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Dedicated to Prof. Suthep Suantai on the occasion of his 60^{th} anniversary

Convergence Analysis of Some Faster Iterative Schemes for G-Nonexpansive Mappings in Convex Metric Spaces Endowed with a Graph

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Abstract We propose two iterative schemes for three G-nonexpansive mappings and present their convergence analysis in the framework of a convex metric space endowed with a directed graph. Some numerical examples are given to support the claim that the proposed iterative schemes converge faster than all of Mann, Ishikawa and Noor iteration schemes. Our results generalize and extend several known results to the setup of a convex metric space endowed with a directed graphic structure, including the results in [S. Suantai, M. Donganont, W. Cholamjiak, Hybrid methods for a countable family of G-nonexpansive mappings in Hilbert spaces endowed with graphs, Mathematics (2019)].

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1. Introduction

A well known Banach contraction principle [1] attracted the attention of several mathematicians due to its applications in physical and engineering sciences. This result has been generalized in several directions either by modifying the topological structure of underlying space or extending the contractive conditions of mappings (see, [2] and the

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references therein). Jachymski [3] proved Banach contraction mapping principle in the setup of a complete metric space endowed with a graph. Consequently, some interesting results for different classes of mappings in the framework of Banach spaces endowed with graphic structure were obtained (see, e.g. [4–10]).

Approximation of fixed point of certain mappings constructing fixed point iterative processes is an important research area (see, [11–21]) and the references therein). The measure of rate of convergence of iterative schemes is an important parameter which helps to prefer one iterative schemes over the other. Abbas and Nazir [22] introduced a new iterative scheme which is faster than all of Picard, Mann and Agarwal *et al.* [23] iterative schemes. In 2019, Okeke [15] introduced the Picard-Ishikawa hybrid iterative process and proved that it converges faster than all of Picard, Krasnoselskii, Mann, Ishikawa, Noor [13], Picard-Mann [24] and Picard-Krasnoselskii [14] iterative schemes.

There are certain mappings which fail to have a fixed point in sets equipped with distance structure only. A rich geometric structure of Banach spaces such as convexity and differentiability of a norm are required to assure the existence of fixed point of such mappings. Consequently, the study of geometric properties of Banach spaces in connection with fixed point theory has become an active research area (see, e.g. [25, 26]). In 1970, Takahashi [27] introduced the concept of convexity in metric spaces. Several authors then obtained some interesting results in the setup of convex metric spaces (see, [18, 25, 28]).

To the best of our knowledge, the study of iterative schemes to approximate fixed point and common fixed points in the setting of convex metric space endowed with a graph is not yet carried out. The purpose of this paper to fill this gap. We propose the modified Picard-Ishikawa hybrid iterative scheme and the modified Abbas-Nazir iterative scheme for three G-nonexpansive mappings and study the convergence analysis of our iterative schemes in the framework of convex metric space endowed with a directed graph. We present some numerical examples to show that the proposed iterative schemes converge faster than all of Mann, Ishikawa and Noor iterations. Our results generalize and extend several known results including the results in [5, 6, 9, 10] among others.

2. Preliminaries

Let X be a metric space, C a nonempty subset of X and $T : C \to C$. A point $x \in C$ is called a *fixed point* of T if $x = Tx$. We denote the set of all fixed points of T by $F(T) := \{x \in C : Tx = x\}$.

Consisted with [3], following definitions will be needed in the sequel.

Suppose $V(G)$ is a set of vertices of a directed graph G and $E(G)$ is the set of edges of G which contains all the loops, that is $(x, x) \in E(G)$ for each $x \in V(G)$. We can identify the graph G with $(V(G), E(G))$, where G has no parallel edges. Denote by G^{-1} the graph obtained from G by reversing the direction of edges, that is, $E(G^{-1}) = \{(x, y) : (y, x) \in E(G)\}$. Suppose $\hat{G}$ is the undirected graph obtained from G by ignoring the direction of edges, that is $E(G) \cup E(G^{-1}) = E(\hat{G})$.

Following are some basic notions given in ([29, 30], and [31]):

Suppose x and y are vertices of a graph G, a *path* in G from x to y of length N ($N \in \mathbb{N} \cup \{0\}$) is a sequence $\{x_i\}_{i=0}^{N}$ of $N + 1$ vertices such that $x_0 = x$, $x_N = y$ and $(x_i, x_{i+1}) \in E(G)$ for $i = 0, 1, \cdots, N-1$. A graph G is *connected* if there is a path between any two vertices.

Let $G = (V(G), E(G))$ be a directed graph. A set $D \subseteq V(G)$ is said to be a *dominating set* if for every $v \in V(G)\backslash D$, there exists $d \in D$ such that $(d, v) \in E(G)$. Let $v \in V(G)$

and set $D \subseteq V(G)$. We say that v is *dominated by* D if $(d, v) \in E(G)$ for any $d \in D$. Let $A, B \subseteq V(G)$. If $(a, b) \in E(G)$ for all $a \in A$ and all $b \in B$, then we say that A *dominates* B. Unless otherwise stated, we shall assume that $E(G)$ contains all loops in this paper.

A graph G is called *transitive* if for each $x, y, z \in V(G)$ such that (x, y) and (y, z) are in $E(G)$, then $(x, z) \in E(G)$.

Let $G = (V(G), E(G))$ be a directed graph such that $V(G) = C$ and $E(G)$ contains all loops, that is, $\Delta = \{(x, x) : x \in C\} \subseteq E(G)$.

A mapping $T : C \to C$ is called a G-*contraction* if T preserves edges of G, that is, for each $x, y \in C$,
(i) $(x, y) \in E(G) \implies (Tx, Ty) \in E(G)$
(ii) T decreases weights of edges of G, that is, there exists $\ell \in (0, 1)$ such that for each $x, y \in C$, we have

$$(x, y) \in E(G) \implies d(Tx, Ty) \leq \ell d(x, y).$$

The mapping $T : C \to C$ is said to be G-*nonexpansive* ([32]) if the following conditions are satisfied:
(i) T preserves edges of G, i.e.

$$(x, y) \in E(G) \implies (Tx, Ty) \in E(G),$$

(ii) T non-increases weights of edges of G, tha is:

$$(x, y) \in E(G) \implies d(Tx, Ty) \leq d(x, y).$$

A mapping $T : C \to C$ is called G-*continuous* if for any given $x \in C$ and a sequence $\{x_n\}_{n\in\mathbb{N}}$ in C,

$$x_n \to x \text{ and } (x_n, x_{n+1}) \in E(G) \text{ imply that } Tx_n \to Tx.$$

Definition 2.1. Mappings $T_i : C \to C$ $(i = 1, 2, 3)$ satisfy *condition* (C) if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for each $r > 0$ such that for all $x \in C$, we have

$$\max\{d(x, T_1x), d(x, T_2x), d(x, T_3x)\} \geq f(d(x, F)),$$

where $F = F(T_1) \cap F(T_2) \cap F(T_3)$ and $d(x, F) = \inf\{d(x, p) : p \in F\}$.

Definition 2.2 ([27])**.** Let (X, d) be a metric space. A mapping $W : X \times X \times [0, 1] \to X$ is said to be a *convex structure on* X if for each $(x, y, \lambda) \in X \times X \times [0, 1]$ and $u \in X$,

$$d(u, W(x, y, \lambda)) \leq \lambda d(u, x) + (1 - \lambda)d(u, y). \tag{2.1}$$

A metric space X together with the convex structure W is called a convex metric space. We denote it by (X, d, W).

From the definition of convex structure W on X, it is obvious that

$$d(u, W(x, y, \lambda)) \geq (1 - \lambda)d(u, y) - \lambda d(u, x), \tag{2.2}$$

for each $x, y, u \in X$ and $\lambda \in [0, 1]$.

A nonvoid subset C of the convex metric space X is said to be convex if $W(x, y, \lambda) \in C$ whenever $(x, y, \lambda) \in C \times C \times [0, 1]$. Takahashi [27] proved that open spheres $B(x, r) = \{y \in X : d(y, x) < r\}$ and closed spheres $B[x, r] = \{y \in X : d(y, x) \leq r\}$ are convex. It is known that every normed space is a convex metric space. However, the converse is not true in general. There are many examples of convex metric spaces which are not embedded in any normed space ([25, 27]).

Let us recall some important concepts which are needed to measure the performances of iterative schemes.

Definition 2.3 ([33]). Let $\{a_n\}_{n=0}^{\infty}$, and $\{b_n\}_{n=0}^{\infty}$ be two sequences of positive numbers that converge to a, and b, respectively. Assume that

$$l = \lim_{n\to\infty} \frac{|a_n - a|}{|b_n - b|}. \tag{2.3}$$

1. If $l = 0$, then the sequence $\{a_n\}_{n=0}^{\infty}$ converges to a faster than the sequence $\{b_n\}_{n=0}^{\infty}$ converges to b;
2. If $0 < l < \infty$, then we say that the sequences $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ have the same rate of convergence.

Suppose that for two fixed point iterative processes $\{x_n\}$ and $\{y_n\}$ converging to the same fixed point z of T, the error estimates $d(x_n, z) \le a_n$ and $d(y_n, z) \le b_n$ for all $n \ge 1$, are available, where $\{a_n\}$ and $\{b_n\}$ are two sequences of positive real numbers converging to zero. Then, in view of above definition the following concept appears to be very natural (see, [28, 34]).

Definition 2.4 ([34]). If $\{a_n\}$ converges faster than $\{b_n\}$, then we say that the fixed point iterative sequence $\{x_n\}$ converges faster than the fixed point iterative sequence $\{y_n\}$ to z.

It has been observed that the comparison of the rate of convergence in the above definition depends on the choice of sequences $\{a_n\}$ and $\{b_n\}$ which are error bounds of $\{x_n\}$ and $\{y_n\}$, respectively. This method of comparison of the rate of convergence of two fixed point iterative sequences seems ambiguous (see, [28, 34]).

In 2013, Phuengrattana and Suantai [34] modified this concept as follows:

Suppose $\{x_n\}$ and $\{y_n\}$ are two iterative sequences converging to the same fixed point z of T, then we say that $\{x_n\}$ converges faster than $\{y_n\}$ to z if

$$\lim_{n\to\infty} \frac{d(x_n, z)}{d(y_n, z)} = 0.$$

Suppose C is a closed convex subset of a convex metric space X and $T_i : C \to C$ $(i = 1, 2, 3)$. Next, we recall the analogues of the following classical fixed point iterative schemes in convex metric spaces.

The Mann iterative sequence $\{u_n\}$ is given by

$$\begin{cases} u_0 = u \in C, \\ u_{n+1} = W(T_1 u_n, u_n, \alpha_n),\ n \ge 1, \end{cases} \tag{2.4}$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$.

The modified Ishikawa iterative sequence $\{p_n\}$ for two mappings is given by

$$\begin{cases} p_0 = p \in C, \\ y_n = W(T_2 p_n, p_n, \beta_n) \\ p_{n+1} = W(T_1 y_n, p_n, \alpha_n),\ n \ge 1, \end{cases} \tag{2.5}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$.

The modified Noor iterative sequence $\{v_n\}$ for three mappings is given by

$$\begin{cases} v_0 = v \in C, \\ z_n = W(T_3 v_n, v_n, \gamma_n) \\ y_n = W(T_2 z_n, v_n, \beta_n) \\ v_{n+1} = W(T_1 y_n, v_n, \alpha_n), \ n \geq 1, \end{cases} \tag{2.6}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0, 1]$.

In 2019, Okeke [15] introduced the Picard-Ishikawa hybrid iterative process and prove that it converges faster than all of Picard, Krasnoselskii, Mann, Ishikawa, Noor [13], Picard-Mann [24] and Picard-Krasnoselskii [14] iterative schemes.

Motivated by [15], we now propose the modified Picard-Ishikawa hybrid iterative scheme $\{s_n\}$ in the framework of convex metric space as follows:

$$\begin{cases} s_0 = s \in C \\ u_n = W(T_3 s_n, s_n, \beta_n) \\ v_n = W(T_2 u_n, s_n, \alpha_n) \\ s_{n+1} = T_1 v_n, \ n \geq 1, \end{cases} \tag{2.7}$$

where $\{\alpha_n\}, \{\beta_n\}$ are real sequences in $[0, 1]$.

We also propose the following iterative scheme for three mappings in convex metric spaces.

$$\begin{cases} x_0 = x \in C, \\ w_n = W(T_3 x_n, x_n, \gamma_n) \\ v_n = W(T_2 w_n, T_3 x_n, \beta_n) \\ x_{n+1} = W(T_1 w_n, T_2 v_n, \alpha_n), \ n \in \mathbb{N}, \end{cases} \tag{2.8}$$

where $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty}, \{\gamma_n\}_{n=0}^{\infty} \in [0, 1]$.

Remark 2.5. Observe that our iterative scheme (2.8) is the modified version of iterative scheme introduced by Abbas and Nazir in [22]. Also, note that an iterative scheme in [22] involve a single mapping whereas our scheme generate a sequence with three mappings. Moreover the scheme in [22] has been extended to the framework of convex metric spaces.

Definition 2.6 ([35])**.** A convex metric space X is called *uniformly convex* if for any $\varepsilon > 0$, there exists $\alpha > 0$ such that $d(z, W(x, y, \frac{1}{2})) \leq r(1 - \alpha) < r$ for all $r > 0$ and $x, y, z \in X$ with $d(z, x) \leq r$, $d(z, y) \leq r$ and $d(x, y) \geq r\varepsilon$.

A closed subset X of the unit ball $S_1(0) = \{x \in H : \|x\| \leq 1\}$ in a Hilbert space H with diameter $\delta(X) \leq \sqrt{2}$, turns out to be a uniformly convex metric space with $d(x, y) = \cos^{-1}\langle x, y\rangle$ for each $x, y \in X$ and $W(x, y, \alpha) = \frac{\alpha x + (1-\alpha)y}{\|\alpha x + (1-\alpha)y\|}$ for each $x, y \in X$ and $\alpha \in I = [0, 1]$ ([36]).

Suppose $\{x_n\}$ is a bounded sequence of a convex metric space X. For $x \in X$, we set

$$r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n).$$

The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in X\}$$

and the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}.$$

Definition 2.7 ([37, 38])**.** A sequence $\{x_n\}$ in X is said to *Δ-converge to* $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case we write $\Delta\text{-}\lim_n x_n = x$ and call x the Δ-limit of $\{x_n\}$.

We use the following notations for the rest of this paper, $w_\Delta(x_n) := \bigcup\{A(\{u_n\})\}$, where the union is taken over all subsequences $\{u_n\}$ of $\{x_n\}$.

The concept of Δ-convergence was introduced in general metric space by Lim [38]. In 2008, Kirk and Panyanak [37] specialized this concept in CAT(0) spaces and proved that several known results in Banach spaces involving weak convergence have precise analogues in the framework of CAT(0) spaces. Moreover, they made a strong case for calling Δ-convergence "weak" convergence at least in the setting of CAT(0) spaces (see, [37]). For more discussion in this direction, we refer to [39], [40] and the references therein.

For sake of completeness, we give the following definition.

Definition 2.8. Suppose $G = (V(G), E(G))$ is a directed graph such that $V(G) = C$. A mapping $T : C \to C$ is said to be *G-demiclosed at 0* if for any sequence $\{x_n\} \subseteq C$, $(x_n, x_{n+1}) \in E(G)$, $\{x_n\}$ Δ-converges to x and $Tx_n \to 0$, then $Tx = 0$.

Suppose (X, d) is a metric space. *A geodesic path joining $x \in X$ to $y \in X$, or a geodesic from x to y is a map c from a closed interval $[0, \ell] \subseteq R$ to X such that $c(0) = x$, $c(\ell) = y$, and $d(c(t), c(t')) = |t - t'|$ for each $t, t' \in [0, \ell]$. In particular, c is an isometry and $d(x, y) = \ell$. The image α of c is known as a geodesic (or metric) segment joining x and y. When it is unique, the geodesic segment is denoted by $[x, y]$. The space (X, d) is said to be a geodesic space if every two points of X are joined by a geodesic segment, and X is said to be uniquely geodesic if there is exactly one geodesic joining x and y for each $x, y \in X$. A subset $Y \subseteq X$ is said to be convex if Y includes every geodesic segment joining any two of its points.*

Remark 2.9 ([41])**.** Let X be a geodesic space. If each pair of geodesics $c_1 : [0, a_1] \to X$ and $c_2 : [0, a_2] \to X$ with $c_1(0) = c_2(0)$ satisfy the inequality $d(c_1(ta_1), c_2(ta_2)) \leq td(c_1(a_1), c_2(a_2))$ for all $t \in [0, 1]$, then one says that the metric on X is convex. It is easy to see that the metric on a CAT(0) space is convex. In general having a convex metric is a weaker property than being $CAT(0)$. There are, however, several important classes of spaces in which convexity of the metric is equivalent to the CAT(0) condition, including Riemannian manifolds and M_k-polyhedral complexes.

Some basic properties of Δ-convergence are as follows:

Proposition 2.10 ([37])**.** *Let X be a complete CAT(0) space.*
(i) If a sequence $\{x_n\}$ in X Δ-converges to $x \in X$, then

$$x \in \bigcap_{k=1}^{\infty} \overline{conv}\{x_k, x_{k+1}, \cdots\},$$

where $\overline{conv}(A) = \bigcap\{B : B \supseteq A$ and B is closed and convex$\}$.
(ii) Every bounded sequence in X has a Δ-convergent subsequence.
(iii) If C is a closed convex subset of X and $\{x_n\}$ is a bounded sequence in C, then the asymptotic center of $\{x_n\}$ is in C.

Proposition 2.11 ([41])**.** *If X is a CAT(0) space, then the distance function $d : X \times X \to \mathbb{R}$ is convex, i.e. given any pair of geodesics $c : [0, 1] \to X$, and $c' : [0, 1] \to X$,* *parameterized proportional to arc length, the following inequality holds for all* $t \in [0,1]$:

$$d(c(t), c'(t)) \leq (1-t)d(c(0), c'(0)) + td(c(1), c'(1)).$$

The following property, called **Property E** will be needed in this paper.

Definition 2.12. Suppose X is a convex metric space and C is a nonempty subset of X. Let $G = (V(G), E(G))$ be a directed graph such that $V(G) = C$. Then C is said to have *Property E* if for each sequence $\{x_n\}$ in C with $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$ which Δ-converges to $x \in C$, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $(x_{n_k}, x) \in E(G)$ for each $k \in \mathbb{N}$.

Remark 2.13. The concept of Property E in Definition 2.8 above is the analogue of the concept of Property G (see, [42]) in the setting of convex metric spaces.

Next, we prove the following lemmas which will be useful in this study.

Lemma 2.14. *Let X be a convex metric space and $\{x_n\}$ a sequence in X. Let $u, v \in X$ be such that $\lim_{n\to\infty} d(x_n, u)$ and $\lim_{n\to\infty} d(x_n, v)$ exist. Suppose that $\{x_{n_k}\}$ and $\{x_{m_k}\}$ are subsequences of $\{x_n\}$ which Δ-converge to u and v, respectively, then $u = v$.*

Proof. By the definition of Δ-convergence, it follows that $\Delta\text{-}\lim_{k\to\infty} x_{n_k} = u$ and $\Delta\text{-}\lim_{k\to\infty} x_{m_k} = v$. We claim that $u = v$. Suppose that $u \neq v$, then we have

$$\begin{array}{rcl}
\lim_{n\to\infty} d(x_n, u) & = & \lim_{k\to\infty} d(x_{n_k}, u) \\
& < & \lim_{k\to\infty} d(x_{n_k}, v) \\
& = & \lim_{n\to\infty} d(x_n, v) \\
& = & \lim_{k\to\infty} d(x_{m_k}, v) \\
& < & \lim_{k\to\infty} d(x_{m_k}, u) \\
& = & \lim_{n\to\infty} d(x_n, u),
\end{array}$$

a contradiction. Thus, $u = v$. ∎

Remark 2.15. Lemma 2.1 is the analogue of ([43], Lemma 2.7) in the framework of convex metric spaces.

Lemma 2.16. *Let X be a convex metric space and C a nonempty closed subset of X having Property E. Let $G = (V(G), E(G))$ be a directed graph with $V(G) = C$ and $T : C \to C$ a G-nonexpansive mapping. Then $I - T$ is demiclosed at zero.*

Proof. Suppose $\{x_n\}$ is a sequence in C which Δ-converges to $x \in C$ with $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$ and $(I-T)x_n \to 0$. Since C has Property E, it there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $(x_{n_k}, x) \in E(G)$ for all $k \in \mathbb{N}$. Since $(I-T)x_n \to 0$, we obtain that

$$\lim_{n\to\infty} d(x_{n_k}, Tx_{n_k}) = 0.$$

If $x \neq Tx$, then by the fact that x is the unique asymptotic center of every subsequence $\{x_{n_k}\}$ of $\{x_n\}$, we have

$$\begin{array}{rcl}
\limsup_{n\to\infty} d(x_{n_k}, x) & < & \limsup_{n\to\infty} d(x_{n_k}, Tx) \\
& \leq & \limsup_{n\to\infty} \left(d(x_{n_k}, Tx_{n_k}) + d(Tx_{n_k}, Tx)\right) \\
& \leq & \limsup_{n\to\infty} d(x_{n_k}, x).
\end{array}$$

a contradiction and hence $(I-T)x = 0$. ∎

Remark 2.17. Lemma 2.2 is the analogue of ([42], Proposition 3.5) in the setting of convex metric spaces.

Lemma 2.18. *Let $\{x_n\}_{n=1}^{\infty}$ be a bounded sequence in a complete CAT(0) space X. If for any Δ-convergent subsequence $\{x_{n_j}\}_{j=1}^{\infty}$ of $\{x_n\}$, both $\{x_{n_j}\}_{j=1}^{\infty}$ and $\{x_{n_{j+1}}\}_{j=1}^{\infty}$ Δ-converges to the same point in X, then the sequence $\{x_n\}_{n=1}^{\infty}$ is Δ-convergent.*

Proof. Since $\{x_n\}_{n=1}^{\infty}$ is a bounded sequence, it follows that the closed convex hull $\overline{conv}(\{x_n\}_{n=1}^{\infty})$ of $\{x_n\}_{n=1}^{\infty}$ is a bounded closed convex subset of X. By Proposition 2.1, we have $\overline{conv}(\{x_n\}_{n=1}^{\infty})$ of $\{x_n\}_{n=1}^{\infty}$ is a bounded closed convex subset of X, which is Δ-convergent in X. Hence, there exists a subsequence $\{x_{n_j}\}_{j=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ which Δ-converges to $x \in \overline{conv}(\{x_n\}_{n=1}^{\infty})$. We can write

$$\{x_n\}_{n=n_1}^{\infty} = \bigcup_{i=0}^{\infty}\{x_{n_{j+i}}\}_{j=1}^{\infty}.$$

We next prove by induction that for all positive integer m, that $\cup_{i=0}^{m}\{x_{n_{j+1}}\}_{j=1}^{\infty}$ Δ-converges to x. Using the hypothesis that $\{x_{n_j}\}$ and $\{x_{n_{j+1}}\}$ Δ-converges to x. It follows that $\cup_{i=0}^{\infty}\{x_{n_{j+i+1}}\}_{j=1}^{\infty}$ Δ-converges to x. This means that $\cup_{i=1}^{m+1}\{x_{n_{j+i}}\}_{j=1}^{\infty}$ Δ-converges to x. Therefore, we have

$$\{x_{n_j}\}_{j=1}^{\infty}\bigcup\left(\bigcup_{i=1}^{m+1}\{x_{n_{j+i}}\}_{j=1}^{\infty}\right) = \bigcup_{i=0}^{m+1}\{x_{n_{j+i}}\}_{j=1}^{\infty}$$

Δ-converges to x. Hence, for any positive integer m, $\cup_{i=0}^{m}\{x_{n_{j+i}}\}_{j=1}^{\infty}$ Δ-converges to x. On taking limit as $m \to \infty$, the sequence $\{x_n\}_{n=n_1}^{\infty}$ Δ-converges to x. ■

Remark 2.19. Lemma 2.3 is the analogue of ([44], Lemma 3.1) in the setting of CAT(0) spaces.

The following lemmas will also be needed in this paper.

Lemma 2.20 ([36])**.** *Let X be a uniformly convex metric space with continuous convex structure W. Let $x \in X$ and $\{a_n\}$ be a sequence in $[b, c]$ for some $b, c \in (0, 1)$. If $\{u_n\}$ and $\{v_n\}$ are sequences in X such that $\limsup_{n\to\infty} d(u_n, x) \leq r$, $\limsup_{n\to\infty} d(v_n, x) \leq r$ and $\lim_{n\to\infty} d(W(u_n, v_n, a_n), x) = r$ for some $r \geq 0$, then $\lim_{n\to\infty} d(u_n, v_n) = 0$.*

Lemma 2.21 ([45])**.** *Let $\{a_n\}$ and $\{t_n\}$ be two sequences of nonnegative real numbers satisfying the inequality:*

$$a_{n+1} \leq a_n + t_n,$$

for each $n \geq 1$. If $\sum_{n=1}^{\infty} t_n < \infty$, then $\lim_{n\to\infty} a_n$ exists.

3. Main Results

Proposition 3.1. *Let C be a closed subset of a convex metric space X endowed with a directed graph G such that $V(G) = C$. Suppose $E(G)$ is convex and G is transitive. Let $T_i : C \to C$, $(i = 1, 2, 3)$ be three G-nonexpansive mappings such that $F = F(T_1) \cap F(T_2) \cap F(T_3) \neq \emptyset$. For an arbitrary $x_0 \in C$, define the sequence $\{x_n\}$ as in (2.8). Let $p_0 \in F$ be such that $(x_0, p_0), (p_0, x_0)$ are in $E(G)$. Then (x_n, p_0), (w_n, p_0), (v_n, p_0), (p_0, x_n), (p_0, w_n), (p_0, v_n), (x_n, v_n), (x_n, w_n) and (x_n, x_{n+1}) are in $E(G)$.*

Proof. We shall prove this result by induction as follows. Using the fact that T_3 is edge-preserving and $(x_0, p_0) \in E(G)$, it follows that $(T_3x_0, p_0) \in E(G)$ so that $(w_0, p_0) \in E(G)$, by the fact that $E(G)$ is convex. Similarly, by edge-preserving of T_2 and $(w_0, p_0) \in E(G)$, we have $(T_2w_0, p_0) \in E(G)$, so that $(v_0, p_0) \in E(G)$. Moreover, since T_1 is edge-preseving and $(v_0, p_0) \in E(G)$, we obtain $(T_1v_0, p_0) \in E(G)$. Using the convexity of $E(G)$ and the fact that $(T_1v_0, p_0), (T_2w_0, p_0) \in E(G)$, we obtain $(x_1, p_0) \in E(G)$. Hence, by edge-preserving of T_3, we have $(T_3x_1, p_0) \in E(G)$. Again, by the convexity of $E(G)$ and $(T_3x_1, p_0), (x_1, p_0) \in E(G)$, we obtain $(w_1, p_0) \in E(G)$. Similarly, using the convexity of $E(G)$ and the fact that $(T_3x_1, p_0), (x_1, p_0) \in E(G)$, we obtain $(w_1, p_0) \in E(G)$. Hence, since T_2 is edge-preserving, $(T_2w_1, p_0) \in E(G)$. Using the convexity of $E(G)$ and $(T_2w_1, p_0), (w_1, p_0) \in E(G)$, we have $(v_1, p_0) \in E(G)$. Therefore, $(T_1v_1, p_0) \in E(G)$.

Next, suppose that $(x_k, p_0) \in E(G)$. Since T_3 is edge-preserving, we have $(T_3x_k, p_0) \in E(G)$. Hence, $(w_k, p_0) \in E(G)$, since $E(G)$ is convex. Hence by the fact that T_2 is edge-preserving and $(w_k, p_0) \in E(G)$, we obtain $(T_2w_k, p_0) \in E(G)$, then $(v_k, p_0) \in E(G)$, by convexity of $E(G)$. Since T_1 is edge-preserving, we obtain $(T_1v_k, p_0) \in E(G)$. Using the convexity of $E(G)$, we obtain $(x_{k+1}, p_0) \in E(G)$. Therefore, by edge-preserving of T_3, we have $(T_3x_{k+1}, p_0) \in E(G)$, so that $(w_{k+1}, p_0) \in E(G)$, since $E(G)$ is convex. Similarly, by edge-preserving of T_2, we get $(T_2w_{k+1}, p_0) \in E(G)$, so that $(v_{k+1}, p_0) \in E(G)$, since $E(G)$ is convex. Hence, (x_n, p_0), (w_n, p_0), $(v_n, p_0) \in E(G)$ for each $n \geq 1$. Using the fact that T_3 is edge-preserving and $(p_0, x_0) \in E(G)$, we obtain $(p_0, T_3x_0) \in E(G)$, so that $(p_0, w_0) \in E(G)$. Similarly, since T_2 is edge-preserving and $(p_0, w_0) \in E(G)$, we obtain $(p_0, T_2x_0) \in E(G)$, so that $(p_0, v_0) \in E(G)$. Therefore, by a similar argument we can prove that (p_0, x_n), (p_0, w_n), $(p_0, v_n) \in E(G)$ for each $n \geq 1$ using the assumption that $(p_0, x_0) \in E(G)$, $(p_0, w_0) \in E(G)$, and $(p_0, v_0) \in E(G)$. Using the transitivity of G, we obtain (x_n, v_n), (x_n, w_n), $(x_n, x_{n+1}) \in E(G)$. ∎

Next, we obtain the following results.

Proposition 3.2. *Let C be a closed subset of a convex metric space X endowed with a directed graph G such that $V(G) = C$. Suppose $E(G)$ is convex and G is transitive. Let $T_i : C \to C$, $(i = 1, 2, 3)$ be three G-nonexpansive mappings such that $F = F(T_1) \cap F(T_2) \cap F(T_3) \neq \emptyset$. For an arbitrary $s_0 \in C$, define the sequence $\{s_n\}$ as in (2.7). Let $p_0 \in F$ be such that $(s_0, p_0), (p_0, s_0)$ are in $E(G)$. Then (s_n, p_0), (u_n, p_0), (v_n, p_0), (p_0, s_n), (p_0, u_n), (p_0, v_n), (s_n, v_n), (s_n, u_n) and (s_n, s_{n+1}) are in $E(G)$.*

Proof. The proof of Proposition 3.2 follows on the similar lines as in the proof of Proposition 3.1. ∎

Lemma 3.3. *Let C be a closed subset of a uniformly convex metric space X endowed with a directed graph G such that $V(G) = C$. Suppose $E(G)$ is convex and G is transitive. Let $T_i : C \to C$, $(i = 1, 2, 3)$ be three G-nonexpansive mappings such that $F = F(T_1) \cap F(T_2) \cap F(T_3) \neq \emptyset$. For arbitrary $x_0 \in C$ and $p_0 \in F$, define the sequence $\{x_n\}$ as in (2.8). Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be real sequences in $[\delta, 1 - \delta]$ for some $\delta \in (0, 1)$ and (x_0, p_0), $(p_0, x_0) \in E(G)$. Then*
(i) $\lim_{n\to\infty} d(x_n, p_0)$ exists;
(ii) $\lim_{n\to\infty} d(x_n, T_1x_n) = \lim_{n\to\infty} d(x_n, T_2x_n) = \lim_{n\to\infty} d(x_n, T_3x_n) = 0$.

Proof. (i) Suppose $p_0 \in F$, it follows from Proposition 3.1 that (x_n, p_0), (v_n, p_0), (w_n, p_0), (x_n, v_n), $(x_n, w_n) \in E(G)$. Since T_i is G-nonexpansive for each $i = 1, 2, 3$, by (2.8), we obtain that

$$\begin{aligned} d(w_n,p_0) &= d(W(T_3x_n,x_n,\gamma_n),p_0)\\ &\leq \gamma_n d(T_3x_n,p_0)+(1-\gamma_n)d(x_n,p_0)\\ &\leq \gamma_n d(x_n,p_0)+(1-\gamma_n)d(x_n,p_0)\\ &= d(x_n,p_0). \end{aligned} \tag{3.1}$$

$$\begin{aligned} d(v_n,p_0) &= d(W(T_2w_n,T_3x_n,\beta_n),p_0)\\ &\leq \beta_n d(T_2w_n,p_0)+(1-\beta_n)d(T_3x_n,p_0)\\ &\leq \beta_n d(w_n,p_0)+(1-\beta_n)d(x_n,p_0). \end{aligned} \tag{3.2}$$

Using (3.1) in (3.2), we get that

$$\begin{aligned} d(v_n,p_0) &\leq \beta_n d(x_n,p_0)+(1-\beta_n)d(x_n,p_0)\\ &= d(x_n,p_0). \end{aligned} \tag{3.3}$$

Also,

$$\begin{aligned} d(x_{n+1},p_0) &= d(W(T_1w_n,T_2v_n,\alpha_n),p_0)\\ &\leq \alpha_n d(T_1w_n,p_0)+(1-\alpha_n)d(T_2v_n,p_0)\\ &\leq \alpha_n d(w_n,p_0)+(1-\alpha_n)d(v_n,p_0). \end{aligned} \tag{3.4}$$

Using (3.1) and (3.3) in (3.4), we get

$$\begin{aligned} d(x_{n+1},p_0) &\leq \alpha_n d(x_n,p_0)+(1-\alpha_n)d(x_n,p_0)\\ &= d(x_n,p_0). \end{aligned} \tag{3.5}$$

Hence, by Lemma 2.5, $\lim_{n\to\infty} d(x_n,p_0)$ exists. In particular, $\{x_n\}$ is bounded.
(ii) Suppose that $\lim_{n\to\infty} d(x_n,p_0)=k$, if $k=0$, then by G-nonexpansiveness of T_i for each $i=1,2,3$, we obtain

$$\begin{aligned} d(x_n,T_ix_n) &\leq d(x_n,p_0)+d(p_0,T_ix_n)\\ &\leq d(x_n,p_0)+d(p_0,x_n). \end{aligned} \tag{3.6}$$

Hence, the results follows. Assume that $k>0$, from (3.1) we have by taking lim sup on the both sides, we have

$$\limsup_{n\to\infty} d(w_n,p_0) \leq \limsup_{n\to\infty} d(x_n,p_0)=k. \tag{3.7}$$

Therefore, by G-nonexpansiveness of T_1, we have $d(T_1w_n,p_0)\leq d(w_n,p_0)$. By taking lim sup of both sides of (3.7), we get

$$\limsup_{n\to\infty} d(T_1w_n,p_0)\leq k. \tag{3.8}$$

Next, taking lim sup on both sides of (3.3), we get

$$\limsup_{n\to\infty} d(v_n,p_0) \leq \limsup_{n\to\infty} d(x_n,p_0)=k. \tag{3.9}$$

By G-nonexpansiveness of T_2, we have $d(T_2v_n,p_0)\leq d(v_n,p_0)$. Taking lim sup of both sides using (3.9), we get

$$\limsup_{n\to\infty} d(T_2v_n,p_0)\leq k. \tag{3.10}$$

Similarly, by (3.7) we have

$$\limsup_{n\to\infty} d(T_2w_n,p_0)\leq k. \tag{3.11}$$

Since $\lim_{n\to\infty} d(x_{n+1},p_0)=k$. On taking limit as $n\to\infty$ in (3.4), we have

$$\lim_{n\to\infty} d(W(T_1w_n,T_2v_n,\alpha_n),p_0)=k. \tag{3.12}$$

By (3.8), (3.10), (3.12) and Lemma 2.4, we have:

$$\lim_{n\to\infty} d(T_1 w_n, T_2 v_n) = 0. \tag{3.13}$$

Using (3.4), $d(x_{n+1}, p_0) \le \alpha_n d(w_n, p_0) + (1-\alpha_n) d(v_n, p_0)$, we have

$$\liminf_{n\to\infty} [\alpha_n d(w_n, p_0) + (1-\alpha_n) d(v_n, p_0)] \ge k. \tag{3.14}$$

Using (3.7), (3.9) and (3.14), we get

$$\lim_{n\to\infty} [\alpha_n d(w_n, p_0) + (1-\alpha_n) d(v_n, p_0)] = k. \tag{3.15}$$

Using (3.1), (3.3) and (3.15), we have

$$\lim_{n\to\infty} d(W(T_3 x_n, x_n, \gamma_n), p_0) = k. \tag{3.16}$$

As $\limsup_{n\to\infty} d(T_3 x_n, p_0) \le \limsup_{n\to\infty} d(x_n, p_0) = k$, by (3.16) and Lemma 2.4, we have:

$$\lim_{n\to\infty} d(T_3 x_n, x_n) = 0. \tag{3.17}$$

Also,

$$\begin{aligned} d(x_{n+1}, p_0) &= d(W(T_1 w_n, T_2 v_n, \alpha_n), p_0) \\ &\le \alpha_n d(T_1 w_n, p_0) + (1-\alpha_n) d(T_2 v_n, p_0) \\ &\le \alpha_n d(T_1 w_n, T_2 v_n) + \alpha_n d(T_2 v_n, p_0) + (1-\alpha_n) d(T_2 v_n, p_0) \\ &= \alpha_n d(T_1 w_n, T_2 v_n) + d(T_2 v_n, p_0). \end{aligned} \tag{3.18}$$

Taking lim inf of both sides using inequality (3.13) and (3.18), we get

$$k \le \liminf_{n\to\infty} d(T_2 v_n, p_0). \tag{3.19}$$

Using (3.10) and (3.19), we obtain $\lim_{n\to\infty} d(T_2 v_n, p_0) = k$. Moreover, we have

$$\begin{aligned} d(T_2 v_n, p_0) &\le d(T_2 v_n, T_1 w_n) + d(T_1 w_n, p_0) \\ &\le d(T_2 v_n, T_1 w_n) + d(w_n, p_0). \end{aligned} \tag{3.20}$$

This implies that

$$k \le \liminf_{n\to\infty} d(w_n, p_0). \tag{3.21}$$

From (3.7) and (3.21), we get

$$\lim_{n\to\infty} d(w_n, p_0) = k. \tag{3.22}$$

Similarly,

$$\lim_{n\to\infty} d(v_n, p_0) = k. \tag{3.23}$$

Using (3.2), (3.22) and (3.23), we have

$$\lim_{n\to\infty} d(v_n, p_0) = \lim_{n\to\infty} d(W(T_2 w_n, T_3 x_n, \beta_n), p_0) = k. \tag{3.24}$$

By (3.11), (3.22), (3.24) and Lemma 2.4, we have

$$\lim_{n\to\infty} d(T_2 w_n, T_3 x_n) = 0. \tag{3.25}$$

Now we have

$$\begin{aligned} d(w_n, x_n) &= d(W(T_3 x_n, x_n, \gamma_n), x_n) \\ &\le \gamma_n d(T_3 x_n, x_n) + (1-\gamma_n) d(x_n, x_n) \longrightarrow 0 \text{ as } n \to \infty. \end{aligned} \tag{3.26}$$

Next, we have

$$\begin{aligned} d(w_n, T_3x_n) &= d(W(T_3x_n, x_n, \gamma_n), T_3x_n) \\ &\leq \gamma_n d(T_3x_n, T_3x_n) + (1-\gamma_n)d(x_n, T_3x_n) \longrightarrow 0 \text{ as } n \to \infty. \quad (3.27) \end{aligned}$$

Using (3.27), we have

$$\begin{aligned} d(v_n, w_n) &= d(W(T_2w_n, T_3x_n, \beta_n), w_n) \\ &\leq \beta_n d(T_2w_n, w_n) + (1-\beta_n)d(T_3x_n, w_n) \longrightarrow 0 \text{ as } n \to \infty. \quad (3.28) \end{aligned}$$

Using (3.26) and G-nonexpansiveness of T_2, we get

$$\begin{aligned} d(T_2x_n, x_n) &= d(T_2x_n, T_2w_n) + d(T_2w_n, x_n) \\ &\leq d(x_n, w_n) + d(T_2w_n, x_n) \\ &\leq d(x_n, w_n) + d(T_2w_n, w_n) + d(w_n, x_n) \longrightarrow 0 \text{ as } n \to \infty. \quad (3.29) \end{aligned}$$

Next, we have

$$\begin{aligned} d(T_1x_n, x_n) &= d(T_1x_n, w_n) + d(w_n, x_n) \\ &\leq d(T_1x_n, T_1w_n) + d(T_1w_n, w_n) + d(w_n, x_n) \\ &\leq d(x_n, w_n) + d(T_1w_n, w_n) + d(w_n, x_n) \longrightarrow 0 \text{ as } n \to \infty. \quad (3.30) \end{aligned}$$

Hence, we obtain

$$\lim_{n\to\infty} d(x_n, T_1x_n) = \lim_{n\to\infty} d(x_n, T_2x_n) = \lim_{n\to\infty} d(x_n, T_3x_n) = 0. \quad (3.31)$$

The proof of Lemma 3.1 is completed. ■

Lemma 3.4. *Let C be a closed subset of a uniformly convex metric space X endowed with a directed graph G such that $V(G) = C$. Suppose $E(G)$ is convex and G is transitive. Let $T_i : C \to C$, $(i = 1, 2, 3)$ be three G-nonexpansive mappings such that $F = F(T_1) \cap F(T_2) \cap F(T_3) \neq \emptyset$. For arbitrary $s_0 \in C$ and $p_0 \in F$, define the sequence $\{s_n\}$ as in (2.7). Let $\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences in $[\delta, 1-\delta]$ for some $\delta \in (0, 1)$ and (s_0, p_0), $(p_0, s_0) \in E(G)$. Then*
(i) $\lim_{n\to\infty} d(s_n, p_0)$ *exists;*
(ii) $\lim_{n\to\infty} d(s_n, T_1s_n) = \lim_{n\to\infty} d(s_n, T_2s_n) = \lim_{n\to\infty} d(s_n, T_3s_n) = 0.$

Proof. The proof follows on the similar lines as in the proof of Lemma 3.1. ■

Next, we prove the following Δ-convergence results in CAT(0) spaces.

Theorem 3.5. *Let C be a closed convex subset of a complete CAT(0) space X endowed with a directed graph G such that $V(G) = C$ and C has Property E. Suppose $E(G)$ is convex and G is transitive. Let $T_i : C \to C$, $(i = 1, 2, 3)$ be three G-nonexpansive mappings such that $F = F(T_1) \cap F(T_2) \cap F(T_3) \neq \emptyset$. For arbitrary $x_0 \in C$ and $p_0 \in F$, define the sequence $\{x_n\}$ as in (2.8). Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be real sequences in $[\delta, 1-\delta]$ for some $\delta \in (0, 1)$ and (x_0, p_0), $(p_0, x_0) \in E(G)$. Then $\{x_n\}$ Δ-converges to a common fixed point of T_1, T_2 and T_3.*

Proof. By Lemma 3.1 (ii), we get

$$\lim_{n\to\infty} d(x_n, T_1x_n) = \lim_{n\to\infty} d(x_n, T_2x_n) = \lim_{n\to\infty} d(x_n, T_3x_n) = 0.$$

It follows from Lemma 3.1 (i) that $\lim_{n\to\infty} d(x_n, p)$ exists for each $p \in F$. Hence, $\{x_n\}$ is bounded. We first show that $w_\Delta(x_n) \subseteq F$. Suppose that $u \in w_\Delta(x_n)$, then there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. By Proposition 2.1 (ii), there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that Δ-$\lim_n v_n = v$ for some $v \in C$. Using Proposition

2.1 (ii), we have $v \in F$. By Lemma 3.1, we get that $\lim_{n\to\infty} d(x_n, v)$ exists. Therefore by Lemma 2.1, we have $u = v$. Hence, $u = v \in F$ so that $w_\Delta(x_n) \subseteq F$. To prove that $\{x_n\}$ Δ-converges to the common fixed point of T_i $(i = 1, 2, 3)$, we show that $w_\Delta(x_n)$ consists of exactly one point. Suppose $\{u_n\}$ is a subsequence of $\{x_n\}$. Since C has Property E. Then by Proposition 2.1, there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that Δ-$\lim_n v_n = v$ for some $v \in C$. Suppose $A(\{u_n\}) = \{u\}$ and $A(\{x_n\}) = \{x\}$. We have already proved that $u = v \in F$. Therefore, by the uniqueness of asymptotic centers and Lemma 2.1, $x = v \in F$. Hence, $w_\Delta(x_n) = \{x\}$. ■

Theorem 3.6. *Let C be a nonempty closed convex subset of a complete CAT(0) space X endowed with a directed graph G such that $V(G) = C$ and C has Property E. Suppose $E(G)$ is convex and G is transitive. Let $T_i : C \to C$, $(i = 1, 2, 3)$ be three G-nonexpansive mappings such that $F = F(T_1) \cap F(T_2) \cap F(T_3) \neq \emptyset$. For arbitrary $x_0 \in C$ and $p_0 \in F$, define the sequence $\{x_n\}$ as in (2.8). Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be real sequences in $[\delta, 1-\delta]$ for some $\delta \in (0, 1)$ and (x_0, p_0), $(p_0, x_0) \in E(G)$. Suppose F is dominated by x_0 and F dominates x_0. Then $\{x_n\}$ Δ-converges to a common fixed point of T_1, T_2 and T_3.*

Proof. Suppose $p_0 \in F$ is such that $(x_0, p_0), (p_0, x_0)$ are in $E(G)$. By Lemma 3.1 (i), we have $\lim_{n\to\infty} d(x_n, p_0)$ exists, so the sequence $\{x_n\}$ is bounded in C. Since C has Property E. We first show that $w_\Delta(x_n) \subseteq F$. If $u \in w_\Delta(x_n)$, then there exists a subsequence $\{u_{n_k}\}$ of $\{x_{n_k}\}$ such that $A(\{u_{n_k}\}) = \{u\}$. Hence, by Proposition 2.1, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $A(\{x_{n_k}\}) = \{x^*\} \in C$. Using Proposition 2.1 (ii), we have that $u \in F$. By Lemma 3.1, we have that $\lim_{n\to\infty} d(x_n, u)$ exists. Therefore by Lemma 2.1, we have that $u = x^*$. Hence, $u = x^* \in F$ so that $w_\Delta(x_n) \subseteq F$.

Using Lemma 3.1 (ii), we get that

$$\lim_{k\to\infty} d(x_{n_k}, T_1 x_{n_k}) = \lim_{k\to\infty} d(x_{n_k}, T_2 x_{n_k}) = \lim_{k\to\infty} d(x_{n_k}, T_3 x_{n_k}) = 0. \tag{3.32}$$

Moreover, we have

$$d(T_1 w_n, w_n) \le d(T_1 w_n, T_2 v_n) + d(T_2 v_n, v_n) + d(v_n, w_n). \tag{3.33}$$

Using (3.13) and (3.28) in (3.33), we get

$$\lim_{n\to\infty} d(T_1 w_n, w_n) = 0. \tag{3.34}$$

Hence, by Lemma 2.2, we have $I - T_1$, $I - T_2$ and $I - T_3$ are G-demiclosed at 0. So, $x^* \in F$.

To prove that $\{x_n\}$ Δ-converges to the common fixed point of T_i $(i = 1, 2, 3)$, we show that $w_\Delta(x_n)$ consists of exactly one point. Suppose $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$. Since C has Property E, by Proposition 2.1, there exists a subsequence $\{u_{n_k}\}$ of $\{x_{n_k}\}$ such that Δ-$\lim_n u_{n_k} = u$ for some $u \in C$. Suppose $A(\{u_{n_k}\}) = \{u\}$ and $A(\{x_{n_k}\}) = \{x^*\}$. We have already proved that $u = x^* \in F$. Therefore, by the uniqueness of asymptotic centers and Lemma 2.1, we have $x^* = u \in F$. Hence, we obtain that $w_\Delta(x_n) = \{x^*\}$. The proof of Theorem 3.2 is completed. ■

Next, we prove the following strong convergence theorems for our proposed iteration (2.8) for three G-nonexpansive mappings in uniformly convex metric spaces endowed with a directed graph.

Theorem 3.7. *Let C be a nonempty closed convex subset of a uniformly convex metric space X endowed with a directed graph G such that $V(G) = C$. Suppose $E(G)$ is convex and G is transitive. Let $T_i : C \to C$, $(i = 1, 2, 3)$ be three G-nonexpansive mappings* *such that T_i satisfies condition (C) and $F = F(T_1) \cap F(T_2) \cap F(T_3) \neq \emptyset$. For arbitrary $x_0 \in C$ and $p \in F$, define the sequence $\{x_n\}$ as in (2.8). Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be real sequences in $[\delta, 1-\delta]$ for some $\delta \in (0,1)$. Suppose F is dominated by x_0 and F dominates x_0. Then $\{x_n\}$ converges strongly to a common fixed point of T_1, T_2 and T_3.*

Proof. By Lemma 3.1 (i), $\lim_{n\to\infty} d(x_n, p)$ exists. Hence, $\lim_{n\to\infty} d(x_n, F)$ exists for each $p \in F$. Moreover, by Lemma 3.1 (ii), we have

$$\lim_{n\to\infty} d(x_n, T_1x_n) = \lim_{n\to\infty} d(x_n, T_2x_n) = \lim_{n\to\infty} d(x_n, T_3x_n) = 0.$$

Hence, from condition (C) we have $\lim_{n\to\infty} f(d(x_n, F)) = 0$. Since $f : [0,\infty) \to [0,\infty)$ is a nondecreasing function satisfying $f(0) = 0$, $f(r) > 0$ for each $r \in (0,\infty)$, it follows that $\lim_{n\to\infty} d(x_n, F) = 0$. Therefore, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and a sequence $\{y_k\} \subset F$ such that $d(x_{n_k}, y_k) \leq \frac{1}{2^k}$. If we put $n_{k+1} = n_{k+j}$ for some $j \geq 1$, then we have

$$d(x_{n_{k+1}}, y_k) \leq d(x_{n_{k+j-1}}, y_k) \leq d(x_{n_k}, y_k) \leq \frac{1}{2^k}.$$

Hence we have $d(y_{k+1}, y_k) \leq \frac{3}{2^{k+1}}$, so $\{y_k\}$ is a Cauchy sequence. Assume that $y_k \to p_0 \in C$ as $k \to \infty$. Since F is closed, we have $p_0 \in F$. Therefore, $\{x_{n_k}\} \to p_0 \in C$ as $k \to \infty$. Using the fact that $\lim_{n\to\infty} d(x_n, p_0)$ exists, it follows that $x_n \to p_0$. The proof of Theorem 3.3 is completed. ∎

Theorem 3.8. *Let C be a nonempty closed convex subset of a uniformly convex metric space X endowed with a directed graph G such that $V(G) = C$ and C has property G. Suppose $E(G)$ is convex and G is transitive. Let $T_i : C \to C$, $(i = 1,2,3)$ be three G-nonexpansive mappings such that $F = F(T_1) \cap F(T_2) \cap F(T_3) \neq \emptyset$. For arbitrary $x_0 \in C$ and $p \in F$, define the sequence $\{x_n\}$ as in (2.8). Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be real sequences in $[\delta, 1-\delta]$ for some $\delta \in (0,1)$. Suppose F is dominated by x_0 and F dominates x_0. If one of T_i is semi-compact, then $\{x_n\}$ converges strongly to a common fixed point of T_1, T_2 and T_3.*

Proof. By Lemma 3.1 , the sequence $\{x_n\}$ is bounded and

$$\lim_{n\to\infty} d(x_n, T_1x_n) = \lim_{n\to\infty} d(x_n, T_2x_n) = \lim_{n\to\infty} d(x_n, T_3x_n) = 0.$$

Since one of T_1, T_2 and T_3 is semi-compact, it follows that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to p \in C$ as $k \to \infty$. As C has property G, by transitivity of graph G, we have $(x_{n_k}, p) \in E(G)$. Observe that for all $i \in \{1,2,3\}$ we have $\lim_{k\to\infty} d(x_{n_k}, T_ix_{n_k}) = 0$. Hence, we have

$$\begin{aligned} d(p, T_ip) &\leq d(p, x_{n_k}) + d(x_{n_k}, T_ix_{n_k}) + d(T_ix_{n_k}, T_ip) \\ &\leq d(p, x_{n_k}) + d(x_{n_k}, T_ix_{n_k}) + d(x_{n_k}, p) \longrightarrow 0 \text{ as } k \to \infty. \end{aligned}$$

Therefore, $p \in F$. Hence, $\lim_{n\to\infty} d(x_n, F)$ exists by Theorem 3.3. Observe that $d(x_{n_k}, F) \leq d(x_{n_k}, p) \to 0$ as $k \to \infty$. Hence, $\lim_{n\to\infty} d(x_n, F) = 0$. Following arguments similar to those given in the proof of Theorem 3.3, we have $\{x_n\}$ converges strongly to a common fixed point of T_1, T_2 and T_3. The proof of Theorem 3.4 is completed. ∎

Next, we obtain the following results.

Theorem 3.9. *Let C be a nonempty closed convex subset of a uniformly convex metric space X endowed with a directed graph G such that $V(G) = C$ and C has property G. Suppose $E(G)$ is convex and G is transitive. Let $T_i : C \to C$, $(i = 1, 2, 3)$ be three G-nonexpansive mappings such that $F = F(T_1) \cap F(T_2) \cap F(T_3) \neq \emptyset$. For arbitrary $s_0 \in C$ and $p \in F$, define the sequence $\{s_n\}$ as in (2.7). Let $\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences in $[\delta, 1-\delta]$ for some $\delta \in (0, 1)$. Suppose F is dominated by s_0 and F dominates s_0. If one of T_i is semi-compact, then $\{s_n\}$ converges strongly to a common fixed point of T_1, T_2 and T_3.*

Proof. The proof follows on the similar lines as in the proof of Theorem 3.4. ∎

4. Numerical Examples

In this section, we provide a number of numerical illustrations to support our results. We compare the speed of convergence of various iterative schemes discussed in this paper, viz a viz: the modified Mann iterative scheme $\{u_n\}$ in (2.4), the modified Ishikawa iterative scheme $\{p_n\}$ in (2.5), the modified Noor iterative scheme $\{v_n\}$ in (2.6), the modified Picard-Ishikawa hybrid iterative scheme $\{s_n\}$ in (2.7) and the modified Abbas-Nazir iteration $\{x_n\}$ in (2.8). We show that the proposed modified Picard-Ishikawa hybrid scheme $\{s_n\}$ and the modified Abbas-Nazir iterative scheme $\{x_n\}$ converge faster than all of the modified Mann iterative scheme $\{u_n\}$, the modified Ishikawa iterative scheme $\{p_n\}$ and the modified Noor iterative scheme $\{v_n\}$. All the codes were written in Matlab (R2010a) and run on PC with Intel(R) Core(TM) i3-4030U CPU @ 1.90 GHz.

We begin with the following example.

Example 4.1. Let $X = [0, 10]$ and the graph G_0 given by $E(G_0) := X \times X$. Let $d(x, y) = |x - y|$, $\forall x, y \in \mathbb{R}$ and $W(x, y, \lambda) = \lambda x + (1 - \lambda)y$, $\forall x, y \in \mathbb{R}$ and $\lambda \in [0, 1]$. Let $T_i : X \to X$ $(i = 1, 2, 3)$ be mappings such that $T_i x = \frac{x}{2}$. Clearly, T_i is G_0-contraction for each $i = 1, 2, 3$, see ([3], Example 2.2) and the common fixed point $F(T_1) = F(T_2) = F(T_3) = \{0\}$. Suppose the first iteration $u_0 = p_0 = v_0 = s_0 = x_0 = 7$ and the number of iterations for each iterative scheme is $n = 100$. Choose $\alpha_n = \frac{n}{7n+1}$, $\beta_n = \frac{1}{17n+1}$ and $\gamma_n = \frac{1}{2n+3}$. We present the numerical results of this example in Table 1 and Figure 1 below.

Step	Abbas-Nazir	Picard-Ishikawa	Noor	Ishikawa	Mann
0	7.0000	7.0000	7.0000	7.0000	7.0000
1	1.9165	3.2630	6.5250	6.5260	6.5625
2	0.5330	1.5178	6.0700	6.0712	6.1250
3	0.1493	0.7055	5.6431	5.6443	5.7074
4	0.0420	0.3279	5.2445	5.2458	5.3138
5	0.0118	0.1523	4.8733	4.8745	4.9448
6	0.0033	0.0708	4.5278	4.5289	4.5998
7	9.4590e-04	0.0329	4.2064	4.2075	4.2778
8	2.6788e-04	0.0153	3.9076	3.9086	3.9776
⋮	⋮	⋮	⋮	⋮	⋮

Table 1. Rate of convergence among various iterations

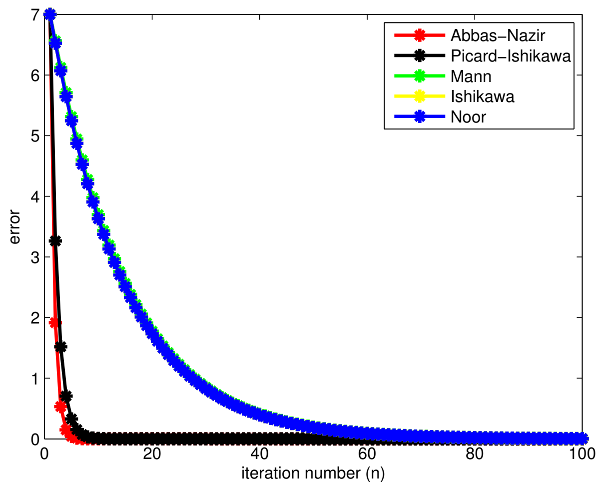

FIGURE 1. Error versus iteration number (n)

The following example in ([5], Example 4) for G-nonexpansive mappings, which is not nonexpansive. We compare the rate of convergence of various iterative processes for this example in the framework of convex metric spaces.

Example 4.2. Suppose $X = \mathbb{R}$ and $C = [0, 2]$. Suppose that $(x, y) \in E(G)$ if and only if $0.4 \leq x, y \leq 1.6$ or $x = y$, where $x, y \in \mathbb{R}$. Let $d(x, y) = |x - y|$, $\forall x, y \in \mathbb{R}$ and $W(x, y, \lambda) = \lambda x + (1 - \lambda)y$, $\forall x, y \in \mathbb{R}$ and $\lambda \in [0, 1]$. Define $T_i : C \to C$ $(i = 1, 2, 3)$ by

$$T_1 x = \sin(\frac{\pi}{2}) \cos(\tan(x - 1)),$$

$$T_2 x = \frac{\ln x}{3} + 1,$$

$$T_3 x = \frac{2}{3} \arcsin(x - 1) + 1,$$

for each $x \in C$. Note that each T_i, $(i = 1, 2, 3)$ is G-nonexpansive with $F(T_1) = F(T_2) = F(T_3) = \{1\}$. However, T_i is not nonexpansive, since for $x = 1.6$, $y = 1.8$, $u = 0.1$, $v = 0.6$, $p = 1.95$ and $q = 1.45$, we have

$$|T_1 x - T_1 y| > 0.21 > |x - y|,$$

$$|T_2 u - T_2 v| > 0.58 > |u - v|,$$

$$|T_3 p - T_3 q| > 0.50 = |p - q|.$$

We consider the following cases for our numerical experiments:

Case I Suppose the first iteration $u_0 = p_0 = v_0 = s_0 = x_0 = 1.5$ and the number of iterations for each iterative scheme is $n = 100$. Choose $\alpha_n = \frac{n}{17n+2}$, $\beta_n = \frac{1}{5n+3}$ and $\gamma_n = \frac{1}{11n+4}$. We present the numerical results of Case I in Table 2 and Figure 2 below.

Case II Suppose the first iteration $u_0 = p_0 = v_0 = s_0 = x_0 = 0.5$ and the number of iterations for each iterative scheme is $n = 100$. Choose $\alpha_n = \frac{n}{17n+2}$, $\beta_n = \frac{1}{5n+3}$ and $\gamma_n = \frac{1}{11n+4}$. We present the numerical results of Case II in Table 3 and Figure 3 below.

Step	Abbas-Nazir	Picard-Ishikawa	Noor	Ishikawa	Mann
0	1.5000	1.5000	1.5000	1.5000	1.5000
1	1.0837	0.9397	1.3279	1.1276	0.8545
2	1.0162	0.9986	1.0515	1.0378	0.9893
3	1.0032	0.9999	1.0101	1.0117	0.9999
4	1.0007	1.0000	1.0020	1.0036	1.0000
5	1.0001	1.0000	1.0004	1.0011	1.0000
6	1.0000	1.0000	1.0001	1.0004	1.0000
7	1.0000	1.0000	1.0000	1.0001	1.0000
8	1.0000	1.0000	1.0000	1.0000	1.0000
⋮	⋮	⋮	⋮	⋮	⋮

TABLE 2. Rate of convergence among various iterations

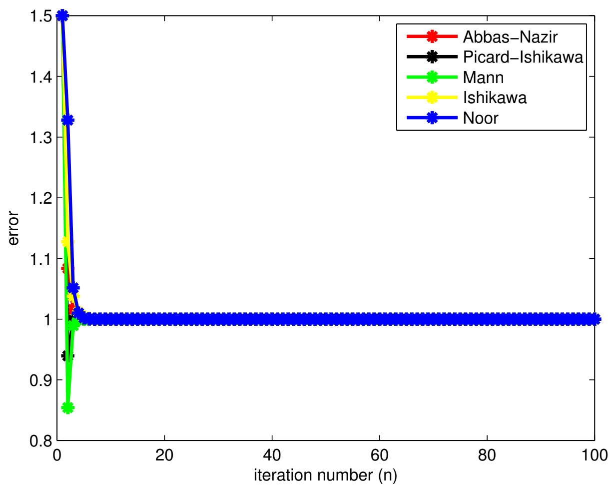

FIGURE 2. Error versus iteration number (n)

Step	Abbas-Nazir	Picard-Ishikawa	Noor	Ishikawa	Mann
0	0.5000	0.5000	0.5000	0.5000	0.5000
1	0.8732	0.9388	0.6664	0.7797	0.8545
2	0.9734	0.9967	0.9186	0.9216	0.9893
3	0.9946	0.9999	0.9832	0.9743	0.9999
4	0.9989	1.0000	0.9966	0.9918	1.0000
5	0.9998	1.0000	0.9993	0.9974	1.0000
6	1.0000	1.0000	0.9999	0.9992	1.0000
7	1.0000	1.0000	1.0000	0.9997	1.0000
8	1.0000	1.0000	1.0000	0.9999	1.0000
⋮	⋮	⋮	⋮	⋮	⋮

TABLE 3. Rate of convergence among various iterations

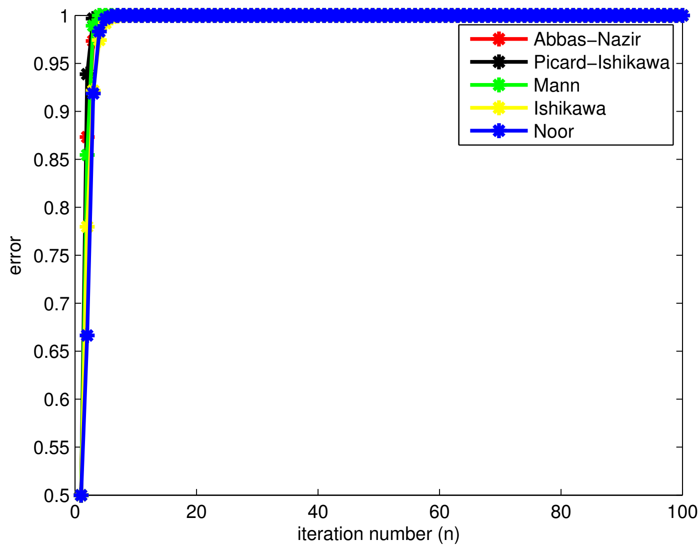

FIGURE 3. Error versus iteration number (n)

The next example in ([46], Example 1) is used for our next numerical experiments as follows:

Example 4.3. Let $X = \mathbb{R}$, $C = [0, 2]$ and $G = (V(G), E(G))$ be a directed graph defined by $V(G) = C$ and $(x, y) \in E(G)$ if and only if $0.50 \leq x \leq y \leq 1.70$. Let $d(x, y) = |x - y|$, $\forall x, y \in \mathbb{R}$ and $W(x, y, \lambda) = \lambda x + (1 - \lambda)y$, $\forall x, y \in \mathbb{R}$ and $\lambda \in [0, 1]$. Define mappings $T_1, T_2, T_3 : C \to C$ by

$$T_1 x = \frac{2}{3} \arcsin(x - 1) + 1,$$
$$T_2 x = \frac{1}{3} \tan(x - 1) + 1,$$
$$T_3 x = \sqrt{x},$$

for each $x \in C$. Note that T_1, T_2 and T_3 are G-nonexpansive. However, T_1, T_2 and T_3 are not nonexpansive since

$$|T_1 x - T_1 y| > 0.50 = |x - y|,$$
$$|T_2 u - T_2 v| > 0.07 = |u - v|$$

and

$$|T_3 p - T_3 q| > 0.45 = |p - q|,$$

when $x = 1.95$, $y = 1.45$, $u = 0.08$, $v = 0.01$, $p = 0.5$ and $q = 0.05$.

Suppose the first iteration $u_0 = p_0 = v_0 = s_0 = x_0 = 1.50$ and the number of iterations for each iterative scheme is $n = 100$. Choose $\alpha_n = \frac{n}{17n+2}$, $\beta_n = \frac{1}{5n+3}$ and $\gamma_n = \frac{1}{11n+4}$. We present the numerical results of this example in Table 4 and Figure 4 below.

Step	Abbas-Nazir	Picard-Ishikawa	Noor	Ishikawa	Mann
0	1.5000	1.5000	1.5000	1.5000	1.5000
1	1.0740	1.1458	1.2202	1.1789	1.3491
2	1.0122	1.0452	1.1025	1.0592	1.2377
3	1.0021	1.0144	1.0490	1.0194	1.1600
4	1.0004	1.0046	1.0237	1.0063	1.1071
5	1.0001	1.0015	1.0116	1.0021	1.0716
6	1.0000	1.0005	1.0056	1.0007	1.0477
7	1.0000	1.0002	1.0028	1.0002	1.0318
8	1.0000	1.0000	1.0014	1.0001	1.0212
⋮	⋮	⋮	⋮	⋮	⋮

TABLE 4. Rate of convergence among various iterations

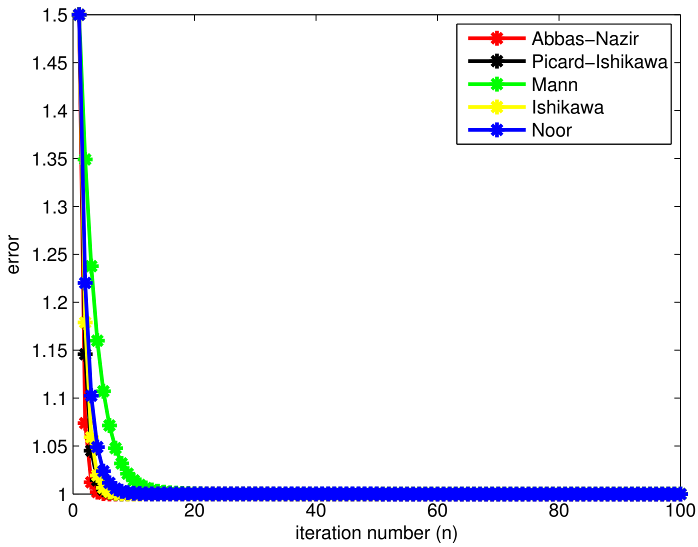

FIGURE 4. Error versus iteration number (n)

Remark 4.4. Clearly, from Table 1 and Figure 1 of Example 4.1, we see that the modified Abbas-Nazir iteration $\{x_n\}$ converges faster than all of modified Picard-Ishikawa iteration $\{s_n\}$, modified Mann iteration $\{u_n\}$, modified Ishikawa iteration $\{p_n\}$ and the modified Noor iteration $\{v_n\}$ to the common fixed point $F(T_1) = F(T_2) = F(T_3) = \{0\}$. Similarly, $\{s_n\}$ converges faster than all of $\{u_n\}$, $\{p_n\}$ and $\{v_n\}$. From Table 2 and Figure 2 of Example 4.2, we see that $\{s_n\}$ and $\{u_n\}$ have the same rate of convergence, and also converges faster than all of $\{x_n\}$, $\{v_n\}$ and $\{p_n\}$ to the common fixed point $F(T_1) = F(T_2) = F(T_3) = \{1\}$. We also have the same conclusion from Table 3 and Figure 3. From Table 4 and Figure 4 of Example 4.3, we see that $\{x_n\}$ converges faster than all of $\{s_n\}$, $\{v_n\}$, $\{p_n\}$ and $\{u_n\}$ to the common fixed point $F(T_1) = F(T_2) = F(T_3) = \{1\}$. Similarly, $\{s_n\}$ converges faster than all of $\{v_n\}$, $\{p_n\}$ and $\{u_n\}$.

Concluding Remarks

In this paper, we extended and unified several known results in the framework of normed spaces endowed with graphs to the setting of CAT(0) spaces endowed with graphs and convex metric spaces endowed with graphs, including the results of [3, 5, 42] and [46].

Open Problems

Is it possible to extend the results of Theorem 3.1 and Theorem 3.2 from the setting of CAT(0) spaces endowed with graphs to the setting of convex metric spaces endowed with graphs?

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References

[1] S. Banach, Sur les opérations dans les ensembles abstraits et leurs applications aux équations intégrales, Fund. Math. 3 (1922) 133–181.

[2] G.A. Okeke, J.O. Olaleru, Fixed points of demicontinuous ϕ-nearly Lipschitzian mappings in Banach spaces, Thai J. Math. 17 (1) (2019) 141–154.

[3] J. Jachymski, The contraction principle for mappings on a metric space with a graph, Proc. Amer. Math. Soc. 136 (4) (2008) 1359–1373.

[4] W. Anakkamatee, B. Tongnoi, Convergence theorems for G-nonexpansive mappings in CAT (0) spaces endowed with graphs, Thai J. Math. 17 (3) (2019) 775–787.

[5] S. Suantai, M. Donganont, W. Cholamjiak, Hybrid methods for a countable family of G-nonexpansive mappings in Hilbert spaces endowed with graphs, Mathematics (2019) doi: 10.3390/math7100936.

[6] R. Suparatulatorn, S. Suantai, W. Cholamjiak, Hybrid methods for a finite family of G-nonexpansive mappings in Hilbert spaces endowed with graphs, AKCE International Journal of Graphs and Combinatorics 14 (2017) 101–111.

[7] J. Tiammee, A. Kaewkhao, S. Suantai, On Browders convergence theorem and Halpern iteration process for G-nonexpansive mappings in Hilbert spaces endowed with graphs, Fixed Point Theory and Applications (2015) https://doi.org/10.1186/s13663-015-0436-9.

[8] J. Tiammee, Fixed point results of generalized almost G-contractions in metric spaces endowed with graphs, Carpathian Journal of Mathematics 34 (3) (2018) 433–439.

[9] O. Tripak, Common fixed points of G-nonexpansive mappings on Banach spaces with a graph, Fixed Point Theory and Applications 2016 (2016) https://doi.org/10.1186/s13663-016-0578-4.

[10] M. Wattanataweekul, C. Klanarong, Convergence theorems for common fixed points of two G-nonexpansive mappings in a Banach space with a directed graph, Thai J. Math. 16 (2) (2018) 503–516.

[11] T. Atsathi, P. Cholamjiak, S. Kesornprom, A. Prasong, S-iteration process for asymptotic pointwise nonexpansive mappings in complete hyperbolic metric spaces, Commun. Korean Math. Soc. 31 (2016) 575–583.

[12] P. Cholamjiak, A new multi-step iteration for solving a fixed point problem of nonexpansive mappings, Fixed Point Theory Appl. 198 (2013) doi: 10.1186/1687-1812-2013-198.

[13] M.A. Noor, New approximation schemes for general variational inequalities, J. Math. Anal. Appl. 251 (2000) 217–229.

[14] G.A. Okeke, M. Abbas, A solution of delay differential equations via Picard-Krasnoselskii hybrid iterative process, Arab. J. Math. 6 (2017) 21–29.

[15] G.A. Okeke, Convergence analysis of the Picard-Ishikawa hybrid iterative process with applications, Afrika Matematika 30 (2019) 817–835.

[16] G.A. Okeke, Iterative approximation of fixed points of contraction mappings in complex valued Banach spaces, Arab J. Math. Sci. 25 (1) (2019) 83–105.

[17] G.A. Okeke, M. Abbas, M. de la Sen, Approximation of the fixed point of multivalued quasi-nonexpansive mappings via a faster iterative process with applications, Discrete Dynamics in Nature and Society 2020 (2020) Article ID 8634050.

[18] G.A. Okeke, Convergence theorems for G-nonexpansive mappings in convex metric spaces with a directed graph, Rendiconti del Circolo Matematico di Palermo Series II (2020) doi: 10.1007/s12215-020-00535-0.

[19] Y. Shehu, P. Cholamjiak, Another look at the split common fixed point problem for demicontractive operators, RACSAM. 110 (2016) 201–218.

[20] R. Suparatulatorn, P. Cholamjiak, The modified S-iteration process for nonexpansive mappings in CAT(k) spaces, Fixed Point Theory Appl. 2016 (2016) doi: 10.1186/s13663-016-0515-6.

[21] R. Suparatulatorn, P. Cholamjiak, S. Suantai, On solving the minimization problem and the fixed point problem for nonexpansive mappings in CAT(0) spaces, Optim. Meth. Softw. 32 (2017) 182–192.

[22] M. Abbas, T. Nazir, A new faster iteration process applied to constrained minimization and feasibility problems, Matematicki Vesnik 66 (2014) 223–234.

[23] R.P. Agarwal, D. O'Regan, D.R. Sahu, Iterative construction of fixed points of nearly asymptotically nonexpansive mappings, J. Nonlinear Convex Anal. 8 (1) (2007) 61–79.

[24] S.H. Khan, A Picard-Mann hybrid iterative process, Fixed Point Theory and Applications 2013 (2013) https://doi.org/10.1186/1687-1812-2013-69.

[25] I. Beg, M. Abbas, J.K. Kim, Convergence theorems of the iterative schemes in convex metric spaces, Nonlinear Funct. Anal. & Appl. 3 (2006) 421–436.

[26] K. Goebel, W.A. Kirk, Topics in Metric Fixed Point Theory, Cambridge Stud. Adv. Math., 28, Cambridge University Press, London, 1990.

[27] W. Takahashi, A convexity in metric spaces and nonexpansive mapping I, Kodai Math. Sem. Rep. 22 (1970) 142–149.

[28] H. Fukhar-ud-din, V. Berinde, Iterative methods for the class of quasi-contractive type operators and comparison of their rate of convergence in convex metric spaces,

Filomat 30 (1) (2016) 223-230.

[29] R. Johnsonbaugh, Discrete Mathematics, New Jersey, 1997.

[30] J. Bang-Jensen, G. Gutin, Digraphs Theory, Algorithms and Applications, Springer Monographs in Mathematics, Springer-Verlag London Ltd. London, 2007.

[31] C. Pang, R. Zhang, Q. Zhang, J. Wang, Dominating sets in directed graphs, Information Sciences 180 (2010) 3647–3652.

[32] M.R. Alfuraidan, M.A. Khamsi, Fixed points of monotone nonexpansive mappings on a hyperbolic metric space with a graph, Fixed Point Theory Appl. 2015 (2015) https://doi.org/10.1186/s13663-015-0294-5.

[33] V. Berinde, Iterative Approximation of Fixed Points, Lecture Notes in Mathematics, Springer-Verlag Berlin Heidelberg, 2007.

[34] W. Phuengrattana, S. Suantai, Comparison of the rate of convergence of various iterative methods for the class of weak contractions in Banach spaces, Thai J. Math 11 (2013) 217–226.

[35] T. Shimizu, W. Takahashi, Fixed points of multivalued mappings in certain convex metric spaces, Topol. Methods Nonlinear Anal. 8 (1) (1996) 197–203.

[36] H. Fukhar-ud-din, One step iterative scheme for a pair of nonexpansive mappings in a convex metric space, Hacettepe Journal of Mathematics and Statistics 44 (5) (2015), 1023–1031.

[37] W.A. Kirk, B. Panyanak, A concept of convergence in geodesic spaces, Nonlinear Anal. 68 (2008) 3689–3696.

[38] T.C. Lim, Remarks on some fixed point theorems, Proc. Amer. Math. Soc. 60 (1976) 179–182.

[39] S. Dhompongsa, B. Panyanak, On Δ-convergence theorems in $CAT(0)$ spaces, Computers and Mathematics with Applications 56 (2008) 2572–2579.

[40] S.H. Khan, M. Abbas, Strong and Δ-convergence of some iterative schemes in $CAT(0)$ spaces, Computers and Mathematics with Applications 61 (2011) 109–116.

[41] M. Bridson, A. Haefliger, Metric Spaces of Non-Positive Curvature, Springer-Verlag, Berlin, Heidelberg, New York, 1999.

[42] P. Sridarat, R. Suparaturatorn, S. Suantai, Y.J. Cho, Convergence analysis of SP-iteration for G-nonexpansive mappings with directed graphs, Bull. Malays. Math. Sci. Soc. (2017) doi:10.1007/s40840-018-0606-0.

[43] S. Suantai, Weak and strong convergence criteria of Noor iterations for asymptotically nonexpansive mappings, J. Math. Anal. Appl. 311 (2005) 506–517.

[44] M.G. Sangago, Convergence of iterative schemes for nonexpansive mappings, Asian-European J. Math. 4 (4) (2011) 671–682.

[45] K.K. Tan, H.K. Xu, Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process, J. Math. Anal. Appl. 178 (1993) 301–308.

[46] D. Yambangwai, S. Aunruean, T. Thianwan, A new modified three-step iteration method for G-nonexpansive mappings in Banach spaces with a graph, Numerical Algorithms (2019) doi: 10.1007/s11075-019-00768-w.