Nonstandard Finite Difference Method Revisited and Application to the Ebola Virus Disease Dynamics Transmission

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Abstract

We provide effective and practical guidelines on the choice of the complex denominator function of the discrete derivative as well as on the choice of the nonlocal approximation of nonlinear terms in the construction of nonstandard finite difference (NSFD) schemes. Firstly, we construct nonstandard one-stage and two-stage theta methods for a general dynamical system defined by a system of autononous ordinary differential equations. We provide a sharp condition, which captures the dynamics of the continuous model. We discuss at length how this condition is pivotal in the construction of the complex denominator function. We show that the nonstandard theta methods are elementary stable in the sense that they have exactly the same fixed-points as the continuous model and they preserve their stability, irrespective of the value of the step size. For more complex dynamical systems that are dissipative, we identify a class of nonstandard theta methods that replicate this property. We apply the first part by considering a dynamical system that models the Ebola Virus Disease (EVD). The formulation of the model involves both the fast/direct and slow/indirect transmission routes. Using the specific structure of the EVD model, we show that, apart from the guidelines in the first part, the nonlocal approximation of nonlinear terms is guided by the productive-destructive structure of the model, whereas the choice of the denominator function is based on the conservation laws and the sub-equations that are associated with the model. We construct a NSFD scheme that is dynamically consistent with respect to the properties of the continuous model such as: positivity and boundedness of solutions; local and or global asymptotic stability

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of disease-free and endemic equilibrium points; dependence of the severity of the infection on self-protection measures. Throughout the paper, we provide numerical simulations that support the theory.

Keywords: Dynamical systems, dissipative systems, nonstandard finite difference schemes, stability; Ebola, environmental transmission.

AMS Subject Classification (2010): 65M06, 65M99, 37M99, 92D30

1 Introduction

The complexity of natural processes leads to challenging mathematical models, specifically in the context of studying problems that are related to the United Nations Sustainable Development Goals (SDGs). Typically, the models are dynamical systems defined by strongly nonlinear differential equations that cannot be completely solved by analytical techniques. Consequently, it is crucial to construct numerical methods that, apart from being convergent, are reliable in that they provide useful information and insight on the solutions of the differential equation model. Seeing that the limit is never reached and that the computations stop after finite number of steps, it is indeed essential to consider numerical schemes that go beyond the usual requirement of convergence in order to guarantee the accuracy of the numerical result.

In this paper, we consider the nonstandard finite difference (NSFD) method initiated by Mickens more than three decades ago. Since the publication of his pioneering monograph [28], the NSFD method has extensively been developed for differential models originating from problems in life sciences, engineering, physics, chemistry, etc. The NSFD method has shown great potential in replicating the rich dynamics of the continuous models, such as their chaotic behavior [18]. In this regard, it is worthwhile mentioning the 2000-2014 edited volumes [20, 21, 29, 30], in which a wide variety of applications have been investigated. See also [34] for an overview on NSFD schemes.

Despite the success of the NSFD approach, Mickens himself acknowledged from the start the following concern that is still not fully addressed: 'the general rules for constructing such schemes are not precisely known at the present time' (see [28]). The purpose of this paper is to revisit some aspects of the NSFD methodology with the aim of providing clear and practical guidelines for the construction of NSFD schemes in the setting described below. Further comments will follow here and there in this introductory section to make the purpose of the paper more precise.

We consider the initial-value problem for the autonomous system of n differential equations in n unknowns

$$\frac{dy}{dt} = g(y), \quad y(0) = y_0, \tag{1}$$

where $y \equiv [{}^1y \; {}^2y \; \cdots \; {}^ny]^T$ and $g \equiv [{}^1g \; {}^2g \; \cdots \; {}^ng]^T$. Eq. (1) is supposed to define a dynamical system on the whole space $\mathbb{R}^n$, and sometimes on a domain $\Omega \subset \mathbb{R}^n$. That is, for any $y_0 \in \mathbb{R}^n$ or Ω, there exists a unique solution $y(t) := S(t)y_0$ in this domain defined at all times $t \geq 0$, where $S(t)$ is the evolution/solution operator. Sufficient conditions that guarantee the well-posedness of the system are well-known (see, e.g [36]). So, in what follows, we assume implicitly that both the function g and the solution y possess all the needed smoothness properties.

We deal in this paper with two qualitative features of the general dynamical system defined by (1). The first property is the basic one in the qualitative analysis of dynamical systems. To describe it, we consider an equilibrium point $\widetilde{y}$ of the differential equation in (1), i.e.

$$g(\widetilde{y}) = 0.$$

We assume that there is a finite number of such points and that all the equilibrium points are hyperbolic, i.e.

$$Re\lambda \neq 0$$

for any $\lambda \in \sigma(J)$, where $\sigma(J)$ is the spectrum of the Jacobian matrix $J \equiv Jg(\widetilde{y})$ of g at $\widetilde{y}$. For simplicity, the matrix J is assumed to be diagonalizable. The said property is contained in the next definition.

Definition 1 *A equilibrium point $\widetilde{y}$ of (1) is called linearly stable if the solution*

$$\varepsilon(t) = e^{tJ}\varepsilon_0$$

of the linearized system

$$\frac{d\varepsilon}{dt} = J\varepsilon; \quad \varepsilon(0) = \varepsilon_0, \tag{2}$$

tends to zero as $t \to \infty$ or, equivalently, $Re\ \lambda < 0$ for all $\lambda \in \sigma(J)$. Otherwise the equilibrium point is called linearly unstable.

The second essential property reads as follows [36]:

Definition 2 *The dynamical system defined by (1) is called dissipative whenever the gross asymptotics of the system are independent of initial conditions with everything ending up inside some absorbing set B. More precisely, there exists a bounded positively invariant subset B of $\mathbb{R}^n$ with the property that to any bounded set $M \subset \mathbb{R}^n$, there corresponds a time $t^* \equiv t^*(B, M) \geq 0$ such that, for any $y_0 \in M$, there holds the inclusion $S(t)y_0 \in B$ whenever $t > t^*$.*

A sufficient condition for the dynamical system to be dissipative is stated in the next theorem, which is proved in [36].

Theorem 3 *Let the function g satisfy the following structural assumption: there exist constants $\alpha \geq 0$ and $\beta > 0$ such that, for every $y \in \mathbb{R}^n$,*

$$\langle g(y); y\rangle \leq \alpha - \beta||y||^2, \tag{3}$$

$||\cdot||$ being the Euclidean norm on $\mathbb{R}^n$ associated with the dot product $\langle\cdot;\cdot\rangle$. Then, the dynamical system defined by (1) is dissipative. The closed ball $\bar{B}\left(0, \sqrt{\alpha/\beta + \epsilon}\right)$ is an absorbing set for every $\epsilon > 0$.

Remark 4 *A linear dynamical system, which has a unique (linearly) stable equilibrium point $\widetilde{y}$ is necessarily dissipative, with the ball $B(\widetilde{y}, \epsilon)$ being an absorbing set for every ϵ. However, the dynamics of a general dissipative system inside the absorbing set can be complex and even chaotic.*

Let y^k denote an approximate solution of the exact solution y at the time $t = t_k$ of the mesh $\{t_k = k\Delta t\}_{k\geq 0}$, the parameter $h = \Delta t > 0$ being the step size. It is assumed that the algorithm that generates the approximate solutions has the general implicit or explicit structure

$$y^{k+1} = D_g(\Delta t;\ y^{k+1}, y^k) \tag{4}$$

where $D \equiv D_g$ denotes the discretization operator; the subscript is used to emphasize how the scheme is constructed from the right-hand side g in Eq. (1).

As said earlier, the aim of this paper is twofold: (a) To design, for (1), numerical methods which replicate the qualitative properties in Definition 1 and Theorem 3; (b) To revisit the nonstandard methodology and provide practical guidelines to construct reliable NSFD schemes. In particular, we will revisit elementary stable schemes in terms of some linearization processes as well as the location in the complex plane of the eigenvalues of involved Jacobian matrices. The general concept behind our specific aim is described in the following definition [2, 6, 8, 30, 31]:

Definition 5 *Let P be some property of the problem (1) or of its exact solution. A numerical scheme of the general form (4) is called (qualitatively) stable or dynamically consistent with respect to P if, for all values of the step size Δt, the scheme or its discrete solutions replicate the property P.*

Similarly to [5], and in an effort to make (4) more precise, our point of departure will be the extensively used schemes

$$\frac{y^{k+1} - y^k}{\Delta t} = \theta g(y^{k+1}) + (1-\theta) g(y^k), \tag{5}$$

$$\frac{y^{k+1} - y^k}{\Delta t} = g[\theta y^{k+1} + (1-\theta) y^k] \tag{6}$$

known as two-stage and one-stage theta methods, respectively [36]. Here, the parameter $\theta \in [0,1]$ is given. Once again, it is implicitly understood in what follows that the function g satisfies sufficient conditions that ensure that, for every Δt, the schemes (5) and (6) generate generalized dynamical systems with generalized evolution maps denoted by S^k (see [36] for the terminology and further details).

Clearly, the classical schemes (5) and (6) fail to preserve the linear stability and the dissipative properties of the dynamical system defined by (1). This can be checked, for instance, for the forward Euler scheme

$$\frac{y^{k+1} - y^k}{\Delta t} = g(y^k), \tag{7}$$

i.e. $\theta = 0$ in (5) and (6), applied to the (scalar) test equation

$$\frac{dy}{dt} = \lambda y; \quad \lambda < 0. \tag{8}$$

Indeed, the forward Euler method produces the discrete solution

$$y^k = (1 - |\lambda|\Delta t)^k y_0,$$

which does not display the same asymptotic behaviour as the solution

$$y(t) = y_0 \exp(\lambda t) \tag{9}$$

of the decay equation (8). On the contrary, the solution of the scheme

$$\frac{y^{k+1} - y^k}{[\exp(\lambda \Delta t) - 1]/\lambda} = \lambda y^k, \; y^0 = y_0, \tag{10}$$

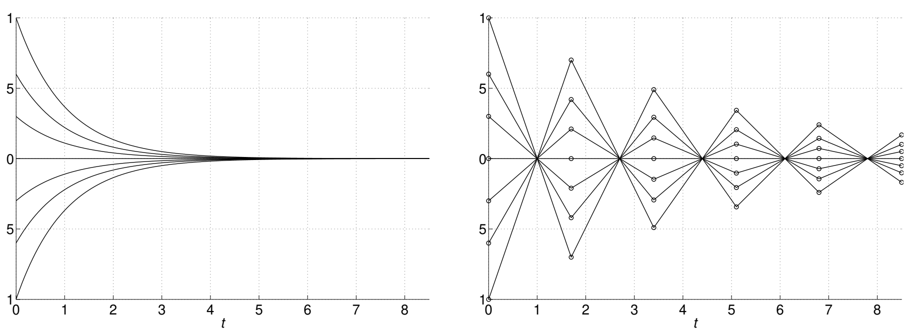
or, equivalently,

$$\frac{y^{k+1} - y^k}{[1 - \exp(-\lambda \Delta t)]/\lambda} = \lambda y^{k+1}, \; y^0 = y_0, \tag{11}$$

is related to the exact solution in (9) by

$$y^k = y(t_k). \tag{12}$$

Eq. (10) or (11) is the so-called exact scheme of the exponential equation (8) (see [28]). The profile of the discrete solutions by the forward Euler and exact schemes is illustrated on Figure 1.



(a) (b)

Figure 1: Dynamic inconsistency between exact scheme (a) and forward Euler method (b) for the decay equation

Likewise for the nonlinear logistic equation

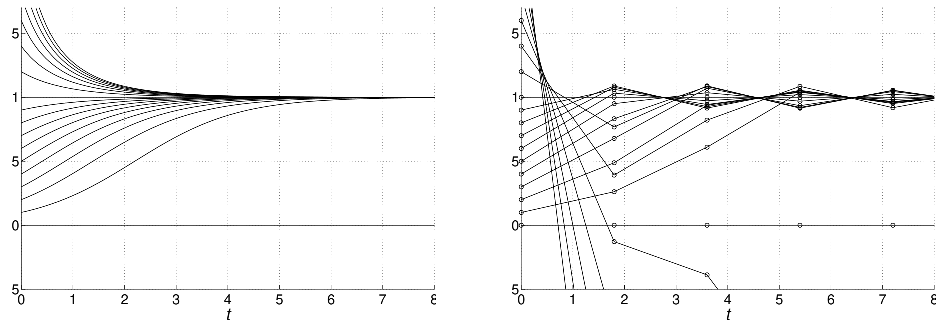
$$\frac{dy}{dt} = y(1 - y), \tag{13}$$

the forward Euler scheme

$$\frac{y^{k+1} - y^k}{\Delta t} = y^k(1 - y^k), \tag{14}$$

behaves differently from its exact scheme [28],

$$\frac{y^{k+1} - y^k}{\exp(\Delta t) - 1} = y^k(1 - y^{k+1}) \text{ or } \frac{y^{k+1} - y^k}{1 - \exp(-\Delta t)} = y^{k+1}(1 - y^k) \tag{15}$$



(a) (b)

Figure 2: Dynamic inconsistency between the exact scheme (a) and the Euler method (b) for the logistic equation

where Eq. (12) holds true. Figure 2 displays the inconsistencies between the exact scheme and the Euler scheme.

We refer the reader to [35] and the references therein for exact schemes of quite a number of differential equations. The main observation about the above two academic examples is that a change in the denominator Δt of the discrete derivatives from (7) and (14) as well as the nonlocal manner in which the nonlinear term y^2 is approximated have resulted into the schemes (10) or (11) and (15), which are qualitatively stable or dynamically consistent with respect to any property of the solutions of the continuous models.

In view of the above comments, our strategy is to modify (5) and (6) in order to avoid as much as possible the inconsistencies between the solution of the continuous models and the discrete solutions. We will construct nonstandard finite difference schemes of the form

$$\frac{y^{k+1}-y^k}{\psi(\Delta t)} = \theta g(y^{k+1}) + (1-\theta) g(y^k), \tag{16}$$

$$\frac{y^{k+1}-y^k}{\psi(\Delta t)} = g[\theta y^{k+1} + (1-\theta) y^k]. \tag{17}$$

A key point of our aim to revisit the NSFD methodology will be to explain how to choose the more complex denominator function $\psi(\Delta t)$, while the following asymptotic behavior is maintained in accordance with Mickens' rules [28] (see also [6]):

$$\psi(\Delta t) = \Delta t + O[(\Delta t)^2]. \tag{18}$$

The process to achieve this will also include an overview of earlier works. In a nutshell, we will construct the function $\psi(\Delta t)$ in such a manner that it reflects the intrinsic properties of the right-hand side of (1). This requirement transpires clearly from (10) or (11) and (15) regarding the specific examples (8) and (13), respectively.

Note that both Eq. (16) and Eq. (17) coupled with Eq. (18) are indeed nonstandard finite difference schemes, as per the formal definition in [6].

Our last objective is to consider the practical setting where Eq (1) models the transmission dynamics of diseases. Though several emerging and re-emerging diseases could be considered, we will focus only on the deadly Ebola Virus Disease (EVD), which is a public health crisis in the Democratic Republic of Congo [37]. The rich dynamics of the model to be explored include: positivity and boundedness of solutions, conservation laws, local and global asymptotic stability of equilibrium points. We will explain how to construct NSFD schemes that preserve these properties.

The rest of this work is organized as follows. In the next section, we revisit some results on the elementary stability of nonstandard finite difference schemes of the general form (4) and of the particular form (16)-(17). Regarding the latter form, the novelty is that we obtain a sharper sufficient condition for the elementary stability and we show that this new condition is equivalent to having the eigenvalues of the involved Jacobian matrices located in specific regions of the complex plane, which are larger than those of the classical schemes. Section 3 is devoted to the study of the dissipative property of the schemes (16)-(17). Numerical experiments that support the theory are presented in Section 4. From Section 5, comes an application of the results obtained in the first part to the transmission dynamics of the Ebola Virus Disease (EVD). We give some background and update on the EVD, and summarize the main theoretical, quantitative and qualitative results. Taking into account the new constructive strategies that are suggested by the EVD model, we revisit in Section 6 the nonstandard methodology and design a NSFD scheme, which is dynamically consistent with the continuous model. In Section 7, we provide numerical simulations that support the theory regarding the dynamic consistency of the NSFD scheme with the continuous model on the following three grounds: positivity and boundnedness of the solutions; local and or global asymptotic stability of the disease-free and endemic equilibrium points, and impact of the interventions on the severity of the infection. The last section deals with some concluding remarks on how our results fit in the literature and on our plan for future research.

2 Elementary Stability

Let $\widetilde{y}$ be an equilibrium point of (1). We assume that $\widetilde{y}$ is also a fixed-point of the scheme (4). In fact we assume that $\widetilde{y}$ is an equilibrium point of (1) if, and only if, $\widetilde{y}$ is a fixed-point of the scheme (4):

$$g(\widetilde{y}) = 0 \Leftrightarrow \widetilde{y} = D_g(\Delta t; \widetilde{y}, \widetilde{y}). \tag{19}$$

To investigate the relevance of a numerical scheme, it is quite natural to implement the scheme for a test equation. The concept of absolute stability [24] of a scheme is based on the linearized equation (2) as a test equation. Since elementary stability is meant to be an extension of the concept of absolute stability, we apply the numerical method (4) to the linearized equation (2). This results into a numerical scheme of the form

$$\varepsilon^{k+1} = D_J(\Delta t; \varepsilon^k, \varepsilon^{k+1}), \tag{20}$$

which can be written as,

$$\varepsilon^{k+1} = DL(\Delta t; \varepsilon^k, \varepsilon^{k+1}), \tag{21}$$

to emphasize that we first linearized the system (1) about the equilibrium $\widetilde{y}$, via the operator L, and subsequently applied the discretization operator D to (2). On the other hand, if the discretization in (4) is linearized about each fixed-point $\widetilde{y}$, we obtain a discrete linear equation of the form

$$\varepsilon^{k+1} = LD_g(\Delta t; \varepsilon^k, \varepsilon^{k+1}) \text{ or } \varepsilon^{k+1} = LD(\Delta t; \varepsilon^k, \varepsilon^{k+1}). \tag{22}$$

where the order of the two operators is reflected. However, Eq. (22) is in general not equivalent to Eq. (20): the discretization operator D and the linearization operator L do not commute. In view of this, the scheme (22) can neither be used to analyze the dynamics of (1) nor to investigate the linear stability of the scheme (4). Consequently, we introduce the two definitions below.

Definition 6

1. *A fixed-point $\widetilde{y}$ of the scheme (4) is called linearly stable if the sequence (ε^k) of solutions of (20) converges to zero as $k \to \infty$ for all $\lambda \in \sigma(J)$. Otherwise the fixed-point is called linearly unstable.*

2. *The numerical scheme (4) is called elementary stable if, for any value of the step size Δt, the condition (19) holds, and each fixed-points $\widetilde{y}$ of the scheme has the same linear stability/instability property as for the differential system in (1).*

Definition 7 *The numerical scheme (4) is called totally elementary stable whenever it is elementary stable and the schemes (20) and (22) are equivalent.*

Remark 8 *PLEASE PROVIDE AN EXAMPLE.*

Coming back to the schemes of the form (16) or (17), which constitute our main focus in this work, we observe that they all satisfy the condition (19). Moreover the linearized equations (20) and (22) reduce to the same equation, which reads as follows:

$$\frac{\varepsilon^{k+1} - \varepsilon^k}{\psi(\Delta t)} = \theta J \varepsilon^{k+1} + (1-\theta) J \varepsilon^k, \ \varepsilon^0 = \varepsilon_0. \tag{23}$$

For this reason, all the results stated below about the elementary stability of the schemes of the form (16) or (17) mean actually that these schemes are totally elementary stable. On the other hand, Eq (23) permits us to characterize the first part of Definition 6, according to the alternative formulation in Definition 1, as follows:

Proposition 9 *A fixed-point $\widetilde{y}$ of the scheme (16) or (17) is linearly stable if, and only if, we have*

$$\left| \frac{1 + \psi(\Delta t)(1-\theta)\lambda}{1 - \psi(\Delta t)\theta\lambda} \right| < 1$$

for all $\lambda \in \sigma(J)$.

Remark 10 *It is easy to check that the trapezoidal rule, i.e. the scheme (5) or (6) where $\theta = 1/2$, is elementary stable. For this reason, we will not consider in what follows nonstandard scheme (16) or (17) for the case when $\theta = 1/2$.*

The important question to be addressed now is the following: how to choose the complex denominator function ψ? In [5], the authors constructed elementary stable nonstandard theta schemes in which the denominator function has the form

$$\psi(\Delta t) = \frac{\phi(q\Delta t)}{q}, \tag{24}$$

for some given real number $q > 0$, where ϕ is a non-negative function satisfying (18) and

$$0 < \phi(z) < 1 \tag{25}$$

for $z > 0$ (a typical example is $\phi(z) = 1 - e^{-z}$). Actually, the investigation of nonstandard finite difference schemes with (24) and (25) as denominator function of the discrete derivative goes back to [28] and earlier works of the authors and this has been exploited extensively in the literature (see, for instance, [6, 17, 26, 30].) The novelty in this paper is an optimal condition on q, as described in the following result, which was announced in the abstract [3] and which, in the particular case of the forward Euler method ($\theta = 0$), is given in [16, 17] :

Theorem 11 *Consider the finite set*

$$E := \cup \ \{\sigma(Jg(\widetilde{y})); \quad \widetilde{y} \in \mathbb{R}^n, \quad g(\widetilde{y}) = 0\}.$$

Then the nonstandard theta method (16) or (17), where ψ is defined by (24) and (25), is elementary stable whenever the number q satisfies the condition

$$q \geq \max\left\{\frac{|\lambda|^2}{2|Re\lambda|}; \ \lambda \in E\right\}. \tag{26}$$

Proof. For $\lambda = \lambda_1 + \imath\lambda_2 \in E$, writing explicitly the quantity that is needed from Proposition 9, we obtain:

$$\begin{aligned} I &\equiv \left|\frac{1 + \psi(\Delta t)(1-\theta)\lambda}{1 - \psi(\Delta t)\theta\lambda}\right|^2 \\ &= \frac{1 + 2\lambda_1\phi(q\Delta t)(1-\theta)/q + |\lambda|^2\phi^2(q\Delta t)(1-\theta)^2/q^2}{1 - 2\lambda_1\phi(q\Delta t)\theta/q + |\lambda|^2\phi^2(q\Delta t)\theta^2/q^2}. \end{aligned} \tag{27}$$

Let $\widetilde{y}$ be an equilibrium point of the differential equation (1). Two cases are possible. Firstly, $\widetilde{y}$ can be linearly stable, which, by Definition 1, implies that $\lambda_1 < 0$ for any eigenvalue $\lambda \in \sigma(Jg(\widetilde{y}))$. Then we have:

$$\begin{aligned} I &= \frac{1 - 2|\lambda_1|\phi(q\Delta t)(1-\theta)/q + |\lambda|^2\phi^2(q\Delta t)(1-\theta)^2/q^2}{1 + 2|\lambda_1|\phi(q\Delta t)\theta/q + |\lambda|^2\phi^2(q\Delta t)\theta^2/q^2} \\ &< 1 - 2|\lambda_1|\phi(q\Delta t)(1-\theta)/q + |\lambda|^2\phi(q\Delta t)(1-\theta)/q^2 \text{ by (25)} \\ &\leq 1 \text{ by (26)}. \end{aligned}$$

This shows that the fixed-point $\widetilde{y}$ is linearly stable for the scheme (16) or (17). Secondly, the equilibrium point $\widetilde{y}$ of (1) can be linearly unstable, i.e., there exists an eigenvalue $\lambda \in \sigma(Jg(\widetilde{y}))$ such that $\lambda_1 > 0$. Working out the above expression of I, we obtain

$$\frac{1 + 2\lambda_1\phi(q\Delta t)(1-\theta)/q + |\lambda|^2\phi^2(q\Delta t)(1-\theta)^2/q^2}{1 - 2\lambda_1\phi(q\Delta t)\theta/q + |\lambda|^2\phi^2(q\Delta t)\theta^2/q^2} > 1$$

$$\Updownarrow$$

$$2\lambda_1 + |\lambda|^2\phi(q\Delta t)/q - 2|\lambda|^2\phi(q\Delta t)\theta/q > 0.$$

But

$$2\lambda_1 + |\lambda|^2\phi(q\Delta t)/q - 2|\lambda|^2\phi(q\Delta t)\theta/q \geq 2\lambda_1 - |\lambda|^2\phi(q\Delta t)/q$$

which, in view of (25), shows that

$$2\lambda_1 - |\lambda|^2\phi(q\Delta t)/q > 0$$

whenever the condition (26) is satisfied. Thus the fixed-point $\widetilde{y}$ is linearly unstable for the scheme (16) or (17). We have thus proved that the schemes (16) and (17) are elementary stable. ■

Our next concern is as follows. While Theorem 11 is theoretically interesting, it is practically difficult to find q that meets the requirement (26) since no low bounds are available in general for $|Re\lambda|$. In what follows, we exploit an idea in [26] where Theorem 11 is implicitly mentioned in Remark 2.8. Following the latter reference, it is convenient to use the identity $Re\lambda = |\lambda| \cos\arg\lambda$ that, in view of (26), yields the relation

$$|\cos\arg\lambda| \geq |\lambda|/(2q) \quad \text{for all } \lambda \in E. \tag{28}$$

The condition (26) in its equivalent form (32) implies a restriction on the location of the eigenvalues in the complex plane in the following precise manner:

Theorem 12 *The condition (26) is equivalent to saying that the eigenvalues of all the matrices $Jg(\widetilde{y})$ are contained in some wedge in the complex plane, i.e.*

$$E \subset W_j^* := \{\lambda \in \mathbb{C};\ |\cos\arg\lambda| \geq j/2\} \tag{29}$$

for some $j \in (0, 2]$.

Proof. If q satisfies (26), then we have, using (32), the inclusion (29) with $j := \min\{|\lambda|;\ \lambda \in E\}/q$. Conversely, if (29) holds, then the number $q := \max\{|\lambda|; \lambda \in E\}/j$ satisfies (26). ■

In view of Theorem 12, Theorem 11 is a rephrasing of the results in [26] for $\theta = 0$, a fact that was observed in [17]. In the following result, we give a somewhat refined version of the inclusion (29); the particular case when $j = 1$ was analyzed in [5] and [26].

Theorem 13 *With a fixed real number $0 < j \leq 2$, we associate the wedges in the left and right hands complex plane defined by*

$$W_j^- := \{\lambda \in \mathbb{C};\ Re\lambda < 0 \text{ and } |\cos\arg\lambda| \geq j/2\} \tag{30}$$

and

$$W_j^+ := \{\lambda \in \mathbb{C};\ Re\lambda > 0 \text{ and } |\cos\arg\lambda| \geq j/2\}. \tag{31}$$

Let the properties of the differential equation be captured by a number q satisfying

$$q \geq \max\{|\lambda|;\ \lambda \in E\}/j. \tag{32}$$

Then, the nonstandard finite difference schemes (16) and (17), with denominator function in (24) and (18), are elementary stable whenever we have the inclusions

$$E \subset W_j^- \cup \{\lambda \in \mathbb{C};\ Re\lambda > 0\} \text{ for } \theta \in [0, 1/2) \tag{33}$$

and

$$E \subset W_j^+ \cup \{\lambda \in \mathbb{C};\ Re\lambda < 0\} \text{ for } \theta \in [1/2, 1]. \tag{34}$$

Proof. The proof works as that of Theorem 11, observing that the condition $|\cos \arg \lambda| \geq j/2$ is used in conjunction with the constraint $\theta \in [0, 1/2)$ or $\theta \in [1/2, 1]$ to show that $I < 1$ or $I > 1$. ■

Remark 14 *We present below some consequences of the previous results regarding the practical choice of q.*

1. *Unlike (26), the choice of the number q in (32) is not so critical if the system is non-stiff. In practice, we may take $jq := max\|J(g)(\widetilde{y})\|_\infty$, where $\|\cdot\|_\infty$ is the matrix norm associated with the supremum norm on $\mathbb{R}^n$.*

2. *If the eigenvalues λ are real numbers, as in the case when the matrix J is symmetric, the choice $q \geq max\|J(g)(\widetilde{y})\|_\infty$, is appropriate.*

3. *For a two-dimensional system with complex eigenvalue λ and its conjugate, their real parts can be expressed in terms of the known trace $Tr(J)$ of the matrix $J(g)(\widetilde{y})$ by $2Re\lambda = Tr(J)$. Therefore, we may take*

$$q \geq \max\{\|J(g)(\widetilde{y})\|_\infty^2 / |Tr(J)|\}.$$

4. *With the definition (30)-(31) of the wedges, the inclusions (33)-(34) for elementary stability of the scheme under consideration are in line with what is done in the classical theory of absolute stability of numerical methods for ordinary differential equations (see [24]). This observation permits to link the extreme case when $j = 0$ in (30) to the classical concept of A-stable schemes (i.e. those that have the whole left complex plane $\{\lambda \in \mathbb{C}; Re\lambda < 0\}$ as region of absolute stability). Furthermore, from the comparative analysis in [26], it follows that the nonstandard theta methods have much larger regions of absolute (elementary) stability than the standard ones.*

3 Dissipative Schemes

The term "dissipative/dissipation" has several meanings in the literature (see for instance [9, 19]. In this paper, we specifically use Definition 2. We are in the setting of Theorem 3 where α and β are given. We start with the following result that is proved in [36]:

Theorem 15 *For $\theta \in (1/2, 1]$, the (generalized) dynamical system on $\mathbb{R}^n$ generated by the classical one-stage theta method (6) is dissipative in a sense, which is similar to Definition 2 with however the discrete variable k and the generalized evolution map S^k in place of the variable t and of the solution operator $S(t)$. Any open ball containing the closed ball $\bar{B}\left(0, \frac{1}{2\theta - 1}\sqrt{\frac{\alpha}{\beta}}\right)$ is an absorbing set. (Often, we will say that a scheme is dissipative to express the dissipative nature of the associated generalized dynamical system.)*

A similar result is stated and proved in the same reference [36] for the classical two-stage method (5). In particular, it follows from these results that within the range $\theta \in [0, 1/2]$, the qualitative stability of the classical schemes (5)-(6) with respect to the dissipative property is not guaranteed.

We now show how the nonstandard approach can help to avoid the above situation in the simple case when the method is explicit i.e., $\theta = 0$. To this end, we assume without loss of generality that

$$\beta < 1 \tag{35}$$

in (3). Furthermore, we assume that there exist positive constants γ and $c > 1$ such that, for every $y \in \mathbb{R}^n$:

$$||g(y)||^2 \leq \gamma + c||y||^2. \tag{36}$$

Remark 16 *The condition (36) is realistic. For instance, (36) holds if the function g is Lipschitz, which is one of the widely used requirement for (1) to define a dynamical system on $\mathbb{R}^n$.*

We have the following result:

Theorem 17 *For $\theta = 0$, the nonstandard finite difference scheme (16) or (17), where ψ is defined by (24) and (25), with $q := c/\beta$, is dissipative.*

Proof. From Eq. (16) or (17) where $\theta = 0$, we have consecutively the following, on using (3), (36), (25) and (24):

$$\begin{aligned}\frac{||y^{k+1}||^2 - ||y^k||^2}{\psi(\Delta t)} &= 2\langle g(y^k); y^k\rangle + \psi(\Delta t)||g(y^k)||^2 \\ &\leq 2\alpha - 2\beta||y^k||^2 + \frac{\beta}{c}\phi(c\Delta t/\beta)(\gamma + c||y^k||^2) \\ &< 2\alpha + \frac{\beta\gamma}{c} - \beta||y^k||^2.\end{aligned}$$

Thus

$$||y^{k+1}||^2 < \left(2\alpha + \frac{\beta\gamma}{c}\right)\psi(\Delta t) + [1 - \beta\psi(\Delta t)]||y^k||^2.$$

Applying the discrete Gronwall inequality [36] yields

$$||y^k||^2 \leq \left(\frac{2\alpha}{\beta} + \frac{\gamma}{c}\right)\left[1 - (1 - \beta\psi(\Delta t))^k\right] + ||y^0||^2(1 - \beta\psi(\Delta t))^k.$$

Thus

$$\limsup_{k\to\infty} ||y^k||^2 \leq \frac{2\alpha}{\beta} + \frac{\gamma}{c}$$

and it follows that the scheme under consideration is dissipative, the closed ball $\bar{B}\left(0, \sqrt{\frac{2\alpha}{\beta} + \frac{\gamma}{c} + \epsilon}\right)$ being an absorbing set for every $\epsilon > 0$. ■

4 Numerical Examples

Example 18 *We consider the dynamical system defined by*

$$\begin{cases} \frac{dy_1}{dt} = 1 + 5y_2 - y_1 \\ \frac{dy_2}{dt} = 1 - 5y_1 - y_2. \end{cases} \tag{37}$$

This dynamical system has a unique equilibrium point $\tilde{y} = (3/13, -2/13)$ and the associated Jacobian matrix

$$J = \begin{bmatrix} -1 & 5 \\ -5 & -1 \end{bmatrix} \tag{38}$$

has eigenvalues $\lambda_1 = -1 + 5\imath$ and $\lambda_2 = -1 - 5\imath$. Thus the equilibrium point is an asymptotically stable spiral point. With $\phi(z) = 1 - \exp(-z)$ and in view of (26) or its equivalent form in Remark 14(3), we take $q \geq 13$, which by Therorem 11 guarantees that the following NSFD scheme is elementary stable:

$$\begin{cases} \frac{y_1^{k+1} - y_1^k}{(1-\exp(-q\Delta t))/q} = 1 + \theta(5y_2^{k+1} - y_1^{k+1}) + (1-\theta)(5y_2^k - y_1^k) \\ \frac{y_2^{k+1} - y_2^k}{(1-\exp(-q\Delta t))/q} = 1 - \theta(5y_1^{k+1} + y_2^{k+1}) - (1-\theta)(5y_1^k + y_2^k). \end{cases} \tag{39}$$

Figure 3 illustrates the elementary stability of the NSFD scheme (39) for $\theta = 1/3$ and $\theta = 0$ with $q = 62$ and $\Delta t = 2$, a value of the step size that is not acceptable for classical theta methods. The profile of the two pictures is similar because the order of convergence of the theta method is 1 for $\theta \neq 1/2$. Even for the small value $\Delta t = 0.01$ of the step size, the instabilty of the classical theta method is apparent on Figure 4.

On the other hand for this example, a straightforward calculation shows that the right hand-side vector function

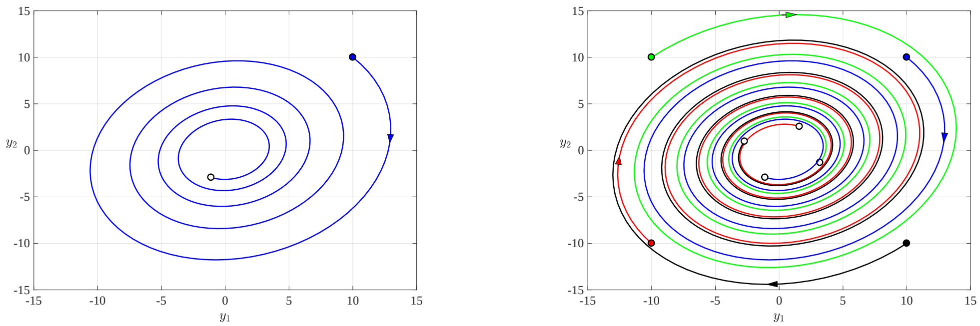
$$g(y) = \begin{pmatrix} 1 + 5y_2 - y_1 \\ 1 - 5y_1 - y_2 \end{pmatrix}$$

satisfies the structural assumption (3) and (35) in the following precise form:

$$\langle g(y), y\rangle \leq 1 - \frac{1}{2}||y||_2^2 \ ,$$

i.e., $\alpha = 1$ and $\beta = \frac{1}{2}$. Furthermore, the norm of $g(y)$ can be estimated as follows:

$$\begin{aligned} ||g(y)||_2^2 &= 2 + 26(y_1^2 + y_2^2) + 8y_2 - 12y_1 \\ &\leq 2 + 26(y_1^2 + y_2^2) + 4(1 + y_2^2) + 6(1 + y_1^2) \\ &\leq 12 + 32||y||_2^2. \end{aligned}$$



(a) (b)

Figure 3: Elementary stable NSFD scheme (39) for $\theta = 1/3$ (a) and $\theta = 0$ (b) in Example 18

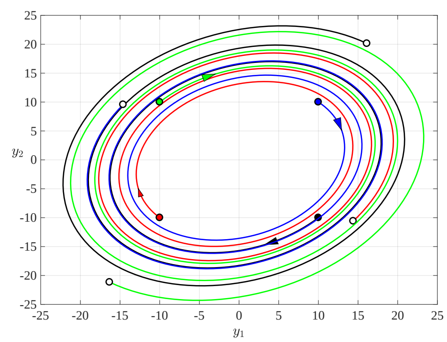


Figure 4: Unstable theta scheme for Example 18

Hence, the requirement (36) is met with $\gamma = 12$ and $c = 32$. The nonstandard scheme considered in Theorem 17 reads

$$\frac{y^{k+1} - y^k}{(1 - \exp(-q\Delta t))/q} = g(y^k), \tag{40}$$

where $q = \dfrac{c}{\beta} = 64$. We take a step size $\Delta t = 0.1$. Figure 5 gives the phase diagrams of the numerical solutions of the system (37) by the nonstandard finite difference scheme (40) using the initial conditions $y(0) = (10, 10)$ and $y(0) = (\pm 10, \pm 10)$, respectively. The dissipativity of the scheme is apparent. For comparison, we apply to the system (37) the standard Euler method (7) with the same step size and initial condition $y(0) = (10, 10)$. The phase diagram of the standard numerical solution given in Figure 6 is not indicative of dissipativity.

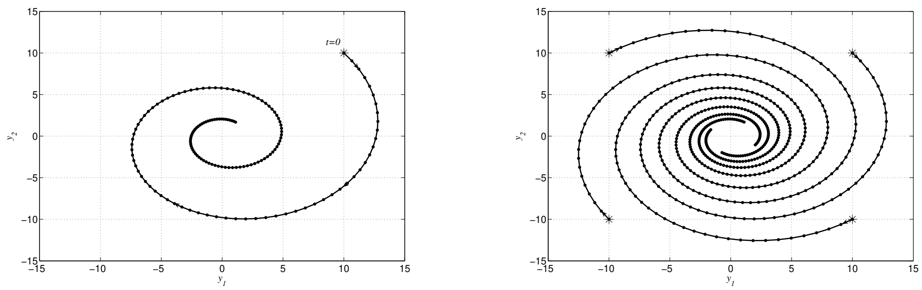
(a)  (b)

Figure 5: Dissipative nonstandard scheme for Example 18 with initial conditions $y(0) = (10, 10)$ on picture (a) and $y(0) = (\pm 10, \pm 10)$ on picture (b)

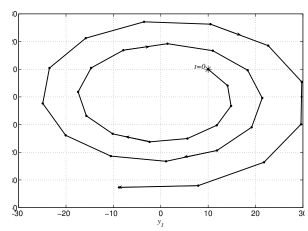


Figure 6: Nondissipative standard scheme for Example 18

Furthermore, we consider the exact scheme of the non homogeneous linear system (37). From the expression given in [6, 28] for homogeneous linear systems, with detail in [25]), we deduce that the required exact scheme is the difference equation

$$\frac{y^{k+1} - \chi y^k}{\eta} = g(y^k), \tag{41}$$

where the general forms of the perturbation function χ in the numerator of the discrete derivative [6] and the complex denominator function η are eventually specified in terms of the example under consideration as follows:

$$\begin{aligned} \eta &= \frac{\exp(\lambda_1 \Delta t) - \exp(\lambda_2 \Delta t)}{\lambda_1 - \lambda_2} = \Delta t + O(\Delta t^2), \\ &= \sin(5\Delta t)\exp(-\Delta t)/5 \end{aligned} \tag{42}$$

and

$$\begin{aligned} \chi &= \frac{\lambda_1 \exp(\lambda_2 \Delta t) - \lambda_2 \exp(\lambda_1 \Delta t)}{\lambda_1 - \lambda_2} = 1 + O(\Delta t^2) \\ &= \exp(-\Delta t)(\cos(5\Delta t) + \frac{1}{5}\sin(5\Delta t)). \end{aligned} \tag{43}$$

The phase diagram of the exact scheme (41) is given on Figure 7. It illustrates the dynamic consistency of the nonstandard schemes on Figure 3, as opposed to

Figure 4, and thus confirming the power of the nonstandard finite difference schemes over the standard ones that have been extensively discussed in the literature (see, for instance, [6, 17, 20, 29, 30] and the references therein).

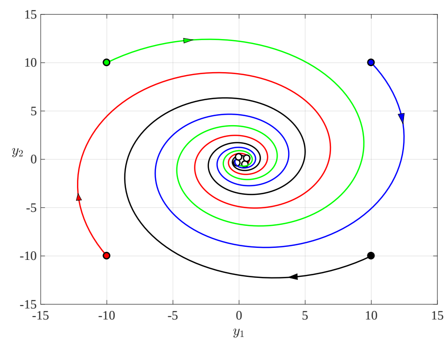


Figure 7: Exact scheme (41) for Example 18

Example 19 *We consider the dynamical system defined by*

$$\begin{cases} y_1' = -y_1 - 5y_2 + \dfrac{y_1}{\sqrt{y_1^2+y_2^2}} \\ y_2' = 5y_1 - y_2 + \dfrac{y_2}{\sqrt{y_1^2+y_2^2}}. \end{cases} \tag{44}$$

It is easy to check that this dynamical system has no equilibrium point. Thus investigating its linear stability is not relevant. However, the conditions (3) and (35) hold. Indeed, for the right hand side

$$g(y) = \begin{pmatrix} -y_1 - 5y_2 + \dfrac{y_1}{\sqrt{y_1^2+y_2^2}} \\ \\ 5y_1 - y_2 + \dfrac{y_2}{\sqrt{y_1^2+y_2^2}} \end{pmatrix},$$

we have

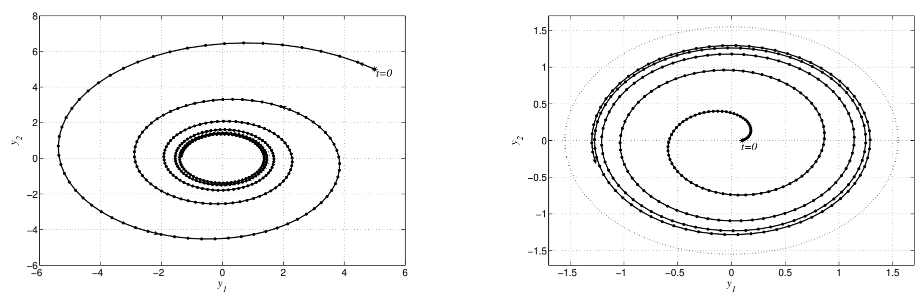
$$\begin{aligned} \langle g(y), y\rangle &= ||y|| - ||y||^2 \\ &\leq \frac{1}{2}(1+||y||^2) - ||y||^2 \\ &= \frac{1}{2} - \frac{1}{2}||y||^2 , \end{aligned}$$

i.e., $\alpha = \frac{1}{2}$ and $\beta = \frac{1}{2}$ in (3). Furthermore,

$$\begin{aligned} ||g(y)||_2^2 &= 1 + 26||y||^2 - ||y|| \\ &\leq 1 + 26||y||^2 \end{aligned}$$

Hence (36) holds with $\gamma = 1$ and $c = 26$. Then the nonstandard scheme considered in Theorem 17 is given by (40) where $q = \dfrac{c}{\beta} = 52$. We take $\Delta t = 0.1$ and $y(0) = (5, 5)$

or $y(0) = (0.1, 0)$. On Figure 8 one can observe that the nonstandard numerical solutions eventually belong the absorbing set $\overline{B}(0, 1.4277... + \varepsilon)$ given in Theorem 17. The ball with radius 1.55 is plotted on the figures by a dotted line. The numerical solution on Figure 8(a) originates outside this ball and enters it after certain number of time steps, while the numerical solution on Figure 8(b) originates inside the ball and does not leave it. We notice from Figure 9 that the standard



(a) (b)

Figure 8: Dissipative nonstandard scheme for Example 19 with initial condition originated from outside the absorbing ball (a) and from inside it (b)

forward Euler method with the same step size and initial condition $y(0) = (0.1, 0)$ is not dissipative.

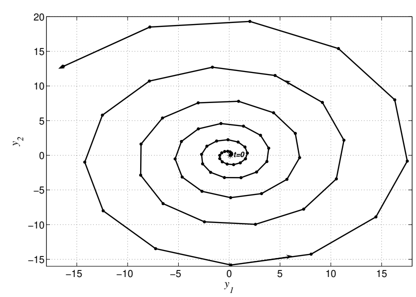


Figure 9: Nondissipative standard scheme for Example 19

Remark 20 *The absorbing sets in Examples 18 and 19 are determined by two different kinds of global attractors. In Example 18, the attractor is a hyperbolic equilibrium point, a case we dealt with through both the linear stability and structural assumptions for dissipativity . However, as mentioned earlier, linear stability does not yield results for Example 19, since the system does not have equilibrium points. In fact, the unit circle is a global attractor for this system. Therefore, it would be interesting to study nonstandard schemes that preserve global attractors as suggested by Figure 8.*

5 Ebola Virus Disease Transmission Dynamics

This second part of the paper is devoted to the application to the epidemiology of infectious diseases of humans of the nonstandard finite difference discretizations presented in the first part. Though the study can be considered for several diseases, we focus on the deadly Ebola Virus Disease (EVD) in Africa. Since the first outbreak of the EVD that occurred in 1976 near the river Ebola in the village Yambuku in the Democratic Republic of Congo (DRC), there has been a recurrence of more than 20 outbreaks, including the deadliest recorderded one in Western Africa from 2013-2015 and the most recent one in DRC, which is ongoing since 2018 (see for instance [1]). As reported by the World Health Organization in its regular external situation reports (e.g. [37]), the challenges experienced since the beginning of the current Ebola outbreak in DRC are much more and more serious than before. They include:

- The spread of the disease in areas with cross-border population flow (e.g. Rwanda, Uganda, etc.);
- The repeated attacks on treatment centres;
- The lack of access to affected communities;
- The infections and deaths among health care workers;
- The increase in number of cases and affected areas: e.g. 3444 cases including 3310 confirmed of which 2264 died as of 1 March 2020 (see [37, 38] and the map on Figure 10).

In view of the growing challenges, it becomes essential and vital to educate the population and to train "legions of disease-fighters", as aggressively promoted by Jean-Jacques Muyembe, the laureate of the 2019 Hideyo Noguchi Africa Prize for Medical Research [23]. Thanks to the educational campaign in place, the outbreak has been confined to a relatively small geographic area in the past few weeks. During this period, the number of new confirmed cases of EVD has been low, with specifically no new cases reported from 24 February to 1 March 2020 (see [38]). This has led the WHO and the DRC Government to put a time line of 42 days before declaring the country Ebola free (see [14]).

By targeting the EVD, we will experience the richness of the properties of the associated dynamical system that must be incorporated into the nonstandard finite difference discretizations if they are to be dynamically consistent. Apart from the contributions [10, 13], our study is mostly based on the combination of two deterministic models that we considered in [11, 12]. For the exposition to be self-contained, we outline below the background and main results in these references, where the following assumptions were made to accommodate both fast/direct and slow/indirect transmission routes as well as the population education:

1. We incorporate the transmission of deceased individuals during funerals and we allow the demographic process (vital dynamics) to take place during the EVD outbreaks [1, 22];

2. We include the infection through the contaminated environment, resulting from African practices and hospitality.

3. We include a provision or recruitment source of Ebola virus linked to the consumption of bats, hunted meat and fruits from rain-forests;

4. We add a vaccinated compartment (note that in DRC, the number of people who receive the rVsV-ZEBOV-GP Ebola vaccine has been increasing gradually: e.g. 283 117 and 299 094 individuals on 2 February and 1 March 2020, respectively. Vaccination with the Ad26.ZEBOV/MVA-BN-Filo vaccine counts less people namely 9715 and 19654, as of 31 January and 1 March 2020, respectively : see [37, 38] and Figure 10);

5. We add a compartment of trained human individuals to take into consideration the awareness of the population and their behavioral change for self-protection [1].

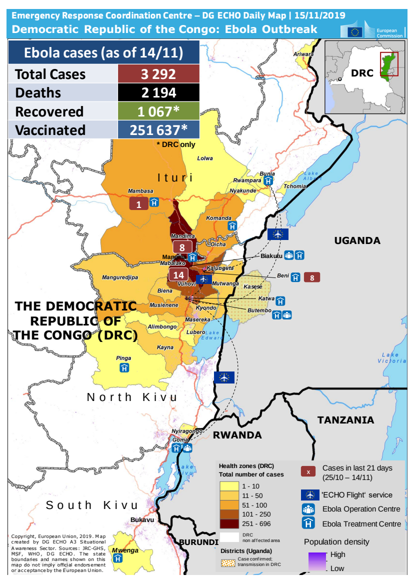


Figure 10: Statistics on EVD cases in DRC

The flow diagram of the transmission (i.e. movements (black arrows) and interations (coloured/lighter arrows) between compartments) of the disease is given in Figure 11, while Table 5 describes the variables and parameters of the model, which consists of the following system of ordinary differential equations:

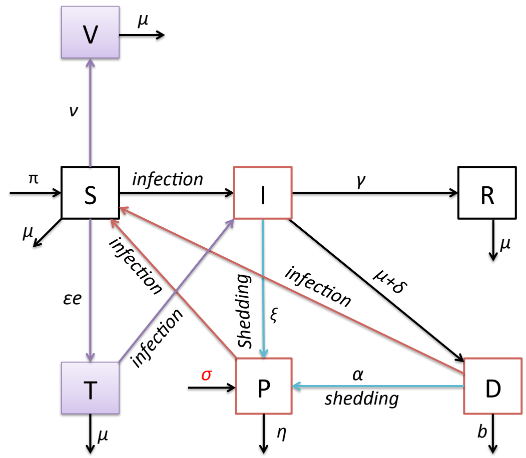


Figure 11: Schematic diagram of EVD transmission dynamics

$$\begin{cases} \dfrac{dS}{dt} = \Pi - (\epsilon e + \lambda + \nu + \mu)S, \\ \dfrac{dI}{dt} = \lambda S - (\gamma + \delta + \mu)I + (1-\tau)\lambda T, \\ \dfrac{dR}{dt} = \gamma I - \mu R, \\ \dfrac{dT}{dt} = \epsilon e S - (1-\tau)\lambda T - \mu T, \\ \dfrac{dV}{dt} = \nu S - \mu V, \\ \dfrac{dD}{dt} = (\delta + \mu)I - bD, \\ \dfrac{dP}{dt} = \sigma + \xi I + \alpha D - \eta P. \end{cases} \tag{45}$$

Here, the force of infection λ is in terms of the mass action in the absence of T and V classes [12]. Otherwise, we have combined standard incidence and mass action incidence [11]

$$\lambda = (\beta_1 I/N) + \beta_2 D + \beta_3 P \tag{46}$$

where

$$N = S + I + R + T + V \tag{47}$$

denotes the total population of human individuals. The awareness function

$$e(\lambda) = \lambda^n/(\lambda_0^n + \lambda^n), \tag{48}$$

is defined by the Hill-type function of order n in terms of the force of infection, λ_0 being the value of the force of infection corresponding to the threshold infectivity in which the individuals start reacting swiftly.

Symbols	Biological descriptions
S	Susceptible human individuals
I	Infectious human individuals
D	Ebola infected and deceased human individuals
R	Recovered human individuals
T	Trained human individuals
V	Vaccinated human individuals
P	Ebola virus pathogens in the environment
Π	Recruitment rate of susceptible human individuals
λ	Force of infection
η	Decay rate of Ebola virus in the environment
ξ	Shedding rate of infectious human individuals
α	Shedding rate of deceased human individuals
δ	Disease-induced death rate of human individuals
β_1	Effective contact rate of infectious human individuals
β_2	Effective contact rate of deceased human individuals
β_3	Effective contact rate of Ebola virus
τ	Average effectiveness of existing self-preventive measures
ν	rate of vaccination
e	Awarness function
ϵ	Rate of dissemination of the information about the disease
γ	Recovered rate of human individuals
μ	Natural death rate of human individuals
$1/b$	Mean caring duration of deceased human individuals
σ	Provision/recruitment rate of Ebola virus in the environment

Table 1: Variables and parameters for model (45)

It is shown in [11, 12] that the model (45) is a dynamical system on the biologically feasible region

$$\mathcal{K}_\sigma = \left\{ X \equiv (S,I,R,T,V,D,P) \in \mathbb{R}^7_+; \begin{array}{l} N \leq \dfrac{\Pi}{\mu}, \quad D \leq \dfrac{(\mu+\delta)\Pi}{b\mu}, \\ P \leq \dfrac{1}{\eta}\left(\dfrac{\sigma+\xi\Pi}{\mu} + \dfrac{\alpha(\mu+\delta)\Pi}{b\mu}\right) \end{array} \right\} \tag{49}$$

and that the positively invariant compact set $\mathcal{K}_\sigma$ is absorbing. Furthermore, it is established in these references that, in the case of the Ebola virus-free environment, i.e. $\sigma = 0$, the model has a unique disease-free equilibrium (DFE) denoted and given by

$$E_0 \equiv (S,I,R,T,V,D,P) = \left(\frac{\Pi}{\mu+\nu}, 0, 0, 0, \frac{\nu\Pi}{\mu(\mu+\nu)}, 0, 0\right). \tag{50}$$

In this case, it follows from the next generation matrix method [39] that the basic reproduction number of the model is

$$\mathcal{R}_0 = \frac{\Pi\beta_1}{(\nu+\mu)(\gamma+\delta+\mu)} + \frac{(\delta+\mu)\beta_2\Pi}{(\nu+\mu)b(\gamma+\delta+\mu)} + \frac{(b\xi+\alpha(\delta+\mu))\,\beta_3\Pi}{b\eta(\nu+\mu)(\gamma+\delta+\mu)}. \tag{51}$$

On the contrary, there is no disease-free equilibrium in the case of the full Ebola virus model with environmental transmission, i.e. $\sigma > 0$.

The main results in [11, 12], which will be at the center of he construction of our nonstandard finite difference discretizations, are summarized in the next theorem.

Theorem 21 *For the Ebola virus-free environment ($\sigma = 0$):*

1. *The disease-free equilibrium E_0 is globally asymptotically stable (GAS) when $\mathcal{R}_0 \leq 1$, and unstable otherwise.*

2. *If $\mathcal{R}_0 > 1$, there exists at least one endemic equilibrium (EE) E^*, which is locally asymptotically stable (LAS) for $\mathcal{R}_0$ close to 1. Furthermore, if there are no classes of vaccinated and trained human individuals, the endemic equilibrium E^* is unique; it is GAS in the absence of shedding ($\alpha = 0$) or manipulation of deceased human individuals before burial ($\xi = 0$).*

3. *The infectious component $I^*(\nu, \epsilon)$ of the EE corresponding to the rate of interventions ν and ϵ is less than the component $I^*(0, 0)$ in the absence of interventions: $I^*(\nu, \epsilon) < I^*(0, 0)$.*

For the full Ebola virus model with environmental transmission ($\sigma > 0$) without V and T classes :

1. *If $\mathcal{R}_0 > 1$, there is a unique EE $E^\#$, which is LAS. It is GAS when $\alpha = 0$ or $\xi = 0$.*

2. *The infectious component $I^\# = I^\#(\sigma)$ of the EE is an increasing function on the interval $0 \leq \sigma < \infty$, with $I^\#(0) = I^*$ denoting the infectious component of the unique EE of the model when $\sigma = 0$. Hence, the severity of the disease decreases with the reduction of the consumption of contaminated bush meat: $I^* < I^\#(\sigma)$.*

Remark 22 *On setting the right-hand side of (45) equal to zero, the existence of an endemic equilibrium $E = (S, I, R, T, V, D, P)$ amounts to showing that the component I or the force of infection λ, evaluated at E, is a positive root of a polynomial whose at least one of the coefficients is proportional to $1 - \mathcal{R}_0$, and to which Descartes rule of signs is applied (see [11, 12]). However, when $\sigma > 0$ and both compartments of vaccinated and trained human individuals are considered in the model, the structure of the involved polynomial, and in fact function, is so complex for the analysis to be carried out. This explains the restriction we made on the second part of the statement of Theorem 21.*

6 NSFD methodology through EVD

Mickens [28] proposed five rules for the construction of discrete models. However, in the formal definition of a nonstandard finite difference scheme proposed by two of the authors [6], only two out of the five rules are retained. As shown in [25], all the other rules can be expressed in terms of Definition 5 of qualitative stability. Note that the said two rules are:

Rule 2 The standard denominator Δt of the discrete derivative must be replaced by a more complex function ψ that satisfies the condition (18).

Rule 3 Nonlinear terms must, in general, be modeled nonlocally on the computational grid or lattice.

At this stage, our first concern is as follows: how to choose a nonlocal approximation of the nonlinear term? A procedure that is based on a possible perturbation on the targeted property of the model is presented in [7]. However, the answer to the question in this paper is guided by the fact that solutions $y(t)$ to biological systems must be nonnegative. Furthermore, the differential equation model enjoys a productive-destructive structure [16] in the sense that the evolution of the i^{th} component of the dependent variable y admits a decomposition of the form

$$\frac{dy_i}{dt} = p_i(y) - d_i(y)y_i \text{ with } p_i(y) \geq 0 \text{ and } d_i(y) \geq 0. \tag{52}$$

Consequently, we consider the following discretization that clearly preserves positivity:

$$\frac{y_i^{k+1} - y_i^k}{\psi} = p_i(y^k) - d_i(y^k)y_i^{k+1}. \tag{53}$$

From the computational perspective, the nonlocal approximation of nonlinear terms in the discrete productive-destructive structure (53) can be combined with the Gauss-Seidel cycle or process. In this regard, we approximate the first, the fourth and the second equations in (45) by

$$\begin{cases} \dfrac{S^{k+1} - S^k}{\psi} = \Pi - (\epsilon e^k + \lambda^k + \mu)S^{k+1} \\ \dfrac{T^{k+1} - T^k}{\psi} = \epsilon e^k S^{k+1} - (1-\tau)\lambda^k T^{k+1} - \mu T^{k+1} \\ \dfrac{I^{k+1} - I^k}{\psi} = \lambda^k S^{k+1} - (\mu + \delta + \gamma)I^{k+1} + (1-\tau)\lambda^k T^{k+1}. \end{cases} \tag{54}$$

Regarding the choice of the complex denominator function ψ, we update the general discussion made in Section 2 by considering sub-equations [28] and conservation laws associated with the model (45). According to [32], a conservation law is the differential equation obtained by adding all the equations of the model under consideration. But, since useful information may result from the sum of some of the dependent variables instead of all of them, we extend the terminology of "conservation law" to the equations satisfied by such partial sums of dependent variables. The process is described below.

- By adding the equations of human individuals in (45), we obtain the conservation law

$$\Pi - (\mu + \delta)N \leq \frac{dN}{dt} \leq \Pi - \mu N. \tag{55}$$

  that the numerical scheme must preserve. In view of the construction done in Section 2, we consider the following discrete conservation law, with a specific denominator function:

$$\Pi - (\mu + \delta)N^{k+1} \leq \frac{N^{k+1} - N^k}{[1 - \exp(-(\mu + \delta)\Delta t)]/(\mu + \delta)} \leq \Pi - \mu N^{k+1}, \tag{56}$$

- Apart from the first equation in (45), each of the remaining differential equations contains a decay sub-equation, a nonstandard scheme of which can be constructed as in Section 2. For example, for the sub-equation

$$\frac{dI}{dt} = -(\mu + \delta + \gamma)I, \tag{57}$$

we consider the NSFD scheme

$$\frac{I^{k+1} - I^k}{[1 - \exp(-(\mu + \delta + \gamma)\Delta t)]/(\mu + \delta + \gamma)} = -(\mu + \delta + \gamma)I^{k+1}. \tag{58}$$

Combining the information on the denominator function in the discrete conservation law (56) with those of each sub-equation of the type (58), we consider the global denominator function

$$\psi = \frac{1 - \exp(-(\mu + \delta + \gamma + \nu + \eta + b)\Delta t)}{\mu + \delta + \gamma + \nu + \eta + b}. \tag{59}$$

Note that the rate b of deceased human individuals who are not directly burried could be excluded from (59) because this number is supposed to be less than or equal to the disease-induced death rate δ that included in the expression of ψ. But, we have included both parameters to simplify the presentation. Thus, we obtain the following NSFD scheme for the model (45) in the Gauss-Seidel cycle:

$$\begin{cases} \dfrac{S^{k+1} - S^k}{\psi} = \Pi - (\epsilon e^k + \lambda^k + \nu + \mu)S^{k+1}, \\ \dfrac{T^{k+1} - T^k}{\psi} = \epsilon e^k S^{k+1} - \big((1-\tau)\lambda^k + \mu\big)\, T^{k+1}, \\ \dfrac{I^{k+1} - I^k}{\psi} = \lambda^k S^{k+1} - (\gamma + \delta + \mu)I^{k+1} + (1-\tau)\lambda^k T^{k+1}, \\ \dfrac{R^{k+1} - R^k}{\psi} = \gamma I^{k+1} - \mu R^{k+1}, \\ \dfrac{V^{k+1} - V^k}{\psi} = \nu S^{k+1} - \mu V^{k+1}, \\ \dfrac{D^{k+1} - D^k}{\psi} = (\mu + \delta)I^{k+1} - bD^{k+1}, \\ \dfrac{P^{k+1} - P^k}{\psi} = \sigma + \xi I^{k+1} + \alpha D^{k+1} - \eta P^{k+1}, \end{cases} \tag{60}$$

Solving sequentially each equation in the NSFD scheme (60) for the unknown discrete solution at the time t_{k+1}, we obtain the following equivalent formulation of the scheme, which is more suitable for computational and programming purposes:

$$\begin{cases} S^{k+1} = \dfrac{\Pi\psi + S^k}{1+\psi(\epsilon e^k + \lambda^k + \nu + \mu)}, \\ T^{k+1} = \dfrac{\psi\epsilon e^k S^{k+1} + T^k}{1+\psi((1-\tau)\lambda^k + \mu)} \\ I^{k+1} = \dfrac{I^k + \psi(\lambda^k S^{k+1} + (1-\tau)\lambda^k T^{k+1})}{1+\psi(\gamma+\delta+\mu)} \\ R^{k+1} = \dfrac{R^k + \psi\gamma I^{k+1}}{1+\psi\mu} \\ V^{k+1} = \dfrac{V^k + \psi\nu S^{k+1}}{1+\psi\mu} \\ D^{k+1} = \dfrac{D^k + \psi(\mu+\delta) I^{k+1}}{1+\psi b} \\ P^{k+1} = \dfrac{P^k + \psi(\sigma + \xi I^{k+1} + \alpha D^{k+1})}{1+\psi\eta}. \end{cases} \tag{61}$$

Theorem 23 *The NSFD scheme (60) is a discrete dynamical system on the very same region $\mathcal{K}_\sigma$ where the continuous model is a dynamical system. Equally, this set is absorbing.*

Proof. Given the vector $X^k \equiv (S^k, I^k, R^k, T^k, V^k, D^k, P^k)$ in $\mathcal{K}_\sigma$, we have to show that the vector $X^{k+1} \equiv (S^{k+1}, I^{k+1}, R^{k+1}, T^{k+1}, V^{k+1}, D^{k+1}, P^{k+1})$ that is defined by (60) is in $\mathcal{K}_\sigma$ as well.

By construction, we have from (61), $X^{k+1} \geq 0$. Now from the inequality on the right-hand side of the conservation law (56) in which the appropriate denominator function ψ given in (59) is used, we obtain

$$N^{k+1} - N^k \leq \psi(\Pi - \mu N^{k+1}). \tag{62}$$

Therefore, we have

$$\begin{aligned} N^{k+1} &\leq (\psi\Pi + N^k)/(1+\psi\mu) \\ &\leq [\psi\Pi + (\Pi/\mu)]/(1+\psi\mu) \text{ as } N^k \leq \Pi/\mu \text{ by definition of } \mathcal{K}_\sigma \\ &= \Pi/\mu. \end{aligned} \tag{63}$$

Next, from the last two equations in (61), we have

$$\begin{aligned} D^{k+1} &\leq [((\mu+\delta)\Pi/b\mu) + (\psi(\mu+\delta)\Pi/\mu)]/(1+\psi b) \text{ by definition of } \mathcal{K}_\sigma \text{ and by (63)} \\ &= (\mu+\delta)\Pi/(b\mu), \end{aligned} \tag{64}$$

and

$$\begin{aligned} P^{k+1} &\leq \frac{(\sigma+\xi\Pi)/(\mu\eta) + \alpha(\mu+\delta)\Pi)/(b\mu\eta) + \psi[\sigma + \xi(\Pi/\mu) + \alpha(\mu+\delta)\Pi/(b\mu)]}{1+\psi\eta} \\ &\quad \text{by definition of } \mathcal{K}_\sigma \text{ as well as by (63) and (64)} \\ &= \sigma/\eta + (\xi\Pi)/\mu\eta) + \alpha(\mu+\delta)\Pi/(b\mu\eta). \end{aligned} \tag{65}$$

Combining $X^{k+1} \geq 0$ with (63), (64) and (65), it follows that $X^{k+1} \in \mathcal{K}_\sigma$ and, thus, the NSFD scheme (60) defines a discrete dynamical system on $\mathcal{K}_\sigma$.

Finally, we prove that $\mathcal{K}_\sigma$ is an absorbing set. To this end, we assume that the sequence (X^k) is in $\mathbf{R}^7_+ \setminus \mathcal{K}_\sigma$ such that for all k, we have for simplicity, $N^k > \Pi/\mu$, $D^k > (\mu+\delta)\Pi/(b\mu)$ and $P^k > \sigma/\eta + (\xi\Pi)/\mu\eta) + \alpha(\mu+\delta)\Pi/(b\mu\eta)$. From the discrete conservation law (62), we deduce on the one hand that the sequence (N^k) is decreasing. On the other hand, the discrete Gronwall inequality [36] applied to this discrete law implies that

$$\Pi/\mu < N^k \leq (\Pi/\mu)(1-(1+\psi\mu)^{-k}) + N^0(1+\psi\mu)^{-k}. \tag{66}$$

This implies that

$$N^k \to \Pi/\mu \text{ as } k \to \infty. \tag{67}$$

The last two equations in (60) lead to two difference inequalities, which owing to (67) are, for k large enough, equivalent to

$$\frac{D^{k+1}-D^k}{\psi} \leq (\mu+\delta)\Pi/\mu - bD^{k+1}, \tag{68}$$

$$\frac{P^{k+1}-P^k}{\psi} \leq \sigma + \xi\Pi/\mu + \alpha D^{k+1} - \eta P^{k+1}. \tag{69}$$

Since by assumption, $D^{k+1} > (\mu+\delta)\Pi/(b\mu)$, Eq. (68) and the discrete Gronwall inequality imply that, for k large enough , the sequence (D^k) is decreasing such that

$$(\mu+\delta)\Pi/(b\mu) < D^k \leq [(\mu+\delta)\Pi/(b\mu)][1-(1+\psi b)^{-k}] + D^0(1+\psi b)^{-k}. \tag{70}$$

This implies that

$$D^k \to (\mu+\delta)\Pi/(b\mu) \text{ as } k \to \infty. \tag{71}$$

We infer from (69) and (71) that

$$\frac{P^{k+1}-P^k}{\psi} \leq \sigma + \xi\Pi/\mu + \alpha(\mu+\delta)\Pi/(b\mu) - \eta P^{k+1} \tag{72}$$

for k large enough. Consequently,

$$P^k \to \sigma/\eta + (\xi\Pi)/\mu\eta) + \alpha(\mu+\delta)\Pi/(b\mu\eta) \text{ as } k \to \infty, \tag{73}$$

because it follows from (72) and the discrete Gronwall inequality that the sequence (P^k) is decreasing such that

$$\begin{cases} \sigma/\eta + (\xi\Pi)/\mu\eta) + \alpha(\mu+\delta)\Pi/(b\mu\eta) < P^k \text{ and} \\ P^k \leq [\sigma/\eta + (\xi\Pi)/\mu\eta) + \alpha(\mu+\delta)\Pi/(b\mu\eta)][1-(1+\psi\eta)^{-k}] + P^0(1+\psi\eta)^{-k}. \end{cases} \tag{74}$$

Eqs. (67), (71) and (73) show that the set $\mathcal{K}_\sigma$ is absorbing and this completes the proof of the theorem. ■

Remark 24 *The NSFD scheme (60) is a particular case ($\theta = 1$ and $\hat{\theta} = 1$) of a new family of nonstandard method depending on two parameters $\theta \in [0,1]$ and $\hat{\theta} \in [0,1]$, which is investigated in [4], for the MSEIR epidemiological model. The additional parameter $\hat{\theta}$ is used to deal with the weighted average of nonlinear tems* *at times t_k and t_{k+1}. For example, the right-hand sides of the first two equations in(60) are extended as follows:*

$$\begin{cases} \Pi - [\hat{\theta}(\epsilon e^k + \lambda^k)S^{k+1} + (1-\hat{\theta})(\epsilon e^{k+1} + \lambda^{k+1})S^k] - (\nu+\mu)[\theta S^{k+1} + (1-\theta)S^k]; \\ [\hat{\theta}\epsilon e^k S^{k+1} + (1-\hat{\theta})\epsilon e^{k+1} S^k] - (1-\tau)[\hat{\theta}\lambda^k T^{k+1} + (1-\hat{\theta})\lambda^{k+1} T^k] - \mu[\theta T^{k+1} + (1-\theta)T^k]. \end{cases} \tag{75}$$

It is possible to consider, for the EVD system (45), a similar NSFD scheme that will then be in the spirit of the NSFD scheme (16), with denominator function chosen carefully in light of (24), (25) and (26). However, the strong nonlinearity of the continuous model (45) can make the implementation of such a scheme difficult, as reflected from Eq. (75). Nevertheless, when the force of infection λ in (46) is a linear function expressed in terms of the mass action incidence [12], and there are no vaccinated as well as trained human individual classes, the analogue of the NSFD scheme in [4] is:

$$\begin{cases} \dfrac{S^{k+1}-S^k}{\psi} = \Pi - [\hat{\theta}\lambda^k S^{k+1} + (1-\hat{\theta})\lambda^{k+1} S^k] - \mu[\theta S^{k+1} + (1-\theta)S^k] \\ \dfrac{I^{k+1}-I^k}{\psi} = [\hat{\theta}\lambda^k S^{k+1} + (1-\hat{\theta})\lambda^{k+1} S^k] - (\gamma+\delta+\mu)[\theta I^{k+1} + (1-\theta)I^k] \\ \dfrac{R^{k+1}-R^k}{\psi} = \gamma[\theta I^{k+1} + (1-\theta)I^k] - \mu[\theta R^{k+1} + (1-\theta)R^k] \\ \dfrac{D^{k+1}-D^k}{\psi} = (\mu+\delta)[\theta I^{k+1} + (1-\theta)I^k] - b[\theta D^{k+1} + (1-\theta)D^k] \\ \dfrac{P^{k+1}-P^k}{\psi} = \sigma + \xi[\theta I^{k+1} + (1-\theta)I^k] + \alpha[\theta D^{k+1} + (1-\theta)D^k] \\ \qquad -\eta[\theta P^{k+1} + (1-\theta)P^k]. \end{cases} \tag{76}$$

The NSFD scheme (76) amounts to solving at each step an equivalent linear sytem of algebraic equations

$$\mathcal{A}\left(S^{k+1}, I^{k+1}, R^{k+1}, D^{k+1}, P^{k+1}\right) = \psi\left(\Pi, 0, 0, 0, \sigma\right) + \mathcal{B}\left(S^k, I^k, D^k, P^k\right), \tag{77}$$

where the matrices $\mathcal{A}$ and $\mathcal{B}$ are given by

$$\begin{bmatrix} 1+\psi\hat{\theta}\lambda^k+\psi\mu\theta & \psi(1-\hat{\theta})\beta_1 S^k & 0 & \psi(1-\hat{\theta})\beta_2 S^k & \psi(1-\hat{\theta})\beta_3 S^k \\ -\psi\hat{\theta}\lambda^k & 1-\psi(1-\hat{\theta})\beta_1 S^k + \psi\theta(\gamma+\delta+\mu) & 0 & -\psi(1-\hat{\theta})\beta_2 S^k & -\psi(1-\hat{\theta})\beta_3 S^k \\ 0 & -\psi\gamma\theta & 1+\psi\mu\theta & 0 & 0 \\ 0 & -\psi(\gamma+\delta)\theta & 0 & 1+\psi b\theta & 0 \\ 0 & -\psi\xi\theta & 0 & -\psi\alpha\theta & 1+\psi\eta\theta \end{bmatrix} \tag{78}$$

and

$$\begin{bmatrix} 1-\psi\mu(1-\theta) & 0 & 0 & 0 & 0 \\ 0 & 1-\psi(1-\theta)(\gamma+\delta+\mu) & 0 & 0 & 0 \\ 0 & \psi\gamma(1-\theta) & 1-\psi\mu(1-\theta) & 0 & 0 \\ 0 & \psi(\mu+\delta)(1-\theta) & 0 & 1-\psi b(1-\theta) & 0 \\ 0 & \psi\xi(1-\theta) & 0 & \psi\alpha(1-\theta) & 1-\psi\eta(1-\theta) \end{bmatrix} \tag{79}$$

respectively. From the expressions of the above matrices, we have to impose the following conditions on the complex denominator ψ in order to solve the system

(77):

$$\psi < [(1-\theta)(\gamma+\delta+\mu+b+\eta)]^{-1} \quad (80)$$
$$\psi < [(\gamma+\xi)\theta + 2(1-\hat{\theta})\beta_1\Pi/\mu]^{-1} \quad (81)$$
$$\psi < [\alpha\theta + 2(1-\hat{\theta})\beta_2\Pi/\mu]^{-1} \quad (82)$$
$$\psi < [2(1-\hat{\theta})\beta_3\Pi/\mu]^{-1}. \quad (83)$$

Condition (80) guarantees that the entries of the matrix $\mathcal{B}$ are nonegative. Combined with Conditions (81)-(83), it follows like in [4] that $\mathcal{A}$ is is an M-matrix because its transpose is strictly diagonally dominant. Consequently, the matrix $\mathcal{A}$ is nosingular, with $\mathcal{A}^{-1} \geq 0$, and thus the discrete solution X^{k+1} of the system is nonegative. In the spirit of (25) and (26) and setting

$$\tilde{q} = \alpha+\xi+\gamma+\delta+\mu+b+\eta,$$

Conditions (80)-(83) can be captured by taking

$$q \equiv q_{\hat{\theta},\theta} \geq \max\{(1-\theta)\tilde{q}, \theta\tilde{q} + 2(1-\hat{\theta})\beta_1\Pi/\mu, \theta\tilde{q} + 2(1-\hat{\theta})\beta_2\Pi/\mu, 2(1-\hat{\theta})\beta_3\Pi/\mu\}, \quad (84)$$

with the denominator function given by (24). The condition (84) on q is stronger than the one on the q considered in (59) because in the latter case, there was no issue to solve the equation (76) in X^{k+1}, as seen from the expression of the solution in (61).

In [4], the elementary stability and the global asymptotic stability of the disease-free equilibrium for the analogue of the NSFD scheme (76) was proved under specific conditions on θ and $\hat{\theta}$ such as $\theta = 0$ or $\theta + \hat{\theta} \geq 1$. We will check this numerically in the next section for the particular case when $\theta = 0$ and $\hat{\theta} = 1$ that is studied in [3].

7 Numerical Simulations for EVD

In this section, we provide numerical simulations for the NSFD scheme (60) to illustrate the results on the dynamics of the EVD model (45), as stated in Theorem 21. As mentioned earlier, proofs of the discrete analogues of these results for the nonstandard method are discussed in or can be deduced from [3, 4, 11, 12]. The values that we use for the parameters are given in Table 2. Some of them are taken from the literature (e.g. [15, 33]) and have served in [11] to evaluate the sensitivity index of the basic reproduction number $\mathcal{R}_0$ with respect to each parameter.

β_1	β_2	β_3	α	ξ	Π	b	η	δ	ν	μ	γ	τ	ϵ	n
0.006	0.012	0.01	0.04	0.04	10	0.8	0.03	0.8	0.005	0.02	0.06	0.5	0.9	2

Table 2: Parameter values of the model (45)

We start by simulations related to whether or not the numerical scheme is a discrete dynamical system on the set $\mathcal{K}_\sigma$. Figure 12 illustrates that, unlike the NSFD scheme, the ode45 implemented in MatLab can produce negative solutions. The same observation is made for the fourth-order Runge-Kutta method that displays negative and unbounded total population N of human individuals, while the NSFD scheme is dynamically consistent with respect to the property $0 \leq N \leq \Pi/\mu = 500$.

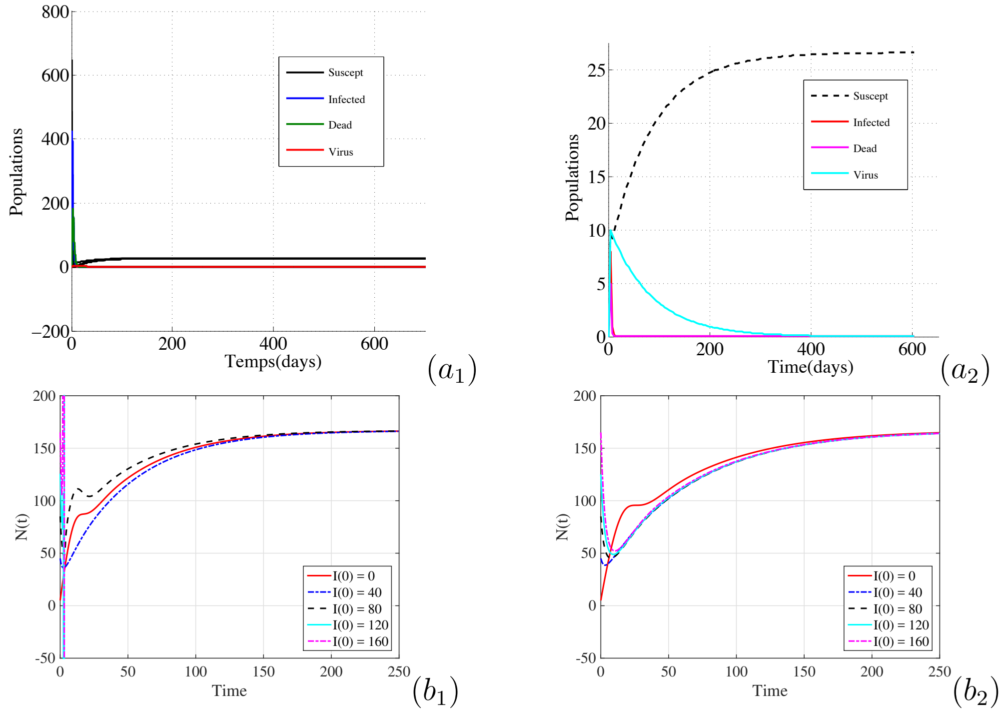


Figure 12: Top pictures are for the model without vaccinated and trained individuals: see (a_1) for ode45 and (a_2) for NSFD scheme. Bottom pictures are for the full model: see (b_1) for the Runge-Kutta method and (b_2) for the NSFD scheme

The latter fact is in line with Theorem 23, which states that the NSFD scheme (60) is a discrete dynamical system on $\mathcal{K}_\sigma$.

The next set of simulations is devoted to the stability of the equilibrium points. In the absence of vaccinated and trained classes and when $\mathcal{R}_0 > 1$, Figure 13 displays the local asymptotic stability of the unique endemic equilibrium point $E^{\#}$ in the full Ebola virus environment characterized by $\sigma > 0$. Note that there is no disease-free equilbrium in this case. Furthermore, we have Figure 14 that deals with the Ebola virus-free environment ($\sigma = 0$), with $\mathcal{R}_0 > 1$ and vaccinated and trained classes present or absent. It is seen from this figure that the endemic equilibrium is locally asymptotically stable in the presence of vaccinated and trained classes, but globally asymptotically stable in their absence. Figure 15 illustrates the global asymptotic stability of the disease-free equilibrium, which exists and is unique when $\sigma = 0$ and $\mathcal{R}_0 \leq 1$.

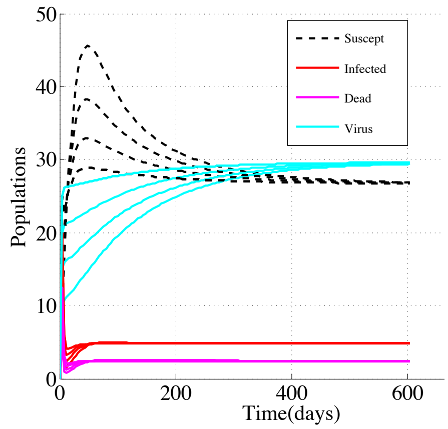


Figure 13: Local asymptotic stability of $E^{\#}$ for $\sigma = 0.6$. See [12] for parameter values that lead to $\mathcal{R}_0 = 17,1$

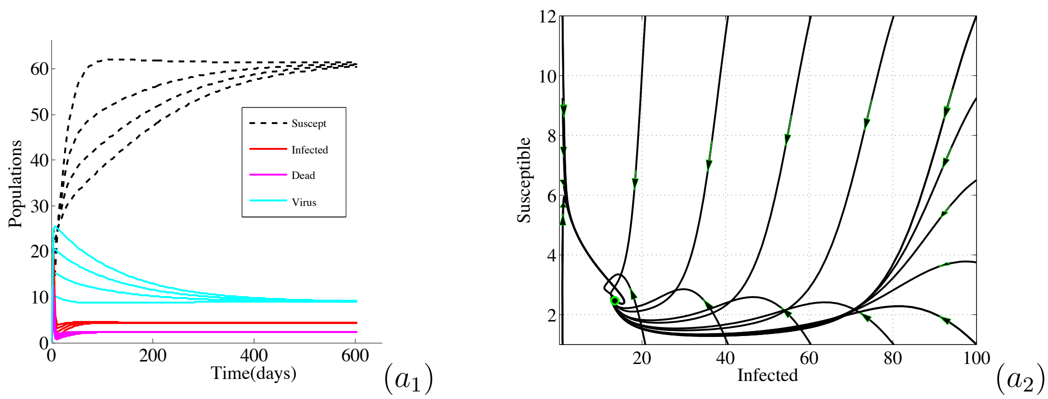


Figure 14: Depending on the absence or presence of the V and T classes, the endemic equilibrium point is GAS (a_1) or LAS (a_2) for $\sigma = 0$, with $\mathcal{R}_0 = 17.1$ in the first case (see [12]) and $\mathcal{R}_0 = 3.23$ in the second case

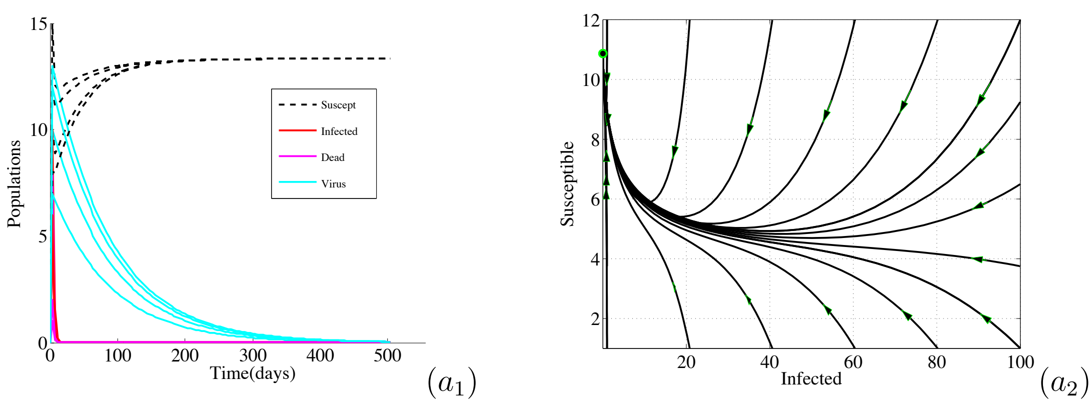


Figure 15: Global asymptotic stability of the disease-free equilibrium in the absence (a_1) or presence (a_2) of the V and T classes for $\sigma = 0$ with $\mathcal{R}_0 = 0.062$ in the first case (see [12]) and $\mathcal{R}_0 = 0.082$ in the second case

Our third set of simulations is concerned with the severity of the infection and the impact of the interventions by comparing the infectious component I of the endemic equilibria in the presence and absence of intervention. For the model without vaccinated and trained individuals, Figure 16(a) shows that $I(0) < I(\sigma)$ for $\sigma = 0.6$, which illustrates that not eating contaminated bush meat would reduce the infection. Figure 16(b,c) is self-explanatory: the implementation of all the interventions or of at least some of them contribute to reduce the severity of the infection. These facts align with Figure 16(d,e), which illustrates the sensitivity analysis that is done in [11], where the parameter β_3 (contact rate of Ebola virus) is the most sensitive one.

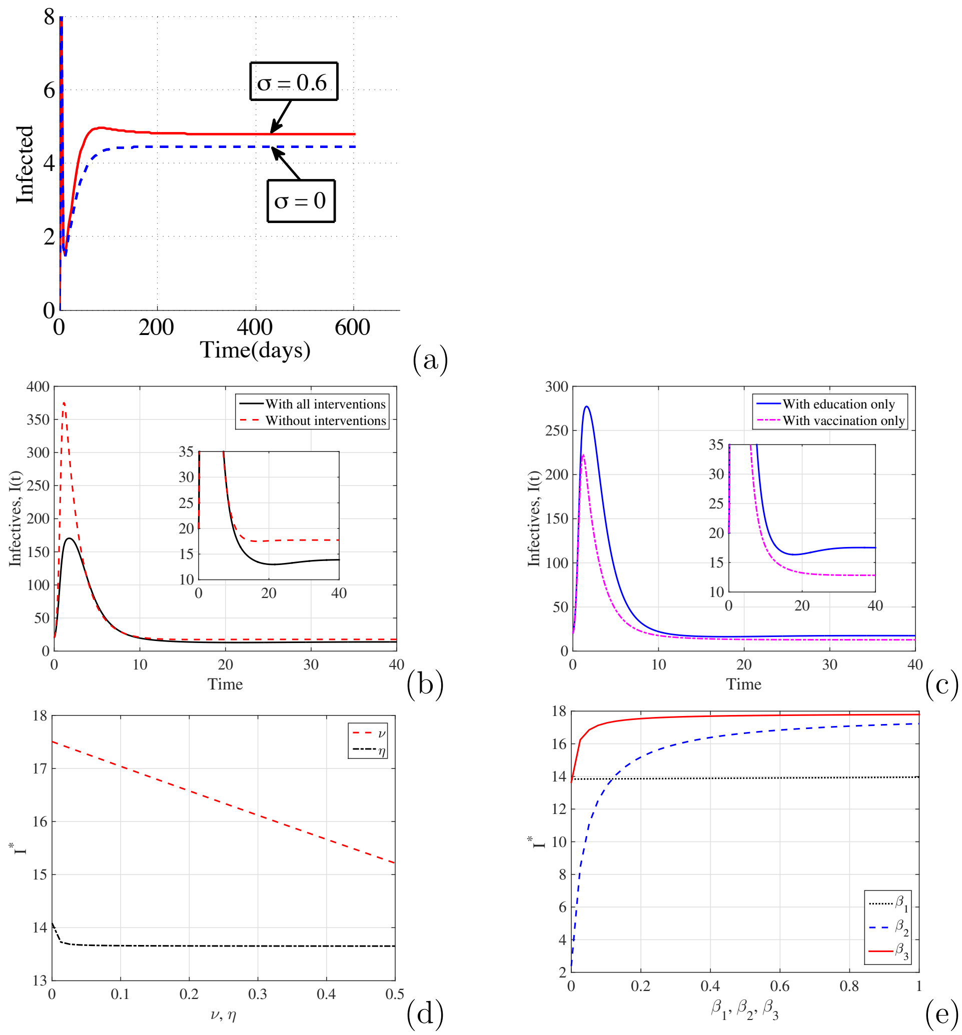


Figure 16: (a) $I^{\#}(0) < I^{\#}(\sigma)$ for $\sigma = 0.6$ (see [12]); (b) Number of infectious with and without interventions; (c) Number of infectious with education and vaccination interventions; (d) Sensitivity analysis wrt ν and η; (e) Sensitivity analysis wrt to β_1, β_2 and β_3.

Finally, we consider, for $\theta = 0$ and $\hat{\theta} = 1$, the phase diagram of the NSFD scheme (76) outlined in Remark 24. In line with the second part of Theorem 21, Figure 17 illustrates that the disease stabilizes at an endemic equilibirium when $\mathcal{R}_0 > 1$ and that its severity increases with the consumption of bats, hunted meat and fruits from rain-forest.

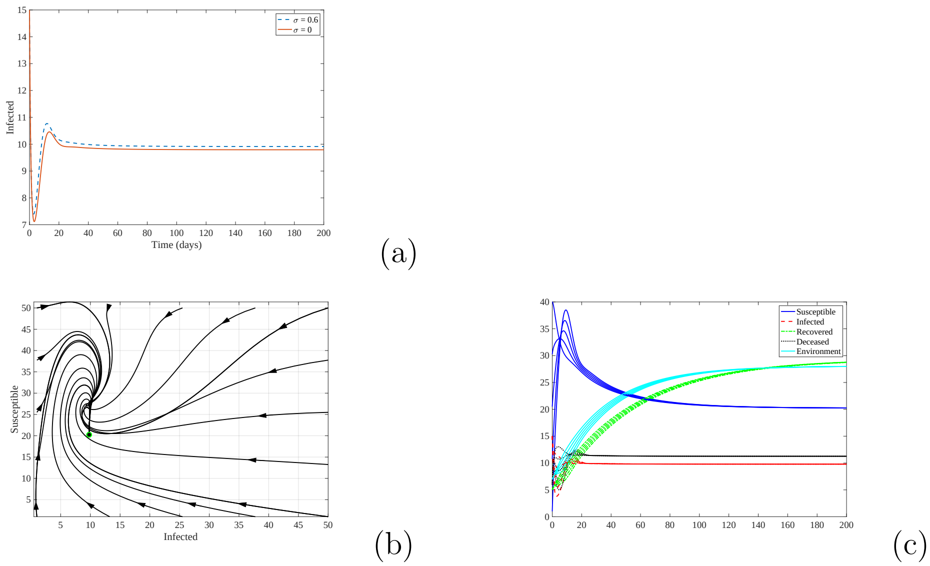


Figure 17: (a): Severity of the disease in terms of the provision σ of Ebola virus ($\sigma = 0.6$); (b) & (c): Stability of the endemic equilibrium for $\mathcal{R}_0 = 17.1$

8 Concluding Remarks

The main purpose of this work is to revisit the nonstandard finite difference (NSFD) method, with the precise aim of answering the double question regarding two of Mickens' rules [28]: how to choose the denominator function of the discrete derivative (Rule 2) and how to perform the nonlocal approximation of nonlinear terms in the right-hand side of the differential equation (Rule 3)? The double question is addressed in the setting of one or more of the following three facts that specifically motivate this work:

- To investigate whether the sharp condition given in [17] for the elementary stability of the nonstandard forward Euler method is related to the location of the eigenvalues of the involved Jacobian matrices in specific regions of the complex plane;
- A follow up to the chapter [25] where the authors announced that other types of dissipative properties of differential equation models would be investigated, apart from the focus in the chapter on the dissipativity of singular perturbed problems, which has a specific meaning in terms of the decay/variation of their solutions in layer regions;
- A classical result on theta methods that restrict their dissipativity as discrete dynamical systems to the range $\theta \in (1/2, 1]$ (see, e.g., [36]).

To address these issues, we have constructed nonstandard theta methods obtained by using Mickens'rule 2, and giving effective and practical guidelines on the choice of the denominator of the discrete derivatives. On the one hand, we showed that the condition in [17] is equally sufficient for the elementary stability in this general setting of nonstandard theta methods. On the other hand, we proved that the stated condition is equivalent to having the eigenvalues of the Jacobian matrices located in some wedges of the complex plane. Unlike our earlier works, the term dissipative is used here to express the fact that the gross asymptotics of a dynamical system are independent of initial conditions with everything ending up inside some absorbing set. We showed that, for θ taking the smallest value 0 in the forbidden interval $[0, 1/2)$, our explicit scheme, i.e. the nonstandard forward Euler scheme replicates the dissipative property of the continuous dynamical system. We also presented some numerical examples that support the theory.

We took the study to a next level by considering the transmission dynamics of the Ebola Virus Disease (EVD). In this context of systems with much richer dynamics, we effectively enriched the methodology of the nonstandard approach for the implementation of Mickens' Rule 2 and Rule 3. This led to the construction of a NSFD scheme that preserves the property of the continuous model that can be paraphrased as follows [11, 12]: The consumption of contaminated bush meat, the funeral practices and the enviromental contamination can explain the recurrence and persistence of EVD outbreaks in Africa, while self-protection measures of the population in terms of vaccination, training people and providing educational programme reduce the severity of the disease.

A natural follow up to the first part of this paper is to investigate the dissipativity of the nonstandard theta methods for any value of the parameter θ. In light of Example 19 and Remark 20, it is also important to construct schemes which are qualitatively stable with respect to global attractors. Furthermore, we are interested in extending this study to the design of dissipative schemes for evolution partial differential differential equations.

Apart from research questions identified in [11] for future, it is worthwile to incorporate the treatment in the transmission dynamics of the EVD model. Indeed, recently discovered drugs such as REGN-EB3 and mAB114 show that people who received either therapy soon after infection have a 90% survival rate [27].

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