Optimal trade credit and replenishment policies for non-instantaneous deteriorating items

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Abstract: The present study presents a fuzzy inventory model for non-instantaneous deteriorating items under conditions of permissible delay in payments. In the current paper, we incorporate the condition in which, the supplier accepts the partial payment at the end of the credit period and the reaming amount after that period under the term and condition. Here, the demand rate is a function of the selling price. Also, it is assumed that shortages are allowed and are fully backlogged. The present paper also considers that the interest earned (I_E) on the fixed deposit amount, i.e., revenue generated by fulfilling the shortage, balance amount, after settling the account is higher than that of usual interest rate (I_e). Hence, the objective of this study is to determine the retailer's optimal policies that maximize the total profit. Also, some theoretical results are obtained, which shows that the optimal solution not only exists, it is unique also. The impact of the new proposed credit policy is investigated on the optimality of the solution for the non- instantaneous deteriorating products. The validation of the proposed model and its solution method is demonstrated through the numerical example. The results indicate that the inventory model, along with the solution method, provides a powerful tool to the retail managers under real-world situations. Results demonstrate that it is essential for the managers to consider the inclusion of new proposed credit policy significantly increases the net annual profit.

Keywords: Inventory; trade credit; pricing; non-instantaneous deterioration; shortages; triangular fuzzy number; function principle and signed distance method.

1. Introduction and Literature Review

Inventory management is essential for the smooth functioning of any firm. Too much of inventory may lead to an addition of a high cost to the company. While on the other hand holding decidedly fewer inventories may lead to stock-out situations and result in loss of potential customers. Inventory theory provides a solution to such problems by addressing the fundamental question of when and how much to order. One of the basic concepts of inventory theory is the economic order quantity (EOQ) formula, which was derived by Harris [1]. Several studies have been done in the past on inventory management. Inventories are primarily classified into two types: perishable and non-perishable. Non-perishable items have a very long lifetime and hence can be used for demand fulfilment over an extended period. Products that degrade in quality and utility with time are called perishable products. Perishable products are primarily of two categories: one, which maintains constant utility throughout the lifetime, for example, blood (which has a fixed lifetime of 21 days with constant utility) and medicines, while the other with exponentially, decaying utility, for example, vegetables, fruits, and fish. Management of perishable items with limited lifetime is a challenge. Inventory models for deteriorating items have attracted considerable interest from researchers in recent decades. Ghare and Schrader [2] that developed an exponentially decaying inventory model firstly tackled the problem of modeling the deterioration process. They observed that certain commodities deteriorate with time by a proportion, which can be approximated by a negative exponential function of time. Successively, Covert and Philip [3], considered a two parameters Weibull deterioration function. Since the work of Ghare and Schrader [2] and Covert and Philip [3], significant works have been done on deteriorating inventory systems that are summarized in Nahmias [4] that presented a review of the early 60s and 70s referred to fixed and random lifetime models. Thus, Raafat [5], dealt with the 70s and 80s about continuously deteriorating items. Goyal and Giri [6] extended the review at the 90s. Finally, Bakker et al. [7], considered the inventory theory with regards to the latest results in such field.

The works cited focused only on those products/items, which starts deteriorated as soon as they enter in the system. However, several items do not start deteriorating instantly. Items like dry fruits, potatoes, yams, and even some fruits and vegetables, etc. have a shelf life and start to deteriorate after a time lag. This phenomenon may be termed as non-instantaneous deterioration. Wu et al. [8] first introduced the phenomenon "non-instantaneous deterioration" and established the optimal replenishment policy for a non-instantaneous deteriorating item with inventory level dependent demand and partial backlogged shortages. Further, Ouyang et al. [9] developed an inventory model for non-instantaneous deteriorating items under trade credits. Other related works in this area have been done by Ouyang et al. [10], Wu et al. [11], Jaggi and Verma [12], Chang et al. [13], Geetha et al. [14], Soni et al. [15], Maihami and Kamalabadi [16, 17], Shah et al. [18], Dye [19] and Tsao [20].

Lately, marketing researchers and practitioners have recognized the phenomenon that the supplier offers a permissible delay to the retailer if the outstanding amount is paid within the permitted fixed settlement period, defined as the trade credit period. During the trade credit period, the retailer can accumulate revenues by selling items and earning interests. As a result, with no incentive for making early payments and earning interest through the accumulated revenue received during the credit period, the retailer postpones payment up to the last moment of the permissible period allowed by the supplier. Therefore, offering trade credit leads to delayed cash inflow and increases the risk of cash flow shortage and bad debt. From the viewpoints of suppliers, they always hope to be able to find a trade credit policy to increase the sale and decrease the risk of cash flow shortage and bad debt. In reality, on the operations management side, a supplier is always willing to provide the retailer, either a cash discount or a permissible delay in payments. In practice, a seller frequently offers his/her buyers a permissible delay in payment (i.e., trade credit) for settling the purchase amount. Usually, there is no interest charged if the outstanding amount is paid within the permissible delay period. However, if the payment is not paid in full by the end of the permissible delay period, then interest is charged on the outstanding amount.

Ever since Goyal [21] first developed an economic order quantity (EOQ) model under the conditions of permissible delay in payments, an increasing interest in the literature dealing with a variety of situations such as allowing of shortage, partial backlogging, credit-linked demand/order quantity, deterioration, etc. has been witnessed. Aggarwal and Jaggi [22] extended Goyal's model for deteriorating items. Jamal et al. [23] further generalized Aggarwal and Jaggi's model to allow for shortages. Teng [24] amended Goyal's model, by considering the difference between the unit price and unit cost, and then established an easy analytical closed-form solution to the problem. Subsequently, Huang [25] proposed an EOQ model in which the supplier offers the retailer a permissible delay, and the retailer, in turn, provides his/ her customers another permissible delay to stimulate demand. Next, Ouyang et al. [26] established an EOQ model for deteriorating items to allow for partial backlogging under trade credit financing. Liao [27] presented an EPQ model for deteriorating items under permissible delay in payments. Teng [28] developed ordering policies for a retailer who offers distinct trade credits to its good and bad credit customers. Hu and Liu [29] presented an EPQ model with the permissible delay in payments and allowable shortages.

Further, Teng et al. [30] extended an EOQ model for stock-dependent demand to supplier's trade credit with a progressive payment scheme. Teng et al. [31] generalized traditional constant demand to non-decreasing demand. Under different financial environments, Cheng et al. [32] developed the proper mathematical models and solved the corresponding optimal order quantity and payoff time for maximizing the retailer's total profit per unit time when a delay in payment is permissible. Lou and Wang [33] proposed an integrated inventory model with trade credit financing in which the vendor decides his/her production lot size while the buyer determines his/her expenditure. Lately, Chen et al. [34] built up an EOQ model when conditionally permissible delay links to order quantity. Saha and Cárdenas-Barrón [35] developed a mathematical model for a product with price and time sensitive demand to maximize the profit functions. They allowed the number of price changes to be a decision variable for policy decisions. Ouyang et al. [36] presented an inventory model under a two-level permissible delay in payments. Sarkar et al. [37] proposed an economic production quantity (EPQ) inventory models for deteriorating items with two-level trade credit for fixed lifetime products. Wu et al. [38] demonstrated a unique replenishment cycle time of the retailer. They developed an inventory model in which the retailer gets an upstream full trade credit from the supplier whereas offers downstream partial trade credit to credit-risk customers. Most recently, Jaggi et al. [39] and Tiwari et al. [40] developed two-warehouse inventory models for non-instantaneous deteriorating items under trade credit policy. They explored the role of permissible delay in payments with shortages on the optimal policy. Tavakoli & Taleizadeh [41] proposed an inventory system for a decaying item by considering the combination of prepayment and partial trade. Taleizadeh et al. [42] proposed an inventory considering prepayment and planned backordering.

Recently, Seifert et al. [43] and Z. Molamohamadi et al. [44] presented an excellent review of trade credit financing. The details about these works have been shown in Table 1.

Table1: Summary of related literature with trade credit

Papers	Deterioration	Demand Rate	Shortages	Trade Credit Policy
Ouyang et al. [9]	Non-instantaneous	Constant	No	Single level
Ouyang et al. [10]	Non-instantaneous	Stock-dependent	Stochastic backorder rate	Single level
Geetha et al. [14]	Non-instantaneous	Constant	Partial backlogging	Single level
Soni et al. [15]	Non-instantaneous	Selling price and stock dependent	No	Single level
Maihami and Kamalabadi [17]	Non-instantaneous	Selling price and time-dependent	Partial backlogging	Single level
Tsao [20]	Non-instantaneous	Constant	No	Single level
Goyal [21]	No	Constant	No	Single level
Jaggi and Aggarwal [22]	Yes	Constant	No	Single level
Jamal et al. [23]	Yes	Constant	Complete backlogging	Single level
Teng [24]	No	Constant	No	Single level
Huang [25]	No	Constant	No	Single level
Ouyang et al. [26]	Yes	Constant	Partial backlogging	Single level
Liao [27]	Yes	Constant	No	Single level
Teng [28]	No	Constant	No	Single level
Hu and Liu [29]	Yes	Constant	Complete backlogging	Single level
Teng et al. [30]	No	Stock-dependent	No	Progressive payment scheme
Teng et al. [31]	No	Time-dependent	No	Single level
Cheng et al. [32]	No	Constant	No	Single level
Lou and Wang [33]	No	Constant	No	Single level
Chen et al. [34]	No	Constant	No	Credit link order quantity
Present Paper	Non-instantaneous (Fuzzy)	Selling price dependent (Fuzzy)	Complete backlogging	Single level (Alternative Approach)

In most of the inventory models, it is assumed that all of the time parameters and relevant data are already exactly known and fixed. Furthermore, in practice, those assumptions are unrealistic since they are generally vague and imprecise, even impossible to get the exact values. That is, they are uncertainty in the real word. Therefore, in order to incorporate the uncertainty of this parameter the inventory models in a fuzzy sense have been studied. Yao and Lee [45] applied the extension principle to solve the inventory model with shortages by fuzzyfying the order quantity. Chang et al. [46] considered the fuzzy problems for the mixture inventory model involving variable lead-time with backorders and lost sales. They used the probabilistic fuzzy set to construct a new random variable for lead-time demand and derive the total expected annual cost in the fuzzy sense. Das et al. [47] studied multi-item stochastic and fuzzy-stochastic inventory models under total budgetary and space constraints. Chen and Ouyang [48] developed a fuzzy inventory model for the deteriorating item under single level credit policy.

The inventory problem of deteriorating items has been extensively studied by researchers. Looking through the inventory models with deteriorating items shows that the deterioration rate is considered constant or some real – valued functions in most of the previous researches. But, in the real world, deterioration rate is not actually constant or some pre-defined function and slightly disturbed from its original crisp value. However, the uncertainties due to deterioration cannot be appropriately treated by using usual probabilistic model. Therefore, it becomes more convenient to deal such problems with fuzzy set theory. Many inventory models are now being developed considering deterioration rate to be imprecise or fuzzy such as: De et al. [49], Roy et al. [50], and Shabani et al. [51].

In real-life inventory situation, it is very difficult to construct a realistic mathematical model, which encodes the information with both precision and certainty. Since, the real business world is full of uncertainties in a non-stochastic sense, which leads to the estimation of different inventory parameters as fuzzy numbers. In practical situations sometimes, the probability distributions of the demands for products are difficult to acquire due to lack of information and historical data. Thus, an inventory system, the demands are approximately estimated by the experts depend on their experience and subjective managerial judgments to tackle the uncertainties, which always fit the real situations.

The main aim of our study is to address the issue of uncertainty – fuzziness, where there may be sufficient or even abundant data – by way of modeling the annual customer demand information as a normally distributed fuzzy random variable where the associated probability density function is also taken to be fuzzy.

Our Contribution:

In the above mention literature on inventory modelling under the conditions of permissible delay in payments, scholars have assumed that the retailers have to settle their accounts at the end of the credit period, i.e., the supplier accepts only full amount at the end of the credit period. However, in reality, either the supplier may accept the partial amount at the end of the credit period and unpaid balance subsequently or the full amount at a fixed point of time after the expiry of the credit period, if the retailer finances the inventory from the supplier itself. This issue motivated us to incorporate the above-mentioned realistic scenario. In the current paper, we incorporate the condition in which, the supplier accepts the partial payment at the end of the credit period and the reaming amount after that period under the term and condition. The main feature of the alternative trade credit approach is the extension of the trade credit period by considering the above realistic possibilities. Based on the situations mentioned above, this paper considers the retailer's optimal policy for non-instantaneous deteriorating items with the permissible delay in payments under different scenarios in a fuzzy environment. The present study discusses all the possible cases, which might arise and yet not considered in the previous inventory models under permissible delay in payments. Here, the demand rate is a function of the selling price. In addition, it is assumed that shortages are allowed and are fully backlogged.

Moreover, the present paper also considers interest earned (I_E) on the fixed deposit amount, i.e., revenue generated by fulfilling the shortage, balance amount after settling the account is higher than that of usual interest rate (I_e). Thus, the revenue, along with interest earned, can be utilized to pay off the amount at the credit period. The whole profit is calculated from the retailer's point of view. In this model, demand, as well as deterioration rates are considered as a triangular fuzzy number. Hence, the objective of this study is to determine the retailer's optimal policies that maximize the total profit.

2. Notations and assumptions

The following notations and assumptions have been used in developing the model.

2.1. Notations

Parameter	Description
$I(t)$	: instantaneous inventory level at time t
Q	: order level
S_1	: positive stock level
$D(p)$ $(= D = a - bp)$	: price dependent demand
$\tilde{D}(p)$ $(= \tilde{D} = \tilde{a} - \tilde{b}p)$	: fuzzy price-dependent demand
A	: replenishment cost (ordering cost) for replenishing the items
C	: unit purchase cost of the retailer
π	: shortage cost per unit per unit time
Θ	: deterioration rate and $0 \le \theta < 1$
$\tilde{\theta}$	: fuzzy deterioration rate
p $(= \mu c)$	: selling price per unit
M	: credit period offered by the supplier
I_e	: rate of interest earned by the retailer ($ per year)
I_E	: rate of interest earned rate on the fixed deposit amount
I_p	: rate of interest payable to the supplier ($ per year)
t_d	: time period during which no deterioration occurs.
B_i	: breakeven point, $i = 1, 2, 3$
$TP_{(.)}(\mu, t_1, T)$	: total profit in case (.)
$TP(\mu, t_1, T)$	: total fuzzy profit
TP (.)	: total profit in combine form for all cases
TP_d (.)	: total profit after defuzzification
Decision Variables	
$\mu (\mu > 1)$	: mark-up rate

t_1	: length of the period with a positive stock of the items
T	: replenishment cycle length

2.2. Assumptions

The mathematical model of the inventory problems is based on the following assumptions:

(i) Replenishment rate is infinite, and lead-time is negligible.

(ii) The planning horizon of the inventory system is infinite.

(iii) Unsatisfied demand/shortages are allowed and fully backlogged.

(iv) Demand rate is assumed to be a function of selling price, i.e., $D(p) = a - bp$ where *a*, *b* are positive constants and $0 < b < a/p$. Further, *a* and *b* are assumed as a triangular fuzzy number. For notational simplicity, $D(p)$ and D will be used interchangeably in this paper.

(v) When $I_e < I_E \le I_p$ and $T \ge M$, the account is settled at *M*. Beyond the fixed credit period, the retailer begins to pay the interest charges on the remaining amount at the rate I_p. Before the settlement of the replenishment account, the retailer can use the sale revenue to earn interest at the annual rate I_e and I_E.

(vi) When $I_e \le I_P < I_E$ and $T \ge M$, the account is settled at *M* if the amount in the account of the retailer is less than the payable amount otherwise pay at the end of the cycle. Beyond the fixed credit period, the retailer begins to pay the interest charges on the remaining amount at the rate I_p. Before the settlement of the replenishment account, the retailer can use the sales revenue to earn interest at the rates I_e and I_E per annum.

(vii) When $I_p < I_e < I_E$ and $T \ge M$, the account is settled at the end of the cycle. Beyond the fixed credit period, the retailer begins to pay the interest charges on the remaining amount at the rate I_p. Before the settlement of the replenishment account, the retailer can use the sale revenue to earn interest at the annual rate I_e and I_E.

(viii) When $T \le M$ the account is settled at *M*, and the retailer does not need to pay any interest charge. Alternatively, the retailer can accumulate revenue and earn interest until the end of the trade credit period.

3. Mathematical Model formulation

3.1. Crisp Model

A graphical representation of the inventory control problem during cycle (*0*, *T*) is shown in Figure 1. Initially, the lot size of Q units enters in the inventory system. Out of Q units, $Q-S_1$ units are fulfilling the shortages, and the remaining S_1 unit will be depleted during the time interval [0, t_1]. During the time interval, $[0,\ t_d]$, there is no deterioration, so the inventory is depleted only due to demand. Further, during the time interval, $[t_d,\ t_1]$ the inventory level is dropping to zero due to the combined effect of demand and deterioration. Moreover, the demand is backlogged in the interval $[t_1,\ T]$.

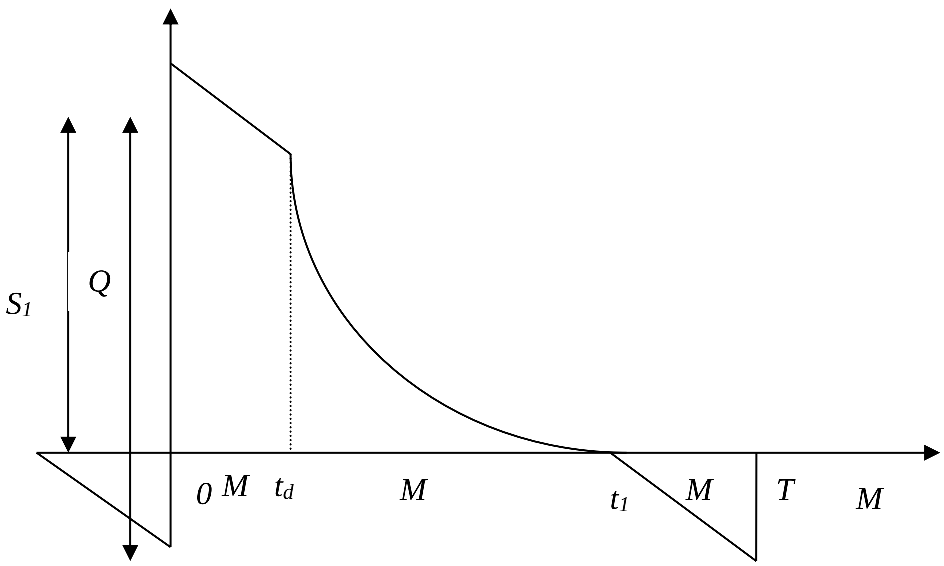


Figure 1: Pictorial representation of inventory level at any time

3. 2. Inventory levels

The differential equations that describe the inventory level at any time *t* over the period (*0*, *T*) are given by:

$$\frac{dI(t)}{dt} = -D, \qquad 0 \le t \le t_d \tag{1}$$

$$\frac{dI(t)}{dt} + \theta I(t) = -D, \qquad t_d < t \le t_1 \tag{2}$$

$$\frac{dI(t)}{dt} = -D, \qquad t_1 < t \le T \tag{3}$$

The solutions of the above three differential equations (1), (2) and (3) with using respective boundary conditions $I(0) = S_1,\ I(t_1) = 0,$ are as follows:

$$I(t) = S_1 - Dt \qquad 0 \le t \le t_d\,, \tag{4}$$

$$I(t)=\frac{D}{\theta}\left(e^{\theta(t_1-t)}-1\right),\quad t_d<t\le t_1 \tag{5}$$

$$I(t)=-D(t-t_1) \qquad t_1\le t\le T\ , \tag{6}$$

Considering continuity of $I(t)$ at $t=t_d$, it follows from equations (4) and (5) that

$$S_1-Dt_d=\frac{D}{\theta}\left(e^{\theta(t_1-t_d)}-1\right)$$

This implies that the maximum inventory level per cycle is given by

$$S_1=D\left(t_d+\frac{1}{\theta}\left(e^{\theta(t_1-t_d)}-1\right)\right) \tag{7}$$

The number of deteriorated units S_1-Dt_1 is

$$=D\left(t_d-t_1+\frac{1}{\theta}\left(e^{\theta(t_1-t_d)}-1\right)\right) \tag{8}$$

The order size Q is $S_1+D(T-t_1)$

$$=D\left(T+t_d-t_1+\frac{1}{\theta}\left(e^{\theta(t_1-t_d)}-1\right)\right) \tag{9}$$

3. 3. Retailer's Profit Components

Now, based on the above-obtained inventory levels, the total profit per cycle is obtained as follows:

a) Ordering cost per cycle = A

b) The inventory holding cost per cycle is given by $=h\left(\int_0^{t_d}I(t)\,dt+\int_{t_d}^{t_1}I(t)\,dt\right)$

$$\mathrm{Hc}=h\left\{S_1t_d-\frac{Dt_d^2}{2}-\frac{D}{\theta}\left(t_1-t_d+\frac{1}{\theta}\left(1-e^{\beta(t_1-t_d)}\right)\right)\right\}$$

c) The shortage cost per cycle is given by $\mathrm{Sc}=\pi\int_{t_1}^{T}I(t)\,dt \;=\; \pi D(T-t_1)^2/2$

d) The purchase cost per cycle $=\ cQ=cD\left(T+t_d-t_1+\frac{1}{\theta}\left(e^{\theta(t_1-t_d)}-1\right)\right)$

3. 4. Interest earned, interest paid and Total Profit

Also, the supplier offers permissible delay in payment to the retailer. Here, in this paper, we consider two types of interest earned rate (namely I_e & I_E). Where I_e is the rate of interest earned by the retailer on continuous sales revenue, I_E is the rate of interest earned by the retailer on sales revenue gating from satisfying the shortages or on the excess amount after settling the account or on from T to M when M > T.

This concept taken from the banking system. In the banking system, there is two rates of interest, i.e., interest earned on the amount, which is in saving account, and interest earned on the fixed deposit amount. These two types of rate of interest are different. I_p is the rate of interest payable to the supplier after the expiry of the credit period. Further, the interest earned, interest paid, and profit functions are computed for different scenarios in each case is discussed in this section. In addition, all the possible cases/sub-cases have been shown in figure 2.

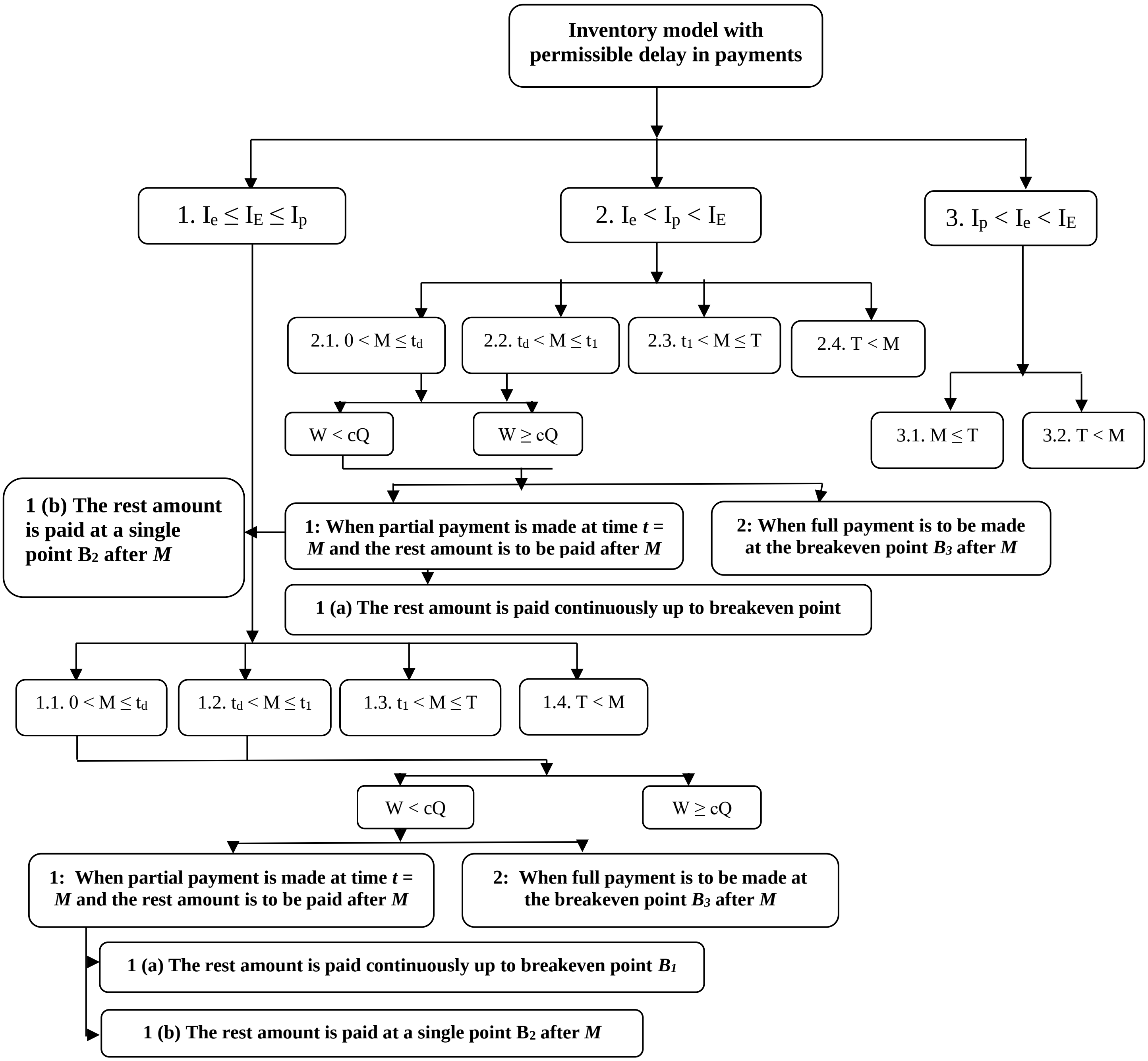


Figure 2: A schematic diagram flow of our model

Case 1: $I_e < I_E \le I_p$

The computation for interest earned and interest payable will depend on the following four possible sub-cases based on the length of t_d, t_1, T, and M:

Subcase 1.1: $0 < M \le t_d$ **; Subcase 1.2:** $t_d < M \le t_1$ **; Subcase 1.3:** $t_1 < M \le T$ **;** & **Subcase 1.4:** $T \le M$ **.**

Subcase 1.1: $0 < M \le t_d < T$

Since both the interest earned rates are less than the interest paid rate, so the retailer would try to pay off the total purchase cost to the supplier as soon as possible. At the expiry of *M*, the retailer will have a certain amount, which is the sum of the sales revenue during the period $[0, M]$ and interest earned on regular sales revenue and fixed deposit. This implies that the amount accumulates after satisfying the shortages during the same time period.

Hence, the total sales revenue during the time period $[0, M]$ is $Dp\{M + (T - t_1)\}$

The interest earned on regular sales revenue during the time period $[0, M]$ is $(1/2)DM^2 pI_e$

The interest earned on the fixed deposit amount during the time period $[0, M]$ is $D(T - t_1)MpI_E$

Therefore, at time $t=M$, the total amount earned by the retailer is

$$Dp\left[(T - t_1) + M\left\{1 + (T - t_1)I_E + (1/2)MI_e\right\}\right] \equiv W_1 \text{(say)} \tag{10}$$

At the end of trade credit period *M*, the retailer wishes to settle his account with the supplier, which gives another two sub-cases viz. Subcase 1.1.1: $W_1 < Qc$ and Subcase 1.1.2: $W_1 \ge Qc$.

Subcase 1.1.1: $W_1 < Qc$

Here, the retailer's earned amount (W_1) is less than the amount payable *(Qc)* to the supplier. In this situation, the supplier may either agree to receive the partial payment or not. Thus, further two scenarios may appear:

Scenario 1.1.1.1: When a partial payment is acceptable at *M*, and the rest amount is to be paid at any time after *M*

Scenario 1.1.1.2: when a partial payment is not acceptable at *M*, but the full payment is acceptable by the supplier at any time after ***M*.**

Scenario 1.1.1.1: When a partial payment is acceptable at *M*, and the rest amount is to be paid at any time after *M*. For this scenario, further two situations may arise, which are discussed below:

Scenario1.1.1.1 (a): When the rest amount is paid continuously up to breakeven point B_1 (say) after M

In this scenario, the retailer pays W_1 the amount at M, and the rest amount $(cQ-W_1)$ along with interest charged will be paid continuously from M to some payoff time B_1 (says).

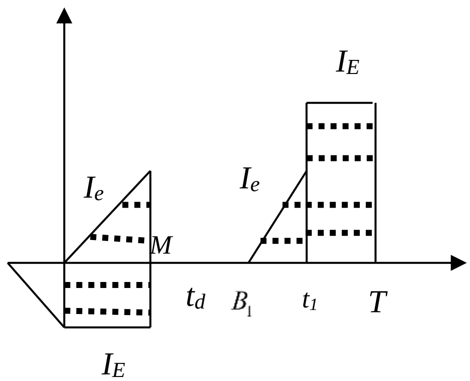


Figure 3: Interest earned in Scenario 1.1.1.1. (a)

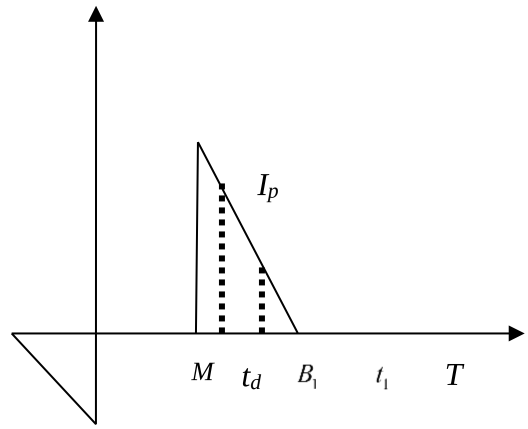


Figure 4: Interest payable in Scenario 1.1.1.1. (a)

The interest payable during the period $[M, B_1] = (1/2)(cQ-W_1)(B_1-M)I_p$ and

The total amount payable during $[M, B_1] = (cQ-W_1)+(1/2)(cQ-W_1)(B_1-M)I_p$

Now, at $t = B_1$, the total amount payable to the supplier = the total amount available to the retailer

$$\Rightarrow (cQ-W_1)+(1/2)(cQ-W_1)(B_1-M)I_p = D(B_1-M)p \tag{11}$$

$$\Rightarrow \qquad B_1 = M + \frac{2(cQ-W_1)}{2Dp-(cQ-W_1)I_p} \tag{12}$$

After the time B_1, the retailer generates revenue $D(t_1-B_1)p$ from the regular sale. He also earns interest on the regular sales revenue during the period $[B_1, t_1]$, which is $(1/2)D(t_1-B_1)^2 pI_e$. At time $t = t_1$ the retailer has the amount $D(t_1-B_1)p+(1/2)D(t_1-B_1)^2 pI_e$. He uses this revenue to earn interest on fix deposit of this amount during the time period $[t_1, T]$.

The interest earned for the time period $[t_1, T]$ is $Dp(t_1-B_1)\{1+(1/2)(t_1-B_1)I_e\}(T-t_1)I_E$

Therefore, the average profit per unit time is given by

$TP_{1.1.1.1.a}(\mu, t_1, T) = \frac{1}{T}$ [<Total selling revenue during $[B_1, t_1]$> + <Interest earned during $[B_1, t_1]$>

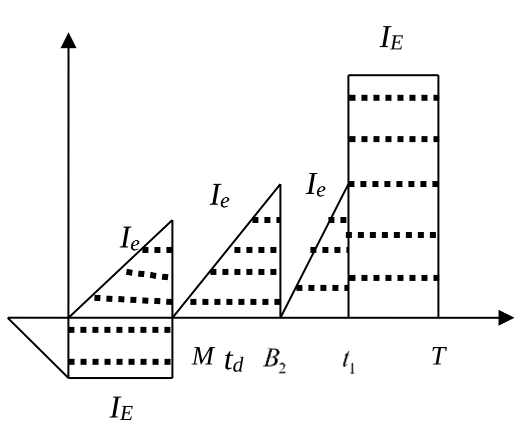
+ <Interest earned during $[t_1, T]$> - <Ordering Cost> - <Holding Cost>

- <Shortage cost>] (13)

$$TP_{1.1.1.1.(a)}(\mu,t_1,T)=\frac{1}{T}\left[D(t_1-B_1)p\left[1+(1/2)(t_1-B_1)I_e\right]\left[1+(T-t_1)I_E\right]-A-Hc-Sc\right] \tag{14}$$

Scenario 1.1.1.1(b): When the rest amount paid at a breakeven point B_2 (say) after M

In this scenario, the supplier accepts the payment only on two instalments first is at time $t = M$ and second is at some payoff time B_2 (says). The retailer pays amount W_1 at time $t = M$, and the rest amount $(cQ-W_1)$ along with interest charged will be paid at a breakeven point $t = B_2$. Now, during the time interval $[M, B_2]$, the retailer would generate an amount of $D(B_2-M)p$ from sales revenue and earn interest from the continuous interest earn on the selling revenue generated during the same.



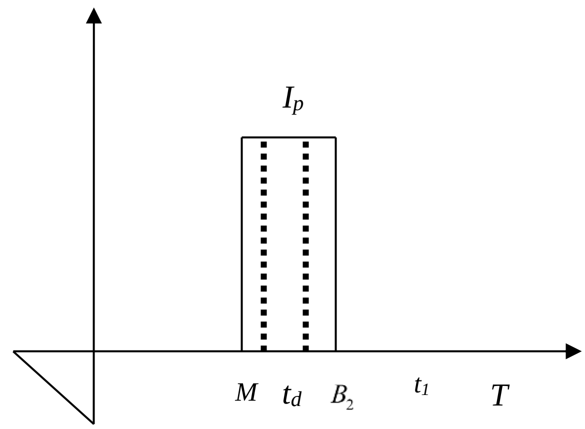


Figure 5: Interest earned in Scenario 1.1.1.1. (b) **Figure 6: Interest payable in Scenario 1.1.1.1. (b)**

The interest payable during the period $[M,B_2]$ is $(cQ-W_1)(B_2-M)I_p$

The interest earned during the period $[M,B_2]$ is $(1/2)D(B_2-M)^2\,pI_e$

The total amount payable at $t = B_2$ is $(cQ-W_1)+(cQ-W_1)(B_2-M)I_p$ and

The total amount earned during the period $[M,B_2]=D(B_2-M)p+(1/2)D(B_2-M)^2\,pI_e$

Now, at $t=B_2$, the total amount payable to the supplier is equal to the total amount available to the retailer

$$\Rightarrow (cQ-W_1)+(cQ-W_1)(B_2-M)I_p=D(B_2-M)p+(1/2)D(B_2-M)^2\,pI_e \tag{15}$$

$$\Rightarrow B_2=\frac{1}{DpI_e}\left\{DpMI_e+QcI_p-Dp-W_1I_p+\left(\left(Dp-QcI_p+W_1I_p\right)^2+2Dp(Qc-W_1)I_e\right)^{\frac{1}{2}}\right\} \tag{16}$$

After the time B_2, the retailer generates revenue $D(t_1-B_2)p$ from the regular sale. He also earns interest on the regular sales revenue during the period $[B_2,t_1]$, which is $(1/2)D(t_1-B_2)^2\,pI_e$. At time $t = t_1$ retailer has the amount $D(t_1 - B_2)\, p + (1/2) D(t_1 - B_2)^2\, pI_e$. He uses this revenue to earn interest on fix deposit of this amount during the time period $[t_1, T]$.

The interest earned for the time period $[t_1, T]$ is $Dp(t_1 - B_2)\{1 + (1/2)(t_1 - B_2) I_e\}(T - t_1) I_E$

Therefore, the total profit per unit time is given by

$TP_{1.1.1.1.b}(\mu, t_1, T) = \frac{1}{T}$ [<Total selling revenue during $[B_1, t_1]$> + <Interest earned during $[B_1, t_1]$>

+ <Interest earned during $[t_1, T]$> - <Ordering Cost> - <Holding Cost>

- <Shortage cost>] (17)

$$TP_{1.1.1.1.(b)}(\mu, t_1, T) = \frac{1}{T}\left[Dp(t_1 - B_2)\left[1 + (1/2)(t_1 - B_2) I_e\right]\left[1 + (T - t_1) I_E\right] - A - Hc - Sc\right] \quad (18)$$

Scenario 1.1.1.2: When full payment is to be made at the breakeven point B_3 (say) after M

In this scenario, **the s**upplier will charge the interest at the rate (I_p) on amount Qc for the period $[M, B_3]$. However, the retailer has W_1 an amount at a time $t = M$, and he will earn interest at the rate (I_E) on fixed deposit of this amount for the period $[M, B_3]$. After M, he also generates the sales revenue as well as earns interest on regular sales revenue during the period $[M, B_3]$.

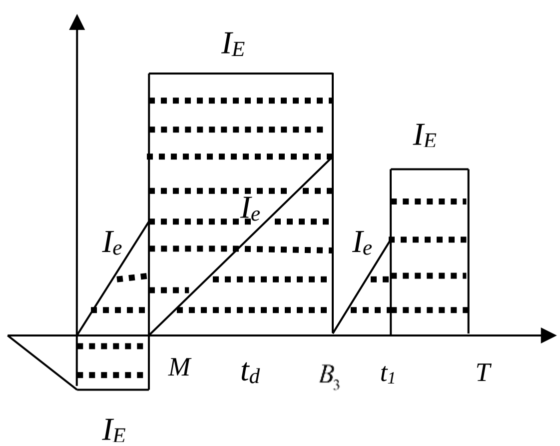


Figure 7: Interest earned in scenario 1.1.1.2

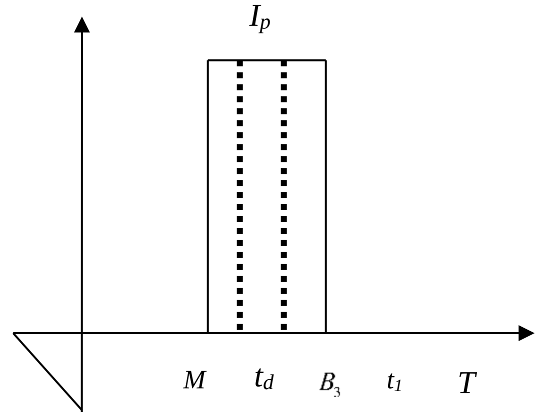


Figure 8: Interest payable in scenario 1.1.1.2

The interest earned from fixed deposit amount W_1 for the time period $[M, B_3]$ is $W_1 I_E (B_3 - M)$ and the interest earned on the continuous sales revenue $D(B_3 - M)\, p$ from the time period $[M, B_3]$ is $(1/2) D(B_3 - M)^2\, pI_e$.

Hence, the total interest earned during the time period $[M, B_3] = W_1 I_E (B_3 - M) + (1/2) D(B_3 - M)^2\, pI_e$

The interest is payable during the same time period = $QcI_p\left(B_3-M\right)$

Again, to determine the value of breakeven point, the total amount payable to the supplier should equal to the total amount available to the retailer i.e.

$$Qc+Qc\left(B_3-M\right)I_p=W_1+D\left(B_3-M\right)p+W_1I_E\left(B_3-M\right)+(1/2)D\left(B_3-M\right)^2pI_e \tag{19}$$

$$\Rightarrow B_3=\frac{1}{DpI_e}\left\{DpMI_e+QcI_p-Dp-W_1I_E+\left(\left(Dp-QcI_p+W_1I_E\right)^2+2DpI_e\left(Qc-W_1\right)\right)^{\frac{1}{2}}\right\} \tag{20}$$

Further, the sales revenue during the time period $\left[B_3,t_1\right]$ is $D\left(t_1-B_3\right)p$ and the interest earned on regular sales revenue during this period is $(1/2)D\left(t_1-B_3\right)^2pI_e$. So that, at time t = t_1 retailer has the amount $D\left(t_1-B_3\right)p+(1/2)D\left(t_1-B_3\right)^2pI_e$. He uses this revenue to earn interest on fixed deposit of this amount during the time period $\left[t_1,T\right]$.

The interest earned on fix deposit amount = $D\left(t_1-B_3\right)p\left(1+(1/2)\left(t_1-B_3\right)I_e\right)\left(T-t_1\right)I_E$

Therefore, the average profit per unit time is given by

$TP_{1.1.1.2}\left(\mu,t_1,T\right)=\frac{1}{T}$ [<Total selling revenue during $\left[B_3,t_1\right]$> + <Interest earned during $\left[B_3,t_1\right]$>

+ <Interest earned during $\left[t_1,T\right]$> - <Ordering cost> - <Holding cost>

- <Shortage cost>] (21)

$$TP_{1.1.1.2}\left(\mu,t_1,T\right)=\frac{1}{T}\left[D\left(t_1-B_3\right)p\left(1+(1/2)\left(t_1-B_3\right)I_e\right)\left[1+\left(T-t_1\right)I_E\right]-A-Hc-Sc\right] \tag{22}$$

Subcase 1.1.2: $W_1\geq Qc$

In this subcase, the retailer has to pay the only Qc amount to the supplier at the time $t=M$ **,** and he will deposit the excess amount $\left(W_1-Qc\right)$ to earn the interest at the rate of *(I_E)* for the time period $\left[M,T\right]$. The interest earned on this amount is equal to $\left(W_1-Qc\right)\left(T-M\right)I_E$**.** Further, after the time $t=M$, the retailer continuously sales the product and uses the revenue to earn interest.

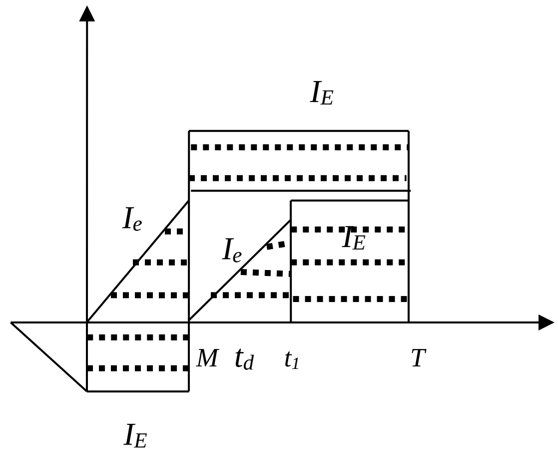


Figure 9: Interest earned in subcase 1.1.2

The interest earned on the regular sales revenue $D(t_1 - M)p$ during the period $[M, t_1]$ is $(1/2)D(t_1 - M)^2 pI_e$. At time $t = t_1$ retailer has the amount $D(t_1 - M)p + (1/2)D(t_1 - M)^2 pI_e$. He uses this revenue to earn interest from the fixed deposit of this amount during the time period $[t_1, T]$, which is $D(t_1 - M)p(1 + (1/2)(t_1 - M)I_e)(T - t_1)I_E$.

Therefore, the average profit per unit time is given by

$TP_{1.1.2}(\mu, t_1, T) = \frac{1}{T}$ [< Total sales revenue during $[M, t_1]$> + <Interest earned on the sales revenue during $[M, t_1]$> + <Interest earned on the sales revenue during $[t_1, T]$> + <Excess amount after paying the amount to the supplier > + <Interest earned on the excess amount during $[M, T]$> - <Ordering cost> - <Holding cost> - < Shortage cost >] (23)

$$TP_{1.1.2}(\mu, t_1, T) = \frac{1}{T}\left[D(t_1 - M)p\left(1 + (1/2)(t_1 - M)I_e\right)\left(1 + (T - t_1)I_E\right) + (W_1 - Qc)\left\{1 + (T - M)I_E\right\} - A - Hc - Sc\right] \quad (24)$$

Case 1.2: $t_d < M \le t_1$

In this case, the permissible delay period M lies between the time t_d at which deterioration start and non-negative stock period time t_1. In this case, the mathematical formulation is the same as of **Case 1.1:** $0 < M \le t_d < t_1 < T$.

Case 1.3: $t_1 < M \le T$

In this case, the trade credit period M offered by the supplier lies between the stock out period t_1 and the replenishment cycle time T. The retailer will pay off the total amount owed to the supplier at the end of the trade credit period M. Therefore, there is no interest payable to the supplier, but the retailer uses the sales revenue to earn interest at the rate of I_e and I_E during the period $[0, M]$.

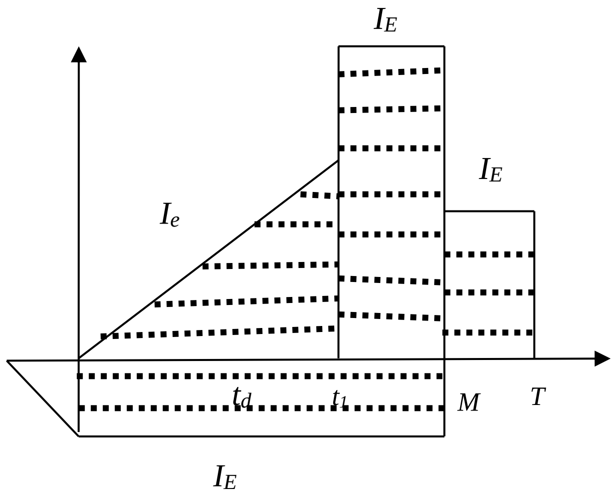


Figure 10: Interest earned in Case 1.3

Hence, the total interest earned by the retailer is calculated in three different cases.

(i) The interest earned at the rate of I_E on the revenue $Dp(T-t_1)$ of shortage items during the period $[0, M]$ is $Dp(T-t_1)MI_E$

(ii) The interest earned interest in continuous sales revenue during the period $[0, t_1]$ is $(1/2)Dpt_1^2I_e$

(iii) The interest earned during the period $[t_1, M]$ is $Dt_1p\left(1+(1/2)t_1I_e\right)I_E\left(M-t_1\right)$

At time $t = M$, the retailer has the amount $Dp(T-t_1)\left(1+MI_E\right)+Dpt_1\left(1+(1/2)t_1I_e\right)\left(1+\left(M-t_1\right)I_E\right)\equiv W_2$ in his account but the retailer settled his account with the supplier at M. He pays Qc amount to the supplier and earns interest on the excess amount $W_2 - Qc$ at the interest rate I_E. The interest earned during the period $[M, T]$ is $\left(W_2 - Qc\right)(T-M)I_E$.

Therefore, the average profit per unit time is given by

$TP_{1.3}(\mu, t_1, T) = \dfrac{1}{T}$ [<Excess amount> + <Interest earned on the excess amount during the period $[M, T]$>

- <Ordering cost> - <Holding cost> - <Shortage cost>] (25)

$$TP_{1.3}(\mu, t_1, T) = \frac{1}{T}\left[\left(W_2 - Qc\right)+\left(W_2 - Qc\right)(T-M)I_E - A - Hc - Sc\right] \tag{26}$$

Case 1.4: $T \le M$

In this case, the trade credit period *M* offer by the supplier is greater than the replenishment cycle time *T*. The retailer will pay off the total amount owed to the supplier at the end of the trade credit period *M*. Therefore, there is no interest payable to the supplier, but the retailer uses the sales revenue to earn the interest at the rate of I_e and I_E during the period $[0, M]$.

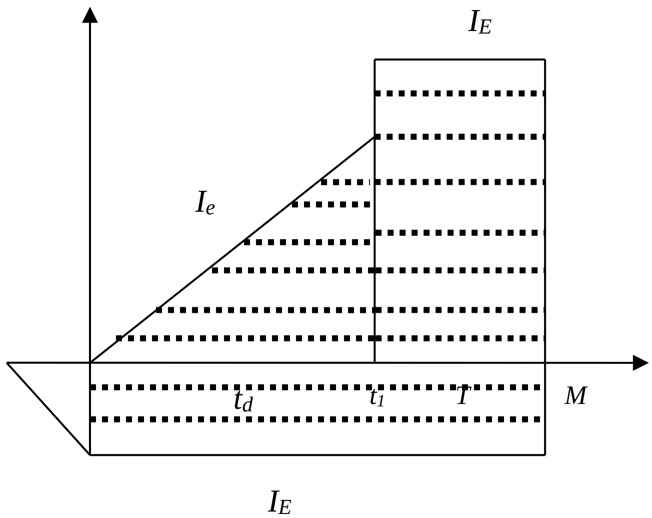


Figure 11: Interest earned in case 1.4

Here, the retailer earns interest as follows:

(i) The interest earned at the rate of I_E on fixed amount which is generated from the shortages revenue $Dp(T-t_1)$ during the period $[0, M]$ is $Dp(T-t_1)MI_E$

(ii) The interest earned interest in sales revenue during the period $[0, t_1]$ is $(1/2)Dpt_1^2 I_e$

(iii) The interest earned during the period $[t_1, M]$ is $Dt_1 p\left(1+(1/2)t_1 I_e\right) I_E (M-t_1)$

At *t* = *M*, *the* retailer has $Dp(T-t_1)(1+MI_E)+Dpt_1\left(1+(1/2)t_1 I_e\right)\left(1+(M-t_1)I_E\right) \equiv W_3$ (say) amount in his account, but the retailer settled his account with the supplier at *M*. He pays *Qc* amount to the supplier. Therefore, the average profit per unit time is given by

$TP_{1.3}(\mu, t_1, T) = \dfrac{1}{T}$ [<Excess amount> - <Ordering cost> - <Holding cost> - <Shortage cost>] (27)

$$TP_{1.4}(\mu, t_1, T) = \frac{1}{T}\left[\left(W_3 - Qc\right) - A - Hc - Sc\right] \tag{28}$$

As earlier one is Case 1: $I_e < I_E \le I_p$. Now we discuss the Case 2: $I_e < I_p \le I_E$

This situation indicates that the interest payable rate (I_p) lies between both interest rates I_e and I_E. Further, depending on the values of t_d, t_1, M, and T the following four sub-cases may arise:

Subcase 2.1: $0 < M \le t_d$ **; Subcase 2.2:** $t_d < M \le t_1$ **; Subcase 2.3:** $t_1 < M \le T$ **;** & **Subcase 2.4:** $T \le M$ **.**

Subcase 2.1: $0 < M \le t_d$

In this case, the retailer would try to pay off the total purchase cost to the supplier as soon as possible. During the period $\left[0, M\right]$, the retailer uses the sales revenue to earn interest. Hence, the total sales revenue from the period $\left[0, M\right]$ is $Dp\{M + (T - t_1)\}$ and the interest earned during the same time period is $DMp\left\{(T - t_1)I_E + (1/2)MI_e\right\}$.

Therefore, the retailer has a total amount at a time $t = M$ is

$$Dp\left[(T - t_1) + M\left\{1 + (T - t_1)I_E + (1/2)M^2 I_e\right\}\right] \equiv W_1 \text{(say)}.$$

However, the retailer owes Qc amounts as the purchase cost from the supplier at time $t = 0$. Based on the difference between W_1 and Qc, further, there may be the following two **subcases 2.1.1:** $W_1 < Qc$ and **subcases 2.1.2:** $W_1 \ge Qc$.

Subcase 2.1.1: $W_1 < Qc$

Since, the fixed amount less than the Qc amount, this implies that the interest earned on the fixed amount is less than the interest paid amount. So, he will pay the amount Qc as soon as possible. The mathematical formulation of this subcase is same as subcase 1.1.1 $W_1 < Qc$ in case 1. So, for this subcase, the profit functions are the same.

Subcase 2.1.2: $W_1 \ge Qc$

In this subcase, the interest earned rate on the fixed amount is greater than the interest payable rate. Therefore, interest in W_1 is greater than the interest payable in one cycle. So, the retailer cannot pay any amount before the cycle length. He pays the total amount along with interest charge at the end of cycle length.

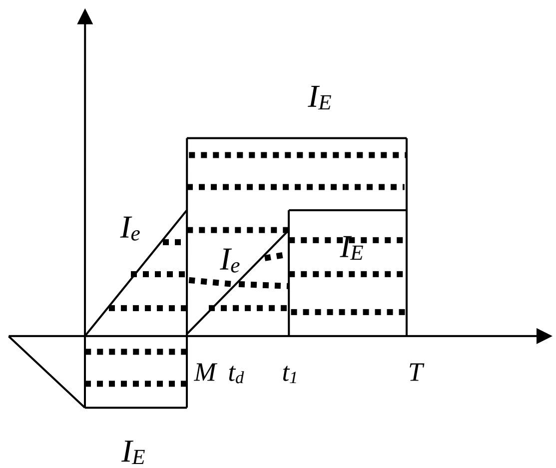


Figure 12: Interest earned in Subcase 2.1.2

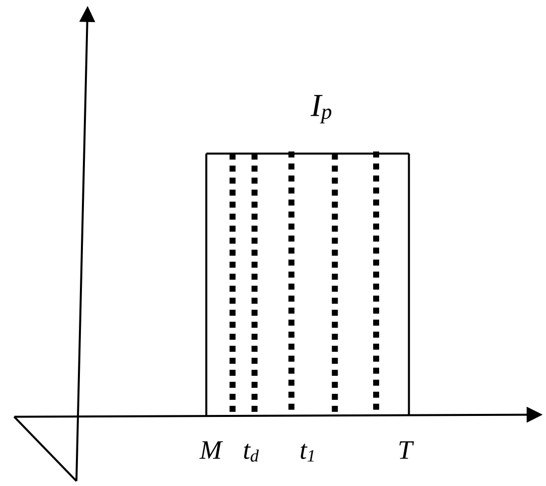


Figure 13: Interest payable in Subcase 2.1.2

The interest payable during the period $[M,T]$ is $Qc(T-M)I_p$

The interest earned on the amount W_1 during the period $[M,T]$ is $W_1 I_E (T-M)$

Further, after time $t=M$, the retailer continuously sales the products and uses the revenue to earn interest. So, interest earned on the sales revenue during the period $\left[M,t_1\right]$ is $(1/2)D\left(t_1-M\right)^2 pI_e$ and also earned interest during the period $\left[t_1,T\right]$ on the revenue $D\left(t_1-M\right)p+(1/2)D\left(t_1-M\right)^2 pI_e$ is $D\left(t_1-M\right)p\left(1+(1/2)\left(t_1-M\right)I_e\right)\left(T-t_1\right)I_E$.

Therefore, the average profit per unit time is given by

$TP_{2.1.2}(\mu,t_1,T)=\dfrac{1}{T}$ [< Total sales revenue during $\left[M,t_1\right]$> + <Interest earned on the sales revenue during $\left[M,t_1\right]$> + <Interest earned on the sales revenue during $\left[t_1,T\right]$> + <Total amount at M i.e. W_1 > + <Interest earned on the amount W_1 during $\left[M,T\right]$> - <Purchasing cost> - <Interest payable > - <Ordering cost> - <Holding cost> - < Shortage cost >] (29)

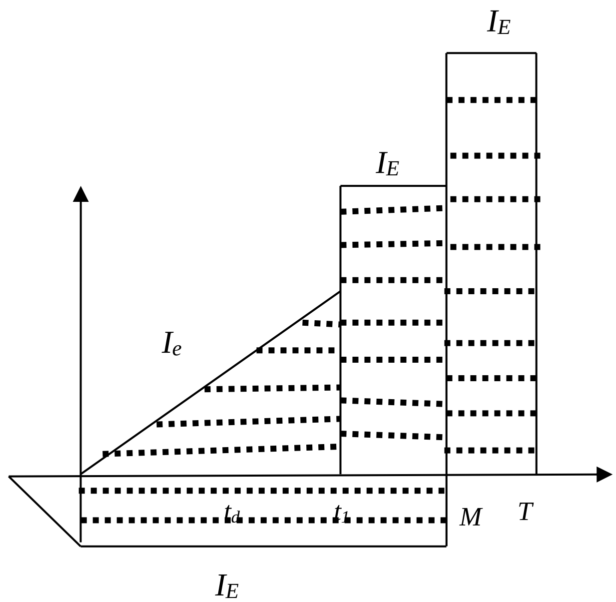
$$TP_{2.1.2}(\mu,t_1,T)=\frac{1}{T}\Big[D\left(t_1-M\right)p\left(1+(1/2)D\left(t_1-M\right)I_e\right)\left[1+\left(T-t_1\right)I_E\right]+W_1\left\{1+I_E(T-M)\right\}-Qc\left\{1+(T-M)I_p\right\}-A-Hc-Sc\Big] \quad (30)$$

Subcase 2.2: $t_d < M \le t_1$

In this case, the permissible delay period M lies between the time t_d at which deterioration starts and non-negative stock period time t_1. In this case, the mathematical formulation is the same as of Case 2.1, i.e. $0 < M \le t_d$ **.** So, the mathematical formulation for this case is not necessitating.

Subcase 2.3: $t_1 < M \le T$

In this case, the trade credit period M offered by the supplier lies between stock out period t_1 and replenishment cycle time T. So, the retailer sells all the products up to the time t_1 and generates sales revenue. He uses this sales revenue to earn interest at the rate of I_e and I_E during the period $[0, M]$.



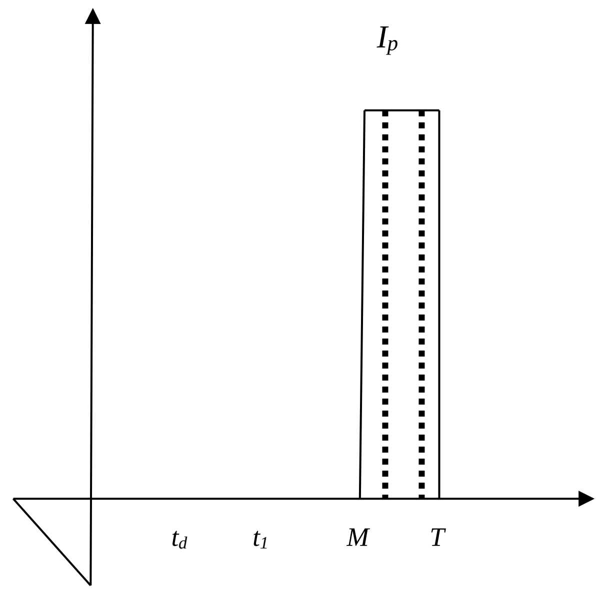


Figure 14: Interest earned in Subcase 2.3 **Figure 15: Interest payable in Subcase 2.3**

The interest earned at the rate of I_E on the shortages revenue $Dp(T-t_1)$ during the period $[0, M]$ is $Dp(T-t_1)MI_E$

The interest earned interest in continuous sales revenue during the period $[0, t_1]$ is $(1/2)Dpt_1^2 I_e$

The interest earned during the period $[t_1, M]$ is $Dt_1 p(1+(1/2)t_1 I_e) I_e (M-t_1)$

At $t = M$, *the* retailer has the amount $Dp(T-t_1)(1+MI_E)+Dpt_1(1+(1/2)t_1 I_e)(1+(M-t_1)I_E) \equiv W_2$ in his account, but the retailer settles his account with the supplier at M. He pays Qc amount to the supplier at M but in this case, only one possibility $W_2 \ge Qc$ **,** because of the sales all product**.** Since the interest earned rate on fixed deposit is higher than interest payable rate. So, he will pay Qc amount with interest payable $Qc(T-M)I_p$ at the end of the cycle length despite at the end of permissible delay in payments. He earns the interest on the amount W_2 by fixed deposit at the rate I_E

The interest earned during the period $[M, T]$ is $W_2(T-M)I_E$.

Therefore, the average profit per unit time is given by

$TP_{2.3}(\mu,t_1,T)=\frac{1}{T}$ [<Total amount at *M*> + <Interest earned on W_2 during the period $[M,T]$>

- <Purchasing cost> - <Interest payable > - <Ordering cost> - <Holding cost>

- <Shortage cost>] (31)

$$TP_{2.3}(\mu,t_1,T)=\frac{1}{T}\left[W_2\left\{1+(T-M)I_E\right\}-Qc\left\{1+(T-M)I_p\right\}-A-Hc-Sc\right] \tag{32}$$

Case 2.4: $T\le M$

In this case, the trade credit period M offered by the supplier is greater than the replenishment cycle time T. The retailer will pay off the total amount owed to the supplier at the end of the trade credit period M. Therefore, there is no interest payable to the supplier, but the retailer uses the sales revenue to earn interest at the rate of I_e and I_E during the period $[0,M]$.

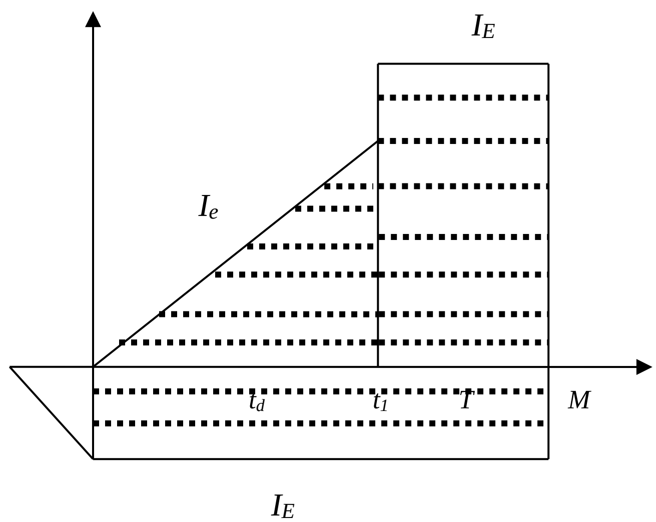


Figure 16: Interest earned in case 2.4

Hence, the retailer the total interest earned is calculated in three different cases.

(i) The interest earned at the rate of I_E on the revenue $Dp(T-t_1)$ of shortage items during the period $[0,M]$ is $Dp(T-t_1)MI_E$

(ii) The interest earned interest in continuous sales revenue during the period $[0,t_1]$ is $(1/2)Dpt_1^2I_e$

(iii) The interest earned during the period $[t_1,M]$ is $Dt_1p\left(1+(1/2)t_1I_e\right)I_E\left(M-t_1\right)$

At $t = M$, *the* retailer has the amount $Dp(T-t_1)\left(1+MI_E\right)+Dpt_1\left(1+(1/2)t_1I_e\right)\left(1+\left(M-t_1\right)I_E\right)\equiv W_3$ (say) in his account, but the retailer settled his account with the supplier at *M*. He pays *Qc* amount to the supplier.

Therefore, the average profit per unit time is given by

$$TP_{2.4}(\mu,t_1,T)=\frac{1}{T}\,[\text{<Excess amount> - <Ordering cost> - <Holding cost> - <Shortage cost>}] \tag{33}$$

$$TP_{2.4}(\mu,t_1,T)=\frac{1}{T}\left[\left(W_3-Qc\right)-A-Hc-Sc\right] \tag{34}$$

Section 3: $I_p \le I_e < I_E$

In this case, both interest earned I_e and I_E are greater than the interest payable I_p . In this case, the retailers cannot pay any amount at the end of permissible delay in payments. He settles his account at the end of cycle length if the permissible delay period is less than cycle length. If permissible delay period is greater than cycle length, then he settles his account at *M*. Further, depending on values of *T* and *M*, two sub-cases may arise which are as follows: **subcases 3.1:** $M \le T$ and **subcases 3.2:** $T < M$.

Subcase 3.1: $M \le T$

This case situation indicates that the replenishment cycle time *T* is greater than or equals to the permissible delay in payments *M*.

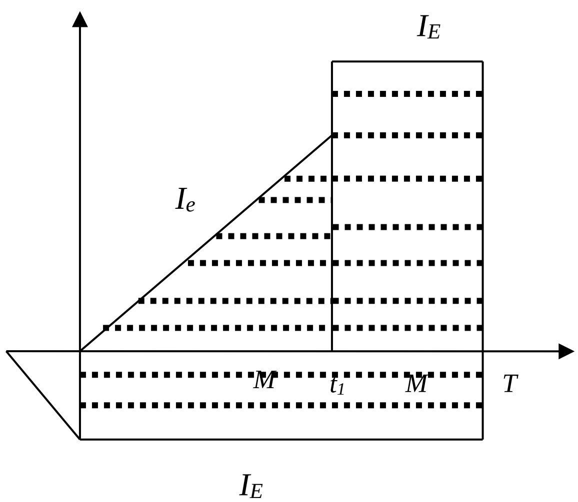


Figure17: Interest earned in Case 3.1

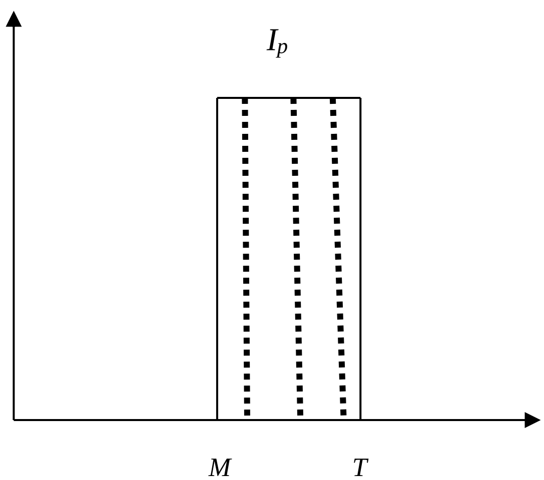


Figure18: Interest payable in Case 3.1

In this case, interest earned is calculated in three parts.

(i) Interest earned on shortages revenue during the period $\left[0,T\right]$ is $D(T-t_1)pTI_E$

(ii) Interest earned on the sale revenue proceeds during the period $\left[0,t_1\right]$ is $(1/2)Dt_1^2 pI_e$

(iii) Interest earned during the period $\left[t_1,T\right]$ is $\left(Dt_1 p+(1/2)Dt_1^2 pI_e\right)\left(T-t_1\right)I_E$

Moreover, the total interest earned in one cycle is

$$D(T-t_1)pTI_E+(1/2)Dt_1^2pI_e+\left(Dt_1p+(1/2)Dt_1^2pI_e\right)\left(T-t_1\right)I_E$$

In this case, the retailer paid total amount Qc as well as interest payable $Qc\left(T-M\right)I_p$ at the end of cycle length. The total payable amount at the end of cycle length is $Qc(1+\left(T-M\right)I_p)$.

Therefore, the average profit per unit time is given by

$TP_{3.1}(\mu,t_1,T)=\dfrac{1}{T}$ [< Total sale revenue during $\left[0,T\right]$> + <Interest earned on the sales revenue during $\left[0,t_1\right]$> + <Interest earned on the sales revenue during $\left[t_1,T\right]$> + <Interest earned on shortages revenue during $\left[0,T\right]$> - < total amount paid as well as interest payable at T > - <Ordering Cost> - <Holding Cost> - <Shortages Cost>] (35)

$$TP_{3.1}(\mu,t_1,T)=\frac{1}{T}\left[DTp+\frac{1}{2}Dt_1^2pI_e+D(T-t_1)pTI_E+\left(Dt_1p+\frac{1}{2}Dt_1^2pI_e\right)\left(T-t_1\right)I_E-Qc\left\{1+\left(T-M\right)I_p\right\}-A-Hc-SC\right] \quad (36)$$

Subcase 3.2: $T<M$

In this case, the replenishment cycle time T is less than or equal to the permissible delay period M. In this situation, the retailer will pay off the total amount owed to the supplier at the end of the trade credit period M. Therefore, there is no interest payable to supplier charge, but the retailer uses the sales revenue to earn interest at the rate of I_e and I_E during the period $\left[0,M\right]$.

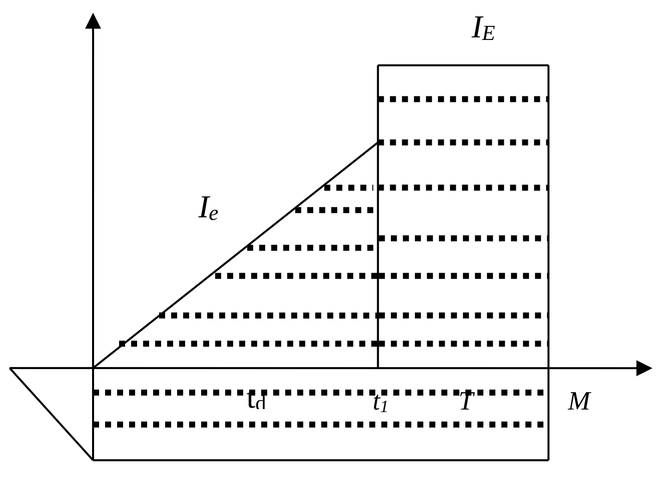


Figure 19: Interest earned in Case 3.2

The interest earned is calculated in three parts:

(i) Interest earned on shortages revenue during the period $\left[0,M\right]$ is $D(T-t_1)pMI_E$

(ii) Interest earned on the sale revenue proceeds during the period $\left[0, t_1\right]$ is $(1/2)Dt_1^2 pI_e$

(iii) Interest earned during the period $\left[t_1, M\right]$ is $\left(Dt_1 p + (1/2)Dt_1^2 pI_e\right)\left(M - t_1\right)I_E$

Hence, the interest earned in one cycle is

$$D(T-t_1)pMI_E + (1/2)Dt_1^2 pI_e + \left(Dt_1 p + (1/2)Dt_1^2 pI_e\right)\left(M - t_1\right)I_E$$

Therefore, the average profit per unit time is given by

$TP_{3.2}(\mu, t_1, T) = \frac{1}{T}$ [<sales revenue> + <interest earned> - <Purchasing cost> - <ordering cost>

- <holding cost> - <shortages cost>] (37)

$$TP_{3.2}(\mu, t_1, T) = \frac{1}{T}\left[DTp + D(T-t_1)pMI_E + (1/2)Dt_1^2 pI_e + Dt_1 p\left(1 + (1/2)t_1 I_e\right)\left(M - t_1\right)I_E - Qc - A - Hc - Sc\right] \quad (38)$$

The average profit can write as in the combined form

$$TP(\mu, t_1, T) = \begin{cases} \mathrm{TP}_{1.1.1.1.(a)}(.) & \text{if } 0<\mathrm{M} \le \mathrm{t_d}, \mathrm{W_1}<\mathrm{Qc}, \mathrm{I_e}<\mathrm{I_E} \le \mathrm{I_p}, \text{ partiallly and rest amount paid continuosly} \\ \mathrm{TP}_{1.1.1.1.(b)}(.) & \text{if } 0<\mathrm{M} \le \mathrm{t_d}, \mathrm{W_1}<\mathrm{Qc}, \mathrm{I_e}<\mathrm{I_E} \le \mathrm{I_p}, \text{ partiallly and rest amount in second shippment} \\ \mathrm{TP}_{1.1.1.2}(.) & \text{if } 0<\mathrm{M} \le \mathrm{t_d}, \mathrm{W_1}<\mathrm{Qc}, \mathrm{I_e}<\mathrm{I_E} \le \mathrm{I_p}, \text{ and full amount made after } \mathrm{t}=\mathrm{M} \\ \mathrm{TP}_{1.1.2}(.) & \text{if } 0<\mathrm{M} \le \mathrm{t_d}, \mathrm{W_1} \ge \mathrm{Qc}, \text{ and } \mathrm{I_e}<\mathrm{I_E} \le \mathrm{I_p} \\ \mathrm{TP}_{1.2.1.1.(a)}(.) & \text{if } \mathrm{t_d}<\mathrm{M} \le \mathrm{t_1}, \mathrm{W_1}<\mathrm{Qc}, \mathrm{I_e}<\mathrm{I_E} \le \mathrm{I_p}, \text{ partiallly and rest amount paid continuosly} \\ \mathrm{TP}_{1.2.1.1.(b)}(.) & \text{if } \mathrm{t_d}<\mathrm{M} \le \mathrm{t_1}, \mathrm{W_1}<\mathrm{Qc}, \mathrm{I_e}<\mathrm{I_E} \le \mathrm{I_p}, \text{ partiallly and rest amount in second shippment} \\ \mathrm{TP}_{1.2.1.2}(.) & \text{if } \mathrm{t_d}<\mathrm{M} \le \mathrm{t_1}, \mathrm{W_1}<\mathrm{Qc}, \mathrm{I_e}<\mathrm{I_E} \le \mathrm{I_p}, \text{ and full amount made after } \mathrm{t}=\mathrm{M} \\ \mathrm{TP}_{1.2.2}(.) & \text{if } \mathrm{t_d}<\mathrm{M} \le \mathrm{t_1}, \mathrm{W_1} \ge \mathrm{Qc}, \text{ and } \mathrm{I_e}<\mathrm{I_E} \le \mathrm{I_p} \\ \mathrm{TP}_{1.3}(.) & \text{if } \mathrm{t_1}<\mathrm{M} \le T, \text{ and } \mathrm{I_e}<\mathrm{I_E} \le \mathrm{I_p} \\ \mathrm{TP}_{1.4}(.) & \text{if } T \le \mathrm{M} \text{ and } \mathrm{I_e}<\mathrm{I_E} \le \mathrm{I_p} \\ \mathrm{TP}_{2.1.1.1.(a)}(.) & \text{if } 0<\mathrm{M} \le \mathrm{t_d}, \mathrm{W_1}<\mathrm{Qc}, \mathrm{I_e}<\mathrm{I_p} \le \mathrm{I_E}, \text{ partiallly and rest amount paid continuosly} \\ \mathrm{TP}_{2.1.1.1.(b)}(.) & \text{if } 0<\mathrm{M} \le \mathrm{t_d}, \mathrm{W_1}<\mathrm{Qc}, \mathrm{I_e}<\mathrm{I_p} \le \mathrm{I_E}, \text{ partiallly and rest amount in second shippment} \\ \mathrm{TP}_{2.1.1.2}(.) & \text{if } 0<\mathrm{M} \le \mathrm{t_d}, \mathrm{W_1}<\mathrm{Qc}, \mathrm{I_e}<\mathrm{I_p} \le \mathrm{I_E}, \text{ and full amount made after } \mathrm{t}=\mathrm{M} \\ \mathrm{TP}_{2.1.2}(.) & \text{if } 0<\mathrm{M} \le \mathrm{t_d}, \mathrm{W_1} \ge \mathrm{Qc}, \text{ and } \mathrm{I_e}<\mathrm{I_p} \le \mathrm{I_E} \\ \mathrm{TP}_{2.2.1.1.(a)}(.) & \text{if } \mathrm{t_d}<\mathrm{M} \le \mathrm{t_1}, \mathrm{W_1}<\mathrm{Qc}, \mathrm{I_e}<\mathrm{I_p} \le \mathrm{I_E}, \text{ partiallly and rest amount paid continuosly} \\ \mathrm{TP}_{2.2.1.1.(b)}(.) & \text{if } \mathrm{t_d}<\mathrm{M} \le \mathrm{t_1}, \mathrm{W_1}<\mathrm{Qc}, \mathrm{I_e}<\mathrm{I_p} \le \mathrm{I_E}, \text{ partiallly and rest amount in second shippment} \\ \mathrm{TP}_{2.2.1.2}(.) & \text{if } \mathrm{t_d}<\mathrm{M} \le \mathrm{t_1}, \mathrm{W_1}<\mathrm{Qc}, \mathrm{I_e}<\mathrm{I_p} \le \mathrm{I_E}, \text{ and full amount made after } \mathrm{t}=\mathrm{M} \\ \mathrm{TP}_{2.2.2} & \text{if } \mathrm{t_d}<\mathrm{M} \le \mathrm{t_1}, \mathrm{W_1} \ge \mathrm{Qc}, \text{ and } \mathrm{I_e}<\mathrm{I_p} \le \mathrm{I_E} \\ \mathrm{TP}_{2.3}(.) & \text{if } \mathrm{t_1}<\mathrm{M} \le T \text{ and } \mathrm{I_e}<\mathrm{I_p} \le \mathrm{I_E} \\ \mathrm{TP}_{2.4}(.) & \text{if } \mathrm{T} \le \mathrm{M} \text{ and } \mathrm{I_e}<\mathrm{I_p} \le \mathrm{I_E} \\ \mathrm{TP}_{3.1}(.) & \text{if } \mathrm{M} \le \mathrm{T} \text{ and } \mathrm{I_p} \le \mathrm{I_e}<\mathrm{I_E} \\ \mathrm{TP}_{3.2}(.) & \text{if } \mathrm{T} \le \mathrm{M} \text{ and } \mathrm{I_p} \le \mathrm{I_e}<\mathrm{I_E} \end{cases} \quad (39)$$

For our convenience, we let twenty-two events as

$E_1 = \{t \,|\, 0 < M \le t_d, W_1 < Qc, I_e < I_E \le I_p, \textit{ partiallly and rest amount paid continuosly}\}$

$E_2 = \{t \,|\, 0 < M \le t_d, W_1 < Qc, I_e < I_E \le I_p, \textit{ partiallly and rest amount } \text{in} \sec\textit{ond shippment}\}$

$$E_3 = \{t \mid if\ 0 < M \le t_d, W_1 < Qc, I_e < I_E \le I_p,\ and\ full\ amount\ made\ after\ t = M\}$$

$$E_4 = \{t \mid 0 < M \le t_d, W_1 \ge Qc,\ and\ I_e < I_E \le I_p\}$$

$$E_5 = \{t \mid if\ t_d < M \le t_1,\ W_1 < Qc, I_e < I_E \le I_p,\ partiallly\ and\ rest\ amount\ paid\ continuosly\}$$

$$E_6 = \{t \mid t_d < M \le t_1, W_1 < Qc, I_e < I_E \le I_p,\ partiallly\ and\ rest\ amount\ in\ \sec ond\ shippment\}$$

$$E_7 = \{t \mid t_d < M \le t_1, W_1 < Qc, I_e < I_E \le I_p,\ and\ full\ amount\ made\ after\ t = M\}$$

$$E_8 = \{t \mid t_d < M \le t_1, W_1 \ge Qc,\ and\ I_e < I_E \le I_p\}$$

$$E_9 = \{t \mid t_1 < M \le T,\ and\ I_e < I_E \le I_p\}$$

$$E_{10} = \{t \mid T \le M\ \ and\ I_e < I_E \le I_p\}$$

$$E_{11} = \{t \mid 0 < M \le t_d, W_1 < Qc, I_e < I_p \le I_E,\ partiallly\ and\ rest\ amount\ paid\ continuosly\}$$

$$E_{12} = \{t \mid 0 < M \le t_d, W_1 < Qc, I_e < I_p \le I_E,\ partiallly\ and\ rest\ amount\ in\ \sec ond\ shippment\}$$

$$E_{13} = \{t \mid 0 < M \le t_d, W_1 < Qc, I_e < I_p \le I_E,\ and\ full\ amount\ made\ after\ t = M\}$$

$$E_{14} = \{t \mid 0 < M \le t_d, W_1 \ge Qc,\ and\ I_e < I_p \le I_E\}$$

$$E_{15} = \{t \mid t_d < M \le t_1,\ W_1 < Qc, I_e < I_p \le I_E,\ partiallly\ and\ rest\ amount\ paid\ continuosly\}$$

$$E_{16} = \{t \mid t_d < M \le t_1, W_1 < Qc, I_e < I_p \le I_E,\ partiallly\ and\ rest\ amount\ in\ \sec ond\ shippment\}$$

$$E_{17} = \{t \mid t_d < M \le t_1, W_1 < Qc, I_e < I_p \le I_E,\ and\ full\ amount\ made\ after\ t = M\}$$

$$E_{18} = \{t \mid t_d < M \le t_1, W_1 \ge Qc,\ and\ I_e < I_p \le I_E\}$$

$$E_{19} = \{t \mid t_1 < M \le T\ \ and\ I_e < I_p \le I_E\}$$

$$E_{20} = \{t \mid T \le M\ \ and\ I_e < I_p \le I_E\}$$

$$E_{21} = \{t \mid M \le T\ \ and\ I_p \le I_e < I_E\}$$

$$E_{22} = \{t \mid T \le M\ \ and\ I_p \le I_e < I_E\} \tag{40}$$

Define the characteristic functions as

$$\phi_j(t) = \begin{cases} 1 & t \in E_j \\ 0 & t \in E_j^c \end{cases} \qquad j = 1, ..., 22, \tag{41}$$

Moreover, let

$$H_1 = \frac{1}{T}\left(A + Hc + Sc\right) \tag{42}$$

$$H_{k+1} = X_k \phi_k(t),\ \ k = 1,\ ..., 22 \tag{43}$$

Where

$$\begin{aligned}&X_1=TP_{1.1.1.1.(a)}(.)-H_1, X_2=TP_{1.1.1.1.(b)}(.)-H_1, X_3=TP_{1.1.1.2}(.)-H_1, X_4=TP_{1.1.2}(.)-H_1, X_5=TP_{1.2.1.1.(a)}(.)-H_1,\\&X_6=TP_{1.2.1.1.(b)}(.)-H_1, X_7=TP_{1.2.1.2}(.)-H_1, X_8=TP_{1.2.2}(.)-H_1, X_9=TP_{1.3}(.)-H_1, X_{10}=TP_{1.4}(.)-H_1,\\&X_{11}=TP_{2.1.1.1.(a)}(.)-H_1, X_{12}=TP_{2.1.1.1.(b)}(.)-H_1, X_{13}=TP_{2.1.1.2}(.)-H_1, X_{14}=TP_{2.1.2}(.)-H_1,\\&X_{15}=TP_{2.2.1.1(a)}(.)-H_1, X_{16}=TP_{2.2.1.1(b)}(.)-H_1, X_{17}=TP_{2.2.1.2}(.)-H_1, X_{18}=TP_{2.2.2}(.)-H_1,\\&X_{19}=TP_{2.3}(.)-H_1, X_{20}=TP_{2.4}(.)-H_1, X_{21}=TP_{3.1}(.)-H_1, X_{22}=TP_{3.2}(.)-H_1\end{aligned} \tag{44}$$

Where,

$$\begin{aligned}&TP_{2.2.1.1(a)}(.)=TP_{2.1.1.1.(a)}(.)=TP_{1.2.1.1.(a)}(.)=TP_{1.1.1.1.(a)}(.), TP_{2.2.1.1(b)}(.)=TP_{2.1.1.1.(b)}(.)=TP_{1.2.1.1.(b)}(.)=TP_{1.1.1.1.(b)}(.),\\&TP_{2.2.1.2}(.)=TP_{2.1.1.2}(.)=TP_{1.2.1.2}(.)=TP_{1.1.1.2}(.), TP_{1.2.2}(.)=TP_{1.1.2}(.),=TP_{1.3}(.),=TP_{1.4}(.), TP_{2.2.2}(.)=TP_{2.1.2}(.)\end{aligned} \tag{45}$$

i.e., $X_{15}=X_{11}=X_5=X_1, X_{16}=X_{12}=X_6=X_2, X_{17}=X_{13}=X_7=X_3, X_8=X_4, X_{18}=X_{14}$

In this paper, we use the similar methodology/approach as of Chen and Ouyang [48]; we can obtain a collective form of the total cost per unit time in all cases as follows:

$$TP(\mu,t_1,T)=\left(\left(\sum_{k=1}^{22}H_{k+1}\right)-H_1\right) \tag{46}$$

3.4. The Proposed Fuzzy Model

In this section, we formulated the fuzzy model of an above-discussed crisp model. In order to show the fuzzy performance rates and the fuzzy availabilities of the components, fuzzy triangular numbers defined as follows are used. Let $\tilde{K}=(k_1,k_2,k_3)$ where $k_1<k_2<k_3$ and defined on $R\in(-\infty,\ \infty)$, is called a triangular fuzzy number if its membership function is

$$\mu_{\tilde{K}}(x)=\begin{cases}\dfrac{x-k_1}{k_2-k_1}, & if \quad k_1\le x\le k_2\\[2mm] \dfrac{k_3-x}{k_3-k_2}, & if \ \ k_2\le x\le k_3\\[2mm] 0, & otherwise\end{cases}$$

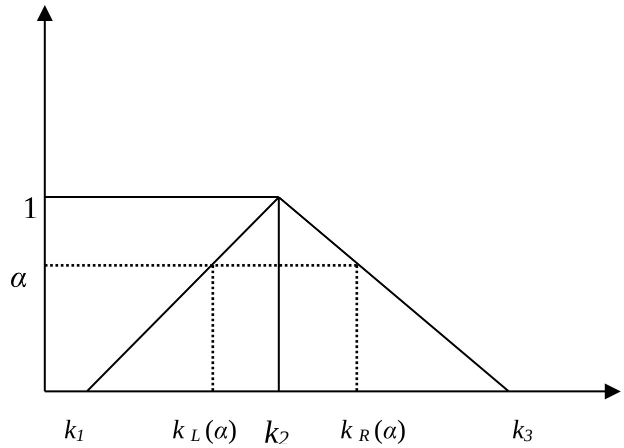


Figure 20: α-cut of a triangular fuzzy number

When $k_1 = k_2 = k_3 = k$, the fuzzy point reduces to $\tilde{k} = (k, k, k)$.

The family of all fuzzy triangular numbers on R is denoted as

$$F_N = \left\{ (k_1, k_2, k_3) \;\middle|\; k_1 < k_2 < k_3 \;\forall\; k_1, k_2, k_3 \in R \right\}$$

The α -cut of $\tilde{K} = (k_1, k_2, k_3) \in F_N$, $0 \le \alpha \le 1$, is $K(\alpha) = \left[K_L(\alpha), K_R(\alpha) \right]$.

Where $K_L(\alpha) = k_1 + (k_2 - k_1)\alpha$ and $K_R(\alpha) = k_3 - (k_3 - k_2)\alpha$ are the left and right endpoints of $K(\alpha)$.

We use the signed distance method to defuzzify the fuzzy triangular numbers. Let $\tilde{K} = (k_1, k_2, k_3)$ is a triangular fuzzy number then the signed distance from $\tilde{K}$ to 0 is defined as

$$d(\tilde{K}\tilde{0}) = \int_0^1 d\left(\left[k_L(\alpha)_\alpha, K_R(\alpha)_\alpha \right], \tilde{0} \right) = \frac{1}{4}\left(k_1 + 2k_2 + k_3 \right)$$

In the real business environment, the decision maker would not easily determine the exact value of the parameters. Thus, the decision maker determines that the approximate value of the parameters, i.e., near about the exact value for that cause we assume two parameters of demand function and deterioration rate in a fuzzy environment, i.e., $D(p) = \tilde{D}(p) = \tilde{a} - \tilde{b}(p)$, and $\theta = \tilde{\theta}$. Put these values, and $\tilde{\theta} = (\theta - \Delta_3,\ \theta,\ \theta + \Delta_3)$ in (43), then the crisp model is converted into a fuzzy model, i.e.

$$TP(\mu, t_1, T) = \left(\left(\sum_{k=1}^{22} \tilde{H}_{k+1} \right) - \tilde{H}_1 \right) \tag{47}$$

Since demand function is a triangular fuzzy number, so that AP is also triangular fuzzy number *i.e.*

$$TP(\mu, t_1, T) = \left(TP_1, TP_2, TP_3 \right) \tag{48}$$

Where $TP_i = \left(\sum_{k=1}^{14} H_{(k+1)_i} \right) - H_{1_{5-i}}$, $i = 1, 2, 3;$

$$\tilde{H}_k = \left(H_{k_1}, H_{k_2}, H_{k_3}\right) \;\&\; k = 1, \ldots, 23 \tag{49}$$

$$H_{1_i} = \frac{1}{T}\left(A + h\left(S_1 t_d - \frac{D_{(4-i)} t_d^2}{2} + \left(-\frac{D_{(4-i)}}{\theta_i}(t_1 - t_d) + \frac{D_i}{\theta_{(4-i)}^2}\left(e^{\theta_i (t_1 - t_d)} - 1\right)\right)\right) + \frac{\pi D_i (T - t_1)^2}{2} \right) \tag{50}$$

$$H_{2_i} = \frac{1}{T}\left[\begin{array}{l} D_i\left(t_1 - B_{1_{(4-i)}}\right)p + \frac{1}{2}D_i\left(t_1 - B_{1_{(4-i)}}\right)^2 pI_e + D_i p\left(t_1 - B_{1_{(4-i)}}\right) \\ \left\{1 + \frac{1}{2}\left(t_1 - B_{1_{(4-i)}}\right)I_e\right\}(T - t_1)I_E \end{array}\right]\phi_1(t) \tag{51}$$

$$H_{3_i} = \frac{1}{T}\left[D_i\left(t_1 - B_{2_{(4-i)}}\right)p + \frac{1}{2}D_i\left(t_1 - B_{2_{(4-i)}}\right)^2 pI_e + D_i p\left(t_1 - B_{2_{(4-i)}}\right)\left(1 + \frac{1}{2}\left(t_1 - B_{2_{(4-i)}}\right)I_e\right)\left(T - t_1\right)I_E \right]\phi_2(t) \tag{52}$$

$$H_{4_i} = \frac{1}{T}\left[\begin{array}{l} D_i\left(t_1 - B_{3_{(4-i)}}\right)p + \frac{1}{2}D_i\left(t_1 - B_{3_{(4-i)}}\right)^2 pI_e + \\ \left(D_i\left(t_1 - B_{3_{(4-i)}}\right)p + \frac{1}{2}D_i\left(t_1 - B_{3_{(4-i)}}\right)^2 pI_e\right)\left(T - t_1\right)I_E \end{array}\right]\phi_3(t) \tag{53}$$

$$H_{5_i} = \frac{1}{T}\left[\begin{array}{l} D_i\left(t_1 - M\right)p + \frac{1}{2}D_i\left(t_1 - M\right)^2 pI_e + \left(W_{1_i} - Q_{(4-i)}c\right)\left\{1 + \left(T - M\right)I_E\right\} \\ + \left(D_i\left(t_1 - M\right)p + \frac{1}{2}D_i\left(t_1 - M\right)^2 pI_e\right)\left(T - t_1\right)I_E \end{array}\right]\varphi_4(t) \tag{54}$$

$$H_{6_i} = \frac{1}{T}\left[\begin{array}{l} D_i\left(t_1 - B_{1_{(4-i)}}\right)p + \frac{1}{2}D_i\left(t_1 - B_{1_{(4-i)}}\right)^2 pI_e + D_i p\left(t_1 - B_{1_{(4-i)}}\right) \\ \left\{1 + \frac{1}{2}\left(t_1 - B_{1_{(4-i)}}\right)I_e\right\}(T - t_1)I_E \end{array}\right]\phi_5(t) \tag{55}$$

$$H_{7_i} = \frac{1}{T}\left[\begin{array}{l} D_i\left(t_1 - B_{2_{(4-i)}}\right)p + \frac{1}{2}D_i\left(t_1 - B_{2_{(4-i)}}\right)^2 pI_e + D_i p\left(t_1 - B_{2_{(4-i)}}\right) \\ \left(1 + \frac{1}{2}\left(t_1 - B_{2_{(4-i)}}\right)I_e\right)\left(T - t_1\right)I_E \end{array}\right]\phi_6(t) \tag{56}$$

$$H_{8_i} = \frac{1}{T}\left[\begin{array}{l} D_i\left(t_1 - B_{3_{(4-i)}}\right)p + \frac{1}{2}D_i\left(t_1 - B_{3_{(4-i)}}\right)^2 pI_e + \\ \left(D_i\left(t_1 - B_{3_{(4-i)}}\right)p + \frac{1}{2}D_i\left(t_1 - B_{3_{(4-i)}}\right)^2 pI_e\right)\left(T - t_1\right)I_E \end{array}\right]\phi_7(t) \tag{57}$$

$$H_{9_i} = \frac{1}{T}\left[\begin{array}{l} D_i\left(t_1 - M\right)p + \frac{1}{2}D_i\left(t_1 - M\right)^2 pI_e + \left(W_{1_i} - Q_{(4-i)}c\right)\left\{1 + \left(T - M\right)I_E\right\} \\ + \left(D_i\left(t_1 - M\right)p + \frac{1}{2}D_i\left(t_1 - M\right)^2 pI_e\right)\left(T - t_1\right)I_E \end{array}\right]\varphi_8(t) \tag{58}$$

$$H_{10_i} = \frac{1}{T}\left[\left(W_{2_i} - Q_{(4-i)}c\right) + \left(W_{2_i} - Q_{(4-i)}c\right)(T - M)I_E\right]\phi_9(t) \tag{59}$$

$$H_{11_i} = \frac{1}{T}\left[\left(W_{3_i} - Q_{(4-i)}c\right)\right]\phi_{11}(t) \tag{60}$$

$$H_{12_i} = \frac{1}{T}\begin{bmatrix} D_i\left(t_1 - B_{1_{(4-i)}}\right)p + \frac{1}{2}D_i\left(t_1 - B_{1_{(4-i)}}\right)^2 pI_e + D_i p\left(t_1 - B_{1_{(4-i)}}\right) \\ \left\{1 + \frac{1}{2}\left(t_1 - B_{1_{(4-i)}}\right)I_e\right\}(T - t_1)I_E \end{bmatrix}\phi_{11}(t) \tag{61}$$

$$H_{13_i} = \frac{1}{T}\begin{bmatrix} D_i\left(t_1 - B_{2_{(4-i)}}\right)p + \frac{1}{2}D_i\left(t_1 - B_{2_{(4-i)}}\right)^2 pI_e + D_i p\left(t_1 - B_{2_{(4-i)}}\right) \\ \left(1 + \frac{1}{2}\left(t_1 - B_{2_{(4-i)}}\right)I_e\right)(T - t_1)I_E \end{bmatrix}\phi_{12}(t) \tag{62}$$

$$H_{14_i} = \frac{1}{T}\begin{bmatrix} D_i\left(t_1 - B_{1_{(4-i)}}\right)p + \frac{1}{2}D_i\left(t_1 - B_{1_{(4-i)}}\right)^2 pI_e + D_i p\left(t_1 - B_{1_{(4-i)}}\right) \\ \left\{1 + \frac{1}{2}\left(t_1 - B_{1_{(4-i)}}\right)I_e\right\}(T - t_1)I_E \end{bmatrix}\phi_{13}(t) \tag{63}$$

$$H_{15_i} = \frac{1}{T}\begin{bmatrix} D_i(t_1 - M)p + \frac{1}{2}D_i(t_1 - M)^2 pI_e + D_i(t_1 - M)p\left(1 + \frac{1}{2}D_i(t_1 - M)I_e\right) \\ (T - t_1)I_E + W_{1_i}\{1 + I_E(T - M)\} - Q_{(4-i)}c\left\{1 + (T - M)I_p\right\} \end{bmatrix}\varphi_{14}(t) \tag{64}$$

$$H_{16_i} = \frac{1}{T}\begin{bmatrix} D_i\left(t_1 - B_{1_{(4-i)}}\right)p + \frac{1}{2}D_i\left(t_1 - B_{1_{(4-i)}}\right)^2 pI_e + D_i p\left(t_1 - B_{1_{(4-i)}}\right) \\ \left\{1 + \frac{1}{2}\left(t_1 - B_{1_{(4-i)}}\right)I_e\right\}(T - t_1)I_E \end{bmatrix}\phi_{15}(t) \tag{65}$$

$$H_{17_i} = \frac{1}{T}\begin{bmatrix} D_i\left(t_1 - B_{2_{(4-i)}}\right)p + \frac{1}{2}D_i\left(t_1 - B_{2_{(4-i)}}\right)^2 pI_e + D_i p\left(t_1 - B_{2_{(4-i)}}\right) \\ \left(1 + \frac{1}{2}\left(t_1 - B_{2_{(4-i)}}\right)I_e\right)(T - t_1)I_E \end{bmatrix}\phi_{16}(t) \tag{66}$$

$$H_{18_i} = \frac{1}{T}\begin{bmatrix} D_i\left(t_1 - B_{3_{(4-i)}}\right)p + \frac{1}{2}D_i\left(t_1 - B_{3_{(4-i)}}\right)^2 pI_e + \\ \left(D_i\left(t_1 - B_{3_{(4-i)}}\right)p + \frac{1}{2}D_i\left(t_1 - B_{3_{(4-i)}}\right)^2 pI_e\right)(T - t_1)I_E \end{bmatrix}\phi_{17}(t) \tag{67}$$

$$H_{19_i} = \frac{1}{T}\begin{bmatrix} D_i(t_1 - M)p + \frac{1}{2}D_i(t_1 - M)^2 pI_e + D_i(t_1 - M)p\left(1 + \frac{1}{2}D_i(t_1 - M)I_e\right) \\ (T - t_1)I_E + W_{1_i}\{1 + I_E(T - M)\} - Q_{(4-i)}c\left\{1 + (T - M)I_p\right\} \end{bmatrix}\varphi_{18}(t) \tag{68}$$

$$H_{20_i} = \left[W_2\left\{1 + (T - M)I_E\right\} - Q_{(4-i)}c\left\{1 + (T - M)I_p\right\}\right]\varphi_{19}(t) \tag{69}$$

$$H_{21_i} = \frac{1}{T}\left(W_{3_i} - Q_i c\right)\varphi_{20}(t) \tag{70}$$

$$H_{22_i} = \frac{1}{T}\left[\begin{array}{l} D_i Tp + \frac{1}{2}D_i t_1^2 pI_e + D_i(T-t_1)pTI_E + \left(D_i t_1 p + \frac{1}{2}D_i t_1^2 pI_e\right)(T-t_1)I_E \\ -Q_{(4-i)}c\left\{1+(T-M)I_p\right\} \end{array}\right]\phi_{21}(t) \tag{71}$$

$$H_{23_i} = \frac{1}{T}\left[D_i Tp + D_i(T-t_1)pMI_E + \frac{1}{2}D_i t_1^2 pI_e + D_i t_1 p\left(1+\frac{1}{2}t_1 I_e\right)(M-t_1)I_E - Q_{(4-i)}c\right]\varphi_{22}(t) \tag{72}$$

$$B_{1_i} = M + \frac{2\left(cQ_i - W_{1_{(4-i)}}\right)}{2D_{(4-i)}p - \left(cQ_i - W_{1_{(4-i)}}\right)I_p} \tag{73}$$

$$B_{2_i} = \frac{1}{D_{(4-i)}pI_e}\left\{D_i pMI_e + Q_i cI_p - D_{(4-i)}p - I_p W_{1_{(4-i)}} + \left(\left(D_i p - Q_{(4-i)}cI_p + W_{1_i}I_e\right)^2 + 2D_i p(Q_i c - W_{1_{(4-i)}})I_e\right)^{\frac{1}{2}}\right\} \tag{74}$$

$$B_{3_i} = \frac{1}{D_{(4-i)}pI_e}\left\{D_i pMI_e + Q_i cI_p - D_{(4-i)}p - W_{1_{(4-i)}}I_E + \left(\left(D_i p + W_{1_i}I_E - Q_{(4-i)}cI_p\right)^2 + 2DpI_e\left(Q_i c - W_{1_{(4-i)}}\right)\right)^{\frac{1}{2}}\right\} \tag{75}$$

$$W_{1_i} = D_i p\left[(T-t_1) + M\left\{1+(T-t_1)I_E + \frac{1}{2}MI_e\right\}\right] \tag{76}$$

$$W_{2_i} = D_i p(T-t_1)(1+MI_E) + D_i pt_1\left(1+\frac{1}{2}t_1 I_e\right)\left(1+(M-t_1)I_E\right) \tag{77}$$

$$W_{3_i} = D_i p(T-t_1)(1+MI_E) + D_i pt_1\left(1+\frac{1}{2}t_1 I_e\right)\left(1+(M-t_1)I_e\right) \tag{78}$$

$$Q_i = D_i\left(T + t_d - t_1 + \frac{1}{\theta_{(4-i)}}\left(e^{\theta_i(t_1-t_d)}-1\right)\right),\ D_i = a_i - pb_{(4-i)} \text{ and } p = \mu c \tag{79}$$

$$Hc_i = h\left(S_1 t_d - \frac{D_{(4-i)}t_d^2}{2} + \left(-\frac{D_{(4-i)}}{\theta_i}(t_1-t_d) + \frac{D_i}{\theta_{(4-i)}^2}\left(e^{\theta_i(t_1-t_d)}-1\right)\right)\right) \tag{80}$$

$$Sc_i = \frac{\pi D_i(T-t_1)^2}{2} \tag{81}$$

Now defuzzify the fuzzy profit function by, using signed distance method, measured from TP to $\tilde{0}$

$$TP_d(\mu,t_1,T) = \frac{1}{4}\left\{TP_1 + 2TP_2 + TP_3\right\} \tag{82}$$

The necessary conditions for the total profit to be maximum is

$$\frac{\partial TP_d(\mu,t_1,T)}{\partial\mu} = 0 \tag{83}$$

$$\frac{\partial TP_d(\mu,t_1,T)}{\partial t_1} = 0 \tag{84}$$

$$\frac{\partial TP_d\left(\mu,t_1,T\right)}{\partial T}=0 \tag{85}$$

Equations (83), (84), and (85) can be solved simultaneously for the optimal values of μ, t_1 and T (say μ^*, t_1^* and T^*) which satisfies the sufficient conditions also.

4. Theoretical Results & Theorems for the optimal solution

Now, in this section, we discuss the theoretical aspects of our proposed model for the crisp case. In this paper, we incorporate a similar concept for proving the optimality as used by Chen et al. [49]. Here, to solve the problem, we apply the existing theoretical result in concave fractional programming. If $f(x)$ is non-negative, differentiable and (strictly) concave, and $g(x)$ is positive, differentiable and convex, then the real-valued function

$$z\left(x\right)=\frac{f\left(x\right)}{g\left(x\right)} \tag{86}$$

Is (strictly) pseudo-concave. For detailed proof, please see Cambini and Martein [52].

For simplicity, let us define

$$J=\left[Dp\left\{\left\{1+(T-t_1)I_E\right\}\frac{d^2B_1}{dT^2}\left\{1+(t_1-B_1)I_e\right\}-I_e\left(\frac{dB_1}{dT}\right)^2\right\}+2I_e\frac{dB_1}{dT}\left\{1+(t_1-B_1)I_e\right\}+\pi D\right]$$

Without loss of generality, we assume that $J>0$. Given μ and t_1, applying the Cambini and Martein [52] result in concave fractional programming, we can prove that the retailer's total annual profit $TP_{(1.1.1.1(a))}(\mu,t_1,T)$ is a strictly pseudo-concave function in T if $J>0$. This implies that there exists a unique global optimal solution T^* such that $TP_{(1.1.1.1(a))}(\mu,t_1,T)$ is maximized.

Theorem 1: Given mark-up rate μ and the time at which inventory level is positive t_1, if $J>0$, then $TP_{(1.1.1.1(a))}(\mu,t_1,T)$ is a strictly pseudo-concave function in T, and there exists a unique solution T^*.

Proof: See Appendix A and B.

Corollary 1: Given the mark-up rate μ and the time at which inventory level is positive t_1,

(i). W_1 is increasing in T

(ii). B_1 is decreasing in T and convex in T.

Proof: See in (A.1), (A.2), (A.6) and (A.7).

Theorem 2: Given replenishment cycle time T, *if* $L<0$, $M<0$, *and* $LM-K^2>0$, *then* $TP_{(1.1.1.1(a))}(\mu, t_1, T)$ is a strictly concave function in both μ and t_1, and hence there exist a unique solution μ^* and t_1^*.

Proof: See Appendix C and D.

Corollary 2: Given replenishment cycle time T,

(i). W_1 is increasing in μ but decreasing in t_1 and

(ii). B_1 is increasing in and concave in both μ and t_1.

Proof: It is clear from (C.24), (C.25), (C.26) & (C.27).

Now, in a similar direction, we can prove the optimality for other cases also.

5. Solution algorithm, numerical example, and sensitivity analysis

5.1. Solution algorithm

The procedure for finding the economic ordering policy in section1, i.e. ($I_e < I_E \leq I_p$) is as follows:

Step 1: For event E_1, determine μ^*, t_1^* and T^* from equation (83), (84) and (85). If μ^*, t_1^* and T^* are in E_1 then calculate TP_d (μ^*, t_1^*, T^*) from (82), this gives $TP_{(d)\,1.1.1.1.(a)}$ (μ^*, t_1^*, T^*). Otherwise, go to step 2.

Step 2: For event E_2, determine μ^*, t_1^* and T^* from equation (83), (84) and (85). If μ^*, t_1^*, and T^* are in E_2 then calculate TP_d (μ^*, t_1^*, T^*) from (82), this gives $TP_{(d)1.1.1.1.(b)}$ (μ^*, t_1^*, T^*). Otherwise, go to step 3.

Step 3: For event E_3, determine μ^*, t_1^* and T^* from equation (83), (84) and (85). If μ^*, t_1^* and T^* are in E_3 then calculate TP_d (μ^*, t_1^*, T^*) from (82), this gives $TP_{(d)1.1.1.2}$ (μ^*, t_1^*, T^*). Otherwise, go to step 4.

Step 4: For event E_4, determine μ^*, t_1^* and T^* from equation (83), (84) and (85). If μ^*, t_1^* and T^* are in E_4 then calculate TP_d (μ^*, t_1^*, T^*) from (82), this gives $TP_{(d)1.1.2}$ (μ^*, t_1^*, T^*). Otherwise, go to step 5.

Step 5: For event E_5, determine μ^*, t_1^* and T^* from equation (83), (84) and (85). If μ^*, t_1^* and T^* are in E_5 then calculate TP_d (μ^*, t_1^*, T^*) from (82), this gives $TP_{(d)1.2}$ (μ^*, t_1^*, T^*). Otherwise, go to step 6.

Step 6: For event E_6, determine μ^*, t_1^* and T^* from equation (83), (84) and (85). If μ^*, t_1^* and T^* are in E_6 then calculate TP_d (μ^*, t_1^*, T^*) from (82), this gives $TP_{(d)1.3}$ (μ^*, t_1^*, T^*). Otherwise, go to step 7.

Step 7: For event E_7, determine μ^*, t_1^* and T^* from equation (83), (84) and (85). If μ^*, t_1^* and T^* are in E_7 then calculate TP_d (μ^*, t_1^*, T^*) from (82), this gives $TP_{(d)1.1.1.2}$ (μ^*, t_1^*, T^*). Otherwise, go to step 8.

Step 8: For event E_8, determine μ^*, t_1^* and T^* from equation (83), (84) and (85). If μ^*, t_1^* and T^* are in E_8 then calculate TP_d (μ^*, t_1^*, T^*) from (82), this gives $TP_{(d)1.1.2}$ (μ^*, t_1^*, T^*). Otherwise, go to step 9.

Step 9: For event E_9, determine μ^*, t_1^* and T^* from equation (83), (84) and (85). If μ^*, t_1^* and T^* are in E_9 then calculate TP_d (μ^*, t_1^*, T^*) from (82), this gives $TP_{(d)1.2}$ (μ^*, t_1^*, T^*). Otherwise, go to step 10.

Step 10: For event E_{10}, determine μ^*, t_1^* and T^* from equation (83), (84), and (85). If μ^*, t_1^* and T^* are in E_{10} then calculate TP_d (μ^*, t_1^*, T^*) from (82), this gives $TP_{(d)1.3}$ (μ^*, t_1^*, T^*). Otherwise, go to step 11.

Step 11: Terminate.

The optimal average profit for Case 1, $TP_{d(1)}$ (μ^*, t_1^*, T^*) is associated with maximum average profit per unit time in getting in Step 1,…,10. Similarly, for section 2 and 3, we get the optimal average profit $TP_{d(2)}$ (μ^*, t_1^*, T^*), and $TP_{d(3)}$ (μ^*, t_1^*, T^*). The optimal solution of the inventory system is associated with the maximum average profits in all sections. Hence, the optimal average profit of the system is given by

TP_d (μ^*,t_1^*, T^*) = max [$TP_{d(1)}$ (μ^*, t_1^*, T^*), $TP_{d(2)}$ (μ^*, t_1^*, T^*), $TP_{d(3)}$ (μ^*,t_1^*, T^*)]

5.2. Numerical examples

The proposed model of the inventory system has been illustrated with the help of two hypothetical numerical examples, and the corresponding data have been depicted in Table 2. Both the examples have been solved by using the proposed algorithm to determine the optimal values of mark-up rate (μ), selling price (p), Breakeven point (B_i), cycle length (T), ordering quantity (Q) along with the optimal profit of the system for all the possible cases and sub-cases.

Table 2: Values of parameters of different examples

Example	A($)	c($)	h($)	π($)	θ	a	b	t_d	I_e (per $/year)	I_E (per $/year)	I_p (per $/year)	M (years)
1. $I_e < I_E \le I_p$	200	100	10	50	0.1	150	0.8	0.2	0.12	0.14	0.15	30/365 =0.082
2. $I_e \le I_p < I_E$	200	100	10	50	0.1	150	0.8	0.2	0.12	0.18	0.15	30/365 =0.082
3. $I_p < I_e < I_E$	200	100	10	50	0.1	150	0.8	0.2	0.18	0.2	0.15	30/365 =0.082

Using the proposed algorithm, the results are as follows:

For $I_e < I_E \le I_p$

μ^*=1.49, t_1^*=0.76 year, T^*=1.47 year, B^*=0.34 year, W^*=3759.71, Q^*= 46 units and Total profit= $1348.76 (Scenario 1.1.1.1.a)

For $I_e \le I_P < I_E$

μ^*=1.46, t_1^*=0.87 year, T^*=1.31 year, B^*=0.45 year, W^*=3105.62, Q^*= 71 units and Total profit= $1291.38 (Scenario 2.1.1.1.a)

For $I_p < I_e < I_E$

μ^*=1.36, t_1^*= 0.082 year = M, T^*= 0.75 year, B^*=0.63 year, Q^*= 46 units and Total profit= $1078.47 (Scenario 3.1)

5.3. Sensitivity analysis

In this subsection, we study the effects of changes in different parameters on the optimal policies. The results of these analyses have been displayed in Table 3.

Table 3: Sensitivity analysis with different parameters of the inventory system

Parameters	%change of parameters	% change in					
		μ	t_1	T	B	Q	Profit
A	-20%	-1.12	-21.07	-25.09	-28.62	-28.45	4.67
	-10%	-0.87	-15.54	-12.78	-14.34	-12.82	1.29
	10%	0.83	21.18	27.18	26.95	14.81	-1.06
	20%	1.57	37.61	39.23	33.61	25.31	-1.95
Δ_1	-20%	-15.23	21.05	26.87	42.23	-35.08	-67.34
	-10%	-6.23	10.57	11.75	23.05	-15.69	-41.56
	10%	6.13	-7.34	-9.23	-15.86	15.02	51.73
	20%	12.53	-11.46	-12.45	-23.34	27.05	92.78

Δ_2	-20%	4.04	38.67	58.53	15.07	53.91	17.89
	-10%	3.09	32.45	34.04	-23.09	32.76	13.57
	10%	-3.18	-31.83	-34.98	-33.11	-33.05	-13.90
	20%	-3.92	-37.97	-57.31	9.87	-50.61	-17.07
h	-20%	-0.14	4.76	6.48	7.47	7.23	3.72
	-10%	-0.08	2.48	3.54	3.58	3.47	1.53
	10%	0.49	-2.36	-2.96	-3.08	-3.26	-1.71
	20%	0.73	-4.73	-6.53	-5.87	-6.47	-3.21
c	-20%	16.47	3.45	5.23	-28.51	32.72	51.87
	-10%	7.42	2.45	2.57	-13.67	15.33	29.25
	10%	-5.83	-4.73	-2.34	13.26	-14.86	-32.84
	20%	-12.55	-6.39	-5.69	27.14	-35.56	-68.15
π	-20%	13.46	12.36	19.64	9.71	31.38	2.89
	-10%	9.25	5.46	9.58	6.83	16.22	1.47
	10%	-0.45	-9.31	-12.68	-5.14	-11.12	-1.28
	20%	-0.69	-13.68	-19.55	-6.60	-17.34	-2.31
M	-20%	0.06	0.87	1.43	1.34	-1.32	-1.07
	-10%	0.03	0.42	0.74	0.67	-0.65	-0.63
	10%	-0.01	-0.56	-0.48	-0.71	0.57	0.51
	20%	-0.02	-1.03	-0.94	-1.47	0.99	1.05
Δ_3	-20%	0.08	9.23	10.17	7.47	-0.081	31.80
	-10%	0.05	5.89	6.90	5.23	-0.078	23.53
	10%	-0.14	-12.45	-13.43	9.55	0.083	-46.40
	20%	-0.18	-17.45	-16.42	11.47	0.086	-58.29
t_d	-20%	-047	-24.73	-31.08	-23.62	-0.15	-18.04
	-10%	-0.53	-33.41	-33.93	-22.45	-0.12	-11.42
	10%	0.61	35.67	36.02	21.33	0.12	13.51
	20%	0.71	41.12	40.23	25.37	0.13	23.36

Following observation & insights have been drawn from Table 3, the following inferences can be made:

- One can easily observe from the Table 3, as the ordering cost (*A*) increases the optimal cycle length (*T*), optimal order quantity (*Q*) increases and there is a significant rise in selling price (*p*) resulting in the decrease of total optimal profit (*TP*).
- If we increase the fuzziness (Δ_1) in the value of (*a*), the total profit (*TP*) increases whereas if there is an increase in the fuzziness (Δ_2) of (*b*), the total profit (*TP*) decreases.

- It can be observed from Table 3 as the cost per unit (c) increases, the optimal cycle length (T) and selling price (p) increases, which results in a decrement of total optimal profit (TP). This reveals the natural trend of cost-profit analysis.
- With the increase in the holding cost, i.e. (h), optimal cycle length (T), optimal order quantity (Q) increases and the selling price (p), the total optimal profit per unit time (TP) decreases but there is an increase in holding cost.
- From Table 3 it is visible that in all cases, as the length of credit period M increases, both optimal order quantity (Q), optimal replenishment cycle time (T) and total optimal profit per unit time (TP) increase. This suggests that if the permissible delay period increases, then it help the retailer to prolong the payments to the supplier without penalty, which indirectly reduces the costs incurred by the retailer, and eventually results in higher profits.
- It is observed from Table 3, as the fraction of deterioration rate (Δ_3) increases, there is significant decrease in total optimal profit (TP) and increase in order quantity (Q) because a rise in deterioration rate (θ) causes an increase in the cost of deteriorated units, which ultimately increase the total cost.
- From the Table 3 it is apparent that, with an increase in the value of non-deteriorating period (t_d), the cycle length (T), order quantity (Q) and total optimal profit (TP) increases. This indicates the positive impact of non-instantaneous deteriorating items in inventory modelling. As the period for non-deterioration (t_d) increases, the deterioration cost for items decreases, which accounts for larger profits for the company.

6. Conclusion and future research direction

In the earlier inventory models on inventory modelling under the conditions of permissible delay in payments, scholars have assumed that the retailers have to settle their accounts at the end of the credit period, i.e., the supplier accepts only full amount at the end of the credit period, which doesn't fit the real circumstances. In reality, either the supplier may accept the partial amount at the end of the credit period and unpaid balance subsequently or the full amount at a fixed point of time after the expiry of the credit period, if the retailer finances the inventory from the supplier itself. This issue motivated us to incorporate the above-mentioned realistic scenario. In the current paper, we incorporate the condition in which, the supplier accepts the partial payment at the end of the credit period and the reaming amount after that period under the term and condition. The main feature of the alternative trade credit approach is the extension of the trade credit period by considering the above realistic possibilities.

In the current paper, we considered the interest earned (I_E) on the fixed deposit amount, i.e., revenue generated by fulfilling the shortage, balance amount, after settling the account is higher than that of usual interest rate (I_e). Hence, the objective of this study is to determine the retailer's optimal policies that maximize the total profit. Also, some theoretical results are obtained, which shows that the optimal solution not only exists, it is unique also. The impact of the new proposed credit policy is investigated on the optimality of the solution for the non-instantaneous deteriorating products. The validation of the proposed model and its solution method is demonstrated through the numerical example. The outcomes suggest significant importance of the proposed inventory model and its solution method to the retail managers under real-world situations. Results demonstrate that it is essential for the managers to consider the inclusion of new proposed credit policy significantly increases the net annual profit.

For future research, it would be interesting to extend the present model under two-level trade credit policy. The model may also be explored for a two warehousing inventory system. Another possible direction may be developed by integrating different forms of trade credit decisions in the present model. The study may be extended by considering the effects of environmental impact during shipment. For future research, it would be interesting to study the present model under different practical parameters like inflation, the multi-product case, and the multi-stage supply chain. The study may be extended for different demand functions viz., price and time-dependent demand, etc. Furthermore, the learning effect may be considered to achieve a more realistic scenario while developing an inventory model.

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Appendix – A:

Given μ & t_1, taking the first and second-order derivative of equations (10) and (12) with respect to *T*.

$$\frac{dW_1}{dT} = Dp\left(1+MI_E\right) > 0 \tag{A.1}$$

$$\frac{d^2W_1}{dT^2} = 0 \tag{A.2}$$

$$\frac{dQ}{dT} = D \tag{A.3}$$

$$\frac{d^2Q}{dT^2} = 0 \tag{A.4}$$

$$\frac{dp}{dT} = 0, \ \& \ \frac{d^2p}{dT^2} = 0 \tag{A.5}$$

$$\frac{dB_1}{dT} = \frac{2\left\{Dc - Dp\left(1+MI_E\right)\right\}}{\left\{2Dp - \left(cQ - W_1\right)I_p\right\}^2}\left\{2Dp + \left(cQ - W_1\right)I_p\right\} < 0 \tag{A.6}$$

$$\frac{d^2B_1}{dT^2} = \frac{2\left\{Dc - Dp\left(1+MI_E\right)\right\}^2}{\left\{2Dp - \left(cQ - W_1\right)I_p\right\}^3} I_p\left\{2Dp + \left(cQ - W_1\right)I_p\right\} > 0 \tag{A.7}$$

$$\frac{dA}{dT} = 0, \ \& \ \frac{d^2A}{dT^2} = 0 \tag{A.8}$$

$$\frac{dS_1}{dT}=0,\ \&\ \frac{d^2S_1}{dT^2}=0 \tag{A.9}$$

$$\frac{dHc}{dT}=0,\ \&\ \frac{d^2Hc}{dT^2}=0 \tag{A.10}$$

$$\frac{dSc}{dT}=\pi D\left(T-t_1\right)>0,\ \&\ \frac{d^2Sc}{dT^2}=\pi D>0 \tag{A.11}$$

From (14), Let

$$y\left(T\right)=D\left(t_1-B_1\right)p\left\{1+(1/2)\left(t_1-B_1\right)I_e\right\}\left\{1+(T-t_1)I_E\right\}-A-Hc-Sc$$

and $g\left(T\right)=T>0$

Consequently, we have

$$q\left(T\right)=\frac{y\left(T\right)}{g\left(T\right)}=TP_{1.1.1.1.(a)}\left(\mu,t_1,T\right)$$

For given value of μ & t_1, taking the first and second-order derivative of $y\left(T\right)$, we get

$$\begin{aligned}y'\left(T\right)=&-Dp\frac{dB_1}{dT}\left\{1+(1/2)\left(t_1-B_1\right)I_e\right\}\left\{1+(T-t_1)I_E\right\}+D(t_1-B_1)p\left(-(1/2)\frac{dB_1}{dT}I_e\right)\left\{1+(T-t_1)I_E\right\}\\&+D(t_1-B_1)p\left\{1+(1/2)\left(t_1-B_1\right)I_e\right\}I_E-\frac{dA}{dT}-\frac{dHc}{dT}-\frac{dSc}{dT}\end{aligned} \tag{A.12}$$

Using (A.8), (A.9), (A.10) & (A.11) in equation (A.12), we get

$$y'\left(T\right)=-Dp\frac{dB_1}{dT}\left\{1+(T-t_1)I_E\right\}\left\{1+(t_1-B_1)I_e\right\}+D(t_1-B_1)pI_E\left\{1+(1/2)\left(t_1-B_1\right)I_e\right\}-\pi D\left(T-t_1\right) \tag{A.13}$$

Now, taking the derivative of (A.13) with respect to *T*, we have

$$y'\left(T\right)=-\left[Dp\left\{\left\{1+(T-t_1)I_E\right\}\left\{\frac{d^2B_1}{dT^2}\left\{1+\left(t_1-B_1\right)I_e\right\}-I_e\left(\frac{dB_1}{dT}\right)^2\right\}+2I_e\frac{dB_1}{dT}\left\{1+\left(t_1-B_1\right)I_e\right\}\right\}+\pi D\right]=-J \tag{A.14}$$

As a result, if $J>0$ then $y''\left(T\right)<0$ and hence $y\left(T\right)$ is non-negative, differentiable and strictly concave. Thus if $J>0$ then $TP_{1.1.1.1.(a)}\left(\mu,t_1,T\right)$ is a strictly pseudo-concave function in *T*, and there exist a unique optimal solution.

Appendix – B

The optimal replenishment cycle time T^*

$$TP_{1.1.1.1.(a)}\left(\mu,t_1,T\right)=\frac{y\left(T\right)}{T}$$

Hence, for the given value of μ & t_1, taking the first order derivative of $TP_{1.1.1.1.(a)}(\mu,t_1,T)$ w.r.t T and setting the result to zero, we get the necessary and sufficient condition to find T^* as follows:

$$\frac{dTP_{1.1.1.1.(a)}(\mu,t_1,T)}{dT}=\frac{y'(T)}{T}-\frac{y(T)}{T^2}=0$$

$$\Rightarrow y'(T)T-y(T)=0 \tag{B.1}$$

Since $J>0$ then necessary and sufficient condition for T^* is

$$\begin{aligned}\Rightarrow&\left[-Dp\frac{dB_1}{dT}\{1+(T-t_1)I_E\}\{1+(t_1-B_1)I_e\}+D(t_1-B_1)pI_E\{1+(1/2)(t_1-B_1)I_e\}-\pi D(T-t_1)\right]T\\&=D(t_1-B_1)p\{1+(1/2)(t_1-B_1)I_e\}\{1+(T-t_1)I_E\}-A-Hc-Sc\end{aligned} \tag{B.2}$$

Appendix – C

For any given value of T, taking the first and second order partial derivatives of equation (14) w.r.t μ & t_1.

$$\frac{\partial p}{\partial \mu}=c,\ \frac{\partial^2 p}{\partial \mu^2}=0 \tag{C.1}$$

$$\frac{\partial D}{\partial \mu}=-b\frac{\partial p}{\partial \mu}=-bc,\ \ \frac{\partial^2 D}{\partial \mu^2}=0 \tag{C.2}$$

$$\begin{aligned}&\frac{dS_1}{dT}=0,\ \frac{d^2S_1}{dT^2}=0\\&\frac{\partial S_1}{\partial t_1}=\frac{D}{\theta^2}e^{\theta(t_1-t_d)},\ \frac{\partial^2 S_1}{\partial t_1^2}=\frac{D}{\theta^3}e^{\theta(t_1-t_d)}\end{aligned} \tag{C.3}$$

$$\frac{\partial Hc}{\partial t_1}=h\left\{(\theta t_d+1)\frac{D}{\theta^3}e^{\theta(t_1-t_d)}-\frac{D}{\theta}\right\} \tag{C.4}$$

$$\frac{\partial^2 Hc}{\partial t_1^2}=h(\theta t_d+1)\frac{D}{\theta^4}e^{\theta(t_1-t_d)} \tag{C.5}$$

$$\frac{\partial Sc}{\partial t_1}=-\pi D(T-t_1),\ \frac{\partial^2 Sc}{\partial t_1^2}=\pi D \tag{C.6}$$

$$\frac{\partial Q}{\partial T}=D,\ \frac{\partial^2 A}{\partial T^2}=0 \tag{C.7}$$

$$\frac{\partial Q}{\partial t_1}=D\left\{-1+\frac{1}{\theta^2}e^{\theta(t_1-t_d)}\right\} \tag{C.8}$$

$$\frac{\partial^2 Q}{\partial t_1^2}=\frac{D}{\theta^3}e^{\theta(t_1-t_d)} \tag{C.9}$$

$$\frac{\partial S_1}{\partial \mu} = -bc\left\{t_d + \frac{1}{\theta}\left(e^{\theta(t_1-t_d)} - 1\right)\right\} \tag{C.10}$$

$$\frac{\partial^2 S_1}{\partial \mu^2} = 0 \tag{C.11}$$

$$\frac{\partial Hc}{\partial \mu} = h\left\{\frac{\partial S_1}{\partial \mu}t_d + \frac{bc}{2}t_d^2 + \frac{bc}{\theta}\left(t_1 - t_d + \frac{1}{\theta}\left(1 - e^{\theta(t_1-t_d)}\right)\right)\right\} \tag{C.12}$$

$$\frac{\partial^2 Hc}{\partial \mu^2} = 0 \tag{C.13}$$

$$\frac{\partial Sc}{\partial \mu} = -\pi bc\frac{\left(T - t_1\right)^2}{2} \tag{C.14}$$

$$\frac{\partial^2 Sc}{\partial \mu^2} = 0 \tag{C.15}$$

$$\frac{\partial^2 S_1}{\partial \mu \partial t_1} = -\frac{bc}{\theta^2}\left(e^{\theta(t_1-t_d)} - 1\right) \tag{C.16}$$

$$\frac{\partial^2 Hc}{\partial \mu \partial t_1} = h\left\{\frac{\partial^2 S_1}{\partial \mu \partial t_1}t_d + \frac{bc}{\theta}\left(1 - \frac{1}{\theta^2}\left(1 - e^{\theta(t_1-t_d)}\right)\right)\right\} \tag{C.17}$$

$$\frac{\partial^2 Sc}{\partial \mu \partial t_1} = \pi bc\left(T - t_1\right) \tag{C.18}$$

$$\frac{\partial W_1}{\partial \mu} = \left(\frac{\partial D}{\partial \mu}p + D\frac{\partial p}{\partial \mu}\right)\left\{\left(T - t_1\right) + M\left\{1 + \left(T - t_1\right)I_E + \frac{1}{2}MI_e\right\}\right\}$$

Using equation (C.1) & (C.2), we get

$$\frac{\partial W_1}{\partial \mu} = \left(-pbc + Dc\right)\left\{\left(T - t_1\right) + M\left\{1 + \left(T - t_1\right)I_E + \frac{1}{2}MI_e\right\}\right\} > 0 \tag{C.19}$$

$$\frac{\partial^2 W_1}{\partial \mu^2} = -2bc^2\left\{\left(T - t_1\right) + M\left\{1 + \left(T - t_1\right)I_E + \frac{1}{2}MI_e\right\}\right\} \tag{C.20}$$

$$\frac{\partial^2 W_1}{\partial \mu \partial t_1} = -\left\{1 + MI_e\right\}\left\{Dc - pbc\right\} \tag{C.21}$$

$$\frac{\partial W_1}{\partial t_1} = -Dp\left\{1 + MI_e\right\} < 0 \tag{C.22}$$

$$\frac{\partial^2 W_1}{\partial t_1^2} = 0 \tag{C.23}$$

$$\frac{\partial B_1}{\partial \mu}=\frac{\left\{2c^2pT\left\{\left(cQ-W_1\right)\left(I_p-1\right)-2Dp\right\}+2\frac{\partial W_1}{\partial \mu}\left\{2\left(cQ-W_1\right)I_p-Dp\right\}+4c\left(cQ-W_1\right)\left(pb+D\right)\right\}}{\left(2Dp-\left(cQ-W_1\right)I_p\right)^2}=X>0 \tag{C.24}$$

$$\frac{\partial B_1}{\partial t_1}=\frac{-4Dp\frac{\partial W_1}{\partial t_1}}{\left(2Dp-\left(cQ-W_1\right)I_p\right)^2}>0 \tag{C.25}$$

$$\frac{\partial^2 B_1}{\partial t_1^2}=\frac{-4Dp\frac{\partial^2 W_1}{\partial t_1^2}\left\{2Dp-\left(cQ-W_1\right)I_p-I_p\right\}}{\left(2Dp-\left(cQ-W_1\right)I_p\right)^3} \tag{C.26}$$

Similarly

$$\frac{\partial^2 B_1}{\partial \mu^2}<0 \tag{C.27}$$

$$\frac{\partial^2 B_1}{\partial t_1\partial \mu}=\frac{4bcp\frac{\partial W_1}{\partial t_1}+4cD\frac{\partial W_1}{\partial t_1}+4Dp\frac{\partial^2 W_1}{\partial t_1\partial \mu}}{\left(2Dp-\left(cQ-W_1\right)I_p\right)^2}+\frac{8Dp\frac{\partial W_1}{\partial t_1}\left\{-2bcp+2cD-c\frac{\partial Q}{\partial \mu}I_p+\frac{\partial W_1}{\partial t_1}I_p\right\}}{\left(2Dp-\left(cQ-W_1\right)I_p\right)^3}>0 \tag{C.28}$$

From $q\left(x\right)=\dfrac{y\left(x\right)}{g\left(x\right)}$

Let $Z\left(\mu,t_1\right)=D\left(t_1-B_1\right)p\left\{1+(1/2)\left(t_1-B_1\right)I_e\right\}\left\{1+(T-t_1)I_E\right\}-A-Hc-Sc$ (C.29)

Consequently, for given T, we have

$$TP_{1.1.1.1.(a)}\left(\mu,t_1,T\right)=\frac{Z\left(\mu,t_1\right)}{T} \tag{C.30}$$

Taking the first-order and second-order partial derivatives of $Z\left(\mu,t_1\right)$, and simplifying terms, we get

$$\frac{\partial Z\left(\mu,t_1\right)}{\partial \mu}=\left[\begin{array}{l}\left\{-bc\left(t_1-B_1\right)p+D\left(t_1-\frac{\partial B_1}{\partial \mu}\right)p+D\left(t_1-B_1\right)c\right\}\\ \left\{1+\left(\frac{1}{2}\right)\left(t_1-B_1\right)I_e\right\}-\frac{1}{2}D\left(t_1-B_1\right)pI_e\frac{\partial B_1}{\partial \mu}\end{array}\right]\left\{1+(T-t_1)I_E\right\}-\frac{\partial Hc}{\partial \mu}-\frac{\partial Sc}{\partial \mu} \tag{C.31}$$

$$\frac{\partial^2 Z\left(\mu,t_1\right)}{\partial \mu^2}=\left[\begin{array}{l}\left\{bc\frac{\partial B_1}{\partial \mu}p-bc^2\left(t_1-B_1\right)-bc\left(t_1-\frac{\partial B_1}{\partial \mu}\right)p-D\frac{\partial^2 B_1}{\partial \mu^2}p+D\left(t_1-\frac{\partial B_1}{\partial \mu}\right)c-bc^2\left(t_1-B_1\right)-Dc\frac{\partial B_1}{\partial \mu}\right\}\\ \left\{1+\frac{1}{2}\left(t_1-B_1\right)I_e\right\}-\left\{-bc\left(t_1-B_1\right)p+D\left(t_1-\frac{\partial B_1}{\partial \mu}\right)p+D\left(t_1-B_1\right)c\right\}\frac{\partial B_1}{\partial \mu}I_e\\ +\frac{1}{2}bc\left(t_1-B_1\right)pI_e\frac{\partial B_1}{\partial \mu}+\frac{1}{2}D\left(\frac{\partial B_1}{\partial \mu}\right)^2pI_e-\frac{1}{2}D\left(t_1-B_1\right)cI_e\frac{\partial B_1}{\partial \mu}-\frac{1}{2}D\left(t_1-B_1\right)pI_e\frac{\partial^2 B_1}{\partial \mu^2}\end{array}\right]\left\{1+\left(T-t_1\right)I_E\right\}=L \tag{C.32}$$

$$\frac{\partial^2 Z(\mu,t_1)}{\partial\mu\partial t_1}=\left[\begin{array}{l}\left\{\left\{-bcp-Dp\frac{\partial^2 B_1}{\partial\mu\partial t_1}+Dc\right\}\left\{1+\frac{1}{2}(t_1-B_1)I_e\right\}+\left\{-bc(t_1-B_1)p+Dp\left(t_1-\frac{\partial B_1}{\partial\mu}\right)+D(t_1-B_1)c\right\}\right\}\\ \left\{\frac{1}{2}\left(1-\frac{\partial B_1}{\partial t_1}\right)I_e\right\}-\frac{1}{2}D\left(1-\frac{\partial B_1}{\partial t_1}\right)pI_e\frac{\partial B_1}{\partial\mu}-\frac{1}{2}D(1-t_1)pI_e\frac{\partial B_1}{\partial\mu\partial t_1}\end{array}\right] \quad \text{(C.33)}$$

$$\{1+(T-t_1)I_E\}-\frac{\partial^2 Hc}{\partial\mu\partial t_1}-\frac{\partial^2 Sc}{\partial\mu\partial t_1}=K$$

$$\frac{\partial Z(\mu,t_1)}{\partial t_1}=\left[\left\{\begin{array}{l}Dp\left(1-\frac{\partial B_1}{\partial t_1}\right)\{1+(t_1-B_1)I_e\}\{1+(T-t_1)I_E\}\\ +Dp(t_1-B_1)I_E\left\{1+\frac{1}{2}(t_1-B_1)I_e\right\}\end{array}\right\}\right]-\frac{\partial Hc}{\partial t_1}-\frac{\partial Sc}{\partial t_1} \quad \text{(C.34)}$$

$$\frac{\partial^2 Z(\mu,t_1)}{\partial t_1^2}=\left[\begin{array}{l}-Dp\frac{\partial^2 B_1}{\partial t_1^2}\{1+(t_1-B_1)I_e\}\{1+(T-t_1)I_E\}+Dp\left(1-\frac{\partial B_1}{\partial t_1}\right)^2 I_e\{1+(T-t_1)I_E\}\\ -Dp\left(1-\frac{\partial B_1}{\partial t_1}\right)I_e\{1+(t_1-B_1)I_e\}I_E-Dp\left(1-\frac{\partial B_1}{\partial t_1}\right)I_E\left\{1+\frac{1}{2}(t_1-B_1)I_e\right\}\\ -\frac{Dp}{2}(t_1-B_1)I_E\left(1-\frac{\partial B_1}{\partial t_1}\right)I_e\end{array}\right]-\frac{\partial^2 Hc}{\partial t_1^2}-\frac{\partial^2 Sc}{\partial t_1^2}=M \quad \text{(C.35)}$$

If $L<0,\ M<0\ and\ LM\ -\ K^2>0,$ then the Hessian Matrix associate with $Z(\mu,t_1)$ is negative definite.

$$H=\begin{bmatrix}\frac{\partial^2 Z(\mu,t_1)}{\partial\mu^2} & \frac{\partial^2 Z(\mu,t_1)}{\partial\mu\partial t_1}\\ \frac{\partial^2 Z(\mu,t_1)}{\partial t_1\partial\mu} & \frac{\partial^2 Z(\mu,t_1)}{\partial t_1^2}\end{bmatrix}=\begin{bmatrix}L & K\\ K & M\end{bmatrix} \quad \text{(C.36)}$$

Consequently, for any given value of $T,\ L<0,\ M<0\ and\ LM\ -\ K^2>0,$ then $TP_{1.1.1.1.(a)}(\mu,t_1,T)$ is strictly concave function in μ & t_1. Hence there exist a unique optimal solution.

Appendix – D

For any given T, substitute (C.4), (C.6), (C.22) & (C.25) into (C.34) and setting the result to zero, we get the necessary and sufficient condition for t_1^* as follows:

$$\left[\left\{\begin{array}{l}Dp\left(1-\frac{4D^2p^2\{1+MI_e\}}{\left(2Dp-(cQ-W_1)I_p\right)^2}\right)\{1+(t_1-B_1)I_e\}\{1+(T-t_1)I_E\}\\ +Dp(t_1-B_1)I_E\left\{1+\frac{1}{2}(t_1-B_1)I_e\right\}\end{array}\right\}\right] \quad \text{(D.1)}$$

$$-h\left\{\frac{D}{\theta^3}(\theta t_d+1)e^{\theta(t_1-t_d)}-\frac{D}{\theta}\right\}-\pi D(T-t_1)=0$$

Similarly, substituting (C.12), (C.14), (C.19) and (C.24) into (C.31) and setting the result to zero, we have the necessary and sufficient condition of μ^* as follows:

$$\begin{bmatrix} \left\{-bc\left(t_1-B_1\right)p+D\left(t_1-X\right)p+D\left(t_1-B_1\right)c\right\} \\ \left\{1+\left(\frac{1}{2}\right)\left(t_1-B_1\right)I_e\right\}-\frac{1}{2}D\left(t_1-B_1\right)pI_eX \end{bmatrix}\left\{1+(T-t_1)I_E\right\}$$
$$-h\left\{-bc\left\{t_d+\frac{1}{\theta}\left(e^{\theta\left(t_1-t_d\right)}-1\right)\right\}t_d+\frac{bc}{2}t_d^2+\frac{bc}{\theta}\left(t_1-t_d+\frac{1}{\theta}\left(1-e^{\theta\left(t_1-t_d\right)}\right)\right)\right\}-\pi bc\frac{\left(T-t_1\right)^2}{2}=0 \tag{D.2}$$