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Author for correspondence:
Jacek Banasiak
e-mail: jacek.banasiak@up.ac.za

Growth–fragmentation–coagulation equations with unbounded coagulation kernels

J. Banasiak [1] and W. Lamb[2]

[1]Department of Mathematics and Applied Mathematics, University of Pretoria & Institute of Mathematics, Łódź University of Technology, & International Scientific Laboratory of Applied Semigroup Research, South Ural State University, e-mail: jacek.banasiak@up.ac.za
[2] Department of Mathematics and Statistics, University of Strathclyde, e-mail: w.lamb@strath.ac.uk

In this paper we prove the global in time solvability of the continuous growth–fragmentation–coagulation equation with unbounded coagulation kernels, in spaces of functions having finite moments of sufficiently high order. The main tool is the recently established result on moment regularization of the linear growth–fragmentation semigroup that allows us to consider coagulation kernels whose growth for large clusters is controlled by how good the regularization is, in a similar manner to the case when the semigroup is analytic.

1. Introduction

Coagulation and fragmentation play a fundamental role in a number of diverse phenomena arising both in natural science and in industrial processes. Specific examples can be found in ecology, human biology, polymer and aerosol sciences, astrophysics and the powder production industry; see [14] for further details and references. A feature shared by these examples is that each involves an identifiable population of inanimate or animate objects that are capable of forming larger or smaller objects through, respectively, coalescence or breakup. The earliest mathematical investigation into processes governed by coagulation or fragmentation was carried out by Smoluchowski in two papers [26,27], published in 1916 and 1917. Smoluchowski introduced, and investigated, a coagulation model in the form of an

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infinite set of ordinary differential equations that describes the time-evolution of a system of particle clusters that, as a result of Brownian motion, become sufficiently close to enable binary coagulation of clusters to occur. In this discrete-size model, it is assumed that the clusters are comprised of a finite number of identical fundamental particles, and so a discrete (positive integer) variable can be used to distinguish between cluster sizes. Over the past one hundred years, the pioneering work of Smoluchowski has been extended considerably, and various models, both deterministic and stochastic, and incorporating both coagulation and fragmentation, have been produced and studied.

In certain applications, such as droplet growth in clouds and fogs [23,24], where it is more realistic to have a continuous particle size variable which can take any positive real value, the standard deterministic coagulation-fragmentation (C-F) model is given by

$$\partial_t f(x,t) = \mathfrak{F} f(x,t) + \mathfrak{K} f(x,t)\,, \quad (x,t) \in \mathbb{R}_+^2; \quad f(0,x) = \mathring{f}(x)\,, \quad x \in \mathbb{R}_+\,, \tag{1.1}$$

where $\mathbb{R}_+ := (0,\infty)$, and

$$\mathfrak{F} f(x,t) = -a(x) f(x,t) + \int_x^\infty a(y) b(x,y) f(y,t)\, \mathrm{d}y\,, \tag{1.2}$$

$$\mathfrak{K} f(x,t) = \frac{1}{2}\int_0^x k(x-y,y) f(x-y,t) f(y,t)\, \mathrm{d}y - f(x,t)\int_0^\infty k(x,y) f(y,t)\, \mathrm{d}y \tag{1.3}$$

model fragmentation and coagulation respectively; see [28]. Here, it is assumed that only a single size variable, such as particle mass, is required to differentiate between the reacting particles, with $f(x,t)$ denoting the density of particles of size $x \in \mathbb{R}_+$ at time $t \geq 0$. The coagulation kernel $k(x,y)$ gives the rate at which particles of size x coalesce with particles of size y, and $a(x)$ represents the overall rate of fragmentation of an x-sized particle. The coefficient $b(x,y)$, often called the fragmentation kernel or daughter distribution function, can be interpreted as giving the number of size x particles produced by the fragmentation of a size y particle; more precisely, it is the distribution function of the sizes of the daughter particles. In most investigations into (1.1), b is assumed to be nonnegative and measurable, with $b(x,y) = 0$ for $x > y$ and

$$\int_0^y x b(x,y)\, \mathrm{d}x = y, \quad \text{for each } y > 0, \tag{1.4}$$

but is otherwise arbitrary. Equation (1.4) can be viewed as a local mass conservation property, as it expresses the fact that, when the size variable is the particle mass, the total mass of all the daughter particles produced by a fragmentation event is the same as that of the parent particle.

In the case of deterministic models, with either discrete or continuous size, two main approaches have been used extensively in their analysis, with one involving weak compactness arguments and the other utilising the theory of operator semigroups. Comprehensive treatments of each are given in [14], and there is also an excellent account in [16, Chapter 36] of the semigroup approach to the discrete fragmentation equation. We focus here on the application of semigroup techniques to continuous C-F models, where the strategy is to express the pointwise initial-value problem (1.1) as a semilinear abstract Cauchy problem (ACP) of the form

$$\frac{d}{dt} f(t) = F f(t) + K f(t),\ t \in \mathbb{R}_+; \quad f(0) = \mathring{f}, \tag{1.5}$$

posed in a physically relevant Banach space X. In (1.5), F and K are operator realisations in X of the formal expressions $\mathfrak{F}$ and $\mathfrak{K}$ defined, respectively, in (1.2) and (1.3).

Initially, only the linear fragmentation part of (1.5) is examined, and a representation F is sought such that F generates a strongly continuous semigroup $(S_F(t))_{t\geq 0}$ on X. If this is possible, then the full abstract C-F problem is recast as the fixed point equation

$$f(t) = S_F(t)\mathring{f} + \int_0^t S_F(t-s) K f(s)\, \mathrm{d}s,\ t \in \mathbb{R}_+, \tag{1.6}$$

to which standard results can be applied to yield the existence and uniqueness of mild and classical solutions $f:[0,\tau_{\max})\to X$. The identification $[f(t)](x)=f(x,t)$ then leads, after some further analysis, to a solution of the pointwise problem (1.1).

Historically, the semigroup approach to C-F problems originated in 1979 with the publication of a seminal paper by Aizenman and Bak [2] for the specific case where the coagulation kernel k is constant, and the fragmentation rate and the fragmentation kernel are given by $a(x)=x$ and $b(x,y)=2/y$. The work presented in [2] was later extended in 1997 to bounded coagulation kernels and more general fragmentation rates and kernels [19,20]. Common to these early semigroup investigations is the use of more tractable, truncated versions of the fragmentation problem to generate a sequence of semigroups that converge, in an appropriate manner, to the semigroup for the original problem; for example, see [20, Sections 3 & 4]. In contrast, the year 2000 saw the introduction, in [3], of a novel approach to the fragmentation problem that relies on the theory of substochastic semigroups. In recent years, this substochastic semigroup approach has been developed further and used to prove many important properties of the fragmentation semigroup such as its analyticity and, in the discrete case, compactness, [11,12]. These properties have made it possible to extend earlier semigroup derived results on the well-posedness of C-F equations to the case where the coagulation kernel may be unbounded; see [6,11,13]. Moreover, it is shown in [6,8] that whenever the semigroup and weak compactness approaches are both applicable to a C-F problem, they both lead to the same solutions.

With regard to the choice of an appropriate space X, the early semigroup (and also weak compactness) analyses of (1.1) used the spaces $X_0:=L_1(\mathbb{R}_+,\mathrm{d}x)$, $X_1:=L_1(\mathbb{R}_+,x\mathrm{d}x)$ and also $X_{0,1}:=L_1(\mathbb{R}_+,(1+x)\mathrm{d}x)$ where, for a nonnegative solution f of (1.5), the respective norms in X_0 and X_1 give the total number of particles in the system and its mass. However, in later investigations it was found that improved results could be obtained by imposing some additional control on the evolution of large particles. A convenient way of introducing such a control is to consider the C-F problem in the more general weighted L_1 spaces $X_m:=L_1(\mathbb{R}_+,x^m\mathrm{d}x)$ and $X_{0,m}:=L_1(\mathbb{R}_+,(1+x^m)\mathrm{d}x)$. The norms on these spaces are defined by

$$\|f\|_{[m]}:=\int_0^\infty |f(x)|x^m\mathrm{d}x;\ \|f\|_{[0,m]}:=\int_0^\infty |f(x)|w_m(x)\mathrm{d}x,\ \text{ where } w_m(x):=1+x^m. \tag{1.7}$$

We shall also use the notation

$$M_m(t):=\int_0^\infty f(x,t)x^m\,\mathrm{d}x;\ \ M_{0,m}(t):=\int_0^\infty f(x,t)w_m(x)\mathrm{d}x, \tag{1.8}$$

when discussing the norms of nonnegative solutions to (1.1). Clearly, $M_m(t)$ and $M_{0.m}(t)$ are finite provided $f(\cdot,t)\in X_m$ and $f(\cdot,t)\in X_{0,m}$.

For ease of exposition, we have restricted our attention in the above discussion to situations involving only the opposing processes of fragmentation and coagulation, and in which the total mass in the system of particles should be a conserved quantity. In many cases, however, these two processes may be complemented by other events which can change the total mass in the system. For example, mass loss can arise due to oxidation, melting, sublimation and dissolution of matter on the exposed particle surfaces. The reverse process of mass gain can also occur due to the precipitation of matter from the environment. Continuous coagulation and fragmentation processes, combined with a mass transport term that leads to either mass loss or mass gain, have also been studied using functional analytic and, in particular, semigroup methods; for example, see [5,7,10] and [14, Section 5.2], or [15,17,22] where, however, the focus is on the long-term behaviour of the linear growth-fragmentation processes. The discrete version of such models have been comprehensively analysed in [8,9]. In the case when the growth rate of a particle of mass x is $r(x)$, the appropriate modified version of (1.1) is

$$\partial_t f(x,t)=-\partial_x[r(x)f(x,t)]+\mathfrak{F}f(x,t)+\mathfrak{K}f(x,t)\ ,\ \ (x,t)\in\mathbb{R}_+^2\ ;\ \ f(0,x)=\mathring{f}(x)\ ,\ \ x\in\mathbb{R}_+\ . \tag{1.9}$$

The main goal of the paper is to prove global classical solvability of (1.9) in the spaces $X_{0,m}$ for sufficiently large m, when the coagulation rate k is unbounded (though controlled by the fragmentation rate). In this way we extend the results of [5], where only bounded coagulation operators are considered. As with (1.5), we re-write (1.9) as (1.6) but with the kernel given by the linear growth–fragmentation semigroup. The main tool is the moment improving property of this semigroup, proven in [15], that makes it a little like an analytic semigroup and allows for an approach similar to that used in [6,11] for pure fragmentation–coagulation problems, where the fragmentation semigroup is indeed analytic. In other words, the growth–fragmentation semigroup retains the moment regularization property of the fragmentation semigroup but it is not regularizing with respect to the differentiation operator, and hence it is not analytic. Thus, while the well-posedness proof for (1.9) follows standard steps, particular estimates must be tailor made for this specific case to yield the desired result. More precisely, while the existence of the mild solution is obtained by a typical fixed point argument, the involved integral operator is weakly singular, in contrast to the standard theory where it is assumed to be continuous, see e.g. [21, Theorem 6.1.2]. Similarly, the proof that the mild solution is a classical solution cannot be obtained, as in other cases where unbounded nonlinearities occur, by using the differentiability of the semigroup, since the growth–fragmentation semigroup is not analytic. Instead, the approach we adopt is to follow [21, Theorem 6.1.5], where a regularity result is established for the case of a continuous nonlinearity, but again we have to show that the result can be extended to an appropriately restricted singular nonlinearity.

The paper is organized as follows. Section 2 deals with the linear growth–fragmentation equation. In particular, we use the Miyadera perturbation theorem to show that the growth–fragmentation operator is the generator of a positive semigroup on $X_{0,m}$ without imposing any restriction on the behaviour of the growth rate r at $x=0$. In this way we improve the corresponding results of [5,15]. Section 3 is devoted to the full equation (1.9). The existence of local mild and classical solutions is proved under quite general conditions, while the global solvability, done along the lines of [6], requires some additional assumptions to control the growth term.

2. Fragmentation with growth

Adopting the semigroup based strategy described in the Introduction, we begin our analysis of equation (1.9) by considering only its linear part. For technical reasons, which will become clear later, we introduce an additional absorption term, $-a_1 f$. This results in the linear equation

$$\begin{aligned} &\partial_t f(x,t) = -\partial_x[r(x)f(x,t)] - q(x)f(x,t) + \int_x^\infty a(y)b(x,y)f(y,t)\,\mathrm{d}y, \quad (x,t)\in\mathbb{R}_+^2, \\ &f(0,x) = \mathring{f}(x), \quad x\in\mathbb{R}_+, \end{aligned} \tag{2.1}$$

where $q(x) = a(x) + a_1(x)$. The aim is to express (2.1) as an ACP of the form

$$\frac{d}{dt}f(t) = T_{0,m}f(t) + B_{0,m}f(t),\ t>0;\ \ f(0) = \mathring{f}, \tag{2.2}$$

where $T_{0,m}$ and $B_{0,m}$, respectively, are operator realisations in $X_{0,m}$ of the formal expressions

$$(\mathcal{T}f)(x) := -\partial_x[r(x)f(x)] - q(x)f(x);\ \ (\mathcal{B}f)(x) := \int_x^\infty a(y)b(x,y)f(y)\,\mathrm{d}y. \tag{2.3}$$

The ACP (2.2) will be well posed in $X_{0,m}$ provided the operator $G_{0,m} := T_{0,m} + B_{0,m}$ generates a C_0-semigroup, $(S_{G_{0,m}}(t))_{t\geq 0}$, on $X_{0,m}$. To show this, we first use the Hille-Yosida theorem to establish that $T_{0,m}$, when defined appropriately, generates a C_0- semigroup, $(S_{T_{0,m}}(t))_{t\geq 0}$ (the absorption semigroup), on $X_{0,m}$. The operator $B_{0,m}$ is then shown to be a Miyadera perturbation of $T_{0,m}$, leading immediately to the existence of $(S_{G_{0,m}}(t))_{t\geq 0}$.

(a) The absorption semigroup

The transport part of the problem is given by

$$\partial_t f(x,t) = -\partial_x[r(x)f(x,t)] - q(x)f(x,t), \ \ (x,t)\in\mathbb{R}_+; \quad f(0,x)=\mathring{f}(x), \ \ x\in\mathbb{R}_+, \tag{2.4}$$

where, as stated above, $q = a + a_1$. Throughout we assume

$$0 \le a \in L_{\infty,loc}([0,\infty)); \tag{2.5}$$

$$1/r \in L_{1,loc}(\mathbb{R}_+) \text{ and } 0 < r(x) \le r_0 + r_1 x \le \widetilde{r}(1+x) \quad \text{on} \quad \mathbb{R}_+, \tag{2.6}$$

for some nonnegative constants r_0, r_1 and $\widetilde{r} = \max\{r_0, r_1\}$. Further,

$$0 \le a_1 \in L_{\infty,loc}([0,\infty)) \text{ and } a_1(x)/a(x) \text{ remains bounded as } x\to\infty. \tag{2.7}$$

On defining operators $A_{0,m}f := -af$ and $A^{(1)}_{0,m}f := -a_1 f$ on their maximal domains in $X_{0,m}$, respectively, $D(A_{0,m}) := \{f \in X_{0,m} : af \in X_{0,m}\}$ and $D(A^{(1)}_{0,m}) := \{f\in X_{0,m} : a_1 f \in X_{0,m}\}$, we see that $D(A_{0,m}) \subseteq D(A^{(1)}_{0,m})$.

In the following treatment of (2.4) we have to distinguish between two distinct cases that may arise due the behaviour of $r(x)$ close to $x = 0$. If we use the symbol $\int_{0+}$ to denote an integral in some right neighbourhood of 0, then we may have either

$$\int_{0+} \frac{\mathrm{d}x}{r(x)} = +\infty, \tag{2.8}$$

or

$$\int_{0+} \frac{\mathrm{d}x}{r(x)} < +\infty. \tag{2.9}$$

When (2.8) is satisfied, the characteristics associated with the transport equation do not reach $x = 0$ and therefore the problem does not require a boundary condition to be specified. This case has been thoroughly researched in [5,14], and, as in *op. cit.*, we define $T_{0,m}$ by

$$T_{0,m}f := \mathcal{T}f; \ \ D(T_{0,m}) := \left\{ f\in X_{0,m} \ : \ rf \in AC(\mathbb{R}_+) \text{ and } \frac{d}{dx}(rf), qf \in X_{0,m} \right\}, \tag{2.10}$$

where $AC(\mathbb{R}_+)$ denotes the class of functions that are absolutely continuous on all compact subintervals of $\mathbb{R}_+$. On the other hand, when (2.9) holds, the characteristics do reach $x = 0$ and therefore a boundary condition is required. Here, following [15], we impose the condition

$$\lim_{x\to 0^+} r(x)f(x,t) = 0 \tag{2.11}$$

but note that more general cases can also be considered, [10]. Then $D(T_{0,m})$ is given by

$$D(T_{0,m}) := \left\{ f\in X_{0,m} \ : \ rf\in AC(\mathbb{R}_+), \ \frac{d}{dx}(rf), qf \in X_{0,m} \ \text{and } r(x)f(x) \to 0 \text{ as } x\to 0^+ \right\}. \tag{2.12}$$

To make the Hille-Yosida theorem applicable, we must determine the resolvent operator, $R(\lambda, T_{0,m})$. Following [14, Section 5.2], we begin by solving

$$\lambda f(x) + \frac{d}{dx}(r(x)f(x)) + q(x)f(x) = g(x), \ \ x\in\mathbb{R}_+, \tag{2.13}$$

where $g \in X_{0,m}$. On introducing antiderivatives of $1/r$ and q/r respectively, defined on $\mathbb{R}_+$, by

$$R(x) := \int_1^x \frac{1}{r(s)}\mathrm{d}s, \qquad Q(x) := \int_1^x \frac{q(s)}{r(s)}\mathrm{d}s, \tag{2.14}$$

we can proceed formally to obtain the general solution of (2.13) in the form

$$f(x) = v_\lambda(x)\int_0^x e^{\lambda R(y)+Q(y)}g(y)\mathrm{d}y + C\,v_\lambda(x), \quad \text{with} \quad v_\lambda(x) = \frac{e^{-\lambda R(x)-Q(x)}}{r(x)}, \ \ x\in\mathbb{R}_+, \tag{2.15}$$

where C is an arbitrary constant. An immediate consequence of (2.6) and (2.14) is that R is strictly increasing (and hence invertible) on $\mathbb{R}_+$, and Q is nondecreasing on $\mathbb{R}_+$. In the sequel, we need the following result that is a slight modification of [10, Lemma 2.1] and [14, Corollary 5.2.9].

Lemma 2.1. *Let $m \geq 1$ be fixed and define $\omega_{r,m} := 2m\tilde{r}$, where $\tilde{r}$ is the positive constant in (2.6). Then, for any $\lambda > \omega_{r,m}$ and $0 < \alpha < \beta \leq \infty$,*

$$I_{0,m}(\alpha,\beta) := \int_\alpha^\beta \frac{e^{-\lambda R(s)}}{r(s)} w_m(s)\mathrm{d}s \leq \frac{e^{-\lambda R(\alpha)}}{\lambda - \omega_{r,m}} w_m(\alpha)\,, \tag{2.16}$$

$$J_{0,m}(\alpha,\beta) := \int_\alpha^\beta \frac{(\lambda + q(s))e^{-\lambda R(s)-Q(s)}}{r(s)} w_m(s)\mathrm{d}s \leq \frac{\lambda e^{-\lambda R(\alpha)-Q(\alpha)}}{\lambda - \omega_{r,m}} w_m(\alpha), \tag{2.17}$$

where, as in (1.7), $w_m(x) = 1 + x^m$.

Proof. Using (2.6) and $\frac{e^{-\lambda R(s)}}{r(s)} = -\frac{d}{ds}e^{-\lambda R(s)}$, integration by parts gives

$$\begin{aligned} I_{0,m}(\alpha,\beta) &= \frac{1}{\lambda}e^{-\lambda R(\alpha)}w_m(\alpha) - \frac{1}{\lambda}e^{-\lambda R(\beta)}w_m(\beta) + \frac{m}{\lambda}\int_\alpha^\beta e^{-\lambda R(s)}s^{m-1}\mathrm{d}s \\ &\leq \frac{1}{\lambda}e^{-\lambda R(\alpha)}w_m(\alpha) + \frac{m\tilde{r}}{\lambda}\int_\alpha^\beta \frac{e^{-\lambda R(s)}}{r(s)}(1+s)s^{m-1}\mathrm{d}s. \end{aligned}$$

The inequality $(1+s)s^{m-1} \leq 2(1+s^m)$, which holds for all $s > 0$ and each fixed $m \geq 1$, yields

$$I_{0,m}(\alpha,\beta) \leq \frac{1}{\lambda}e^{-\lambda R(\alpha)}w_m(\alpha) + \frac{2m\tilde{r}}{\lambda}I_{0,m}(\alpha,\beta), \tag{2.18}$$

and (2.16) follows.

Inequality (2.17) follows then from (2.16) as in [14, Corollary 5.2.9]. □

Lemma 2.2. *Let $\lambda > 0$ and let v_λ be as defined in (2.15).*
(a) If (2.9) holds, then v_λ does not satisfy (2.11).
(b) If (2.8) holds, then $v_\lambda \notin X_{0,m}$ for any $m \geq 1$.

Proof. (a) For $0 < x < 1$ we have $r(x)v_\lambda(x) = e^{\int_x^1 \frac{\lambda+q(s)}{r(s)}\mathrm{d}s}$, and so $r(x)v_\lambda(x)$ does not converge to 0 as $x \to 0^+$.

(b) Let (2.8) be satisfied. Then, for each $\lambda > 0$,

$$\lim_{x\to 0^+} e^{-\lambda R(x)} = \lim_{x\to 0^+} e^{\int_x^1 \frac{\lambda}{r(s)}\mathrm{d}s} = \infty. \tag{2.19}$$

Consequently, since $e^{-Q(x)} \geq 1$ for $x \in [0,1]$, and $R(1) = 0$, we obtain

$$\int_0^\infty v_\lambda(x)w_m(x)\mathrm{d}x \geq \int_0^1 \frac{e^{-\lambda R(x)}}{r(x)}\mathrm{d}x = -\lambda\int_0^1 \frac{d}{dx}e^{-\lambda R(x)}\mathrm{d}x = \lambda(\lim_{x\to 0^+} e^{-\lambda R(x)} - 1),$$

and, from (2.19), it follows that $v_\lambda \notin X_{0,m}$. □

Motivated by (2.15) and Lemma 2.2, we are led, as in [14, Section 5.2.2], to

$$[\mathcal{R}(\lambda)g](x) := \frac{e^{-\lambda R(x)-Q(x)}}{r(x)}\int_0^x e^{\lambda R(y)+Q(y)}g(y)\mathrm{d}y \tag{2.20}$$

as a natural candidate for the resolvent, $R(\lambda, T_{0,m})$, of $T_{0,m}$. Then the proof of the following theorem follows exactly as the proof of [14, Theorem 5.2.11].

Theorem 2.1. *Let (2.5), (2.6) and (2.7) be satisfied. Then, for each $m \geq 1$ and $\lambda > \omega_{r,m}$, the resolvent of $(T_{0,m}, D(T_{0,m}))$ (in both cases (2.10) and (2.12)) is given by $R(\lambda, T_{0,m}) = \mathcal{R}(\lambda)$. Moreover,*

$$\|R(\lambda, T_{0,m})g\|_{[0,m]} \leq \frac{1}{\lambda - \omega_{r,m}} \|g\|_{[0,m]}, \text{ for all } g \in X_{0,m}. \tag{2.21}$$

Hence $(T_{0,m}, D(T_{0,m}))$ is the generator of a positive C_0-semigroup, $(S_{T_{0,m}}(t))_{t\geq 0}$, on $X_{0,m}$ satisfying

$$\|S_{T_{0,m}}(t)f\|_{[0,m]} \leq e^{\omega_{r,m}t}\|f\|_{[0,m]}\,, \text{ for all } f \in X_{0,m}.$$

(b) The growth-fragmentation semigroup

We now consider the growth–fragmentation equation (2.1). In addition to the restrictions (2.5), (2.6) and (2.7) imposed on a, r and a_1 respectively, we assume that the fragmentation kernel, b, satisfies (1.4) and further, for each $m \geq 0$, we define

$$n_m(y) = \int_0^y b(x,y)x^m \mathrm{d}x, \quad N_m(y) = y^m - n_m(y). \tag{2.22}$$

The local mass conservation condition in (1.4) then leads to

$$n_0(y) > 1; \qquad N_m(y) > 0, \quad m > 1; \qquad N_1(y) = 0; \qquad N_m(y) < 0, \quad 0 \leq m < 1; \tag{2.23}$$

see [14, Eqns. (2.2.53) & (2.3.16)]. The function n_0 is also assumed to satisfy

$$n_0(y) \leq b_0(1 + y^l)\,, \qquad y \in \mathbb{R}_+\,, \tag{2.24}$$

for constants $b_0 > 0$ and $l \geq 0$. A crucial role in the analysis is played by the further assumption that there exists $m_0 > 1$ such that

$$\liminf_{y\to\infty} \frac{N_{m_0}(y)}{y^{m_0}} > 0. \tag{2.25}$$

It follows, [6, Theorem 2.2], that for any fixed $y > 0$, $(1,\infty) \ni m \mapsto \frac{N_m(y)}{y^m}$ is an increasing and concave function. Hence, if (2.25) holds for some $m_0 > 1$, then

$$\liminf_{y\to\infty} \frac{N_m(y)}{y^m} > 0, \tag{2.26}$$

for all $m > 1$. For a given $m > 0$, (2.26) yields the existence of $y_m > 0$ and $c_m < 1$ such that

$$n_m(y) \leq c_m y^m, \quad y \geq y_m. \tag{2.27}$$

We note that (2.26) is satisfied for a large class of kernels b, including the homogeneous ones used in [15]; there are, however, cases when it does not hold, [14, Example 5.1.51].

Henceforth, we assume that

$$m > \max\{1, l\}, \tag{2.28}$$

and for each m we define an operator realisation, $B_{0,m}$, of the formal expression $\mathcal{B}$ in (2.3) by

$$(B_{0,m}f)(x) := \int_x^\infty a(y)b(x,y)f(y,t)dy,\ x \in \mathbb{R}_+;\ D(B_{0,m}) := \{f \in X_{0,m} : B_{0,m}f \in X_{0,m}\}. \tag{2.29}$$

Theorem 2.2. *Let (2.24), (2.26), (2.28) and the assumptions of Theorem 2.1 be satisfied. Then $(G_{0,m}, D(T_{0,m})) = (T_{0,m} + B_{0,m}, D(T_{0,m}))$ generates a positive C_0-semigroup, $(S_{G_{0,m}}(t))_{t\geq 0}$, on $X_{0,m}$.*

Proof. We use a version, [7, Lemma 5.12], of a theorem due to Desch that is applicable to positive operators in L_1 spaces. Thus, we must prove that $\|B_{0,m}R(\lambda, T_{0,m})\| < 1$ for some $\lambda > \omega_{r,m}$. Since $B_{0,m}R(\lambda, T_{0,m})$ is positive, we need only establish that $\|B_{0,m}R(\lambda, T_{0,m})f\|_{[0,m]} < \|f\|_{[0,m]}$ for all f in the positive cone, $X_{0,m,+}$, and some $\lambda > \omega_{r,m}$; see [7, Proposition 2.67]. Let $f \in X_{0,m,+}$ and $\lambda > \omega_{r,m}$. Then, using the Fubini–Tonelli theorem and (2.22), we get

$$\|B_{0,m}R(\lambda, T_{0,m})f\|_{[0,m]} = \int_0^\infty a(y)[R(\lambda, T_{0,m})f](y)(n_0(y) + n_m(y))\mathrm{d}y.$$

On setting $a_\rho = \operatorname{ess\,sup}_{x\in[0,\rho]} a(x)$ for each fixed $\rho > 0$, we obtain, by (2.24), (2.23) and (2.21),

$$\int_0^\rho a(y)[R(\lambda, T_{0,m})f](y)(n_0(y) + n_m(y))\mathrm{d}y \le a_\rho \int_0^\infty [R(\lambda, T_{0,m})f](y)(b_0 w_l(y) + y^m)\mathrm{d}y$$
$$\le C_m a_\rho \int_0^\infty [R(\lambda, T_{0,m})f](y) w_m(y)\mathrm{d}y \le \frac{C_m a_\rho}{\lambda - \omega_{r,m}} \|f\|_{[0,m]},$$

where $C_m := \sup_{0\le y<\infty} b_0 \frac{w_l(y)}{w_m(y)} + \frac{y^m}{w_m(y)} \le 2b_0 + 1$. To obtain a suitable estimate on the integral over $[\rho, \infty)$, we now use (2.27). Since $\rho > y_m$ can be chosen sufficiently large for $b_0 \frac{w_l(y)}{w_m(y)} < \delta$ for all $y \ge \rho$, where $c_m + \delta < 1$, we can argue as in Lemma 2.1 to obtain

$$\int_\rho^\infty a(y)[R(\lambda, T_{0,m})f](y)(n_0(y) + n_m(y))\mathrm{d}y \le (\delta + c_m) \int_0^\infty a(y)[R(\lambda, T_{0,m})f](y) w_m(y)\mathrm{d}y$$
$$= (\delta + c_m) \int_0^\infty \left(e^{\lambda R(x)+Q(x)} \int_x^\infty \frac{w_m(y)a(y)e^{-\lambda R(y)-Q(y)}}{r(y)}\mathrm{d}y \right) f(x)\mathrm{d}x$$
$$\le (\delta + c_m) \int_0^\infty e^{\lambda R(x)+Q(x)} J_{0,m}(x,\infty) f(x)\mathrm{d}x \le \frac{\lambda(\delta + c_m)}{\lambda - \omega_{r,m}} \|f\|_{[0,m]}.$$

Hence

$$\|B_{0,m} R(\lambda, T_{0,m})f\|_{[0,m]} \le \left(\frac{C_m a_\rho}{\lambda - \omega_{r,m}} + \frac{\lambda}{\lambda - \omega_{r,m}}(\delta + c_m) \right) \|f\|_{[0,m]}.$$

Since $\frac{\lambda}{\lambda - \omega_{r,m}}(\delta + c_m) \to \delta + c_m < 1$ and $\frac{C_m a_\rho}{\lambda - \omega_{r,m}} \to 0$ as $\lambda \to \infty$, there is λ_0 such that

$$\frac{C_m a_\rho}{\lambda - \omega_{r,m}} + \frac{\lambda}{\lambda - \omega_{r,m}}(\delta + c_m) < 1$$

for $\lambda > \lambda_0$. Therefore $B_{0,m}$ is a Miyadera perturbation of $T_{0,m}$, and the stated result follows. □

Under the conditions of Theorem 2.2, it follows that constants $C(m)$ and $\theta(m)$ exist such that

$$\|S_{G_{0,m}}(t)f\|_{[0,m]} \le C(m) e^{\theta(m)t} \|f\|_{[0,m]}, \quad \text{for all } f \in X_{0,m} \text{ and } t \ge 0. \tag{2.30}$$

Moreover, an alternative, but equivalent, representation of the generator $G_{0,m}$ is

$$G_{0,m} := T^0_{0,m} + A^{(1)}_{0,m} + A_{0,m} + B_{0,m} = T^0_{0,m} + A^{(1)}_{0,m} + F_{0,m}, \tag{2.31}$$

where $A^{(1)}_{0,m}$, $A_{0,m}$ were defined in subsection (a) and

$$[T^0_{0,m}f](x) := -\partial_x[r(x)f(x)]\,; \quad D(T^0_{0,m}) := \left\{ f \in X_{0,m} \,:\, rf \in AC(\mathbb{R}_+) \text{ and } \frac{d}{dx}(rf) \in X_{0,m} \right\}.$$

As with the operator $T_{0,m}$, the homogeneous boundary condition must also be incorporated in the above definition of $D(T^0_{0,m})$ when (2.9) holds.

Next we show that the regularising property of the growth-fragmentation semigroup, established in [15, Lemma 2.7] for growth rates satisfying (2.9) and homogeneous fragmentation kernels, holds also in the current setting. The presented proof, while using the better characterization of the generator obtained in Theorem 2.2, essentially follows the lines of *op.cit*. We shall need the adjoint semigroup, $\left(S^*_{G_{0,m}}(t)\right)_{t\ge 0}$, defined on the dual space $X^*_{0,m}$, where the latter can be identified, via the duality pairing

$$\langle f, g\rangle := \int_0^\infty f(x)g(x)\mathrm{d}x, \quad f \in X^*_{0,m},\ g \in X_{0,m},$$

with the space of measurable functions f such that $f w_m^{-1}$ is essentially bounded on $\mathbb{R}_+$.

Since $w_m \in X^*_{0,m}$, we can define

$$\Psi_m(x,t) := [S^*_{G_{0,m}}(t) w_m](x), \ (x,t) \in \mathbb{R}^2_+. \tag{2.32}$$

Theorem 2.3. *In addition to the conditions required for Theorem 2.2 to hold, assume that positive constants a_0, γ_0 and x_0 exist such that*

$$a(x) \geq a_0 x^{\gamma_0}, \quad \text{for all } x \geq x_0. \tag{2.33}$$

Then, for any n, p and m satisfying $\max\{1, l\} < n < p < m$, there are constants $C = C(m,n,p) > 0$ and $\theta = \theta(m,n) > 0$ such that

$$\|S_{G_{0,p}}(t)\mathring{f}\|_{[0,m]} \leq Ce^{\theta t} t^{\frac{n-m}{\gamma_0}} \|f\|_{[0,p]}, \quad \text{for all } f \in X_{0,p}. \tag{2.34}$$

Proof. First we note that $X_{0,m} \hookrightarrow X_{0,p} \hookrightarrow X_{0,n}$, where $\hookrightarrow$ denotes a continuous embedding. Moreover, for each $j = n, p, m$, the operator $G_{0,j}$ generates a positive C_0-semigroup $(S_{G_{0,j}}(t))_{t\geq 0}$ on $X_{0,j}$. Assume initially that $\mathring{f} \in D(T_{0,m})_+ = D(G_{0,m})_+$. Then, for all $t \geq 0$,

$$f(\cdot,t) := [S_{G_{0,m}}(t)\mathring{f}](\cdot) = [S_{G_{0,p}}(t)\mathring{f}](\cdot) \in D(G_{0,m}) = D(T^0_{0,m}) \cap D(A_{0,m}).$$

Consequently, we can multiply (2.1) by $w_m(x) = 1 + x^m$, integrate term by term and use $a_1 \geq 0$ to obtain, as in [14, Lemma 5.2.17],

$$\frac{d}{dt} M_{0,m}(t) \leq \int_0^\infty \left(mr(x)x^{m-1} - (N_0(x) + N_m(x))a(x) \right) f(x,t)\mathrm{d}x =: \int_0^\infty \Phi_m(x) f(x,t)\mathrm{d}x. \tag{2.35}$$

Recalling from (2.27) that $n_m(y) \leq c_m y^m$ for all $y \geq y_m$, where $0 < c_m < 1$, we choose a positive constant $R_m > \max\{1, x_0, y_m\}$ such that

$$(b_0(1 + x^l) - 1) - (1 - c_m)x^m \leq 0, \quad \text{for all } x \geq R_m. \tag{2.36}$$

It then follows from (2.6), (2.24), (2.27) and (2.33) that, for any $R \geq R_m$ and for all $x \geq R$, we have

$$\begin{aligned} \Phi_m(x) &\leq m\widetilde{r}(1+x)x^{m-1} + (b_0(1+x^l) - 1) - (1-c_m)x^m)a_0 R^{\gamma_0} \\ &\leq (2m\widetilde{r} - (1-c_m)a_0 R^{\gamma_0})w_m(x) + (b_0 w_l(x) - c_m)a_0 R^{\gamma_0}. \end{aligned}$$

If we now impose the further restriction that R_m is also chosen so that $2m\widetilde{r} - (1-c_m)a_0 R^{\gamma_0} \leq -d_m R^{\gamma_0}$ for each $R \geq R_m$, where $d_m > 0$, then, for any x and R satisfying $x \geq R \geq R_m$, we have

$$\Phi_m(x) \leq -d_m R^{\gamma_0} w_m(x) + b_0 a_0 R^{\gamma_0} w_n(x). \tag{2.37}$$

Turning to the case when $x \leq R$, from (2.23) we have $N_m(x) \geq 0$ for all x, and we know also that (2.36) holds for $x \in [R_m, R]$. Consequently, on setting $a_{R_m} = \operatorname*{ess\,sup}_{x\in[0,R_m]} a(x)$, we obtain, for $0 < x \leq R$,

$$\begin{aligned} \Phi_m(x) &\leq 2m\widetilde{r}w_m(x) + (b_0(1 + R_m^l) - 1)a_{R_m} \\ &\leq -d_m R^{\gamma_0} w_m(x) + \left((d_m R^{\gamma_0} + 2m\widetilde{r})(1 + R^{m-n}) + \frac{(b_0(1+R_m^l) - 1)a_{R_m}}{w_n(x)} \right) w_n(x), \end{aligned}$$

where we have used the inequality $w_m(x)/w_n(x) \leq 1 + x^{m-n}$, $x > 0$. Combining the above inequality with (2.37), for any fixed $R \geq R_m$, there exist positive constants d_m and D_m such that

$$\Phi_m(x) \leq -d_m R^{\gamma_0} w_m(x) + D_m R^{\gamma_0+m-n} w_n(x), \quad \text{for all } x \in \mathbb{R}_+,$$

and therefore, from (2.35),

$$\frac{d}{dt} M_{0,m}(t) \leq -d_m R^{\gamma_0} M_{0,m}(t) + D_m R^{\gamma_0+m-n} M_{0,n}(t). \tag{2.38}$$

Since Theorem 2.2 ensures that $M_{0,n}(t) \leq C(n)e^{\theta(n)t}\|\mathring{f}\|_{[0,n]} =: \sigma_n(t)\|\mathring{f}\|_{[0,n]}$, (2.38) leads to

$$\frac{d}{dt}(e^{d_m R^{\gamma_0} t} M_{0,m}(t)) \leq D_m C(n) R^{\gamma_0+m-n} e^{(d_m R^{\gamma_0} + \theta(n))t} \|\mathring{f}\|_{[0,n]}.$$

Integrating, using the definitions of $\left(S^*_{G_{0,m}}(t)\right)_{t\geq 0}$ and Ψ_m, see (2.32), we get for some $D'_{m,n}>0$,

$$\begin{aligned}\int_0^\infty \Psi_m(x,t)\mathring{f}(x)\mathrm{d}x = M_{0,m}(t) &\leq e^{-d_m R^{\gamma_0}t}\|\mathring{f}\|_{[0,m]} + D'_{m,n}R^{m-n}\sigma_n(t)\|\mathring{f}\|_{[0,n]}\\ &\leq \int_0^\infty (e^{-d_m R^{\gamma_0}t}w_m(x) + D'_{m,n}R^{m-n}\sigma_n(t)w_n(x))\mathring{f}(x)\mathrm{d}x.\end{aligned} \tag{2.39}$$

Since all positive $C_0^\infty(\mathbb{R}_+)$ functions are in $D(G_{0,m})_+$, this leads to

$$\Psi_m(x,t) \leq e^{-d_m R^{\gamma_0}t}w_m(x) + D'_m R^{m-n}\sigma_n(t)w_n(x), \tag{2.40}$$

for almost any $x>0$ and each $R\geq R_m$. Since t, x and R are independent, for fixed t and x, with $x\geq e^{\frac{d_m R_m^{\gamma_0}t}{m-n}}$, we can define $R := \left(\frac{m-n}{d_m}\frac{\log x}{t}\right)^{1/\gamma_0}$. It then follows from (2.40) that

$$\Psi_m(x,t) \leq x^{n-m}w_m(x) + D'_m\left(\frac{m-n}{d_m}\right)^{1/\gamma_0} t^{\frac{n-m}{\gamma_0}}\log^{\frac{n-m}{\gamma_0}} x\sigma_n(t)w_n(x) \leq D_{m,n,p}\,\widehat{\sigma}_n(t)t^{\frac{n-m}{\gamma_0}}w_p(x),$$

where p is any number bigger than n, the function $\widehat{\sigma}_n(t)$ is bounded as $t\to 0^+$ and exponentially bounded as $t\to\infty$, and $D_{m,n,p}$ is a constant depending on m,n,p. For $x<e^{\frac{d_m R_m^{\gamma_0}t}{m-n}}$, we take $R=R_m$ and use the fact that $w_m(x)$ and $w_n(x)$ are increasing functions to obtain from (2.40)

$$\Psi_m(x,t) \leq e^{-d_m R_m^{\gamma_0}t}w_m\left(e^{\frac{d_m R_m^{\gamma_0}t}{m-n}}\right) + D'_m R_m^{m-n}\sigma_n(t)w_n\left(e^{\frac{d_m R_m^{\gamma_0}t}{m-n}}\right) \leq D_{m,n}\widetilde{\sigma}_n(t)e^{\frac{md_m R_m^{\gamma_0}t}{m-n}}w_p(x).$$

Thus, there are constants $C=C(m,n,p)$ and $\theta=\theta(m,n)$ such that, for almost all $x>0$ and $t>0$,

$$\Psi_m(x,t) \leq Ce^{\theta t}t^{\frac{n-m}{\gamma_0}}w_p(x)$$

and hence, using (2.39), for $\mathring{f}\in X_{0,m,+}$,

$$\|S_{G_{0,p}}(t)\mathring{f}\|_{[0,m]} \leq Ce^{\theta t}t^{\frac{n-m}{\gamma_0}}\int_0^\infty \mathring{f}(x)w_p(x)\mathrm{d}x.$$

The inequality can be extended to $\mathring{f}\in X_{0,p}$ by linearity and density. □

Corollary 2.1. *Under the assumptions of Theorem 2.3, $S_{G_{0,p}}(t): D(G_{0,p})\to D(G_{0,m})$ for all $t>0$.*

Proof. Let m,n and p be as in Theorem 2.3. Since f and $G_{0,p}f$ belong to $X_{0,p}$, both $S_{G_{0,p}}(t)f$ and $S_{G_{0,p}}(t)G_{0,p}f$ are in $X_{0,m}$ for $t>0$, and therefore we can evaluate

$$\frac{S_{G_{0,m}}(h)-I}{h}S_{G_{0,p}}(t)f = \frac{S_{G_{0,p}}(h)-I}{h}S_{G_{0,p}}(t)f = S_{G_{0,p}}(t)\frac{S_{G_{0,p}}(h)-I}{h}f.$$

It then follows from Theorem 2.3 that

$$\begin{aligned}&\left\|\frac{S_{G_{0,m}}(h)-I}{h}S_{G_{0,p}}(t)f - S_{G_{0,p}}(t)G_{0,p}f\right\|_{[0,m]} = \left\|S_{G_{0,p}}(t)\left(\frac{S_{G_{0,p}}(h)-I}{h}f - G_{0,p}f\right)\right\|_{[0,m]}\\ &\leq Ce^{\theta t}t^{-\frac{m-n}{\gamma_0}}\left\|\frac{S_{G_{0,p}}(h)-I}{h}f - G_{0,p}f\right\|_{[0,p]} \to 0,\end{aligned} \tag{2.41}$$

as $h\to 0^+$, which establishes that $S_{G_{0,p}}(t)f\in D(G_{0,m})$ for all $t>0$. □

Corollary 2.2. *Assume that (2.6), (2.24), (2.26), (2.28) and (2.33) are all satisfied, and let $p>\max\{1,l\}$. Then, for each $\mathring{f}\in X_{0,m}\cap D(G_{0,p})$, problem (2.1) has a classical solution in $X_{0,m}$.*

Proof. Let $f(t)=S_{G_{0,m}}(t)\mathring{f}$. We can assume that $p<m$, as otherwise $\mathring{f}\in D(G_{0,m})$. Then, for all $t>0$, $S_{G_{0,p}}(t)\mathring{f} = S_{G_{0,m}}(t)\mathring{f}$ and, by Corollary 2.1, $S_{G_{0,p}}(t)\mathring{f}\in D(G_{0,m})$ so that, as in (2.41),

$$\lim_{h\to 0^+}\frac{f(t+h)-f(t)}{h} = \lim_{h\to 0}\frac{S_{G_{0,p}}(h)-I}{h}S_{G_{0,p}}(t)\mathring{f} = G_{0,p}S_{G_{0,p}}(t)\mathring{f} = G_{0,m}S_{G_{0,m}}(t)\mathring{f}$$

in $X_{0,m}$, where the last equality follows from Corollary 2.1. □

3. Coagulation-fragmentation with growth

The results obtained in the previous section can now be exploited to establish the well-posedness of the initial value problem (IVP) (1.9). The restrictions placed on r, a and b for Theorem 2.3 to hold continue to be assumed, but now we specify that $a_1(x) := \beta(1 + x^\alpha)$, where β is a constant that will be determined later, and $0 < \alpha < \gamma_0$, with γ_0 defined in (2.33), so that $a_1(x)/a(x)$ remains bounded as $x \to \infty$.

The coagulation kernel is required to satisfy

$$k(x, y) \leq k_0(1 + x^\alpha)(1 + y^\alpha), \tag{3.1}$$

for some positive constant k_0. It is convenient to express (1.9) in the form

$$\partial_t f(x,t) = -\partial_x[r(x)f(x,t)] - a_1(x)f(x,t) + \mathfrak{F}f(x,t) + \mathfrak{K}_\beta f(x,t)\ ,\ (x,t) \in \mathbb{R}^2_+, \tag{3.2}$$

$$f(0,x) = \mathring{f}(x)\ ,\quad x \in \mathbb{R}_+\ , \tag{3.3}$$

where $\mathfrak{K}_\beta f(x,t) := a_1(x)f(x,t) + \mathfrak{K}f(x,t)$, and $\mathfrak{F}$, $\mathfrak{K}$ are given by (1.2) and (1.3) respectively. Denoting the generator of the growth-fragmentation semigroup in this case by $G^{(\beta)}_{0,m}$, the corresponding abstract formulation of the IVP (3.2) - (3.3) can be written as

$$\frac{d}{dt}f(t) = G^{(\beta)}_{0,m}f(t) + K^{(\beta)}_{0,m}f(t),\ t > 0;\ \ f(0) = \mathring{f}, \tag{3.4}$$

where the operator $K^{(\beta)}_{0,m}$ is the realisation of $\mathfrak{K}_\beta$ on $X_{0,m}$.

The following inequalities will often be used. For $x \geq 0$

$$(1 + x^\delta) \leq 2(1 + x^\eta),\quad 0 \leq \delta \leq \eta,\quad \text{and}\quad (1 + x^\delta)(1 + x^\eta) \leq 4(1 + x^{\delta+\eta}),\quad 0 \leq \delta \leq \eta. \tag{3.5}$$

(a) Local Existence

We begin by proving the local (in time) existence and uniqueness of a mild solution to (3.4).

Theorem 3.1. *Let r, a, b be such that Theorem 2.3 holds, k satisfy (3.1) and, in addition, let $m > \alpha + \max\{1, l\}$. Then, for each $\mathring{f} \in X_{0,m,+}$, the semilinear ACP (3.4) has a unique nonnegative mild solution $f \in C([0, \tau_{\max}), X_{0,m})$ defined on its maximal interval of existence $[0, \tau_{\max})$, where $\tau_{max} = \tau_{\max}(\mathring{f})$. If $\tau_{\max} < \infty$, then $\|f(t)\|_{[0,m]}$ is unbounded as $t \to \tau_{\max}^-$.*

Proof. Let p be defined by

$$p := m - \alpha, \tag{3.6}$$

and, noting that $(m - n)/\gamma_0 = (p - n)/\gamma_0 + \alpha/\gamma_0$, we are then able to choose $n < p$ such that $(m - n)/\gamma_0 < 1$ and $n > \max\{1, l\}$, so that m, n, p satisfy assumptions of Theorem 2.3. We begin by showing that the bilinear form $\mathcal{K}^{(\beta)}_{0,m}$, defined by

$$[\mathcal{K}^{(\beta)}_{0,m}(f, g)](x) := \beta(1 + x^\alpha)f(x) - f(x)\int_0^\infty k(x, y)g(y)\mathrm{d}y + \frac{1}{2}\int_0^x k(x - y, y)f(x - y)g(y)\mathrm{d}y, \tag{3.7}$$

is continuous from $X_{0,m} \times X_{0,m}$ into $X_{0,p}$. From (3.1), (3.5) and (3.6), we obtain, for $f, g \in X_{0,m}$,

$$\beta\int_0^\infty (1 + x^\alpha)|f(x)|w_p(x)\mathrm{d}x \leq 4\beta\|f\|_{[0,m]}, \tag{3.8}$$

$$\int_0^\infty |f(x)|\left(\int_0^\infty k(x, y)|g(y)|\mathrm{d}y\right) w_p(x)\mathrm{d}x \leq 4k_0\|f\|_{[0,m]}\|g\|_{[0,m]}, \tag{3.9}$$

$$\frac{1}{2}\int_0^\infty\left(\int_0^x k(x - y, y)|f(y)||g(x - y)|\mathrm{d}y\right) w_p(x)\mathrm{d}x \leq 2^{p+3}k_0\|f\|_{[0,m]}\|g\|_{[0,m]}. \tag{3.10}$$

Hence,

$$\|\mathcal{K}^{(\beta)}_{0,m}(f,g)\|_{[0,p]} \le \left(4\beta + (4+2^{p+3})k_0\|g\|_{[0,m]}\right)\|f\|_{[0,m]}, \quad \text{for all } f,g\in X_{0,m}. \tag{3.11}$$

Since $K^{(\beta)}_{0,m}f = \mathcal{K}^{(\beta)}_{0,m}(f,f)$, it follows that $K^{(\beta)}_{0,m}$ is a continuous mapping from $X_{0,m}$ into $X_{0,p}$. Consequently, the integral equation that arises as the mild formulation of (3.4) can be written as

$$f(t) = S_{G^{(\beta)}_{0,m}}(t)\mathring{f} + \int_0^t S_{G^{(\beta)}_{0,p}}(t-s)K^{(\beta)}_{0,m}f(s)\mathrm{d}s. \tag{3.12}$$

Next consider the set

$$\mathcal{U} := \{f\in X_{0,m,+} : \|f\|_{[0,m]} \le 1+b\}, \tag{3.13}$$

for an arbitrary $b>0$. For each $f\in\mathcal{U}$, we can use (3.1), (3.5) and the fact that $\alpha<m$, to obtain

$$\int_0^\infty k(x,y)f(y)\mathrm{d}y \le 2k_0(1+x^\alpha)\|f\|_{[0,m]} \le \beta(1+x^\alpha), \quad \text{for all } x>0,$$

where we now define $\beta := 2k_0(1+b)$ and therefore, with this choice of β,

$$(K^{(\beta)}_{0,m}f)(x) \ge \frac{1}{2}\int_0^x k(x-y,y)f(x-y)f(y)\mathrm{d}y \ge 0. \tag{3.14}$$

Also, from (3.11), we have, for all $f,g\in\mathcal{U}$,

$$\|K^{(\beta)}_{0,m}f\|_{[0,p]} \le K(\mathcal{U}), \quad \text{and} \quad \|K^{(\beta)}_{0,m}f - K^{(\beta)}_{0,m}g\|_{[0,p]} \le L(\mathcal{U})\|f-g\|_{[0,m]}, \tag{3.15}$$

where $K(\mathcal{U}) = \frac{\beta^2}{k_0}(2+(1+2^{p+1}))$ and, by the definition of β, $L(\mathcal{U}) = 8\beta(1+2^p)$.

Then, for $\mathring{f}\in X_{0,m,+}$ satisfying $\|\mathring{f}\|_{0,m}\le b$, we define the operator

$$Tf(t) = S_{G^{(\beta)}_{0,m}}(t)\mathring{f} + \int_0^t S_{G^{(\beta)}_{0,p}}(t-s)K^{(\beta)}_{0,m}f(s)\mathrm{d}s \tag{3.16}$$

in $Y_m := C([0,\tau],\mathcal{U})$, with $\mathcal{U}$ defined by (3.13) and τ to be determined so that T is a contraction on Y_m, when Y_m is equipped with the metric induced by the norm from $C([0,\tau],X_{0,m})$. First, observe that $Tf\in C([0,\tau],X_{0,m,+})$ for all $f\in Y_m$. Indeed, for any $t\ge 0$ and $h>0$, with $t+h\le\tau$,

$$\|Tf(t+h)-Tf(t)\|_{[0,m]} \le \int_t^{t+h}\left\|S_{G^{(\beta)}_{0,p}}(t+h-s)K^{(\beta)}_{0,m}f(s)\right\|_{[0,m]}\mathrm{d}s$$
$$+\int_0^t\left\|S_{G^{(\beta)}_{0,p}}(t-s)(S_{G^{(\beta)}_{0,p}}(h)-I)K^{(\beta)}_{0,m}f(s)\right\|_{[0,m]}\mathrm{d}s =: I_1(h)+I_2(h).$$

Now, by (2.34) and (3.15),

$$\|S_{G^{(\beta)}_{0,p}}(t+h-s)K^{(\beta)}_{0,m}f(s)\|_{[0,m]} \le C(m,n,p)e^{\theta(m,n)(t+h-s)}(t+h-s)^{\frac{n-m}{\gamma_0}}K(\mathcal{U}).$$

Since $(n-m)/\gamma_0 > -1$, it follows that

$$\int_t^{t+h}(t+h-s)^{\frac{n-m}{\gamma_0}}\mathrm{d}s = \int_0^h \sigma^{\frac{n-m}{\gamma_0}}\mathrm{d}\sigma \to 0 \ \text{ as } h\to 0^+,$$

and therefore $\lim_{h\to 0^+} I_1(h) = 0$. Similarly,

$$\|S_{G^{(\beta)}_{0,p}}(t-s)(S_{G^{(\beta)}_{0,p}}(h)-I)K^{(\beta)}_{0,m}f(s)\|_{[0,m]}$$
$$\le C(m,n,p)e^{\theta(m,n)(t-s)}(t-s)^{\frac{n-m}{\gamma_0}}\|(S_{G^{(\beta)}_{0,p}}(h)-I)K^{(\beta)}_{0,m}f(s)\|_{[0,p]}$$
$$\le C(m,n,p,\tau)(t-s)^{\frac{n-m}{\gamma_0}}, \tag{3.17}$$

where $C(m,n,p,\tau)$ is a constant that is independent of h. Thus, from the first inequality we see that the integrand in $I_2(h)$ converges to zero as $h\to 0^+$ for each $0\le s<t$ and the second ascertains that this convergence is dominated by an integrable function. Hence, an application

of the Lebesgue dominated convergence theorem shows that Tf is right continuous at t for all $t \in [0,\tau)$. Similarly, when $0<h<t\le\tau$ and $t-h\ge 0$, we have

$$\|Tf(t-h)-Tf(t)\|_{[0,m]} \le \int_{t-h}^{t} \left\| S_{G_{0,p}^{(\beta)}}(t-s)K_{0,m}^{(\beta)}f(s)\right\|_{[0,m]} \mathrm{d}s + \int_0^{t-h} \left\| S_{G_{0,p}^{(\beta)}}(t-h-s)(I-S_{G_{0,p}^{(\beta)}}(h))K_{0,m}^{(\beta)}f(s)\right\|_{[0,m]} \mathrm{d}s =: I_1'(h)+I_2'(h).$$

Arguing as before, we obtain $\lim_{h\to 0^+} I_1'(h)=0$. As for $I_2'(h)$, we rewrite it as

$$I_2'(h)=\int_0^t \chi_{[h,t]}\left\| S_{G_{0,p}^{(\beta)}}(t-\sigma)(I-S_{G_{0,p}^{(\beta)}}(h))K_{0,m}^{(\beta)}f(\sigma-h)\right\|_{[0,m]} \mathrm{d}\sigma, \tag{3.18}$$

where χ_Ω is the characteristic function of Ω. Since $t\to K_{0,m}^{(\beta)}f(t)$ is a continuous function in $X_{0,p}$, $\lim\limits_{h\to 0^+} K_{0,m}^{(\beta)}f(\sigma-h)=K_{0,m}^{(\beta)}f(\sigma)$ for each $\sigma>0$. Then, on account of the local uniform boundedness of $(S_{G_{0,p}^{(\beta)}}(t))_{t\ge 0}$, a corollary of the Banach-Steinhaus theorem ensures that $\lim\limits_{h\to 0^+}(I-S_{G_{0,p}^{(\beta)}}(h))K_{0,m}^{(\beta)}f(\sigma-h)=0$ for any fixed $\sigma>0$ and we see that the integrand in (3.18) converges to zero on $[0,t]$. Moreover, from (3.17),

$$\|S_{G_{0,p}^{(\beta)}}(t-\sigma)(I-S_{G_{0,p}^{(\beta)}})(h))K_{0,m}^{(\beta)}f(\sigma-h)\|_{[0,m]} \le C(m,n,p,\tau)(t-\sigma)^{\frac{n-m}{\gamma_0}}$$

for all $\sigma\in[h,t]$, where, by (3.15), $K_{0,m}^{(\beta)}f(\sigma-h)$ is estimated by the Y_m norm of f, and this is independent of h. Consequently, by the Lebesgue dominated convergence theorem, we obtain $\lim_{h\to 0^+} I_2'(h)=0$. Further, thanks to (3.14), $Tf(t)\ge 0$ since $f(s)\ge 0$ for all $s\in[0,\tau]$.

As in the proof of [14, Theorem 8.1.2], we establish the existence of $\tau=\tau(\mathcal{U})$ such that $\|Tf(t)\|_{[0,m]}\le 1+b$ for $0\le t\le\tau$ and moreover T is a contractive mapping on Y_m. Hence, in the usual way, we can extend the solution to the maximal interval $[0,\tau_{\max})$. The last statement of the theorem follows since τ is uniform on bounded subsets of $X_{0,m}$. □

The next objective is to prove that the mild solution of the previous theorem is a classical solution of (3.4) under an additional restriction on $\mathring{f}$. We require the following three lemmas.

Lemma 3.1. $K_{0,m}^{(\beta)}: X_{0,m}\to X_{0,p}$ *is continuously Fréchet differentiable.*

Proof. Recall that $K_{0,m}^{(\beta)}f=\mathcal{K}_{0,m}^{(\beta)}(f,f)$, see (3.7). Using (3.9) and (3.10), we see that $K_{0,m}^{(\beta)}$ is Fréchet differentiable at each $f\in X_{0,m}$, with Fréchet derivative given by

$$[\partial K_{0,m}^{(\beta)}f]h := \beta w_\alpha h+\mathcal{K}_{0,m}^{(0)}(f,h)+\mathcal{K}_{0,m}^{(0)}(h,f)\,,\quad h\in X_{0,m}.$$

Moreover, again by (3.9) and (3.10), for any $f,g,h\in X_{0,m}$,

$$\|[\partial K_{0,m}^{(\beta)}f]h-[\partial K_{0,m}^{(\beta)}g]h\|_{[0,p]}\le 8\beta(1+2^p)\|h\|_{[0,m]}\,\|f-g\|_{[0,m]}\to 0$$

as $\|f-g\|_{[0,m]}\to 0$, uniformly in $\|h\|_{[0,m]}\le 1$. Hence, the Fréchet derivative is continuous. □

Lemma 3.2. *Assume that* $1<p<m$ *and* $T\in\mathbb{R}_+$. *Let* $(S(t))_{t\ge 0}$ *be a* C_0*-semigroup on* $X_{0,p}$ *and* $\{P(t)\}_{t\in[0,T]}$ *be bounded linear operators from* $X_{0,m}$ *to* $X_{0,p}$ *such that, for all* $u\in X_{0,p}$ *and* $f\in X_{0,m}$,

$$\|S(t)u\|_{[0,m]}\le M(t)t^{-\kappa}\|u\|_{[0,p]}\,,\qquad \|P(t)f\|_{[0,p]}\le L(t)\|f\|_{[0,m]},$$

where $M,L\in L_\infty([0,T])$ *and* $0<\kappa<1$. *Moreover, let* $g\in C((0,T],X_{0,m})$ *be such that* $\|g(t)\|_{[0,m]}\le G(t)t^{-\delta}$, *where* $G\in L_\infty([0,T])$ *and* $0<\delta<1$. *Then the integral equation*

$$f(t)=g(t)+\int_0^t S(t-s)P(s)f(s)\mathrm{d}s \tag{3.19}$$

has a unique solution $f\in C((0,T],X_{0,m})$ *that satisfies* $\|f(t)\|_{[0,m]}\le F(t)t^{-\delta}$ *for some* $F\in L_\infty([0,T])$.

Proof. We use some ideas from [8, Lemma 3.2]. Denoting Laplace convolution by $*$ and defining $\theta_r(t) := t^{-r}$, a simple argument shows that $\theta_\delta * \theta_\kappa$ exists for any choice of $\delta < 1$ and $\kappa < 1$, with

$$(\theta_\delta * \theta_\kappa)(t) = B(1-\delta, 1-\kappa)\, t^{1-\delta-\kappa} = B(1-\delta, 1-\kappa)\, \theta_{\delta+\kappa-1}(t), \tag{3.20}$$

where B is the Beta function. Let us denote by M_T, L_T and G_T the suprema of the respective functions M, L and G on $[0,T]$. Then, defining for all $t \in (0,T]$,

$$g_1(t) = \int_0^t S(t-s)P(s)g(s)\mathrm{d}s, \qquad g_n(t) = \int_0^t S(t-s)P(s)g_{n-1}(s)\mathrm{d}s, \quad n \geq 1$$

we have, by induction,

$$\|g_n(t)\|_{[0,m]} \leq (M_T L_T)^n G_T \prod_{i=1}^{n} B(1-\kappa, i-(i-1)\kappa-\delta) t^{-(n\kappa-n+\delta)},$$

for all $n \in \mathbb{N}$. Since $n - n\kappa - \delta = n(1-\kappa-\delta/n)$, there exists $n_0 \in \mathbb{N}$ such that $n - n\kappa - \delta > 0$ for all $n \geq n_0$. Then, denoting $g(t) = g_0(t)$, we can re-write (3.19) as

$$f(t) - \sum_{i=0}^{n_0-1} g_i(t) = g_{n_0}(t) + \int_0^t S(t-s)\left(f(s) - \sum_{i=0}^{n_0-1} g_i(s)\right)\mathrm{d}s \tag{3.21}$$

where we have $g_{n_0} \in C([0,T], X_{0,m})$. Consider now an operator on $C([0,T], X_{0,m})$ given by

$$Qu(t) = g_{n_0}(t) + \int_0^t S(t-s)P(s)u(s)\mathrm{d}s.$$

The argument used in Theorem 3.1 to prove the continuity of the operator T can be applied again to establish the continuity of Q. Then, for $u, v \in C([0,T], X_{0,m})$ we obtain

$$\|Qu(t) - Qv(t)\|_{[0,m]} \leq M_T L_T \sup_{s\in[0,T]} \|u(s)-v(s)\|_{[0,m]} B(1-\kappa, 1) t^{1-\kappa}$$

and, again by induction,

$$\|Q^k u(t) - Q^k v(t)\|_{[0,m]} \leq M_T^k L_T^k \sup_{s\in[0,T]} \|u(s)-v(s)\|_{[0,m]} \prod_{i=0}^{k-1} B(1-\kappa, i+1-i\kappa) t^{-i(\kappa-1)}.$$

Now, using the fact that $B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ and $\frac{\Gamma(y)}{\Gamma(x+y)} \leq c_y y^{-x}, x > 0, y \to \infty,$, [1, Inequality 6.1.47], we see, on account of $i+1-i\kappa = i(1-\kappa)+1 \geq 1$ for $i \geq 0$, that

$$\prod_{i=0}^{k-1} B(1-\kappa, i+1-i\kappa) \leq \Gamma^k(1-\kappa) c_{1-\kappa}^k \left(\frac{1}{1-\kappa}\right)^{k(1-\kappa)} \left(\frac{1}{(k-1)!}\right)^{1-\kappa}.$$

Hence, for some constant C_T,

$$\sup_{t\in[0,T]} \|Q^k u(t) - Q^k v(t)\|_{[0,m]} \leq \left(\frac{C_T^{\frac{k}{1-\kappa}}}{(k-1)!}\right)^{1-\kappa} \sup_{s\in[0,T]} \|u(s)-v(s)\|_{[0,m]}$$

and therefore there exists k such that Q^k is a contraction. Thus, the equation $u = Qu$ has a unique solution $u \in C([0,T], X_{0,m})$ (the uniqueness follows from the Gronwall-Henry inequality, see [8, Lemma 3.2])

$$f(t) = u(t) + \sum_{i=0}^{n_0-1} g_i(t)$$

is a unique solution to (3.19) satisfying the stipulated growth condition at $t = 0$. □

We now give the following lemma which seems to belong to mathematical folklore.

Lemma 3.3. *Let X, Y be Banach spaces, K be a continuously Fréchet differentiable operator from X to Y with the derivative $\partial K \in C(X, \mathcal{L}(X, Y))$ and let the remainder $\omega : X \times X \mapsto Y$ be defined by*

$$K(x+h) - K(x) - \partial K(x)h = \omega(h, x), \quad x, h \in X.$$

Then the function $\omega_0(h, x) := \frac{\omega(h,x)}{\|h\|_X}$ if $h \neq 0$ and $\omega_0(0, x) := 0$ otherwise, is continuous.

In the next theorem we address the issue of differentiability of the mild solution constructed in Theorem 3.1 and it being a classical solution to (3.4). The result is similar to that for analytic semigroups in that the mild solution in a smaller space (here $X_{0,m}$) is a classical solution in a bigger space (here $X_{0,p}$), see [18, Definitions 7.0.1 & 7.0.2] or [25, Section 4.7.1].

Theorem 3.2. *Let the assumptions of Theorem 3.1 hold and assume also that $\mathring{f} \in X_{0,m} \cap D(G^{(\beta)}_{0,p})$, where $p = m - \alpha$. Then the mild solution f, defined on its maximal interval of existence $[0, \tau_{\max})$, satisfies $f \in C([0, \tau_{\max}), X_{0,m}) \cap C^1((0, \tau_{\max}), X_{0,m}) \cap C((0, \tau_{\max}), D(G^{(\beta)}_{0,p}))$ and is a classical solution to (3.4) in $X_{0,p}$.*

Proof. The proof follows the lines of [21, Theorem 6.1.5] but additional steps are required due to the unboundedness of the nonlinear term. First we observe that it suffices to prove the regularity on $(0, \tau)$ of the local solution constructed in Theorem 3.1 as the proof can then be repeated for each local solution until we reach $\tau_{\max}$.

As in the proof of Theorem 3.1, we choose n so that $\kappa := \frac{m-n}{\gamma_0} \in (0, 1)$. Since $\mathring{f} \in D(G^{(\beta)}_{0,p})$, the mild solution f satisfies the integral equation

$$f(t) = S_{G^{(\beta)}_{0,m}}(t)\mathring{f} + \int_0^t S_{G^{(\beta)}_{0,p}}(t-s)K^{(\beta)}_{0,m}f(s)\mathrm{d}s = S_{G^{(\beta)}_{0,p}}(t)\mathring{f} + \int_0^t S_{G^{(\beta)}_{0,p}}(s)K^{(\beta)}_{0,m}f(t-s)\mathrm{d}s. \tag{3.22}$$

We first consider the Lipschitz continuity of f. Let $t > 0$ and $h > 0$. We have

$$\begin{aligned}\frac{f(t+h) - f(t)}{h} &= \frac{1}{h}\left(S_{G^{(\beta)}_{0,m}}(h) - I\right) S_{G^{(\beta)}_{0,m}}(t)\mathring{f} + \frac{1}{h}\int_0^h S_{G^{(\beta)}_{0,p}}(t+h-s)K^{(\beta)}_{0,m}f(s)\mathrm{d}s \\ &+ \frac{1}{h}\int_0^t S_{G^{(\beta)}_{0,p}}(t-s)(K^{(\beta)}_{0,m}f(s+h) - K^{(\beta)}_{0,m}f(s))\mathrm{d}s =: I_1(h) + I_2(h) + I_3(h).\end{aligned}$$

Arguing as in Corollary 2.1, we have

$$\left\|\frac{1}{h}\left(S_{G^{(\beta)}_{0,m}}(h) - I\right) S_{G^{(\beta)}_{0,m}}(t)\mathring{f}\right\|_{[0,m]} \leq C_1(\tau)t^{-\kappa}\|G^{(\beta)}_{0,p}\mathring{f}\|_{[0,p]},$$

where $C_1(\tau) = Ce^{\theta\tau} \max_{0 \leq t \leq \tau} \|S_{G^{(\beta)}_{0,p}}(t)\|_{[0,p]}$, see (2.34). Next, using (3.15),

$$\frac{1}{h}\int_0^h \|S_{G^{(\beta)}_{0,p}}(t+h-s)K^{(\beta)}_{0,m}f(s)\|_{[0,m]}\mathrm{d}s \leq C(\tau)K(\mathcal{U})\frac{1}{h}\int_0^h (t+h-s)^{-\kappa}\mathrm{d}s \leq C(\tau)K(\mathcal{U})t^{-\kappa}.$$

Finally, using the second inequality in (3.15),

$$\begin{aligned}&\frac{1}{h}\int_0^t \|S_{G^{(\beta)}_{0,p}}(t-s)(K^{(\beta)}_{0,m}f(s+h) - K^{(\beta)}_{0,m}f(s))\|_{[0,m]}\mathrm{d}s \\ \leq\ & M(\tau)L(\mathcal{U})\int_0^t (t-s)^{-\kappa}\frac{\|f(s+h) - f(s)\|_{[0,m]}}{h}\mathrm{d}s.\end{aligned}$$

Thus, for some constants C_1, C_2,

$$\frac{\|f(t+h)-f(t)\|_{[0,m]}}{h} \leq \frac{C_1}{t^\kappa} + C_2\int_0^t (t-s)^{-\kappa}\frac{\|f(s+h)-f(s)\|_{[0,m]}}{h}\mathrm{d}s$$

and, by the Gronwall–Henry inequality [14, Lemma 7.1], for some constant C_3,

$$\frac{\|f(t+h)-f(t)\|_{[0,m]}}{h} \leq C_3 t^{-\kappa}. \tag{3.23}$$

In the estimates above, we can use the same bounds for $f(t)$ and $f(t+h)$ as the function $t \mapsto f(t+h)$ can be treated as the solution for the initial value $f(h)$ which is in $\mathcal{U}$ for h small enough.

For the differentiability of f, first we observe that formally differentiating (3.22) gives, for $0<t<\tau$,

$$\partial_t f(t) = S_{G^{(\beta)}_{0,p}}(t)G^{(\beta)}_{0,p}\mathring{f} + S_{G^{(\beta)}_{0,p}}(t)K^{(\beta)}_{0,m}\mathring{f} + \int_0^t S_{G^{(\beta)}_{0,p}}(t-s)\partial K^{(\beta)}_{0,m}f(s)\partial_s f(s)\mathrm{d}s. \tag{3.24}$$

On defining $g(t) := G^{(\beta)}_{0,p}S_{G^{(\beta)}_{0,p}}(t)\mathring{f} + S_{G^{(\beta)}_{0,p}}(t)K^{(\beta)}_{0,m}\mathring{f}$ and $P(s) = \partial K^{(\beta)}_{0,m}f(s)$, we see that the derivative of f, if it exists, satisfies the linear integral equation

$$w(t) = g(t) + \int_0^t S_{G^{(\beta)}_{0,m}}(t-s)P(s)w(s)\mathrm{d}s. \tag{3.25}$$

Now, for $t>0, h>0$,

$$\|g(t+h)-g(t)\|_{[0,m]} \leq C(\tau)t^{-\kappa}\|(S_{G^{(\beta)}_{0,p}}(h)-I)(G^{(\beta)}_{0,p}\mathring{f} + K^{(\beta)}_{0,m}\mathring{f})\|_{[0,p]}$$

and, analogously, for left-hand limits. Hence the function $t \mapsto g(t)$ is in $C((0,\tau), X_{0,m})$ and is $O(t^{-\kappa})$ close to $t=0$. Next, by Lemma 3.1, $s \mapsto P(s)$ is a continuous function that takes values in $\mathcal{L}(X_{0,m}, X_{0,p})$. Hence, Lemma 3.2 yields the existence of a solution $w \in C((0,T], X_{0,m})$ to (3.25) for any $0<T<\tau$, with $\|w(t)\|_{[0,m]} = O(t^{-\kappa})$ as $t \to 0^+$.

Next, we prove that f is differentiable in $X_{0,m}$ for $0<t<\tau$. From (3.12), we obtain

$$\frac{f(t+h)-f(t)}{h} - w(t) = J_1(h) + J_2(h) + J_3(h),$$

where

$$J_1(h) := \frac{1}{h}\left(S_{G^{(\beta)}_{0,p}} - I\right)S_{G^{(\beta)}_{0,p}}(t)\mathring{f} - S_{G^{(\beta)}_{0,p}}(t)G^{(\beta)}_{0,p}\mathring{f},$$

$$J_2(h) := \frac{1}{h}\int_0^h \left(S_{G^{(\beta)}_{0,p}}(t+h-s)K^{(\beta)}_{0,m}f(s) - S_{G^{(\beta)}_{0,p}}(t)K^{(\beta)}_{0,m}\mathring{f}\right)\mathrm{d}s,$$

$$J_3(h) := \frac{1}{h}\int_0^t S_{G^{(\beta)}_{0,p}}(t-s)(K^{(\beta)}_{0,m}f(s+h) - K^{(\beta)}_{0,m}f(s))\mathrm{d}s - \int_0^t S_{G^{(\beta)}_{0,p}}(t-s)P(s)w(s)\mathrm{d}s.$$

Clearly $\lim_{h\to 0^+} J_1(h) = 0$ by (2.41). For $J_2(h)$, we take $t>0$ and $0 \leq s \leq h \leq t/2$. Then

$$\begin{aligned}
&\|S_{G^{(\beta)}_{0,p}}(t+h-s)K^{(\beta)}_{0,m}f(s) - S_{G^{(\beta)}_{0,p}}(t)K^{(\beta)}_{0,m}\mathring{f}\|_{[0,m]} \\
&\leq \|S_{G^{(\beta)}_{0,p}}(t-s)(S_{G^{(\beta)}_{0,p}}(h)-I)K^{(\beta)}_{0,m}f(s)\|_{[0,m]} + \|S_{G^{(\beta)}_{0,p}}(t-s)(K^{(\beta)}_{0,m}f(s) - K^{(\beta)}_{0,m}\mathring{f})\|_{[0,m]} \\
&\quad + \|(S_{G^{(\beta)}_{0,p}}(t-s) - S_{G^{(\beta)}_{0,p}}(t))K^{(\beta)}_{0,m}\mathring{f}\|_{[0,m]} =: \mathcal{J}_1(s,h) + \mathcal{J}_2(s) + \mathcal{J}_3(s).
\end{aligned}$$

Now

$$\mathcal{J}_1(s,h) \leq Ce^{\theta(t-s)}(t-s)^{-\kappa}\|(S_{G^{(\beta)}_{0,p}}(h)-I)K^{(\beta)}_{0,m}f(s)\|_{[0,p]}.$$

Since $t \mapsto S_{G^{(\beta)}_{0,p}}(t)$ is strongly continuous in $X_{0,p}$, it is uniformly continuous on compact sets of $X_{0,p}$. Moreover, as the function $s \mapsto K^{(\beta)}_{0,m}f(s) \in C([0,\tau), X_{0,p})$ for any $f \in C([0,\tau), X_{0,m})$ and the continuous image of the compact interval $[0, \frac{t}{2}]$ is compact, we see that for any $\epsilon > 0$ there is $h_0 < \frac{t}{2}$ such that for all $0 < h \leq h_0$ we have $\mathcal{J}_1(s,h) \leq \epsilon$, uniformly in $s \in [0, h_0]$.

Similarly, by (2.34) and (3.15),

$$\mathcal{J}_2(s) \leq Ce^{\theta(t-s)}(t-s)^{-\kappa} L(\mathcal{U}) \|f(s) - \mathring{f}\|_{[0,m]}$$

and for any ϵ there is $0 < h_0 < \frac{t}{2}$ such that for any $0 \leq s \leq h \leq h_0$ we have $\mathcal{J}_2(s) \leq \epsilon$. Finally, as with $\mathcal{J}_1$,

$$\mathcal{J}_3(s) \leq \|S_{G^{(\beta)}_{0,p}}(t-s)(S_{G^{(\beta)}_{0,p}}(s) - I)K^{(\beta)}_{0,m}\mathring{f}\|_{[0,m]} = Ce^{\theta(t-s)}(t-s)^{-\kappa}\|(S_{G_p}(s) - I)K^{(\beta)}_{0,m}\mathring{f}\|_{[0,p]}$$

and hence $\mathcal{J}_3$ is a continuous function at 0 and therefore $\lim\limits_{h\to 0^+} \frac{1}{h}\int_0^h \mathcal{J}_3(s)\mathrm{d}s = 0$. Summarizing,

$$\lim_{h\to 0^+} J_2(h) = \lim_{h\to 0^+} \frac{1}{h}\int_0^h S_{G^{(\beta)}_{0,p}}(t+h-s)K^{(\beta)}_{0,m}f(s)\mathrm{d}s - S_{G^{(\beta)}_{0,p}}(t)K^{(\beta)}_{0,m}\mathring{f} = 0.$$

Finally, by Lemma 3.1, and with ω defined as in Lemma 3.3,

$$K^{(\beta)}_{0,m}f(s+h) - K^{(\beta)}_{0,m}f(s) - P(s)(f(s+h) - f(s)) = \omega(f(s+h) - f(s), f(s)). \tag{3.26}$$

Now

$$\frac{\|\omega(f(s+h) - f(s))\|_{[0,p]}}{h} = \frac{\|\omega(f(s+h) - f(s), f(s))\|_{[0,p]}}{\|f(s+h) - f(s)\|_{[0,m]}} \frac{\|f(s+h) - f(s)\|_{[0,m]}}{h}.$$

By Lemma 3.3, the function $(h,s) \mapsto \frac{\|\omega(f(s+h)-f(s),f(s))\|_{[0,p]}}{\|f(s+h)-f(s)\|_{[0,m]}}$ is continuous on $[0,h_0] \times [0,s']$ for any $s' < s$ and hence it is uniformly continuous. Thus, for any $\epsilon > 0$ there is h_0 such that for any $0 < h < h_1 \leq h_0, s \in [0,s']$

$$\frac{\|\omega(f(s+h) - f(s), f(s))\|_{[0,p]}}{\|f(s+h) - f(s)\|_{[0,m]}} \leq \epsilon.$$

Hence, by (3.23), (3.20) and (3.26),

$$\begin{aligned}
&\left\| \frac{1}{h}\int_0^t S_{G^{(\beta)}_{0,p}}(t-s)(K^{(\beta)}_{0,m}f(s+h) - K^{(\beta)}_{0,m}f(s))\mathrm{d}s - \int_0^t S_{G^{(\beta)}_{0,p}}(t-s)P(s)w(s)\mathrm{d}s \right\|_{[0,m]} \\
&\leq C_1 \int_0^t (t-s)^{-\kappa} \left\| \frac{\omega(f(s+h) - f(s), f(s))}{h} \right\|_{[0,p]} \mathrm{d}s \\
&\quad + C_2 \int_0^t (t-s)^{-\kappa} \left\| \frac{f(s+h) - f(s)}{h} - w(s) \right\|_{[0,m]} \mathrm{d}s \\
&\leq C_1 C_3 B(1-\kappa, 1-\kappa)\epsilon t^{1-2\kappa} + C_2 \int_0^t (t-s)^{-\kappa} \left\| \frac{f(s+h) - f(s)}{h} - w(s) \right\|_{[0,m]} \mathrm{d}s.
\end{aligned}$$

Since for small t we have $t^{1-2\kappa} \leq t^{-\kappa}$, it follows that, on any time interval $(0,s')$ where $s' < s$, and for any $\epsilon > 0$, there is h_0 such that for any $0 < h < h_0$

$$\left\| \frac{f(t+h) - f(t)}{h} - w(t) \right\|_{[0,m]} \leq \epsilon t^{-\kappa} C_5 + C_2 \int_0^t (t-s)^{-\kappa} \left\| \frac{f(s+h) - f(s)}{h} - w(s) \right\|_{[0,m]} \mathrm{d}s$$

and thus, by [8, Lemma 3.2],

$$\left\| \frac{f(t+h) - f(t)}{h} - w(t) \right\|_{[0,m]} \leq \epsilon t^{-\kappa} C_6.$$

Hence the right-hand derivative of f exists on $(0,\tau)$, and satisfies (3.24). As in the proof of Theorem 3.1, the right-hand side of (3.24) is continuous on $(0,s)$ and thus the left-hand derivative also exists. Hence $f \in C^1((0,\tau), X_{0,m})$.

To show that $f(t) \in D(G_{0,p}^{(\beta)})$ for $t>0$, we evaluate

$$\frac{1}{h}(S_{G_{0,p}^{(\beta)}}(h)-I)f(t) = \frac{1}{h}S_{G_{0,p}^{(\beta)}}(t)(S_{G_{0,p}^{(\beta)}}(h)-I)\mathring{f} + \frac{1}{h}\int_0^h S_{G_{0,p}^{(\beta)}}(t-s)K_{0,m}^{(\beta)}f(s)\mathrm{d}s$$
$$-\frac{1}{h}\int_{t-h}^t S_{G_{0,p}^{(\beta)}}(t-s)K_{0,m}^{(\beta)}f(s+h)\mathrm{d}s + \frac{1}{h}\int_0^t S_{G_{0,p}^{(\beta)}}(t-s)(K_{0,m}^{(\beta)}f(s+h)-K_{0,m}^{(\beta)}f(s))\mathrm{d}s$$
$$=: L_1(h)+L_2(h)+L_3(h)+L_4(h).$$

Using again (2.41), $L_1(h)\to S_{G_{0,p}^{(\beta)}}(t)G_{0,p}^{(\beta)}\mathring{f}$ in $X_{0,m}$ for $t>0$. Also, as above,

$$\lim_{h\to 0+} L_2(h) = S_{G_{0,m}^{(\beta)}}(t)K_{0,m}^{(\beta)}\mathring{f}, \qquad \lim_{h\to 0+} L_4(h) = \int_0^t S_{G_{0,p}^{(\beta)}}(t-s)\partial K_{0,m}^{(\beta)}f(s)\partial_s f(s)\mathrm{d}s.$$

While the above limits hold in $X_{0,m}$ thanks to the regularizing effect of $(S_{G_{0,p}^{(\beta)}}(t))_{t\geq 0}$ for $t>0$, in L_3 we must use the continuity of the integrand at $t=0$ so we are only able to pass to the limit in $X_{0,p}$. Then, in the same way as for L_2, we have

$$\lim_{h\to 0+} L_3(h) = -K_{0,m}^{(\beta)}f(t),$$

in $X_{0,p}$. Hence $f(t)\in D(G_{0,p}^{(\beta)})$ for $t>0$ and

$$G_{0,p}^{(\beta)}f(t) = S_{G_{0,p}^{(\beta)}}(t)G_{0,p}^{(\beta)}\mathring{f} + S_{G_{0,p}^{(\beta)}}(t)K_{0,m}^{(\beta)}\mathring{f} - K_{0,m}^{(\beta)}f(t) + \int_0^t S_{G_{0,p}^{(\beta)}}(t-s)\partial K_{0,m}^{(\beta)}f(s)\partial_s f(s)\mathrm{d}s$$
$$= -K_{0,m}^{(\beta)}f(t) + \frac{d}{dt}f(t). \tag{3.27}$$

□

(b) Global solvability

To establish the existence of global (in time) solutions to the growth C-F equation we must impose the more restrictive condition

$$k(x,y) \leq k_0(1+x^\alpha+y^\alpha) \tag{3.28}$$

on the coagulation kernel. As in (3.1), k_0 is a positive constant and $0<\alpha<\gamma_0$, where γ_0 is given in (2.33). Also, the inclusion of the term $a_1(x)=\beta(1+x^\alpha)$ being required only to prove the nonnegativity of mild solutions in Theorem 3.1, we now set $\beta=0$, in which case, from (2.31) and Theorem 3.2, there exists a unique solution f to

$$\frac{d}{dt}f(t) = T_{0,p}^0 f(t) + A_{0,p}f(t) + B_{0,p}f(t) + K_{0,m}f(t), \qquad f(0)=\mathring{f}\in X_{0,m}\cap D(G_{0,p}), \tag{3.29}$$

in $C([0,\tau_{\max}),X_{0,m})\cap C^1((0,\tau_{\max}),X_{0,m})\cap C((0,\tau_{\max}),D(G_{0,p}))$, where $K_{0,m}:=K_{0,m}^{(0)}$ and $G_{0,p}:=G_{0,p}^{(0)}$. We emphasize that, once α is given, we can use an arbitrary $p>\max\{1,l\}$ and then take $m=p+\alpha$.

Theorem 3.3. *Let the assumptions of Theorem 3.1 be satisfied with (3.1) replaced by (3.28). If either*
a) there are constants m_0 and m_1 such that for all $x\geq 0$ $(n_0(x)-1)a(x)\leq m_0+m_1x$, or
b) $r_0=0$; that is, $r(x)\leq \tilde{r}x$,
then the solutions of Theorem 3.1 are global in time.

Proof. The proof follows the lines of the proof of [6, Theorem 5.1] but the technicalities are slightly different. Using the classical identities and estimates, [14, Eqn. (8.1.22) & Lemma 7.4.2] and (3.28),

$$\int_0^\infty x^i \mathcal{K}f(x)\mathrm{d}x \leq K_i(\|f\|_{[1]}\|f\|_{[i-1]} + \|f\|_{[1]}\|f\|_{[\alpha+i-1]} + \|f\|_{[\alpha+1]}\|f\|_{[i-1]}), \quad i\geq 1, \tag{3.30}$$

for some constants K_i. For the fragmentation terms, let us first recall a_0, γ_0 and x_0, defined in (2.33). From [6, Theorem 2.2] we have that if $N_{m_0}(x)/x^{m_0} \geq \delta'_{m_0}$ holds for some $m_0 > 1$, δ'_{m_0} and $x \geq x_0$, then for any $i > 1$ there is $\delta'_i > 0$ such that $N_i(x)/x^i \geq \delta'_i > 0$ for any $x \geq x_0$. Hence,

$$\int_0^\infty N_i(x)a(x)f(x)\mathrm{d}x = \int_0^{x_0} a(x)N_i(x)f(x)\mathrm{d}x + \int_{x_0}^\infty a(x)f(x)x^i \frac{N_i(x)}{x^i}\mathrm{d}x \geq \delta_i \|f\|_{[i+\gamma_0]} - \nu_i \|f\|_{[i]}, \tag{3.31}$$

where $\delta_i = \delta'_i a_0$ and $\nu_i = \delta_i \mathrm{ess\,sup}_{0 \leq x \leq x_0} a(x)$. If we take $\mathring{f}$ to be a $C^\infty(\mathbb{R}_+)$ function with bounded support, then $\mathring{f} \in D(G_{0,i})$ for any i and, if additionally $i > \max\{1, l\}$, then, by Theorem 3.2, the corresponding solution $(0, \tau_{\max}) \ni t \mapsto f(t) = f(t, \mathring{f})$ is differentiable in any such X_i. First, let us consider an integer $i \geq 2$. Then, from (2.35), and using (3.30) and (3.31),

$$\frac{d}{dt}M_0(t) \leq \int_0^\infty (n_0(x) - 1)a(x)f(x,t)\mathrm{d}x, \tag{3.32}$$

$$\frac{d}{dt}M_1(t) = \int_0^\infty r(x)f(x,t)\mathrm{d}x \leq \widetilde{r}M_0(t) + \widetilde{r}M_1(t), \tag{3.33}$$

$$\begin{aligned}\frac{d}{dt}M_i(t) &\leq \widetilde{r}M_{i-1}(t) + (\nu_i + \widetilde{r})M_i(t) - \delta_i M_{i+\gamma_0}(t)\\ &\quad + K_i(M_1(t)M_{i-1}(t) + M_1(t)M_{\alpha+i-1}(t) + M_{\alpha+1}(t)M_{i-1}(t)), \quad i > 1.\end{aligned} \tag{3.34}$$

To simplify (3.34), we use the following auxiliary inequalities. For $i \geq 2$ and $1 \leq r \leq i-1$, we apply Hölder's inequality with $p = \gamma_0/\alpha$ and $q = \gamma_0/(\gamma_0 - \alpha)$ to obtain, as in [14, Eqn (8.1.59)],

$$\|f\|_{[r+\alpha]} \leq c_\alpha \|f\|_{[1]} + \|f\|_{[i-1]}^{\frac{\gamma_0-\alpha}{\gamma_0}} \|f\|_{[i+\gamma_0]}^{\frac{\gamma_0}{\alpha}}, \tag{3.35}$$

for some constant c_α. Then Young's inequality gives

$$\|f\|_{[i+\alpha-1]}\|f\|_{[1]} \leq c_\alpha \|f\|_{[1]}^2 + \|f\|_{[1]} \left(\frac{\gamma_0 - \alpha}{\gamma_0} \epsilon^{\frac{\gamma_0}{\alpha-\gamma_0}} \|f\|_{[i-1]} + \frac{\alpha}{\gamma_0}\epsilon^{\frac{\gamma_0}{\alpha}} \|f\|_{[i+\gamma_0]} \right) \tag{3.36}$$

and

$$\|f\|_{[i-1]}\|f\|_{[1+\alpha]} \leq c_\alpha \|f\|_{[1]}\|f\|_{[i-1]} + \left(\frac{\gamma_0 - \alpha}{\gamma_0} \epsilon^{\frac{\gamma_0}{\alpha-\gamma_0}} \|f\|_{[i-1]}^{\frac{2\gamma_0-\alpha}{\gamma_0-\alpha}} + \frac{\alpha}{\gamma_0}\epsilon^{\frac{\gamma_0}{\alpha}} \|f\|_{[i+\gamma_0]} \right). \tag{3.37}$$

We now apply these inequalities to the solution $t \mapsto f(t)$, transforming (3.34) into

$$\begin{aligned}\frac{d}{dt}M_i(t) &\leq r_0 M_{i-1}(t) + (\nu_i + r_1)M_i(t) - \delta_i M_{i+\gamma_0}(t)\\ &\quad + K_i \left(M_1(t)M_{i-1}(t) + c_\alpha M_1^2(t) + M_1(t) \left(\frac{\gamma_0 - \alpha}{\gamma_0}\epsilon^{\frac{\gamma_0}{\alpha-\gamma_0}} M_{i-1}(t) + \frac{\alpha}{\gamma_0}\epsilon^{\frac{\gamma_0}{\alpha}} M_{i+\gamma_0}(t) \right) \right.\\ &\quad \left. + c_\alpha M_1(t)M_{i-1}(t) + \left(\frac{\gamma_0 - \alpha}{\gamma_0}\epsilon^{\frac{\gamma_0}{\alpha-\gamma_0}} M_{i-1}^{\frac{2\gamma_0-\alpha}{\gamma_0-\alpha}}(t) + \frac{\alpha}{\gamma_0}\epsilon^{\frac{\gamma_0}{\alpha}} M_{i+\gamma_0}(t) \right) \right).\end{aligned} \tag{3.38}$$

The problem is that the estimates derived above require some control of $M_1(t)$. It is standard in pure coagulation-fragmentation models, as then $M_1 = \|\mathring{f}\|_{[1]}$ is constant on $[0, \tau_{\max})$. Here, however, the second inequality of (3.34) shows that $M_1(t)$ is coupled with $M_0(t)$ and the latter in general depends on higher order moments. There are two easy ways to remedy this situation, related to assumptions a) and b), respectively. If a) is satisfied, then

$$\frac{d}{dt}M_0(t) \leq \int_0^\infty (n_0(x) - 1)a(x)f(x,t)\mathrm{d}x \leq m_0 M_0(t) + m_1 M_1(t),$$

which, together with (3.33), yield $M_0(t) \leq \mathring{M}_0 e^{\mu t}$ and $M_1(t) \leq \mathring{M}_1 e^{\mu t}$ for some constant μ. Thus neither moment blows up in finite time. If b) is satisfied, then $M_1(t) \leq \mathring{M}_1 e^{\widetilde{r}t}$ and the inequalities for the moments of order greater than one become decoupled from M_0. In both cases $M_1(t) \leq M_{1,\tau_{\max}}$ on $[0, \tau_{max})$ and, by choosing ϵ so that $\frac{\alpha}{\gamma_0}\epsilon^{\frac{\gamma_0}{\alpha}} K_i(M_{1,\tau_{\max}} + 1) \leq \delta_i$, we see that there are positive constants $D_{0,i}, D_{1,i}, D_{2,i}, D_{3,i}$ such that (3.38) can be written as

$$\frac{d}{dt}M_i(t) \leq D_{0,i} + D_{1,i}M_i(t) + D_{2,i}M_{i-1}(t) + D_{3,i}M_{i-1}^{\frac{2\gamma_0-\alpha}{\gamma_0-\alpha}}(t), \quad t\in[0,\tau_{\max}). \tag{3.39}$$

In particular, we immediately see that $t\mapsto M_2(t)$ is bounded on $[0,\tau(\max))$. Then we can use (3.39) to proceed inductively to establish the boundedness of $t\mapsto M_i(t)$ for any integer i (for the chosen initial condition) and, for noninteger $i>1$ we use the estimate $\|f\|_{[i]} \leq \|f\|_{[1]} + \|f\|_{[\lfloor i\rfloor+1]}$.

It remains to prove that $t\mapsto M_0(t)$ is bounded on $[0,\tau_{\max})$ (in case b)). Let us fix an integer $i>\max\{1,l\}$. Using the fact that $\int_0^\infty \mathcal{K}f(x,t)\mathrm{d}x \leq 0$ and, by (2.24),

$$\int_0^\infty \mathcal{F}f(x,t)\mathrm{d}x \leq 2b_0\int_0^\infty a(y)f(y,t)w_i(y)\mathrm{d}y \leq \widetilde{a}\int_0^{x_0} f(y,t)\mathrm{d}y + 2b_0R(t), \tag{3.40}$$

on $[0,\tau_{\max})$, where $\widetilde{a} = 2b_0\mathrm{ess\,sup}_{y\in[0,x_0]}\, a(y)w_i(y)$, for the zeroth moment we have

$$\frac{d}{dt}M_0(t) \leq a_1M_0(t) + 2b_0R(t),$$

where we denoted $R(t) = \int_{x_0}^\infty a(x)f(x,t)w_i(x)\mathrm{d}x$. Hence

$$M_0(t) \leq e^{\widetilde{a}t}\left(\|\mathring{f}\|_{[0]} + 2b_0\int_0^t R(s)\mathrm{d}s\right). \tag{3.41}$$

We have the estimate

$$\int_0^t R(s)\mathrm{d}s = \int_0^t\int_{x_0}^\infty a(x)f(x,s)w_i(x)\mathrm{d}x\mathrm{d}s \leq (1+x_0^{-i})\int_0^t\int_{x_0}^\infty a(x)f(x,s)x^i\mathrm{d}x\mathrm{d}s. \tag{3.42}$$

Now, similarly to (3.31), we can write,

$$\begin{aligned}\int_0^\infty [\mathcal{F}f](x)x^i\mathrm{d}x &\leq -\frac{\delta_i'}{2}\int_{x_0}^\infty a(x)f(x)x^i\mathrm{d}x - \frac{\delta_i'}{2}\int_0^\infty a(x)f(x)x^i\mathrm{d}x + \frac{\delta_i'}{2}\int_0^{x_0} a(x)x^if(x)\mathrm{d}x\\ &\leq -\frac{\delta_i'}{2}\int_{x_0}^\infty a(x)f(x)x^i\mathrm{d}x - \frac{\delta_i}{2}\|f\|_{[i+\gamma_0]} + \nu_i\|f\|_{[i]},\end{aligned} \tag{3.43}$$

where δ_i and ν_i were defined previously. Now, by selecting ϵ so that $\frac{\alpha}{\gamma_0}\epsilon^{\frac{\gamma_0}{\alpha}}K_i(M_{1,\tau_{\max}}+1) \leq \frac{\delta_i}{2}$ and knowing that all lower order moments are finite on $[0,\tau_{\max})$, we can write (3.39) as

$$\frac{d}{dt}M_i(t) \leq -\frac{\delta_i'}{2}\int_{x_0}^\infty a(x)f(x,t)x^i\mathrm{d}x + D_{0,i} + D_{1,i}M_i(t) + \Theta(t), \tag{3.44}$$

where Θ is bounded on $t\in[0,\tau_{\max})$. This can be re-written as

$$\frac{d}{dt}\Phi(t) \leq D_{0,i} + D_{1,i}\Phi(t) + \Theta(t),$$

where $\Phi(t) = M_i(t) + \frac{\delta_i'}{2}\int_0^t\int_{x_0}^\infty a(x)f(x,s)x^i\mathrm{d}x\mathrm{d}s$. Integrating,

$$\Phi(t) \leq e^{D_{1,i}t}\left(\Phi(0) + \frac{D_{0,i}}{D_{1,i}}(1-e^{-D_{1,i}t}) + \int_0^t \Theta(s)e^{-D_{1,i}s}\mathrm{d}s\right)$$

and we see that neither Φ, nor $t\mapsto\int_0^t\int_{x_0}^\infty a(x)f(x,s)x^i\mathrm{d}x\mathrm{d}s$ can blow up at $t=\tau_{\max}$. Hence, by (3.42) and (3.41), neither can $t\mapsto M_0(t)$.

This shows that solutions emanating from compactly supported differentiable initial conditions are global in time. Consider now $\mathring{f}\in X_{0,m,+}$ and a sequence of such regular initial conditions $(\mathring{f}_k)_{k\geq1}$ approximating $\mathring{f}$ and assume that the corresponding solution $t\to f(t,\mathring{f})$ has a finite time blow up at $\tau_{\max}$. By the moment estimates above, the bounds of $\|f(t,\mathring{f}_k)\|_{0,m}$ over any finite time interval depend continuously on $\mathring{f}_k$ and thus are uniform in k on $[0,\tau_{\max}]$. On the other hand, there is a sequence $(t_n)_{n\geq1}$ such that $t_n\to\tau_{\max}, n\to\infty$ and $\|f(t_n,\mathring{f})\|_{[0,m]}$ is unbounded; that is, the distance between $f(t_n,\mathring{f})$ and all $f(t_n,\mathring{f}_k)$ becomes arbitrarily large. This contradicts the continuous dependence of solutions on the initial conditions following, on each $[0, t_n]$, from the Gronwall–Henry inequality, [8, Lemma 3.2]), see also [14, Theorem 8.1.1]. □

Remark 3.1. *The additional restrictions in Theorem 3.3 are due to the fact that, in the general case, we cannot control the production of particles; that is, the zeroth moment. In principle, there is a positive feedback loop in which M_0 contributes to M_1 which, in turn, amplifies, in a nonlinear way, higher order moments that determine the rate of growth of M_0. The adopted assumptions, which postulate that either M_0 is controlled by M_1, or that the evolution of mass is not influenced by other mechanisms ($r_0 \neq 0$ implies that there is a production of mass independent of the existing one), although technical, seem to be the simplest ones that break this cycle. We do not claim that these assumptions are optimal but at present we do not have any examples of a finite time blow up of solutions in this setting. It is, however, worthwhile to note that there are known cases of a finite time blow up of solutions to growth–fragmentation–coagulation equations even with bounded coagulation kernels but with the renewal boundary condition, [4].*

4. Conclusion

In this paper we have applied results from the theory of C_0-semigroups to prove the global in time classical solvability of the growth-fragmentation-coagulation equation with an unbounded coagulation kernel.

Authors' Contributions. The paper is based on techniques developed by both authors.

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