UNIVERSAL ALGEBRAIC METHODS FOR NON-CLASSICAL LOGICS

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1. Introduction

This expository article is a compilation of universal algebraic prerequisites and tools for the analysis of non-classical logics, with particular (but not exclusive) reference to substructural logics. As such, it is selective and not a fully representative précis of contemporary universal algebra, although it does recount some quite recent developments. It is influenced by standard texts, including [8, 25, 58, 59, 115], and by the more specialized monographs [50, 68, 81].

The history of universal algebra is entwined with that of mathematical logic. Although A.N. Whitehead's 1898 treatise [146] gave the subject its name, it conceived of the common algebra of groups, vector spaces and their relatives in terms of formal deductive systems, and it was only in the 1930s that the first significant results of universal algebra appeared. They include Garrett Birkhoff's characterization of varieties as equationally axiomatized classes and were followed before long by his subdirect decomposition theorem. The promotion of model theory in the middle decades of the twentieth century—particularly by Alfred Tarski in the West and by A.I. Maltsev in the Soviet Union—narrowed the gap between between algebra and logic again, as did Tarski's revival of the algebraic theory of binary relations (see [100]) and his investigations of general deductive systems. By the early 1960s, however, universal algebraists were pursuing problems outside the methodological scope of model theory. Congruence lattices had become an indispensable tool, and results of Bjarni Jónsson had illuminated the structure of congruence distributive varieties. The investigation of general commutators began in the 1970s, with tame congruence theory following in the 1980s. The classification of varieties via Maltsev conditions was by now a major organizational theme. A much fuller history of universal algebra up to the mid-1980s can be found in the introduction to [115].

In parallel with these developments, the diverse assortment of non-classical logics invented during the twentieth century was beginning to undergo some mathematical unification. Kripke's semantics brought order to the study of modal logics in the 1960s and, among other contributions, Ono and Komori's seminal paper [124] of 1985 heralded a general theory of substructural logics for which the most recent advances of universal algebra were once again ripe investigative tools.

By the late 1980s, the discourse of universal algebra had been largely transformed by the work of Ralph McKenzie and his students and collaborators; their influence continues to dominate the subject. One question from the 1950s has remained at the forefront of research, however: *which finite algebras have a finitely axiomatizable equational theory*? This problem is connected in various ways with the sizes of subdirectly irreducible algebras in varieties, another enduring theme of the discipline. A significant new application emerged around the turn of the 21st century, when the theory of Maltsev conditions began to throw fresh light on the tractability of *constraint satisfaction problems* (see for instance [6, 92]). Conversely, the study of such problems has provoked new insights in universal algebra itself, some of which will be mentioned here.

2. Basic Concepts and Results

An *algebra* $\boldsymbol{A} = \langle A; F\rangle$ comprises a non-empty set A (its *universe*) and an indexed family $F = \{f^{\boldsymbol{A}} : f \in \mathcal{F}\}$ of finitary *basic operations* on A. Thus, $\mathcal{F}$ is itself a set (whose elements are called *operation symbols*) and each $f^{\boldsymbol{A}}$ is a function from the cartesian power A^n to A, for some non-negative integer n, called the *rank* of $f^{\boldsymbol{A}}$. By *n–ary*, we mean 'of rank n'. Nullary (i.e., 0–ary) operations are identified with distinguished elements of A, because we identify A^0 with $\{\emptyset\}$.

The function sending each $f \in \mathcal{F}$ to the rank of $f^{\boldsymbol{A}}$ is called the *type* of $\boldsymbol{A}$, and algebras with the same type are said to be *similar*. A *(similarity) type* can be defined without reference to particular algebras as any function from a set $\mathcal{F}$ into the set ω of non-negative integers. Finite types are often abbreviated, e.g., a group $\langle G; \cdot, ^{-1}, e\rangle$ has type $\langle 2, 1, 0\rangle$.

A *subuniverse* of an algebra $\boldsymbol{A}$ is a subset of A, closed under the basic operations of $\boldsymbol{A}$. It becomes a *subalgebra* of $\boldsymbol{A}$ when equipped with the appropriate restrictions of these operations, provided it is not empty, and $\boldsymbol{A}$ is then called an *extension* of this subalgebra. Arbitrary intersections of subuniverses are again subuniverses. *Reducts* of $\boldsymbol{A}$ arise by discarding basic operations, and *subreducts* are subalgebras of indicated reducts. We call $\boldsymbol{A}$ an *expansion* of each of its reducts.

The *direct product* $\prod_{i\in I} \boldsymbol{A}_i$ of a family $\{\boldsymbol{A}_i : i \in I\}$ of similar algebras is their cartesian product, on which the appropriate basic operations are defined in terms of those of the algebras $\boldsymbol{A}_i$ in the obvious co-ordinatewise fashion. Products of empty families are understood to have universe $\{\emptyset\}$.

A *homomorphism* $h\colon \boldsymbol{A} \longrightarrow \boldsymbol{B}$ between similar algebras is a function that preserves the basic operations $f^{\boldsymbol{A}}$ of $\boldsymbol{A}$, in the sense that

$$h(f^{\boldsymbol{A}}(\vec{a})) = f^{\boldsymbol{B}}(h(a_1), \dots, h(a_n)) \text{ for all } \vec{a} = a_1, \dots, a_n \in A,$$

where n is the rank of $f^{\boldsymbol{A}}$. We call h an *embedding* if it is also injective, and an *isomorphism* if it is bijective. As usual, $h\colon \boldsymbol{A} \cong \boldsymbol{B}$ signifies that h is an isomorphism. The target of a surjective [bijective] homomorphism is called a *homomorphic* [*isomorphic*] *image* of the domain. An *endomorphism* [*automorphism*] of $\boldsymbol{A}$ is a homomorphism [isomorphism] $h\colon \boldsymbol{A} \longrightarrow \boldsymbol{A}$. If $h\colon \boldsymbol{A} \longrightarrow \boldsymbol{B}$ is a homomorphism, and $\boldsymbol{C}$ and $\boldsymbol{D}$ are subalgebras of $\boldsymbol{A}$ and $\boldsymbol{B}$, respectively, then $h[C]$ and $h^{-1}[D]$ are subuniverses of $\boldsymbol{B}$ and $\boldsymbol{A}$, respectively. The corresponding subalgebras are denoted by $h[\boldsymbol{C}]$ and $h^{-1}[\boldsymbol{D}]$.

A class K of similar algebras is called a *variety* if it is closed under the class operators $\mathbb{H}$, $\mathbb{S}$ and $\mathbb{P}$ (standing for homomorphic images, subalgebras and direct products). The smallest such class containing K is $\mathbb{HSP}(\mathsf{K})$ [138]. The class operator for isomorphic images will be denoted by $\mathbb{I}$. The earliest result of universal algebra *not* predicted by classical algebra is *Birkhoff's Theorem* below; its proof will be discussed in Section 3.

Theorem 2.1. (Birkhoff [12]) *A class of similar algebras is a variety iff it can be axiomatized by equations.*

For example, groups $\langle G; \cdot, ^{-1}, e\rangle$ form a variety, because they are exactly the algebras of type $\langle 2, 1, 0\rangle$ satisfying the equations

$$x \cdot (y \cdot z) \approx (x \cdot y) \cdot z, \quad x \cdot e \approx x \approx e \cdot x, \quad x \cdot x^{-1} \approx e \approx x^{-1} \cdot x.$$

Here, $\approx$ is just formalized equality; it prevents notational clashes with $=$.

An algebra $\boldsymbol{A}$ is *generated* by a subset X of A if no proper subalgebra of $\boldsymbol{A}$ contains X. In this case, if m is any cardinal $\geq |X|$, we say that $\boldsymbol{A}$ is *m–generated* (so that 'finitely generated' means 'm–generated for some finite m'). If X generates $\boldsymbol{A}$ and $h\colon \boldsymbol{A} \longrightarrow \boldsymbol{B}$ is a homomorphism, then $h[X]$ generates $h[\boldsymbol{A}]$.

An algebra is said to be *finite* if it has a finite universe. A variety K is *locally finite* if each of its finitely generated members is finite. A surprising amount of information about the structure of K can then be inferred if we know that, for some polynomial p and all $m \in \omega$, K contains at most $p(m)$, or even at most $2^{p(m)}$, non-isomorphic m–generated algebras, see [11, 70].

Given a class K of algebras similar to $\boldsymbol{A}$, we say that $\boldsymbol{A}$ is K*–free over* X if X generates $\boldsymbol{A}$ and every function from X into an algebra $\boldsymbol{B} \in \mathsf{K}$ can be extended to a homomorphism from $\boldsymbol{A}$ to $\boldsymbol{B}$. (The extension is then unique.) In this case, we call X a K*–free generating set* for $\boldsymbol{A}$. Provided that $X \neq \emptyset$ or that some nullary basic operations are available, there *is* a K–free algebra $\boldsymbol{A}$ over X, with $\boldsymbol{A} \in \mathbb{ISP}(\mathsf{K})$ (see Section 3). Any bijection from X to a K–free generating set for another K–free algebra $\boldsymbol{C} \in \mathbb{HSP}(\mathsf{K})$ extends to a unique isomorphism from $\boldsymbol{A}$ to $\boldsymbol{C}$. We therefore denote $\boldsymbol{A}$ by $\boldsymbol{F}_{\mathsf{K}}(X)$, or by $\boldsymbol{F}_{\mathsf{K}}(m)$ if $m = |X|$. It follows that $\boldsymbol{F}_{\mathsf{K}}(X) \in \mathsf{K}$ whenever K is a variety.

If we employ the elements of $\boldsymbol{B} \in \mathsf{K}$ (or those of a generating set for $\boldsymbol{B}$) as free generators for a K–free algebra $\boldsymbol{F} \in \mathbb{ISP}(\mathsf{K})$ and then map these back to themselves, we obtain a surjective homomorphism $\boldsymbol{F} \longrightarrow \boldsymbol{B}$. This yields the following result, which is used in the proof of Birkhoff's Theorem.

Theorem 2.2. ([12]) *Let* K *be a variety. Then every algebra in* K *is a homomorphic image of a* K–free *algebra in* K. *In fact, for any cardinal* *m, every m–generated algebra in* K *is a homomorphic image of* $\boldsymbol{F}_{\mathsf{K}}(m)$*, provided that* $\boldsymbol{F}_{\mathsf{K}}(m)$ *exists.*

A variety is said to be *finitely generated* if it has the form $\mathbb{HSP}(\boldsymbol{A})$ for some finite algebra $\boldsymbol{A}$ (or equivalently, $\mathbb{HSP}(\mathsf{M})$ for some finite set M of finite algebras). Boolean algebras, distributive lattices and semilattices (i.e., idempotent commutative semigroups) form finitely generated varieties, generated in each case by the unique 2–element member of the class. If $\mathsf{K} = \mathbb{HSP}(\boldsymbol{A})$, then $\boldsymbol{F}_{\mathsf{K}}(m)$ can be embedded into the direct power $\boldsymbol{A}^{A^m} = \prod_{i \in A^m} \boldsymbol{A}$, for every cardinal m. This, with Theorem 2.2, yields:

Corollary 2.3. *Every finitely generated variety is locally finite.*

A *congruence* (relation) on an algebra $\boldsymbol{A}$ is the *kernel*

$$\{\langle a, b\rangle \in A^2 : h(a) = h(b)\}$$

of a homomorphism h with domain $\boldsymbol{A}$, i.e., it is an equivalence relation θ on A, *compatible* with each basic operation $f^{\boldsymbol{A}}$ of $\boldsymbol{A}$ in the sense that whenever $a_k \equiv_\theta b_k$ (i.e., $\langle a_k, b_k\rangle \in \theta$) for $k = 1, \dots, n$, then

$$f^{\boldsymbol{A}}(a_1, \dots, a_n) \equiv_\theta f^{\boldsymbol{A}}(b_1, \dots, b_n),$$

n being the rank of $f^{\boldsymbol{A}}$. This compatibility demand means that θ is a subuniverse of the direct square $\boldsymbol{A}^2 = \boldsymbol{A} \times \boldsymbol{A}$. Thus, the set A/θ of equivalence classes a/θ ($a \in A$) becomes a (factor) algebra $\boldsymbol{A}/\theta$ of the same type as $\boldsymbol{A}$, under the unambiguous natural definition of the basic operations: $f^{\boldsymbol{A}/\theta}(a_1/\theta, \dots, a_n/\theta) = f^{\boldsymbol{A}}(a_1, \dots, a_n)/\theta$. Moreover:

Theorem 2.4. *If* $h \colon \boldsymbol{A} \longrightarrow \boldsymbol{B}$ *is a homomorphism, then* $\boldsymbol{A}/\theta \cong h[\boldsymbol{A}]$*, where* θ *is the kernel of* h*. The isomorphism identifies each* a/θ *with* $h(a)$*.*

This is the *Homomorphism Theorem* (or *First Isomorphism Theorem*) of universal algebra.

An algebra $\boldsymbol{A}$ is said to be *simple* if it has just two congruences. These must be the identity relation $\mathrm{id}_A = \{\langle a, a\rangle : a \in A\}$ and the total relation A^2. In this case, $\boldsymbol{A}$ has no nontrivial homomorphic image, other than isomorphic images, by the previous result. (An algebra is called *trivial* if its universe is a singleton. A class of algebras is *nontrivial* if it has a nontrivial member.) Every nontrivial variety contains a simple algebra [101].

Theorem 2.5. (Second Isomorphism Theorem) *If* θ *and* φ *are congruences of an algebra* $\boldsymbol{A}$*, with* $\theta \subseteq \varphi$*, then* $\varphi/\theta := \{\langle a/\theta, b/\theta\rangle : \langle a, b\rangle \in \varphi\}$ *is a congruence of* $\boldsymbol{A}/\theta$ *and* $(\boldsymbol{A}/\theta)/(\varphi/\theta) \cong \boldsymbol{A}/\varphi$*.*

This follows from Theorem 2.4, because $a/\theta \mapsto a/\varphi$ ($a \in A$) is a well-defined homomorphism from $\boldsymbol{A}/\theta$ onto $\boldsymbol{A}/\varphi$, whose kernel is φ/θ.

Arbitrary intersections of congruences are again congruences, so the set of congruences of an algebra $\boldsymbol{A}$ becomes a lattice $\mathbf{Con}\,\boldsymbol{A} = \langle \mathrm{Con}\,\boldsymbol{A}; \wedge, \vee\rangle$, when ordered by inclusion. The meet $\theta \wedge \varphi$ of two congruences is $\theta \cap \varphi$, while their join $\theta \vee \varphi$ is the transitive closure of $\theta \cup \varphi$. In fact, $\mathbf{Con}\,\boldsymbol{A}$ is a complete sublattice of the lattice of equivalence relations on A.

Theorem 2.6. (Correspondence Theorem) *For any congruence* θ *of an algebra* $\boldsymbol{A}$, *the interval sublattice* $[\![\theta, A^2]\!] := \{\varphi \in \mathrm{Con}\,\boldsymbol{A} : \theta \subseteq \varphi\}$ *of* $\mathbf{Con}\,\boldsymbol{A}$ *is isomorphic to* $\mathbf{Con}\,\boldsymbol{A}/\theta$ *under the map* $\varphi \mapsto \varphi/\theta$.

An abiding insight of universal algebra is that the congruence lattices of algebras in a well-behaved variety contain much structural information about the algebras themselves. The smallest congruence of $\boldsymbol{A}$ containing a set $X \subseteq A^2$ is denoted by $\Theta^{\boldsymbol{A}}X$. We write $\Theta^{\boldsymbol{A}}\{\langle a,b\rangle\}$ as $\Theta^{\boldsymbol{A}}(a,b)$.

Lemma 2.7. *Every* finitely generated *congruence*

$$\theta = \Theta^{\boldsymbol{A}}\{\langle a_1, b_1\rangle, \ldots, \langle a_n, b_n\rangle\}$$

of an algebra $\boldsymbol{A}$ *is* compact *in* $\mathbf{Con}\,\boldsymbol{A}$, *i.e., whenever* $\theta \subseteq \Theta^{\boldsymbol{A}}X$, *then* $\theta \subseteq \Theta^{\boldsymbol{A}}Y$ *for some finite* $Y \subseteq X$.

Conversely, compact congruences are finitely generated, so $\mathbf{Con}\,\boldsymbol{A}$ is an *algebraic* lattice—i.e., a complete lattice in which every element is the join of a set of compact elements.

A *subdirect product* $\boldsymbol{B}$ of a family $\{\boldsymbol{A}_i : i \in I\}$ of similar algebras is a subalgebra of their direct product, such that each of the natural projection homomorphisms $\pi_j \colon \prod_{i\in I} \boldsymbol{A}_i \longrightarrow \boldsymbol{A}_j$ $(j \in I)$ restricts to a surjection from B to A_j (so each $\boldsymbol{A}_j$ is a homomorphic image of $\boldsymbol{B}$). An embedding $h \colon \boldsymbol{A} \longrightarrow \prod_{i\in I} \boldsymbol{A}_i$ is called a *subdirect embedding* if $h[\boldsymbol{A}]$ is a subdirect product of $\{\boldsymbol{A}_i : i \in I\}$.

An algebra $\boldsymbol{A}$ is *subdirectly irreducible* if its identity congruence is not the intersection of any family of non-identity congruences of $\boldsymbol{A}$—i.e., id_A is *completely meet-irreducible* in the lattice $\mathbf{Con}\,\boldsymbol{A}$. This amounts to the demand that any subdirect embedding h of $\boldsymbol{A}$ into a product $\prod_{i\in I} \boldsymbol{A}_i$ is trivial—in the sense that $\pi_j h \colon \boldsymbol{A} \cong \boldsymbol{A}_j$ for some $j \in I$. A subdirectly irreducible algebra is therefore nontrivial; its smallest non-identity congruence is called its *monolith*.

Obviously, all simple algebras—and in particular, all 2–element algebras—are subdirectly irreducible. A variety is said to be *semisimple* if its subdirectly irreducible members are simple. This applies to the varieties of Boolean algebras, of distributive lattices and of semilattices. In these three varieties, an algebra $\boldsymbol{A}$ is subdirectly irreducible iff $|A| = 2$.

Birkhoff's *subdirect decomposition theorem* is the following result (which relies on the axiom of choice).

Theorem 2.8. (Birkhoff [13]) *Every algebra* $\boldsymbol{A}$ *is isomorphic to a subdirect product of subdirectly irreducible algebras.*

As the subdirectly irreducible algebras in this statement are homomorphic images of $\boldsymbol{A}$, they belong to every variety containing $\boldsymbol{A}$. This yields:

Corollary 2.9. *Every variety* K *is determined by its class* K_{si} *of subdirectly irreducible members—in fact* $\mathsf{K} = \mathbb{IP}_{\mathbb{S}}(\mathsf{K}_{\mathrm{si}})$.

Here, $\mathbb{P}_{\mathbb{S}}$ is the class operator corresponding to subdirect products. For any class K of similar algebras, we have $\mathbb{SP}(\mathsf{K}) \subseteq \mathbb{P}_{\mathbb{S}}\mathbb{S}(\mathsf{K})$. Less obviously, $\mathbb{S}(\mathsf{K}) \subseteq \mathbb{HP}_{\mathbb{S}}(\mathsf{K})$, whence $\mathbb{HSP}(\mathsf{K}) = \mathbb{HP}_{\mathbb{S}}(\mathsf{K})$ (see [90] and [29, p. 171]).

The subdirect decomposition theorem is important because it is not generally possible to decompose algebras as *direct* products of *directly* indecomposable algebras. Instead, the building blocks of a variety are its *subdirectly* irreducible members. In fact, Theorem 2.8 instantiates a purely lattice-theoretic result: in an algebraic lattice (such as $\mathbf{Con}\,\boldsymbol{A}$), every element is the meet of a set of completely meet-irreducible elements.

3. Terms and Term Operations

Universal algebraists are typically more concerned with *compositions* of operations than with basic operations. Let $\rho\colon \mathcal{F} \longrightarrow \omega$ be a similarity type and X a set of objects called *variables*. The set $T = T_\rho(X)$ of all *terms of type* ρ (over X) is defined recursively as follows: $X \subseteq T$ and if $f \in \mathcal{F}$ with $\rho(f) = n$ and $t_1, \dots, t_n \in T$, then $f(t_1, \dots, t_n) \in T$. Here, $f(t_1, \dots, t_n)$ is just a formal string of symbols. However, T is naturally the universe of an algebra $\boldsymbol{T} = \boldsymbol{T}_\rho(X)$, the *term algebra* over X, whose basic operations are the functions $f^{\boldsymbol{T}}\colon \langle t_1, \dots, t_n\rangle \mapsto f(t_1, \dots, t_n)$, $f \in \mathcal{F}$. Nullary operation symbols of ρ are identified with elements of T, so $\boldsymbol{T}$ exists unless $X = \emptyset$ and ρ includes no nullary symbol. In fact, $\boldsymbol{T} \cong \boldsymbol{F}_{\mathsf{K}_\rho}(X)$, where K_ρ is the class of all algebras of type ρ, and we sometimes denote $\boldsymbol{T}$ as $\boldsymbol{T}(k)$, where $k = |X|$.

For $m \in \omega$ and $\vec{x} = x_1, \dots, x_m \in X$, the expression '$t(\vec{x}) \in T$' signifies that $t \in T$ and that the variables occurring in t are *among* $x_1, \dots, x_m$. Every such expression gives rise, in each algebra $\boldsymbol{A}$ of type ρ, to an m–ary *term operation* $t(\vec{x})^{\boldsymbol{A}}\colon A^m \longrightarrow A$ (abbreviated as $t^{\boldsymbol{A}}$ when $\vec{x}$ is understood), which is also defined recursively: if t is x_i, then $t^{\boldsymbol{A}}$ is the i th projection $\pi_i\colon A^m \longrightarrow A$; if $t_j^{\boldsymbol{A}}\colon A^m \longrightarrow A$ is defined for $j = 1, \dots, n$ and t is $f(t_1, \dots, t_n) \in T$, where $f \in \mathcal{F}$, then $t^{\boldsymbol{A}}(\vec{a}) := f^{\boldsymbol{A}}(t_1^{\boldsymbol{A}}(\vec{a}), \dots, t_n^{\boldsymbol{A}}(\vec{a}))$ for all $\vec{a} \in A^m$. The expression $t(\vec{x})$ is sometimes called an *m–ary term*, even if $x_1, \dots, x_m$ don't all occur in t. In an Abelian group $\boldsymbol{G}$, for instance, if $z + ((x + y) + (-z))$ is written as $t(x, y, z, w)$, then the corresponding term operation $t^{\boldsymbol{G}}\colon G^4 \longrightarrow G$ is given by $t^{\boldsymbol{G}}(a, b, c, d) = a +^{\boldsymbol{G}} b$.

The compatibility of congruences and homomorphisms with basic operations extends inductively to term operations. In any algebra $\boldsymbol{A}$, the smallest subuniverse containing a subset B consists of all $t^{\boldsymbol{A}}(\vec{b})$ such that $t^{\boldsymbol{A}}$ is a term operation of $\boldsymbol{A}$ and $\vec{b}$ a tuple of elements of B, whose length is the rank of $t^{\boldsymbol{A}}$. (Here and whenever the set of variables is not specified, it is assumed to be infinite.)

For any nontrivial class K of algebras of type ρ and any set X for which $\boldsymbol{T} = \boldsymbol{T}_\rho(X)$ exists, we define

$$\Phi_{\mathsf{K}}(X) := \{\varphi \in \mathbf{Con}\,\boldsymbol{T} : \boldsymbol{T}/\varphi \in \mathbb{IS}(\mathsf{K})\} \text{ and } \theta = \theta_{\mathsf{K}}(X) := \bigcap \Phi_{\mathsf{K}}(X).$$

The map $x \mapsto \overline{x} := x/\theta$ ($x \in X$) is injective, as K is nontrivial. In fact, we may identify $\boldsymbol{F} = \boldsymbol{F}_{\mathsf{K}}(|X|)$ with $\boldsymbol{T}/\theta$, because $\boldsymbol{T}/\theta$ can be shown K–free over $\{\overline{x} : x \in X\}$ and it belongs to $\mathbb{ISP}(\mathsf{K})$ (apply Theorem 2.4 to the map $t \mapsto \langle t/\varphi : \varphi \in \Phi_{\mathsf{K}}(X)\rangle$). Then, given $t(x_1, \ldots, x_m) \in T$, we write $\overline{t}$ for the element t/θ of $\boldsymbol{F}$, i.e., $\overline{t} = t^{\boldsymbol{F}}(\overline{x}_1, \ldots, \overline{x}_m)$. For $s, t \in T$, we can show that

(1) $\overline{s} = \overline{t}$ iff (every algebra in) K satisfies the equation $s \approx t$,

using the fact that $\boldsymbol{F}$ is K–free over $\{\overline{x} : x \in X\}$.

The nontrivial half of Birkhoff's Theorem 2.1 can now be explained as follows. Let K be a variety and Σ the *equational theory* of K, i.e., the set of equations (over some infinite set of variables) that are satisfied by K. Let K' be the class of all *models of* Σ, i.e., all algebras of K's type satisfying Σ. (Of course, any two infinite sets of variables produce the same K' from K.) Now $\mathsf{K} \subseteq \mathsf{K}'$ and these two classes satisfy the same equations. So, if we construct free algebras from term algebras in the above manner, then (1) ensures that $\boldsymbol{F}_{\mathsf{K}'}(X) = \boldsymbol{F}_{\mathsf{K}}(X)$ for every infinite X. Thus, each $\boldsymbol{A} \in \mathsf{K}'$ belongs to $\mathbb{H}(\boldsymbol{F}_{\mathsf{K}}(X))$ for a sufficiently large X, by Theorem 2.2. But $\boldsymbol{F}_{\mathsf{K}}(X) \in \mathbb{ISP}(\mathsf{K})$, and so $\mathsf{K}' \subseteq \mathbb{HSP}(\mathsf{K}) = \mathsf{K}$, i.e., $\mathsf{K} = \mathsf{K}'$. This means that K is axiomatized by the equations in Σ.

The following generalization of (1) is useful in establishing 'Maltsev conditions' (see the proofs of Theorems 4.1 and 5.9 below).

Lemma 3.1. *A variety* K *satisfies a quasi-equation*

(2) $$(s_1(\vec{x}) \approx t_1(\vec{x}) \;\&\; \ldots \;\&\; s_m(\vec{x}) \approx t_m(\vec{x})) \implies s(\vec{x}) \approx t(\vec{x})$$

iff $\Theta^{\boldsymbol{F}}(\overline{s}, \overline{t}) \subseteq \Theta^{\boldsymbol{F}}\{\langle \overline{s}_i, \overline{t}_i\rangle : i = 1, \ldots, m\}$, *where* $\boldsymbol{F}$ *is as above and* X *includes the variables* $\vec{x}$. *In this case, for every* $\boldsymbol{A} \in \mathsf{K}$ *and* $\vec{a} \in A$, *we have*

$$\Theta^{\boldsymbol{A}}(s^{\boldsymbol{A}}(\vec{a}), t^{\boldsymbol{A}}(\vec{a})) \subseteq \Theta^{\boldsymbol{A}}\{\langle s_i^{\boldsymbol{A}}(\vec{a}), t_i^{\boldsymbol{A}}(\vec{a})\rangle : i = 1, \ldots, m\}.$$

Note that quasi-equations have finite length. In fact, (2) is really a universally quantified first order sentence, with suppressed quantifiers on the left. (Readers are assumed to have encountered the basic definitions of first order logic—such as that a sentence is a formula with no free variable. Alternatively, see the concise account in [25, Sec. V.1].)

When ordered by inclusion, the varieties of type ρ form a lattice $\boldsymbol{\Lambda}(\rho)$, in which the meet of two varieties K and M is their intersection; the join is $\mathbb{HSP}(\mathsf{K} \cup \mathsf{M})$. In fact, $\boldsymbol{\Lambda}(\rho)$ is isomorphic to the dual of the (algebraic) lattice of *fully invariant congruences* of $\boldsymbol{T} = \boldsymbol{T}_\rho(\aleph_0)$—i.e., congruences θ such that whenever $s \equiv_\theta t$ then $h(s) \equiv_\theta h(t)$ for every endomorphism h of $\boldsymbol{T}$. For this reason, $\boldsymbol{\Lambda}(\rho)$ can be treated as a set. The isomorphism identifies each fully invariant $\theta \in \operatorname{Con} \boldsymbol{T}$ with $\mathbb{HSP}(\boldsymbol{T}/\theta)$. It allows us to show that every nontrivial variety has a *minimal* nontrivial subvariety (Zorn's Lemma applies), and that a variety with m operation symbols has at most $2^{\max\{m,\aleph_0\}}$ subvarieties, where m is any cardinal.

The *polynomial operations* of an algebra $\boldsymbol{A} = \langle A; F\rangle$ are the term operations of $\langle A; F \cup F_0\rangle$, where F_0 consists of the elements of A, considered as nullary operations. In particular, constant functions $A^k \longrightarrow A$ are polynomial operations of $\boldsymbol{A}$, for every $k \in \omega$. Of course, $\boldsymbol{A}$ and $\langle A; F \cup F_0 \rangle$ have the same congruences.

Definition 3.2. A finite nontrivial algebra $\boldsymbol{A}$ is said to be *primal* [*functionally complete*] if, for each $k \in \omega$, every function $f \colon A^k \longrightarrow A$ is a term [polynomial] operation of $\boldsymbol{A}$.

If $\boldsymbol{A}$ is primal, then $\boldsymbol{F}_{\mathbb{HSP}(\boldsymbol{A})}(n) \cong \boldsymbol{A}^{|A|^n}$ for all $n \in \omega$. The canonical example is the 2–element Boolean algebra, cf. Theorem 13.4. The n–valued Post algebras are also primal, see [25, pp. 26, 174].

Definition 3.3. The *discriminator* $d \colon A^3 \longrightarrow A$ of a set A is defined by

$$d(a,b,c) = a \text{ if } a \neq b, \text{ and } d(a,a,c) = c.$$

A *discriminator term* for a class K of similar algebras is a term $t(x,y,z)$ such that for every $\boldsymbol{A} \in \mathsf{K}$, the operation $t^{\boldsymbol{A}}$ is the discriminator of A. In this case, $\mathbb{HSP}(\mathsf{K})$ is called a *discriminator variety.*

Clearly, any primal algebra $\boldsymbol{A}$ generates a discriminator variety; in fact, $\mathbb{HSP}(\boldsymbol{A}) = \mathbb{IP}_{\mathbb{S}}(\boldsymbol{A})$ [45]. The class of cylindic algebras of any fixed finite dimension is another example of a discriminator variety, see [25, p. 165]. In a class of algebras with a discriminator term, many quantifier-free first order formulas are equivalent to equations, and this makes the concept useful (see for instance [77]). Discriminator varieties are semisimple; actually, their members are 'Boolean products' of simple ones. The Boolean product construction will not be discussed here, but is described in [25, Sec. IV.8–9].

A finite nontrivial algebra is functionally complete iff its discriminator is a polynomial operation, in which case, if it has no proper subalgebra, then it has a discriminator term (see [8, Chap. 7]). We shall return to these notions in Theorems 4.5, 6.10, 6.11 and 8.4. For more information about polynomial operations, see [78].

4. Permutability of Congruences

The *relational product* $\theta \circ \varphi$ of binary relations θ and φ on the universe of an algebra $\boldsymbol{A}$ is defined as follows. For $a, b \in A$,

$$a \equiv_{\theta \circ \varphi} b \text{ iff } a \equiv_\theta c \text{ and } c \equiv_\varphi b \text{ for some } c \in A.$$

If θ and φ are congruences, then $\theta \circ \varphi$ is a reflexive subuniverse of $\boldsymbol{A} \times \boldsymbol{A}$ and the following conditions are equivalent:

$$\theta \circ \varphi = \varphi \circ \theta, \quad \theta \circ \varphi \subseteq \varphi \circ \theta, \quad \theta \circ \varphi = \theta \vee \varphi,$$

where $\theta \vee \varphi$ is the join $\Theta^{\boldsymbol{A}}(\theta \cup \varphi)$ in the lattice $\mathbf{Con}\,\boldsymbol{A}$. We say that $\boldsymbol{A}$ is *congruence* (2–) *permutable* if these conditions hold for all $\theta, \varphi \in \mathbf{Con}\,\boldsymbol{A}$. It is *congruence n–permutable* if, instead, the congruences of $\boldsymbol{A}$ satisfy

$$\theta \circ \varphi \circ \theta \circ \varphi \circ \ldots \;=\; \varphi \circ \theta \circ \varphi \circ \theta \circ \ldots$$

(with n factors on each side). Note that n–permutability implies $(n+1)$–permutability. A class of algebras is *congruence* $[n\text{–}]$ *permutable* if its members are. Congruence permutable varieties are often called *Maltsev varieties*, because of the next result, which is the prototype for 'Maltsev conditions'.

Theorem 4.1. (Maltsev [103]) *A variety* K *is congruence permutable iff there is a term* $q(x,y,z)$ *such that* K *satisfies the equations*

$$q(x,y,y) \approx x \text{ and } q(x,x,y) \approx y.$$

The implication from left to right illustrates a general method. Because K is a variety, $\boldsymbol{F} = \boldsymbol{F}_{\mathsf{K}}(3) = \boldsymbol{T}_\rho(X)/\theta_{\mathsf{K}}(X) \in \mathsf{K}$, where $X = \{x,y,z\}$. Therefore, $\boldsymbol{F}$ is congruence permutable. Set $\theta = \Theta^{\boldsymbol{F}}(\overline{x},\overline{y})$ and $\varphi = \Theta^{\boldsymbol{F}}(\overline{y},\overline{z})$. Then $\langle \overline{x},\overline{z}\rangle \in \theta \circ \varphi = \varphi \circ \theta$, so there exists $q(x,y,z) \in T_\rho(X)$ (and $\overline{q} = q^{\boldsymbol{F}}(\overline{x},\overline{y},\overline{z})$ in $\boldsymbol{F}$) such that $\langle \overline{x},\overline{q}\rangle \in \varphi$ and $\langle \overline{q},\overline{z}\rangle \in \theta$. By Lemma 3.1, K satisfies both $y \approx z \implies x \approx q(x,y,z)$ and $x \approx y \implies q(x,y,z) \approx z$. That is, K satisfies $q(x,y,y) \approx x$ and $q(x,x,z) \approx z$.

Conversely, if $a \equiv_\theta c \equiv_\varphi b$ for some $\theta,\varphi \in \operatorname{Con}\boldsymbol{A}$, where $\boldsymbol{A} \in \mathsf{K}$, then $a = q^{\boldsymbol{A}}(a,c,c) \equiv_\varphi q^{\boldsymbol{A}}(a,c,b) \equiv_\theta q^{\boldsymbol{A}}(c,c,b) = b$.

The term q in Theorem 4.1 is called a *minority term* (or *Maltsev term*) for K. The variety of groups is congruence permutable, with minority term $q(x,y,z) = xy^{-1}z$, or more accurately, $(x(y^{-1}))z$.

Congruence $(n{+}1)$–permutable varieties can be characterized in the spirit of Theorem 4.1, using a more general scheme of equations (see [65]):

$$\begin{aligned} x &\approx q_1(x,y,y) \\ q_i(x,x,y) &\approx q_{i+1}(x,y,y) \quad (i=1,\dots,n-1) \\ q_n(x,x,y) &\approx y. \end{aligned}$$

Theorem 4.2. *If* θ *is a congruence of a* 3*–permutable algebra* $\boldsymbol{A}$ *and* $h\colon \boldsymbol{A} \longrightarrow \boldsymbol{B}$ *is a homomorphism, then* $\{\langle h(a),h(a')\rangle : \langle a,a'\rangle \in \theta\}$ *is a congruence of* $\boldsymbol{B}$.

The 3–permutability demand cannot be dropped, although the proof uses only the fact that $\theta\circ\varphi\circ\theta = \varphi\circ\theta\circ\varphi$, where φ is the kernel of h.

Congruence permutable varieties have a number of desirable properties, of which two are stated below. The first is known as *Fleischer's Lemma*; it is involved in the proof of Theorem 4.5.

Theorem 4.3. ([42]) *Let* $h\colon \boldsymbol{A} \longrightarrow \boldsymbol{A}_1 \times \boldsymbol{A}_2$ *be a subdirect embedding, where* $\boldsymbol{A}$ *belongs to a congruence permutable variety. Then there exist an algebra* $\boldsymbol{C}$ *and surjective homomorphisms* $h_i\colon \boldsymbol{A}_i \longrightarrow \boldsymbol{C}$ $(i=1,2)$ *such that*

$$h[A] = \{\langle a_1,a_2\rangle \in A_1 \times A_2 : h_1(a_1) = h_2(a_2)\}.$$

Theorem 4.4. ([47]) *Let* K *be a congruence permutable variety, and* $\boldsymbol{A} \in \mathsf{K}$. *If a subdirect embedding* $h\colon \boldsymbol{A} \longrightarrow \boldsymbol{A}_1 \times \dots \times \boldsymbol{A}_n$ *(*n *finite) exists, where each* $\boldsymbol{A}_i$ *is simple, then* $\boldsymbol{A} \cong \boldsymbol{A}_{m_1} \times \dots \times \boldsymbol{A}_{m_k}$ *for some* $m_1,\dots,m_k \in \{1,\dots,n\}$ *(with* k *finite).*

Theorem 4.5. ([46]) *A finite algebra $\boldsymbol{A}$ is primal iff $\mathbb{HSP}(\boldsymbol{A})$ is congruence permutable and $\boldsymbol{A}$ is simple and has no proper subalgebra and no automorphism other than the identity map.*

The following is a variant of *Maltsev's Lemma.* It characterizes congruence generation in arbitrary algebras.

Lemma 4.6. *Let $a_1, \ldots, a_m, b_1, \ldots, b_m, c, d$ be elements of an algebra $\boldsymbol{A}$. Then $\langle c, d\rangle \in \Theta^{\boldsymbol{A}}\{\langle a_1, b_1\rangle, \ldots, \langle a_m, b_m\rangle\}$ iff there exist finitely many $2m$–ary polynomial operations $p_1, \ldots, p_k$ of $\boldsymbol{A}$ such that*

$$\begin{gathered} c = p_1(a_1, \ldots, a_m, b_1, \ldots, b_m) \\ p_i(b_1, \ldots, b_m, a_1, \ldots, a_m) = p_{i+1}(a, \ldots, a_m, b_1, \ldots, b_m) \quad (i = 1, \ldots, k-1) \\ p_k(b_1, \ldots, b_m, a_1, \ldots, a_m) = d. \end{gathered}$$

If, moreover, $\boldsymbol{A}$ belongs to a congruence $(n+1)$–permutable variety, then there are m–ary polynomial operations $p_1, \ldots, p_k$, with $k \leq n$, such that

$$\begin{gathered} c = p_1(a_1, \ldots, a_m) \\ p_i(b_1, \ldots, b_m) = p_{i+1}(a_1, \ldots, a_m) \quad (i = 1, \ldots, k-1) \\ p_k(b_1, \ldots, b_m) = d. \end{gathered}$$

5. Variants of Distributivity

Recall that a lattice is said to be *distributive* or *modular* or *meet semi-distributive* if it satisfies the respective law (3) or (4) or (5) below.

$$x \wedge (y \vee z) \approx (x \wedge y) \vee (x \wedge z) \tag{3}$$

$$y \preccurlyeq x \implies x \wedge (y \vee z) \approx y \vee (x \wedge z) \tag{4}$$

$$x \wedge y \approx x \wedge z \implies x \wedge (y \vee z) \approx x \wedge y \tag{5}$$

In fact, a lattice is distributive iff it is modular and meet semi-distributive. Modular lattices form a variety, as (4) is equivalent to an equation in $\wedge, \vee$. Meet semi-distributive lattices do not form a variety [51].

Theorem 5.1. *A lattice $\boldsymbol{L}$ is modular iff, for each $a, b \in L$, the sublattice $[\![a \wedge b, a]\!] := \{c \in L : a \wedge b \leq c \leq a\}$ is isomorphic to the sublattice $[\![b, a \vee b]\!]$ under the map $x \mapsto x \vee b$.*

An algebra is said to be *congruence* [*meet semi-*] *distributive* or *modular* if it has a [meet semi-] distributive or modular congruence lattice, respectively. These adjectives are also applied to classes of algebras if all members of the class have the indicated property. Much information about congruence distributive varieties can be found in the survey [77].

Theorem 5.2. ([73]) *Every congruence 3–permutable algebra is congruence modular.*

Examples 5.3. Lattices are congruence distributive (Example 5.7 below) but need not be congruence n–permutable for any integer $n \geq 2$, while semilattices are congruence meet semi-distributive but need not be congruence modular [51], and groups are congruence permutable but not generally distributive. *Implication algebras* $\langle A; \rightarrow \rangle$ (i.e., the $\rightarrow$ subreducts of Boolean algebras) form a congruence distributive and 3–permutable variety, which is not congruence permutable. Many familiar logics are modeled by congruence distributive varieties. The $\leftrightarrow$ fragments of classical and intuitionistic logic are exceptions; they correspond to congruence modular varieties that are not distributive (see for instance [71]).

Theorem 5.4. *Let* $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *be algebras in a congruence distributive variety. Then every congruence of* $\boldsymbol{A} \times \boldsymbol{B}$ *has the form*

$$\theta \odot \varphi := \{\langle\langle a, b\rangle, \langle a', b'\rangle\rangle : \langle a, a'\rangle \in \theta \text{ and } \langle b, b'\rangle \in \varphi\}$$

for suitable congruences θ, φ *of* $\boldsymbol{A}, \boldsymbol{B}$, *respectively, so* $\mathbf{Con}\,\boldsymbol{A} \times \mathbf{Con}\,\boldsymbol{B}$ *is naturally isomorphic to* $\mathbf{Con}\,(\boldsymbol{A} \times \boldsymbol{B})$, *under the map* $\langle \theta, \varphi \rangle \mapsto \theta \odot \varphi$.

This consequence of congruence distributivity is known as the *Fraser-Horn property*, after [48]. It is not a characterization, because it obtains also in the variety of rings with identity, which is not congruence distributive.

An algebra (or class of algebras) is called *arithmetical* if it is both congruence distributive and congruence permutable. The name 'arithmetical' comes from the following variant of the Chinese Remainder Theorem:

Theorem 5.5. *An algebra* $\boldsymbol{A}$ *is arithmetical iff the following is true for every positive integer* n: *any simultaneous system of relations*

$$x \equiv_{\theta_i} a_i, \quad i = 1, \ldots, n$$

(*with each* $a_i \in A$ *and each* $\theta_i \in \operatorname{Con} \boldsymbol{A}$) *can be solved for* x *in* A, *provided that* $a_i \equiv_{\theta_i \vee \theta_j} a_j$ *whenever* $i \neq j$.

In the next result, the second item should be compared with Theorem 4.1. Its proof is in the same spirit.

Theorem 5.6. (Pixley [127]) *Let* K *be a variety.*

(i) *If there is a term* $m(x, y, z)$ *such that* K *satisfies*

$$m(x, x, y) \approx m(x, y, x) \approx m(y, x, x) \approx x,$$

then K *is congruence distributive.*

(ii) K *is arithmetical iff there is a term* $t(x, y, z)$ *such that* K *satisfies*

$$t(x, y, y) \approx t(y, y, x) \approx t(x, y, x) \approx x.$$

We call m and t a *majority term* and a $\frac{2}{3}$-*minority term* (or *Pixley term*) for K, respectively. Clearly, the latter is also a minority term. Although the converse of (i) is false, every arithmetical variety has a majority term. If m and q are majority and minority terms (respectively) for a variety K, then $t(x, y, z) = q(x, m(x, y, z), z)$ is a $\frac{2}{3}$-minority term for K.

Example 5.7. The variety of lattices is congruence distributive, because $m(x, y, z) = (x \wedge y) \vee (x \wedge z) \vee (y \wedge z)$ is a majority term for this class. Likewise, expansions of lattices are congruence distributive. The variety of residuated lattices $\langle A; \cdot, \backslash, /, \wedge, \vee, 1\rangle$ (see [56]) is arithmetical, because $q(x, y, z) = (x/((z\backslash y) \wedge 1)) \wedge (z/((x\backslash y) \wedge 1))$ is a minority term for these algebras, whence $q(x, m(x, y, z), z)$ is a $\frac{2}{3}$-minority term.

Although congruence distributivity (for varieties) is not characterized by the existence of a majority term, it is still a Maltsev condition, and so is congruence modularity. The respective 'Maltsev schemes' were provided by Jónsson [74] and by Day [35] (see [62] also). The one for distributivity postulates ternary terms $t_0, \dots, t_n$ such that the variety satisfies

$$t_0(x, y, z) \approx x \approx t_i(x, y, x) \text{ for all } i \leq n, \quad t_n(x, y, z) \approx z,$$
$$t_i(x, x, z) \approx t_{i+1}(x, x, z) \text{ for even } i < n,$$
$$t_i(x, z, z) \approx t_{i+1}(x, z, z) \text{ for odd } i < n.$$

Modularity will be discussed in Section 6. Even congruence meet semi-distributivity turns out to be a Maltsev condition [83, 94]. This claim and the precise meaning of 'Maltsev condition' will be elaborated in Section 12.

An algebra $\boldsymbol{A}$ is said to be *congruence regular* provided that its congruences are determined by any single congruence class, i.e., whenever $a \in A$ and $\varphi, \psi \in \operatorname{Con} \boldsymbol{A}$ with $a/\varphi = a/\psi$, then $\varphi = \psi$. When $\boldsymbol{A}$ has a nullary basic operation c, we can consider a more widely applicable variant of this definition: for all $\varphi, \psi \in \operatorname{Con} \boldsymbol{A}$, if $c^{\boldsymbol{A}}/\varphi = c^{\boldsymbol{A}}/\psi$, then $\varphi = \psi$. This is called *c–regularity* (or *point regularity* when c is understood). A class of algebras is *congruence regular* or *c–regular* if its members are.

Lemma 5.8. ([63]) *If a variety is congruence regular or point regular, then it is congruence modular and congruence n–permutable for some integer n.*

Theorem 5.9. ([30, 40, 41]) *A variety K is congruence regular iff there are ternary terms $d_1(x, y, z), \dots, d_m(x, y, z)$ such that K satisfies*

$$(d_1(x, y, z) \approx z \;\&\; \dots \;\&\; d_m(x, y, z) \approx z) \iff x \approx y.$$

It is c–regular iff there are binary terms $d_1(x, y), \dots, d_m(x, y)$ such that K satisfies

$$(d_1(x, y) \approx c \;\&\; \dots \;\&\; d_m(x, y) \approx c) \iff x \approx y.$$

In the case of c–regularity, the argument from left to right uses the algebra $\boldsymbol{F} = \boldsymbol{F}_{\mathsf{K}}(2) \in \mathsf{K}$, freely generated by $\overline{x}, \overline{y}$. Let

$$\varphi = \Theta^{\boldsymbol{F}}(\overline{x}, \overline{y}) \text{ and } \psi = \Theta^{\boldsymbol{F}}((\overline{c}/\varphi) \times \{\overline{c}\}).$$

Then $\overline{c}/\varphi \subseteq \overline{c}/\psi$, i.e., $\overline{c}/\varphi = \overline{c}/(\varphi \cap \psi)$, so $\varphi \subseteq \psi$, by the c–regularity of $\boldsymbol{F}$. That is, $\Theta^{\boldsymbol{F}}(\overline{x}, \overline{y}) \subseteq \Theta^{\boldsymbol{F}}\{\langle \overline{d}, \overline{c}\rangle : \overline{d} \in \overline{c}/\varphi\}$. As $\mathbf{Con}\, \boldsymbol{F}$ is an algebraic lattice, there is a finite subset $\{\overline{d}_1, \dots, \overline{d}_m\}$ of $\overline{c}/\varphi$ such that

$$\Theta^{\boldsymbol{F}}(\overline{x}, \overline{y}) \subseteq \Theta^{\boldsymbol{F}}\{\langle \overline{d}_i, \overline{c}\rangle : i = 1, \dots, m\},$$

by Lemma 2.7. Then K satisfies $(\&_i \; d_i(x,y) \approx c) \implies x \approx y$, by Lemma 3.1. And for each i, we have $\langle \overline{d}_i, \overline{c} \rangle \in \Theta^{\boldsymbol{F}}(\overline{x}, \overline{y})$, so by the same lemma, K satisfies $x \approx y \implies d_i(x,y) \approx c$, i.e., $d_i(x,x) \approx c$.

Example 5.10. The variety of groups is congruence regular, as it satisfies $xy^{-1}z \approx z \iff x \approx y$. MV–algebras form a congruence regular variety [7] and so do quasigroups. Discriminator varieties (and in particular, Boolean algebras) are congruence regular. The variety of residuated lattices is 1–regular, as it satisfies $((x \backslash y) \wedge 1 \approx 1 \;\&\; (y \backslash x) \wedge 1 \approx 1) \iff x \approx y$. It is not congruence regular. The algebraic counterparts of algebraizable logics (in the sense of [21]) visibly generalize point regular varieties, as they are characterized by quasi-equations of the form

$$(\&_{i,j} \; s_i(d_j(x,y)) \approx t_i(d_j(x,y))) \iff x \approx y.$$

Using (1) and Lemmas 3.1, 4.6 and 5.8, we can replace the quasi-equations in Theorem 5.9 by equational schemes. In the case of point regularity, we get a Maltsev condition consisting of $d_j(x,x) \approx c$ $(j = 1, \ldots, m)$ and the following, where k is less than the variety's degree of permutability:

$$\begin{aligned} x \;&\approx\; t_1(x,y,d_1(x,y),\ldots,d_m(x,y)) \\ t_i(x,y,c,\ldots,c) \;&\approx\; t_{i+1}(x,y,d_1(x,y),\ldots,d_m(x,y)) \quad (1 \le i < k) \\ t_k(x,y,c,\ldots,c) \;&\approx\; y. \end{aligned}$$

Actually, a more complex scheme of this kind (corresponding to the first claim in Lemma 4.6, rather than the second) can be used to prove Lemma 5.8, which in turn justifies the simpler form of the above scheme.

Example 5.11. Integral residuated lattices satisfy $x \preccurlyeq 1$ and $x \preccurlyeq y/(x \backslash y)$ and $x/1 \approx x$ and $x \backslash x \approx 1 \approx x/x$. Consequently, they satisfy

$$\begin{aligned} x \;&\approx\; x \vee [x/((y \backslash x)/(\underline{y \backslash x}))] \\ x \vee [x/((y \backslash x)/\underline{1})] \;&\approx\; x \vee y \vee (x/(\underline{y \backslash x})) \vee [y/((x \backslash y)/(\underline{x \backslash y}))] \\ x \vee y \vee (x/\underline{1}) \vee [y/((x \backslash y)/\underline{1})] \;&\approx\; y \vee (y/(\underline{x \backslash y})) \\ y \vee (y/\underline{1}) \;&\approx\; y. \end{aligned}$$

These equations witness the 1–regularity and congruence 4–permutability of any variety of $\backslash, /, \vee, 1$ subreducts of integral residuated lattices. The class of all such subreducts is itself a variety [145] and is not 3–permutable [130].

6. Abelian Algebras

Recall that a group $\boldsymbol{G}$ is Abelian iff $[G,G]$ is trivial, where $[\,,]$ is the commutator operation on normal subgroups. A general *commutator theory* for all congruence permutable varieties (including a notion of Abelianness) was provided in the 1970s by J.D.H. Smith [137]. It was extended to congruence modular varieties by Hagemann and Herrmann [64]. Alternative approaches to the modular theory were then provided by Gumm [62] and by Freese and

McKenzie [50]. The main concepts are recounted here. Some of the theory's striking applications are Theorems 6.3, 9.10(i) and 10.4(i) below.

Suppose $\vec{c} = c_1, \ldots, c_n$ and $\vec{d} = d_1, \ldots, d_n$ are elements of an arbitrary algebra $\boldsymbol{A}$, where $n \in \omega$. Given $\theta \in \operatorname{Con} \boldsymbol{A}$, we write $\vec{c} \equiv_\theta \vec{d}$ if we want to signify that $c_i \equiv_\theta d_i$ for $i = 1, \ldots, n$.

Definition 6.1. The *centralizer* θ^* of $\theta \in \operatorname{Con} \boldsymbol{A}$ is the binary relation on A which identifies elements a, b iff, for every term $t(x, \vec{y})$ of $\boldsymbol{A}$'s type and for all tuples $\vec{c}, \vec{d} \in A$ of the same length as $\vec{y}$, such that $\vec{c} \equiv_\theta \vec{d}$, we have

$$t^{\boldsymbol{A}}(a, \vec{c}) = t^{\boldsymbol{A}}(a, \vec{d}) \text{ iff } t^{\boldsymbol{A}}(b, \vec{c}) = t^{\boldsymbol{A}}(b, \vec{d}). \tag{6}$$

It can be shown that $\theta^* \in \operatorname{Con} \boldsymbol{A}$. The *centre* $\zeta_{\boldsymbol{A}}$ of $\boldsymbol{A}$ is the congruence $(A^2)^*$. We say that $\boldsymbol{A}$ is *Abelian* if $\zeta_{\boldsymbol{A}} = A^2$, i.e., if (6) holds for all t and all $a, b, \vec{c}, \vec{d} \in A$. A class of algebras is said to be *Abelian* if its members are.

Examples 6.2. If θ is a congruence of a group $\boldsymbol{G} = \langle G; \cdot, ^{-1}, e \rangle$ and H is the normal subgroup e/θ, then $e/\theta^* = \{a \in G : ah = ha \text{ for all } h \in H\}$ (see for instance [8, Ex. 7.18]), and $\boldsymbol{G}$ is Abelian in the above sense iff it satisfies $xy \approx yx$. Likewise, all modules over rings are Abelian. This motivates the terminology, but the notions make sense for arbitrary algebras. For instance, if θ is a congruence of a lattice $\boldsymbol{L}$, then θ^* is the *pseudocomplement* of θ in $\mathbf{Con}\,\boldsymbol{L}$, i.e., it is the largest $\varphi \in \operatorname{Con} \boldsymbol{L}$ for which $\theta \cap \varphi = \mathrm{id}_L$. And, for an ideal $I = 0/\theta$ of a ring $\boldsymbol{R}$, the annihilator $\{r \in R : Ir = \{0\} = rI\}$ is $0/\theta^*$.

Two algebras are said to be *polynomially equivalent* if they have the same universe and the same polynomial operations. Note that the polynomial operations of a left module $\boldsymbol{M}$ over a ring $\boldsymbol{R}$ all have the simple form

$$\langle x_1, \ldots, x_n \rangle \mapsto m + \sum\nolimits_{i=1}^{n} r_i x_i \quad (n \in \omega,\, m \in M,\, r_1, \ldots, r_n \in R).$$

The following result of C. Herrmann is known as the *Fundamental Theorem of Abelian Algebras.*

Theorem 6.3. ([67]) *In a congruence modular variety* K, *every Abelian algebra* $\boldsymbol{A}$ *is polynomially equivalent to a module over a ring with identity.*

The proof in [50] begins by showing that $\boldsymbol{A}$ is *affine*, i.e., there is a term $d(x, y, z)$ of K and an Abelian group structure $+, -, 0$ on A such that

$$d^{\boldsymbol{A}}(a, b, c) = a - b + c \text{ and } s^{\boldsymbol{A}}(\vec{a} - \vec{b} + \vec{c}) = s^{\boldsymbol{A}}(\vec{a}) - s^{\boldsymbol{A}}(\vec{b}) + s^{\boldsymbol{A}}(\vec{c})$$

for all $a, b, c \in A$, all term operations $s^{\boldsymbol{A}}$ of $\boldsymbol{A}$ and all n–tuples $\vec{a}, \vec{b}, \vec{c} \in A$, where n is the rank of $s^{\boldsymbol{A}}$. This implies that $d^{\boldsymbol{A}}$ is a homomorphism from $\boldsymbol{A}^3$ to $\boldsymbol{A}$, but 0 need not be definable by means of term operations of $\boldsymbol{A}$. (Conversely, affine algebras are always Abelian and generate congruence permutable—hence modular—varieties. It is easy to see that modules are affine.) The production of d is nontrivial; it is described after Definition 6.6. An affine algebra $\boldsymbol{A}$, as above, is polynomially equivalent to the natural (unital) left $\boldsymbol{R}$–module expansion of $\langle A; +, -, 0 \rangle$, where $\boldsymbol{R}$ is the subring (with identity) of the endomorphism ring of $\langle A; +, -, 0\rangle$ consisting of all unary polynomials p of $\boldsymbol{A}$ such that $p(0) = 0$, cf. [115, Thm. 4.155].

In Chapter 9 of [50], a stronger result than Theorem 6.3 is proved: for every congruence modular variety K, the Abelian algebras in K constitute a subvariety M, and there is a *single* ring $\boldsymbol{R}$ with identity such that M is equivalent, in a strong sense, to a variety of $\boldsymbol{R}$–modules.

Definition 6.4. Let θ, φ, ψ be congruences of an algebra $\boldsymbol{A}$. We say that *φ centralizes ψ modulo θ* if $\theta \subseteq \varphi \cap \psi$ and $\varphi/\theta \subseteq (\psi/\theta)^*$ in $\mathbf{Con}\,\boldsymbol{A}/\theta$. The smallest $\theta \in \mathrm{Con}\,\boldsymbol{A}$ for which this is true is called the *commutator of* $\langle\varphi, \psi\rangle$ and denoted by $[\varphi, \psi]$.

This congruence always exists—it is the intersection of all congruences modulo which φ centralizers ψ. Thus, $[\varphi, \psi] \subseteq \varphi \cap \psi$, and $[\,,]$ preserves $\subseteq$ in both arguments. Clearly, an algebra $\boldsymbol{A}$ is Abelian iff $[A^2, A^2] = \mathrm{id}_A$, iff $[\varphi, \psi] = \mathrm{id}_A$ for all $\varphi, \psi \in \mathrm{Con}\,\boldsymbol{A}$. Less obviously, $\boldsymbol{A}$ is Abelian iff id_A is an equivalence class of some congruence on $\boldsymbol{A} \times \boldsymbol{A}$ [115, p. 253]. There are attendant notions of solvability and nilpotence (see [80, 81] in particular). In congruence modular or meet semi-distributive varieties, we always have $[\varphi, \psi] = [\psi, \varphi]$, and the operation $[\,,]$ turns congruence lattices into residuated lattice-ordered groupoids [50]. In the meet semi-distributive case, $[\varphi, \psi] = \varphi \cap \psi$ (see Theorem 12.10).

Examples 6.5. If congruences θ, φ correspond to normal subgroups H, K of a group $\boldsymbol{G}$, then $[\theta, \varphi]$ corresponds to the normal subgroup generated by $\{aba^{-1}b^{-1} : a \in H \text{ and } b \in K\}$, so $[\theta, \varphi] = \mathrm{id}_G$ means that the elements of H commute with those of K. In a ring, the commutator of congruences corresponding to ideals I, J corresponds itself to $IJ + JI$.

Definition 6.6. By a *difference* [*weak difference*] *term* for a variety K, we mean a term $d(x, y, z)$ such that whenever $\boldsymbol{A} \in \mathsf{K}$ and $\theta \in \mathrm{Con}\,\boldsymbol{A}$ with $\langle a, b\rangle \in \theta$, then $d^{\boldsymbol{A}}(a, a, b) = b$ [$d^{\boldsymbol{A}}(a, a, b) \equiv_{[\theta,\theta]} b$] and $d^{\boldsymbol{A}}(a, b, b) \equiv_{[\theta,\theta]} a$.

The accounts of Theorem 6.3 in [62, 50] reveal that any congruence modular variety K has a difference term—which can serve as the term d in the above proof-sketch of 6.3. It can be obtained from Day's Maltsev scheme for congruence modularity [35], which postulates 4–ary terms $m_0, \ldots, m_n$, such that K satisfies

$$m_0(x, y, z, u) \approx x \approx m_i(x, y, y, x) \text{ for all } i \leq n, \quad m_n(x, y, z, u) \approx u,$$
$$m_i(x, x, y, y) \approx m_{i+1}(x, x, y, y) \text{ for all even } i < n,$$
$$m_i(x, y, y, z) \approx m_{i+1}(x, y, y, z) \text{ for all odd } i < n.$$

We define ternary terms $q_0, \ldots, q_n$ by $q_0(x, y, z) = z$ and

$$q_{i+1}(x, y, z) \;=\; \begin{cases} m_{i+1}(q_i(x, y, z), y, x, q_i(x, y, z)) & i \text{ odd} \\ m_{i+1}(q_i(x, y, z), x, y, q_i(x, y, z)) & i \text{ even.} \end{cases}$$

It can then be shown that q_n is a difference term for K.

Congruence meet semi-distributive varieties have a much simpler difference term, viz. $d(x, y, z) = z$. In fact, Abelian algebras are affine in any variety possessing a weak difference term [83].

A class K of similar algebras has the *congruence extension property* (CEP) if every congruence on a subalgebra $\boldsymbol{B}$ of a member of K is the restriction $B^2 \cap \varphi$ of some congruence φ on the parent algebra. In this case, $\mathbb{HS}(\mathsf{K}) \subseteq \mathbb{SH}(\mathsf{K})$ and any nontrivial subalgebra of a simple algebra in K is simple. Abelian groups, modules, semilattices, distributive lattices and commutative residuated lattices all have the CEP. The CEP is closely connected with 'local deduction theorems' in abstract algebraic logic [31, 43, 44].

Theorem 6.7. (Kiss [88, 89]) *Let K be a congruence modular variety.*

(i) *If K has the congruence extension property and $\boldsymbol{A} \in \mathsf{K}$, then*

$$[\theta, \varphi] = \theta \cap \varphi \cap [A^2, A^2] \text{ for all } \theta, \varphi \in \mathbf{Con}\, \boldsymbol{A}.$$

(ii) *The displayed condition in* (i) *holds for all $\boldsymbol{A} \in \mathsf{K}$ iff every non-Abelian subdirectly irreducible algebra $\boldsymbol{A} \in \mathsf{K}$ has the property that $[\theta, \varphi]$ is a non-identity congruence of $\boldsymbol{A}$ whenever θ and φ are.*

Theorem 6.8. ([50, Thm. 8.5]) *The following conditions on a congruence modular variety K are equivalent.*

(i) K *has the Fraser-Horn property* (*see Theorem* 5.4).
(ii) K *contains no nontrivial Abelian algebra.*
(iii) *The centre $\zeta_{\boldsymbol{A}}$ of each $\boldsymbol{A} \in \mathsf{K}$ is the identity relation.*
(iv) $[\theta, A^2] = \theta$ *whenever* $\boldsymbol{A} \in \mathsf{K}$ *and* $\theta \in \mathrm{Con}\, \boldsymbol{A}$.
(v) $[A^2, A^2] = A^2$ *for all* $\boldsymbol{A} \in \mathsf{K}$.

Two varieties K_1 and K_2 of the same type are said to be *independent* if there is a term $t(x_1, x_2)$ such that K_i satisfies $t(x_1, x_2) \approx x_i$ for $i = 1, 2$. If K_1 and K_2 are subvarieties of a variety K, we write $\mathsf{K} = \mathsf{K}_1 \otimes \mathsf{K}_2$ to signify that K_1 and K_2 are independent and $\mathsf{K} = \mathbb{HSP}(\mathsf{K}_1 \cup \mathsf{K}_2)$. We then call K the *varietal product* of $\mathsf{K}_1, \mathsf{K}_2$. In this case, every $\boldsymbol{A} \in \mathsf{K}$ is isomorphic to an algebra of the form $\boldsymbol{A}_1 \times \boldsymbol{A}_2$ with $\boldsymbol{A}_i \in \mathsf{K}_i$ for $i = 1, 2$, and $\mathbf{Con}\, \boldsymbol{A}$ is naturally isomorphic to $\mathbf{Con}\, \boldsymbol{A}_1 \times \mathbf{Con}\, \boldsymbol{A}_2$.

Theorem 6.9. (Herrmann [67]) *Let $\mathsf{M} = \mathbb{HSP}(\mathsf{D} \cup \mathsf{A})$, where M is a congruence modular variety, D is a congruence distributive variety and A is an Abelian variety. Then $\mathsf{M} = \mathsf{D} \otimes \mathsf{A}$.*

The genesis and proof of the next theorem are described in [8, Chap. 7].

Theorem 6.10. *A finite nontrivial algebra in a congruence permutable variety is functionally complete iff it is simple and non-Abelian.*

A nontrivial variety is said to be *minimal* if it has no nontrivial proper subvariety. An algebra is said to be *strictly simple* if it is finite and simple and has no nontrivial proper subalgebra. Clearly, every locally finite minimal variety is generated by a strictly simple algebra.

Theorem 6.11. ([67]) *Let $\boldsymbol{A}$ be a strictly simple algebra in a congruence modular variety* K, *and assume that* $\mathbb{HSP}(\boldsymbol{A})$ *is not Abelian. Then* $\mathbb{HSP}(\boldsymbol{A})$ *is congruence distributive, and if* K *is congruence permutable, then $\boldsymbol{A}$ has a discriminator term.*

Corollary 6.12. *Every locally finite congruence modular* [*permutable*] *minimal variety is congruence distributive* [*a discriminator variety*] *or Abelian.*

7. Filtered Products

For consistency, the results of this section are stated for algebras, but they can be extended to structures with relations (as well as operations), except for Theorem 7.4.

Recall that a *filter over* a set I is a lattice-filter $\mathcal{U}$ of the Boolean algebra of all subsets of I, i.e., it is a non-empty set of subsets of I, closed under taking supersets and finite intersections. It is an *ultrafilter* over I if it excludes $\emptyset$ and is not properly contained in any filter over I, except for the filter of all subsets of I. In this case, for any $J, J' \subseteq I$, if $J \cup J' \in \mathcal{U}$, then $J \in \mathcal{U}$ or $J' \in \mathcal{U}$ (in particular, just one of J, $I \backslash J$ belongs to $\mathcal{U}$). Note that there is no ultrafilter over $\emptyset$.

Given a family $\{\boldsymbol{A}_i : i \in I\}$ of similar algebras and a filter $\mathcal{U}$ over I, the relation $\theta_{\mathcal{U}}$ identifies all pairs $a, b \in \prod_{i \in I} A_i$ such that

$$a(i) = b(i) \text{ for all } i \text{ in some member of } \mathcal{U}.$$

It is a congruence of $\prod_{i \in I} \boldsymbol{A}_i$. The factor algebra $\left(\prod_{i \in I} \boldsymbol{A}_i\right)/\theta_{\mathcal{U}}$ is called a *reduced product* of the algebras $\boldsymbol{A}_i$. It is an *ultraproduct* if $\mathcal{U}$ is an ultrafilter over I (whence $I \neq \emptyset$). In this case, roughly speaking, first order properties of tuples hold modulo $\theta_{\mathcal{U}}$ iff they hold co-ordinatewise throughout some element of $\mathcal{U}$. We use the term *ultrapower* when the algebras $\boldsymbol{A}_i$ are all the same. If $\boldsymbol{B}$ is an ultrapower $\boldsymbol{A}^I/\theta_{\mathcal{U}}$ of $\boldsymbol{A}$, then $\boldsymbol{A}$ is called an *ultraroot* of $\boldsymbol{B}$. Any algebra can be embedded into each of its ultrapowers by the obvious map $a \mapsto \langle a, a, a, \dots \rangle/\theta_{\mathcal{U}}$.

Lemma 7.1. ([25, p. 146]) *If* K *is a finite set of finite algebras, then any ultraproduct of members of* K *is isomorphic to a member of* K.

A class of algebras is called *elementary* if it can be axiomatized by a set of first order sentences, and *strictly elementary* if it can be axiomatized by one such sentence. The following result combines contributions of Łoś [97], Keisler [86] and Shelah [135].

Theorem 7.2. *Let* $\mathsf{K} \subseteq \mathsf{K}_\rho$, *where* K_ρ *is the class of all algebras of type* ρ *and* K *is closed under isomorphic images.*

(i) K *is elementary iff it is closed under ultraproducts and ultraroots.*
(ii) K *is strictly elementary iff both* K *and* $\mathsf{K}_\rho \backslash \mathsf{K}$ *are closed under ultraproducts.*
(iii) *Two algebras satisfy the same first order sentences iff they have isomorphic ultrapowers.*

Theorem 7.2(i) is related to the *Compactness Theorem* of first order logic, which says (in the case of algebras) that a set Σ of first order sentences has a model in K_ρ provided that every finite subset of Σ does (see [25, p. 212]). Equivalently, for any set $\Sigma \cup \{\Phi\}$ of first order sentences, if every model of Σ in K_ρ is a model of Φ, then the same is true for some finite $\Sigma' \subseteq \Sigma$.

The symbols $\mathbb{P}_\mathbb{R}$ and $\mathbb{P}_\mathbb{U}$ stand for closure under reduced products and ultraproducts, respectively. A class K of similar algebras is called a *quasivariety* if it is closed under $\mathbb{I}, \mathbb{S}, \mathbb{P}$ and $\mathbb{P}_\mathbb{U}$. The smallest such class containing K is $\mathsf{Q} = \mathbb{I}\mathbb{S}\mathbb{P}\mathbb{P}_\mathbb{U}(\mathsf{K})$ (which coincides with $\mathbb{I}\mathbb{S}\mathbb{P}_\mathbb{R}(\mathsf{K})$). Every algebra in Q is isomorphic to a subdirect product of algebras that embed into ultraproducts of members of K, because $\mathbb{S}\mathbb{P}(\mathsf{M}) \subseteq \mathbb{P}_\mathbb{S}\mathbb{S}(\mathsf{M})$ for every M.

Theorem 7.3. ([104, 61]) *A class of similar algebras is a quasivariety iff it can be axiomatized by a set of quasi-equations.*

If Σ is a set of [quasi-] equations axiomatizing a [quasi] variety K, then the [quasi-] equations satisfied by K can be derived from Σ using a fixed set of purely [quasi-] equational inference rules. (See [12] or [25, Sec. II.14] in the case of varieties and [134] or [58, Sec. 2.2] for quasivarieties.) If, moreover, K is strictly elementary, then it can be axiomatized by a finite set of [quasi-] equations. This follows from the Compactness Theorem (in conjunction with Theorems 2.1 and 7.3).

Theorem 7.4. ([9]) *Every locally finite congruence modular minimal variety is minimal as a quasivariety.*

An algebra $\boldsymbol{A} = \langle A; F \rangle$ is said to be *locally embeddable into* a class K of algebras of the same type if every finite subset B of A can be extended to an isomorphic copy $\boldsymbol{C}$ of an algebra from K, in such a way that all partial $\boldsymbol{A}$–operations on elements of B are preserved—i.e., whenever $f^{\boldsymbol{A}}(\vec{b}), \vec{b} \in B$ then $f^{\boldsymbol{C}}(\vec{b}) = f^{\boldsymbol{A}}(\vec{b})$.

Theorem 7.5. (cf. [58, pp. 15–17]) *$\boldsymbol{A}$ is locally embeddable into K iff $\boldsymbol{A}$ can be embedded into an ultraproduct of members of K.*

The proof makes use of the Compactness Theorem and the *diagram* Δ_B of each $B \subseteq A$, which consists of the true atomic and negated atomic sentences of the expansion $\boldsymbol{A}_B$ of $\boldsymbol{A}$ by nullary operations c_b corresponding to the elements $b \in B$. Note that Δ_A holds in an algebra $\boldsymbol{E}$ of same type as $\boldsymbol{A}_A$ iff $a \mapsto c_a^{\boldsymbol{E}}$ is an embedding of $\boldsymbol{A}$ into the appropriate reduct of $\boldsymbol{E}$.

Corollary 7.6. *Every algebra can be embedded into an ultraproduct of finitely generated subalgebras of itself.*

An algebra in a variety K is said to be *finitely presented* (in K) if it is isomorphic to $\boldsymbol{F}_\mathsf{K}(m)/\theta$ for some $m \in \omega$ and some finitely generated congruence θ. For example, in the variety G of groups, the dihedral groups $\boldsymbol{D}_n$ are isomorphic to $\boldsymbol{F}/\Theta^{\boldsymbol{F}}\{\langle a^n, e\rangle, \langle b^2, e\rangle, \langle ba, a^{-1}b\rangle\}$, where $\boldsymbol{F} = \boldsymbol{F}_\mathsf{G}(2)$ and a, b are its free generators. More generally, every finite algebra of finite type is finitely presented in any variety containing it. Every algebra in a variety K is locally embeddable into the class K_{fp} of all finitely presented algebras in K [58, Prop. 2.1.18], whence $\mathsf{K} = \mathbb{ISP}_{\mathbb{U}}(\mathsf{K}_{\mathrm{fp}})$, by Theorem 7.5.

Let $\mathsf{K}_{\mathrm{fin}}$ denote the class of all finite algebras in a variety K. If each member of K is locally embeddable into $\mathsf{K}_{\mathrm{fin}}$ then K is said to have the *finite embeddability property* (FEP). To establish this, it suffices to show that each subdirectly irreducible algebra in K is locally embeddable into $\mathsf{K}_{\mathrm{fin}}$, see for instance [16, Lem. 3.7]. Clearly, every locally finite variety has the FEP.

The next result is essentially due to Evans [39] (also see [23]).

Theorem 7.7. *The following conditions on a variety* K *are equivalent.*

(i) K *has the finite embeddability property.*
(ii) *Every finitely presented algebra in* K *is isomorphic to a subdirect product of finite algebras.*
(iii) K *is generated as a quasivariety by its class of finite members, i.e.,* $\mathsf{K} = \mathbb{ISPP}_{\mathbb{U}}(\mathsf{K}_{\mathrm{fin}})$.

If these conditions hold and K *is finitely axiomatized and has finite type, then it has a decidable universal theory, i.e., the set of universally quantified first order sentences satisfied by* K *is recursive.*

The last claim instantiates a principle—often called *Harrop's Theorem*, after [66]—which may be stated as follows: if S is a recursive set of sentences in the language of a finitely formalized first order theory $\mathcal{T}$, where the proper axioms of $\mathcal{T}$ belong to S, then the set of theorems of $\mathcal{T}$ belonging to S is also recursive, provided that every non-theorem of $\mathcal{T}$ within S has a *finite* counter-model. The finiteness assumptions ensure that both the theorems of $\mathcal{T}$ within S and the finite models of $\mathcal{T}$ can be enumerated effectively, and this yields an obvious decision procedure. In Theorem 7.7, the FEP guarantees the existence of the finite counter-models when S consists of universally quantified sentences. On the same grounds, if a variety K of finite type is finitely axiomatized and generated by its finite members (as a variety, i.e., $\mathsf{K} = \mathbb{HSP}(\mathsf{K}_{\mathrm{fin}})$), then K has a decidable *equational* theory.

Example 7.8. The variety of all commutative residuated lattices (CRLs) lacks the FEP, although it is generated by its finite members [123] and hence has a decidable equational theory. The variety of CRLs satisfying $x^n \leq x^m$ has the FEP whenever $m, n \in \omega$ with $m \neq n > 0$ [144]. These facts relate to the decision and deducibility problems of various substructural logics.

To demand that a variety has a decidable (full) first order theory is very restrictive. The structure of locally finite varieties with this property has been largely determined by McKenzie and Valeriote [116], building on earlier work of Burris and McKenzie [24] in the congruence modular case. The arguments make extensive use of *tame congruence theory*, which was developed in [68] by Hobby and McKenzie.

8. Definable Principal Congruences

A congruence of an algebra $\boldsymbol{A}$ is said to be *principal* if it has the form $\Theta^{\boldsymbol{A}}(a,b)$. We say that a variety V has *definable principal congruences* (DPC) if there is a first order formula $\Phi[x,y,z,w]$ with four free variables such that for every $\boldsymbol{A} \in \mathsf{V}$ and $a,b,c,d \in A$, we have

$$\langle c,d\rangle \in \Theta^{\boldsymbol{A}}(a,b) \text{ iff } \Phi[a,b,c,d] \text{ is true in } \boldsymbol{A}. \tag{7}$$

If, in addition, we can choose for Φ a (finite) conjunction of equations, then V is said to have *equationally definable principal congruences* (EDPC).

Even in the case of DPC, Maltsev's Lemma 4.6 reveals much about the logical form that Φ can be assumed to take, but we shall not pursue this detail here. It is easy to see that a variety V with EDPC has the *principal CEP*, i.e., every principal congruence $\Theta^{\boldsymbol{B}}(a,b)$ of a subalgebra $\boldsymbol{B}$ of an algebra $\boldsymbol{A} \in \mathsf{V}$ is the restriction $B^2 \cap \Theta^{\boldsymbol{A}}(a,b)$ of a principal congruence of $\boldsymbol{A}$. By a result of Day [36], the principal CEP entails the CEP in any variety, so varieties with EDPC have the CEP.

A *join semilattice with* 0 is a semilattice $\langle S;\vee\rangle$ having a least element, where we define $u \leq v$ iff $u \vee v = v$. Its *ideals* are its non-empty downward-closed subuniverses, and they form an algebraic lattice when ordered by inclusion, in which meets are intersections. We say that $\langle S;\vee\rangle$ is *dually Brouwerian* if, for any $a,b \in S$, there is a smallest $c \in S$ such that $a \leq b \vee c$. In this case, the ideal lattice of $\langle S;\vee\rangle$ is distributive. The set of compact (i.e., finitely generated) congruences of any algebra $\boldsymbol{A}$ is naturally a join semilattice with 0, whose order is set inclusion.

Theorem 8.1. (Köhler & Pigozzi [91]) *A variety has EDPC iff each of its members has a dually Brouwerian semilattice of compact congruences.*

Every algebraic lattice (in particular, every congruence lattice) is naturally isomorphic to the ideal lattice of its own join semilattice of compact elements [59, p. 22], so it follows from Theorem 8.1 that every variety with EDPC is congruence distributive.

In a variety V with DPC, the class V_{s} [V_{si}] of all simple [subdirectly irreducible] algebras in V is elementary, because V is. Indeed, an algebra $\boldsymbol{A} \in \mathsf{V}$ belongs to V_{s} [V_{si}] iff there exist distinct $a,b \in A$ such that for all distinct $c,d \in A$, we have $\Theta^{\boldsymbol{A}}(a,b) = \Theta^{\boldsymbol{A}}(c,d)$ [$\Theta^{\boldsymbol{A}}(a,b) \subseteq \Theta^{\boldsymbol{A}}(c,d)$], so the claim follows from (7). In particular, if V is a finitely axiomatized variety with DPC, then V_{s} and V_{si} are strictly elementary. The first item in the next theorem follows from these observations and Theorem 7.2. The second was proved in [5], and the third combines results from [5, 52, 91]; its forward implication has been explained above.

Theorem 8.2. *Let V be a variety.*

(i) *If V has DPC, then V_{s} and V_{si} are both closed under ultraproducts.*
(ii) *Every locally finite variety with the CEP has DPC.*

(iii) V *has EDPC iff it is congruence distributive and has both DPC and the CEP.*

Examples 8.3. The variety of distributive lattices has EDPC, since in these algebras we always have

$$c \equiv_{\Theta(a,b)} d \ \text{ iff } \ (a \wedge b \wedge c = a \wedge b \wedge d \ \text{ and } \ a \vee b \vee c = a \vee b \vee d\,).$$

The variety of commutative residuated lattices $\langle A; \cdot, \rightarrow, \wedge, \vee, 1\rangle$ satisfying $(x \wedge 1)^2 \approx x \wedge 1$ has EDPC. Here, reading $u \leq v$ as $u \wedge v = u$, we have

$$c \equiv_{\Theta(a,b)} d \ \text{ iff } \ (a \leftrightarrow b) \wedge 1 \ \leq \ (c \leftrightarrow d) \wedge 1,$$

where $a \leftrightarrow b$ abbreviates $(a \rightarrow b) \wedge (b \rightarrow a)$. The variety of interior algebras (a.k.a. closure algebras) has EDPC as well. In this case

$$c \equiv_{\Theta(a,b)} d \ \text{ iff } \ c \wedge \Box(a \leftrightarrow b) \ = \ d \wedge \Box(a \leftrightarrow b).$$

In commutative rings with identity, we have

$$c \equiv_{\Theta(a,b)} d \ \text{ iff } \ \exists x\,(c - d = x(a-b)),$$

so this variety has DPC. It does not have EDPC, as it is not congruence distributive. For similar reasons, the variety of modules over a finite ring with identity has DPC but lacks EDPC. The same applies to semilattices—in fact, every variety generated by a 2–element algebra has DPC [10].

An algebra $\boldsymbol{A}$ is said to be *finitely subdirectly irreducible* if id_A is not the intersection of any two non-identity congruences, i.e., it is *meet-irreducible* in $\mathbf{Con}\,\boldsymbol{A}$. The class of all such algebras in a variety V is denoted by $\mathsf{V}_{\mathrm{fsi}}$. Thus, $\mathsf{V}_{\mathrm{s}} \subseteq \mathsf{V}_{\mathrm{si}} \subseteq \mathsf{V}_{\mathrm{fsi}}$. (Trivial algebras in V belong to $\mathsf{V}_{\mathrm{fsi}}$ but not to V_{si}.) The property of *not* being finitely subdirectly irreducible persists in ultraproducts, as a consequence of Maltsev's Lemma 4.6. Thus, for a finitely axiomatized variety V, if $\mathsf{V}_{\mathrm{fsi}}$ is an elementary class, then it is strictly elementary, by Theorem 7.2(ii).

A variety V is said to be *filtral* if every congruence θ on a subdirect product $\boldsymbol{A}$ of subdirectly irreducible algebras in V is determined by a suitable filter $\mathcal{U}$ over the index set I of the product—that is to say,

$$\theta = \{\langle a, b\rangle \in A^2 : \{i \in I : a(i) = b(i)\} \in \mathcal{U}\}.$$

Theorem 8.4. *Let* V *be a variety.*

(i) ([52, 53]) V *is filtral iff it is semisimple and has EDPC. In this case, every nontrivial finitely subdirectly irreducible algebra in* V *is simple.*
(ii) ([18, 53]) V *is congruence permutable and filtral iff it is a discriminator variety.*

For example, the variety of distributive lattices is filtral, but not a discriminator variety. The idea of filtrality originates with Magari [102].

For varieties that are the algebraic counterparts of algebraizable logics, EDPC corresponds to the existence of 'deduction theorems' [22]. This discovery of Blok and Pigozzi was suggested by Theorem 8.1. For analogous connections between 'inconsistency lemmas', filtrality and dually pseudo-complemented semilattices of compact congruences, see [131, 26].

9. Controlling Irreducible Algebras

In the light of Birkhoff's subdirect decomposition theorem, the following definitions are natural.

Definition 9.1. A variety V is said to be *residually* $< m$, where m is a cardinal, if $|A| < m$ for every $\boldsymbol{A} \in \mathsf{V}_{\mathrm{si}}$. In this case, the smallest such m is called the *residual bound* of V. We say that V is *residually small* if it has a residual bound, and *residually finite* if it is residually $< \aleph_0$.

Every residually finite variety has the FEP, by Theorem 7.7. Every variety generated by a 2–element algebra has a finite residual bound, but a variety generated by a 3–element algebra of type $\langle 2 \rangle$ need not be residually small [141]. In principle, it is easier to analyze a variety if its residual bound exists and is not too large. On a finite set, there are only finitely many operations of any given finite rank. Thus, any variety of finite type with a finite residual bound is finitely generated, by Corollary 2.9, as it has only finitely many non-isomorphic subdirectly irreducible members, each of which is finite. The next theorem provides two contrasting results about residual size. A more detailed breakdown of possibilities is given in [112].

Theorem 9.2. *Let* V *be a variety with* m *operation symbols, where* m *is a (possibly infinite) cardinal.*

- (i) (Taylor [139]) *If* V *has a subdirectly irreducible member of cardinality greater than* $2^{\max\{m,\aleph_0\}}$, *then* V *has arbitrarily large subdirectly irreducible members.*
- (ii) (Quackenbush [129]) *Suppose* V *is locally finite. If* V *contains only finitely many finite subdirectly irreducible algebras, up to isomorphism, then* V *contains no infinite subdirectly irreducible algebra (and therefore has a finite residual bound).*

To see that (ii) holds, note that every variety V is generated as a quasi-variety by the finitely generated members of V_{si}, by Corollaries 2.9 and 7.6. So, when V is locally finite, it is generated as a quasivariety by the class K of finite members of V_{si}, but in (ii), K is assumed to be the isomorphic closure of a finite set of finite algebras. Thus,

$$\mathsf{V} = \mathbb{ISPP}_{\mathbb{U}}(\mathsf{K}) = \mathbb{IP}_{\mathbb{S}}\mathbb{SP}_{\mathbb{U}}(\mathsf{K}) \subseteq \mathbb{IP}_{\mathbb{S}}\mathbb{S}(\mathsf{K}),$$

by Lemma 7.1, whence all subdirectly irreducible algebras in V belong to $\mathbb{IS}(\mathsf{K})$ and are therefore finite.

The converse of Theorem 9.2(ii) (ignoring the parenthetical text) is false [5]. Refuting the so-called *RS Conjecture*, McKenzie [112] showed that it is false even for finitely generated varieties. It is an open problem, however, for finitely generated varieties of finite type. This is the

Restricted Quackenbush Problem: *Must a finitely generated and residually finite variety of finite type have a finite residual bound?*

Some partial answers are given below. First, for varieties with DPC, the answer is positive. In fact, the following is true.

Theorem 9.3. (Baldwin & Berman [5]) *Let* V *be a variety with DPC.*

(i) *If* V *is residually small, then it has a finite residual bound.*
(ii) *If* V *has finite type and contains infinitely many non-isomorphic finite subdirectly irreducible algebras, then it contains subdirectly irreducible algebras of every infinite cardinality.*

In the absence of DPC, it is not generally easy to identify the subdirectly irreducible members of a variety $\mathbb{HSP}(\mathsf{K})$, even if we fully understand K. The situation improves in congruence modular (and especially distributive) varieties, as the next two results show.

Theorem 9.4. ([50]) *Let* K *be a class of algebras in a congruence modular variety, let* $\boldsymbol{A}$ *be a subdirectly irreducible algebra in* $\mathbb{HSP}(\mathsf{K})$, *and let* θ *be the centralizer of the monolith* μ *of* $\boldsymbol{A}$. *Then* $\boldsymbol{A}/\theta \in \mathbb{HSP}_{\mathbb{U}}(\mathsf{K})$. *In particular, if* $[\mu,\mu] \neq \mathrm{id}_A$, *then* $\boldsymbol{A} \in \mathbb{HSP}_{\mathbb{U}}(\mathsf{K})$.

The following result of B. Jónsson pre-dates Theorem 9.4. Its proof used *Jónsson's Lemma*, which says that, for any finitely subdirectly irreducible factor algebra $\boldsymbol{C}/\varphi$ of a congruence distributive subalgebra $\boldsymbol{C}$ of a product $\prod_{i\in I} \boldsymbol{A}_i$, we have $\varphi \supseteq C^2 \cap \theta_{\mathcal{U}}$ (see page 17) for some ultrafilter $\mathcal{U}$ over I.

Theorem 9.5. (Jónsson [74]) *Let* K *be a class of algebras in a congruence distributive variety, and let* $\boldsymbol{A}$ *be a finitely subdirectly irreducible algebra in* $\mathbb{HSP}(\mathsf{K})$. *Then* $\boldsymbol{A} \in \mathbb{HSP}_{\mathbb{U}}(\mathsf{K})$. *Thus,* $\mathbb{HSP}(\mathsf{K}) = \mathbb{IP}_{\mathbb{S}}\mathbb{HSP}_{\mathbb{U}}(\mathsf{K})$.

If K *consists of finitely many finite algebras, then* $\boldsymbol{A} \in \mathbb{HS}(\mathsf{K})$, *and so* $\mathbb{HSP}(\mathsf{K}) = \mathbb{IP}_{\mathbb{S}}\mathbb{HS}(\mathsf{K})$.

The restriction of Theorem 9.5 to subdirectly irreducible algebras $\boldsymbol{A}$ follows from Theorem 9.4, as the commutator of two congruences is just their intersection in the congruence distributive case. The last claim in Theorem 9.5 follows from the first and Lemma 7.1, and it implies the following.

Corollary 9.6. ([74]) *A finitely generated congruence distributive variety* V *has a finite residual bound, and only finitely many subvarieties. Moreover, two non-isomorphic subdirectly irreducible algebras in* V *cannot satisfy the same equations.*

In fact, if $\boldsymbol{A}$ is a subdirectly irreducible algebra in a locally finite congruence distributive variety $\mathbb{HSP}(\mathsf{K})$, then $\boldsymbol{A} \in \mathbb{ISP}_{\mathbb{U}}\mathbb{HS}(\mathsf{K})$, see [50, Cor. 10.3].

It is natural to ask whether the distributivity demand in Corollary 9.6 can be relaxed. Before doing so, we should note that for residually small varieties, the assumption of congruence *modularity* is less restrictive than it appears. To this end, we make the next definition.

Definition 9.7. A *congruence equation* is a formal equation in the binary symbols $\wedge, \vee, \circ$. It is *satisfied* by an algebra $\boldsymbol{A}$ if it becomes true whenever we interpret the variables of the equation as congruence relations of $\boldsymbol{A}$, and for arbitrary binary relations θ and φ on A, we interpret $\theta \wedge \varphi$, $\theta \vee \varphi$ and $\theta \circ \varphi$ as $\theta \cap \varphi$, $\Theta^{\boldsymbol{A}}(\theta \cup \varphi)$ and the relational product, respectively. A congruence equation is *satisfied* by a class of algebras if it is satisfied by every member of the class. It is *nontrivial* if some algebra fails to satisfy it.

It follows that a congruence equation involving $\wedge, \vee$ only (not $\circ$) is nontrivial iff it fails in some lattice. It is possible to exhibit infinitely many inequivalent congruence equations of this kind, all weaker than congruence modularity, see [37] and [81, Sec. 1.2]. Strikingly, however, they collapse in residually small varieties:

Theorem 9.8. (Kearnes & Kiss [81]) *If a residually small variety satisfies a nontrivial congruence equation in $\wedge, \vee$ only, then it is congruence modular (and conversely, of course).*

This recent result should be borne in mind when reading Theorems 9.10(i), 9.11, 9.16 and 10.4(i), and Corollary 9.12 (which were proved much earlier). Moreover, for each integer $n \geq 2$, every congruence n–permutable variety satisfies a nontrivial congruence equation in $\wedge, \vee$ only [93, 95], so Theorem 9.8 specializes as follows.

Corollary 9.9. *Every residually small congruence n–permutable variety is congruence modular.*

The following partial generalizations of Corollary 9.6 have been obtained.

Theorem 9.10. *Let V be a variety.*

(i) (Freese & McKenzie [49]) *If V is finitely generated, congruence modular and residually small, then it has a finite residual bound.*
(ii) (Kearnes & Willard [85]) *If V is congruence meet semi-distributive, residually finite and of finite type, then it has a finite residual bound.*

As an instance of (i), the variety generated by a finite nontrivial module over a finite ring with identity has a finite residual bound (cf. Example 9.17 below); it is not congruence distributive. Non-distributive witnesses of (ii) include the variety of semilattices and certain varieties generated by tournaments [106]. The proof of (i) uses the next result.

Theorem 9.11. (Freese & McKenzie [49]) *Let V be a congruence modular variety. If V is residually small, then for all $\boldsymbol{A} \in \mathsf{V}$ and $\theta, \varphi \in \operatorname{Con} \boldsymbol{A}$,*

$$\theta \cap [\varphi, \varphi] \subseteq [\theta, \varphi] \quad (\textit{equivalently, } \theta \cap [\varphi, \varphi] = [\theta \cap \varphi, \varphi]). \tag{8}$$

The converse holds if V is finitely generated.

Corollary 9.12. *Let V be a locally finite variety with the CEP.*

(i) *If V is finitely generated and congruence modular, then it is residually small.*

(ii) *If* V *is residually small, then it has a finite residual bound.*

Here, (i) follows from Theorems 6.7(i) and 9.11, as the condition displayed in 6.7(i) entails (8), while (ii) follows from Theorems 8.2(ii) and 9.3(i).

In summary, the restricted Quackenbush problem has a positive solution for all varieties that have DPC or are congruence meet semi-distributive or satisfy a nontrivial congruence equation in $\wedge, \vee$ only—and hence for all n–permutable varieties and all varieties with the CEP.

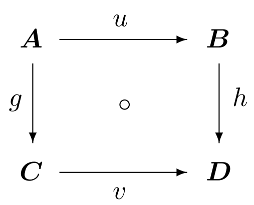
Injectivity and Amalgamation.

An algebra $\boldsymbol{C}$ in a variety V is said to be V–*injective* if, whenever $\boldsymbol{A} \in \mathsf{V}$ and $\boldsymbol{B} \in S(\boldsymbol{A})$, then each homomorphism from $\boldsymbol{B}$ into $\boldsymbol{C}$ can be extended to $\boldsymbol{A}$. Clearly, every such $\boldsymbol{C}$ is an *absolute retract* in V, i.e., whenever $\boldsymbol{D} \in \mathsf{V}$ is an extension of $\boldsymbol{C}$, then there is a surjective homomorphism $h\colon \boldsymbol{D} \longrightarrow \boldsymbol{C}$ such that $h|_C = \mathrm{id}_C$.

Definition 9.13. We say that a variety V *has enough injectives* if every algebra in V can be embedded into a V–injective member of V.

In this case, each $\boldsymbol{D} \in \mathsf{V}$ has an *injective hull*, i.e., it has a V–injective extension $\boldsymbol{E} \in \mathsf{V}$ such that no proper extension of $\boldsymbol{D}$ within $\boldsymbol{E}$ is V–injective, except for $\boldsymbol{E}$ itself. This $\boldsymbol{E}$ is unique up to an isomorphism that fixes $\boldsymbol{D}$, so it is denoted by $\boldsymbol{E}(\boldsymbol{D})$; it may also be characterized as a maximal *essential extension* of $\boldsymbol{D}$ within V, where 'essential' signifies that non-identity congruences of the extension restrict to non-identity congruences of $\boldsymbol{D}$. Moreover, $|E| \leq 2^{\max\{|D|,\aleph_0,m\}}$, where m is the number of operation symbols of V [139].

Definition 9.14. We say that *injections are transferable* in a class K of similar algebras if, for any embedding $u\colon \boldsymbol{A} \longrightarrow \boldsymbol{B}$ and homomorphism $g\colon \boldsymbol{A} \longrightarrow \boldsymbol{C}$, with $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C} \in \mathsf{K}$, there exist an embedding $v\colon \boldsymbol{C} \longrightarrow \boldsymbol{D}$ and a homomorphism $h\colon \boldsymbol{B} \longrightarrow \boldsymbol{D}$, with $\boldsymbol{D} \in \mathsf{K}$, such that $v \circ g = h \circ u$.



The *amalgamation property* for K is the variant of this demand in which g and h are also embeddings.

The connection between these properties and residual size is as follows.

Theorem 9.15. *Let* V *be any variety.*

(i) ([139]) V *has enough injectives iff it is residually small and injections are transferable in* V.
(ii) ([1, 139]) *Injections are transferable in* V *iff* V *has the congruence extension property and the amalgamation property.*

Theorem 9.16. (Kearnes [79]) *Every residually small congruence modular variety with the amalgamation property has the congruence extension property, and therefore has enough injectives.*

Example 9.17. The variety of all left modules over a ring $\boldsymbol{R}$ with identity witnesses Theorem 9.16. It is residually $< m$, where m is any strict upper bound on the (set of) cardinalities of the injective hulls $\boldsymbol{E}(\boldsymbol{R}/A)$ of the cyclic modules $\boldsymbol{R}/A$ got from the maximal left ideals A of $\boldsymbol{R}$. Indeed, every subdirectly irreducible module $\boldsymbol{M}$ has a smallest nonzero submodule $\boldsymbol{N}$ (corresponding to the monolith under the isomorphism $\theta \mapsto 0/\theta$ between congruences and submodules), and of course $\boldsymbol{N}$ is simple, hence cyclic, and therefore isomorphic to one such $\boldsymbol{R}/A$. But $\boldsymbol{M}$ is an essential extension of $\boldsymbol{N}$, and injective hulls are maximal essential extensions, so $\boldsymbol{M}$ can be embedded into $\boldsymbol{E}(\boldsymbol{N})$.

The following theorem combines results in [60, 117].

Theorem 9.18. *Let* V *be a variety with the CEP. If* $\mathsf{V}_{\mathrm{fsi}}$ *or* V_{si} *is closed under nontrivial subalgebras and has the amalgamation property, then* V *itself has the amalgamation property.*

Actually, the results in [60, 117] are formally stronger, as they weaken the hypothesis that $\mathsf{V}_{[\mathrm{f}]\mathrm{si}}$ be amalgamable: it is enough that the algebra $\boldsymbol{D}$ in the definition should belong to V (as opposed to $\mathsf{V}_{[\mathrm{f}]\mathrm{si}}$).

For varieties with the CEP that are algebraic counterparts of algebraizable logics, the amalgamation property corresponds to the 'deductive interpolation property', see for instance [33, 87]. (There are analogous correspondences between the surjectivity of epimorphisms and various Beth-style definability properties, see [17, 118].)

Examples 9.19. The variety of commutative residuated lattices (CRLs) is not residually small, but it has the amalgamation property and the CEP, so injections are transferable in this variety (see [56]). In the variety V generated by the class C of all totally ordered CRLs, we have $\mathsf{V}_{\mathrm{fsi}} = \mathsf{C}$ (partly on account of Theorem 9.5), and C is obviously closed under subalgebras. Theorem 9.18 is used in [117] to show, inter alia, that V has the amalgamation property, because C does. The same principle has been used to determine completely the amalgamable varieties of Heyting and interior algebras [54], of commutative GMV-algebras [117] and of Sugihara monoids [105]. These families encompass the algebraic counterparts of super-intuitionistic logics, normal extensions of the modal logic **S4**, Łukasiewicz logics, and axiomatic extensions of the relevance logic $\mathbf{RM^t}$, respectively.

Direct Decomposition.

An algebra $\boldsymbol{A}$ is said to be *directly indecomposable* if $|A| > 1$ and whenever $\boldsymbol{A} \cong \boldsymbol{B} \times \boldsymbol{C}$, then $|B|$ or $|C|$ is 1. For an arbitrary algebra, a decomposition into directly indecomposable direct factors need not exist, and when it exists, it need not be essentially unique. This problem is analyzed in detail in [115, Chap. 5]. One positive result in this connection is the following.

Theorem 9.20. ([27]) *Let* $h\colon \boldsymbol{B} = \prod_{i\in I} \boldsymbol{B}_i \cong \prod_{j\in J} \boldsymbol{C}_j$, *where all* $\boldsymbol{B}_i$ *and* $\boldsymbol{C}_j$ *are directly indecomposable. If* $\boldsymbol{B}$ *is congruence distributive, then there is a bijection* $\lambda\colon J \longrightarrow I$ *and a family of isomorphisms* $h_j\colon \boldsymbol{B}_{\lambda(j)} \cong \boldsymbol{C}_j$, $j \in J$, *such that* $h(b) = \langle h_j(b_{\lambda(j)}) : j \in J\rangle$ *for all* $b = \langle b_i : i \in I\rangle \in B$.

Obviously, every finite algebra is isomorphic to the direct product of a finite family of directly indecomposable (finite) algebras.

Definition 9.21. A variety is said to be *directly representable* if it is finitely generated and contains only finitely many finite directly indecomposable algebras, up to isomorphism.

It follows from Quackenbush's Theorem 9.2(ii) that a directly representable variety has a finite residual bound. All proper subvarieties of the variety of Abelian groups are directly representable. It is not known for which finite rings $\boldsymbol{R}$ (with identity) the variety of all $\boldsymbol{R}$–modules is directly representable but, modulo this question, the class of directly representable varieties has been completely described by McKenzie [110]. In particular:

Theorem 9.22.

(i) *A directly representable variety is congruence permutable and has DPC, and its subdirectly irreducible members are simple or Abelian.*
(ii) *A finitely generated congruence meet semi-distributive variety is directly representable iff it is semisimple and arithmetical.*

The forward implication in (ii) follows from (i); its converse uses Theorem 4.4. The distributive case of (ii) was established by Burris [25, p. 189], before the publication of [110].

10. Some Finite Basis Theorems

Suppose M is a finite set of finite algebras of the same finite type. Every isomorphically-closed subclass of $\mathbb{HS}(\mathsf{M})$ is of course strictly elementary, but the variety $\mathbb{HSP}(\mathsf{M})$ need not be finitely axiomatizable [99]. In 1993, McKenzie settled *Tarski's finite basis problem* by providing a construction that assigns to each Turing machine T a finite algebra $\boldsymbol{A}(T)$ of finite type such that T halts iff $\mathbb{HSP}(\boldsymbol{A}(T))$ is finitely axiomatizable [113]. Thus, *it is undecidable whether a set* M *of the above kind generates a finitely axiomatizable variety*; the corresponding question for quasivarieties is still open.

In contrast, Theorems 10.3, 10.4 and 10.10 below supply sufficient conditions for $\mathbb{HSP}(\mathsf{M})$ or $\mathbb{ISP}(\mathsf{M})$ to be finitely axiomatized, and these conditions have all been shown to be algorithmically verifiable (from M). They, together with Theorems 10.1 and 10.5, support a speculation known as

Park's Conjecture ([125]): *Every variety of finite type with a finite residual bound is finitely axiomatized.*

Theorem 10.1. (McKenzie [109]) *Every residually small variety* V *of finite type with DPC is finitely axiomatized (by equations).*

The idea of the proof is as follows. Let Φ be as in condition (7) of Section 8. By Theorem 9.3(i), V has a finite residual bound, so V_{si} is the isomorphic closure of a finite set $\{\boldsymbol{S}_1, \dots, \boldsymbol{S}_n\}$ of finite algebras. There is therefore a first order sentence Ψ which says (of an algebra) that (7) holds for all elements a, b, c, d and if the algebra is subdirectly irreducible, then it is isomorphic to one of $\boldsymbol{S}_1, \dots, \boldsymbol{S}_n$. As V satisfies Ψ, the Compactness Theorem shows that, for some finite subset Σ of the equational theory of V, every model of Σ is a model of Ψ. Then Σ axiomatizes V, by Corollary 2.9, because the subdirectly irreducible algebras satisfying Σ will satisfy Ψ, and will therefore belong to V.

Every directly representable variety of finite type is finitely axiomatized, by Theorems 9.22(i) and 10.1. Another consequence of 10.1 is that every 2–element algebra of finite type generates a finitely axiomatized variety, but this was known earlier [98]. In contrast, the variety generated by a 3–element algebra of type $\langle 2 \rangle$ need not be finitely axiomatizable [119]. (As for infinite types, even a 2–element lattice with infinitely many additional nullary operations generates a variety that is clearly not finitely axiomatizable.)

The following finite basis theorem of Jónsson does not impose local finiteness conditions, although it was motivated by Theorem 10.3, which does. Its proof uses the Maltsev condition for congruence distributivity on page 12.

Theorem 10.2. (Jónsson [75]) *Let V be a congruence distributive variety of finite type, such that the class $\mathsf{V}_{\mathrm{fsi}}$ is strictly elementary. Then V is finitely axiomatized.*

From Theorems 9.5 and 10.2, we immediately obtain *Baker's Finite Basis Theorem* (which pre-dates 10.2):

Theorem 10.3. (Baker [2]) *Every finitely generated congruence distributive variety of finite type is finitely axiomatized.*

This has been generalized beyond the congruence distributive case:

Theorem 10.4. *Let V be a variety of finite type.*

(i) (McKenzie [111]) *If V is finitely generated, congruence modular and residually small, then it is finitely axiomatized.*
(ii) (Willard [148]) *If V is congruence meet semi-distributive and residually finite, then it is finitely axiomatized.*

The residual finiteness demand in (ii) has been weakened in [3]. The following common generalization of (i) and (ii) has been established recently by Kearnes, Szendrei and Willard. (Recall Definition 6.6.)

Theorem 10.5. ([84]) *If a variety of finite type has a difference term and a finite residual bound, then it is finitely axiomatized.*

A common generalization of Theorem 10.1 and Baker's Theorem has been obtained by Baker and Wang: a variety V with *definable principal subcongruences* is finitely axiomatized, provided that V_{si} is strictly elementary. See [4] (or [8]) for the relevant definitions and the proof.

A class of algebras is said to be *universal* if it is axiomatizable by a set of universally quantified first order sentences—equivalently, it is closed under $\mathbb{I}$, $\mathbb{S}$ and $\mathbb{P}_{\mathbb{U}}$. In the context of algebras, a *positive universal sentence* is a universally quantified disjunction of conjunctions of equations (of finite total length). Up to logical equivalence, these are just the sentences that persist under $\mathbb{H}$, $\mathbb{S}$ and $\mathbb{P}_{\mathbb{U}}$ (see [28]). Like Theorem 10.2, the next three results make no assumption about residual size or local finiteness. All of them use ideas occurring in the proof of 10.2 and the first is actually a special case.

Theorem 10.6. ([19]) *Let* V *be a finitely axiomatized variety with EDPC. If a subclass* K *of* V *is axiomatized by finitely many positive universal sentences, then the variety* $\mathbb{HSP}(\mathsf{K})$ *is finitely axiomatized.*

A variety V is said to have *equationally definable principal meets* (EDPM) if it has finitely many pairs $\langle u_i(x,y,z,w), v_i(x,y,z,w)\rangle$, $i \in I$, of 4–ary terms such that for all $\boldsymbol{A} \in \mathsf{V}$ and $a,b,c,d \in A$,

$$\Theta^{\boldsymbol{A}}(a,b) \cap \Theta^{\boldsymbol{A}}(c,d) = \Theta^{\boldsymbol{A}}\{\langle u_i^{\boldsymbol{A}}(a,b,c,d), v_i^{\boldsymbol{A}}(a,b,c,d)\rangle : i \in I\}.$$

Theorem 10.7. ([20, 32]) *For a variety* V, *the following are equivalent.*

(i) V *has EDPM.*
(ii) V *is congruence distributive and* $\mathsf{V}_{\mathrm{fsi}}$ *is a universal class.*
(iii) V *is congruence distributive and, for all* $\boldsymbol{A} \in \mathsf{V}$, *the intersection of any two compact congruences of* $\boldsymbol{A}$ *is compact.*
(iv) *There are finitely many pairs* $\langle u_i, v_i\rangle$, $i \in I$, *of* 4*–ary terms such that* $\mathsf{V}_{\mathrm{fsi}}$ *satisfies*

$$(\&_{i\in I}\; u_i(x,y,z,w) \approx v_i(x,y,z,w)) \iff (x \approx y \text{ or } z \approx w).$$

Moreover, a variety V *of finite type with EDPM is finitely axiomatized iff* $\mathsf{V}_{\mathrm{fsi}}$ *is strictly elementary.*

The pairs that witness (iv) also witness EDPM, and vice versa. From Theorems 8.2(i),(iii), 8.4(i) and 10.7, we deduce:

Corollary 10.8. *A filtral variety has EDPM, and if it has finite type, then it is finitely axiomatized iff its class of simple members is strictly elementary.*

Example 10.9. Commutative residuated lattices $\boldsymbol{A} = \langle A; \cdot, \to, \wedge, \vee, 1\rangle$ have EDPM. For $a, b \in A$, we have $\Theta^{\boldsymbol{A}}(a,b) = \Theta^{\boldsymbol{A}}((a \leftrightarrow b) \wedge 1, 1)$ and if $a, b \leq 1$, then $\Theta^{\boldsymbol{A}}\{\langle a,1\rangle, \langle b,1\rangle\} = \Theta^{\boldsymbol{A}}(a \wedge b, 1)$ and $\Theta^{\boldsymbol{A}}(a,1) \cap \Theta^{\boldsymbol{A}}(b,1) = \Theta^{\boldsymbol{A}}(a \vee b, 1)$. This follows from Lemma 3.1 and basic properties of the algebras [56]. In fact, $\boldsymbol{A}$ is finitely subdirectly irreducible iff 1 is join-irreducible in the lattice $\langle A; \wedge, \vee\rangle$. That demand can be expressed, as in Theorem 10.7(iv), by

$$((x \leftrightarrow y) \wedge 1) \vee ((z \leftrightarrow w) \wedge 1) \approx 1 \iff (x \approx y \text{ or } z \approx w)\,.$$

A quasivariety Q is *relatively congruence meet semi-distributive* if every $\boldsymbol{A} \in \mathsf{Q}$ has a meet semi-distributive lattice of *relative congruences*—these are the congruences θ such that $\boldsymbol{A}/\theta \in \mathsf{Q}$, and they always form an algebraic lattice in which meets are intersections.

Theorem 10.10. ([38]) *For any finite set* M *of finite algebras of the same finite type, if the quasivariety* $\mathbb{ISP}(\mathsf{M})$ *is relatively congruence meet semi-distributive, then it is finitely axiomatized (by quasi-equations).*

The distributive case of this result was proved earlier, by D. Pigozzi [126]. It is not currently known whether Theorem 10.10 remains true when we replace 'meet semi-distributive' by 'modular' in its statement. (This will be the case if, in addition, $\mathbb{HSP}(\mathsf{M})$ is congruence modular and residually small [38].) Quasivarietal analogues of Theorems 10.1, 10.2, 10.6 and 10.7 can be found in [32, 121, 122], and further finite basis theorems for quasivarieties in [38, 58, 107].

11. Lattices of Subvarieties

Universal algebraic methods are particularly useful when it comes to analyzing the lattice of axiomatic extensions of a given logic, as these often correspond to the subvarieties of a congruence distributive variety.

Theorem 11.1. ([74]) *For any two subvarieties* V, W *of a congruence distributive variety, we have*

$$(\mathbb{HSP}(\mathsf{V} \cup \mathsf{W}))_{\mathrm{si}} = \mathsf{V}_{\mathrm{si}} \cup \mathsf{W}_{\mathrm{si}} \quad \textit{and} \quad (\mathbb{HSP}(\mathsf{V} \cup \mathsf{W}))_{\mathrm{fsi}} = \mathsf{V}_{\mathrm{fsi}} \cup \mathsf{W}_{\mathrm{fsi}}.$$

This follows from Theorem 9.5, as $\mathbb{P}_{\mathbb{U}}(\mathsf{V} \cup \mathsf{W}) \subseteq \mathsf{V} \cup \mathsf{W}$. Indeed, $\mathsf{V} \cup \mathsf{W}$ is an elementary class, because V and W are. As varieties are determined by their subdirectly irreducible members and $\cap$ distributes over $\cup$, we can infer:

Corollary 11.2. ([74]) *The lattice of subvarieties of a congruence distributive variety is itself distributive.*

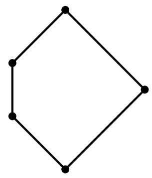
From Theorems 10.2 and 11.1, we obtain:

Corollary 11.3. *Let* V *be a congruence distributive variety of finite type, such that* $\mathsf{V}_{\mathrm{fsi}}$ *is strictly elementary. Then the join* $\mathbb{HSP}(\mathsf{V}_1 \cup \mathsf{V}_2)$ *of any two finitely axiomatized subvarieties* $\mathsf{V}_1, \mathsf{V}_2$ *of* V *is itself finitely axiomatized.*

Example 11.4. ([55]) By Example 10.9 and Corollary 11.3, the varietal join of two finitely axiomatized varieties of commutative residuated lattices is still finitely axiomatized.

In a lattice $\boldsymbol{M}$, we denote by $[\![x)$ the set of all upper bounds of an element x (including x itself), and by $(x]\!]$ the set of all lower bounds. We say that x *splits* $\boldsymbol{M}$ if $M = [\![x) \mathbin{\dot\cup} (y]\!]$ for some element y, where $\dot\cup$ indicates *disjoint* union (hence $x \nleq y$). This (unique) y is called the *splitting conjugate* of x.

Example 11.5. Recall that a lattice is modular iff its sublattices do not include the (subdirectly irreducible) *pentagon* $\boldsymbol{N}_5$, depicted below.



Thus, every variety of lattices either contains $\mathbb{HSP}(\boldsymbol{N}_5)$ or is contained in the class of modular lattices (and not both). Because the class of modular lattices is itself a variety, it is a splitting conjugate for $\mathbb{HSP}(\boldsymbol{N}_5)$ in the lattice of all varieties of lattices, hence $\mathbb{HSP}(\boldsymbol{N}_5)$ splits this lattice.

We express this by saying that in the variety of lattices, $\boldsymbol{N}_5$ is a *splitting algebra* and its *conjugate variety* is the variety of modular lattices. In extending this terminology to arbitrary varieties, it is convenient to insist that splitting algebras be subdirectly irreducible. In the absence of this demand, we could prove that every splitting algebra generates the same variety as a subdirectly irreducible splitting algebra, so there is no loss of generality.

Theorem 11.6. (cf. [108]) *If a congruence distributive variety* V *is generated (as a variety) by its finite members, then every splitting algebra in* V *is finite, and any two non-isomorphic splitting algebras have distinct conjugate varieties.*

A partial converse is supplied by the next theorem:

Theorem 11.7. ([19]) *In a variety* V *with EDPC, every finitely presented subdirectly irreducible algebra* $\boldsymbol{A}$ *is a splitting algebra and its conjugate variety is just the class of all* $\boldsymbol{B} \in \mathsf{V}$ *such that* $\boldsymbol{A} \notin \mathbb{SH}(\boldsymbol{B})$.

Corollary 11.8. *Let* V *be a variety of finite type with EDPC. If* V *is generated by its finite members, then the splitting algebras in* V *are exactly the finite subdirectly irreducible algebras in* V.

The assumptions of Corollary 11.8 apply to interior algebras, to Heyting algebras and to commutative residuated lattices satisfying $(x \wedge 1)^2 \approx x \wedge 1$.

Results of this kind can be useful when proving that a variety V of finite type has (no fewer than) $2^{\aleph_0}$ subvarieties. This happens, for instance, when V has EDPC and a sequence $\boldsymbol{A}_n$ of finite simple members, none of which embeds into any other. Let W_n be the conjugate variety of $\boldsymbol{A}_n$, which exists by Theorem 11.7. For $m \neq n$, it is given that $\boldsymbol{A}_n \notin \mathbb{IS}(\boldsymbol{A}_m)$, so $\boldsymbol{A}_n \notin \mathbb{SH}(\boldsymbol{A}_m)$, because $\boldsymbol{A}_m$ is simple. Thus, $\mathbb{HSP}\{\boldsymbol{A}_m : m \neq n\} \subseteq \mathsf{W}_n$, by Theorem 11.7, and so $\mathbb{HSP}(\boldsymbol{A}_n) \nsubseteq \mathbb{HSP}\{\boldsymbol{A}_m : m \neq n\}$. It follows that there are as many distinct (semisimple) subvarieties of V as there are subsets of ω.

The concept of splitting was introduced by Whitman [147]. It occurs in McKenzie's analysis of varieties of lattices [108], in Jankov's work on extensions of intuitionistic logic [72] and in Blok's determination of the degrees of incompleteness of all normal modal logics [14, 15] (also see [96, 132, 133]).

12. Maltsev Conditions

Our discussion of Maltsev conditions can be made more systematic. In what follows, it is convenient to assume that no algebra under discussion has a nullary basic operation. This is not restrictive, as nullary operations can be construed as unary operations f for which $f(x) \approx f(y)$ holds.

An *idempotent variety* is one that satisfies $f(x, x, \ldots, x) \approx x$ for each of its basic operation symbols f. More generally, a term t is *idempotent over* a variety V if V satisfies $t(x, x, \ldots, x) \approx x$.

Definition 12.1. A variety U of type $\rho\colon \mathcal{F} \longrightarrow \omega$ can be *interpreted into* a variety V of type $\tau\colon \mathcal{G} \longrightarrow \omega$ if there is a function $D\colon \mathcal{F} \longrightarrow T_\tau(\aleph_0)$ such that for every $\boldsymbol{A} \in \mathsf{V}$, the algebra $\boldsymbol{A}^D := \langle A; \{D(f)^{\boldsymbol{A}} : f \in \mathcal{F}\}\rangle$ belongs to U. When this is true, we write $\mathsf{U} \leq_I \mathsf{V}$. The relation $\leq_I$ is transitive. If there is also an interpretation E of V into U such that $\boldsymbol{A}^{DE} = \boldsymbol{A}$ and $\boldsymbol{B}^{ED} = \boldsymbol{B}$ for all $\boldsymbol{A} \in \mathsf{V}$ and $\boldsymbol{B} \in \mathsf{U}$, we say that U and V are *termwise equivalent.*

Example 12.2. The variety of semilattices $\langle A; \vee\rangle$ can be interpreted into the variety of implication algebras $\langle A; \to\rangle$, using the definition

$$x \vee y = (x \to y) \to y.$$

The respective varieties of Boolean algebras and Boolean rings (with identity) are termwise equivalent.

Definition 12.3. If U is a finitely axiomatized variety of finite type, then the class of varieties V with $\mathsf{U} \leq_I \mathsf{V}$ is called the *strong Maltsev class* defined by U, and the condition $\mathsf{U} \leq_I \mathsf{V}$ (on V) is called a *strong Maltsev condition.*

If U_n $(n \in \omega)$ is a decreasing sequence (with respect to $\leq_I$) of finitely axiomatized varieties of finite type, then the class of varieties

$$\{\mathsf{V} : \mathsf{U}_n \leq_I \mathsf{V} \text{ for some } n \in \omega\}$$

is called the *Maltsev class* defined by this sequence, and the associated condition on varieties V is called a *Maltsev condition.* It is said to be *idempotent* if the varieties U_n are idempotent, and *nontrivial* if it fails for some variety V (equivalently, for the variety of sets). It is *linear* if the axioms $t_1 \approx t_2$ for each U_n can be chosen linear, i.e., each t_i contains at most one occurrence of a basic operation symbol. Strong Maltsev conditions can be thought of as Maltsev conditions defined by constant sequences.

A congruence permutable variety K with minority term q satisfies a nontrivial linear idempotent strong Maltsev condition, because the idempotent variety of algebras $\langle A; q^{\boldsymbol{A}}\rangle$ of type $\langle 3\rangle$ satisfying $q(x, y, y) \approx x$ and $q(x, x, y) \approx y$ is interpretable into K. Most of the familiar varieties of classical algebra or logic satisfy nontrivial idempotent Maltsev conditions. An exception is the variety of semigroups. Maltsev classes are characterized by their closure properties, as in the next result, whose formulation is taken from [76].

Theorem 12.4. ([140, 120]) *A class $\mathcal{C}$ of varieties is a Maltsev class iff it is upward-closed with respect to $\leq_I$ and closed under the binary varietal product operation $\otimes$ and every member of $\mathcal{C}$ is contained in a finitely axiomatized member of $\mathcal{C}$.*

Our definition of $\otimes$ as a partial binary operation on varieties of the same type (in Section 6) is adequate for the statement of Theorem 12.4. Up to term equivalence, however, $\otimes$ is the restriction of a total binary operation on varieties of all possible types, under which Maltsev classes are still closed. For the details, see [76] or [59, p. 357].

The following characterization of varieties satisfying nontrivial *idempotent* Maltsev conditions combines contributions in [81, 128, 142, 149].

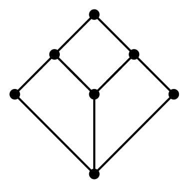
Theorem 12.5. *For any variety* V, *the following conditions are equivalent.*

(i) V *satisfies a nontrivial congruence equation in* $\wedge, \circ$ *only* (*see Definition* 9.7).
(ii) V *satisfies a nontrivial idempotent Maltsev condition.*
(iii) *There exist an integer* $n > 1$, *an* n*-ary term* t *that is idempotent over* V *and a choice of (not necessarily distinct) variables* x_{ij}, y_{ij} $(1 \leq i, j \leq n)$ *such that* $x_{ii} \neq y_{ii}$ *for each* i *and* V *satisfies*

$$\begin{array}{rcl} t(x_{11}, \ldots, x_{1n}) & \approx & t(y_{11}, \ldots, y_{1n}) \\ & \cdots\cdots & \\ t(x_{n1}, \ldots, x_{nn}) & \approx & t(y_{n1}, \ldots, y_{nn}). \end{array}$$

The equivalence of (ii) and (iii) follows from a result of Taylor [142, Cor. 5.3]. When (iii) holds, we call t a *Taylor term* for V, and there is no loss of generality in assuming that every x_{ij} and y_{ij} belongs to $\{x, y\}$. The implication (i) $\Rightarrow$ (ii) is essentially given by an algorithm of Wille [149] and Pixley [128]; see [81, Sec. 4]. The converse is implicit in the implication (1) $\Rightarrow$ (5) of [81, Thm. 4.12]. [1]

The lattice $\boldsymbol{D}_1$ of all convex subsets of a totally ordered 3–element set is depicted below. (It is neither modular nor meet semi-distributive.)



Theorem 12.6. (Kearnes & Kiss [81]) *A variety* V *satisfies no nontrivial idempotent Maltsev condition iff* $\boldsymbol{D}_1$ *is a sublattice of the congruence lattice of some member of* V.

[1] There, for a given V, the term on the right of the displayed inclusion in (5) can be replaced by a $\wedge, \circ$ term; cf. the proof of [81, Thm. 4.7]. Further characterizations of the varieties in Theorem 12.5 are provided in [81], one of which is the satisfaction of a nontrivial congruence equation in the infinitary language of meet-continuous lattices.

For locally finite varieties, Theorem 12.6 was proved by Hobby and McKenzie [68] in the context of tame congruence theory. Kearnes and Kiss show that if an arbitrary variety V satisfies a nontrivial idempotent Maltsev condition then the congruence lattices of all algebras in V satisfy

$$(x \wedge y \approx w \ \& \ x \wedge z \approx w \ \& \ x_{[2]} \approx w) \implies x \wedge (y \vee z) \approx w,$$

where $x_{[2]} := x \wedge (y \vee z) \wedge [(y \wedge (x \vee z)) \vee (z \wedge (x \vee y))]$. To see that $\boldsymbol{D}_1$ violates this quasi-equation, take $\{x, y, z\}$ to be the unique 3–element generating set for $\boldsymbol{D}_1$, with x as the meet-reducible generator. It is also shown in [81] that various other small lattices can play the role of $\boldsymbol{D}_1$ in Theorem 12.6.

Any variety with a weak difference term (Definition 6.6) satisfies a nontrivial idempotent Maltsev condition; the converse holds for locally finite varieties [68], but not generally—see [81, Sec. 6.1]. The Fundamental Theorem of Abelian Algebras extends in a weaker form to varieties of this kind:

Theorem 12.7. (Kearnes & Szendrei [83]) *In a variety satisfying a nontrivial idempotent Maltsev condition, every Abelian algebra is a subreduct of an algebra polynomially equivalent to a module over a ring with identity.*

In the present century, interest in idempotent Maltsev conditions has been stimulated by their connection with the tractability of *constraint satisfaction problems*, see [6, 92]. In the statement of Theorem 12.5, the rank n of the Taylor term t ostensibly depends on the variety V. For locally finite varieties, however, the rank is uniformly bounded. This was shown by Siggers [136], who exhibited a 6–ary Taylor term. After seeing that result, Kearnes, Markovic and McKenzie were able to prove the following.

Theorem 12.8. ([82]) *A locally finite variety* V *satisfies a nontrivial idempotent Maltsev condition iff it has an idempotent term* $t(x, y, z, w)$ *such that* V *satisfies* $t(x, y, z, y) \approx t(y, z, x, x)$.

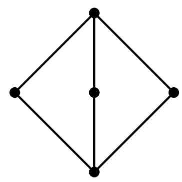
The order-dual of the lattice $\boldsymbol{D}_1$ is denoted by $\boldsymbol{D}_2$. Extending results about locally finite varieties in [68], Kearnes and Kiss [81] have shown:

Theorem 12.9. *The following conditions on a variety* V *are equivalent.*

(i) V *satisfies a nontrivial congruence equation in* $\wedge, \vee$ *only.*
(ii) V *satisfies an idempotent Maltsev condition that fails in the variety of semilattices.*
(iii) *The lattice* $\boldsymbol{D}_2$ *cannot be embedded into the congruence lattice of any algebra from* V.

The varieties satisfying these conditions have weak difference terms [83]. As noted earlier, they include all congruence n–permutable varieties. Recently, Valeriote and Willard [143] have shown that an idempotent variety U is n–permutable for some n iff $\mathsf{U} \not\leq_I \mathsf{DL}$, where DL is the variety of distributive lattices. Thus, an arbitrary variety is n–permutable for some n iff it satisfies an idempotent Maltsev condition that fails in DL.

The smallest modular non-distributive lattice $\boldsymbol{M}_3$ is depicted below. The next theorem combines results from [81, 83, 94, 148].



Theorem 12.10. *For a variety* V, *the following conditions are equivalent.*

(i) V *is congruence meet semi-distributive.*
(ii) V *is* congruence neutral, *i.e., the commutator equation* $[\varphi, \psi] = \varphi \cap \psi$ *holds for all* $\varphi, \psi \in \operatorname{Con} \boldsymbol{A}$, *whenever* $\boldsymbol{A} \in \mathsf{V}$.
(iii) $\boldsymbol{M}_3$ *cannot be embedded into the congruence lattice of any member of* V.
(iv) V *satisfies an idempotent Maltsev condition that fails in any non-trivial variety of modules.*
(v) *There exists a finite family* $\{\langle s_i(x,y,z), t_i(x,y,z)\rangle : i \in I\}$ *of pairs of ternary terms such that* V *satisfies*

$$s_i(x,y,x) \approx t_i(x,y,x) \quad (i \in I);$$
$$x \approx y \iff \&_{i \in I}\, [s_i(x,x,y) \approx t_i(x,x,y) \iff s_i(x,y,y) \approx t_i(x,y,y)].$$

The idempotent Maltsev conditions mentioned in Theorems 12.9 and 12.10 can be chosen linear. If a variety satisfies the conditions in one of those theorems, then it clearly witnesses Theorem 12.5. The variety of semilattices witnesses 12.10, but obviously not 12.9. The variety generated by the 2–element group exhibits the reverse behavior.

Theorem 12.11. ([81]) *A variety satisfies the conditions of Theorems* 12.9 *and* 12.10 *iff it is congruence* join *semi-distributive, i.e., the congruence lattices of its members satisfy* $x \vee y \approx x \vee z \implies x \vee (y \wedge z) \approx x \vee y$.

13. Categorical Equivalence

Varieties can obviously be considered as (concrete) categories. Termwise equivalent varieties are then categorically equivalent, but not conversely. We have noted in passing that various metalogical properties of deductive systems have algebraic counterparts such as congruence extensibility, EDPC and certain amalgamation or epimorphism-surjectivity properties. As it happens, these are categorical properties, i.e., they will persist under any category equivalence between two varieties of possibly different type—in which case they may be more easily established in one variety than in the other. It is therefore desirable to have purely algebraic criteria for the existence of such category equivalences.

An algebraic characterization of categorical equivalence for arbitrary pairs of varieties (among other classes) is provided in McKenzie's paper [114]. In principle, it allows us to establish an equivalence without producing two explicit functors. The characterization involves two constructions: idempotent images and matrix powers. We recall the definitions here.

Given an algebra $\boldsymbol{A}$ and a positive integer k, let $T_k(\boldsymbol{A})$ be the set of all k-ary terms of $\boldsymbol{A}$'s type, and let $T(\boldsymbol{A}) = \bigcup_{0<n\in\omega} T_n(\boldsymbol{A})$. For a unary term $\sigma \in T_1(\boldsymbol{A})$, the *$\sigma$-image* of $\boldsymbol{A}$ is the algebra

$$\boldsymbol{A}(\sigma) \;=\; \langle \sigma^{\boldsymbol{A}}[A]; \{t_\sigma : t \in T(\boldsymbol{A})\}\rangle,$$

where, for each positive n and each $t \in T_n(\boldsymbol{A})$,

$$t_\sigma(a_1, \dots, a_n) = \sigma^{\boldsymbol{A}}(t^{\boldsymbol{A}}(a_1, \dots, a_n)) \text{ for } a_1, \dots, a_n \in \sigma^{\boldsymbol{A}}[A].$$

Thus, every term of $\boldsymbol{A}$ gives rise to a *basic* operation of $\boldsymbol{A}(\sigma)$.

For each positive n, the n-th *matrix power* $\boldsymbol{A}^{[n]}$ of $\boldsymbol{A}$ is the algebra with universe A^n whose basic operations are all conceivable operations on n-tuples that can be defined using the term operations of $\boldsymbol{A}$. More precisely,

$$\boldsymbol{A}^{[n]} = \langle A^n; \{m_t : t \in (T_{kn}(\boldsymbol{A}))^n \text{ for some positive } k \in \omega\}\rangle,$$

where, for each $t = \langle t_1, \dots, t_n\rangle \in (T_{kn}(\boldsymbol{A}))^n$, we define $m_t \colon (A^n)^k \longrightarrow A^n$ as follows: if $a_j = \langle a_{j1}, \dots, a_{jn}\rangle \in A^n$ for $j = 1, \dots, k$, then

$$\pi_i(m_t(a_1, \dots, a_k)) = t_i^{\boldsymbol{A}}(a_{11}, \dots, a_{1n}, \dots, a_{k1}, \dots, a_{kn})$$

for each of the n projections $\pi_i \colon A^n \longrightarrow A$.

Let K be a class of similar algebras. A unary term σ of K is said to be *externally idempotent in* K if K satisfies $\sigma(\sigma(x)) \approx \sigma(x)$, and *invertible in* K if K satisfies

$$x \approx t(\sigma(t_1(x)), \dots, \sigma(t_r(x)))$$

for some positive integer r, some unary terms $t_1, \dots, t_r$ and some r-ary term t. Let $\mathsf{K}(\sigma)$ and $\mathsf{K}^{[n]}$ denote the isomorphic closures of $\{\boldsymbol{A}(\sigma) : \boldsymbol{A} \in \mathsf{K}\}$ and $\{\boldsymbol{A}^{[n]} : \boldsymbol{A} \in \mathsf{K}\}$, respectively. If K is a variety then so are $\mathsf{K}(\sigma)$ and $\mathsf{K}^{[n]}$, provided that σ is externally idempotent in K.

McKenzie's result, restricted to varieties, is as follows.

Theorem 13.1. (McKenzie [114]) *Two varieties K and M are categorically equivalent iff there is a positive integer n and an invertible externally idempotent unary term σ of $\mathsf{K}^{[n]}$ such that M is termwise equivalent to $\mathsf{K}^{[n]}(\sigma)$.*

The equivalence functor $\mathsf{K} \longrightarrow \mathsf{M}$ sends each $\boldsymbol{A} \in \mathsf{K}$ to a reduct of $\boldsymbol{A}^{[n]}(\sigma)$.

Example 13.2. An *IUML-algebra* is a bounded residuated distributive lattice-ordered commutative idempotent monoid with an order-reversing involution $\neg$ that fixes the monoid identity 1. A *Gödel algebra* is a subdirect product of totally ordered Heyting algebras. Both classes algebraize significant logics. It is shown in [57] that, among other correspondences, the varieties of IUML-algebras and Gödel algebras are categorically equivalent.

The proof uses Theorem 13.1. From left to right, the externally idempotent term σ is $x \wedge 1$, with $n = 1$. It is invertible because IUML-algebras satisfy

$$x \approx (x \wedge 1) \cdot \neg((\neg x) \wedge 1) \approx t(\sigma(t_1(x)), \sigma(t_2(x))),$$

where $t_1(x)$ is x and $t_2(x)$ is $\neg x$ and $t(x, y)$ is $x \cdot \neg y$ (and $\cdot$ is the monoid operation). The result is used in [57] to derive new properties of the logic algebraized by IUML-algebras.

A similar term σ can be used, with Theorem 13.1, to confirm the well-known category equivalence between lattice-ordered Abelian groups and cancellative hoops (cf. [16]). Morita-equivalent rings are another specialization.

The following two results pre-date [114], but elegant proofs based on Theorem 13.1 can be found in [114].

Theorem 13.3. ([34]) *A category equivalence between varieties preserves all linear Maltsev conditions.*

Theorem 13.4. ([69, 34]) *A variety* K *is categorically equivalent to the variety of Boolean algebras iff* $\mathsf{K} = \mathbb{HSP}(\boldsymbol{A})$ *for some primal algebra* $\boldsymbol{A}$.

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