *Applied Numerical Mathematics*, 62:1411–1427, 2012.

[LMR16] C. Lovadina, D. Mora, and R. Rodríguez. Finite Element Analysis of a Bending Moment Formulation for the Vibration Problem of a Non-homogeneous Timoshenko Beam. *J. Sci. Comput.*, 66:825–848, 2016.

[LL91] J.E. Lagnese, G. Leugering. Uniform stabilization of a nonlinear beam by nonlinear boundary feedback. *Journal of Differential Equations*, 91(2): 355-388, 1991.

[LVV09] A. Labuschagne, N.F.J. van Rensburg, A.J. van der Merwe. Comparison of linear beam theories.*Mathematical and Computer Modelling*, 49:20-30, 2009.

[OR76] J.T. Oden, and J.N. Reddy. *An Introduction to the Mathematical theory of Finite Elements*. Wiley, 1976.

[PK20] J. Peradze, Z. Kalichava. A numerical algorithm for the nonlinear Timoshenko beam system. *Numer Methods Partial Differential Eq.*, 36: 1318-1347, 2020.

[Rud64] W. Rudin. Principles of Mathematical Analysis. Mc Graw-Hill, 2nd edition, 1964.

[Sem94] B. Semper. Semi-discrete and fully discrete Galerkin methods for the vibrating Timoshenko beam. *Computer methods in applied mechanics and engineering*, 117:353-360, 1994.

[SF73] G. Strang, G.J. Fix. *An Analysis of the Finite Element Method*. Prentice-Hall Inc., Englewood Cliffs, New Jersey, 1972.

[Sho77] R. E. Showalter. *Hilbert Space Methods for Partial Differential Equations*. Pitman, London, 1977.

[SR79] M. H. Sapir, E. L. Reiss. Dynamic buckling of a nonlinear Timoshenko beam. *SIAM. J. Appl. Math.*,37(2):290–301, 1979.

[Str66] W.A. Strauss. On continuity of functions with values in various banach spaces. *Pacific Journal of Mathematics*, 19(3):543-551, 1966.

[Tim37] H. Timoshenko. *Vibration Problems in Engineering*. D van Nostrand Company, Inc., Second edition, New-York, 1937.

[VDB16] N.F.J. van Rensburg, S. Du Toit, M. Basson. Locally linear Timoshenko beam model. Technical report UPWT 2016/06, University of Pretoria, Pretoria, 2016.

[VDL21] N.F.J. van Rensburg, S. du Toit, M. Labuschagne. Local linear Timoshenko rod. *Acta Mechanica*, DOI 10.1007/s00707-021-03048-8, 2021.

[VS19] N.F.J. Van Rensburg , B. Stapelberg. Existence and uniqueness of solutions of a general linear second-order hyperbolic problem. *IMA Journal of Applied Mathematics*, 84(1): 1-22, 2019.

[VV02] N.F.J. Van Rensburg, A.J. Van der Merwe. Analysis of the Solvability of Linear Vibration Models. *Applicable Analysis*, 81: 1143-1159, 2002.

[VV06] N.F.J. Van Rensburg, A.J. Van der Merwe. Natural frequencies and modes of a Timoshenko beam. *Wave Motion*, 44:58-69, 2006.

[VZV09] N.F.J. Van Rensburg, L. Zietsman, A.J. Van der Merwe. Solvability of a Reissner–Mindlin–Timoshenko plate-beam vibration model. *IMA Journal of Applied Mathematics*, 74(1):149–162, 2009.

[Zei95] E. Zeidler. *Applied Functional Analysis. Applications to Mathematical Physics*. Springer-Verlag. New York Berlin London Paris, 1995.