The compounding method for finding bivariate noncentral distributions

by

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Declaration

I, Johannes Theodorus Ferreira, declare that this dissertation, which I hereby submit for the degree Master of Science (Mathematical Statistics) at the University of Pretoria, is my own work and has not previously been submitted by me for a degree at this or any other tertiary institution.

SIGNATURE: ....
DATE: ....

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Summary

The univariate and bivariate central chi-square- and F distributions have received a decent amount of attention in the literature during the past few decades; the noncentral counterparts of these distributions have been much less present. This study enriches the existing literature by proposing bivariate noncentral chi-square and F distributions via the employment of the compounding method with Poisson probabilities. This method has been used to a limited extent in the field of distribution theory to obtain univariate noncentral distributions; this study extends some results in literature to the corresponding bivariate setting. The process which is followed to obtain such bivariate noncentral distributions is systematically described and motivated. Some distributions of composites (univariate functions of the dependent components of the bivariate distributions) are derived and studied, in particular the product, ratio, and proportion. The benefit of introducing these bivariate noncentral distributions and their respective composites is demonstrated by graphical representations of their probability density functions. Furthermore, an example of possible application is given and discussed to illustrate the versatility of the proposed models.

Keywords : bivariate, composites, compounding, noncentral, Poisson

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"Totally mad," he said, "utter nonsense. But we'll do it because it's brilliant nonsense."

~ Douglas Adams

The Restaurant at the End of the Universe (1980)

Contents

Chapter 1

Introduction

1.1 Background and motivation

Noncentral distributions have always been a subject of interest due to its inadmissible role in hypothesis testing and construction of power of tests. Speaking in general of a distribution, one is usually implicitly assuming referral to the central distribution – this is the distribution that is most commonly used for testing null hypotheses. However, under the alternative hypothesis, the pivotal quantity does not conform to a central distribution, but rather to its noncentral counterpart, due to some shift observed in the alternative hypothesis. This is of special importance in determining the power of the test, since it is only under the alternative hypothesis one would like to determine the test's ability to pick up departures from the null hypothesis.

Furthermore, extensions from univariate distribution theory – which has been saturated for some time now – to bivariate, trivariate, and in general, multivariate cases, has been the topic of study by many authors for the past few decades. This has been done mainly to attempt to accommodate the changing of statistical paradigms to scenarios where a multitude of variables and data are becoming more easily available. Of course, should these variables be independent, the mathematics involved in constructing joint probability density functions (pdfs) from which power of tests, critical values, and other measures are computed, becomes greatly simplified due to the product property of independent variables (see Bain & Engelhardt (1992), p. 150). However, in practice, it is rare that one would observe the case of independence between variables, and construction of correlation structures in such an environment have become eminent.

This task of constructing dependent multivariate distributions is no mean feat. Introducing correlation coefficient(s) between variables often result in cumbersome and clumsy expressions which leaves little room for any reasonable measure of inference. Be that as it may, due to the past few year's technological advances, performing analyses such as these even with correlation structures has become much more manageable computation-wise, and motivates sufficiently for still maintaining this branch of distribution theory.

The genesis for this study originated from the papers by Yunus & Khan (2011) and by Van den Berg et. al. (2013), the last of which was a result from Van den Berg (2010). The methods used in this dissertation are well-known in the statistical universe, and is rephrased here within the following descriptions.

Description 1.1 *(Kotz et. al. (2006b), p. 4880) A pdf $f(x)$ is said to have a mixture or compound pdf if it has the general form*

$$f(x) = \int f(x|\theta)h(\theta)\, d\theta$$

where $f(x|\theta)$ is a conditional pdf depending on the parameter θ, itself subject to chance variation described by the pdf $h(\theta)$, the mixing- or compounding probability density. When $h(\theta)$ is discrete, the integral is replaced by a sum to give a mixture- or compound pdf of the form

$$f(x) = \sum_i f(x|\theta_i)h(\theta_i)\,. \tag{1.1}$$

Strictly speaking, the terms $h(\theta_i)$ are called the mixing proportions, subject to the constraint that $\sum_i h(\theta_i) = 1$. In this study, the terms $h(\theta_i)$ are referred to as the compounding factors. In literature, the terms mixing- and compound are often used interchangeably – in this study, an equation of the likes as (1.1) will be termed a compound pdf – the method is termed the compounding method. Marshall & Olkin (1990) described this method in similar fashion. ▲

In this study this compounding method is used to:

(i) structure existing models in forms such as in (1.1);

(ii) to derive noncentral counterparts for the considered bivariate central distributions;

(iii) derive distributions of functions of the dependent components (termed *composites*) of the considered bivariate distributions.

The term *composite* is rephrased in the form of the following description:

Description 1.2 *If X_1 and X_2 have a bivariate distribution, then a univariate function of the components X_1 and X_2 is said to be a composite.* ▲

The main focus will be on bivariate chi-square models following from Van den Berg (2010), as well as Yunus & Khan (2011). The journey is then extended to new bivariate F distributions. It is the first time, to the knowledge of the author, that the role of the compounding method is systematically described in these existing and new bivariate models.

1.2 Literature review

1.2.1 From the noncentral to the bivariate chi-square and F

For the univariate universe, several known central- and noncentral distributions and computation methods are available in the literature: Patnaik (1949) showed that compounding a central chi-square distribution with Poisson probabilities results in its noncentral counterpart. Fraser et. al. (1998) proposed an approximation for the noncentral chi-square distribution using third-order asymptotic methods, which is heavily restricted in its parametrization of the degrees of freedom being linked to the noncentrality parameter. Robertson (1969) provided a formula to calculate accurate values of the cumulative noncentral chi-square distribution over a range of parameters, but proved to be computationally inefficient.

A limited number of work is available on the bivariate noncentral chi-square distribution. Yunus & Khan (2011) extended the approach of Patnaik (1949) for only isolated cases of the bivariate chi-square distribution. Some interesting extensions to various bivariate distributions e.g. Poisson, negative binomial, and chi square was proposed by Marshall & Olkin (1990) via mixtures of convolution and product methods. As mentioned, Marshall & Olkin (1990) provided an example of a noncentral bivariate chi-square distribution but only considering one noncentrality parameter for the joint probability density function. Yunus & Khan (2011) discussed the lack of available noncentral bivariate chi-square distributional forms that is suitable for computation, stemming from their proposed distribution based on a central bivariate chi-square probability density function originally proposed by Krishnaiah et. al. (1963). Nadarajah (2010) provides simple expressions which acts as substitutes for other known results in the bivariate chi-square literature.

The F distribution enjoys widespread use in statistical inference due to its extensive use in, amongst others, analysis of variance. It is common knowledge that the univariate F distribution is a ratio function of two independent chi-square distributed random variables and their degrees of freedom. Some years ago Mudholkar et. al. (1976) provided some approximations for the noncentral (univariate) F distribution, Tiku (1966) also provided some suggestions of approximations for a similar distribution.

Bivariate representations have been studied to some extent (see Balakrishnan & Lai (2009) and Nadarajah (2008)), but bivariate noncentral F distributions have not been widely available or studied. A pioneering paper in the literature regarding bivariate F distributions is the paper by El-Bassiouny & Jones (2009), where special emphasis was laid upon the marginal components and their respective distributions. Furthermore, Krishnaia (1964) studied the central bivariate F distribution - however, very little has been mentioned anywhere in the literature regarding a noncentral counterpart. It seems evident that the literature has not been provided with sufficient studies on bivariate noncentral F distributions.

1.2.2 Composites

The derivation and study of univariate functions from bivariate (and multivariate) distributions is not new in the literature. These composites provide distributions which can often be used in a comparative sense: for example, the case $\frac{X_1}{X_2}$. Such a univariate distribution is often used in a stress-strength environment, to investigate a certain random strength X_1 subject to a random stress X_2 - which, jointly, follows some bivariate distribution. Al-Ruzaiza & El-Gohary (2008) mentions that such univariate distributions are interesting in statistics and has several important applications; ranging from reliability- and industrial engineering, to computer systems. In that paper the authors specifically mention the bivariate beta distribution - whose univariate distributions of the composites have special significance in a Bayesian context, since beta distributions provide a family of conjugate prior distributions for binomial distributions. In addition, Nadarajah & Kotz (2006) motivates the study of univariate distributions of composites from a reliability perspective.

Another example is the proportion $\frac{X_1}{X_1+X_2}$. This distribution has domain on $(0, 1)$, and has relevant applications in medicine and biostatistics due to this bounded domain feature (Cepeda-Cuervo et. al. (2014)), and Nadarajah & Kotz (2007) studies the same ratio for a bivariate gamma distribution, mentioning application in genetics, nuclear physics, and meteorology. Finally, Nadarajah (2008) mentions that composites of random variables arise naturally in many hydrological problems. It is by these motivations that the obtained bivariate noncentral distributions' composites are derived and studied. Specifically, the distributions of the product (denoted by W_1), the ratio (denoted by W_2), and the proportion (denoted by W_3) are of interest in this study. The composites W_2 and W_3 will be referred to as ratios of type II- and I respectively.

1.3 Aims and outlay of study

The aim of this study is to sufficiently:

- Follow a systematic approach in building up bivariate noncentral distributions (specifically chi-square and F) using the compounding method as presented in Description 1.1;
- Illustrate that the use of the compounding method is appropriate to obtain noncentral counterparts to the considered bivariate central distributions – by using the moment generating function (mgf);
- Derive the distributions of the composites of the proposed bivariate models;
- Highlight the advantage of using this compounding method to obtain more elegant representations of bivariate noncentral distributions than before in literature;
- Develop new bivariate noncentral F distributions;
- Investigate the use of these new models with specific focus on an example of possible application.

The statistical property that will be focussed on is the moment generating function in Chapter 2. It is shown in Chapter 2 that it sheds light on the nature of the distributions and aids in revealing the relationship of the obtained distributions to other known distributions, as well as revealing the dependence structure and the independence of the components of the bivariate case as special cases. It is by using the mgf that the moments of the distributions could be determined and studied; say, to investigate the correlation structure of the distribution – this however falls outside the scope of this dissertation and could be ground for future work (see Chapter 5).

The outline of the study is described below, and summarized in Figure 1.1.

In Chapter 2, a brief overview of the univariate case of the noncentral chi-square distributions is presented with some corresponding properties. Subsequently the bivariate case is considered for two versions of the bivariate chi-square distribution: a generalized case, and a compound extended case. Both of these distributions are further generalized to their noncentral counterparts by using the compounding method, and distributions of composites are derived. For each of these distributions, the moment generating function is determined. A shape analysis of the newly obtained bivariate noncentral distributions and their corresponding composite distributions follows and concludes the chapter.

Next, in Chapter 3, a similar approach as that of Chapter 2 is followed but for the F distribution. A short description of the univariate noncentral case is presented after which bivariate central F distributions follow via a transformation technique using bivariate central densities from Chapter 2. These derived bivariate central F distributions are then generalized to bivariate noncentral F distributions by using the same compounding method as described earlier and employed in Chapter 2. Expressions are derived for the product of the random variables of the F distributions. The distributions of the composites of the new bivariate distributions are derived, and a shape analysis of these distributions follow.

Chapter 4 sees a possible application to drought data for some of the obtained distributions of the composites from Chapter 3.

Finally, Chapter 5 consists of a summary of obtained results, whereafter the Appendix follows which explains the notation used in this study, along with some useful results.

As a final conclusion, the Bibliography is contained at the end of this dissertation.

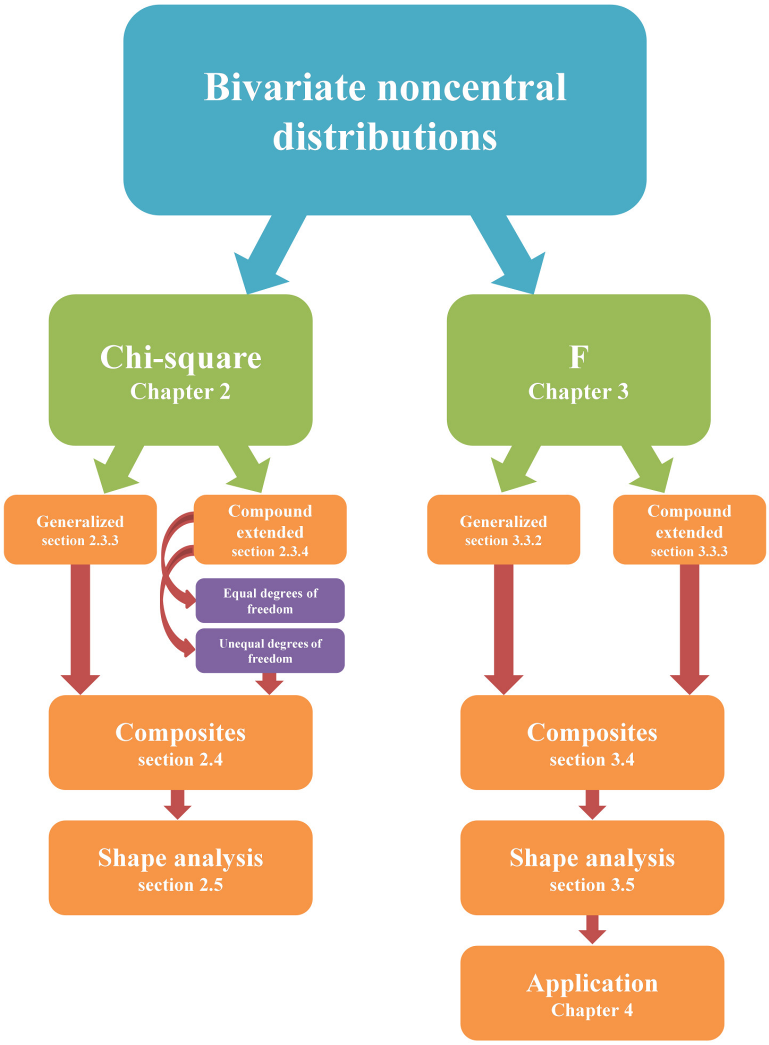

Figure 1.1 Outline of this study.

Chapter 2

Bivariate noncentral chi-square distributions

2.1 Introduction

As an introduction to this chapter, a brief background on univariate noncentral chi-square distributions being considered is given to set the platform for the bivariate case. Figure 2.1 provides a schematic overview of the basic elements of these noncentral chi-square distributions, and is followed by a discussion in sections 2.1.1 and 2.1.2 (note how the compounding method is used to obtain the noncentral cases, as well as the compound extended chi-square distribution).

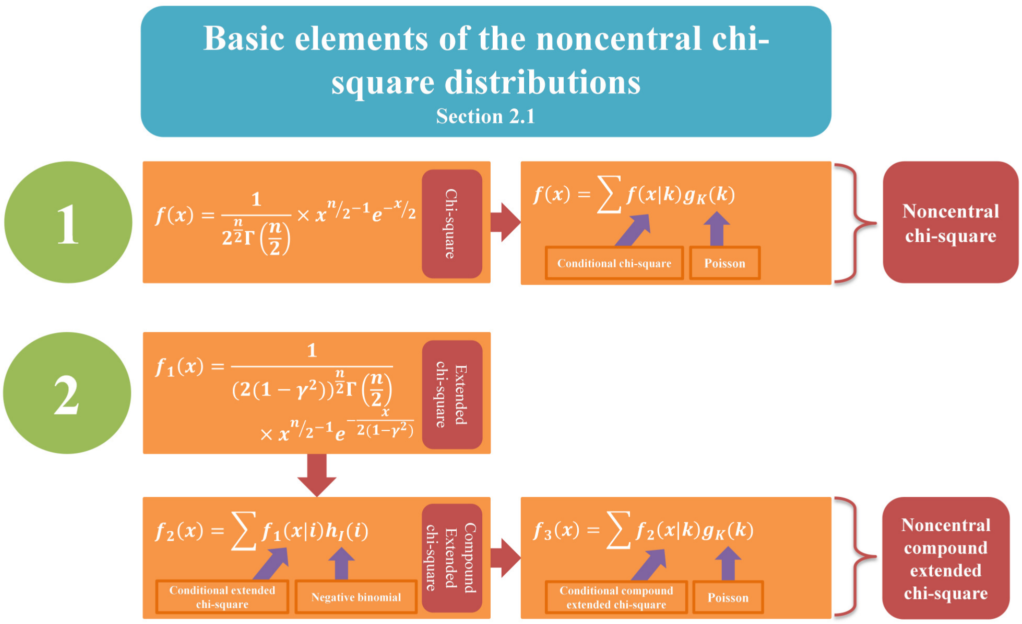

Figure 2.1 The basic elements of the noncentral chi-square distributions in this study.

2.1.1 Noncentral chi-square distribution

Patnaik (1949) showed that the *noncentral* chi-square distribution with n degrees of freedom and noncentrality parameter θ can be represented as a weighted sum of univariate chi-square probabilities with weights equal to the probabilities of a Poisson distribution with expected value $\frac{\theta}{2}$ (see Result C.22). In the text by Johnson et. al. (1995) (p. 433), the noncentral chi-square distribution is presented and discussed, and it is rephrased here in the form of the following descriptions and properties.

Description 2.1 *Let* $X = \sum_{i=1}^{n} U_i^2$, *where* $U_1, U_2, ..., U_n$ *are independent random variables and* U_i *is normally distributed with mean* μ_i *and unit variance. Then the pdf of a noncentral chi-square distributed random variable* X *with* n *degrees of freedom, and noncentrality parameter* $\theta = \sum_{i=1}^{n} \mu_i^2$ *is given by*

$$f_X(x) = \sum_{k=0}^{\infty} \frac{x^{\frac{n}{2}+k-1} e^{-\frac{1}{2}x}}{2^{\frac{n}{2}+k}\Gamma\left(\frac{n}{2}+k\right)} \frac{e^{-\frac{\theta}{2}}\left(\frac{\theta}{2}\right)^k}{k!}, \quad x > 0 \tag{2.1}$$

where $n > 0$ *and* $\theta > 0$ *(Patnaik (1949)). The distribution is denoted as* $X \sim \chi^2(n, \theta)$. *Johnson et. al. (1995) notes that this distribution is useful since it represents the distribution of a sample variance from a normal distribution with unstable expected value - in light of this reasoning, the use of the distribution, its extensions, and its applications, remains relevant.* ▲

Property 2.1 If X has pdf (2.1), then the mgf of X, $M_X(t)$, is given by

$$M_X(t) = (1-2t)^{-\frac{n}{2}} e^{\theta t(1-2t)^{-1}}, \quad t > 0 \tag{2.2}$$

where $n > 0$ and $\theta > 0$ is the noncentrality parameter. ▲

Proof. From (2.1), consider the following:

$$\begin{aligned}
M_X(t) &= E(e^{Xt}) \\
&= \sum_{k=0}^{\infty} \int_0^{\infty} e^{-x\left(\frac{1}{2}-t\right)} x^{\frac{n}{2}+k-1} dx \frac{1}{2^{\frac{n}{2}+k}\Gamma\left(\frac{n}{2}+k\right)} \frac{e^{-\frac{\theta}{2}}\left(\frac{\theta}{2}\right)^k}{k!} \\
&= \sum_{k=0}^{\infty} \frac{\Gamma\left(\frac{n}{2}+k\right)}{\left(\frac{1}{2}-t\right)^{\frac{n}{2}+k}} \frac{1}{2^{\frac{n}{2}+k}\Gamma\left(\frac{n}{2}+k\right)} \frac{e^{-\frac{\theta}{2}}\left(\frac{\theta}{2}\right)^k}{k!}
\end{aligned}$$

by applying Result C.12. It follows then by using the series expansion of e (see Result

C.8) and some algebraic simplification that

$$\begin{aligned}
M_X(t) &= \sum_{k=0}^{\infty} \frac{1}{(1-2t)^{\frac{n}{2}+k}} \frac{e^{-\frac{\theta}{2}}\left(\frac{\theta}{2}\right)^k}{k!} \\
&= (1-2t)^{-\frac{n}{2}} e^{-\frac{\theta}{2}} \sum_{k=0}^{\infty} \frac{\left(\frac{\theta}{2(1-2t)}\right)^k}{k!} \\
&= (1-2t)^{-\frac{n}{2}} e^{-\frac{\theta}{2}} e^{\frac{\theta}{2(1-2t)}}
\end{aligned}$$

which leaves the desired result. ■

Remark 2.1 *When $\theta = 0$, the mgf in (2.2) simplifies to that of a central chi-square distribution with n degrees of freedom (see Result C.19).*

Property 2.2 The pdf (2.1) is a compound pdf (see (1.1)), namely

$$f_X(x) = \sum_{k=0}^{\infty} f_X(x|k) g_K(k)$$

such that $f_X(x|k) = \frac{x^{\frac{n}{2}+k-1} e^{-\frac{1}{2}x}}{2^{\frac{n}{2}+k}\Gamma\left(\frac{n}{2}+k\right)}$, $x > 0$, and $g_K(k) = \frac{e^{-\frac{\theta}{2}}\left(\frac{\theta}{2}\right)^k}{k!}$, $k = 0, 1, 2, 3, ...$, i.e. $X|\,(K = k) \sim \chi^2(n + 2k)$ (see Result C.18) and $K \sim Poi\left(\frac{\theta}{2}\right)$ (see Result C.22). ▲

2.1.2 Noncentral compound extended chi-square distribution

In this section an extended chi-square distribution as well as the compound extended chi-square distributions, together with the corresponding noncentral counterpart, is described.

Description 2.2 *A random variable X has an extended chi-square distribution with n degrees of freedom if it has a pdf given by*

$$f_1(x) = \frac{x^{\frac{n}{2}-1} e^{-\frac{x}{2(1-\gamma^2)}}}{\left(2\left(1-\gamma^2\right)\right)^{\frac{n}{2}} \Gamma\left(\frac{n}{2}\right)}, \quad x > 0$$

where $n > 0$ and $-1 < \gamma < 1$. The parameter γ is an additional parameter, termed the extension parameter. ▲

Description 2.3 *A random variable $X|\,(I = i)$ has a conditional extended chi-square distribution if it has a pdf given by*

$$f_1(x|i) = \frac{x^{\frac{n}{2}+i-1} e^{-\frac{x}{2(1-\gamma^2)}}}{\left(2\left(1-\gamma^2\right)\right)^{\frac{n}{2}+i} \Gamma\left(\frac{n}{2}+i\right)}, \quad x > 0, i \geq 0 \tag{2.3}$$

where $n > 0$ *and* $-1 < \gamma < 1$. ▲

Remark 2.2 *When* $\gamma = 0$, *the pdf in (2.3) reduces to that of a conditional central chi-square distribution with* $n + 2i$ *degrees of freedom:*

$$f_1(x|i) = \frac{x^{\frac{n}{2}+i-1} e^{-\frac{1}{2}x}}{2^{\frac{n}{2}+i}\Gamma\left(\frac{n}{2}+i\right)}, \quad x > 0, i \geq 0$$

thus, $X|\,(I = i) \sim \chi^2(n + 2i)$ *(see Result C.18).* ▲

Description 2.4 *The random variable* X *is said to have the compound extended chi-square distribution if* $X|\,(I = i)$ *has the conditional extended chi-square distribution (see (2.3)), and* I *has the negative binomial distribution, i.e.* $I \sim NB(\gamma^2, n)$ *(see Result C.21). The pdf of* X *can be obtained as:*

$$\begin{aligned} f_2(x) &= \sum_{i=0}^{\infty} f_1(x|i) h_I(i) \\ &= \sum_{i=0}^{\infty} \frac{x^{\frac{n}{2}+i-1} e^{-\frac{x}{2(1-\gamma^2)}}}{\left(2\left(1-\gamma^2\right)\right)^{\frac{n}{2}+i}\Gamma\left(\frac{n}{2}+i\right)} \frac{\Gamma\left(\frac{n}{2}+i\right)}{\Gamma\left(\frac{n}{2}\right) i!} \gamma^{2i}(1-\gamma^2)^{\frac{n}{2}}, \quad x > 0 \end{aligned} \tag{2.4}$$

where $n > 0$ *and* $-1 < \gamma < 1$. ▲

From (2.4) a conditional distribution can be defined.

Description 2.5 *A random variable* $X|\,(K = k)$ *has a conditional compound extended chi-square distribution if it has the pdf given by*

$$f_2(x|k) = \sum_{i=0}^{\infty} \frac{x^{\frac{n}{2}+i+k-1} e^{-\frac{x}{2(1-\gamma^2)}}}{\left(2\left(1-\gamma^2\right)\right)^{\frac{n}{2}+i+k}\Gamma\left(\frac{n}{2}+i+k\right)} \frac{\Gamma\left(\frac{n}{2}+i\right)}{\Gamma\left(\frac{n}{2}\right) i!} \gamma^{2i}(1-\gamma^2)^{\frac{n}{2}}, \quad x > 0, k \geq 0$$

where $n > 0$, $-1 < \gamma < 1$. ▲

The condition here has been imposed on the degrees of freedom, to arrive at a similar unconditional expression as in (2.1) (or rather compound pdf (see (1.1))).

Description 2.6 *A random variable X is said to have the noncentral compound extended chi-square distribution if it has the pdf given by*

$$
\begin{aligned}
f_3(x) &= \sum_{k=0}^{\infty} f_2(x|k) g_K(k) \\
&= \sum_{i=0}^{\infty}\sum_{k=0}^{\infty} \frac{x^{\frac{n}{2}+i+k-1} e^{-\frac{x}{2(1-\gamma^2)}}}{\left(2\left(1-\gamma^2\right)\right)^{\frac{n}{2}+i+k} \Gamma\left(\frac{n}{2}+i+k\right)} \\
&\quad \times \frac{\Gamma\left(\frac{n}{2}+i\right)}{\Gamma\left(\frac{n}{2}\right) i!} \gamma^{2i} (1-\gamma^2)^{\frac{n}{2}} \frac{e^{-\frac{\theta}{2}}\left(\frac{\theta}{2}\right)^k}{k!}, \quad x>0, k \geq 0 \qquad (2.5)
\end{aligned}
$$

where $n > 0$, $-1 < \gamma < 1$, $\theta > 0$ represents the noncentrality parameter, and $g_K(k)$ the pmf of a Poisson distribution with expected value $\frac{\theta}{2}$ (see Result C.22). ▲

Property 2.3 If X has pdf (2.5), then the mgf, $M_X(t)$, is given by

$$
\begin{aligned}
M_X(t) &= \left(1-\gamma^2\right)^{\frac{n}{2}} \left(1-2\left(1-\gamma^2\right)t\right)^{-\frac{n}{2}} \\
&\quad \times \left(1 - \frac{\gamma^2}{\left(1-2\left(1-\gamma^2\right)t\right)}\right)^{-\frac{n}{2}} e^{\theta t (1-2t)^{-1}}, \quad t>0 \qquad (2.6)
\end{aligned}
$$

where $n > 0$, $\theta > 0$ is the noncentrality parameter, and $-1 < \gamma < 1$ the extension parameter. ▲

Proof. From (2.5), consider the following:

$$
\begin{aligned}
M_X(t) &= E\left(e^{Xt}\right) \\
&= \sum_{i=0}^{\infty}\sum_{k=0}^{\infty} \frac{\Gamma\left(\frac{n}{2}+i\right)}{\Gamma\left(\frac{n}{2}\right) i!} \gamma^{2i} (1-\gamma^2)^{\frac{n}{2}} \frac{1}{\left(2\left(1-\gamma^2\right)\right)^{\frac{n}{2}+i+k} \Gamma\left(\frac{n}{2}+i+k\right)} \\
&\quad \times \frac{e^{-\frac{\theta}{2}}\left(\frac{\theta}{2}\right)^k}{k!} \int_0^{\infty} e^{-x\left(\frac{1}{2(1-\gamma^2)}-t\right)} x^{\frac{n}{2}+i+k-1} dx \\
&= \sum_{i=0}^{\infty}\sum_{k=0}^{\infty} \frac{\Gamma\left(\frac{n}{2}+i\right)}{\Gamma\left(\frac{n}{2}\right) i!} \gamma^{2i} (1-\gamma^2)^{\frac{n}{2}} \frac{1}{\left(2\left(1-\gamma^2\right)\right)^{\frac{n}{2}+i+k} \Gamma\left(\frac{n}{2}+i+k\right)} \\
&\quad \times \frac{e^{-\frac{\theta}{2}}\left(\frac{\theta}{2}\right)^k}{k!} \frac{\Gamma\left(\frac{n}{2}+i+k\right)}{\left(\frac{1}{2(1-\gamma^2)}-t\right)^{\frac{n}{2}+i+k}}
\end{aligned}
$$

by applying Result C.12. By reordering the expression accordingly and rearranging constant terms:

$$\begin{aligned}
M_X(t) &= \sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\frac{\Gamma\left(\frac{n}{2}+i\right)}{\Gamma\left(\frac{n}{2}\right)i!}\gamma^{2i}(1-\gamma^2)^{\frac{n}{2}} \\
&\quad\times\frac{1}{(1-2\left(1-\gamma^2\right)t)^{\frac{n}{2}+i+k}}\frac{e^{-\frac{\theta}{2}}\left(\frac{\theta}{2}\right)^k}{k!} \\
&= (1-\gamma^2)^{\frac{n}{2}}\left(1-2\left(1-\gamma^2\right)t\right)^{-\frac{n}{2}} \\
&\quad\times\sum_{i=0}^{\infty}\frac{\Gamma\left(\frac{n}{2}+i\right)}{\Gamma\left(\frac{n}{2}\right)i!}\left(\frac{\gamma^2}{(1-2\left(1-\gamma^2\right)t)}\right)^i \\
&\quad\times e^{-\frac{\theta}{2}}\sum_{k=0}^{\infty}\frac{\left(\frac{\theta}{2(1-2(1-\gamma^2)t)}\right)^k}{k!} \\
&= (1-\gamma^2)^{\frac{n}{2}}\left(1-2\left(1-\gamma^2\right)t\right)^{-\frac{n}{2}} \\
&\quad\times\left(1-\frac{\gamma^2}{(1-2\left(1-\gamma^2\right)t)}\right)^{-\frac{n}{2}} e^{-\frac{\theta}{2}}e^{\frac{\theta}{2(1-2(1-\gamma^2)t)}}
\end{aligned}$$

by considering the binomial expansion, and the series expansion for e (see Result C.9 and C.8). By simplifying the terms further the final result is obtained. ■

Property 2.4 It is readily seen that when $\gamma = 0$ in (2.6), it reduces to that of the mgf given in (2.2). ▲

The focus now shifts in section 2.2 to the bivariate chi-square distribution, followed by its noncentral counterparts in section 2.3.

2.2 Bivariate chi-square distributions

2.2.1 Introduction

The chi-square distribution in its bivariate form is of immense importance as it is used to construct power of hypothesis test which is applicable in almost any branch of science (Saleh & Singh (1983), Royen (1995)). In the literature various authors used the mgf in deriving the bivariate chi-square pdf, and in theoretical statistics, by using the Laplace transform, the mgf uniquely determines the distribution. The focus will be on the bivariate chi-square distributions following from Van den Berg (2010) as well as Yunus & Khan (2011).

2.2.2 Bivariate chi-square distribution

Kibble (1941) defined the mgf of a bivariate gamma distribution with correlated gamma marginals as

$$M_{X_1,X_2}(t_1,t_2) = (1-t_1)^{-\alpha}(1-t_2)^{-\alpha}\left(1-\frac{4\xi^2 t_1 t_2}{(1-t_1)(1-t_2)}\right)^{-\alpha}.$$

The joint pdf of X_1 and X_2, in terms of Laguerre polynomials (see Result C.13), is given by

$$f_{X_1,X_2}(x_1,x_2) = f_{X_1}(x_1) f_{X_2}(x_2) \sum_{j=0}^{\infty} \frac{\xi^{2j}\Gamma(\alpha)\Gamma(j+1)}{\Gamma(\alpha+j)} L_j^{\alpha-1}(x_1)\, L_j^{\alpha-1}(x_2)$$

where $f_{X_v}(x_v) = \frac{1}{\Gamma(\alpha)} x_v^{\alpha-1} e^{-x_v}, x_v > 0,\ v = 1,2.$ Van den Berg (2010) investigated this result further for the case of the bivariate chi-square distribution by considering the following mgf:

$$M_{X_1,X_2}(t_1,t_2) = (1-2t_1)^{-\frac{n}{2}}(1-2t_2)^{-\frac{n}{2}}\left(1-\frac{4\xi^2 t_1 t_2}{(1-2t_1)(1-2t_2)}\right)^{-\frac{n}{2}},$$

and subsequently deriving the corresponding joint pdf of X_1 and X_2, given by

$$\begin{aligned} f_{X_1,X_2}(x_1,x_2) &= \frac{2^{-\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} x_1^{\frac{n}{2}-1} e^{-\frac{1}{2}x_1} \frac{2^{-\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} x_2^{\frac{n}{2}-1} e^{-\frac{1}{2}x_2} \sum_{j=0}^{\infty} \frac{j!\,\xi^{2j}\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n}{2}+j\right)} \\ &\quad \times L_j^{\frac{n}{2}-1}\left(\frac{1}{2}x_1\right) L_j^{\frac{n}{2}-1}\left(\frac{1}{2}x_2\right), \quad x_1, x_2 > 0, \end{aligned} \tag{2.7}$$

where $n > 0$, $-1 \leq \xi \leq 1$, and $L_j^n(\cdot)$ is the Laguerre polynomial as defined in Result C.13. Here, ξ is a component of the coefficient of correlation, but does not represent the correlation coefficient itself.

Remark 2.3 *This pdf (2.7) is the pdf of a bivariate chi-square distribution with correlated chi-square marginals - where these marginals have chi-square distributions with n equal degrees of freedom (see Van den Berg (2010)).*

2.2.3 Bivariate generalized chi-square distribution

Van den Berg (2010) defined a joint mgf of X_1 and X_2 in which an additional parameter (as component of the correlation) was introduced, namely r, and the corresponding pdf was derived by using an inverse Laplace transform. The mgf and corresponding pdf is given below.

Description 2.7 *Van den Berg (2010) defined a joint mgf of X_1 and X_2 of the form*

$$M_{X_1,X_2}(t_1,t_2) = (1-2t_1)^{-\frac{n_1}{2}} (1-2t_2)^{-\frac{n_2}{2}} \left(1 - \frac{4\xi^2 t_1 t_2}{(1-2t_1)(1-2t_2)}\right)^{-\frac{r}{2}}.$$

The pdf of this distribution is given by

$$\begin{aligned} f_{X_1,X_2}(x_1,x_2) &= x_1^{\frac{n_1}{2}-1} e^{-\frac{1}{2}x_1} x_2^{\frac{n_2}{2}-1} e^{-\frac{1}{2}x_2} \sum_{j=0}^{\infty} \frac{\left(\frac{r}{2}\right)_j \; j!}{\Gamma\left(\frac{n_1}{2}+j\right)\Gamma\left(\frac{n_2}{2}+j\right)} \\ &\quad \times \frac{\xi^{2j}}{2^{\frac{1}{2}(n_1+n_2)}} L_j^{\frac{n_1}{2}-1}\left(\frac{1}{2}x_1\right) L_j^{\frac{n_2}{2}-1}\left(\frac{1}{2}x_2\right), \quad x_1,x_2>0 \qquad (2.8) \end{aligned}$$

where $n_1, n_2, r > 0$, $-1 \leq \xi \leq 1$, and $L_j^n(\cdot)$ is the Laguerre polynomial as defined in Result C.13. The joint distribution of X_1 and *X_2 is referred to as the bivariate generalized chi-square distribution.* ▲

The following results were derived in Van den Berg (2010), and is stated here without proof. Reference will be made to these results furtheron.

Result A *If X_1 and X_2 are jointly distributed according to (2.8), the pdf of $W_1 = X_1X_2$ is given by*

$$\begin{aligned} f_{W_1}(w_1) &= \sum_{j=0}^{\infty}\sum_{l_1=0}^{j}\sum_{l_2=0}^{j} K_\tau\left(\sqrt{w_1}\right) w_1^{\frac{\frac{n_1}{2}+\frac{n_2}{2}+l_1+l_2-2}{2}} \left(\frac{\xi^{2j}\left(\frac{r}{2}\right)_j (-1)^{l_1+l_2}\binom{j}{l_1}\binom{j}{l_2}}{j!\Gamma\left(\frac{n_1}{2}+l_1\right)\Gamma\left(\frac{n_2}{2}+l_2\right)}\right) \\ &\quad \times \left(\frac{1}{2}\right)^{\frac{1}{2}(n_1+n_2+2l_1+2l_2-2)}, \qquad w_1>0 \end{aligned}$$

where $n_1, n_2, r > 0$, $-1 \leq \xi \leq 1$, $\tau = \frac{n_2-n_1}{2} + l_2 - l_1$, *and* $K_\tau(\cdot)$ *the modified Bessel function of the second kind (see Result C.10).* ■

Result B *If* X_1 *and* X_2 *are jointly distributed according to (2.8), the pdf of* $W_2 = \frac{X_1}{X_2}$ *is given by*

$$f_{W_2}(w_2) = \sum_{j=0}^{\infty}\sum_{l_1=0}^{j}\sum_{l_2=0}^{j} \frac{w_2^{\frac{n_1}{2}+l_1-1}}{(1+w_2)^{\frac{n_1+n_2}{2}+l_1+l_2}} \left(\frac{\xi^{2j}\left(\frac{r}{2}\right)_j (-1)^{l_1+l_2}\binom{j}{l_1}\binom{j}{l_2}}{j!B\left(\frac{n_1}{2}+l_1, \frac{n_2}{2}+l_2\right)} \right), \quad w_2 > 0$$

where $n_1, n_2, r > 0$, $-1 \leq \xi \leq 1$, *and* $B(\cdot,\cdot)$ *is the beta function (see Result C.4).* ■

Result C *If* X_1 *and* X_2 *are jointly distributed according to (2.8), the pdf of* $W_3 = \frac{X_1}{X_1+X_2}$ *is given by*

$$f_{W_3}(w_3) = \sum_{j=0}^{\infty}\sum_{l_1=0}^{j}\sum_{l_2=0}^{j} w_3^{\frac{n_1}{2}+l_1-1}(1-w_3)^{\frac{n_2}{2}+l_2-1} \left(\frac{\xi^{2j}\left(\frac{r}{2}\right)_j (-1)^{l_1+l_2}\binom{j}{l_1}\binom{j}{l_2}}{j!B\left(\frac{n_1}{2}+l_1, \frac{n_2}{2}+l_2\right)} \right), \quad 0<w_3<1$$

where $n_1, n_2, r > 0$, $-1 \leq \xi \leq 1$, *and* $B(\cdot,\cdot)$ *is the beta function (see Result C.4).* ■

2.2.4 Bivariate compound extended chi-square distribution

In this section a systematic development of the bivariate compound extended chi-square distribution for the equal-, as well as unequal degrees of freedom cases is presented, as this development is not covered in literature.

2.2.4.1 Equal degrees of freedom

The bivariate central chi-square pdf (i.e. the bivariate case of (2.4)) as presented by Yunus & Khan (2011) was originally posed by Krishnaiah et. al. (1963), and is rephrased in a compounding description next.

Description 2.8 *Krishnaiah et. al. (1963) defined the bivariate compound extended chi-square with pdf as*

$$f_{X_1,X_2}(x_1,x_2) = \sum_{i=0}^{\infty} f_{X_1}(x_1|i) f_{X_2}(x_2|i) h_I(i)$$

where $f_{X_v}(x_v|i) = \frac{x_v^{\frac{n}{2}+i-1} e^{-\frac{x_v}{2(1-\gamma^2)}}}{(2(1-\gamma^2))^{\frac{n}{2}+i}\Gamma\left(\frac{n}{2}+i\right)}$, $v = 1,2$ *(the pdf of the extended chi-square distribution (see (2.3))), and* $h_I(i) = \frac{\Gamma\left(\frac{n}{2}+i\right)}{\Gamma\left(\frac{n}{2}\right)i!}\gamma^{2i}\left(1-\gamma^2\right)^{\frac{n}{2}}$ *the pmf of a negative binomial distribution with parameters* γ^2 $(-1 < \gamma < 1)$ *and* $n > 0$ *(see Result C.21). Thus, the* *joint pdf of* X_1 *and* X_2 *can be given as*

$$f_{X_1,X_2}(x_1,x_2) = \sum_{i=0}^{\infty} \frac{(x_1x_2)^{\frac{n}{2}+i-1} e^{-\frac{x_1+x_2}{2(1-\gamma^2)}}}{\left((2(1-\gamma^2))^{\frac{n}{2}+i}\Gamma\left(\frac{n}{2}+i\right)\right)^2} \frac{\Gamma\left(\frac{n}{2}+i\right)}{\Gamma\left(\frac{n}{2}\right)i!}\gamma^{2i}\left(1-\gamma^2\right)^{\frac{n}{2}}, \quad x_1,x_2>0 \tag{2.9}$$

This central pdf (2.9) can be considered a compound pdf (see (1.1)) with the correlation structure inherently incorporated via the negative binomial distribution. $f_{X_1,X_2|I}(x_1,x_2|i)$ *is defined to be the joint pdf of* X_1 *and* X_2 *conditioned on the correlation structure, which is induced in the negative binomial pdf with index* i. ▲

Property 2.5 The joint mgf of X_1 and X_2 with distribution in (2.9) is given by:

$$\begin{aligned} M_{X_1,X_2}(t_1,t_2) &= \left(1-\gamma^2\right)^{\frac{n}{2}}\left(\left(1-2\left(1-\gamma^2\right)t_1\right)\left(1-2\left(1-\gamma^2\right)t_2\right)\right)^{-\frac{n}{2}}, \\ &\times\left(1-\frac{\gamma^2}{(1-2(1-\gamma^2)t_1)(1-2(1-\gamma^2)t_2)}\right)^{-\frac{n}{2}}, \quad t_1,t_2>0. \end{aligned}$$

▲

Proof. The proof follows similarly as in property 2.3, section 2.1.2. ■

Remark 2.4 *In the mgf above, if* $\gamma=0$, *the mgf reduces to the product of two independent univariate central chi-square mgfs (see Result C.19) each with* n *degrees of freedom.*

2.2.4.2 Unequal degrees of freedom

The central pdf for the unequal degrees of freedom case as presented by Yunus & Khan (2011) (originally by Wright & Kennedy (2002)) is rephrased and structured within the following description.

Description 2.9 *The pdf of the bivariate compound extended chi-square of Wright and Kennedy (2002) is given by*

$$\begin{aligned}
f_{X_1,X_2}(x_1,x_2) &= \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty} f_{X_1}(x_1|i_1)f_{X_2}(x_2|i_2)h_{I_1}(i_1)h_{I_2}(i_2) \\
&= \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty} \frac{\Gamma\left(\frac{n_1}{2}+i_1\right)}{\Gamma\left(\frac{n_1}{2}\right)i_1!}\frac{\Gamma\left(\frac{n_2}{2}+i_2\right)}{\Gamma\left(\frac{n_2}{2}\right)i_2!}\gamma^{2(i_1+i_2)}\left(1-\gamma^2\right)^{\frac{n_1+n_2}{2}} \\
&\quad\times \frac{x_1^{\frac{n_1}{2}+i_1-1}e^{-\frac{x_1}{2(1-\gamma^2)}}}{\left(\left(2\left(1-\gamma^2\right)\right)^{\frac{n_1}{2}+i_1}\Gamma\left(\frac{n_1}{2}+i_1\right)\right)}\frac{x_2^{\frac{n_2}{2}+i_2-1}e^{-\frac{x_2}{2(1-\gamma^2)}}}{\left(\left(2\left(1-\gamma^2\right)\right)^{\frac{n_2}{2}+i_2}\Gamma\left(\frac{n_2}{2}+i_2\right)\right)}
\end{aligned} \tag{2.10}$$

where $f_{X_v}(x_v|i) = \frac{1}{(2(1-\gamma^2))^{\frac{n_v}{2}+i}\Gamma\left(\frac{n_v}{2}+i\right)}x_v^{\frac{n_v}{2}+i-1}e^{-\frac{x_v}{2(1-\gamma^2)}}$, $v=1,2$ *(the pdf of the extended chi-square distribution (see (2.3))), and* $h_{I_v}(i)=\frac{\Gamma\left(\frac{n_v}{2}+i\right)}{\Gamma\left(\frac{n_v}{2}\right)i!}\gamma^{2i}\left(1-\gamma^2\right)^{\frac{n_v}{2}}$, $v=1,2$, *are the pdfs of a negative binomial distribution with parameters* γ^2 $(-1<\gamma<1)$ *and* $n_v>0$ *(see Result C.21).* ▲

This pdf (2.10), as a central bivariate compound extended chi-square distribution with unequal degrees of freedom, differs to some extent from (2.9) which is for equal degrees of freedom to provide for the unequal degrees of freedom scenario. In (2.10), the joint pdf $f_{X_1,X_2}(x_1,x_2|i_1,i_2)=f_{X_1}(x_1|i_1)f_{X_2}(x_2|i_2)$ is compounded by two negative binomial distributions – which is different compared to (2.9), where the joint pdf is only compounded by one negative binomial distribution. In (2.10), the negative binomial pdfs $h_{I_1}(i_1)$ and $h_{I_2}(i_2)$ have parameters γ^2, and n_1 and n_2 respectively, where $-1<\gamma<1$ and $n_1,n_2>0$.

Property 2.6 The joint mgf of X_1 and X_2 with distribution in (2.10) is given by:

$$\begin{aligned}
M_{X_1,X_2}(t_1,t_2) &= \left(1-\gamma^2\right)^{\frac{n_1+n_2}{2}}\left(1-2\left(1-\gamma^2\right)t_1\right)^{-\frac{n_1}{2}}\left(1-2\left(1-\gamma^2\right)t_2\right)^{-\frac{n_2}{2}} \\
&\quad\times\left(1-\frac{\gamma^2}{\left(1-2\left(1-\gamma^2\right)t_1\right)}\right)^{-\frac{n_1}{2}}\left(1-\frac{\gamma^2}{\left(1-2\left(1-\gamma^2\right)t_2\right)}\right)^{-\frac{n_2}{2}}, \quad t_1,t_2>0.
\end{aligned}$$

▲

Proof. The proof follows similarly as in property 2.3, section 2.1.2. ■

Remark 2.5 *In the mgf above, if* $\gamma=0$*, the mgf reduces to the product of two independent univariate central chi-square mgfs (see Result C.19) with* n_1 *and* n_2 *degrees of freedom respectively.*

2.3 Bivariate noncentral chi-square distributions

2.3.1 Introduction

As mentioned in section 2.1.1, it seems evident that a noncentral chi-square distribution can be obtained for certain central chi-square counterparts when one compounds on the degrees of freedom (i.e. parameters) with Poisson probabilities. This is the predominant idea present in this section to obtain bivariate noncentral distributions. Yunus & Khan (2011) mentions that there is a lack of available theory of the bivariate noncentral chi-square distribution, where a vast array of papers fails to provide suitable critical values for applicational value (see for example, Royen (1995), Marshall & Olkin (1990)). In Figure 2.2, an overview of the core concepts of the remainder of this chapter is illustrated (given below).

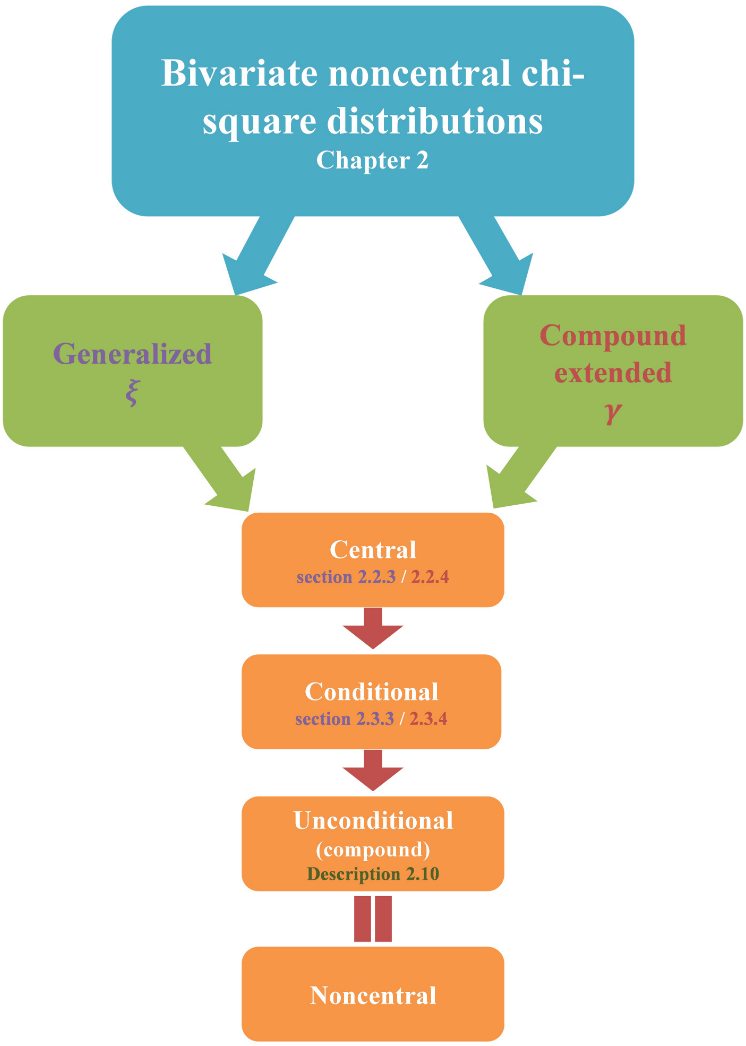

Figure 2.2 Diagram of core concepts of Chapter 2 - section 2.3.

The compounding method used by Yunus & Khan (2011) is phrased within the following description.

Description 2.10 *An (unconditional) bivariate noncentral chi-square pdf can be obtained from a (conditional) bivariate central chi-square pdf in the following manner:*

$$f_{X_1,X_2}(x_1,x_2) = \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} f_{X_1,X_2}(x_1,x_2|k_1,k_2)g_{K_1}(k_1)g_{K_2}(k_2). \tag{2.11}$$

In this study, $g_{K_v}(k_v) = \frac{e^{-\frac{\theta_v}{2}}\left(\frac{\theta_v}{2}\right)^{k_v}}{k_v!}, v = 1,2$ *are the compounding factors - in this case, Poisson probabilities where* θ_v *are the noncentrality parameters, and* $f(x_1,x_2|k_1,k_2)$ *being the pdf of some suitable bivariate central chi-square distribution for* X_1 *and* X_2*.* ▲

2.3.2 Bivariate noncentral generalized chi-square distribution

Along with the bivariate central generalized chi-square mgf that Van den Berg (2010) defined, an mgf for a bivariate noncentral generalized chi-square distribution was also defined. This distribution also contained the additional parameter r, and the corresponding pdf was derived by using an inverse Laplace transform. The form of this distribution is given below.

Description 2.11 *Van den Berg (2010) defined the mgf of a bivariate noncentral generalized chi-square distribution as*

$$\begin{aligned} M_{X_1,X_2}(t_1,t_2) &= (1-2t_1)^{-\frac{n_1}{2}}(1-2t_2)^{-\frac{n_2}{2}}\left(1-\frac{4\xi^2 t_1 t_2}{(1-2t_1)(1-2t_2)}\right)^{-\frac{r}{2}} \\ &\quad \times e^{\theta_1 t_1(1-2t_1)^{-1}} e^{\theta_2 t_2 (1-2t_2)^{-1}}, \qquad t_1,t_2>0. \end{aligned} \tag{2.12}$$

The pdf of this distribution is given by

$$\begin{aligned} f_{X_1,X_2}(x_1,x_2) &= x_1^{\frac{n_1}{2}-1}e^{-\frac{1}{2}x_1}x_2^{\frac{n_2}{2}-1}e^{-\frac{1}{2}x_2}\sum_{j=0}^{\infty}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\frac{\left(\frac{r}{2}\right)_j(-\theta_1)^{k_1}(-\theta_2)^{k_2}}{\Gamma\left(\frac{n_1}{2}+j+k_1\right)\Gamma\left(\frac{n_2}{2}+j+k_2\right)} \\ &\quad \times \frac{(j+k_1)!\,(j+k_2)!}{j!k_1!k_2!}\frac{\xi^{2j}}{2^{\frac{1}{2}(n_1+n_2)+k_1+k_2}}L_{j+k_1}^{\frac{n_1}{2}-1}\left(\frac{1}{2}x_1\right)L_{j+k_2}^{\frac{n_2}{2}-1}\left(\frac{1}{2}x_2\right), \quad x_1,x_2>0 \end{aligned} \tag{2.13}$$

where $n_1,n_2,r>0$*, and* $L_j^n(\cdot)$ *is the Laguerre polynomial as defined in Result C.13. The parameter* ξ*, where* $-1\le\xi\le1$*, is a component of the product-moment correlation* *between X_1 and X_2. The parameters $\theta_1, \theta_2 > 0$ are the noncentrality parameters respectively, and the marginal distribution of X_1 and X_2 are univariate noncentral chi-square distributions with parameters (n_1, θ_1) and (n_2, θ_2) (see Van den Berg (2010)).* ▲

2.3.3 An alternative representation of the bivariate noncentral generalized chi-square distribution

In this section an alternative representation of (2.13) is given. However, to make use of the methodology as given in Description 2.10, a conditional bivariate central chi-square distribution from (2.8) is defined first.

Description 2.12 *Let $(X_1, X_2| (K_1 = k_1, K_2 = k_2))$ have a conditional bivariate generalized chi-square distribution (see (2.8)) with pdf given by*

$$\begin{aligned} f_{X_1,X_2}(x_1, x_2|k_1, k_2) &= x_1^{\frac{n_1}{2}+k_1-1} e^{-\frac{1}{2}x_1} x_2^{\frac{n_2}{2}+k_2-1} e^{-\frac{1}{2}x_2} \sum_{j=0}^{\infty} \left(\frac{\left(\frac{r}{2}\right)_j j! \xi^{2j}}{2^{\frac{1}{2}(n_1+n_2+2k_1+2k_2)}} \right) \\ &\quad \times \frac{L_j^{\frac{n_1}{2}+k_1-1}\left(\frac{1}{2}x_1\right) L_j^{\frac{n_2}{2}+k_2-1}\left(\frac{1}{2}x_2\right)}{\Gamma\left(\frac{n_1}{2}+j+k_1\right)\Gamma\left(\frac{n_2}{2}+j+k_2\right)}, \qquad x_1, x_2 > 0 \qquad (2.14) \end{aligned}$$

where $n_1, n_2, r > 0$, and $L_j^n(\cdot)$ is the Laguerre polynomial as defined in Result C.13. As previously, ξ (where $-1 \leq \xi \leq 1$) is a parameter which is a component of the product-moment correlation between X_1 and X_2. The conditional values k_1 and k_2 have domain such that $k_v \geq 0$, $v = 1, 2$. ▲

A conditional bivariate chi-squared distribution is defined with pdf in (2.14), which now fulfills the role of the conditional pdf as required by (2.11) in order to obtain an unconditional bivariate noncentral chi-square pdf with the compounding method. By applying (2.11), an alternative representation to the bivariate noncentral generalized chi-square distribution is obtained (see (2.13)); originally derived by Van den Berg (2010), see section 2.3.2), and is defined next.

Definition 1 *The joint pdf of X_1 and X_2, that is an alternative representation of the bivariate noncentral generalized chi-square pdf (2.13), is proposed by substituting (2.14) in (2.11), with the following result:*

$$\begin{aligned} f_{X_1,X_2}(x_1,x_2) &= f_{X_1,X_2}(x_1,x_2|k_1,k_2)g_{K_1}(k_1)g_{K_2}(k_2) \\ &= \sum_{j=0}^{\infty}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} x_1^{\frac{n_1}{2}+k_1-1}e^{-\frac{1}{2}x_1}x_2^{\frac{n_2}{2}+k_2-1}e^{-\frac{1}{2}x_2} \\ &\times\left(\frac{\left(\frac{r}{2}\right)_j j!\xi^{2j}}{2^{\frac{1}{2}(n_1+n_2+2k_1+2k_2)}}\right)\frac{L_j^{\frac{n_1}{2}+k_1-1}\left(\frac{1}{2}x_1\right)L_j^{\frac{n_2}{2}+k_2-1}\left(\frac{1}{2}x_2\right)}{\Gamma\left(\frac{n_1}{2}+j+k_1\right)\Gamma\left(\frac{n_2}{2}+j+k_2\right)} \\ &\times\frac{e^{-\frac{\theta_1}{2}}\left(\frac{\theta_1}{2}\right)^{k_1}}{k_1!}\frac{e^{-\frac{\theta_2}{2}}\left(\frac{\theta_2}{2}\right)^{k_2}}{k_2!}, \qquad x_1,x_2>0 \end{aligned} \tag{2.15}$$

where $n_1, n_2, r > 0$, $-1 \le \xi \le 1$, and both noncentrality parameters $\theta_1, \theta_2 > 0$.

Evidently, the Poisson probability factors, namely $g_{K_v}(k_v) = \frac{e^{-\frac{\theta_v}{2}}\left(\frac{\theta_v}{2}\right)^{k_v}}{k_v!}$, $v = 1, 2$, isolates the noncentrality parameters as suggested by equation (2.11) in a mathematically convenient way. In the following section, a discussion follows as to why this distribution defined in (2.15) is an alternative representation of the bivariate noncentral generalized chi-square distribution given in (2.13), i.e. (2.13) $\equiv$ (2.15).

Joint moment generating function

In this section, the joint mgf of X_1 and X_2 is derived for the distribution with pdf (2.15).

Property 2.7 *If X_1 and X_2 are jointly distributed with pdf (2.15), then the joint mgf is given by*

$$\begin{aligned} M_{X_1,X_2}(t_1,t_2) &= (1-2t_1)^{-\frac{n_1}{2}}(1-2t_2)^{-\frac{n_2}{2}}\left(1-\frac{4\xi^2t_1t_2}{(1-2t_1)(1-2t_2)}\right)^{-\frac{r}{2}} \\ &\times e^{\theta_1t_1(1-2t_1)^{-1}}e^{\theta_2t_2(1-2t_2)^{-1}} \qquad t_1,t_2>0. \end{aligned}$$

where $n_1, n_2 > 0$, $-1 \le \xi \le 1$, and both noncentrality parameters $\theta_1, \theta_2 > 0$. ▲

Proof. From (2.15), consider the following:

$$
\begin{aligned}
M_{X_1,X_2}(t_1,t_2) &= E\left(e^{t_1X_1+t_2X_2}\right) \\
&= \int_0^\infty\int_0^\infty \sum_{j=0}^{\infty}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} e^{-\left(\frac{1}{2}-t_1\right)x_1} x_1^{\frac{n_1}{2}+k_1-1} e^{-\left(\frac{1}{2}-t_2\right)x_2} x_2^{\frac{n_2}{2}+k_2-1} \\
&\quad \times \left(\frac{\left(\frac{r}{2}\right)_j j!\xi^{2j}}{2^{\frac{1}{2}(n_1+n_2+2k_1+2k_2)}}\right) \frac{L_j^{\frac{n_1}{2}+k_1-1}\left(\frac{1}{2}x_1\right) L_j^{\frac{n_2}{2}+k_2-1}\left(\frac{1}{2}x_2\right)}{\Gamma\left(\frac{n_1}{2}+k_1+j\right)\Gamma\left(\frac{n_2}{2}+k_2+j\right)} \\
&\quad \times \frac{e^{-\frac{\theta_1}{2}}\left(\frac{\theta_1}{2}\right)^{k_1}}{k_1!}\frac{e^{-\frac{\theta_2}{2}}\left(\frac{\theta_2}{2}\right)^{k_2}}{k_2!} dx_1 dx_2 \\
&= \sum_{j=0}^{\infty}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\int_0^\infty e^{-\left(\frac{1}{2}-t_1\right)x_1} x_1^{\frac{n_1}{2}+k_1-1} L_j^{\frac{n_1}{2}+k_1-1}\left(\frac{1}{2}x_1\right) dx_1 \\
&\quad \times \int_0^\infty e^{-\left(t_2-\frac{1}{2}\right)x_2} x_2^{\frac{n_2}{2}+k_2-1} L_j^{\frac{n_2}{2}+k_2-1}\left(\frac{1}{2}x_2\right) dx_2 \frac{1}{2^{\frac{1}{2}(n_1+n_2+2k_1+2k_2)}} \\
&\quad \times \frac{\left(\frac{r}{2}\right)_j j!\xi^{2j}}{\Gamma\left(\frac{n_1}{2}+k_1+j\right)\Gamma\left(\frac{n_2}{2}+k_2+j\right)} \frac{e^{-\frac{\theta_1}{2}}\left(\frac{\theta_1}{2}\right)^{k_1}}{k_1!}\frac{e^{-\frac{\theta_2}{2}}\left(\frac{\theta_2}{2}\right)^{k_2}}{k_2!}.
\end{aligned}
$$

By applying Result C.15 to both integrals above, one obtains the following:

$$\begin{aligned}
M_{X_1,X_2}(t_1,t_2) &= \sum_{j=0}^{\infty}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} \frac{\Gamma\left(\frac{n_1}{2}+k_1+j\right)\left(\frac{1}{2}-t_1-\frac{1}{2}\right)^j}{j!\left(\frac{1}{2}-t_1\right)^{\frac{n_1}{2}+k_1+j}} \\
&\quad\times\frac{\Gamma\left(\frac{n_2}{2}+k_2+j\right)\left(\frac{1}{2}-t_2-\frac{1}{2}\right)^j}{j!\left(\frac{1}{2}-t_2\right)^{\frac{n_2}{2}+k_2+j}} \\
&\quad\times\frac{1}{2^{\frac{1}{2}(n_1+n_2+2k_1+2k_2)}}\frac{\left(\frac{r}{2}\right)_j j!\xi^{2j}}{\Gamma\left(\frac{n_1}{2}+k_1+j\right)\Gamma\left(\frac{n_2}{2}+k_2+j\right)} \\
&\quad\times\frac{e^{-\frac{\theta_1}{2}}\left(\frac{\theta_1}{2}\right)^{k_1}}{k_1!}\frac{e^{-\frac{\theta_2}{2}}\left(\frac{\theta_2}{2}\right)^{k_2}}{k_2!} \\
&= \sum_{j=0}^{\infty}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\frac{\xi^{2j}\left(\frac{r}{2}\right)_j}{j!2^{\frac{1}{2}(n_1+n_2+2k_1+2k_2)}}\frac{(t_1t_2)^j}{\left(\frac{1}{2}-t_1\right)^{\frac{n_1}{2}+k_1+j}\left(\frac{1}{2}-t_2\right)^{\frac{n_2}{2}+k_2+j}} \\
&\quad\times\frac{e^{-\frac{\theta_1}{2}}\left(\frac{\theta_1}{2}\right)^{k_1}}{k_1!}\frac{e^{-\frac{\theta_2}{2}}\left(\frac{\theta_2}{2}\right)^{k_2}}{k_2!} \\
&= \sum_{j=0}^{\infty}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\frac{\xi^{2j}\left(\frac{r}{2}\right)_j}{j!2^{\frac{1}{2}(n_1+n_2+2k_1+2k_2)}}\frac{e^{-\frac{\theta_1}{2}}\left(\frac{\theta_1}{2}\right)^{k_1}}{k_1!}\frac{e^{-\frac{\theta_2}{2}}\left(\frac{\theta_2}{2}\right)^{k_2}}{k_2!} \\
&\quad\times\frac{(t_1t_2)^j}{\left(\frac{1}{2}\right)^{\frac{n_1}{2}+k_1+j}(1-2t_1)^{\frac{n_1}{2}+k_1+j}\left(\frac{1}{2}\right)^{\frac{n_2}{2}+k_2+j}(1-2t_2)^{\frac{n_2}{2}+k_2+j}} \\
&= \sum_{j=0}^{\infty}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\frac{\xi^{2j}\left(\frac{r}{2}\right)_j}{j!2^{\frac{1}{2}(n_1+n_2+2k_1+2k_2)}}\left(\frac{1}{2}\right)^{-\frac{1}{2}(n_1+2k_1+n_2+2k_2)}2^{2j} \\
&\quad\times\frac{(t_1t_2)^j}{(1-2t_1)^{\frac{n_1}{2}+k_1+j}(1-2t_2)^{\frac{n_2}{2}+k_2+j}}\frac{e^{-\frac{\theta_1}{2}}\left(\frac{\theta_1}{2}\right)^{k_1}}{k_1!}\frac{e^{-\frac{\theta_2}{2}}\left(\frac{\theta_2}{2}\right)^{k_2}}{k_2!} \\
&= \sum_{j=0}^{\infty}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\frac{\xi^{2j}\left(\frac{r}{2}\right)_j}{j!}\left(\frac{(4t_1t_2)}{(1-2t_1)(1-2t_2)}\right)^j \\
&\quad\times\frac{1}{(1-2t_1)^{\frac{n_1}{2}+k_1}(1-2t_2)^{\frac{n_2}{2}+k_2}}\frac{e^{-\frac{\theta_1}{2}}\left(\frac{\theta_1}{2}\right)^{k_1}}{k_1!}\frac{e^{-\frac{\theta_2}{2}}\left(\frac{\theta_2}{2}\right)^{k_2}}{k_2!}.
\end{aligned}$$

Now, by reordering the expression accordingly and rearranging constant terms:

$$\begin{aligned}
M_{X_1,X_2}(t_1,t_2) &= (1-2t_1)^{-\frac{n_1}{2}}(1-2t_2)^{-\frac{n_2}{2}}\sum_{j=0}^{\infty}\frac{\left(\frac{r}{2}\right)_j}{j!}\left(\frac{\left(4\xi^2 t_1 t_2\right)}{(1-2t_1)(1-2t_2)}\right)^j \\
&\quad\times\sum_{k_1=0}^{\infty}\frac{1}{(1-2t_1)^{k_1}}\frac{e^{-\frac{\theta_1}{2}}\left(\frac{\theta_1}{2}\right)^{k_1}}{k_1!}\sum_{k_2=0}^{\infty}\frac{1}{(1-2t_2)^{k_2}}\frac{e^{-\frac{\theta_2}{2}}\left(\frac{\theta_2}{2}\right)^{k_2}}{k_2!} \\
&= (1-2t_1)^{-\frac{n_1}{2}}(1-2t_2)^{-\frac{n_2}{2}}\left(1-\frac{4\xi^2 t_1 t_2}{(1-2t_1)(1-2t_2)}\right)^{-\frac{r}{2}} \\
&\quad\times\sum_{k_1=0}^{\infty}e^{-\frac{\theta_1}{2}}\frac{\left(\frac{\theta_1}{2(1-2t_1)}\right)^{k_1}}{k_1!}\sum_{k_2=0}^{\infty}e^{-\frac{\theta_2}{2}}\frac{\left(\frac{\theta_2}{2(1-2t_2)}\right)^{k_2}}{k_2!} \\
&= (1-2t_1)^{-\frac{n_1}{2}}(1-2t_2)^{-\frac{n_2}{2}}\left(1-\frac{4\xi^2 t_1 t_2}{(1-2t_1)(1-2t_2)}\right)^{-\frac{r}{2}} \\
&\quad\times e^{-\frac{\theta_1}{2}}e^{\frac{\theta_1}{2(1-2t_1)}}e^{-\frac{\theta_2}{2}}e^{\frac{\theta_2}{2(1-2t_2)}}
\end{aligned}$$

by using the binomial expansion (see Result C.9), and the series expansion for e (see Result C.8). Finally, after some simplification, it follows from above that

$$\begin{aligned}
M_{X_1,X_2}(t_1,t_2) &= (1-2t_1)^{-\frac{n_1}{2}}(1-2t_2)^{-\frac{n_2}{2}}\left(1-\frac{4\xi^2 t_1 t_2}{(1-2t_1)(1-2t_2)}\right)^{-\frac{r}{2}} \\
&\quad\times e^{\theta_1 t_1(1-2t_1)^{-1}}e^{\theta_2 t_2(1-2t_2)^{-1}}
\end{aligned}$$

which is the same as the mgf for the bivariate noncentral generalized chi-square distribution as defined by Van den Berg (2010) (see (2.12)). ■

Remark 2.6 *This result is highly significant since it shows that by taking a bivariate central chi-square distribution (such as (2.8)) and compounding it on the degrees of freedom with Poisson probabilities, it results in a well-defined bivariate noncentral chi-square distribution as if it were derived from, say, a characteristic function (such as Van den Berg (2010)). This method intuitively seems easier to obtain noncentral bivariate chi-square distributions since no transform is required to obtain a noncentral pdf from a characteristic function or a mgf (for example), but one can simply consider a bivariate central chi-square distribution (albeit conditional), and after compounding on the parameters with the compounding factors, consider the resulting pdf as its noncentral counterpart – even more so significant due to the elegant construction of the pdf of this bivariate noncentral chi-square distribution.*

2.3.4 Bivariate noncentral compound extended chi-square distribution

Yunus & Khan (2011) used the compounding method for constructing bivariate noncentral chi-square distributions. This approach is rephrased in this section in a compounding sense for two cases, namely equal- and unequal degrees of freedom.

2.3.4.1 Equal degrees of freedom

First consider the bivariate central compound extended chi-square distribution as a conditional distribution, with pdf given in (2.9). Thereafter, an unconditional *bivariate noncentral compound extended chi-square* distribution will be proposed (see Description 2.10).

Description 2.13 *Let* $(X_1, X_2| (K_1 = k_1, K_2 = k_2))$ *have a conditional bivariate compound extended chi-square distribution with equal degrees of freedom (see (2.9)) and pdf given by*

$$\begin{aligned} f_{X_1,X_2}(x_1, x_2|k_1, k_2) &= \sum_{i=0}^{\infty} \frac{\Gamma\left(\frac{n}{2}+i\right)}{\Gamma\left(\frac{n}{2}\right) i!} \gamma^{2i} \left(1-\gamma^2\right)^{\frac{n}{2}} \\ &\quad \times \frac{x_1^{\frac{n}{2}+i+k_1-1} e^{-\frac{x_1}{2(1-\gamma^2)}}}{\left(2\left(1-\gamma^2\right)\right)^{\frac{n}{2}+i+k_1} \Gamma\left(\frac{n}{2}+i+k_1\right)} \frac{x_2^{\frac{n}{2}+i+k_2-1} e^{-\frac{x_2}{2(1-\gamma^2)}}}{\left(2\left(1-\gamma^2\right)\right)^{\frac{n}{2}+i+k_2} \Gamma\left(\frac{n}{2}+i+k_2\right)} \\ &= \sum_{i=0}^{\infty} C x_1^{\frac{n}{2}+i+k_1-1} e^{-\frac{x_1}{2(1-\gamma^2)}} x_2^{\frac{n}{2}+i+k_2-1} e^{-\frac{x_2}{2(1-\gamma^2)}}, \qquad x_1, x_2 > 0 \quad (2.16) \end{aligned}$$

where

$$\begin{aligned} C &= \frac{\Gamma\left(\frac{n}{2}+i\right)}{\Gamma\left(\frac{n}{2}\right) i!} \gamma^{2j} \left(1-\gamma^2\right)^{\frac{n}{2}} \frac{1}{\left(2\left(1-\gamma^2\right)\right)^{\frac{n}{2}+i+k_1} \Gamma\left(\frac{n}{2}+i+k_1\right)} \\ &\quad \times \frac{1}{\left(2\left(1-\gamma^2\right)\right)^{\frac{n}{2}+i+k_2} \Gamma\left(\frac{n}{2}+i+k_2\right)} \qquad (2.17) \end{aligned}$$

and $n > 0$. *As previously,* γ *(where* $-1 < \gamma < 1$*) is an extension parameter. The conditional values* k_1 *and* k_2 *have domain such that* $k_v \geq 0$, $v = 1, 2$. ▲

Yunus & Khan (2011) substituted (2.16) into equation (2.11) to obtain the noncentral case of (2.9). It is rephrased in the following description.

Description 2.14 *The unconditional distribution of (2.9) can be obtained by substituting (2.16) into (2.11):*

$$\begin{aligned}
f_{X_1,X_2}(x_1,x_2) &= \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} f_{X_1,X_2}(x_1,x_2|k_1,k_2)g_K(k_1)g_K(k_2) \\
&= \sum_{i=0}^{\infty}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} \frac{\Gamma\left(\frac{n}{2}+i\right)}{\Gamma\left(\frac{n}{2}\right)i!}\gamma^{2i}\left(1-\gamma^2\right)^{\frac{n}{2}} \\
&\quad\times\frac{x_1^{\frac{n}{2}+i+k_1-1}e^{-\frac{x_1}{2(1-\gamma^2)}}}{\left(2\left(1-\gamma^2\right)\right)^{\frac{n}{2}+i+k_1}\Gamma\left(\frac{n}{2}+i+k_1\right)}\frac{x_2^{\frac{n}{2}+i+k_2-1}e^{-\frac{x_2}{2(1-\gamma^2)}}}{\left(2\left(1-\gamma^2\right)\right)^{\frac{n}{2}+i+k_2}\Gamma\left(\frac{n}{2}+i+k_2\right)} \\
&\quad\times\frac{e^{-\frac{\theta_1}{2}}\left(\frac{\theta_1}{2}\right)^{k_1}}{k_1!}\frac{e^{-\frac{\theta_2}{2}}\left(\frac{\theta_2}{2}\right)^{k_2}}{k_2!}, \qquad x_1,x_2>0
\end{aligned} \tag{2.18}$$

where $n>0$, $-1<\gamma<1$, and noncentrality parameters $\theta_1,\theta_2>0$. ▲

Again, due to the construction of the noncentrality the Poisson probability factors, namely $g_{K_v}(k_v) = \frac{e^{-\frac{\theta_v}{2}}\left(\frac{\theta_v}{2}\right)^{k_v}}{k_v!}$, $v=1,2$, isolates the noncentrality parameters in a mathematical convenient way.

Joint moment generating function

Next, the joint mgf of X_1 and X_2 is derived for the distribution with pdf (2.18).

Property 2.8 *If X_1 and X_2 are jointly distributed with pdf (2.18), then the mgf is given by*

$$\begin{aligned}
M_{X_1,X_2}(t_1,t_2) &= \left(1-\gamma^2\right)^{\frac{n}{2}}\left(1-2\left(1-\gamma^2\right)t_1\right)^{-\frac{n}{2}}\left(1-2\left(1-\gamma^2\right)t_2\right)^{-\frac{n}{2}} \\
&\quad\times\left(1-\frac{\gamma^2}{\left(1-2\left(1-\gamma^2\right)t_1\right)\left(1-2\left(1-\gamma^2\right)t_2\right)}\right)^{-\frac{n}{2}} \\
&\quad\times e^{\theta_1 t_1\left(\frac{1}{1-\gamma^2}-2t_1\right)^{-1}}e^{\theta_2 t_2\left(\frac{1}{1-\gamma^2}-2t_2\right)^{-1}}, \qquad t_1,t_2>0
\end{aligned}$$

where $n>0$, $-1<\gamma<1$, and both noncentrality parameters $\theta_1,\theta_2>0$. ▲

Proof. From (2.18), consider the following:

$$\begin{aligned}
M_{X_1,X_2}(t_1,t_2) &= E\left(e^{t_1X_1+t_2X_2}\right) \\
&= \int_0^\infty\int_0^\infty e^{t_1x_1+t_2x_2}\sum_{i=0}^{\infty}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\frac{\Gamma\left(\frac{n}{2}+i\right)}{\Gamma\left(\frac{n}{2}\right)i!}\gamma^{2i}\left(1-\gamma^2\right)^{\frac{n}{2}} \\
&\quad\times\frac{x_1^{\frac{n}{2}+i+k_1-1}e^{-\frac{x_1}{2(1-\gamma^2)}}}{\left(2\left(1-\gamma^2\right)\right)^{\frac{n}{2}+i+k_1}\Gamma\left(\frac{n}{2}+i+k_1\right)}\frac{x_2^{\frac{n}{2}+i+k_2-1}e^{-\frac{x_2}{2(1-\gamma^2)}}}{\left(2\left(1-\gamma^2\right)\right)^{\frac{n}{2}+i+k_2}\Gamma\left(\frac{n}{2}+i+k_2\right)} \\
&\quad\times\frac{e^{-\frac{\theta_1}{2}}\left(\frac{\theta_1}{2}\right)^{k_1}}{k_1!}\frac{e^{-\frac{\theta_2}{2}}\left(\frac{\theta_2}{2}\right)^{k_2}}{k_2!}dx_1dx_2 \\
&= \sum_{i=0}^{\infty}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\frac{\Gamma\left(\frac{n}{2}+i\right)}{\Gamma\left(\frac{n}{2}\right)i!}\gamma^{2i}\left(1-\gamma^2\right)^{\frac{n}{2}}\int_0^\infty x_1^{\frac{n}{2}+i+k_1-1}e^{-\left(\frac{1}{2(1-\gamma^2)}-t_1\right)x_1}dx_1 \\
&\quad\times\int_0^\infty x_2^{\frac{n}{2}+i+k_2-1}e^{-\left(\frac{1}{2(1-\gamma^2)}-t_2\right)x_2}dx_2\frac{1}{\left(2\left(1-\gamma^2\right)\right)^{\frac{n}{2}+i+k_1}\Gamma\left(\frac{n}{2}+i+k_1\right)} \\
&\quad\times\frac{1}{\left(2\left(1-\gamma^2\right)\right)^{\frac{n}{2}+i+k_2}\Gamma\left(\frac{n}{2}+i+k_2\right)}\frac{e^{-\frac{\theta_1}{2}}\left(\frac{\theta_1}{2}\right)^{k_1}}{k_1!}\frac{e^{-\frac{\theta_2}{2}}\left(\frac{\theta_2}{2}\right)^{k_2}}{k_2!}.
\end{aligned}$$

By applying Result C.12 to both integrals above, one obtains the following:

$$\begin{aligned}
M_{X_1,X_2}(t_1,t_2) &= \sum_{i=0}^{\infty}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\frac{\Gamma\left(\frac{n}{2}+i\right)}{\Gamma\left(\frac{n}{2}\right)i!}\gamma^{2i}\left(1-\gamma^2\right)^{\frac{n}{2}}\frac{\Gamma\left(\frac{n_1}{2}+i+k_1\right)}{\left(\frac{1}{2(1-\gamma^2)}-t_1\right)^{\frac{n}{2}+i+k_1}} \\
&\quad\times\frac{\Gamma\left(\frac{n}{2}+i+k_1\right)}{\left(\frac{1}{2(1-\gamma^2)}-t_2\right)^{\frac{n}{2}+i+k_1}}\frac{1}{\left(2\left(1-\gamma^2\right)\right)^{\frac{n}{2}+i+k_1}\Gamma\left(\frac{n}{2}+i+k_1\right)} \\
&\quad\times\frac{1}{\left(2\left(1-\gamma^2\right)\right)^{\frac{n}{2}+i+k_2}\Gamma\left(\frac{n}{2}+i+k_2\right)}\frac{e^{-\frac{\theta_1}{2}}\left(\frac{\theta_1}{2}\right)^{k_1}}{k_1!}\frac{e^{-\frac{\theta_2}{2}}\left(\frac{\theta_2}{2}\right)^{k_2}}{k_2!} \\
&= \sum_{i=0}^{\infty}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\frac{\Gamma\left(\frac{n}{2}+i\right)}{\Gamma\left(\frac{n}{2}\right)i!}\gamma^{2i}\left(1-\gamma^2\right)^{\frac{n}{2}}\left(1-2\left(1-\gamma^2\right)t_1\right)^{-\left(\frac{n}{2}+i+k_1\right)} \\
&\quad\times\left(1-2\left(1-\gamma^2\right)t_2\right)^{-\left(\frac{n}{2}+i+k_1\right)}\frac{e^{-\frac{\theta_1}{2}}\left(\frac{\theta_1}{2}\right)^{k_1}}{k_1!}\frac{e^{-\frac{\theta_2}{2}}\left(\frac{\theta_2}{2}\right)^{k_2}}{k_2!} \\
&= \left(1-\gamma^2\right)^{\frac{n}{2}}\left(1-2\left(1-\gamma^2\right)t_1\right)^{-\frac{n}{2}}\left(1-2\left(1-\gamma^2\right)t_2\right)^{-\frac{n}{2}} \\
&\quad\times\sum_{i=0}^{\infty}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\frac{\Gamma\left(\frac{n}{2}+i\right)}{\Gamma\left(\frac{n}{2}\right)i!}\left(\frac{\gamma^2}{\left(1-2\left(1-\gamma^2\right)t_1\right)\left(1-2\left(1-\gamma^2\right)t_2\right)}\right)^i \\
&\quad\times\frac{e^{-\frac{\theta_1}{2}}\left(\frac{\theta_1}{2(1-2(1-\gamma^2)t_1)}\right)^{k_1}}{k_1!}\frac{e^{-\frac{\theta_2}{2}}\left(\frac{\theta_2}{2(1-2(1-\gamma^2)t_2)}\right)^{k_2}}{k_2!}.
\end{aligned}$$

Now, by reordering the expression accordingly and rearranging constant terms:

$$\begin{aligned}
M_{X_1,X_2}(t_1,t_2) &= \left(1-\gamma^2\right)^{\frac{n}{2}}\left(1-2\left(1-\gamma^2\right)t_1\right)^{-\frac{n}{2}}\left(1-2\left(1-\gamma^2\right)t_2\right)^{-\frac{n}{2}} \\
&\quad\times\sum_{i=0}^{\infty}\frac{\left(\frac{n}{2}\right)_i}{i!}\left(\frac{\gamma^2}{\left(1-2\left(1-\gamma^2\right)t_1\right)\left(1-2\left(1-\gamma^2\right)t_2\right)}\right)^i \\
&\quad\times e^{-\frac{\theta_1}{2}}\left(\sum_{k_1=0}^{\infty}\frac{\left(\frac{\theta_1}{2(1-2(1-\gamma^2)t_1)}\right)^{k_1}}{k_1!}\right)e^{-\frac{\theta_2}{2}}\left(\sum_{k_2=0}^{\infty}\frac{\left(\frac{\theta_2}{2(1-2(1-\gamma^2)t_2)}\right)^{k_2}}{k_2!}\right) \\
&= \left(1-\gamma^2\right)^{\frac{n}{2}}\left(1-2\left(1-\gamma^2\right)t_1\right)^{-\frac{n}{2}}\left(1-2\left(1-\gamma^2\right)t_2\right)^{-\frac{n}{2}} \\
&\quad\times\left(1-\frac{\gamma^2}{\left(1-2\left(1-\gamma^2\right)t_1\right)\left(1-2\left(1-\gamma^2\right)t_2\right)}\right)^{-\frac{n}{2}} \\
&\quad\times e^{-\frac{\theta_1}{2}}e^{\frac{\theta_1}{2(1-2(1-\gamma^2)t_1)}}e^{-\frac{\theta_2}{2}}e^{\frac{\theta_2}{2(1-2(1-\gamma^2)t_2)}}
\end{aligned}$$

by using the binomial expansion (see Result C.9), and the series expansion for e (see Result C.8). Finally, after some simplification, it follows that

$$\begin{aligned}
M_{X_1,X_2}(t_1,t_2) &= \left(1-\gamma^2\right)^{\frac{n}{2}}\left(1-2\left(1-\gamma^2\right)t_1\right)^{-\frac{n}{2}}\left(1-2\left(1-\gamma^2\right)t_2\right)^{-\frac{n}{2}} \\
&\quad\times\left(1-\frac{\gamma^2}{\left(1-2\left(1-\gamma^2\right)t_1\right)\left(1-2\left(1-\gamma^2\right)t_2\right)}\right)^{-\frac{n}{2}} \\
&\quad\times e^{\theta_1 t_1\left(\frac{1}{(1-\gamma^2)}-2t_1\right)^{-1}}e^{\theta_2 t_2\left(\frac{1}{(1-\gamma^2)}-2t_2\right)^{-1}}.
\end{aligned}$$

■

Remark 2.7 *When $\gamma = 0$ in above mgf, then*

$$M_{X_1,X_2}(t_1,t_2) = \left(1-2t_1\right)^{-\frac{n}{2}}e^{\theta_1 t_1(1-2t_1)^{-1}}\left(1-2t_2\right)^{-\frac{n}{2}}e^{\theta_2 t_2(1-2t_2)^{-1}}$$

or alternatively, the product of two mgfs of independent noncentral chi-square distributions with n degrees of freedom and noncentrality parameters $\theta_1, \theta_2 > 0$ (see (2.2)). Furthermore, if $\theta_1 = \theta_2 = 0$, then this mgf further simplifies to the product of two mgfs of independent chi-square distributions with n degrees of freedom (see Result C.19).

2.3.4.2 Unequal degrees of freedom

As before (see section 2.3.1), a conditional central pdf is defined from (2.10) (see Description 2.10).

Description 2.15 *Let $(X_1, X_2|K_1 = k_1, K_2 = k_2)$ have a conditional bivariate compound extended chi-square distribution with n_1 and n_2 degrees of freedom (see (2.10)) with pdf given by*

$$\begin{aligned} f_{X_1,X_2}(x_1, x_2|k_1, k_2) &= \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty} \frac{\Gamma\left(\frac{n_1}{2}+i_1\right)}{\Gamma\left(\frac{n_1}{2}\right) i_1!} \frac{\Gamma\left(\frac{n_2}{2}+i_2\right)}{\Gamma\left(\frac{n_2}{2}\right) i_2!} \gamma^{2(i_1+i_2)} \left(1-\gamma^2\right)^{\frac{n_1+n_2}{2}} \\ &\quad \times \frac{x_1^{\frac{n_1}{2}+i_1+k_1-1} e^{-\frac{x_1}{2(1-\gamma^2)}}}{\left(2\left(1-\gamma^2\right)\right)^{\frac{n_1}{2}+i_1+k_1} \Gamma\left(\frac{n_1}{2}+i_1+k_1\right)} \\ &\quad \times \frac{x_2^{\frac{n_2}{2}+i_2+k_2-1} e^{-\frac{x_2}{2(1-\gamma^2)}}}{\left(2\left(1-\gamma^2\right)\right)^{\frac{n_2}{2}+i_2+k_2} \Gamma\left(\frac{n_2}{2}+i_2+k_2\right)} \\ &= \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty} C x_1^{\frac{n_1}{2}+i_1+k_1-1} x_2^{\frac{n_2}{2}+i_2+k_2-1} e^{-\frac{x_1+x_2}{2(1-\gamma^2)}}, \quad x_1, x_2 > 0 \end{aligned} \tag{2.19}$$

where

$$\begin{aligned} C &= \frac{\Gamma\left(\frac{n_1}{2}+i_1\right)}{\Gamma\left(\frac{n_1}{2}\right) i_1!} \frac{\Gamma\left(\frac{n_2}{2}+i_2\right)}{\Gamma\left(\frac{n_2}{2}\right) i_2!} \gamma^{2(i_1+i_2)} \left(1-\gamma^2\right)^{\frac{n_1+n_2}{2}} \\ &\quad \times \left(\left(2\left(1-\gamma^2\right)\right)^{\frac{n_1}{2}+i_1+k_1} \Gamma\left(\frac{n_1}{2}+i_1+k_1\right)\right)^{-1} \\ &\quad \times \left(\left(2\left(1-\gamma^2\right)\right)^{\frac{n_2}{2}+i_2+k_2} \Gamma\left(\frac{n_2}{2}+i_2+k_2\right)\right)^{-1} \end{aligned} \tag{2.20}$$

and $n_1, n_2 > 0$. As previously, γ (where $-1 < \gamma < 1$) is an extension parameter. The conditional values k_1 and k_2 have domain such that $k_v \geq 0$, $v = 1, 2$. ▲

Yunus & Khan (2011) substituted (2.19) into equation (2.11) to obtain the noncentral case of (2.10). It is rephrased in the following description.

Description 2.16 *The unconditional distribution of (2.10) can be obtained by substituting (2.19) into (2.11)*

$$\begin{aligned} f_{X_1,X_2}(x_1,x_2) &= \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} f_{X_1,X_2}(x_1,x_2|k_1,k_2)g_{K_1}(k_1)g_{K_2}(k_2) \\ &= \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} \frac{\Gamma\left(\frac{n_1}{2}+i_1\right)}{\Gamma\left(\frac{n_1}{2}\right)i_1!}\frac{\Gamma\left(\frac{n_2}{2}+i_2\right)}{\Gamma\left(\frac{n_2}{2}\right)i_2!}\gamma^{2(i_1+i_2)}\left(1-\gamma^2\right)^{\frac{n_1+n_2}{2}} \\ &\quad\times\frac{x_1^{\frac{n_1}{2}+i_1+k_1-1}e^{-\frac{x_1}{2(1-\gamma^2)}}}{\left(2\left(1-\gamma^2\right)\right)^{\frac{n_1}{2}+i_1+k_1}\Gamma\left(\frac{n_1}{2}+i_1+k_1\right)}\frac{x_2^{\frac{n_2}{2}+i_2+k_2-1}e^{-\frac{x_2}{2(1-\gamma^2)}}}{\left(2\left(1-\gamma^2\right)\right)^{\frac{n_2}{2}+i_2+k_2}\Gamma\left(\frac{n_2}{2}+i_2+k_2\right)} \\ &\quad\times\frac{e^{-\frac{\theta_1}{2}}\left(\frac{\theta_1}{2}\right)^{k_1}}{k_1!}\frac{e^{-\frac{\theta_2}{2}}\left(\frac{\theta_2}{2}\right)^{k_2}}{k_2!}, \qquad x_1,x_2>0 \end{aligned} \tag{2.21}$$

where $n_1, n_2 > 0$, $-1 < \gamma < 1$, *and noncentrality parameters* $\theta_1, \theta_2 > 0$. ▲

This pdf can be regarded as a *bivariate noncentral compound extended chi-square distribution* with degrees of freedom n_1, n_2, γ, and noncentrality parameters $\theta_1, \theta_2 > 0$. Again, due to the construction of the noncentrality the Poisson probability factors, namely $g_{K_v}(k_v) = \frac{e^{-\frac{\theta_v}{2}}\left(\frac{\theta_v}{2}\right)^{k_v}}{k_v!}$, $v = 1, 2$, isolates the noncentrality parameters.

Joint moment generating function

Next, the joint mgf of X_1 and X_2 is derived for the distribution with pdf (2.21).

Property 2.9 *If* X_1 *and* X_2 *are jointly distributed with pdf (2.21), then the mgf is given by*

$$\begin{aligned} M_{X_1,X_2}(t_1,t_2) &= \left(1-2\left(1-\gamma^2\right)t_1\right)^{-\frac{n_1}{2}}\left(1-2\left(1-\gamma^2\right)t_2\right)^{-\frac{n_2}{2}}\left(1-\gamma^2\right)^{\frac{n_1+n_2}{2}} \\ &\quad\times\left(1-\frac{\gamma^2}{\left(1-2\left(1-\gamma^2\right)t_1\right)}\right)^{-\frac{n_1}{2}}\left(1-\frac{\gamma^2}{\left(1-2\left(1-\gamma^2\right)t_2\right)}\right)^{-\frac{n_2}{2}} \\ &\quad\times e^{\theta_1 t_1\left(\frac{1}{(1-\gamma^2)}-2t_1\right)^{-1}} e^{\theta_2 t_2\left(\frac{1}{(1-\gamma^2)}-2t_2\right)^{-1}} \end{aligned}$$

where $t_1, t_2 > 0$, $n_1, n_2 > 0$, $-1 < \gamma < 1$, *and both noncentrality parameters* $\theta_1, \theta_2 > 0$.

▲

Proof. From (2.21), consider the following:

$$
\begin{aligned}
M_{X_1,X_2}(t_1,t_2) &= E\left(e^{t_1X_1+t_2X_2}\right) \\
&= \int_0^\infty\int_0^\infty e^{t_1x_1+t_2x_2}\sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\frac{\Gamma\left(\frac{n_1}{2}+i_1\right)}{\Gamma\left(\frac{n_1}{2}\right)i_1!}\frac{\Gamma\left(\frac{n_2}{2}+i_2\right)}{\Gamma\left(\frac{n_2}{2}\right)i_2!}\gamma^{2(i_1+i_2)}\left(1-\gamma^2\right)^{\frac{n_1+n_2}{2}} \\
&\quad\times\frac{x_1^{\frac{n_1}{2}+i_1+k_1-1}e^{-\frac{x_1}{2(1-\gamma^2)}}}{\left(2\left(1-\gamma^2\right)\right)^{\frac{n_1}{2}+i_1+k_1}\Gamma\left(\frac{n_1}{2}+i_1+k_1\right)}\frac{x_2^{\frac{n_2}{2}+i_2+k_2-1}e^{-\frac{x_2}{2(1-\gamma^2)}}}{\left(2\left(1-\gamma^2\right)\right)^{\frac{n_2}{2}+i_2+k_2}\Gamma\left(\frac{n_2}{2}+i_2+k_2\right)} \\
&\quad\times\frac{e^{-\frac{\theta_1}{2}}\left(\frac{\theta_1}{2}\right)^{k_1}}{k_1!}\frac{e^{-\frac{\theta_2}{2}}\left(\frac{\theta_2}{2}\right)^{k_2}}{k_2!}dx_1dx_2 \\
&= \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\frac{\Gamma\left(\frac{n_1}{2}+i_1\right)}{\Gamma\left(\frac{n_1}{2}\right)i_1!}\frac{\Gamma\left(\frac{n_2}{2}+i_2\right)}{\Gamma\left(\frac{n_2}{2}\right)i_2!}\gamma^{2(i_1+i_2)}\left(1-\gamma^2\right)^{\frac{n_1+n_2}{2}} \\
&\quad\times\int_0^\infty x_1^{\frac{n_1}{2}+i_1+k_1-1}e^{-\left(\frac{1}{2(1-\gamma^2)}-t_1\right)x_1}dx_1\int_0^\infty x_2^{\frac{n_2}{2}+i_2+k_2-1}e^{-\left(\frac{1}{2(1-\gamma^2)}-t_2\right)x_2}dx_2 \\
&\quad\times\frac{1}{\left(2\left(1-\gamma^2\right)\right)^{\frac{n}{2}+i_1+k_1}\Gamma\left(\frac{n_1}{2}+i_1+k_1\right)}\frac{1}{\left(2\left(1-\gamma^2\right)\right)^{\frac{n}{2}+i_2+k_2}\Gamma\left(\frac{n_2}{2}+i_2+k_2\right)} \\
&\quad\times\frac{e^{-\frac{\theta_1}{2}}\left(\frac{\theta_1}{2}\right)^{k_1}}{k_1!}\frac{e^{-\frac{\theta_2}{2}}\left(\frac{\theta_2}{2}\right)^{k_2}}{k_2!}.
\end{aligned}
$$

By applying Result C.12 to both integrals above, one obtains the following:

$$\begin{aligned}
M_{X_1,X_2}(t_1,t_2) &= \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} \frac{\Gamma\left(\frac{n_1}{2}+i_1\right)}{\Gamma\left(\frac{n_1}{2}\right)i_1!}\frac{\Gamma\left(\frac{n_2}{2}+i_2\right)}{\Gamma\left(\frac{n_2}{2}\right)i_2!}\gamma^{2(i_1+i_2)}\left(1-\gamma^2\right)^{\frac{n_1+n_2}{2}} \\
&\times \frac{\Gamma\left(\frac{n_1}{2}+i_1+k_1\right)}{\left(\frac{1}{2(1-\gamma^2)}-t_1\right)^{\frac{n_1}{2}+i_1+k_1}}\frac{\Gamma\left(\frac{n_2}{2}+i_2+k_2\right)}{\left(\frac{1}{2(1-\gamma^2)}-t_2\right)^{\frac{n_2}{2}+i_2+k_2}} \\
&\times \frac{1}{\left(2\left(1-\gamma^2\right)\right)^{\frac{n_1}{2}+i_1+k_1}\Gamma\left(\frac{n_1}{2}+i_1+k_1\right)}\frac{1}{\left(2\left(1-\gamma^2\right)\right)^{\frac{n_2}{2}+i_2+k_2}\Gamma\left(\frac{n_2}{2}+i_2+k_2\right)} \\
&\times \frac{e^{-\frac{\theta_1}{2}}\left(\frac{\theta_1}{2}\right)^{k_1}}{k_1!}\frac{e^{-\frac{\theta_2}{2}}\left(\frac{\theta_2}{2}\right)^{k_2}}{k_2!} \\
&= \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} \frac{\Gamma\left(\frac{n_1}{2}+i_1\right)}{\Gamma\left(\frac{n_1}{2}\right)i_1!}\frac{\Gamma\left(\frac{n_2}{2}+i_2\right)}{\Gamma\left(\frac{n_2}{2}\right)i_2!}\gamma^{2(i_1+i_2)}\left(1-\gamma^2\right)^{\frac{n_1+n_2}{2}} \\
&\times \left(1-2\left(1-\gamma^2\right)t_1\right)^{-\left(\frac{n_1}{2}+i_1+k_1\right)}\left(1-2\left(1-\gamma^2\right)t_2\right)^{-\left(\frac{n_2}{2}+i_2+k_2\right)} \\
&\times \frac{e^{-\frac{\theta_1}{2}}\left(\frac{\theta_1}{2}\right)^{k_1}}{k_1!}\frac{e^{-\frac{\theta_2}{2}}\left(\frac{\theta_2}{2}\right)^{k_2}}{k_2!} \\
&= \left(1-\gamma^2\right)^{\frac{n_1+n_2}{2}}\left(1-2\left(1-\gamma^2\right)t_1\right)^{-\frac{n_1}{2}}\left(1-2\left(1-\gamma^2\right)t_2\right)^{-\frac{n_2}{2}} \\
&\times \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} \frac{\Gamma\left(\frac{n_1}{2}+i_1\right)}{\Gamma\left(\frac{n_1}{2}\right)i_1!}\frac{\Gamma\left(\frac{n_2}{2}+i_2\right)}{\Gamma\left(\frac{n_2}{2}\right)i_2!}\left(\frac{\gamma^2}{\left(1-2\left(1-\gamma^2\right)t_1\right)}\right)^{i_1} \\
&\times \left(\frac{\gamma^2}{\left(1-2\left(1-\gamma^2\right)t_2\right)}\right)^{i_2}\frac{e^{-\frac{\theta_1}{2}}\left(\frac{\theta_1}{2(1-2(1-\gamma^2)t_1)}\right)^{k_1}}{k_1!}\frac{e^{-\frac{\theta_2}{2}}\left(\frac{\theta_2}{2(1-2(1-\gamma^2)t_2)}\right)^{k_2}}{k_2!}.
\end{aligned}$$

Now, by reordering the expression accordingly and rearranging constant terms:

$$\begin{aligned}
M_{X_1,X_2}(t_1,t_2) &= \left(1-\gamma^2\right)^{\frac{n_1+n_2}{2}}\left(1-2\left(1-\gamma^2\right)t_1\right)^{-\frac{n_1}{2}}\left(1-2\left(1-\gamma^2\right)t_2\right)^{-\frac{n_2}{2}} \\
&\times \sum_{i_1=0}^{\infty}\frac{\left(\frac{n_1}{2}\right)_{i_1}}{i_1!}\left(\frac{\gamma^2}{\left(1-2\left(1-\gamma^2\right)t_1\right)}\right)^{i_1}\sum_{i_2=0}^{\infty}\frac{\left(\frac{n_2}{2}\right)_{i_2}}{i_2!}\left(\frac{\gamma^2}{\left(1-2\left(1-\gamma^2\right)t_2\right)}\right)^{i_2} \\
&\times \sum_{k_1=0}^{\infty}e^{-\frac{\theta_1}{2}}\frac{\left(\frac{\theta_1}{2(1-2(1-\gamma^2)t_1)}\right)^{k_1}}{k_1!}\sum_{k_2=0}^{\infty}e^{-\frac{\theta_2}{2}}\frac{\left(\frac{\theta_2}{2(1-2(1-\gamma^2)t_2)}\right)^{k_2}}{k_2!} \\
&= \left(1-\gamma^2\right)^{\frac{n_1+n_2}{2}}\left(1-2\left(1-\gamma^2\right)t_1\right)^{-\frac{n_1}{2}}\left(1-2\left(1-\gamma^2\right)t_2\right)^{-\frac{n_2}{2}} \\
&\times \left(1-\frac{\gamma^2}{\left(1-2\left(1-\gamma^2\right)t_1\right)}\right)^{-\frac{n_1}{2}}\left(1-\frac{\gamma^2}{\left(1-2\left(1-\gamma^2\right)t_2\right)}\right)^{-\frac{n_2}{2}} \\
&\times e^{-\frac{\theta_1}{2}}e^{\frac{\theta_1}{2(1-2(1-\gamma^2)t_1)}}e^{-\frac{\theta_2}{2}}e^{\frac{\theta_2}{2(1-2(1-\gamma^2)t_2)}}
\end{aligned}$$

by using the binomial expansion (see Result C.9), and the series expansion for e (see Result C.8). Finally, after some simplification:

$$\begin{aligned} M_{X_1,X_2}(t_1,t_2) &= \left(1-2\left(1-\gamma^2\right)t_1\right)^{-\frac{n_1}{2}}\left(1-2\left(1-\gamma^2\right)t_2\right)^{-\frac{n_2}{2}}\left(1-\gamma^2\right)^{\frac{n_1+n_2}{2}} \\ &\quad \times\left(1-\frac{\gamma^2}{\left(1-2\left(1-\gamma^2\right)t_1\right)}\right)^{-\frac{n_1}{2}}\left(1-\frac{\gamma^2}{\left(1-2\left(1-\gamma^2\right)t_2\right)}\right)^{-\frac{n_2}{2}} \\ &\quad \times e^{\theta_1 t_1\left(\frac{1}{(1-\gamma^2)}-2t_1\right)^{-1}} e^{\theta_2 t_2\left(\frac{1}{(1-\gamma^2)}-2t_2\right)^{-1}}. \end{aligned}$$

■

Remark 2.8 *When $\gamma = 0$ in above mgf, then*

$$M_{X_1,X_2}(t_1,t_2) = (1-2t_1)^{-\frac{n_1}{2}} e^{\theta_1 t_1(1-2t_1)^{-1}} (1-2t_2)^{-\frac{n_2}{2}} e^{\theta_2 t_2(1-2t_2)^{-1}}$$

or alternatively, the product of two mgfs of independent noncentral chi-square distributions with n_1 and n_2 degrees of freedom respectively and noncentrality parameters $\theta_1, \theta_2 > 0$ (see (2.2)). Furthermore, if $\theta_1 = \theta_2 = 0$, then this mgf further simplifies to the product of two mgfs of independent chi-square distributions with n_1 and n_2 degrees of freedom respectively (see Result C.19).

2.4 Distributions of composites

In this section, the univariate distributions of the composites as discussed in Chapter 1 are systematically derived for the pdfs as proposed in (2.15) and (2.21). The focus will be around the following composites: the product of the bivariate components $W_1 = X_1X_2$, the ratio of the bivariate components (ratio of type II) $W_2 = \frac{X_1}{X_2}$, and the proportion of the bivariate components (ratio of type I) $W_3 = \frac{X_1}{X_1+X_2}$.

2.4.1 An alternative representation of the bivariate noncentral generalized chi-square distribution

2.4.1.1 The probability density function of the product

In this section, the pdf of the product of the components of the bivariate distribution proposed in (2.14) is given. Subsequently the noncentral case is proposed by using the compounding method (see (1.1)). Let $W_1 = X_1X_2$.

Description 2.17 *If X_1 and X_2 are jointly distributed according to equation (2.14), the pdf of the conditional distribution of $W_1|\left(K_1 = k_1, K_2 = k_2\right)$ is given by*

$$\begin{aligned} f_{W_1}(w_1|k_1,k_2) &= \sum_{j=0}^{\infty}\sum_{l_1=0}^{j}\sum_{l_2=0}^{j} K_{\tau}\left(\sqrt{w_1}\right) w_1^{\frac{\frac{n_1}{2}+\frac{n_2}{2}+l_1+l_2+k_1+k_2-2}{2}} \left(\frac{\left(\frac{r}{2}\right)_j \xi^{2j}}{j!}\right) \\ &\quad \times \frac{(-1)^{l_1+l_2}\binom{j}{l_1}\binom{j}{l_2}}{\Gamma\left(\frac{n_1}{2}+l_1+k_1\right)\Gamma\left(\frac{n_2}{2}+l_2+k_2\right)} \\ &\quad \times \left(\frac{1}{2}\right)^{\frac{1}{2}(n_1+n_2+2l_1+2l_2+2k_1+2k_2-2)}, \qquad w_1 > 0 \qquad (2.22) \end{aligned}$$

where $n_1, n_2, r > 0$, $-1 \leq \xi \leq 1$, $\tau = \frac{n_2-n_1}{2}+l_2-l_1+k_2-k_1$, and $K_{\tau}(\cdot)$ the modified Bessel function of the second kind (see Result C.11). $B(\cdot,\cdot)$ is the beta function as defined in Result C.4 and the conditional values k_1 and k_2 have domain such that $k_v \geq 0$, $v = 1, 2$. ▲

Proof. See Result A, section 2.2.3. ■

Description 2.18 *Upon taking the pdf in equation (2.22) one can now obtain the (unconditional) noncentral distribution of $W_1 = X_1X_2$ by substituting the Poisson probabilities and the corresponding summation operators (stemming from (2.11)):*

$$\begin{aligned} f_{W_1}(w_1) &= f_{W_1}(w_1|k_1,k_2)g_{K_1}(k_1)g_{K_2}(k_2) \\ &= \sum_{j=0}^{\infty}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{l_1=0}^{j}\sum_{l_2=0}^{j} K_\tau\left(\sqrt{w_1}\right) w_1^{\frac{\frac{n_1}{2}+\frac{n_2}{2}+l_1+l_2+k_1+k_2-2}{2}} \left(\frac{\left(\frac{r}{2}\right)_j \xi^{2j}}{j!}\right) \\ &\quad \times \frac{(-1)^{l_1+l_2}\binom{j}{l_1}\binom{j}{l_2}}{\Gamma\left(\frac{n_1}{2}+l_1+k_1\right)\Gamma\left(\frac{n_2}{2}+l_2+k_2\right)}\left(\frac{1}{2}\right)^{\frac{1}{2}(n_1+n_2+2l_1+2l_2+2k_1+2k_2-2)} \\ &\quad \times \frac{e^{-\frac{\theta_1}{2}}\left(\frac{\theta_1}{2}\right)^{k_1}}{k_1!}\frac{e^{-\frac{\theta_2}{2}}\left(\frac{\theta_2}{2}\right)^{k_2}}{k_2!}, \qquad w_1>0 \end{aligned} \tag{2.23}$$

where $n_1, n_2, r > 0$, $-1 \le \xi \le 1$, $\tau = \frac{n_2-n_1}{2} + l_2 - l_1 + k_2 - k_1$, $\theta_1, \theta_2 > 0$, and $g_{K_v}(k_v) = \frac{e^{-\frac{\theta_v}{2}}\left(\frac{\theta_v}{2}\right)^{k_v}}{k_v!}$, $v = 1,2$. ▲

2.4.1.2 The probability density function of the ratio of type II

In this section, the pdf of the ratio of type II of the components of the bivariate distribution proposed in (2.14) is given. Subsequently the noncentral case is proposed by using the compounding method (see (1.1)). Let $W_2 = \frac{X_1}{X_2}$.

Description 2.19 *If X_1 and X_2 are jointly distributed according to equation (2.14), the pdf of the conditional distribution of $W_2|(K_1 = k_1, K_2 = k_2)$ is given by*

$$\begin{aligned} f_{W_2}(w_2|k_1,k_2) &= \sum_{j=0}^{\infty}\sum_{l_1=0}^{j}\sum_{l_2=0}^{j} \frac{w_2^{\frac{n_1}{2}+l_1+k_1-1}}{(1+w_2)^\tau}\left(\frac{\left(\frac{r}{2}\right)_j \xi^{2j}}{j!}\right) \\ &\quad \times \frac{(-1)^{l_1+l_2}\binom{j}{l_1}\binom{j}{l_2}}{B\left(\frac{n_1}{2}+l_1+k_1, \frac{n_2}{2}+l_2+k_2\right)}, \qquad w_2>0 \end{aligned} \tag{2.24}$$

where $n_1, n_2, r > 0$, $-1 \le \xi \le 1$, and $\tau = \frac{n_1}{2} + \frac{n_2}{2} + l_1 + l_2 + k_1 + k_2$. $B(\cdot,\cdot)$ is the beta function (see Result C.4) and the conditional values k_1 and k_2 have domain such that $k_v \ge 0$, $v = 1,2$. ▲

Proof. See Result B, section 2.2.3. ■

Description 2.20 *Upon taking the pdf in equation (2.24) one can now obtain the (unconditional) noncentral distribution of* $W_2 = \frac{X_1}{X_2}$ *by substituting the Poisson probabilities and the corresponding summation operators (stemming from (2.11)):*

$$\begin{aligned} f_{W_2}(w_2) &= f_{W_2}(w_2|k_1,k_2)g_{K_1}(k_1)g_{K_2}(k_2) \\ &= \sum_{j=0}^{\infty}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{l_1=0}^{j}\sum_{l_2=0}^{j} \frac{w_2^{\frac{n_1}{2}+l_1+k_1-1}}{(1+w_2)^{\tau}} \left(\frac{\left(\frac{r}{2}\right)_j \xi^{2j}}{j!} \right) \\ &\quad \times \frac{(-1)^{l_1+l_2}\binom{j}{l_1}\binom{j}{l_2}}{B\left(\frac{n_1}{2}+l_1+k_1, \frac{n_2}{2}+l_2+k_2\right)} \\ &\quad \times \frac{e^{-\frac{\theta_1}{2}}\left(\frac{\theta_1}{2}\right)^{k_1}}{k_1!} \frac{e^{-\frac{\theta_2}{2}}\left(\frac{\theta_2}{2}\right)^{k_2}}{k_2!}, \qquad w_2 > 0 \end{aligned} \tag{2.25}$$

where $n_1, n_2, r > 0$, $-1 \le \xi \le 1$, $\tau = \frac{n_1}{2} + \frac{n_2}{2} + l_1 + l_2 + k_1 + k_2$, $\theta_1, \theta_2 > 0$, *and* $g_{K_v}(k_v) = \frac{e^{-\frac{\theta_v}{2}}\left(\frac{\theta_v}{2}\right)^{k_v}}{k_v!}$, $v = 1, 2$. ▲

2.4.1.3 The probability density function of the ratio of type I

In this section, the pdf of the ratio of type I of the components of the bivariate distribution proposed in (2.14) is given. Subsequently the noncentral case is proposed by using the compounding method (see (1.1)). Let $W_3 = \frac{X_1}{X_1+X_2}$.

Description 2.21 *If* X_1 *and* X_2 *are jointly distributed according to equation (2.14), the pdf of the conditional distribution of* $W_3|(K_1 = k_1, K_2 = k_2)$ *is given by*

$$\begin{aligned} f_{W_3}(w_3|k_1,k_2) &= \sum_{j=0}^{\infty}\sum_{l_1=0}^{j}\sum_{l_2=0}^{j} w_3^{\frac{n_1}{2}+l_1+k_1-1}(1-w_3)^{\frac{n_2}{2}+l_1+k_2-1} \left(\frac{\left(\frac{r}{2}\right)_j \xi^{2j}}{j!} \right) \\ &\quad \times \frac{(-1)^{l_1+l_2}\binom{j}{l_1}\binom{j}{l_2}}{B\left(\frac{n_1}{2}+l_1+k_1, \frac{n_2}{2}+l_2+k_2\right)}, \qquad 0 < w_3 < 1 \end{aligned} \tag{2.26}$$

where $n_1, n_2, r > 0$, *and* $-1 \le \xi \le 1$. $B(\cdot,\cdot)$ *is the beta function (see Result C.4) and the conditional values* k_1 *and* k_2 *have domain such that* $k_v \ge 0$, $v = 1, 2$. ▲

Proof. See Result C, section 2.2.3. ■

Description 2.22 *Upon taking the pdf in equation (2.26) one can now obtain the (unconditional) noncentral distribution of* $W_3 = \frac{X_1}{X_1+X_2}$ *by substituting the Poisson probabilities and the corresponding summation operators (stemming from (2.11)):*

$$\begin{aligned} f_{W_3}(w_3) &= f_{W_3}(w_3|k_1,k_2)g_{K_1}(k_1)g_{K_2}(k_2) \\ &= \sum_{j=0}^{\infty}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{l_1=0}^{j}\sum_{l_2=0}^{j} w_3^{\frac{n_1}{2}+l_1+k_1-1}(1-w_3)^{\frac{n_2}{2}+l_1+k_2-1} \\ &\quad\times\left(\frac{\left(\frac{r}{2}\right)_j \xi^{2j}}{j!}\right)\frac{(-1)^{l_1+l_2}\binom{j}{l_1}\binom{j}{l_2}}{B\left(\frac{n_1}{2}+l_1+k_1,\frac{n_2}{2}+l_2+k_2\right)} \\ &\quad\times\frac{e^{-\frac{\theta_1}{2}}\left(\frac{\theta_1}{2}\right)^{k_1}}{k_1!}\frac{e^{-\frac{\theta_2}{2}}\left(\frac{\theta_2}{2}\right)^{k_2}}{k_2!}, \qquad 0<w_3<1 \end{aligned} \tag{2.27}$$

where $n_1,n_2,r>0$, $-1\le\xi\le 1$, $\theta_1,\theta_2>0$, *and* $g_{K_v}(k_v)=\frac{e^{-\frac{\theta_v}{2}}\left(\frac{\theta_v}{2}\right)^{k_v}}{k_v!}$, $v=1,2$. ▲

2.4.2 Bivariate noncentral compound extended chi-square distribution: equal degrees of freedom

2.4.2.1 The probability density function of the product

In this section, the pdf of the product of the components of the bivariate distribution proposed in (2.16) is derived. Subsequently the noncentral case is proposed by using the compounding method (see (1.1)). Let $W_1 = X_1X_2$.

Theorem 2.1 *If* X_1 *and* X_2 *are jointly distributed according to equation (2.16), the pdf of the conditional distribution of* $W_1|\left(K_1=k_1,K_2=k_2\right)$ *is given by*

$$f_{W_1}(w_1|k_1,k_2)=\sum_{i=0}^{\infty}2Cw_1^{\frac{n}{2}+i+\frac{k_2+k_1}{2}-1}K_\tau\left(\frac{\sqrt{w_1}}{(1-\gamma^2)}\right), \qquad w_1>0 \tag{2.28}$$

where $n>0$, $-1<\gamma<1$, $\tau=k_2-k_1$, $K_\tau(\cdot)$ *the modified Bessel function of the second kind (see Result C.10), and* C *the value as given in (2.17). The conditional values* k_1 *and* k_2 *have domain such that* $k_v\ge 0$, $v=1,2$.

Proof. The Jacobian of the transformation is given by

$$J(x_1,x_2\to w_1,x_2)=\begin{vmatrix}\frac{dx_1}{dw_1} & \frac{dx_1}{dx_2}\\ \frac{dx_2}{dw_1} & \frac{dx_2}{dx_2}\end{vmatrix}=\begin{vmatrix}\frac{1}{x_2} & -\frac{w_1}{x_2^2}\\ 0 & 1\end{vmatrix}=\frac{1}{x_2}$$

and thus the joint pdf of W_1 and X_2 is given by

$$\begin{aligned}
f_{W_1,X_2}(w_1, x_2|k_1, k_2) &= \frac{1}{x_2} f_{X_1,X_2}\left(\frac{w_1}{x_2}, x_2|k_1, k_2\right) \\
&= \frac{1}{x_2}\sum_{i=0}^{\infty} Ce^{-\frac{w_1}{x_2}\left(\frac{1}{2(1-\gamma^2)}\right)} \left(\frac{w_1}{x_2}\right)^{\frac{n}{2}+i+k_1-1} e^{-\frac{x_2}{2(1-\gamma^2)}} x_2^{\frac{n}{2}+i+k_2-1} \\
&= \sum_{i=0}^{\infty} Ce^{-\frac{w_1}{x_2}\left(\frac{1}{2(1-\gamma^2)}\right)-\frac{x_2}{2(1-\gamma^2)}} w_1^{\frac{n}{2}+i+k_1-1} x_2^{k_2-k_1-1}.
\end{aligned}$$

Now, since $x_2 > 0$; the pdf of W_1 is given by

$$\begin{aligned}
f_{W_1}(w_1|k_1, k_2) &= \int_0^{\infty} f_{W_1,X_2}(w_1, x_2|k_1, k_2)dx_2 \\
&= \sum_{i=0}^{\infty} Cw_1^{\frac{n}{2}+i+k_1-1} \int_0^{\infty} e^{-\frac{w_1}{x_2}\left(\frac{1}{2(1-\gamma^2)}\right)-\frac{x_2}{2(1-\gamma^2)}} x_2^{k_2-k_1-1} dx_2
\end{aligned}$$

where this last integral is evaluated using Result C.11: setting $\tau = k_2 - k_1$, $\gamma = \frac{1}{2(1-\gamma^2)}$ and $\beta = \frac{w_1}{2(1-\gamma^2)}$, one obtains

$$\begin{aligned}
f_{W_1}(w_1|k_1, k_2) &= \sum_{i=0}^{\infty} Cw_1^{\frac{n}{2}+i+k_1-1} 2w_1^{\frac{k_2-k_1}{2}} K_\tau\left(2\sqrt{\frac{w_1}{4(1-\gamma^2)^2}}\right) \\
&= \sum_{i=0}^{\infty} 2Cw_1^{\frac{n}{2}+i+\frac{k_2+k_1}{2}-1} K_\tau\left(\frac{\sqrt{w_1}}{(1-\gamma^2)}\right)
\end{aligned}$$

which leaves the final result. ■

Description 2.23 *Upon taking the pdf in equation (2.28) one can now obtain the (unconditional) noncentral distribution of $W_1 = X_1X_2$ by substituting the Poisson probabilities and the corresponding summation operators (stemming from (2.11)):*

$$\begin{aligned}
f_{W_1}(w_1) &= f_{W_1}(w_1|k_1, k_2) g_{K_1}(k_1) g_{K_2}(k_2) \\
&= \sum_{i=0}^{\infty}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} 2Cw_1^{\frac{n}{2}+i+\frac{k_2+k_1}{2}-1} K_\tau\left(\frac{\sqrt{w_1}}{(1-\gamma^2)}\right) \\
&\quad \times \frac{e^{-\frac{\theta_1}{2}}\left(\frac{\theta_1}{2}\right)^{k_1}}{k_1!} \frac{e^{-\frac{\theta_2}{2}}\left(\frac{\theta_2}{2}\right)^{k_2}}{k_2!}, \qquad w_1 > 0 \qquad (2.29)
\end{aligned}$$

where $n > 0$, $-1 < \gamma < 1$, $\tau = k_2 - k_1$, $\theta_1, \theta_2 > 0$, and $g_{K_v}(k_v) = \frac{e^{-\frac{\theta_v}{2}}\left(\frac{\theta_v}{2}\right)^{k_v}}{k_v!}$, $v = 1, 2$. ▲

2.4.2.2 The probability density function of the ratio of type II

In this section, the pdf of the ratio of type II of the components of the bivariate distribution proposed in (2.16) is derived. Subsequently the noncentral case is proposed by using the compounding method (see (1.1)). Let $W_2 = \frac{X_1}{X_2}$.

Theorem 2.2 *If X_1 and X_2 are jointly distributed according to equation (2.16), the pdf of the conditional distribution of $W_2|\left(K_1 = k_1, K_2 = k_2\right)$ is given by*

$$\begin{aligned} f_{W_2}(w_2|k_1,k_2) &= \sum_{i=0}^{\infty} \frac{\Gamma\left(\frac{n}{2}+i\right)}{\Gamma\left(\frac{n}{2}\right)i!}\gamma^{2i}\left(1-\gamma^2\right)^{\frac{n}{2}} \frac{w_2^{\frac{n}{2}+i+k_1-1}}{(1+w_2)^{n+2i+k_2+k_1}} \\ &\quad \times \left(B\left(\frac{n}{2}+i+k_1, \frac{n}{2}+i+k_2\right)\right)^{-1}, \qquad w_2 > 0 \qquad (2.30) \end{aligned}$$

where $n > 0$, and $-1 < \gamma < 1$. $B(\cdot,\cdot)$ is the beta function (see Result C.4) and the conditional values k_1 and k_2 have domain such that $k_v \geq 0$, $v = 1,2$.

Proof. Let C be the constant as given in (2.17). The Jacobian of the transformation is given by

$$J(x_1, x_2 \to w, x_2) = \begin{vmatrix} \frac{dx_1}{dw} & \frac{dx_1}{dx_2} \\ \frac{dx_2}{dw} & \frac{dx_2}{dx_2} \end{vmatrix} = \begin{vmatrix} x_2 & w \\ 0 & 1 \end{vmatrix} = x_2$$

and thus the joint pdf of W_2 and X_2 is given by

$$\begin{aligned} f_{W_2,X_2}(w_2,x_2|k_1,k_2) &= x_2 f_{X_1,X_2}\left(w_2x_2, x_2|k_1,k_2\right) \\ &= x_2\sum_{i=0}^{\infty} Ce^{-\frac{w_2x_2}{2(1-\gamma^2)}}\left(w_2x_2\right)^{\frac{n}{2}+i+k_1-1} e^{-\frac{x_2}{2(1-\gamma^2)}} x_2^{\frac{n}{2}+i+k_2-1} \\ &= \sum_{i=0}^{\infty} Ce^{-\frac{w_2x_2}{2(1-\gamma^2)}-\frac{x_2}{2(1-\gamma^2)}} w_2^{\frac{n}{2}+i+k_1-1} x_2^{n+2i+k_2+k_1-1} \\ &= \sum_{i=0}^{\infty} Ce^{-x_2\left(\frac{1+w_2}{2(1-\gamma^2)}\right)} w_2^{\frac{n}{2}+i+k_1-1} x_2^{n+2i+k_2+k_1-1}. \end{aligned}$$

Now, since $x_2 > 0$; the pdf of W_2 is given by

$$\begin{aligned} f_{W_2}(w_2|k_1,k_2) &= \int_0^{\infty} f_{W_2,X_2}(w_2,x_2|k_1,k_2)dx_2 \\ &= \sum_{i=0}^{\infty} Cw_2^{\frac{n}{2}+i+k_1-1} \int_0^{\infty} e^{-x_2\left(\frac{1+w_2}{2(1-\gamma^2)}\right)} x_2^{n+2i+k_2+k_1-1} dx_2 \end{aligned}$$

where this last integral is evaluated using Result C.12: setting $\alpha = \frac{w_2+1}{2(1-\gamma^2)}$ and $\beta = n + k_2 + k_1 + 2i$, one obtains

$$f_{W_2}(w_2|k_1,k_2) = \sum_{i=0}^{\infty} C w_2^{\frac{n}{2}+i+k_1-1} \left(\frac{1+w_2}{2(1-\gamma^2)}\right)^{-(n+2i+k_2+k_1)} \Gamma(n+2i+k_2+k_1)$$

which leaves the final result. ■

Description 2.24 *Upon taking the pdf in equation (2.30) one can now obtain the (unconditional) noncentral distribution of $W_2 = \frac{X_1}{X_2}$ by substituting the Poisson probabilities and the corresponding summation operators (stemming from (2.11)):*

$$\begin{aligned} f_{W_2}(w_2) &= f_{W_2}(w_2|k_1,k_2) g_{K_1}(k_1) g_{K_2}(k_2) \\ &= \sum_{i=0}^{\infty}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} \frac{\Gamma\left(\frac{n}{2}+i\right)}{\Gamma\left(\frac{n}{2}\right) i!} \gamma^{2i} \left(1-\gamma^2\right)^{\frac{n}{2}} \frac{w_2^{\frac{n}{2}+i+k_1-1}}{(1+w_2)^{n+2i+k_2+k_1}} \\ &\quad \times \left(B\left(\frac{n}{2}+i+k_1, \frac{n}{2}+i+k_2\right)\right)^{-1} \\ &\quad \times \frac{e^{-\frac{\theta_1}{2}}\left(\frac{\theta_1}{2}\right)^{k_1}}{k_1!} \frac{e^{-\frac{\theta_2}{2}}\left(\frac{\theta_2}{2}\right)^{k_2}}{k_2!}, \qquad w_2 > 0 \end{aligned} \tag{2.31}$$

where $n > 0$, $-1 < \gamma < 1$, $\theta_1, \theta_2 > 0$, and $g_{K_v}(k_v) = \frac{e^{-\frac{\theta_v}{2}}\left(\frac{\theta_v}{2}\right)^{k_v}}{k_v!}$, $v = 1, 2$. ▲

2.4.2.3 The probability density function of the ratio of type I

Here the pdf of the ratio of type I of the components of the bivariate distribution proposed in (2.16) is derived. Subsequently the noncentral case is proposed by using the compounding method (see (1.1)). Let $W_3 = \frac{X_1}{X_1+X_2}$.

Theorem 2.3 *If X_1 and X_2 are jointly distributed according to equation (2.16), the pdf of the conditional distribution of $W_3|\left(K_1 = k_1, K_2 = k_2\right)$ is given by*

$$\begin{aligned} f_{W_3}(w_3|k_1,k_2) &= \sum_{i=0}^{\infty} \frac{\Gamma\left(\frac{n}{2}+i\right)}{\Gamma\left(\frac{n}{2}\right) i!} \gamma^{2i} \left(1-\gamma^2\right)^{\frac{n}{2}} w_3^{\frac{n}{2}+i+k_1-1} (1-w_3)^{\frac{n}{2}+i+k_2-1} \\ &\quad \times \left(B\left(\frac{n}{2}+i+k_1, \frac{n}{2}+i+k_2\right)\right)^{-1}, \qquad 0 < w_3 < 1 \end{aligned} \tag{2.32}$$

where $n > 0$, and $-1 < \gamma < 1$. $B(\cdot,\cdot)$ is the beta function as defined in Result C.4 and the conditional values k_1 and k_2 have domain such that $k_v \geq 0$, $v = 1, 2$.

Proof. By using Result C.1, letting $C^* = \frac{\Gamma\left(\frac{n}{2}+i\right)}{\Gamma\left(\frac{n}{2}\right)i!}\gamma^{2i}\left(1-\gamma^2\right)^{\frac{n}{2}}\left(B\left(\frac{n}{2}+i+k_1, \frac{n}{2}+i+k_2\right)\right)^{-1}$ and via substitution, the following is obtained from (2.30):

$$\begin{aligned} f_{W_3}(w_3|k_1,k_2) &= \sum_{i=0}^{\infty} C^* \left(\frac{w_3}{1-w_3}\right)^{\frac{n}{2}+i+k_1-1}\left(\frac{1}{1+\frac{w_3}{1-w_3}}\right)^{n+2i+k_1+k_2} \frac{1}{(1-w_3)^2} \\ &= \sum_{i=0}^{\infty} C^* w_3^{\frac{n}{2}+i+k_1-1}\left(1-w_3\right)^{\frac{n}{2}+i+k_2-1} \end{aligned}$$

which leaves the final result. ■

Description 2.25 *Upon taking the pdf in equation (2.32) one can now obtain the (unconditional) noncentral distribution of* $W_3 = \frac{X_1}{X_1+X_2}$ *by substituting the Poisson probabilities and the corresponding summation operators (stemming from (2.16)):*

$$\begin{aligned} f_{W_3}(w_3) &= f_{W_3}(w_3|k_1,k_2)g_{K_1}(k_1)g_{K_2}(k_2) \\ &= \sum_{i=0}^{\infty}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} C^* w_3^{\frac{n}{2}+i+k_1-1}\left(1-w_3\right)^{\frac{n}{2}+i+k_2-1} \\ &\quad \times \frac{e^{-\frac{\theta_1}{2}}\left(\frac{\theta_1}{2}\right)^{k_1}}{k_1!}\frac{e^{-\frac{\theta_2}{2}}\left(\frac{\theta_2}{2}\right)^{k_2}}{k_2!}, \qquad 0 < w_3 < 1 \end{aligned} \tag{2.33}$$

where $n > 0$, $-1 < \gamma < 1$, $\theta_1, \theta_2 > 0$, $g_{K_v}(k_v) = \frac{e^{-\frac{\theta_v}{2}}\left(\frac{\theta_v}{2}\right)^{k_v}}{k_v!}$, $v = 1, 2$, *and* C^* *the value as defined above.* ▲

2.4.3 Bivariate noncentral compound extended chi-square distribution: unequal degrees of freedom

2.4.3.1 The probability density function of the product

In this section, the pdf of the product of the components of the bivariate distribution proposed in (2.19) is derived. Subsequently the noncentral case is proposed by using the compounding method (see (1.1)). Let $W_1 = X_1X_2$.

Theorem 2.4 *If* X_1 *and* X_2 *are jointly distributed according to equation (2.19), the pdf of the conditional distribution of* $W_1|\left(K_1 = k_1, K_2 = k_2\right)$ *is given by*

$$f_{W_1}(w_1|k_1,k_2) = \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty} 2Cw_1^{\frac{n_1+n_2}{4}+\frac{i_1+i_2+k_1+k_2}{2}-1} K_\tau\left(\frac{\sqrt{w_1}}{(1-\gamma^2)}\right), \qquad 0 < w_3 < 1 \tag{2.34}$$

where $n_1, n_2 > 0$, $-1 < \gamma < 1$, $\tau = \frac{n_2}{2} - \frac{n_1}{2} + i_2 - i_1 + k_2 - k_1$, $K_\tau(\cdot)$ *the modified Bessel function of the second kind (see Result C.10), and* C *the value defined in (2.20). The* *conditional values k_1 and k_2 have domain such that $k_v \geq 0$, $v = 1, 2$.*

Proof. The Jacobian of the transformation is again given by $\frac{1}{x_2}$, and thus the joint pdf of W_1 and X_2 is given by

$$\begin{aligned} f_{W_1,X_2}(w_1, x_2|k_1, k_2) &= \frac{1}{x_2} f_{X_1,X_2}\left(\frac{w_1}{x_2}, x_2|k_1, k_2\right) \\ &= \frac{1}{x_2} \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty} C e^{-\frac{w_1}{x_2}\left(\frac{1}{2(1-\gamma^2)}\right)} \left(\frac{w_1}{x_2}\right)^{\frac{n_1}{2}+i_1+k_1-1} e^{-\frac{x_2}{2(1-\gamma^2)}} x_2^{\frac{n_2}{2}+i_2+k_2-1} \\ &= \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty} C e^{-\frac{w_1}{x_2}\left(\frac{1}{2(1-\gamma^2)}\right)-\frac{x_2}{2(1-\gamma^2)}} w_1^{\frac{n_1}{2}+i_1+k_1-1} x_2^{\frac{n_2}{2}-\frac{n_1}{2}+i_2-i_1+k_2-k_1-1}. \end{aligned}$$

Now, since $x_2 > 0$; the pdf of W_1 is given by

$$\begin{aligned} f_{W_1}(w_1|k_1, k_2) &= \int_0^{\infty} f_{W_1,X_2}(w_1, x_2|k_1, k_2)dx_2 \\ &= \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty} C w_1^{\frac{n_1}{2}+i_1+k_1-1} \int_0^{\infty} e^{-\frac{w_1}{x_2}\left(\frac{1}{2(1-\gamma^2)}\right)-\frac{x_2}{2(1-\gamma^2)}} x_2^{\frac{n_2}{2}-\frac{n_1}{2}+i_2-i_1+k_2-k_1-1} dx_2 \end{aligned}$$

where this last integral is evaluated using Result C.11: setting $\tau = \frac{n_2}{2} - \frac{n_1}{2} + i_2 - i_1 + k_2 - k_1$, $\gamma = \frac{1}{2(1-\gamma^2)}$ and $\beta = \frac{w_1}{2(1-\gamma^2)}$, one obtains

$$\begin{aligned} f_{W_1}(w_1|k_1, k_2) &= \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty} C w_1^{\frac{n_1}{2}+i_1+k_1-1} 2 w_1^{\frac{\tau}{2}} K_\tau\left(2\sqrt{\frac{w_1}{4\left(1-\gamma^2\right)^2}}\right) \\ &= \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty} 2 C w_1^{\frac{n_1+n_2}{4}+\frac{i_1+i_2+k_1+k_2}{2}-1} K_\tau\left(\frac{\sqrt{w_1}}{(1-\gamma^2)}\right) \end{aligned}$$

which leaves the final result. ∎

Description 2.26 *Upon taking the pdf in equation (2.34) one can now obtain the (unconditional) noncentral distribution of $W_1 = X_1 X_2$ by substituting the Poisson probabilities and the corresponding summation operators (stemming from (2.11)):*

$$\begin{aligned} f_{W_1}(w_1) &= f_{W_1}(w_1|k_1, k_2) g_{K_1}(k_1) g_{K_2}(k_2) \\ &= \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} 2 C w_1^{\frac{n_1+n_2}{4}+\frac{i_1+i_2+k_1+k_2}{2}-1} K_\tau\left(\frac{\sqrt{w_1}}{(1-\gamma^2)}\right) \\ &\quad \times \frac{e^{-\frac{\theta_1}{2}}\left(\frac{\theta_1}{2}\right)^{k_1}}{k_1!} \frac{e^{-\frac{\theta_2}{2}}\left(\frac{\theta_2}{2}\right)^{k_2}}{k_2!}, \qquad w_1 > 0 \end{aligned} \tag{2.35}$$

where $n_1, n_2 > 0$, $-1 < \gamma < 1$, $\tau = \frac{n_2}{2} - \frac{n_1}{2} + i_2 - i_1 + k_2 - k_1$, $\theta_1, \theta_2 > 0$, $g_{K_v}(k_v) = \frac{e^{-\frac{\theta_v}{2}}\left(\frac{\theta_v}{2}\right)^{k_v}}{k_v!}$, $v = 1, 2$, *and* C *the value defined in (2.20).* ▲

2.4.3.2 The probability density function of the ratio of type II

Here the pdf of the ratio of type II of the components of the bivariate distribution proposed in (2.19) is derived. Subsequently the noncentral case is proposed by using the compounding method (see (1.1)). Let $W_2 = \frac{X_1}{X_2}$.

Theorem 2.5 *If* X_1 *and* X_2 *are jointly distributed according to equation (2.19), the pdf of the conditional distribution of* $W_2|\left(K_1 = k_1, K_2 = k_2\right)$ *is given by*

$$\begin{aligned} f_{W_2}(w_2|k_1,k_2) &= \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\frac{\Gamma\left(\frac{n_1}{2}+i_1\right)}{\Gamma\left(\frac{n_1}{2}\right)i_1!}\frac{\Gamma\left(\frac{n_2}{2}+i_2\right)}{\Gamma\left(\frac{n_2}{2}\right)i_2!}\gamma^{2(i_1+i_2)}\left(1-\gamma^2\right)^{\frac{n_1+n_2}{2}}\frac{w_2^{\frac{n_1}{2}+i_1+k_1-1}}{(1+w_2)^{\frac{n_1+n_2}{2}+i_1+i_2+k_1+k_2}} \\ &\quad \times\left(B\left(\frac{n_1}{2}+i_1+k_1, \frac{n_2}{2}+i_2+k_2\right)\right)^{-1}, \qquad w_2 > 0 \end{aligned} \tag{2.36}$$

where $n_1, n_2 > 0$, *and* $-1 < \gamma < 1$. $B(\cdot,\cdot)$ *is the beta function as defined in Result C.4 and the conditional values* k_1 *and* k_2 *have domain such that* $k_v \geq 0$, $v = 1, 2$.

Proof. Let C be the value as defined in (2.20). The Jacobian of the transformation is again given by x_2, and thus the joint pdf of W_2 and X_2 is given by

$$\begin{aligned} f_{W_2,X_2}(w_2,x_2|k_1,k_2) &= x_2 f_{X_1,X_2}\left(w_2x_2,x_2|k_1,k_2\right) \\ &= x_2\sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}C\left(w_2x_2\right)^{\frac{n_1}{2}+i_1+k_1-1}x_2^{\frac{n_2}{2}+i_2+k_2-1}e^{-\frac{w_2x_2+x_2}{2(1-\gamma^2)}} \\ &= \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}Ce^{-x_2\left(\frac{1+w_2}{2(1-\gamma^2)}\right)}w_2^{\frac{n_1}{2}+i_1+k_1-1}x_2^{\frac{n_1+n_2}{2}+i_1+i_2+k_1+k_2-1}. \end{aligned}$$

Now, since $x_2 > 0$; the pdf of W_2 is given by

$$\begin{aligned} f_{W_2}(w_2|k_1,k_2) &= \int_0^{\infty} f_{W_2,X_2}(w_2,x_2|k_1,k_2)dx_2 \\ &= \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}Cw_2^{\frac{n_1}{2}+i_1+k_1-1}\int_0^{\infty}e^{-x_2\left(\frac{1+w_2}{2(1-\gamma^2)}\right)}x_2^{\frac{n_1+n_2}{2}+i_1+i_2+k_1+k_2-1}dx_2 \end{aligned}$$

where this last integral is evaluated using Result C.12: setting $\alpha = \frac{w_2+1}{2(1-\gamma^2)}$ and $\beta = \frac{n_1+n_2}{2} + i_1 + i_2 + k_1 + k_2$, one obtains

$$\begin{aligned} f_{W_2}(w_2|k_1,k_2) &= \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty} C w_2^{\frac{n_1}{2}+i_1+k_1-1}\left(\frac{1+w_2}{2\left(1-\gamma^2\right)}\right)^{-\left(\frac{n_1+n_2}{2}+i_1+i_2+k_1+k_2\right)} \\ &\quad \times\Gamma\left(\frac{n_1+n_2}{2}+i_1+i_2+k_1+k_2\right) \end{aligned}$$

which leaves the final result. ■

Description 2.27 *Upon taking the pdf in equation (2.36) one can now obtain the (unconditional) noncentral distribution of* $W_2 = \frac{X_1}{X_2}$ *by substituting the Poisson probabilities and the corresponding summation operators (stemming from (2.11)):*

$$\begin{aligned} f_{W_2}(w_2) &= f_{W_2}(w_2|k_1,k_2)g_{K_1}(k_1)g_{K_2}(k_2) \\ &= \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} \frac{\Gamma\left(\frac{n_1}{2}+i_1\right)}{\Gamma\left(\frac{n_1}{2}\right)i_1!}\frac{\Gamma\left(\frac{n_2}{2}+i_2\right)}{\Gamma\left(\frac{n_2}{2}\right)i_2!}\gamma^{2(i_1+i_2)}\left(1-\gamma^2\right)^{\frac{n_1+n_2}{2}} \\ &\quad \times\frac{w_2^{\frac{n_1}{2}+i_1+k_1-1}}{\left(1+w_2\right)^{\frac{n_1+n_2}{2}+i_1+i_2+k_1+k_2}}\left(B\left(\frac{n_1}{2}+i_1+k_1,\frac{n_2}{2}+i_2+k_2\right)\right)^{-1} \\ &\quad \times\frac{e^{-\frac{\theta_1}{2}}\left(\frac{\theta_1}{2}\right)^{k_1}}{k_1!}\frac{e^{-\frac{\theta_2}{2}}\left(\frac{\theta_2}{2}\right)^{k_2}}{k_2!}, \qquad w_2>0 \end{aligned} \tag{2.37}$$

where $n_1,n_2>0$, $-1<\gamma<1$, $\theta_1,\theta_2>0$ *and* $g_{K_v}(k_v) = \frac{e^{-\frac{\theta_v}{2}}\left(\frac{\theta_v}{2}\right)^{k_v}}{k_v!}$, $v=1,2$. ▲

2.4.3.3 The probability density function of the ratio of type I

Here the pdf of the ratio of type I of the components of the bivariate distribution proposed in (2.19) is derived. Subsequently the noncentral case is proposed by using the compounding method (see (1.1)). Let $W_3 = \frac{X_1}{X_1+X_2}$.

Theorem 2.6 *If* X_1 *and* X_2 *are jointly distributed according to equation (2.19), the pdf of the conditional distribution of* $W_3|\left(K_1=k_1,K_2=k_2\right)$ *is given by*

$$\begin{aligned} f_{W_3}(w_3|k_1,k_2) &= \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\frac{\Gamma\left(\frac{n_1}{2}+i_1\right)}{\Gamma\left(\frac{n_1}{2}\right)i_1!}\frac{\Gamma\left(\frac{n_2}{2}+i_2\right)}{\Gamma\left(\frac{n_2}{2}\right)i_2!}\gamma^{2(i_1+i_2)}\left(1-\gamma^2\right)^{\frac{n_1+n_2}{2}} \\ &\quad \times\frac{w_3^{\frac{n_1}{2}+i_1+k_1-1}\left(1-w_3\right)^{\frac{n_2}{2}+i_2+k_2-1}}{B\left(\frac{n_1}{2}+i_1+k_1,\frac{n_2}{2}+i_2+k_2\right)}, \qquad 0<w_3<1 \end{aligned} \tag{2.38}$$

where $n_1,n_2>0$, *and* $-1<\gamma<1$. $B(\cdot,\cdot)$ *is the beta function as defined in Result C.4 and the conditional values* k_1 *and* k_2 *have domain such that* $k_v \geq 0$, $v=1,2$.

Proof. Let $C^* = \frac{\Gamma\left(\frac{n_1}{2}+i_1\right)}{\Gamma\left(\frac{n_1}{2}\right)i_1!}\frac{\Gamma\left(\frac{n_2}{2}+i_2\right)}{\Gamma\left(\frac{n_2}{2}\right)i_2!}\gamma^{2(i_1+i_2)}\left(1-\gamma^2\right)^{\frac{n_1+n_2}{2}}\left(B\left(\frac{n_1}{2}+i_1+k_1, \frac{n_2}{2}+i_2+k_2\right)\right)^{-1}$. By using Result C.1 and via substitution, the following is obtained from (2.36):

$$\begin{aligned} f_{W_3}(w_3|k_1,k_2) &= \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty} C^*\left(\frac{w_3}{1-w_3}\right)^{\frac{n_1}{2}+i_1+k_1-1}\left(1+\frac{w_3}{1-w_3}\right)^{-\left(\frac{n_1+n_2}{2}+i_1+i_2+k_1+k_2\right)}\frac{1}{(1-w_3)^2} \\ &= \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty} C^* w_3^{\frac{n_1}{2}+i_1+k_1-1}\left(1-w_3\right)^{\frac{n_2}{2}+i_2+k_2-1} \end{aligned}$$

which leaves the final result. ■

Description 2.28 *Upon taking the pdf in equation (2.38) one can now obtain the (unconditional) noncentral distribution of* $W_3 = \frac{X_1}{X_1+X_2}$ *by substituting the Poisson probabilities and the corresponding summation operators (stemming from (2.11)):*

$$\begin{aligned} f_{W_3}(w_3) &= f_{W_3}(w_3|k_1,k_2)g_{K_1}(k_1)g_{K_2}(k_2) \\ &= \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\frac{\Gamma\left(\frac{n_1}{2}+i_1\right)}{\Gamma\left(\frac{n_1}{2}\right)i_1!}\frac{\Gamma\left(\frac{n_2}{2}+i_2\right)}{\Gamma\left(\frac{n_2}{2}\right)i_2!}\gamma^{2(i_1+i_2)}\left(1-\gamma^2\right)^{\frac{n_1+n_2}{2}} \\ &\quad \times\frac{w_3^{\frac{n_1}{2}+i_1+k_1-1}\left(1-w_3\right)^{\frac{n_2}{2}+i_2+k_2-1}}{B\left(\frac{n_1}{2}+i_1+k_1, \frac{n_2}{2}+i_2+k_2\right)} \\ &\quad \times\frac{e^{-\frac{\theta_1}{2}}\left(\frac{\theta_1}{2}\right)^{k_1}}{k_1!}\frac{e^{-\frac{\theta_2}{2}}\left(\frac{\theta_2}{2}\right)^{k_2}}{k_2!}, \qquad 0 < w_3 < 1 \end{aligned} \tag{2.39}$$

where $n_1, n_2 > 0$, $-1 < \gamma < 1$, $\theta_1, \theta_2 > 0$, *and* $g_{K_v}(k_v) = \frac{e^{-\frac{\theta_v}{2}}\left(\frac{\theta_v}{2}\right)^{k_v}}{k_v!}$, $v = 1, 2$. ▲

2.5 Shape analysis

In this section, a visual representation is provided on the forms of the pdf of the alternative bivariate noncentral generalized chi-square distribution (see (2.15)), the bivariate noncentral compound extended chi-square distribution for unequal degrees of freedom (see (2.21)), and their derived univariate composites, for arbitrary parameter values.

2.5.1 An alternative bivariate noncentral generalized chi-square distribution

2.5.1.1 The joint probability density function (2.15)

The pdf (2.15) is illustrated here for arbitrary parameter values: $n_1 = 10$, $n_2 = 12$, and $r = 2$. In Figure 2.3, the value of ξ varies, with $\theta_1 = \theta_2 = 3$, and in Figure 2.4, $\theta_1 = 3$ remains fixed together with $\xi = 0.5$, whilst θ_2 varies.

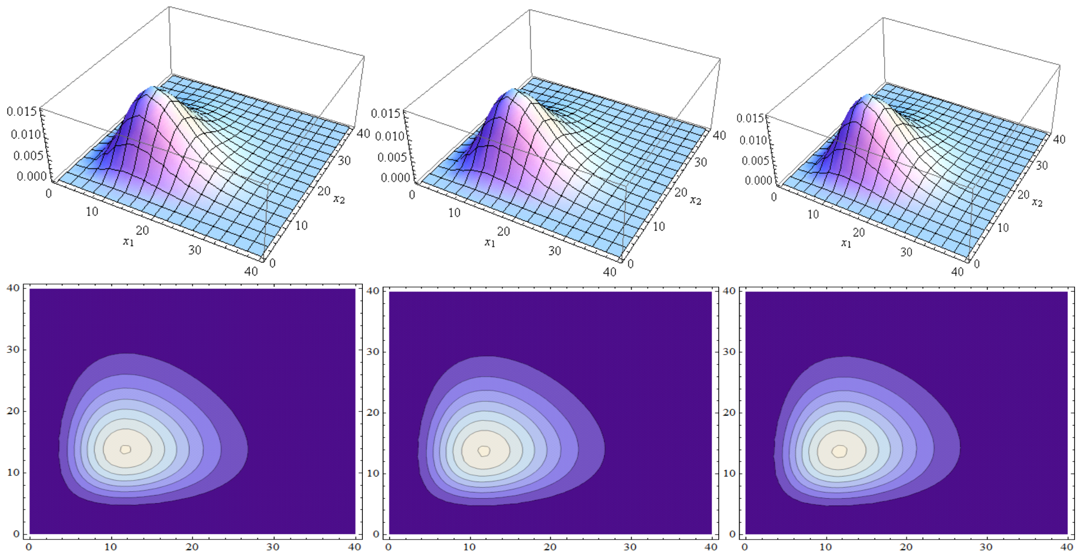

Figure 2.3 f.l.t.r., (2.15) for $\xi = 0.2$, 0.5, and 0.7.

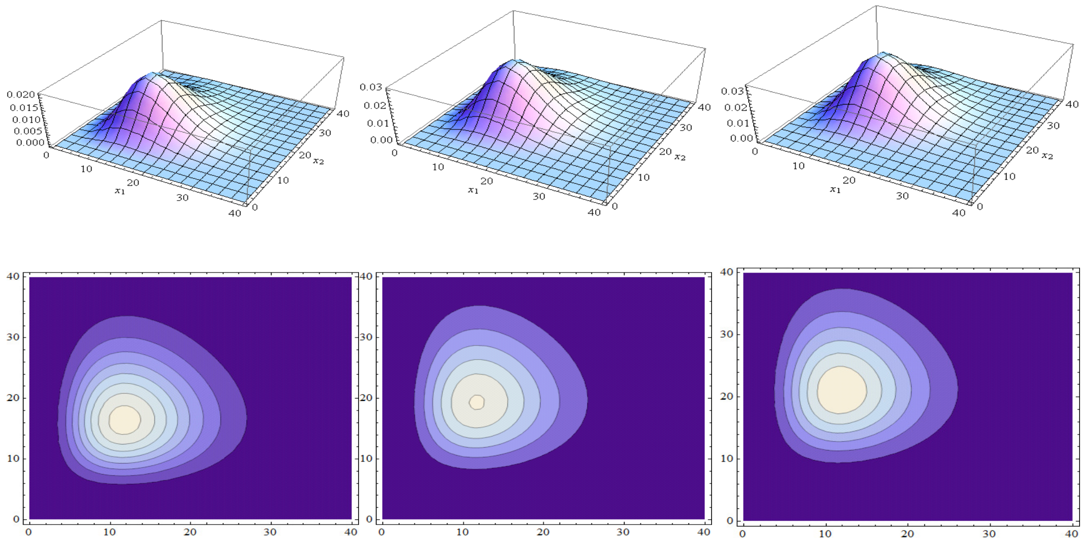

Figure 2.4 f.l.t.r., (2.15) for $\theta_1 = 3$, and $\theta_2 = 5$, 8, and 11.

It is seen that the parameter ξ has very little influence on the shape of the pdf (2.15). It is observed to marginally increase the contours near the top of the distribution, but not other obvious effects are observed. For the change in θ_2, it is seen that as θ_2 increases, the pdf moves away from the axis of variable X_1 - which is to be expected, as θ_2 represents the noncentrality of variable X_2. It is noted here that the same effect would be observed for changes in θ_1, but moving away from the axis of variable X_2, because of the symmetric nature of the noncentrality components (Poisson probabilities).

2.5.1.2 The composites

For the purposes of this study, the effect of the correlation component ξ will be considered, as well as the effect of the noncentrality parameters θ_1 and θ_2. In this section specifically, the univariate distributions of the composites given in (2.23), (2.25), and (2.27), are considered respectively. Similar as before, $n_1 = 10$, $n_2 = 12$, and $r = 2$. In Figure 2.5, 2.6, and 2.7, the value of ξ varies, with $\theta_1 = \theta_2 = 3$ (left), and $\theta_1 = 3$ remains fixed together with $\xi = 0.5$, whilst θ_2 varies (right).

Probability density function of the product

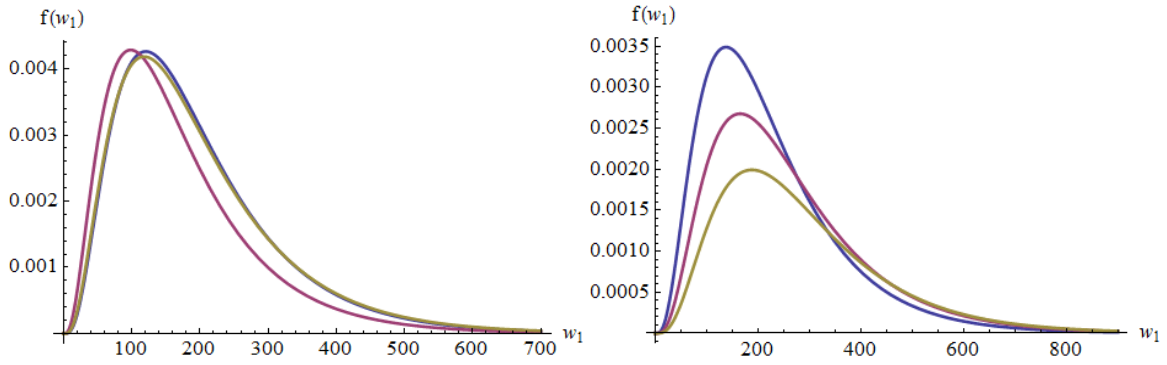

Figure 2.5 (2.23) for $\xi = 0.2$ (blue), 0.5 (purple), and 0.7 (green) (left), and for $\theta_1 = 3$, and $\theta_2 = 5$ (blue), 8 (purple), and 11 (green) (right).

Probability density function of the ratio of type II

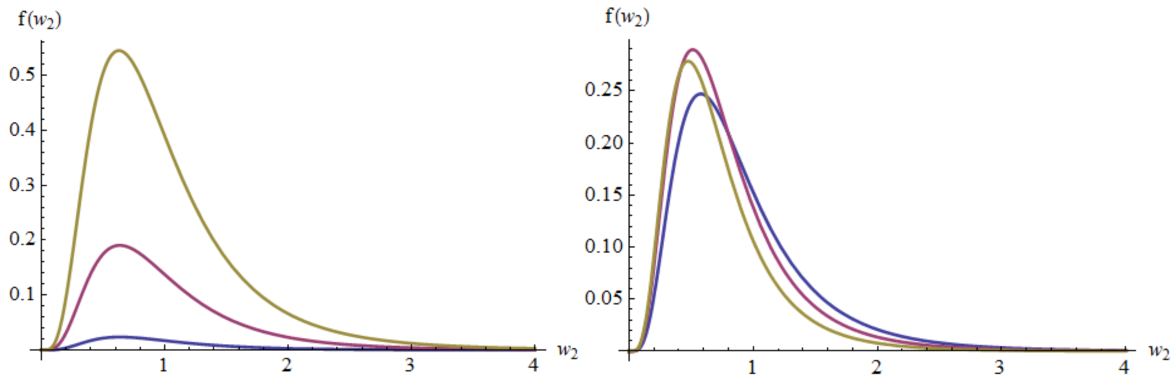

Figure 2.6 (2.25) for $\xi = 0.2$ (blue), 0.5 (purple), and 0.7 (green) (left), and for $\theta_1 = 3$, and $\theta_2 = 5$ (blue), 8 (purple), and 11 (green) (right).

Probability density function of the ratio of type I

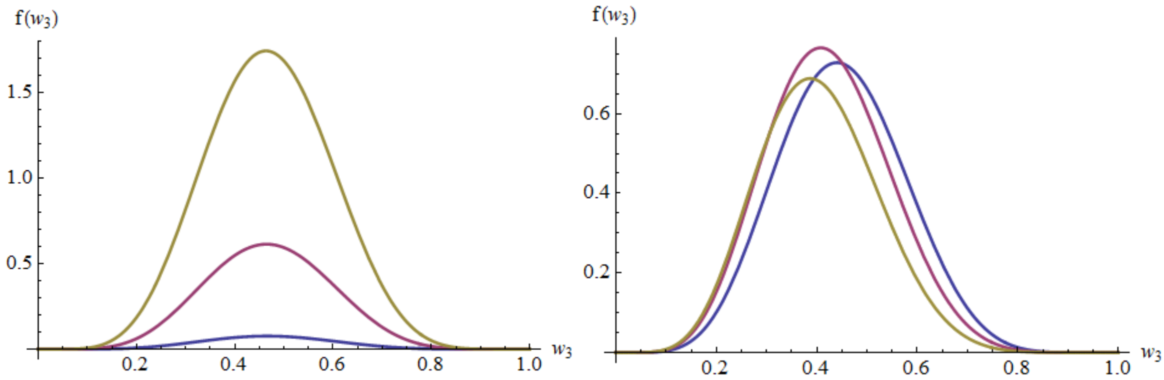

Figure 2.7 (2.27) for $\xi = 0.2$ (blue), 0.5 (purple), and 0.7 (green) (left), and for $\theta_1 = 3$, and $\theta_2 = 5$ (blue), 8 (purple), and 11 (green) (right).

It is interesting to observe the trend in the changes of ξ when considering the distributions of the ratios of type II and type I respectively: as ξ increases, the pdf of the distribution becomes more leptokurtic. As for the change in θ_2, it seems that as θ_2 increases, the variability of the distributions of the ratios of type II and type I seem to decrease marginally.

2.5.2 Bivariate noncentral compound extended chi-square distribution: unequal degrees of freedom

2.5.2.1 The joint probability density function (2.21)

In this section, the bivariate noncentral compound extended chi-square distribution with unequal degrees of freedom (2.21) will be considered. The distribution with equal degrees of freedom (2.18) will not be considered, due to the similarities between these two. The pdf (2.21) is illustrated here for arbitrary parameter values: $n_1 = 10$, $n_2 = 12$, and $r = 2$. In Figure 2.8, the value of γ varies, with $\theta_1 = \theta_2 = 3$, and in Figure 2.9, $\theta_1 = 3$ remains fixed together with $\gamma = 0.5$, whilst θ_2 varies.

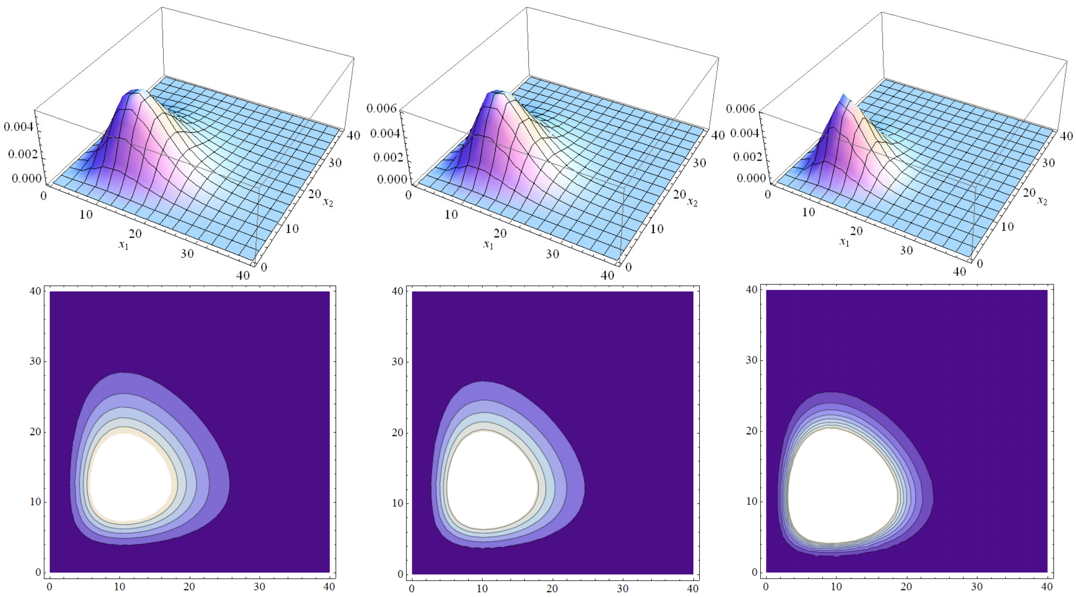

Figure 2.8 f.l.t.r., (2.21) for $\gamma = 0.2$, 0.5, and 0.7.

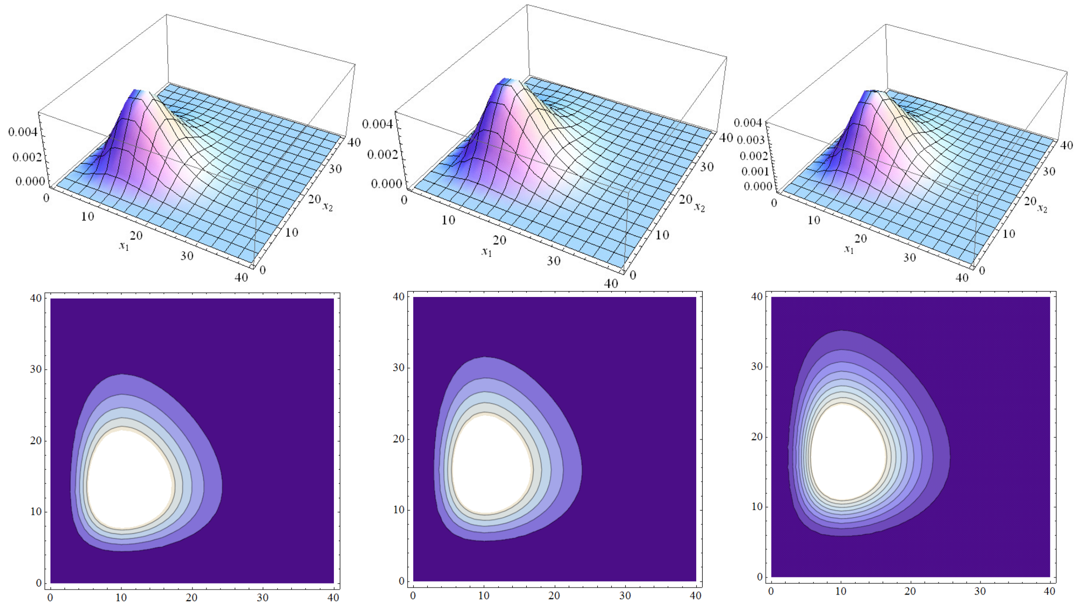

Figure 2.9 f.l.t.r., (2.21) for $\theta_1 = 3$, and $\theta_2 = 5$, 8, and 11.

Here it is seen that the parameter γ has a significant influence on the shape of the pdf (2.21). When γ increases, the density's contour plots become stacked more closely together. For the change in θ_2, a similar trend as in Figure 2.4 is observed: as θ_2 increases, the pdf moves away from the axis of variable X_1 - which is again what is to be expected.

2.5.2.2 The composites

For the purposes of this study, the effect of the extension parameter γ will be considered, as well as the effect of the noncentrality parameters θ_1 and θ_2. In this section specifically, the univariate distributions of the composites given in (2.35), (2.37), and (2.39), are considered respectively. Similar as before, $n_1 = 10$, $n_2 = 12$, and $r = 2$. In Figure 2.10, 2.11, and 2.12, the value of γ varies, with $\theta_1 = \theta_2 = 3$ (left), and $\theta_1 = 3$ remains fixed together with $\gamma = 0.5$, whilst θ_2 varies (right).

Probability density function of the product

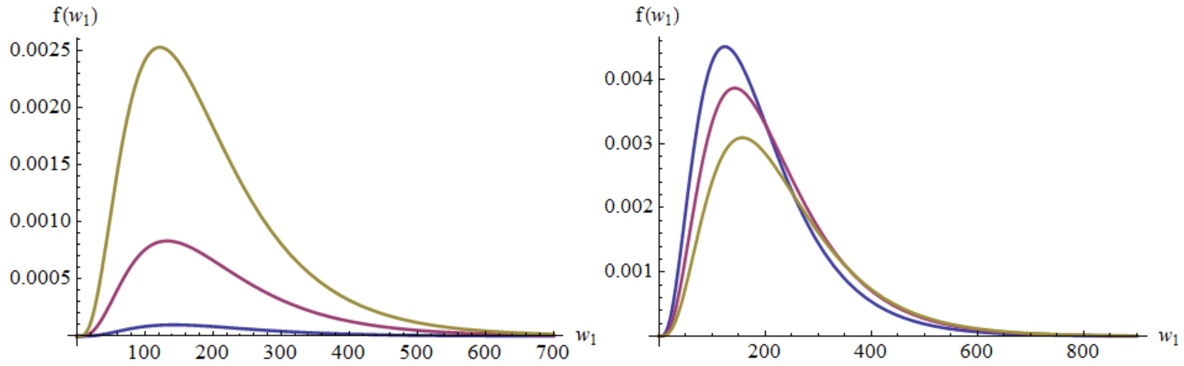

Figure 2.10 (2.35) for $\gamma = 0.2$ (blue), 0.5 (purple), and 0.7 (green) (left), and for $\theta_1 = 3$, and $\theta_2 = 5$ (blue), 8 (purple), and 11 (green) (right).

Probability density function of the ratio of type II

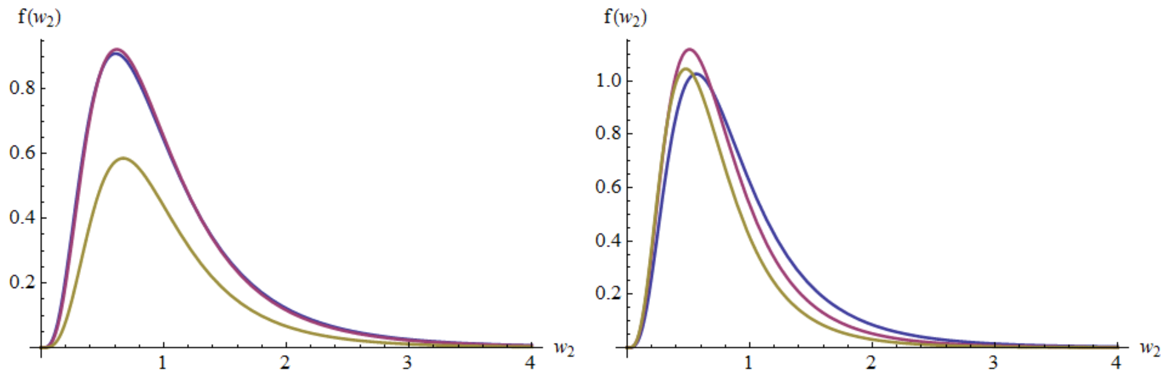

Figure 2.11 (2.37) for $\gamma = 0.2$ (blue), 0.5 (purple), and 0.7 (green) (left), and for $\theta_1 = 3$, and $\theta_2 = 5$ (blue), 8 (purple), and 11 (green) (right).

Probability density function of the ratio of type I

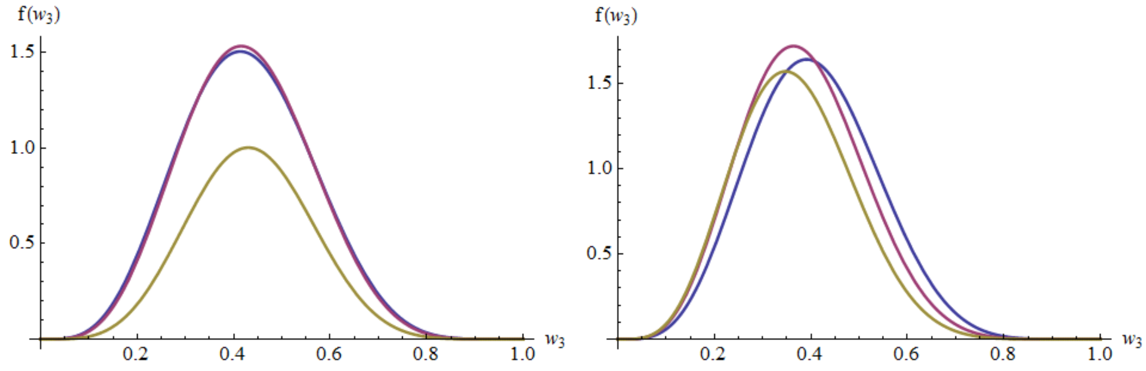

Figure 2.12 (2.39) for $\gamma = 0.2$ (blue), 0.5 (purple), and 0.7 (green) (left), and for $\theta_1 = 3$, and $\theta_2 = 5$ (blue), 8 (purple), and 11 (green) (right).

Similar trends as the generalized case (see section 2.5.1.2) is observed in the changes of γ when considering the distributions of the ratios of type II and type I respectively; however, the converse holds: as γ increases, the pdf of the distribution becomes more platykurtic. As for the change in θ_2, it seems that as θ_2 increases, the variability of the distributions of the ratios of type II and type I again seem to decrease marginally.

2.5.3 Some percentage points for the distribution of the ratio of type II

Certain percentage points w_α of the distribution of $W_2 = \frac{X_1}{X_2}$ are obtained numerically by solving the equation $\int_0^{w_\alpha} f_{W_2}(w_2)dw_2 = \alpha$. By considering both distributions (2.25) and (2.37), some lower percentage points are calculated for arbitrary parameters. Similar tabulations can be obtained for other values of the parameters. The calculated values are given below in Tables 2.1 and 2.2. This distribution is considered since it may offer an alternative approach to the well-known stress-strength model in the context of reliability, where the lifetime of a random component with strength X_2 is subjected to a random stress X_1. The measure $P(X_1 < X_2)$ is thus of interest, and translates to $P(\frac{X_1}{X_2} < 1) = P(W_2 < 1)$ - thereby revealing the relevance of this specific distribution.

ξ	θ_1	θ_2	$\alpha = 0.01$	0.025	0.05	0.1
	3	5	0.171788	0.221699	0.273741	0.346185
	3	8	0.148046	0.190774	0.235201	0.296845
0.5	3	11	0.132922	0.171042	0.210582	0.265314

Table 2.1 Percentage points for W_2 (2.25), for $n_1 = 10$, $n_2 = 12$, and $r = 2$

γ	θ_1	θ_2	$\alpha = 0.01$	0.025	0.05	0.1
	3	5	0.181303	0.232991	0.286627	0.360968
	3	8	0.166272	0.213473	0.262394	0.330156
0.5	3	11	0.163029	0.209744	0.258485	0.326712

Table 2.2 Percentage points for W_2 (2.37), for $n_1 = 10$, and $n_2 = 12$

2.6 Conclusion

This chapter saw a brief overview of the univariate noncentral chi-square distribution with focus on:

(a) the well-known chi-square distribution as a conditional distribution;

(b) the compound extended chi-square distribution as a conditional distribution;

and by using the Poisson probabilities as compounding factors to achieve noncentrality (see (1.1)). Subsequently, an alternative representation of a known bivariate noncentral generalized chi-square distribution, introduced by Van den Berg (2010), was proposed by employing Poisson probabilities via the compounding method. These distributions were shown to be equivalent by showing their respective mgfs to be the same. The proposed compounding method is very desirable since the method to obtain a noncentral distribution is quite straightforward and the resulting pdfs are mathematically friendly, in the sense that the noncentrality parameters remain isolated in known form - in this case, as Poisson probabilities.

Furthermore, a bivariate noncentral compound extended chi-square distribution was redescribed for both cases of equal- and unequal degrees of freedom, via the same route as that of the generalized distribution. Thereafter, univariate distributions of functions of X_1 and X_2 having these bivariate noncentral chi-square distributions were derived. The bivariate noncentral chi-square distributions were then visually studied for arbitrary parameter choices, together with their corresponding univariate distributions of the composites. Some percentage points were provided for a special composite case.

Chapter 3

Bivariate noncentral F distributions

3.1 Introduction

As an introduction, some standard theory from literature is presented to introduce the univariate case of the *noncentral* F distribution. Figure 3.1 provides a schematic overview of the noncentral F distribution, given below.

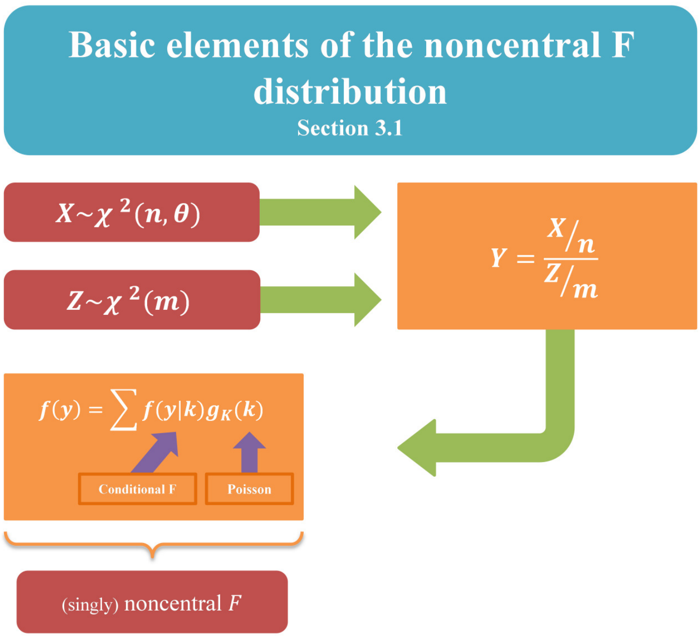

Figure 3.1 The basic elements of the noncentral F distribution in this study.

As in the *noncentral* chi-square environment, the text of Johnson et. al. (1995) (p. 480) provides a concise overview of the univariate *noncentral* F distribution, and it is rephrased here in the form of the following descriptions and properties.

Description 3.1. *Let* $X \sim \chi^2(n, \theta)$ *and* $Z \sim \chi^2(m)$. *If* X *and* Z *are independently distributed, then the distribution of the ratio*

$$Y = \frac{X/n}{Z/m}$$

is called the noncentral F distribution with n *and* m *degrees of freedom, and noncentrality parameter* θ. *The pdf is given by*

$$f_Y(y) \;=\; \sum_{k=0}^{\infty} \frac{\Gamma\left(\frac{n+m}{2}+k\right)}{\Gamma\left(\frac{n}{2}+k\right)\Gamma\left(\frac{m}{2}\right)} \left(\frac{n}{m}\right)^{\frac{n}{2}+k} y^{\frac{n}{2}+k-1} \left(1+\frac{n}{m}y\right)^{-\left(\frac{n+m}{2}+k\right)} \frac{e^{-\frac{\theta}{2}}\left(\frac{\theta}{2}\right)^k}{k!}, \quad y > 0 \tag{3.1}$$

where $n, m > 0$, *and* $\theta > 0$. *This distribution is denoted by* $Y \sim F(n, m; \theta)$. ▲

Property 3.1. The pdf (3.1) is a compound pdf (*1.1*), namely

$$f_Y(y) = \sum_{k=0}^{\infty} f_Y(y|k) g_K(k)$$

such that $f_Y(y|k) = \frac{\Gamma\left(\frac{n+m}{2}+k\right)}{\Gamma\left(\frac{n}{2}+k\right)\Gamma\left(\frac{m}{2}\right)} \left(\frac{n}{m}\right)^{\frac{n}{2}+k} y^{\frac{n}{2}+k-1} \left(1+\frac{n}{m}y\right)^{-\left(\frac{n+m}{2}+k\right)}$, $y > 0$ and $g_K(k) = \frac{e^{-\frac{\theta}{2}}\left(\frac{\theta}{2}\right)^k}{k!}$, $k = 0, 1, 2, 3...$, i.e. $Y|\left(K = k\right) \sim F(n+2k, m)$ (see Result C.20) and $K \sim Poi\left(\frac{\theta}{2}\right)$ (see Result C.22). ▲

The focus now shifts in section 3.2 to the bivariate F distribution, followed by its noncentral counterparts in section 3.3.

3.2 Bivariate F distributions

3.2.1 Introduction

As an introduction, the well-known standard bivariate F distribution (as given in Balakrishnan & Lai (2009), p. 367) is described below.

Description 3.2. *Let $X_1 \sim \chi^2(n_1)$, $X_2 \sim \chi^2(n_2)$, and $Z \sim \chi^2(m)$. If X_1, X_2, and Z are independently distributed, then the bivariate distribution of the ratios*

$$Y_1 = \frac{X_1/n_1}{Z/m}, \quad Y_2 = \frac{X_2/n_2}{Z/m}$$

is called the bivariate F distribution with n_1, n_2 and m degrees of freedom. The pdf is given by

$$\begin{aligned} f_{Y_1,Y_2}(y_1,y_2) &= \frac{\Gamma\left(\frac{n_1+n_2+m}{2}\right)}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)\Gamma\left(\frac{m}{2}\right)}\left(\frac{n_1}{m}\right)^{\frac{n_1}{2}}\left(\frac{n_2}{m}\right)^{\frac{n_2}{2}} \\ &\quad \times y_1^{\frac{n_1}{2}-1} y_2^{\frac{n_2}{2}-1}\left(1+\frac{n_1}{m}y_1+\frac{n_2}{m}y_2\right)^{-\left(\frac{n_1+n_2+m}{2}\right)}, \qquad y_1,y_2>0 \end{aligned}$$

where $n_1, n_2, m > 0$. ▲

In the following two sections (section 3.2.2 and section 3.2.3), two new bivariate F distributions are prepared with the building blocks being the bivariate generalized chi-square distribution (2.8) and the bivariate extended chi-square distribution (2.10), together with the (independent) chi-square distribution (see Result C.18).

3.2.2 Bivariate generalized F distribution

A bivariate generalized F distribution is derived in this section via a transformation approach, by using the bivariate generalized chi-square distribution in (2.8).

Theorem 3.1 *Let X_1 and X_2 be jointly distributed with pdf (2.8) with n_1 and n_2 degrees of freedom respectively, and let $Z \sim \chi^2(m)$ be an independent chi-square distributed random variable with m degrees of freedom. Let $Y_1 = \frac{X_1/n_1}{Z/m}$ and $Y_2 = \frac{X_2/n_2}{Z/m}$. The joint pdf of Y_1 and Y_2 is given by*

$$\begin{aligned} f_{Y_1,Y_2}(y_1,y_2) &= \sum_{j=0}^{\infty}\sum_{l_1=0}^{j}\sum_{l_2=0}^{j}\left(\frac{\xi^{2j}\left(\frac{r}{2}\right)_j(-1)^{l_1+l_2}\binom{j}{l_1}\binom{j}{l_2}}{j!}\right) \\ &\quad\times\left(\frac{\Gamma\left(\frac{n_1+n_2+m}{2}+l_1+l_2\right)}{\Gamma\left(\frac{n_1}{2}+l_1\right)\Gamma\left(\frac{n_2}{2}+l_2\right)\Gamma\left(\frac{m}{2}\right)}\right)\left(\frac{n_1}{m}\right)^{\frac{n_1}{2}+l_1}\left(\frac{n_2}{m}\right)^{\frac{n_2}{2}+l_2} \\ &\quad\times y_1^{\frac{n_1}{2}+l_1-1}y_2^{\frac{n_2}{2}+l_2-1}\left(1+\frac{n_1}{m}y_1+\frac{n_2}{m}y_2\right)^{-\left(\frac{n_1+n_2+m}{2}+l_1+l_2\right)}, \quad y_1,y_2>0 \end{aligned} \tag{3.2}$$

where $n_1, n_2, r, m > 0$, and $-1 \le \xi \le 1$.

Proof. First, from (2.8) and Result C.18, consider the joint pdf of X_1, X_2 and Z:

$$\begin{aligned} f_{X_1,X_2,Z}(x_1,x_2,z) &= f_{X_1,X_2}(x_1,x_2)\,f_Z(z) \\ &= \sum_{j=0}^{\infty} e^{-\frac{1}{2}x_1}x_1^{\frac{n_1}{2}-1}e^{-\frac{1}{2}x_2}x_2^{\frac{n_2}{2}-1}\frac{\left(\frac{r}{2}\right)_j\ j!}{\Gamma\left(\frac{n_1}{2}+j\right)\Gamma\left(\frac{n_2}{2}+j\right)}\frac{\xi^{2j}}{2^{\frac{1}{2}(n_1+n_2)}} \\ &\quad\times L_j^{\frac{n_1}{2}-1}\left(\frac{1}{2}x_1\right)L_j^{\frac{n_2}{2}-1}\left(\frac{1}{2}x_2\right)\frac{z^{\frac{m}{2}-1}e^{-\frac{z}{2}}}{\Gamma\left(\frac{m}{2}\right)2^{\frac{m}{2}}} \end{aligned}$$

where $n_1, n_2, r > 0$, and $L_j^n(\cdot)$ is the Laguerre polynomial as defined in Result C.13. Let $Y_1 = \frac{X_1/n_1}{Z/m}$ and $Y_2 = \frac{X_2/n_2}{Z/m}$, which gives the inverse transformation $X_1 = \frac{Y_1 n_1 Z}{m}$, and $X_2 = \frac{Y_2 n_2 Z}{m}$. The Jacobian of the transformation is given by

$$\begin{aligned} J(y_1,y_2,z\to x_1,x_2,z) &= \begin{vmatrix} \frac{n_1 z}{m} & 0 & \frac{y_1 n_1}{m} \\ 0 & \frac{n_2 z}{m} & \frac{n_2 y_2}{m} \\ -\frac{x_1 m}{n_1 y_1^2} & -\frac{x_2 m}{n_2 y_2^2} & 0 \end{vmatrix} \\ &= \left(\frac{z}{m}\right)^2 n_1 n_2. \end{aligned}$$

By using the obtained Jacobian and applying the transformation, the following is obtained:

$$
\begin{aligned}
f_{Y_1,Y_2,Z}\left(y_1,y_2,z\right) &= \sum_{j=0}^{\infty} e^{-\frac{1}{2}\frac{y_1n_1z}{m}}\left(\frac{y_1n_1z}{m}\right)^{\frac{n_1}{2}-1} e^{-\frac{1}{2}\frac{y_2n_2z}{m}}\left(\frac{y_2n_2z}{m}\right)^{\frac{n_2}{2}-1} \frac{\left(\frac{r}{2}\right)_j\ j!}{\Gamma\left(\frac{n_1}{2}+j\right)\Gamma\left(\frac{n_2}{2}+j\right)\Gamma\left(\frac{m}{2}\right)} \\
&\quad \times\frac{\xi^{2j}}{2^{\frac{1}{2}(n_1+n_2)}2^{\frac{m}{2}}} L_j^{\frac{n_1}{2}-1}\left(\frac{1}{2}\frac{y_1n_1z}{m}\right) L_j^{\frac{n_2}{2}-1}\left(\frac{1}{2}\frac{y_2n_2z}{m}\right) z^{\frac{m}{2}-1}e^{-\frac{z}{2}}\left(\frac{z}{m}\right)^2 n_1n_2.
\end{aligned}
$$

By expanding the Laguerre polynomial (see Result C.13) and by further algebraic simplification, the following is obtained:

$$
\begin{aligned}
f_{Y_1,Y_2,Z}\left(y_1,y_2,z\right) &= \sum_{j=0}^{\infty}\sum_{l_1=0}^{j}\sum_{l_2=0}^{j} e^{-z\left(\frac{y_1n_1+y_2n_2}{2m}+\frac{1}{2}\right)} z^{\frac{n_1+n_2+m}{2}+l_1+l_2}\left(\frac{n_1}{m}\right)^{\frac{n_1}{2}+l_1}\left(\frac{n_2}{m}\right)^{\frac{n_2}{2}+l_2} \\
&\quad \times y_1^{\frac{n_1}{2}+l_1-1} y_2^{\frac{n_2}{2}+l_2-1}\left(\frac{\left(\frac{r}{2}\right)_j \xi^{2j}\left(-1\right)^{l_1+l_2}\binom{j}{l_1}\binom{j}{l_2}}{j!\Gamma\left(\frac{n_1}{2}+l_1\right)\Gamma\left(\frac{n_2}{2}+l_2\right)\Gamma\left(\frac{m}{2}\right)2^{\frac{1}{2}(n_1+n_2+m+2l_1+2l_2)}}\right).
\end{aligned}
$$

Now, since $z>0$, the pdf of $f_{Y_1,Y_2}(y_1,y_2)$ is given by

$$
\begin{aligned}
f_{Y_1,Y_2}(y_1,y_2) &= \int_0^{\infty} f_{Y_1,Y_2,Z}\left(y_1,y_2,z\right)dz \\
&= \sum_{j=0}^{\infty}\sum_{l_1=0}^{j}\sum_{l_2=0}^{j}\left(\frac{n_1}{m}\right)^{\frac{n_1}{2}+l_1}\left(\frac{n_2}{m}\right)^{\frac{n_2}{2}+l_2} y_1^{\frac{n_1}{2}+l_1-1} y_2^{\frac{n_2}{2}+l_2-1} \\
&\quad \times\left(\frac{\left(\frac{r}{2}\right)_j \xi^{2j}\left(-1\right)^{l_1+l_2}\binom{j}{l_1}\binom{j}{l_2}}{j!\Gamma\left(\frac{n_1}{2}+l_1\right)\Gamma\left(\frac{n_2}{2}+l_2\right)\Gamma\left(\frac{m}{2}\right)2^{\frac{1}{2}(n_1+n_2+m+2l_1+2l_2)}}\right) \\
&\quad \times\int_0^{\infty} e^{-z\left(\frac{y_1n_1+y_2n_2}{2m}+\frac{1}{2}\right)} z^{\frac{n_1+n_2+m}{2}+l_1+l_2}dz
\end{aligned}
$$

and by applying Result C.12 to the integral, the following form is obtained:

$$
\begin{aligned}
f_{Y_1,Y_2}(y_1,y_2) &= \sum_{j=0}^{\infty}\sum_{l_1=0}^{j}\sum_{l_2=0}^{j}\left(\frac{\left(-1\right)^{l_1+l_2}\binom{j}{l_1}\binom{j}{l_2}\xi^{2j}\left(\frac{r}{2}\right)_j}{j!}\frac{\Gamma\left(\frac{n_1+n_2+m}{2}+l_1+l_2\right)}{\Gamma\left(\frac{n_1}{2}+l_1\right)\Gamma\left(\frac{n_2}{2}+l_2\right)\Gamma\left(\frac{m}{2}\right)}\right) \\
&\quad \times\left(\frac{n_1}{m}\right)^{\frac{n_1}{2}+l_1}\left(\frac{n_2}{m}\right)^{\frac{n_2}{2}+l_2} y_1^{\frac{n_1}{2}+l_1-1} y_2^{\frac{n_2}{2}+l_2-1}\left(1+\frac{n_1}{m}y_1+\frac{n_2}{m}y_2\right)^{-\left(\frac{n_1+n_2+m}{2}+l_1+l_2\right)}
\end{aligned}
$$

which completes the proof. ∎

3.2.3 Bivariate compound extended F distribution

3.2.3.1 Unequal degrees of freedom

In this section, a bivariate compound extended F distribution is derived, also via a transformation approach, by using the bivariate compound extended chi-square distribution in (2.10). Note that the distribution under consideration is for the unequal degrees of freedom case, as to provide for greater flexibility.

Theorem 3.2 *Let X_1 and X_2 be jointly distributed with pdf (2.10) with n_1 and n_2 degrees of freedom respectively, and let $Z \sim \chi^2(m)$ be an independent chi-square distributed random variable with m degrees of freedom. Let $Y_1 = \frac{X_1/n_1}{Z/m}$ and $Y_2 = \frac{X_2/n_2}{Z/m}$. The joint pdf of Y_1 and Y_2 is given by*

$$
\begin{aligned}
f_{Y_1,Y_2}(y_1,y_2) &= \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\frac{\Gamma\left(\frac{n_1}{2}+i_1\right)}{\Gamma\left(\frac{n_1}{2}\right)i_1!}\frac{\Gamma\left(\frac{n_2}{2}+i_2\right)}{\Gamma\left(\frac{n_2}{2}\right)i_2!}\gamma^{2(i_1+i_2)}\left(1-\gamma^2\right)^{\frac{n_1+n_2}{2}}\left(\frac{n_1}{m}\right)^{\frac{n_1}{2}+i_1}\left(\frac{n_2}{m}\right)^{\frac{n_2}{2}+i_2} \\
&\quad \times y_1^{\frac{n_1}{2}+i_1-1}y_2^{\frac{n_2}{2}+i_2-1}\left(\frac{y_1n_1+y_2n_2}{2m\left(1-\gamma^2\right)}+\frac{1}{2}\right)^{-\left(\frac{n_1+n_2+m}{2}+i_1+i_2\right)}\left(\frac{1}{2}\right)^{\frac{n_1+n_2+m}{2}+i_1+i_2} \\
&\quad \times\left(\frac{1}{\left(1-\gamma^2\right)}\right)^{\frac{n_1+n_2}{2}+i_1+i_2}\frac{\Gamma\left(\frac{n_1+n_2+m}{2}+i_1+i_2\right)}{\Gamma\left(\frac{n_1}{2}+i_1\right)\Gamma\left(\frac{n_2}{2}+i_2\right)\Gamma\left(\frac{m}{2}\right)} \qquad (3.3) \\
&= \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\gamma^{2(i_1+i_2)}\left(1-\gamma^2\right)^{\frac{n_1+n_2}{2}}\frac{\Gamma\left(\frac{n_1+n_2+m}{2}+i_1+i_2\right)}{\Gamma\left(\frac{n_1}{2}\right)i_1!\Gamma\left(\frac{n_2}{2}\right)i_2!\Gamma\left(\frac{m}{2}\right)} \\
&\quad \times\left(\frac{n_1}{m}\right)^{\frac{n_1}{2}+i_1}\left(\frac{n_2}{m}\right)^{\frac{n_2}{2}+i_2}y_1^{\frac{n_1}{2}+i_1-1}y_2^{\frac{n_2}{2}+i_2-1}\left(\frac{1}{\left(1-\gamma^2\right)}\right)^{\frac{n_1+n_2}{2}+i_1+i_2} \\
&\quad \times\left(1+\frac{n_1}{m\left(1-\gamma^2\right)}y_1+\frac{n_2}{m\left(1-\gamma^2\right)}y_2\right)^{-\left(\frac{n_1+n_2+m}{2}+i_1+i_2\right)}, \qquad y_1,y_2>0 \qquad (3.4)
\end{aligned}
$$

where $n_1, n_2, m > 0$, and $-1 < \gamma < 1$. As previously (see Chapter 2), γ is an extension parameter.

Proof. First, from (2.10) and Result C.18, consider the joint pdf of X_1, X_2 and Z:

$$\begin{aligned}
f_{X_1,X_2,Z}(x_1,x_2,z) &= f_{X_1,X_2}(x_1,x_2)\, f_Z(z) \\
&= \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty} \frac{\Gamma\left(\frac{n_1}{2}+i_1\right)}{\Gamma\left(\frac{n_1}{2}\right) i_1!}\frac{\Gamma\left(\frac{n_2}{2}+i_2\right)}{\Gamma\left(\frac{n_2}{2}\right) i_2!}\gamma^{2(i_1+i_2)}\left(1-\gamma^2\right)^{\frac{n_1+n_2}{2}} \\
&\quad\times\frac{x_1^{\frac{n_1}{2}+i_1-1}e^{-\frac{x_1}{2(1-\gamma^2)}}}{\left(\Gamma\left(\frac{n_1}{2}+i_1\right)\left(2\left(1-\gamma^2\right)\right)^{\frac{n_1}{2}+i_1}\right)}\frac{x_2^{\frac{n_2}{2}+i_2-1}e^{-\frac{x_2}{2(1-\gamma^2)}}}{\left(\Gamma\left(\frac{n_2}{2}+i_2\right)\left(2\left(1-\gamma^2\right)\right)^{\frac{n_2}{2}+i_2}\right)} \\
&\quad\times\frac{z^{\frac{m}{2}-1}e^{-\frac{z}{2}}}{\Gamma\left(\frac{m}{2}\right)2^{\frac{m}{2}}}
\end{aligned}$$

where $n_1, n_2 > 0$, and $-1 < \gamma < 1$. Let $Y_1 = \frac{X_1/n_1}{Z/m}$ and $Y_2 = \frac{X_2/n_2}{Z/m}$, which gives the inverse transformation $X_1 = \frac{Y_1 n_1 Z}{m}$, and $X_2 = \frac{Y_2 n_2 Z}{m}$. The Jacobian of the transformation is given by

$$\begin{aligned}
J\left(y_1,y_2,z \rightarrow x_1,x_2,z\right) &= \begin{vmatrix} \frac{n_1 z}{m} & 0 & \frac{y_1 n_1}{m} \\ 0 & \frac{n_2 z}{m} & \frac{n_2 y_2}{m} \\ -\frac{x_1 m}{n_1 y_1^2} & -\frac{x_2 m}{n_2 y_2^2} & 0 \end{vmatrix} \\
&= \left(\frac{z}{m}\right)^2 n_1 n_2.
\end{aligned}$$

By using the obtained Jacobian, applying the transformation, and algebraic simplification, the following is obtained:

$$\begin{aligned}
&f_{Y_1,Y_2,Z}(y_1,y_2,z) \\
&= \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty} \frac{\Gamma\left(\frac{n_1}{2}+i_1\right)}{\Gamma\left(\frac{n_1}{2}\right) i_1!}\frac{\Gamma\left(\frac{n_2}{2}+i_2\right)}{\Gamma\left(\frac{n_2}{2}\right) i_2!}\gamma^{2(i_1+i_2)}\left(1-\gamma^2\right)^{\frac{n_1+n_2}{2}} \\
&\quad\times\frac{\left(\frac{y_1 n_1 z}{m}\right)^{\frac{n_1}{2}+i_1-1}e^{-\frac{y_1 n_1 z}{2m(1-\gamma^2)}}}{\left(\Gamma\left(\frac{n_1}{2}+i_1\right)\left(2\left(1-\gamma^2\right)\right)^{\frac{n_1}{2}+i_1}\right)}\frac{\left(\frac{y_2 n_2 z}{m}\right)^{\frac{n_2}{2}+i_2-1}e^{-\frac{y_2 n_2 z}{2(1-\gamma^2)}}}{\left(\Gamma\left(\frac{n_2}{2}+i_2\right)\left(2\left(1-\gamma^2\right)\right)^{\frac{n_2}{2}+i_2}\right)} \\
&\quad\times\frac{z^{\frac{m}{2}-1}e^{-\frac{z}{2}}}{\Gamma\left(\frac{m}{2}\right)2^{\frac{m}{2}}}\left(\frac{z}{m}\right)^2 n_1 n_2 \\
&= \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty} \frac{\Gamma\left(\frac{n_1}{2}+i_1\right)}{\Gamma\left(\frac{n_1}{2}\right) i_1!}\frac{\Gamma\left(\frac{n_2}{2}+i_2\right)}{\Gamma\left(\frac{n_2}{2}\right) i_2!}\gamma^{2(i_1+i_2)}\left(1-\gamma^2\right)^{\frac{n_1+n_2}{2}} z^{\frac{n_1+n_2+m}{2}+i_1+i_2-1} \\
&\quad\times e^{-z\left(\frac{y_1 n_1+y_2 n_2}{2m(1-\gamma^2)}+\frac{1}{2}\right)}\left(\frac{n_1}{m}\right)^{\frac{n_1}{2}+i_1}\left(\frac{n_2}{m}\right)^{\frac{n_2}{2}+i_2} y_1^{\frac{n_1}{2}+i_1-1} y_2^{\frac{n_2}{2}+i_2-1} \\
&\quad\times\frac{1}{\Gamma\left(\frac{n_1}{2}+i_1\right)\Gamma\left(\frac{n_2}{2}+i_2\right)\Gamma\left(\frac{m}{2}\right)}\left(\frac{1}{2}\right)^{\frac{n_1+n_2+m}{2}+i_1+i_2}\left(\frac{1}{(1-\gamma^2)}\right)^{\frac{n_1+n_2}{2}+i_1+i_2}.
\end{aligned}$$

Now, since $z > 0$, the pdf of $f_{Y_1,Y_2}(y_1, y_2)$ is given by

$$\begin{aligned}
f_{Y_1,Y_2}(y_1,y_2) &= \int_0^{\infty} f_{Y_1,Y_2,Z}\left(y_1,y_2,z\right) dz \\
&= \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty} \frac{\Gamma\left(\frac{n_1}{2}+i_1\right)}{\Gamma\left(\frac{n_1}{2}\right) i_1!} \frac{\Gamma\left(\frac{n_2}{2}+i_2\right)}{\Gamma\left(\frac{n_2}{2}\right) i_2!} \gamma^{2(i_1+i_2)} \left(1-\gamma^2\right)^{\frac{n_1+n_2}{2}} \left(\frac{n_1}{m}\right)^{\frac{n_1}{2}+i_1} \left(\frac{n_2}{m}\right)^{\frac{n_2}{2}+i_2} \\
&\quad \times y_1^{\frac{n_1}{2}+i_1-1} y_2^{\frac{n_2}{2}+i_2-1} \left(\frac{1}{2}\right)^{\frac{n_1+n_2+m}{2}+i_1+i_2} \left(\frac{1}{(1-\gamma^2)}\right)^{\frac{n_1+n_2}{2}+i_1+i_2} \\
&\quad \times \frac{1}{\Gamma\left(\frac{n_1}{2}+i_1\right)\Gamma\left(\frac{n_2}{2}+i_2\right)\Gamma\left(\frac{m}{2}\right)} \int_0^{\infty} z^{\frac{n_1+n_2+m}{2}+i_1+i_2-1} e^{-z\left(\frac{y_1n_1+y_2n_2}{2m(1-\gamma^2)}+\frac{1}{2}\right)} dz
\end{aligned}$$

and by applying Result C.12 to the integral, the following form is obtained:

$$\begin{aligned}
& f_{Y_1,Y_2}(y_1,y_2) \\
= \; & \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty} \frac{\Gamma\left(\frac{n_1}{2}+i_1\right)}{\Gamma\left(\frac{n_1}{2}\right) i_1!} \frac{\Gamma\left(\frac{n_2}{2}+i_2\right)}{\Gamma\left(\frac{n_2}{2}\right) i_2!} \gamma^{2(i_1+i_2)} \left(1-\gamma^2\right)^{\frac{n_1+n_2}{2}} \left(\frac{n_1}{m}\right)^{\frac{n_1}{2}+i_1} \left(\frac{n_2}{m}\right)^{\frac{n_2}{2}+i_2} \\
& \times y_1^{\frac{n_1}{2}+i_1-1} y_2^{\frac{n_2}{2}+i_2-1} \left(\frac{y_1n_1+y_2n_2}{2m\left(1-\gamma^2\right)}+\frac{1}{2}\right)^{-\left(\frac{n_1+n_2+m}{2}+i_1+i_2\right)} \left(\frac{1}{2}\right)^{\frac{n_1+n_2+m}{2}+i_1+i_2} \\
& \times \left(\frac{1}{(1-\gamma^2)}\right)^{\frac{n_1+n_2}{2}+i_1+i_2} \frac{\Gamma\left(\frac{n_1+n_2+m}{2}+i_1+i_2\right)}{\Gamma\left(\frac{n_1}{2}+i_1\right)\Gamma\left(\frac{n_2}{2}+i_2\right)\Gamma\left(\frac{m}{2}\right)}
\end{aligned}$$

which completes the proof. ■

3.3 Bivariate noncentral F distributions

3.3.1 Introduction

As in the case of bivariate noncentral chi-square distributions (see Chapter 2), an important concept used in this section is that of using some conditional bivariate pdf and compounding it with Poisson probabilities to obtain an unconditional noncentral bivariate pdf. Before the discussion commences, Figure 3.2 provides an overview of core concepts of this section 3.3.

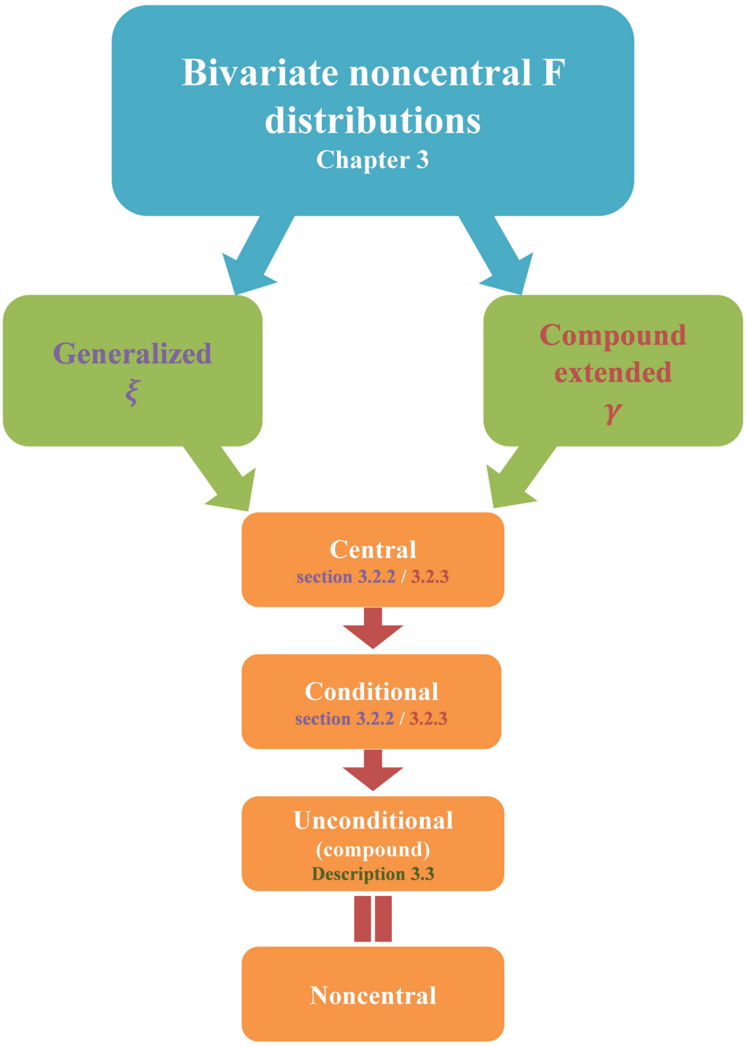

Figure 3.2 Diagram of core concepts of Chapter 3 - section 3.3.

It is described in the F distribution setting in the following description.

Description 3.3 *A(n) (unconditional) bivariate noncentral F pdf can be obtained from a (conditional) bivariate central F pdf in the following manner:*

$$f_{Y_1,Y_2}(y_1,y_2) = \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} f_{Y_1,Y_2}(y_1,y_2|k_1,k_2)g_{K_1}(k_1)g_{K_2}(k_2). \tag{3.5}$$

As before in Description 2.10 (see equation (2.11)), $g_{K_v}(k_v) = \frac{e^{-\frac{\theta_v}{2}}\left(\frac{\theta_v}{2}\right)^{k_v}}{k_v!}$, $v = 1,2$ *are the compounding factors, which is here Poisson probabilities with* θ_v *the noncentrality parameters. Also,* $f_{Y_1,Y_2}(y_1,y_2|k_1,k_2)$ *represents the (conditional) pdf of some suitable bivariate central F distribution.* ▲

3.3.2 Bivariate noncentral generalized F distribution

In this section a noncentral version of (3.2) will be proposed. First, a conditional bivariate generalized F distribution from (3.2) is defined next.

Description 3.4 *Let* $(Y_1,Y_2|\,(K_1=k_1,K_2=k_2))$ *have a conditional bivariate generalized F distribution (see (3.2)) with pdf given by*

$$\begin{aligned}
&f_{Y_1,Y_2}(y_1,y_2|k_1,k_2)\\
=\ &\sum_{j=0}^{\infty}\sum_{l_1=0}^{j}\sum_{l_2=0}^{j}\left(\frac{\xi^{2j}\left(\frac{r}{2}\right)_j(-1)^{l_1+l_2}\binom{j}{l_1}\binom{j}{l_2}}{j!}\right)\left(\frac{\Gamma\left(\frac{n_1+n_2+m}{2}+l_1+l_2+k_1+k_2\right)}{\Gamma\left(\frac{n_1}{2}+l_1+k_1\right)\Gamma\left(\frac{n_2}{2}+l_2+k_2\right)\Gamma\left(\frac{m}{2}\right)}\right)\\
&\times\left(\frac{n_1}{m}\right)^{\frac{n_1}{2}+l_1+k_1}\left(\frac{n_2}{m}\right)^{\frac{n_2}{2}+l_2+k_2} y_1^{\frac{n_1}{2}+l_1+k_1-1}\, y_2^{\frac{n_2}{2}+l_2+k_2-1}\\
&\times\left(1+\frac{n_1}{m}y_1+\frac{n_2}{m}y_2\right)^{-\left(\frac{n_1+n_2+m}{2}+l_1+l_2+k_1+k_2\right)}, \qquad y_1,y_2>0
\end{aligned} \tag{3.6}$$

where $n_1,n_2,r,m>0$. *As previously,* ξ *(where* $-1\le\xi\le 1$*) is a parameter which is a component of the product-moment correlation between* X_1 *and* X_2*. The conditional values* k_1 *and* k_2 *have domain such that* $k_v \ge 0$, $v=1,2$. ▲

A conditional bivariate generalized F distribution is defined with pdf in (3.6), which now fulfills the role of the conditional pdf as described by (3.5). Similar as in section 2.3.3, a bivariate *noncentral* generalized F distribution is obtained by applying (3.5) together with the respective Poisson probabilities, and is described next.

Description 3.5 *The joint pdf of Y_1 and Y_2, which represent the bivariate noncentral generalized F distribution is proposed by substituting (3.6) in (3.5), with the following result:*

$$\begin{aligned} f_{Y_1,Y_2}(y_1,y_2) &= f_{Y_1,Y_2}(y_1,y_2|k_1,k_2)g_{K_1}(k_1)g_{K_2}(k_2) \\ &= \sum_{j=0}^{\infty}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{l_1=0}^{j}\sum_{l_2=0}^{j} C y_1^{\frac{n_1}{2}+l_1+k_1-1} y_2^{\frac{n_2}{2}+l_2+k_2-1} \\ &\quad \times \left(1+\frac{n_1}{m}y_1+\frac{n_2}{m}y_2\right)^{-\left(\frac{n_1+n_2+m}{2}+l_1+l_2+k_1+k_2\right)} \\ &\quad \times \frac{e^{-\frac{\theta_1}{2}}\left(\frac{\theta_1}{2}\right)^{k_1}}{k_1!}\frac{e^{-\frac{\theta_2}{2}}\left(\frac{\theta_2}{2}\right)^{k_2}}{k_2!}, \qquad y_1,y_2>0 \end{aligned} \tag{3.7}$$

where

$$\begin{aligned} C &= \left(\frac{(-1)^{l_1+l_2}\binom{j}{l_1}\binom{j}{l_2}\xi^{2j}\left(\frac{r}{2}\right)_j}{j!}\right)\left(\frac{\Gamma\left(\frac{n_1+n_2+m}{2}+l_1+l_2+k_1+k_2\right)}{\Gamma\left(\frac{n_1}{2}+l_1+k_1\right)\Gamma\left(\frac{n_2}{2}+l_2+k_2\right)\Gamma\left(\frac{m}{2}\right)}\right) \\ &\quad \times \left(\frac{n_1}{m}\right)^{\frac{n_1}{2}+l_1+k_1}\left(\frac{n_2}{m}\right)^{\frac{n_2}{2}+l_2+k_2} \end{aligned} \tag{3.8}$$

and $n_1, n_2, r, m > 0$, and $\theta_1, \theta_2 > 0$. Again, due to the construction of the noncentrality, the Poisson probability factors, namely $g_{K_v}(k_v) = \frac{e^{-\frac{\theta_v}{2}}\left(\frac{\theta_v}{2}\right)^{k_v}}{k_v!}, v = 1,2,$ isolates the noncentrality parameters in a mathematically convenient way. ▲

To ensure transparency of the noncentrality components, it must be noted that since the conditional distribution defined in (3.6) was conditioned on k_1 and k_2 (relating to the degrees of freedom n_1 and n_2 of variables Y_1 and Y_2) that the noncentrality parameters θ_1 and θ_2 refers to the noncentrality of the variables Y_1 and Y_2.

Remark 3.1 *In this section a bivariate noncentral generalized F distribution was proposed (see (3.7)) by using the compounding method (see (3.5)) after defining a conditional bivariate generalized F distribution in (3.6). It is noted here that the same bivariate distribution (3.7) would've been obtained as a joint distribution for Y_1 and Y_2 with a transformation approach had the joint pdf of X_1 and X_2 be the bivariate noncentral generalized chi-square distribution as given in (2.15), together with $Z \sim \chi^2(m)$ and $Y_1 = \frac{X_1/n_1}{Z/m}$ and $Y_2 = \frac{X_2/n_2}{Z/m}$.*

3.3.2.1 Product moments of random variables

In the following theorem, an expression for the product moments is derived.

Theorem 3.3 *If Y_1 and Y_2 are jointly distributed according to (3.7), then the product moment, i.e. $E\left(Y_1^q Y_2^s\right)$, is given by*

$$\begin{aligned}
E\left(Y_1^q Y_2^s\right) &= \sum_{j=0}^{\infty}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{l_1=0}^{j}\sum_{l_2=0}^{j}\left(\frac{(-1)^{l_1+l_2}\binom{j}{l_1}\binom{j}{l_2}\xi^{2j}\left(\frac{r}{2}\right)_j}{j!}\right) \\
&\quad\times\left(\frac{\Gamma\left(\frac{n_1+n_2+m}{2}+l_1+l_2+k_1+k_2\right)}{\Gamma\left(\frac{n_1}{2}+l_1+k_1\right)\Gamma\left(\frac{n_2}{2}+l_2+k_2\right)\Gamma\left(\frac{m}{2}\right)}\right)\left(\frac{n_1}{m}\right)^{-q} \\
&\quad\times\left(\frac{n_2}{m}\right)^{-s} B\left(s+\frac{n_2}{2}+l_2+k_2,\frac{n_1+m}{2}+l_1+k_1-s\right) \\
&\quad\times B\left(q+\frac{n_1}{2}+l_1+k_1,\frac{m}{2}-q-s\right)\frac{e^{-\frac{\theta_1}{2}}\left(\frac{\theta_1}{2}\right)^{k_1}}{k_1!}\frac{e^{-\frac{\theta_2}{2}}\left(\frac{\theta_2}{2}\right)^{k_2}}{k_2!}
\end{aligned}$$

where $n_1, n_2, m, r > 0$, $-1 \leq \xi \leq 1$, and $\theta_1, \theta_2 > 0$. $B(\cdot,\cdot)$ is the beta function as defined in Result C.4 with values of argument such that $B(\cdot,\cdot)$ is well-defined.

Proof. From (3.7), the expected value of $Y_1^q Y_2^s$ is taken (and C is the value defined in (3.8)):

$$\begin{aligned}
E\left(Y_1^q Y_2^s\right) &= \int_0^\infty\int_0^\infty y_1^q y_2^s \sum_{j=0}^{\infty}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{l_1=0}^{j}\sum_{l_2=0}^{j} C y_1^{\frac{n_1}{2}+l_1+k_1-1} y_2^{\frac{n_2}{2}+l_2+k_2-1} \\
&\quad \times \left(1+\frac{n_1}{m}y_1+\frac{n_2}{m}y_2\right)^{-\left(\frac{n_1+n_2+m}{2}+l_1+l_2+k_1+k_2\right)} \\
&\quad \times \frac{e^{-\frac{\theta_1}{2}}\left(\frac{\theta_1}{2}\right)^{k_1}}{k_1!}\frac{e^{-\frac{\theta_2}{2}}\left(\frac{\theta_2}{2}\right)^{k_2}}{k_2!} dy_2 dy_1 \\
&= \sum_{j=0}^{\infty}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{l_1=0}^{j}\sum_{l_2=0}^{j} C\frac{e^{-\frac{\theta_1}{2}}\left(\frac{\theta_1}{2}\right)^{k_1}}{k_1!}\frac{e^{-\frac{\theta_2}{2}}\left(\frac{\theta_2}{2}\right)^{k_2}}{k_2!}\int_0^\infty y_1^{q+\frac{n_1}{2}+l_1+k_1-1} \\
&\quad \times \int_0^\infty y_2^{s+\frac{n_2}{2}+l_2+k_2-1}\left(1+\frac{n_1}{m}y_1+\frac{n_2}{m}y_2\right)^{-\left(\frac{n_1+n_2+m}{2}+l_1+l_2+k_1+k_2\right)} dy_2 dy_1 \\
&= \sum_{j=0}^{\infty}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{l_1=0}^{j}\sum_{l_2=0}^{j} C\frac{e^{-\frac{\theta_1}{2}}\left(\frac{\theta_1}{2}\right)^{k_1}}{k_1!}\frac{e^{-\frac{\theta_2}{2}}\left(\frac{\theta_2}{2}\right)^{k_2}}{k_2!}\left(\frac{m}{n_2}\right)^{\frac{n_1+n_2+m}{2}+l_1+l_2+k_1+k_2} \\
&\quad \times \int_0^\infty y_1^{q+\frac{n_1}{2}+l_1+k_1-1}\int_0^\infty y_2^{s+\frac{n_2}{2}+l_2+k_2-1}\left(\frac{m}{n_2}+\frac{n_1}{n_2}y_1+y_2\right)^{-\left(\frac{n_1+n_2+m}{2}+l_1+l_2+k_1+k_2\right)} dy_2 dy_1 \\
&= \sum_{j=0}^{\infty}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{l_1=0}^{j}\sum_{l_2=0}^{j} C\frac{e^{-\frac{\theta_1}{2}}\left(\frac{\theta_1}{2}\right)^{k_1}}{k_1!}\frac{e^{-\frac{\theta_2}{2}}\left(\frac{\theta_2}{2}\right)^{k_2}}{k_2!}B\left(s+\frac{n_2}{2}+l_2+k_2;\frac{n_1+m}{2}+l_1+k_1-s\right) \\
&\quad \times \left(\frac{m}{n_2}\right)^{\frac{n_1+n_2+m}{2}+l_1+l_2+k_1+k_2}\int_0^\infty y_1^{q+\frac{n_1}{2}+l_1+k_1-1}\left(\frac{m+n_1y_1}{n_2}\right)^{-\left(\frac{n_1+m}{2}+l_1+k_1-s\right)} dy_1
\end{aligned}$$

which follows by applying Result C.16: setting $\alpha = s+\frac{n_2}{2}+l_2+k_2$, $\rho = \frac{n_1+n_2+m}{2}+l_1+l_2+k_1+k_2$, and $z = \frac{m}{n_2}+\frac{n_1}{n_2}y_1$. The remaining integral is solved by applying the same Result C.16: setting then $\alpha = q+\frac{n_1}{2}+l_1+k_1$, $\rho = \frac{n_1+m}{2}+l_1+k_1-s$, and $z = \frac{m}{n_1}$ - and then, after some algebraic manipulation:

$$
\begin{aligned}
E\left(Y_1^q Y_2^s\right) &= \sum_{j=0}^{\infty}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{l_1=0}^{j}\sum_{l_2=0}^{j} C\frac{e^{-\frac{\theta_1}{2}}\left(\frac{\theta_1}{2}\right)^{k_1}}{k_1!}\frac{e^{-\frac{\theta_2}{2}}\left(\frac{\theta_2}{2}\right)^{k_2}}{k_2!}\left(\frac{m}{n_2}\right)^{\frac{n_1+n_2+m}{2}+l_1+l_2+k_1+k_2} \\
&\quad \times B\left(s+\frac{n_2}{2}+l_2+k_2;\frac{n_1+m}{2}+l_1+k_1-s\right) \\
&\quad \times \int_0^\infty y_1^{q+\frac{n_1}{2}+l_1+k_1-1}\left(\frac{n_2}{n_1}\right)^{\frac{n_1+m}{2}+l_1+k_1-s}\left(\frac{m}{n_1}+y_1\right)^{-\left(\frac{n_1+m}{2}+l_1+k_1-s\right)} dy_1 \\
&= \sum_{j=0}^{\infty}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{l_1=0}^{j}\sum_{l_2=0}^{j} C\frac{e^{-\frac{\theta_1}{2}}\left(\frac{\theta_1}{2}\right)^{k_1}}{k_1!}\frac{e^{-\frac{\theta_2}{2}}\left(\frac{\theta_2}{2}\right)^{k_2}}{k_2!}\left(\frac{m}{n_2}\right)^{\frac{n_1+n_2+m}{2}+l_1+l_2+k_1+k_2} \\
&\quad \times B\left(s+\frac{n_2}{2}+l_2+k_2;\frac{n_1+m}{2}+l_1+k_1-s\right)\left(\frac{n_2}{n_1}\right)^{\frac{n_1+m}{2}+l_1+k_1-s} \\
&\quad \times \int_0^\infty y_1^{q+\frac{n_1}{2}+l_1+k_1-1}\left(\frac{m}{n_1}+y_1\right)^{-\left(\frac{n_1+m}{2}+l_1+k_1-s\right)} dy_1 \\
&= \sum_{j=0}^{\infty}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{l_1=0}^{j}\sum_{l_2=0}^{j} C\frac{e^{-\frac{\theta_1}{2}}\left(\frac{\theta_1}{2}\right)^{k_1}}{k_1!}\frac{e^{-\frac{\theta_2}{2}}\left(\frac{\theta_2}{2}\right)^{k_2}}{k_2!}\left(\frac{m}{n_2}\right)^{\frac{n_1+n_2+m}{2}+l_1+l_2+k_1+k_2} \\
&\quad \times B\left(s+\frac{n_2}{2}+l_2+k_2;\frac{n_1+m}{2}+l_1+k_1-s\right)\left(\frac{n_2}{n_1}\right)^{\frac{n_1+m}{2}+l_1+k_1-s} \\
&\quad \times \left(\frac{m}{n_1}\right)^{q+s-\frac{m}{2}} B\left(q+\frac{n_1}{2}+l_1+k_1;\frac{m}{2}-q-s\right)
\end{aligned}
$$

where $n_1, n_2, m, r > 0$, $-1 \leq \xi \leq 1$, and $\theta_1, \theta_2 > 0$. After further simplification, the proof is complete. ■

Remark 3.2 *The moments of the random variables can be calculated with above expression with the aim to investigate the correlation structure for this bivariate noncentral generalized F distribution (see Chapter 5).*

3.3.3 Bivariate noncentral compound extended F distribution

By using the pdf in (3.4) and from it define a conditional pdf, a bivariate noncentral compound extended F distribution can subsequently be obtained using (3.5). To provide more flexibility, only the case of unequal degrees of freedom will be considered. Similar as before (see section 2.2.3.1), a conditional central pdf is defined from (3.4).

Description 3.6 *Let $(Y_1, Y_2| (K_1 = k_1, K_2 = k_2))$ have a conditional bivariate compound extended F distribution (see (3.4)) with pdf given by*

$$\begin{aligned}
& f_{Y_1,Y_2}(y_1, y_2|k_1, k_2) \\
= & \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty} \frac{\Gamma\left(\frac{n_1}{2}+i_1\right)}{\Gamma\left(\frac{n_1}{2}\right)i_1!}\frac{\Gamma\left(\frac{n_2}{2}+i_2\right)}{\Gamma\left(\frac{n_2}{2}\right)i_2!}\gamma^{2(i_1+i_2)}\left(1-\gamma^2\right)^{\frac{n_1+n_2}{2}} \\
& \times \frac{\Gamma\left(\frac{n_1+n_2+m}{2}+i_1+i_2+k_1+k_2\right)}{\Gamma\left(\frac{n_1}{2}+i_1+k_1\right)\Gamma\left(\frac{n_2}{2}+i_2+k_2\right)\Gamma\left(\frac{m}{2}\right)}\left(\frac{n_1}{m}\right)^{\frac{n_1}{2}+i_1+k_1}\left(\frac{n_2}{m}\right)^{\frac{n_2}{2}+i_2+k_2} \\
& \times \left(\frac{1}{(1-\gamma^2)}\right)^{\frac{n_1+n_2}{2}+i_1+i_2+k_1+k_2} y_1^{\frac{n_1}{2}+i_1+k_1-1} y_2^{\frac{n_2}{2}+i_2+k_2-1} \\
& \times \left(1+\frac{n_1}{m\left(1-\gamma^2\right)}y_1+\frac{n_2}{m\left(1-\gamma^2\right)}y_2\right)^{-\left(\frac{n_1+n_2}{2}+i_1+i_2+k_1+k_2\right)}, \quad y_1, y_2 > 0 \quad (3.9)
\end{aligned}$$

and $n_1, n_2, m > 0$, and $-1 < \gamma < 1$. The conditional values k_1 and k_2 have domain such that $k_v \geq 0$, $v = 1, 2$. ▲

By substituting (3.9) into (3.5) together with the corresponding Poisson probabilities, an (unconditional) bivariate *noncentral* compound extended F distribution can be obtained. This is described next.

Description 3.7 *The joint pdf of Y_1 and Y_2, which represent the bivariate noncentral compound extended F distribution is proposed by substituting (3.9) in (3.5), with the following result:*

$$\begin{aligned}
& f_{Y_1,Y_2}(y_1, y_2) \\
= & f_{Y_1,Y_2}(y_1, y_2|k_1, k_2)g_{K_1}(k_1)g_{K_2}(k_2) \\
= & \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} C y_1^{\frac{n_1}{2}+i_1+k_1-1} y_2^{\frac{n_2}{2}+i_2+k_2-1} \\
& \times \left(1+\frac{n_1}{m\left(1-\gamma^2\right)}y_1+\frac{n_2}{m\left(1-\gamma^2\right)}y_2\right)^{-\left(\frac{n_1+n_2+m}{2}+k_1+k_2+i_1+i_2\right)} \frac{e^{-\frac{\theta_1}{2}}\left(\frac{\theta_1}{2}\right)^{k_1}}{k_1!}\frac{e^{-\frac{\theta_2}{2}}\left(\frac{\theta_2}{2}\right)^{k_2}}{k_2!}, \quad y_1, y_2>0
\end{aligned}$$

(3.10)

where

$$\begin{aligned} C &= \frac{\Gamma\left(\frac{n_1}{2}+i_1\right)}{\Gamma\left(\frac{n_1}{2}\right)i_1!}\frac{\Gamma\left(\frac{n_2}{2}+i_2\right)}{\Gamma\left(\frac{n_2}{2}\right)i_2!}\gamma^{2(i_1+i_2)}\left(\frac{1}{(1-\gamma^2)}\right)^{i_1+i_2+k_1+k_2} \\ &\quad\times\frac{\Gamma\left(\frac{n_1+n_2+m}{2}+i_1+i_2+k_1+k_2\right)}{\Gamma\left(\frac{n_1}{2}+i_1+k_1\right)\Gamma\left(\frac{n_2}{2}+i_2+k_2\right)\Gamma\left(\frac{m}{2}\right)}\left(\frac{n_1}{m}\right)^{\frac{n_1}{2}+i_1+k_1}\left(\frac{n_2}{m}\right)^{\frac{n_2}{2}+i_2+k_2} \end{aligned} \tag{3.11}$$

and $n_1, n_2, m > 0$. *Again, due to the construction of the noncentrality the Poisson probability factors, namely* $g_{K_v}(k_v) = \frac{e^{-\frac{\theta_v}{2}}\left(\frac{\theta_v}{2}\right)^{k_v}}{k_v!}$, $v = 1, 2$, *isolates the noncentrality parameters.*

▲

Remark 3.3 *In this section a bivariate noncentral compound extended F distribution was proposed (see (3.10)) by using the compounding method (see (3.5)) after defining a conditional bivariate compound extended F distribution in (3.9). It is noted here that the same bivariate distribution (3.10) would've been obtained as a joint distribution for* Y_1 *and* Y_2 *with a transformation approach had the joint pdf of* X_1 *and* X_2 *be the bivariate noncentral compound extended chi-square distribution as given in (2.21), together with* $Z \sim \chi^2(m)$ *and* $Y_1 = \frac{X_1/n_1}{Z/m}$ *and* $Y_2 = \frac{X_2/n_2}{Z/m}$.

3.3.3.1 Product moments of random variables

In the following theorem, the product moments of the distribution in (3.10) is derived.

Theorem 3.4 *If* Y_1 *and* Y_2 *are jointly distributed according to (3.10), then the product moment, i.e.* $E\left(Y_1^q Y_2^s\right)$, *is given by*

$$\begin{aligned} E\left(Y_1^q Y_2^s\right) &= \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\frac{\Gamma\left(\frac{n_1}{2}+i_1\right)}{\Gamma\left(\frac{n_1}{2}\right)i_1!}\frac{\Gamma\left(\frac{n_2}{2}+i_2\right)}{\Gamma\left(\frac{n_2}{2}\right)i_2!}\gamma^{2(i_1+i_2)}\left(1-\gamma^2\right)^{\frac{n_1+n_2}{2}+q+s} \\ &\quad\times\frac{\Gamma\left(\frac{n_1+n_2+m}{2}+i_1+i_2+k_1+k_2\right)}{\Gamma\left(\frac{n_1}{2}+i_1+k_1\right)\Gamma\left(\frac{n_2}{2}+i_2+k_2\right)\Gamma\left(\frac{m}{2}\right)}\left(\frac{n_1}{m}\right)^{-q}\left(\frac{n_2}{m}\right)^{-s} \\ &\quad\times B\left(s+\frac{n_2}{2}+i_2+k_2;\frac{n_1+m}{2}+i_1+k_1-s\right)B\left(q+\frac{n_1}{2}+i_1+k_1;\frac{m}{2}-q-s\right) \end{aligned}$$

where $n_1, n_2, m > 0$, $-1 < \gamma < 1$, *and* $\theta_1, \theta_2 > 0$. $B(\cdot,\cdot)$ *is the beta function as defined in Result C.4 with values of argument such that* $B(\cdot,\cdot)$ *is well-defined.*

Proof. From (3.10), the expected value of $Y_1^q Y_2^s$ is taken (and C is the value defined in (3.11)):

$$
\begin{aligned}
E\left(Y_1^q Y_2^s\right) &= \int_0^\infty \int_0^\infty y_1^q y_2^s \sum_{i_1=0}^\infty \sum_{i_2=0}^\infty \sum_{k_1=0}^\infty \sum_{k_2=0}^\infty C y_1^{\frac{n_1}{2}+i_1+k_1-1} y_2^{\frac{n_2}{2}+i_2+k_2-1} \\
&\quad \times \left(1+\frac{n_1}{m\left(1-\gamma^2\right)}y_1+\frac{n_2}{m\left(1-\gamma^2\right)}y_2\right)^{-\left(\frac{n_1+n_2+m}{2}+i_1+i_2+k_1+k_2\right)} \\
&\quad \times \frac{e^{-\frac{\theta_1}{2}}\left(\frac{\theta_1}{2}\right)^{k_1}}{k_1!} \frac{e^{-\frac{\theta_2}{2}}\left(\frac{\theta_2}{2}\right)^{k_2}}{k_2!} dy_2 dy_1 \\
&= \sum_{i_1=0}^\infty \sum_{i_2=0}^\infty \sum_{k_1=0}^\infty \sum_{k_2=0}^\infty C \frac{e^{-\frac{\theta_1}{2}}\left(\frac{\theta_1}{2}\right)^{k_1}}{k_1!} \frac{e^{-\frac{\theta_2}{2}}\left(\frac{\theta_2}{2}\right)^{k_2}}{k_2!} \int_0^\infty y_1^{q+\frac{n_1}{2}+i_1+k_1-1} \\
&\quad \times \int_0^\infty y_2^{s+\frac{n_2}{2}+i_2+k_2-1} \left(1+\frac{n_1}{m\left(1-\gamma^2\right)}y_1+\frac{n_2}{m\left(1-\gamma^2\right)}y_2\right)^{-\left(\frac{n_1+n_2+m}{2}+i_1+i_2+k_1+k_2\right)} dy_2 dy_1 \\
&= \sum_{i_1=0}^\infty \sum_{i_2=0}^\infty \sum_{k_1=0}^\infty \sum_{k_2=0}^\infty C \left(\frac{m\left(1-\gamma^2\right)}{n_2}\right)^{\frac{n_1+n_2+m}{2}+i_1+i_2+k_1+k_2} \int_0^\infty y_1^{q+\frac{n_1}{2}+i_1+k_1-1} \\
&\quad \times \int_0^\infty y_2^{s+\frac{n_2}{2}+i_2+k_2-1} \left(\frac{m\left(1-\gamma^2\right)}{n_2}+\frac{n_1}{n_2}y_1+y_2\right)^{-\left(\frac{n_1+n_2+m}{2}+i_1+i_2+k_1+k_2\right)} dy_2 dy_1 \\
&\quad \times \frac{e^{-\frac{\theta_1}{2}}\left(\frac{\theta_1}{2}\right)^{k_1}}{k_1!} \frac{e^{-\frac{\theta_2}{2}}\left(\frac{\theta_2}{2}\right)^{k_2}}{k_2!} \\
&= \sum_{i_1=0}^\infty \sum_{i_2=0}^\infty \sum_{k_1=0}^\infty \sum_{k_2=0}^\infty C \frac{e^{-\frac{\theta_1}{2}}\left(\frac{\theta_1}{2}\right)^{k_1}}{k_1!} \frac{e^{-\frac{\theta_2}{2}}\left(\frac{\theta_2}{2}\right)^{k_2}}{k_2!} \left(\frac{m\left(1-\gamma^2\right)}{n_2}\right)^{\frac{n_1+n_2+m}{2}+i_1+i_2+k_1+k_2} \\
&\quad \times \int_0^\infty y_1^{q+\frac{n_1}{2}+i_1+k_1-1} \left(\frac{m\left(1-\gamma^2\right)+n_1 y_1}{n_2}\right)^{-\left(\frac{n_1+m}{2}+i_1+k_1-s\right)} dy_1 \\
&\quad \times B\left(s+\frac{n_2}{2}+i_2+k_2; \frac{n_1+m}{2}+i_1+k_1-s\right)
\end{aligned}
$$

which follows by applying Result C.16: setting $\alpha = s+\frac{n_2}{2}+i_2+k_2$, $\rho = \frac{n_1+n_2+m}{2}+i_1+i_2+k_1+k_2$, and $z = \frac{m\left(1-\gamma^2\right)}{n_2}+\frac{n_1}{n_2}y_1$. The remaining integral is solved by applying the same Result C.16: setting then $\alpha = q+\frac{n_1}{2}+i_1+k_1$, $\rho = \frac{n_1+m}{2}+i_1+k_1-s$, and $z = \frac{m\left(1-\gamma^2\right)}{n_1}$ -

and then, after some algebraic manipulation:

$$\begin{aligned}
E\left(Y_1^q Y_2^s\right) &= \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} C\frac{e^{-\frac{\theta_1}{2}}\left(\frac{\theta_1}{2}\right)^{k_1}}{k_1!}\frac{e^{-\frac{\theta_2}{2}}\left(\frac{\theta_2}{2}\right)^{k_2}}{k_2!}\left(\frac{m\left(1-\gamma^2\right)}{n_2}\right)^{\frac{n_1+n_2+m}{2}+i_1+i_2+k_1+k_2} \\
&\quad \times B\left(s+\frac{n_2}{2}+i_2+k_2;\frac{n_1+m}{2}+i_1+k_1-s\right) \\
&\quad \times \int_0^{\infty} y_1^{q+\frac{n_1}{2}+i_1+k_1-1}\left(\frac{n_2}{n_1}\right)^{\frac{n_1+m}{2}+i_1+k_1-s}\left(\frac{m\left(1-\gamma^2\right)}{n_1}+y_1\right)^{-\left(\frac{n_1+m}{2}+i_1+k_1-s\right)} dy_1 \\
&= \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} C\frac{e^{-\frac{\theta_1}{2}}\left(\frac{\theta_1}{2}\right)^{k_1}}{k_1!}\frac{e^{-\frac{\theta_2}{2}}\left(\frac{\theta_2}{2}\right)^{k_2}}{k_2!}\left(\frac{m\left(1-\gamma^2\right)}{n_2}\right)^{\frac{n_1+n_2+m}{2}+i_1+i_2+k_1+k_2} \\
&\quad \times B\left(s+\frac{n_2}{2}+i_2+k_2;\frac{n_1+m}{2}+i_1+k_1-s\right)\left(\frac{n_2}{n_1}\right)^{\frac{n_1+m}{2}+i_1+k_1-s} \\
&\quad \times \int_0^{\infty} y_1^{q+\frac{n_1}{2}+i_1+k_1-1}\left(\frac{m\left(1-\gamma^2\right)}{n_1}+y_1\right)^{-\left(\frac{n_1+m}{2}+i_1+k_1-s\right)} dy_1 \\
&= \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} C\frac{e^{-\frac{\theta_1}{2}}\left(\frac{\theta_1}{2}\right)^{k_1}}{k_1!}\frac{e^{-\frac{\theta_2}{2}}\left(\frac{\theta_2}{2}\right)^{k_2}}{k_2!}\left(\frac{m\left(1-\gamma^2\right)}{n_2}\right)^{\frac{n_1+n_2+m}{2}+i_1+i_2+k_1+k_2} \\
&\quad \times B\left(s+\frac{n_2}{2}+i_2+k_2;\frac{n_1+m}{2}+i_1+k_1-s\right)\left(\frac{n_2}{n_1}\right)^{\frac{n_1+m}{2}+i_1+k_1-s} \\
&\quad \times \left(\frac{m\left(1-\gamma^2\right)}{n_1}\right)^{q+s-\frac{m}{2}} B\left(q+\frac{n_1}{2}+i_1+k_1;\frac{m}{2}-q-s\right)
\end{aligned}$$

where $n_1, n_2, m > 0$, $-1 < \gamma < 1$, and $\theta_1, \theta_2 > 0$. After further simplification, the proof is completed. ∎

Remark 3.4 *The moments of the random variables can be calculated with above expression with the aim to investigate the correlation structure for this bivariate noncentral compound extended F distribution (see Chapter 5).*

3.4 Distributions of composites

3.4.1 Bivariate noncentral generalized F distribution

3.4.1.1 The probability density function of the product

In this section, the pdf of the product of the components of the bivariate distribution proposed in (3.6) is derived. Subsequently the noncentral case is proposed by using the compounding method (see (1.1)). Let $W_1 = Y_1 Y_2$.

Theorem 3.5 *If Y_1 and Y_2 are jointly distributed according to equation (3.6), the pdf of the conditional distribution of $W_1 | (K_1 = k_1, K_2 = k_2)$ is given by*

$$\begin{aligned} & f_{W_1}(w_1|k_1,k_2) \\ = & \sum_{j=0}^{\infty}\sum_{l_1=0}^{j}\sum_{l_2=0}^{j} C w_1^{-\left(\frac{m}{4}+1\right)} \left(\frac{n_1}{m}\right)^{-\left(\frac{n_1}{2}+\frac{m}{4}+l_1+k_1\right)} \left(\frac{n_2}{m}\right)^{-\left(\frac{n_2}{2}+\frac{m}{4}+l_2+k_2\right)} \\ & \times {}_2F_1\left(\frac{n_2}{2}+\frac{m}{4}+l_2+k_2, \frac{n_1}{2}+\frac{m}{4}+l_1+k_1; \frac{n_1+n_2+m+1}{2}+l_1+l_2+k_1+k_2; 1-\frac{m^2}{4w_1 n_1 n_2}\right), \quad w_1>0 \end{aligned} \tag{3.12}$$

where $n_1, n_2, m > 0$, $-1 \leq \xi \leq 1$, and ${}_2F_1(\cdot,\cdot;\cdot;\cdot)$ is the Gauss hypergeometric function (see Result C.7) with $\left|1 - \frac{m^2}{4w_1 n_1 n_2}\right| < 1$. The conditional values k_1 and k_2 have domain such that $k_v \geq 0$, $v = 1, 2$, and C is the value as given in (3.8).

Proof. The Jacobian of the transformation is given by $\frac{1}{y_2}$, and thus from (3.6) the joint pdf of W_1 and Y_2 is given by

$$
\begin{aligned}
& f_{W_1,Y_2}(w_1, y_2|k_1, k_2) \\
= \; & \frac{1}{y_2} f_{Y_1,Y_2}\left(\frac{w_1}{y_2}, y_2|k_1, k_2\right) \\
= \; & \frac{1}{y_2} \sum_{j=0}^{\infty} \sum_{l_1=0}^{j} \sum_{l_2=0}^{j} C \left(\frac{w_1}{y_2}\right)^{\frac{n_1}{2}+l_1+k_1-1} y_2^{\frac{n_2}{2}+l_2+k_2-1} \left(1 + \frac{n_1 \frac{w_1}{y_2} + n_2 y_2}{m}\right)^{-\left(\frac{n_1+n_2+m}{2}+l_1+l_2+k_1+k_2\right)} \\
= \; & \sum_{j=0}^{\infty} \sum_{l_1=0}^{j} \sum_{l_2=0}^{j} C w_1^{\frac{n_1}{2}+l_1+k_1-1} y_2^{\frac{n_2}{2}+l_2+k_2-1-\left(\frac{n_1}{2}+l_1+k_1-1\right)-1} \\
& \times \left(\left(\frac{1}{y_2}\right)\left(\frac{n_1 w_1 + n_2 y_2^2}{m} + y_2\right)\right)^{-\left(\frac{n_1+n_2+m}{2}+l_1+l_2+k_1+k_2\right)} \\
= \; & \sum_{j=0}^{\infty} \sum_{l_1=0}^{j} \sum_{l_2=0}^{j} C w_1^{\frac{n_1}{2}+l_1+k_1-1} y_2^{\frac{n_2-n_1}{2}+l_2-l_1+k_2-k_1-1+\frac{n_1+n_2+m}{2}+l_1+l_2+k_1+k_2} \\
& \times \left(\frac{n_1 w_1 + n_2 y_2^2}{m} + y_2\right)^{-\left(\frac{n_1+n_2+m}{2}+l_1+l_2+k_1+k_2\right)} \\
= \; & \sum_{j=0}^{\infty} \sum_{l_1=0}^{j} \sum_{l_2=0}^{j} C w_1^{\frac{n_1}{2}+l_1+k_1-1} \frac{y_2^{n_2+\frac{m}{2}+2l_2+2k_2-1}}{\left(\frac{n_2}{m} y_2^2 + y_2 + \frac{n_1 w_1}{m}\right)^{\frac{n_1+n_2+m}{2}+l_1+l_2+k_1+k_2}}
\end{aligned}
$$

Now, since $y_2 > 0$; the pdf of W_1 is given by

$$
\begin{aligned}
& f_{W_1}(w_1|k_1, k_2) \\
= \; & \int_0^{\infty} f_{W_1,Y_2}(w_1, y_2|k_1, k_2) dy_2 \\
= \; & \sum_{j=0}^{\infty} \sum_{l_1=0}^{j} \sum_{l_2=0}^{j} C w_1^{\frac{n_1}{2}+l_1+k_1-1} \int_0^{\infty} \frac{y_2^{n_2+\frac{m}{2}+2l_2+2k_2-1}}{\left(\frac{n_2}{m} y_2^2 + y_2 + \frac{n_1 w_1}{m}\right)^{\frac{n_1+n_2+m}{2}+l_1+l_2+k_1+k_2}} dy_2
\end{aligned}
$$

where this last integral is evaluated using Result C.14: setting $\rho = \frac{n_1+n_2+m}{2}+l_1+l_2+k_1+k_2$, $\alpha = n_2 + \frac{m}{2} + 2l_2 + 2k_2$, $a = \frac{n_2}{m}$, $b = \frac{1}{2}$, and $c = \frac{n_1 w_1}{m}$, one obtains

$$
\begin{aligned}
& f_{W_1}(w_1|k_1,k_2) \\
= & \sum_{j=0}^{\infty}\sum_{l_1=0}^{j}\sum_{l_2=0}^{j} C w_1^{\frac{n_1}{2}+l_1+k_1-1}\left(\frac{n_2}{m}\right)^{-\left(\frac{n_2}{2}+\frac{m}{4}+l_2+k_2\right)}\left(\frac{n_1 w_1}{m}\right)^{-\left(\frac{n_1}{2}+\frac{m}{4}+l_1+k_1\right)} \\
& \times {}_2F_1\left(\frac{n_2}{2}+\frac{m}{4}+l_2+k_2, \frac{n_1}{2}+\frac{m}{4}+l_1+k_1; \frac{n_1+n_2+m+1}{2}+l_1+l_2+k_1+k_2; 1-\frac{\left(\frac{1}{2}\right)^2}{\left(\frac{w_1 n_1 n_2}{m^2}\right)}\right) \\
= & \sum_{j=0}^{\infty}\sum_{l_1=0}^{j}\sum_{l_2=0}^{j} C w_1^{-\left(\frac{m}{4}+1\right)}\left(\frac{n_1}{m}\right)^{-\left(\frac{n_1}{2}+\frac{m}{4}+l_1+k_1\right)}\left(\frac{n_2}{m}\right)^{-\left(\frac{n_2}{2}+\frac{m}{4}+l_2+k_2\right)} \\
& \times\ {}_2F_1\left(\frac{n_2}{2}+\frac{m}{4}+l_2+k_2, \frac{n_1}{2}+\frac{m}{4}+l_1+k_1; \frac{n_1+n_2+m+1}{2}+l_1+l_2+k_1+k_2; 1-\frac{m^2}{4w_1 n_1 n_2}\right)
\end{aligned}
$$

which completes the proof. ■

Description 3.8 *Upon taking the pdf in equation (3.12) one can now obtain the (unconditional) noncentral distribution of $W_1 = Y_1 Y_2$ by substituting the Poisson probabilities and the corresponding summation operators (stemming from (3.5)):*

$$
\begin{aligned}
& f_{W_1}(w_1) \\
= & f_{W_1}(w_1|k_1,k_2) g_{K_1}(k_1) g_{K_2}(k_2) \\
= & \sum_{j=0}^{\infty}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{l_1=0}^{j}\sum_{l_2=0}^{j} C\left(\frac{n_1}{m}\right)^{-\left(\frac{n_1}{2}+\frac{m}{4}+l_1+k_1\right)}\left(\frac{n_2}{m}\right)^{-\left(\frac{n_2}{2}+\frac{m}{4}+l_2+k_2\right)} w_1^{-\left(\frac{m}{4}+1\right)} \\
& \times\ {}_2F_1\left(\frac{n_2}{2}+\frac{m}{4}+l_2+k_2, \frac{n_1}{2}+\frac{m}{4}+l_1+k_1; \frac{n_1+n_2+m+1}{2}+l_1+l_2+k_1+k_2; 1-\frac{m^2}{4w_1 n_1 n_2}\right) \\
& \times \frac{e^{-\frac{\theta_1}{2}}\left(\frac{\theta_1}{2}\right)^{k_1}}{k_1!}\frac{e^{-\frac{\theta_2}{2}}\left(\frac{\theta_2}{2}\right)^{k_2}}{k_2!}, \qquad w_1 > 0
\end{aligned}
\tag{3.13}
$$

where $n_1, n_2, m > 0$, $-1 \leq \xi \leq 1$, $\theta_1, \theta_2 > 0$, and $g_{K_v}(k_v) = \frac{e^{-\frac{\theta_v}{2}}\left(\frac{\theta_v}{2}\right)^{k_v}}{k_v!}$ for $v = 1, 2$, and $\left|1 - \frac{m^2}{4w_1 n_1 n_2}\right| < 1$. ▲

3.4.1.2 The probability density function of the ratio of type II

Here the pdf of the ratio of type II of the components of the bivariate distribution proposed in (3.6) is derived. Subsequently the noncentral case is proposed by using the compounding method (see (1.1)). Let $W_2 = \frac{Y_1}{Y_2}$.

Theorem 3.6 *If Y_1 and Y_2 are jointly distributed according to equation (3.6), the pdf of the conditional distribution of $W_2|\,(K_1 = k_1, K_2 = k_2)$ is given by*

$$
\begin{aligned}
f_{W_2}(w_2|k_1,k_2) &= \sum_{j=0}^{\infty}\sum_{l_1=0}^{j}\sum_{l_2=0}^{j} C w_2^{\frac{n_1}{2}+l_1+k_1-1}\left(\frac{w_2 n_1 + n_2}{m}\right)^{\frac{n_1+n_2}{2}+l_1+l_2+k_1+k_2} \\
&\quad \times B\left(\frac{n_1+n_2}{2}+l_1+l_2+k_1+k_2, \frac{m}{2}\right), \qquad w_2 > 0 \qquad (3.14)
\end{aligned}
$$

where $n_1, n_2, m > 0$, and $-1 \leq \xi \leq 1$. The conditional values k_1 and k_2 have domain such that $k_v \geq 0$, $v = 1, 2$, C is the value as given in (3.8), and $B(\cdot,\cdot)$ is the beta function as defined in Result C.4.

Proof. The Jacobian of the transformation is again given by y_2, and thus from (3.6) the joint pdf of W_2 and Y_2 is given by

$$
\begin{aligned}
f_{W_2,Y_2}(w_2,y_2|k_1,k_2) &= y_2 f_{Y_1,Y_2}\left(w_2 y_2, y_2|k_1,k_2\right) \\
&= y_2 \sum_{j=0}^{\infty}\sum_{l_1=0}^{j}\sum_{l_2=0}^{j} C\left(w_2 y_2\right)^{\frac{n_1}{2}+l_1+k_1-1} y_2^{\frac{n_2}{2}+l_2+k_2-1} \\
&\quad \times \left(1+\frac{n_1 w_2 y_2 + n_2 y_2}{m}\right)^{-\left(\frac{n_1+n_2+m}{2}+l_1+l_2+k_1+k_2\right)} \\
&= \sum_{j=0}^{\infty}\sum_{l_1=0}^{j}\sum_{l_2=0}^{j} C w_2^{\frac{n_1}{2}+l_1+k_1-1} y_2^{\frac{n_1+n_2}{2}+l_1+l_2+k_1+k_2-1} \\
&\quad \times \left(1+\frac{n_1 w_2 + n_2}{m} y_2\right)^{-\left(\frac{n_1+n_2+m}{2}+l_1+l_2+k_1+k_2\right)}.
\end{aligned}
$$

Now, since $y_2 > 0$; the pdf of W_2 is given by

$$\begin{aligned}
f_{W_2}(w_2|k_1,k_2) &= \int_0^\infty f_{W_2,Y_2}(w_2,y_2|k_1,k_2)dy_2 \\
&= \sum_{j=0}^{\infty}\sum_{l_1=0}^{j}\sum_{l_2=0}^{j} C w_2^{\frac{n_1}{2}+l_1+k_1-1} \\
&\quad \times \int_0^\infty y_2^{\frac{n_1+n_2}{2}+l_1+l_2+k_1+k_2-1}\left(1+\frac{n_1w_2+n_2}{m}y_2\right)^{-\left(\frac{n_1+n_2+m}{2}+l_1+l_2+k_1+k_2\right)} dy_2.
\end{aligned}$$

Let $I = \int_0^\infty y_2^{\frac{n_1+n_2}{2}+l_1+l_2+k_1+k_2-1}\left(1+\frac{n_1w_2+n_2}{m}y_2\right)^{-\left(\frac{n_1+n_2+m}{2}+l_1+l_2+k_1+k_2\right)} dy_2$. This integral is simplified by applying Result C.17: setting $\beta = \frac{n_1w_2+n_2}{m}$, $\mu = \frac{n_1+n_2}{2}+l_1+l_2+k_1+k_2$, $\kappa = \frac{n_1+n_2+m}{2}+l_1+l_2+k_1+k_2$, and by substituting the result back, one obtains

$$\begin{aligned}
f_{W_2}(w_2|k_1,k_2) &= \sum_{j=0}^{\infty}\sum_{l_1=0}^{j}\sum_{l_2=0}^{j} C w_2^{\frac{n_1}{2}+l_1+k_1-1}\left(\frac{n_1w_2+n_2}{m}\right)^{-\left(\frac{n_1+n_2}{2}+l_1+l_2+k_1+k_2\right)} \\
&\quad \times B\left(\frac{n_1+n_2}{2}+l_1+l_2+k_1+k_2,\frac{m}{2}\right)
\end{aligned}$$

which completes the proof. ■

Description 3.9 *Upon taking the pdf in equation (3.14) one can now obtain the (unconditional) noncentral distribution of* $W_2 = \frac{Y_1}{Y_2}$ *by substituting the Poisson probabilities and the corresponding summation operators (stemming from (3.5)):*

$$\begin{aligned}
f_{W_2}(w_2) &= f_{W_2}(w_2|k_1,k_2)g_{K_1}(k_1)g_{K_2}(k_2) \\
&= \sum_{j=0}^{\infty}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{l_1=0}^{j}\sum_{l_2=0}^{j} C w_2^{\frac{n_1}{2}+l_1+k_1-1}\left(\frac{n_1w_2+n_2}{m}\right)^{-\left(\frac{n_1+n_2}{2}+l_1+l_2+k_1+k_2\right)} \\
&\quad \times B\left(\frac{n_1+n_2}{2}+l_1+l_2+k_1+k_2,\frac{m}{2}\right) \\
&\quad \times \frac{e^{-\frac{\theta_1}{2}}\left(\frac{\theta_1}{2}\right)^{k_1}}{k_1!}\frac{e^{-\frac{\theta_2}{2}}\left(\frac{\theta_2}{2}\right)^{k_2}}{k_2!}, \qquad w_2 > 0 \qquad (3.15)
\end{aligned}$$

where $n_1, n_2, m > 0$, $-1 \le \xi \le 1$, $\theta_1, \theta_2 > 0$, *and* $g_{K_v}(k_v) = \frac{e^{-\frac{\theta_v}{2}}\left(\frac{\theta_v}{2}\right)^{k_v}}{k_v!}$ *for* $v = 1, 2$. ▲

3.4.1.3 The probability density function of the ratio of type I

Here the pdf of the ratio of type I of the components of the bivariate distribution proposed in (3.6) is derived. Subsequently the noncentral case is proposed by using the compounding method (see (1.1)). Let $W_3 = \frac{Y_1}{Y_1+Y_2}$.

Theorem 3.7 *If Y_1 and Y_2 are jointly distributed according to equation (3.6), the pdf of the conditional distribution of $W_3|\,(K_1 = k_1, K_2 = k_2)$ is given by*

$$\begin{aligned} & f_{W_3}(w_3|k_1,k_2) \\ = & \sum_{j=0}^{\infty}\sum_{l_1=0}^{j}\sum_{l_2=0}^{j} C w_3^{\frac{n_1}{2}+l_1+k_1-1}(1-w_3)^{-\left(\frac{n_1}{2}+l_1+k_1+1\right)}\left(\frac{n_1\frac{w_3}{1-w_3}+n_2}{m}\right)^{\frac{n_1+n_2}{2}+l_1+l_2+k_1+k_2} \\ & \times B\left(\frac{n_1+n_2}{2}+l_1+l_2+k_1+k_2,\frac{m}{2}\right), \qquad 0<w_3<1 \end{aligned} \tag{3.16}$$

where $n_1, n_2, m > 0$, and $-1 \leq \xi \leq 1$. C is the value as given in (3.8), $B(\cdot,\cdot)$ is the beta function as defined in Result C.4, and the conditional values k_1 and k_2 have domain such that $k_v \geq 0$, $v = 1, 2$.

Proof. By using Result C.1 and via substitution, the following is obtained:

$$\begin{aligned} & f_{W_3}(w_3|k_1,k_2) \\ = & \sum_{j=0}^{\infty}\sum_{l_1=0}^{j}\sum_{l_2=0}^{j} C\left(\frac{w_3}{1-w_3}\right)^{\frac{n_1}{2}+l_1+k_1-1}\left(\frac{n_1\frac{w_3}{1-w_3}+n_2}{m}\right)^{\frac{n_1+n_2}{2}+l_1+l_2+k_1+k_2}\frac{1}{(1-w_3)^2} \\ & \times B\left(\frac{n_1+n_2}{2}+l_1+l_2+k_1+k_2,\frac{m}{2}\right) \\ = & \sum_{j=0}^{\infty}\sum_{l_1=0}^{j}\sum_{l_2=0}^{j} C w_3^{\frac{n_1}{2}+l_1+k_1-1}(1-w_3)^{-\left(\frac{n_1}{2}+l_1+k_1+1\right)}\left(\frac{n_1\frac{w_3}{1-w_3}+n_2}{m}\right)^{\frac{n_1+n_2}{2}+l_1+l_2+k_1+k_2} \\ & \times B\left(\frac{n_1+n_2}{2}+l_1+l_2+k_1+k_2,\frac{m}{2}\right) \end{aligned}$$

which completes the proof. ■

Description 3.10 *Upon taking the pdf in equation (3.16) one can now obtain the (unconditional) noncentral distribution of $W_3 = \frac{Y_1}{Y_1+Y_2}$ by substitution the Poisson probabilities* *and the corresponding summation operators (stemming from (3.5)):*

$$\begin{aligned}
& f_{W_3}(w_3) \\
= \; & f_{W_3}(w_3|k_1,k_2)g_{K_1}(k_1)g_{K_2}(k_2) \\
= \; & \sum_{j=0}^{\infty}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{l_1=0}^{j}\sum_{l_2=0}^{j} C w_3^{\frac{n_1}{2}+l_1+k_1-1}(1-w_3)^{-\left(\frac{n_1}{2}+l_1+k_1+1\right)}\left(\frac{n_1\frac{w_3}{1-w_3}+n_2}{m}\right)^{\frac{n_1+n_2}{2}+l_1+l_2+k_1+k_2} \\
& \times B\left(\frac{n_1+n_2}{2}+l_1+l_2+k_1+k_2,\frac{m}{2}\right)\frac{e^{-\frac{\theta_1}{2}}\left(\frac{\theta_1}{2}\right)^{k_1}}{k_1!}\frac{e^{-\frac{\theta_2}{2}}\left(\frac{\theta_2}{2}\right)^{k_2}}{k_2!}, \qquad 0<w_3<1 \qquad (3.17)
\end{aligned}$$

where $n_1, n_2, m > 0$, $-1 \le \xi \le 1$, $\theta_1, \theta_2 > 0$, and $g_{K_v}(k_v) = \frac{e^{-\frac{\theta_v}{2}}\left(\frac{\theta_v}{2}\right)^{k_v}}{k_v!}$ for $v = 1, 2$. ▲

3.4.2 Bivariate noncentral compound extended F distribution: unequal degrees of freedom

3.4.2.1 The probability density function of the product

In this section, the pdf of the product of the components of the bivariate distribution proposed in (3.9) is derived. Subsequently the noncentral case is proposed by using the compounding method (see (1.1)). Let $W_1 = Y_1Y_2$.

Theorem 3.8 *If Y_1 and Y_2 are jointly distributed according to equation (3.9), the pdf of the conditional distribution of $W_1|\,(K_1 = k_1, K_2 = k_2)$ is given by*

$$\begin{aligned}
& f_{W_1}(w_1|k_1,k_2) \\
= \; & \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\frac{\Gamma\left(\frac{n_1}{2}+i_1\right)}{\Gamma\left(\frac{n_1}{2}\right)i_1!}\frac{\Gamma\left(\frac{n_2}{2}+i_2\right)}{\Gamma\left(\frac{n_2}{2}\right)i_2!}\gamma^{2(i_1+i_2)}\left(1-\gamma^2\right)^{\frac{n_1+n_2+m}{2}} \\
& \times\frac{\Gamma\left(\frac{n_1+n_2+m}{2}+k_1+k_2+i_1+i_2\right)}{\Gamma\left(\frac{n_1}{2}+k_1+i_1\right)\Gamma\left(\frac{n_2}{2}+k_2+i_2\right)\Gamma\left(\frac{m}{2}\right)}w_1^{-\left(\frac{m}{4}+1\right)}\left(\frac{n_1}{m}\right)^{-\frac{m}{4}}\left(\frac{n_2}{m}\right)^{-\frac{m}{4}} \\
& \times\,{}_2F_1\left(\frac{n_2}{2}+\frac{m}{4}+i_2+k_2,\frac{n_1}{2}+\frac{m}{4}+i_1+k_1;\frac{n_1+n_2+m+1}{2}+i_1+i_2+k_1+k_2;1-\frac{\left(m\left(1-\gamma^2\right)\right)^2}{4w_1n_1n_2}\right),\; w_1>0
\end{aligned}$$
$$(3.18)$$

where $n_1, n_2, m > 0$, $-1 < \gamma < 1$, and ${}_2F_1(\cdot,\cdot;\cdot;\cdot)$ is the Gauss hypergeometric function (see Result C.7) where $\left|1-\frac{\left(m\left(1-\gamma^2\right)\right)^2}{4w_1n_1n_2}\right| < 1$. The conditional values k_1 and k_2 have domain such that $k_v \ge 0$, $v = 1, 2$.

Proof. The Jacobian of the transformation is given by $\frac{1}{y_2}$, and thus from ((3.9), with C the value defined in (3.11)) the joint pdf of W_1 and Y_2 is given by

$$\begin{aligned}
& f_{W_1,Y_2}(w_1, y_2|k_1,k_2) \\
= \;& \frac{1}{y_2} f_{Y_1,Y_2}\left(\frac{w_1}{y_2}, y_2|k_1,k_2\right) \\
= \;& \frac{1}{y_2}\sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty} C\left(\frac{w_1}{y_2}\right)^{\frac{n_1}{2}+i_1+k_1-1} y_2^{\frac{n_2}{2}+i_2+k_2-1}\left(\frac{n_1\frac{w_1}{y_2}+n_2y_2}{m\left(1-\gamma^2\right)}+1\right)^{-\left(\frac{n_1+n_2+m}{2}+i_1+i_2+k_1+k_2\right)} \\
= \;& \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty} C w_1^{\frac{n_1}{2}+k_1+i_1-1} y_2^{\frac{n_2}{2}+k_2+i_2-1-\left(\frac{n_1}{2}+k_1+i_1-1\right)-1} \\
& \times\left(\left(\frac{1}{y_2}\right)\left(\frac{n_1w_1+n_2y_2^2}{m\left(1-\gamma^2\right)}+y_2\right)\right)^{-\left(\frac{n_1+n_2+m}{2}+i_1+i_2+k_1+k_2\right)} \\
= \;& \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty} C w_1^{\frac{n_1}{2}+i_1+k_1-1} y_2^{\frac{n_2-n_1}{2}+i_2-i_1+k_2-k_1-1+\frac{n_1+n_2+m}{2}+i_1+i_2+k_1+k_2} \\
& \times\left(\frac{n_1w_1+n_2y_2^2}{m\left(1-\gamma^2\right)}+y_2\right)^{-\left(\frac{n_1+n_2+m}{2}+i_1+i_2+k_1+k_2\right)} \\
= \;& \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty} C w_1^{\frac{n_1}{2}+i_1+k_1-1} \frac{y_2^{n_2+\frac{m}{2}+2i_2+2k_2-1}}{\left(\frac{n_2}{m(1-\gamma^2)}y_2^2+y_2+\frac{w_1n_1}{m(1-\gamma^2)}\right)^{\frac{n_1+n_2+m}{2}+i_1+i_2+k_1+k_2}}
\end{aligned}$$

Now, since $y_2 > 0$; the pdf of W_1 is given by

$$\begin{aligned}
& f_{W_1}(w_1|k_1,k_2) \\
= \;& \int_0^{\infty} f_{W_1,Y_2}(w_1,y_2|k_1,k_2)dy_2 \\
= \;& \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty} C w_1^{\frac{n_1}{2}+i_1+k_1-1}\int_0^{\infty}\frac{y_2^{n_2+\frac{m}{2}+2i_2+2k_2-1}}{\left(\frac{n_2}{m(1-\gamma^2)}y_2^2+y_2+\frac{n_1w_1}{m(1-\gamma^2)}\right)^{\frac{n_1+n_2+m}{2}+i_1+i_2+k_1+k_2}}dy_2
\end{aligned}$$

where this last integral is evaluated using Result C.14: setting $\rho = \frac{n_1+n_2+m}{2}+i_1+i_2+k_1+k_2$, $\alpha = n_2 + \frac{m}{2} + 2i_2 + 2k_2$, $a = \frac{n_2}{m(1-\gamma^2)}$, $b = \frac{1}{2}$, and $c = \frac{n_1 w_1}{m(1-\gamma^2)}$, one obtains

$$
\begin{aligned}
& f_{W_1}(w_1|k_1,k_2) \\
= \; & \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty} C w_1^{\frac{n_1}{2}+i_1+k_1-1}\left(\frac{n_2}{m\left(1-\gamma^2\right)}\right)^{-\left(\frac{n_2}{2}+\frac{m}{4}+i_2+k_2\right)}\left(\frac{n_1 w_1}{m\left(1-\gamma^2\right)}\right)^{-\left(\frac{n_1}{2}+\frac{m}{4}+i_1+k_1\right)} \\
& \times {}_2F_1\left(\frac{n_2}{2}+\frac{m}{4}+i_2+k_2, \frac{n_1}{2}+\frac{m}{4}+i_1+k_1; \frac{n_1+n_2+m+1}{2}+i_1+i_2+k_1+k_2; 1-\frac{\left(\frac{1}{2}\right)^2}{\left(\frac{w_1 n_1 n_2}{\left(m(1-\gamma^2)\right)^2}\right)}\right) \\
= \; & \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty} C w_1^{-\left(\frac{m}{4}+1\right)}\left(\frac{n_1}{m\left(1-\gamma^2\right)}\right)^{-\left(\frac{n_1}{2}+\frac{m}{4}+i_1+k_1\right)}\left(\frac{n_2}{m\left(1-\gamma^2\right)}\right)^{-\left(\frac{n_2}{2}+\frac{m}{4}+i_2+k_2\right)} \\
& \times {}_2F_1\left(\frac{n_2}{2}+\frac{m}{4}+i_2+k_2, \frac{n_1}{2}+\frac{m}{4}+i_1+k_1; \frac{n_1+n_2+m+1}{2}+i_1+i_2+k_1+k_2; 1-\frac{\left(m\left(1-\gamma^2\right)\right)^2}{4w_1 n_1 n_2}\right) \\
= \; & \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty} \frac{\Gamma\left(\frac{n_1}{2}+i_1\right)}{\Gamma\left(\frac{n_1}{2}\right) i_1!}\frac{\Gamma\left(\frac{n_2}{2}+i_2\right)}{\Gamma\left(\frac{n_2}{2}\right) i_2!}\gamma^{2(i_1+i_2)}\left(1-\gamma^2\right)^{\frac{n_1+n_2+m}{2}} \\
& \times \frac{\Gamma\left(\frac{n_1+n_2+m}{2}+i_1+i_2+k_1+k_2\right)}{\Gamma\left(\frac{n_1}{2}+i_1+k_1\right)\Gamma\left(\frac{n_2}{2}+i_2+k_2\right)\Gamma\left(\frac{m}{2}\right)}\left(\frac{n_1}{m}\right)^{-\frac{m}{4}}\left(\frac{n_2}{m}\right)^{-\frac{m}{4}} w_1^{-\left(\frac{m}{4}+1\right)} \\
& \times {}_2F_1\left(\frac{n_2}{2}+\frac{m}{4}+i_2+k_2, \frac{n_1}{2}+\frac{m}{4}+i_1+k_1; \frac{n_1+n_2+m+1}{2}+i_1+i_2+k_1+k_2; 1-\frac{\left(m\left(1-\gamma^2\right)\right)^2}{4w_1 n_1 n_2}\right)
\end{aligned}
$$

which completes the proof. ∎

Description 3.11 *Upon taking the pdf in equation (3.18) one can now obtain the (unconditional) noncentral distribution of $W_1 = Y_1 Y_2$ by substituting the Poisson probabilities and the corresponding summation operators (stemming from (3.5)):*

$$
\begin{aligned}
& f_{W_1}(w_1) \\
= \; & f_{W_1}(w_1|k_1,k_2) g_{K_1}(k_1) g_{K_2}(k_2) \\
= \; & \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} \frac{\Gamma\left(\frac{n_1}{2}+i_1\right)}{\Gamma\left(\frac{n_1}{2}\right) i_1!}\frac{\Gamma\left(\frac{n_2}{2}+i_2\right)}{\Gamma\left(\frac{n_2}{2}\right) i_2!}\gamma^{2(i_1+i_2)}\left(1-\gamma^2\right)^{\frac{n_1+n_2+m}{2}} \\
& \times \frac{\Gamma\left(\frac{n_1+n_2+m}{2}+i_1+i_2+k_1+k_2\right)}{\Gamma\left(\frac{n_1}{2}+i_1+k_1\right)\Gamma\left(\frac{n_2}{2}+i_2+k_2\right)\Gamma\left(\frac{m}{2}\right)}\left(\frac{n_1}{m}\right)^{-\frac{m}{4}}\left(\frac{n_2}{m}\right)^{-\frac{m}{4}} w_1^{-\left(\frac{m}{4}+1\right)} \\
& \times {}_2F_1\left(\frac{n_2}{2}+\frac{m}{4}+i_2+k_2, \frac{n_1}{2}+\frac{m}{4}+i_1+k_1; \frac{n_1+n_2+m+1}{2}+i_1+i_2+k_1+k_2; 1-\frac{\left(m\left(1-\gamma^2\right)\right)^2}{4w_1 n_1 n_2}\right) \\
& \times \frac{e^{-\frac{\theta_1}{2}}\left(\frac{\theta_1}{2}\right)^{k_1}}{k_1!}\frac{e^{-\frac{\theta_2}{2}}\left(\frac{\theta_2}{2}\right)^{k_2}}{k_2!}, \qquad w_1 > 0
\end{aligned}
\tag{3.19}
$$

where $n_1, n_2, m > 0$, $-1 < \gamma < 1$, $\theta_1, \theta_2 > 0$, $g_{K_v}(k_v) = \frac{e^{\frac{\theta_v}{2}}\left(\frac{\theta_v}{2}\right)^{k_v}}{k_v!}$ *for* $v = 1, 2$, *and* $\left|1 - \frac{\left(m\left(1-\gamma^2\right)\right)^2}{4w_1 n_1 n_2}\right| < 1$. ▲

3.4.2.2 The probability density function of the ratio of type II

Here the pdf of the ratio of type II of the components of the bivariate distribution proposed in (3.9) is given. Subsequently the noncentral case is proposed by using the compounding method (see (1.1)). Let $W_2 = \frac{Y_1}{Y_2}$.

Theorem 3.9 *If* Y_1 *and* Y_2 *are jointly distributed according to equation (3.9), the pdf of the conditional distribution of* $W_2|\left(K_1 = k_1, K_2 = k_2\right)$ *is given by*

$$\begin{aligned}
& f_{W_2}(w_2|k_1,k_2) \\
= & \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty} C w_2^{\frac{n_1}{2}+i_1+k_1-1}\left(\frac{n_2 w_2 + n_1}{m\left(1-\gamma^2\right)}\right)^{\frac{n_1+n_2}{2}+i_1+i_2+k_1+k_2} \\
& \times B\left(\frac{n_1+n_2}{2}+i_1+i_2+k_1+k_2, \frac{m}{2}\right), \qquad w_2 > 0
\end{aligned} \tag{3.20}$$

where $n_1, n_2, m > 0$, *and* $-1 < \gamma < 1$. C *is the value as given in (3.11),* $B(\cdot,\cdot)$ *is the beta function as defined in Result C.4, and the conditional values* k_1 *and* k_2 *have domain such that* $k_v \geq 0$, $v = 1, 2$.

Proof. The Jacobian of the transformation is again given by y_2, and thus from ((3.9), with C the value defined in (3.11)) the joint pdf of W_2 and Y_2 is given by

$$\begin{aligned}
f_{W_2,Y_2}(w_2,y_2|k_1,k_2) &= y_2 f_{Y_1,Y_2}\left(w_2 y_2, y_2|k_1,k_2\right) \\
&= y_2 \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty} C\left(w_2 y_2\right)^{\frac{n_1}{2}+i_1+k_1-1} y_2^{\frac{n_2}{2}+i_2+k_2-1} \\
&\quad \times \left(1+\frac{n_2 w_2 y_2 + n_1 y_2}{m\left(1-\gamma^2\right)}\right)^{-\left(\frac{n_1+n_2+m}{2}+i_1+i_2+k_1+k_2\right)} \\
&= \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty} C w_2^{\frac{n_1}{2}+i_1+k_1-1} y_2^{\frac{n_1+n_2}{2}+i_1+i_2+k_1+k_2-1} \\
&\quad \times \left(1+\frac{n_2 w_2 + n_1}{m\left(1-\gamma^2\right)} y_2\right)^{-\left(\frac{n_1+n_2+m}{2}+i_1+i_2+k_1+k_2\right)}.
\end{aligned}$$

Now, since $y_2 > 0$; the pdf of W_2 is given by

$$
\begin{aligned}
& f_{W_2}(w_2|k_1,k_2) \\
= & \int_0^\infty f_{W_2,Y_2}(w_2,y_2|k_1,k_2)dy_2 \\
= & \sum_{i_1=0}^\infty \sum_{i_2=0}^\infty C w_2^{\frac{n_1}{2}+i_1+k_1-1} \int_0^\infty y_2^{\frac{n_1+n_2}{2}+i_1+i_2+k_1+k_2-1} \left(1+\frac{n_2w_2+n_1}{m\left(1-\gamma^2\right)}y_2\right)^{-\left(\frac{n_1+n_2+m}{2}+i_1+i_2+k_1+k_2\right)} dy_2.
\end{aligned}
$$

Let $I = \int_0^\infty y_2^{\frac{n_1+n_2}{2}+i_1+i_2+k_1+k_2-1} \left(1+\frac{n_2w_2+n_1}{m(1-\gamma^2)}y_2\right)^{-\left(\frac{n_1+n_2+m}{2}+i_1+i_2+k_1+k_2\right)} dy_2$. This integral is simplified by applying Result C.17: setting $\beta = \frac{n_2w_2+n_1}{m(1-\gamma^2)}$, $\mu = \frac{n_1+n_2}{2}+i_1+i_2+k_1+k_2$, $\kappa = \frac{n_1+n_2+m}{2}+i_1+i_2+k_1+k_2$, and by substituting the result back, one obtains

$$
\begin{aligned}
f_{W_2}(w_2|k_1,k_2) \quad = \quad & \sum_{i_1=0}^\infty \sum_{i_2=0}^\infty C w_2^{\frac{n_1}{2}+i_1+k_1-1} \left(\frac{n_2w_2+n_1}{m\left(1-\gamma^2\right)}\right)^{-\left(\frac{n_1+n_2}{2}+i_1+i_2+k_1+k_2\right)} \\
& \times B\left(\frac{n_1+n_2}{2}+i_1+i_2+k_1+k_2, \frac{m}{2}\right)
\end{aligned}
$$

which completes the proof. ■

Description 3.12 *Upon taking the pdf in equation (3.20) one can now obtain the (unconditional) noncentral distribution of* $W_2 = \frac{Y_1}{Y_2}$ *by substituting the Poisson probabilities and the corresponding summation operators (stemming from (3.5)):*

$$
\begin{aligned}
f_{W_2}(w_2) \quad = \quad & f_{W_2}(w_2|k_1,k_2)g_{K_1}(k_1)g_{K_2}(k_2) \\
= \quad & \sum_{i_1=0}^\infty \sum_{i_2=0}^\infty \sum_{k_1=0}^\infty \sum_{k_2=0}^\infty C w_2^{\frac{n_1}{2}+i_1+k_1-1} \left(\frac{n_2w_2+n_1}{m\left(1-\gamma^2\right)}\right)^{-\left(\frac{n_1+n_2}{2}+i_1+i_2+k_1+k_2\right)} \\
& \times B\left(\frac{n_1+n_2}{2}+i_1+i_2+k_1+k_2, \frac{m}{2}\right) \\
& \times \frac{e^{-\frac{\theta_1}{2}}\left(\frac{\theta_1}{2}\right)^{k_1}}{k_1!} \frac{e^{-\frac{\theta_2}{2}}\left(\frac{\theta_2}{2}\right)^{k_2}}{k_2!}, \qquad w_2 > 0
\end{aligned}
\tag{3.21}
$$

where $n_1, n_2, m > 0$, $-1 < \gamma < 1$, $\theta_1, \theta_2 > 0$ *and* $g_{K_v}(k_v) = \frac{e^{-\frac{\theta_v}{2}}\left(\frac{\theta_v}{2}\right)^{k_v}}{k_v!}$ *for* $v = 1, 2$. ▲

3.4.2.3 The probability density function of the ratio of type I

Here the pdf of the proportion of the components of the bivariate distribution proposed in (3.9) is derived. Subsequently the noncentral case is proposed by using the compounding method (see (1.1)). Let $W_3 = \frac{Y_1}{Y_1+Y_2}$.

Theorem 3.10 *If Y_1 and Y_2 are jointly distributed according to equation (3.9), the pdf of the conditional distribution of $W_3|\left(K_1 = k_1, K_2 = k_2\right)$ is given by*

$$\begin{aligned} f_{W_3}(w_3|k_1,k_2) &= \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty} C w_3^{\frac{n_1}{2}+i_1+k_1-1}(1-w_3)^{-\left(\frac{n_1}{2}+i_1+k_1+1\right)}\left(\frac{n_2\frac{w_3}{1-w_3}+n_1}{m\left(1-\gamma^2\right)}\right)^{-\left(\frac{n_1+n_2}{2}+i_1+i_2+k_1+k_2\right)} \\ &\quad \times B\left(\frac{n_1+n_2}{2}+i_1+i_2+k_1+k_2, \frac{m}{2}\right), \qquad 0 < w_3 < 1 \end{aligned} \tag{3.22}$$

where $n_1, n_2, m > 0$, and $-1 < \gamma < 1$. C is the value as given in (3.11), $B(\cdot,\cdot)$ is the beta function as defined in Result C.4, and the conditional values k_1 and k_2 have domain such that $k_v \geq 0$, $v = 1, 2$.

Proof. By using Result C.1 and via substitution, the following is obtained:

$$\begin{aligned} f_{W_3}(w_3|k_1,k_2) &= \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty} C\left(\frac{w_3}{1-w_3}\right)^{\frac{n_1}{2}+i_1+k_1-1}\left(\frac{n_2\frac{w_3}{1-w_3}+n_1}{m\left(1-\gamma^2\right)}\right)^{\frac{n_1+n_2}{2}+i_1+i_2+k_1+k_2}\frac{1}{(1-w_3)^2} \\ &\quad \times B\left(\frac{n_1+n_2}{2}+i_1+i_2+k_1+k_2, \frac{m}{2}\right) \\ &= \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty} C w_3^{\frac{n_1}{2}+i_1+k_1-1}(1-w_3)^{-\left(\frac{n_1}{2}+i_1+k_1+1\right)}\left(\frac{n_2\frac{w_3}{1-w_3}+n_1}{m\left(1-\gamma^2\right)}\right)^{-\left(\frac{n_1+n_2}{2}+i_1+i_2+k_1+k_2\right)} \\ &\quad \times B\left(\frac{n_1+n_2}{2}+i_1+i_2+k_1+k_2, \frac{m}{2}\right) \end{aligned}$$

which completes the proof. ■

Description 3.13 *Upon taking the pdf in equation (3.22) one can now obtain the (unconditional) noncentral distribution of $W_3 = \frac{Y_1}{Y_1+Y_2}$ by substituting the Poisson probabilities and the corresponding summation operators (stemming from (3.5)):*

$$\begin{aligned} & f_{W_3}(w_3) \\ =\ & f_{W_3}(w_3|k_1,k_2)g_{K_1}(k_1)g_{K_2}(k_2) \\ =\ & \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} C w_3^{\frac{n_1}{2}+i_1+k_1-1}(1-w_3)^{-\left(\frac{n_1}{2}+i_1+k_1+1\right)}\left(\frac{n_2\frac{w_3}{1-w_3}+n_1}{m\left(1-\gamma^2\right)}\right)^{-\left(\frac{n_1+n_2}{2}+i_1+i_2+k_1+k_2\right)} \\ & \times B\left(\frac{n_1+n_2}{2}+i_1+i_2+k_1+k_2, \frac{m}{2}\right)\frac{e^{-\frac{\theta_1}{2}}\left(\frac{\theta_1}{2}\right)^{k_1}}{k_1!}\frac{e^{-\frac{\theta_2}{2}}\left(\frac{\theta_2}{2}\right)^{k_2}}{k_2!}, \qquad 0 < w_3 < 1 \end{aligned} \tag{3.23}$$

where $n_1, n_2, m > 0$, $-1 < \gamma < 1$, $\theta_1, \theta_2 > 0$, and $g_{K_v}(k_v) = \frac{e^{-\frac{\theta_v}{2}}\left(\frac{\theta_v}{2}\right)^{k_v}}{k_v!}$ for $v = 1, 2$. ▲

3.5 Shape analysis

In this section, a visual representation is given on the forms of the pdfs of the bivariate noncentral generalized F distribution (see (3.7)), and the bivariate noncentral compound extended F distribution for unequal degrees of freedom (see (3.10)), and their derived univariate composites, for arbitrary parameter values.

3.5.1 Bivariate noncentral generalized F distribution

3.5.1.1 The joint probability density function (3.7)

The pdf (3.7) is illustrated here for arbitrary parameter values: $n_1 = 10$, $n_2 = 12$, and $r = 2$. In Figure 3.3, the value of ξ varies, with $\theta_1 = \theta_2 = 3$, and in Figure 3.4, $\theta_1 = 3$ remains fixed together with $\xi = 0.5$, whilst θ_2 varies.

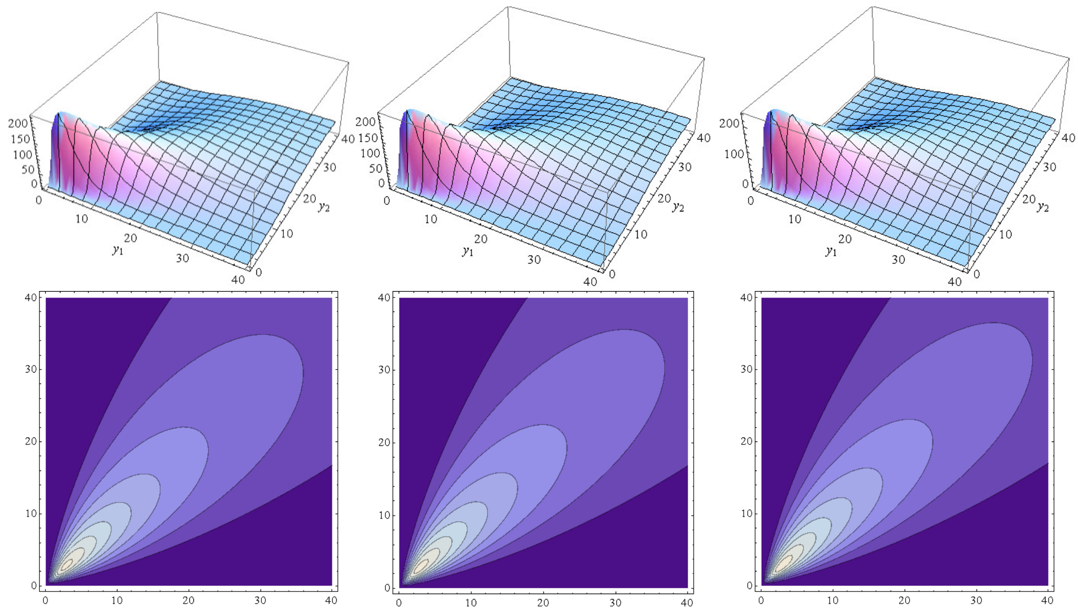

Figure 3.3 f.l.t.r., (3.7) for $\xi = 0.2$, 0.5, and 0.7.

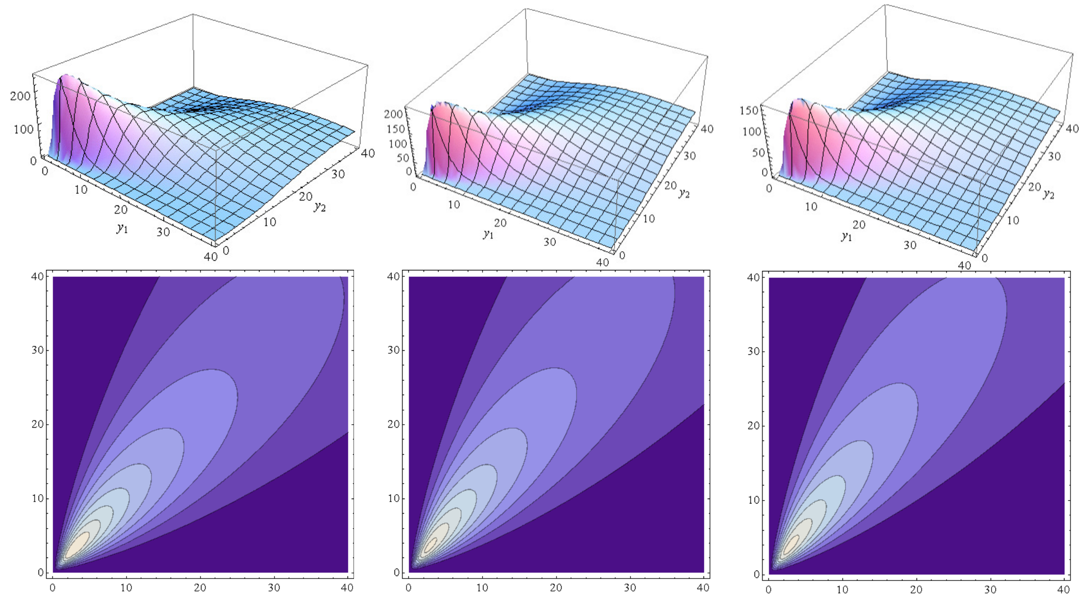

Figure 3.4 f.l.t.r., (3.7) for $\theta_1 = 8$, and $\theta_2 = 10$, 15, and 20.

It is seen that the parameter ξ has very little influence on the shape of the pdf. It is observed to marginally result in more spread out contours near the top right of the pdf, but not other obvious effects are observed. For the change in θ_2, it is seen that as θ_2 increases, the pdf becomes more attracted to the axis of the corresponding variable X_2. Similar as the shape analysis of the analogous bivariate chi-square distribution in Chapter 2, this is to be expected.

3.5.1.2 The composites

For the purposes of this study, the effect of the correlation component ξ will be considered, as well as the effect of the noncentrality parameters θ_1 and θ_2. In this section specifically, the univariate distributions of the composites given in (3.13), (3.15), and (3.17), are considered respectively. Similar as before, $n_1 = 10$, $n_2 = 12$, and $r = 2$. In Figure 3.5, 3.6, and 3.7, the value of ξ varies, with $\theta_1 = \theta_2 = 3$ (left), and $\theta_1 = 3$ remains fixed together with $\xi = 0.5$, whilst θ_2 varies (right).

Probability density function of the product

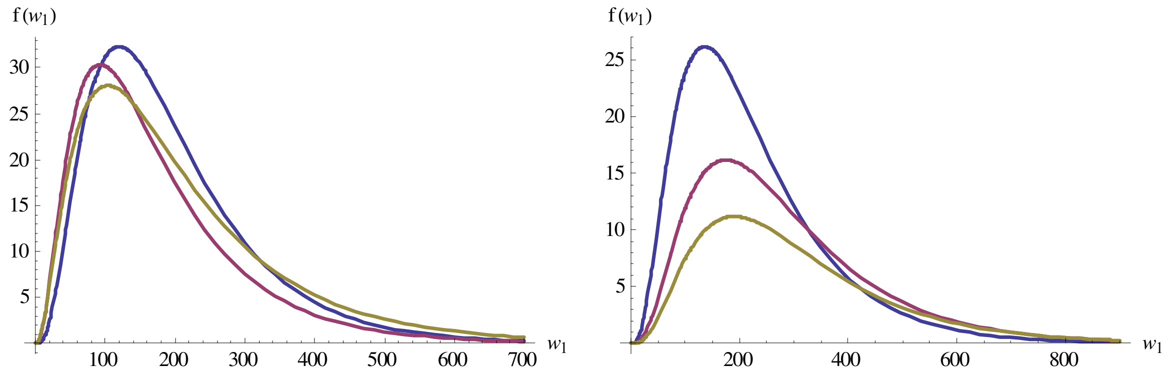

Figure 3.5 (3.13) for $\xi = 0.2$ (blue), 0.5 (purple), and 0.7 (green) (left), and for $\theta_1 = 3$, and $\theta_2 = 5$ (blue), 8 (purple), and 11 (green) (right).

Probability density function of the ratio of type II

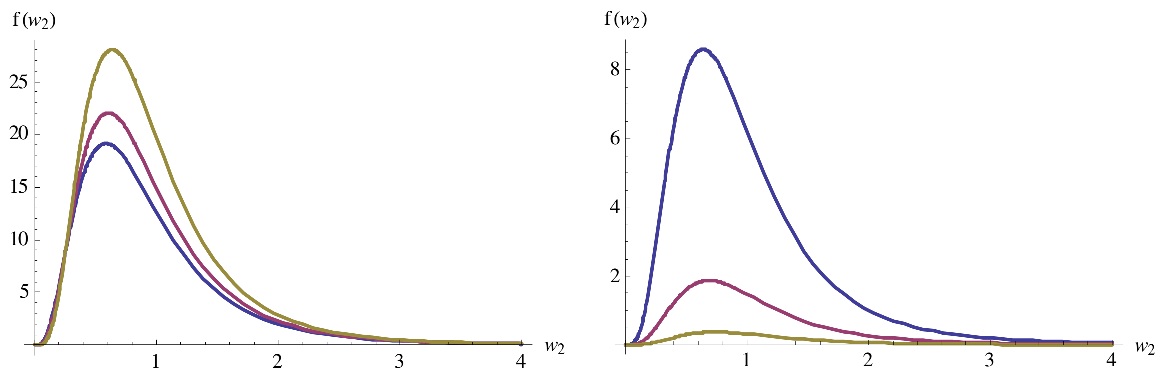

Figure 3.6 (3.15) for $\xi = 0.2$ (blue), 0.5 (purple), and 0.7 (green) (left), and for $\theta_1 = 3$, and $\theta_2 = 5$ (blue), 8 (purple), and 11 (green) (right).

Probability density function of the ratio of type I

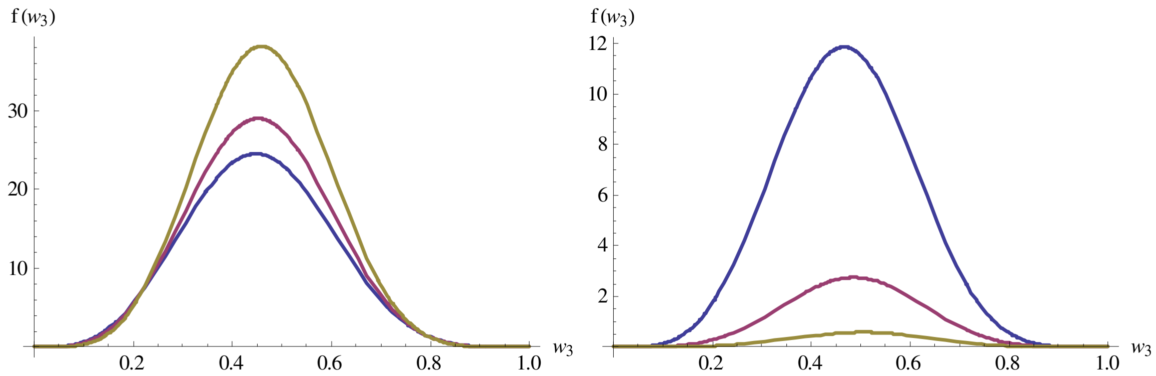

Figure 3.7 (3.17) for $\xi = 0.2$ (blue), 0.5 (purple), and 0.7 (green) (left), and for $\theta_1 = 3$, and $\theta_2 = 5$ (blue), 8 (purple), and 11 (green) (right).

As in the case of the bivariate noncentral generalized chi-square distribution's composites, the trend of the changes of ξ when considering the distributions of the ratios of type II and type I respectively, is observed: as ξ increases, the pdf of the distribution becomes more leptokurtic. Again, for the change in θ_2, it seems that as θ_2 increases, the variability of the distributions of the ratios of type II and type I seem to decrease marginally.

3.5.2 Bivariate noncentral compound extended F distribution: unequal degrees of freedom

3.5.2.1 The joint probability density function (3.10)

In this section, the bivariate noncentral compound extended F distribution with unequal degrees of freedom as given in (3.10) will be considered. The pdf (3.10) is illustrated here for arbitrary parameter values: $n_1 = 10$, $n_2 = 12$, and $r = 2$. In Figure 3.8, the value of γ varies, with $\theta_1 = \theta_2 = 3$, and in Figure 3.9, $\theta_1 = 3$ remains fixed together with $\gamma = 0.5$, whilst θ_2 varies.

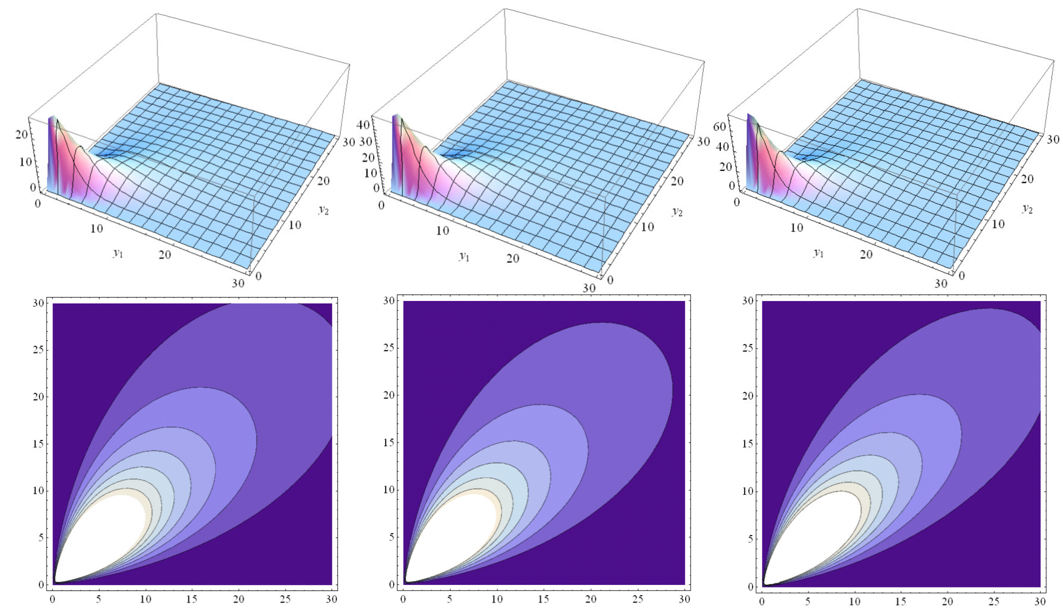

Figure 3.8 f.l.t.r., (3.10) for $\gamma = 0.2$, 0.5, and 0.7.

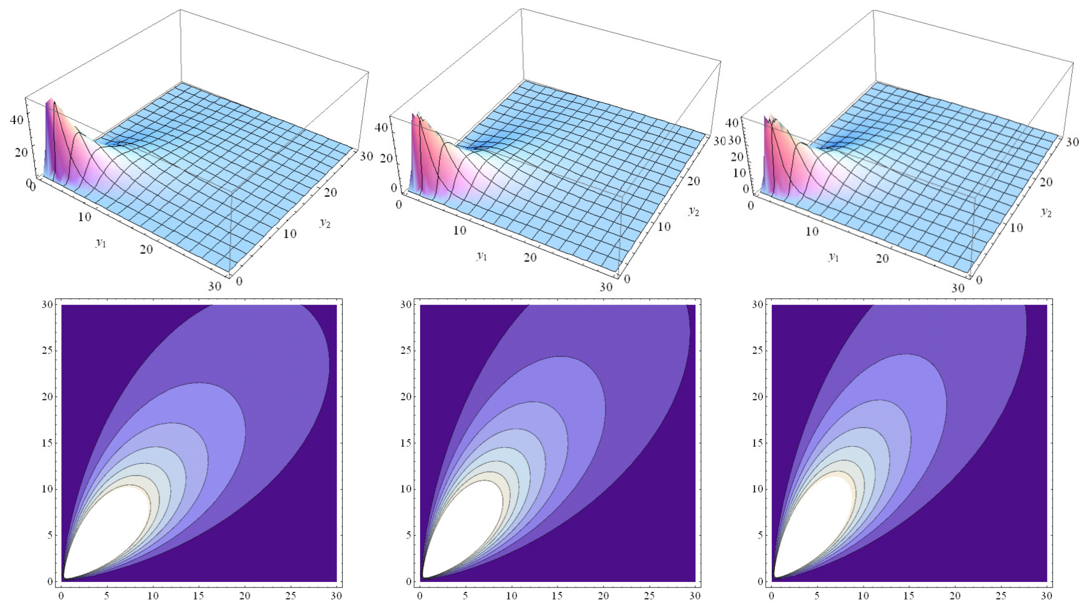

Figure 3.9 f.l.t.r., (3.10) for $\theta_1 = 3$, and $\theta_2 = 5$, 8, and 11.

Here it is seen that the parameter γ greatly affects the contours of this bivariate distribution, with larger values resulting in these contours being more spread out. For the change in θ_2, a similar trend as in Figure 3.4 is observed: as θ_2 increases, the density moves away from the axis of variable X_1 - which is again what is to be expected.

3.5.2.2 The composites

For the purposes of this study, the effect of the extension parameter γ will be considered, as well as the effect of the noncentrality parameters θ_1 and θ_2. In this section specifically, the univariate distributions of the composites given in (3.19), (3.21), and (3.23), are considered respectively. Similar as before, $n_1 = 10$, $n_2 = 12$, and $r = 2$. In Figure 3.10, 3.11, and 3.12, the value of γ varies, with $\theta_1 = \theta_2 = 3$ (left), and $\theta_1 = 3$ remains fixed together with $\xi = 0.5$, whilst θ_2 varies (right).

Probability density function of the product

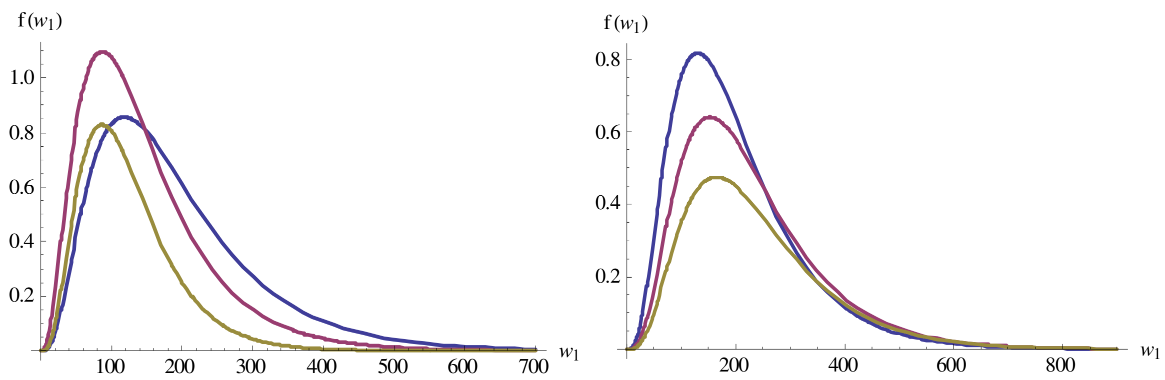

Figure 3.10 (3.19) for $\gamma = 0.2$ (blue), 0.5 (purple), and 0.7 (green) (left), and for $\theta_1 = 3$, and $\theta_2 = 5$ (blue), 8 (purple), and 11 (green) (right).

Probability density function of the ratio of type II

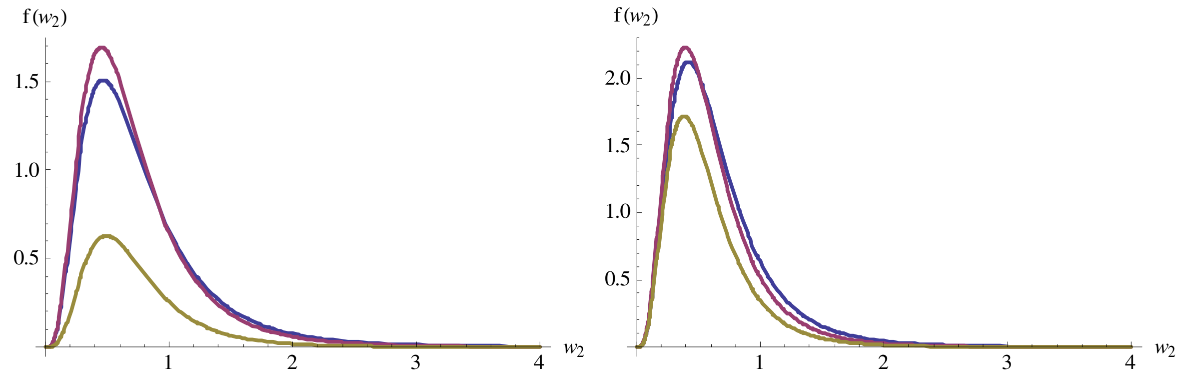

Figure 3.11 (3.21) for $\gamma = 0.2$ (blue), 0.5 (purple), and 0.7 (green) (left), and for $\theta_1 = 3$, and $\theta_2 = 5$ (blue), 8 (purple), and 11 (green) (right).

Probability density function of the ratio of type I

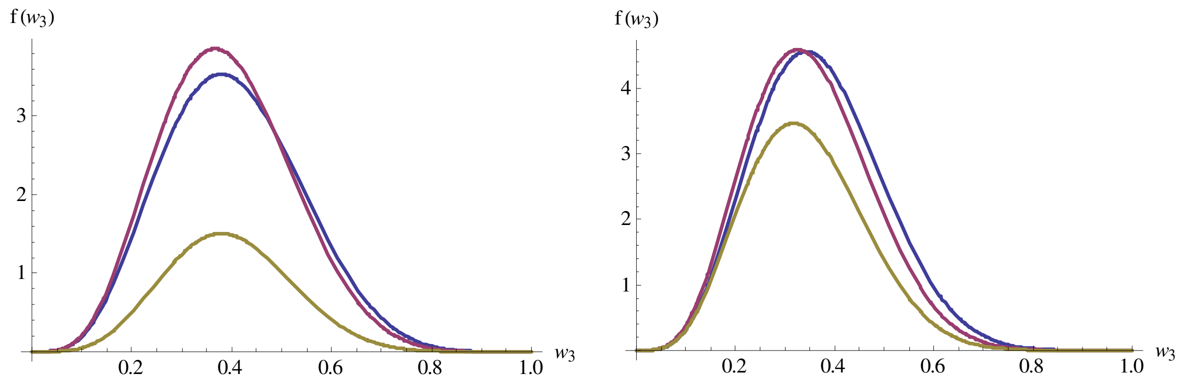

Figure 3.12 (3.23) for $\gamma = 0.2$ (blue), 0.5 (purple), and 0.7 (green) (left), and for $\theta_1 = 3$, and $\theta_2 = 5$ (blue), 8 (purple), and 11 (green) (right).

Similar trends as the generalized case (see section 3.5.1.2) is observed in the changes of γ when considering the distributions of the ratios of type II and type I respectively; however, the converse holds: as γ increases, the pdf of the distribution becomes more platykurtic (similar to the compound extended case for the corresponding distributions in Chapter 2). Also, for the change in θ_2, it seems that as θ_2 increases the variability of the distributions of the ratios of type II and type I again seem to decrease marginally.

3.6 Conclusion

In this chapter, fresh contributions to the bivariate noncentral distribution literature were made in the form of bivariate noncentral F distributions. The approach was similar as that of Chapter 2, as illustrated in the roadmap of Chapter 3 in section 3.3. A brief discussion was given of the existing univariate noncentral F distribution and was observed to be similar as that of the chi-square distribution (see section 2.1.1, property 2.3 and section 3.1.1, property 3.1). In light of these similarities, two bivariate *central* F distributions were derived first via a transformation approached, the bivariate generalized F distribution, and the bivariate compound extended F distribution from their respective chi-square counterparts (from Chapter 2). Subsequently, conditional versions of these bivariate F distributions were defined, from where it was proceeded to obtain their corresponding unconditional distributions – which is equivalent to their bivariate *noncentral* F distributions (see Remark 3.2 and Remark 3.3). See Figure 3.13 which displays the mathematical construction of the new bivariate F distributions. Finally, distributions of composites were derived as discussed in Chapter 1, and the shape analysis of both newly obtained bivariate noncentral F distributions and their respective composite distributions followed.

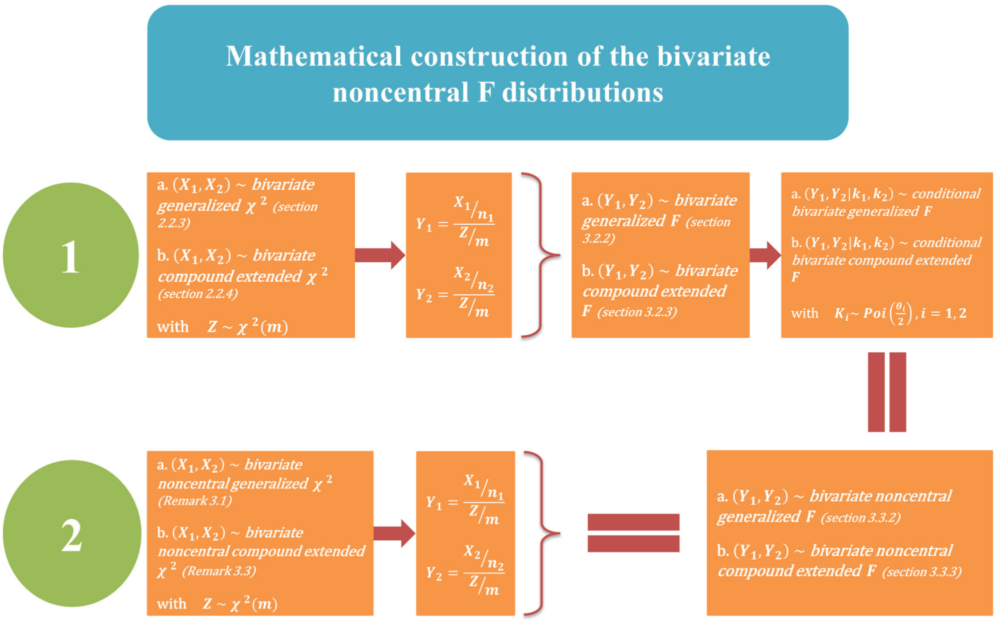

Figure 3.13 Summary of the mathematical construction of the bivariate noncentral F distributions in this study.

Chapter 4

Application

4.1 Introduction

In this chapter, a possible application of some results from Chapter 3 is presented and discussed. As mentioned in Chapter 1, the bivariate F distribution has known applications in meteorology and hydrological processes (see Nadarajah (2008)), and an application is demonstrated here for some results obtained in this study. The data used is drought data from the state of Nebraska, USA, obtained freely from the website *http://lwf.ncdc.noaa.gov.oa/climate/onlineprod/drought/xmgrg3.html*, which consists of drought- and nondrought duration (in months) for eight climate divisions of Nebraska from January 1895 to December 2004. Nadarajah (2008) showed that a bivariate F distribution is suitable for modeling this data - in this dissertation, the analysis is extended to allow for the more flexible bivariate F distributions such as the bivariate noncentral generalized F distribution; and the bivariate noncentral compound extended F distribution. Specifically, some composite distributions will be fitted to the data and subsequently be compared to each other.

4.1.1 Description of the data

Figure 4.1 Climate divisions of Nebraska, USA.

Nebraska is one of the fifty states of the United States of America (USA), and is a member state of the Midwestern United States (as defined by the United States Census Bureau) – which is one of the four main US geographic regions. The state itself is divided into eight climate divisions - as depicted by Figure 4.1 (there is no defined fourth division). Considering the state as a whole, Nebraska is considered to have two major climate divisions: the eastern half of the state has a humid, continental climate (i.e. divisions 3, 6, and 9), and the western half of state which has a semi-arid climate (i.e. divisions 1, 2, and 7). Overall, it is known that the entire state encounters wide seasonal changes in temperature and precipitation throughout the year.

Due to these known differences between the climate divisions from the eastern- and western half of the state, one would expect the number of dry months between, say, division 1 and division 9, to be independent from each other. This is mainly attributed to the fact that these divisions fall within different climate regions, each with their own climatic structure. This is under the assumption that Y_1 = number of dry months for division 1, and Y_2 = number of dry months for division 9, is jointly distributed according to either of the bivariate noncentral F distributions under consideration. It is intended to show that the correlation component in the bivariate noncentral generalized F distribution's considered composite (see (3.15)), as well as the extension parameter in the bivariate noncentral compound extended F distribution's corresponding considered composite (see (3.21)) will exhibit this independence in the estimation of the parameters, by being negligibly small. The divisions that will be considered for this purpose are regions 1 and 7, and 1 and 9.

Furthermore, letting Y_1 = number of dry months of division t, and Y_2 = number of nondry months of division t, the distribution of W_2 would give an indication of the degree of dryness experienced by the division in question – if $W_2 > 1$, it inherently implies that $Y_1 > Y_2$, or rather, that the division seems to statistically exhibit more dry months than otherwise. This is investigated for divisions 1, 2, and 8.

4.1.2 Model description and parameter estimation

The distributions of particular interest here are the following cases:

1. the ratio of drought duration of division t to drought duration of division s (W_2) = drought duration of division t / drought duration of division s $\left(\equiv \frac{Y_1}{Y_2}\right)$ ($t = 1$, $s = 7, 9$).
2. the ratio of drought duration of division t to nondrought duration of division t (W_2) = drought duration of division t / nondrought duration of division t $\left(\equiv \frac{Y_1}{Y_2}\right)$ ($t = 1, 2, 8$).

By using the method of maximum likelihood for estimation of the parameters, the parameters of the distributions of W_2 was calculated for the described cases 1 and 2. Thus, the log-likelihood function is required - due to the nature of the pdfs whose parameters are to be estimated, the log-likelihood function does not simplify to a convenient optimizable function; see (3.15). For instance:

$$\begin{aligned}
& L(m, r, \xi, \theta_1, \theta_2) \\
= & \sum_{i=1}^{N} \ln \left(f_{W_2}(w_{2_i}) \right) \\
= & \sum_{i=1}^{N} \ln \sum_{j=0}^{\infty} \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \sum_{l_1=0}^{j} \sum_{l_2=0}^{j} C w_{2_i}^{\frac{n_1}{2}+l_1+k_1-1} \left(\frac{n_2 w_{2_i}+n_1}{m} \right)^{-\left(\frac{n_1+n_2}{2}+l_1+l_2+k_1+k_2\right)} \\
& \times B\left(\frac{n_1+n_2}{2}+l_1+l_2+k_1+k_2, \frac{m}{2} \right) \frac{e^{-\frac{\theta_1}{2}} \left(\frac{\theta_1}{2}\right)^{k_1}}{k_1!} \frac{e^{-\frac{\theta_2}{2}} \left(\frac{\theta_2}{2}\right)^{k_2}}{k_2!}.
\end{aligned}$$

Regardless, the required log-likelihood functions were optimized by using the SAS/IML call **nlpnra** - this call finds the maximum value of the provided function via a Newton-Rhapson algorithm. All parameters except ξ and γ needed to be positive (where $-1 \leq \xi \leq 1$ and $-1 < \gamma < 1$), and this **nlpnra** call provides for user-defined parameter constraints.

Finally, the fitted distributions were compared according to the different distributions for W_2 fitted per climate division, according the the AIC (Akaike Information Criterion) of the fitted distribution. This criterion is computed as

$$AIC = 2d - 2\ln(likelihood)$$

where d is the number of estimated parameters, and ($likelihood$) is the value of the log-likelihood function evaluated at the maximized parameters. When two models are compared with this criterion, the model with the smaller AIC is said to be preferable.

4.2 Fitted distributions: results

For the ratio of drought duration of division 1 to drought duration of division 7 and 9, the data was fitted to the distributions given in (3.15) and (3.21) respectively by using the above mentioned call in SAS/IML (code in the Appendix). Afterwards, the corresponding pdfs were drawn by substituting the parameter estimates into (3.15) and (3.21). These pdfs are given in Figures 4.1 and 4.2. Note that the parameters n_1 and n_2 was assumed to be known as it corresponds to the respective sample sizes, and was not estimated.

	Climate division	
Parameter estimates	1, 7	1, 9
n_1	82	74
n_2	82	74
$\hat{m}$	19.548878	20.004631
$\hat{r}$	7.8525326	7.2786409
$\hat{\xi}$	1.271×10^{-8}	-8.42×10^{-8}
$\hat{\theta}_1$	1.997×10^{-17}	1.642×10^{-17}
$\hat{\theta}_2$	0.2914278	3.8882968
AIC	2361.257	2352.4459

Table 4.1 Parameter estimates of W_2, (3.15) for case 1

	Climate division	
Parameter estimates	1, 7	1, 9
n_1	82	74
n_2	82	74
$\hat{m}$	19.989758	19.134776
$\hat{\gamma}$	-0.000178	2.466×10^{-8}
$\hat{\theta}_1$	3.375×10^{-17}	6.91×10^{-17}
$\hat{\theta}_2$	0.2911099	3.0481582
AIC	2358.9257	2345.5089

Table 4.2 Parameter estimates of W_2, (3.21) for case 1

By considering the parameters, one immediately picks up that both the correlation component in (3.15) and the extension parameter in (3.21) is negligibly small. The expected independence mentioned in section 4.1.1 seems to be confirmed by these results. When considering at the AIC as criterion to compare models, it is seen in both cases that the distribution of W_2 under (3.21) yields a marginally better fit than that of the distribution of W_2 under (3.15). The estimated pdfs of these distributions are given below. The graph of the distribution of ((3.21) - blue) neglects to sufficiently highlight the long tails of this distribution.

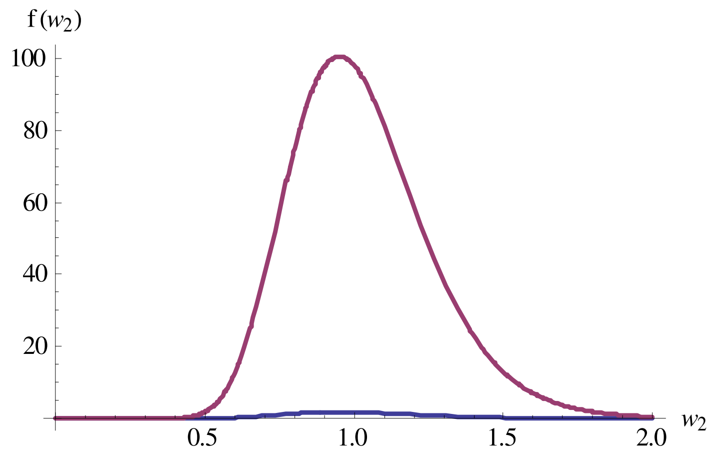

Figure 4.2 Estimated distribution of climate division 1 and 7 for W_2 - (3.21) (blue) and (3.15) (purple).

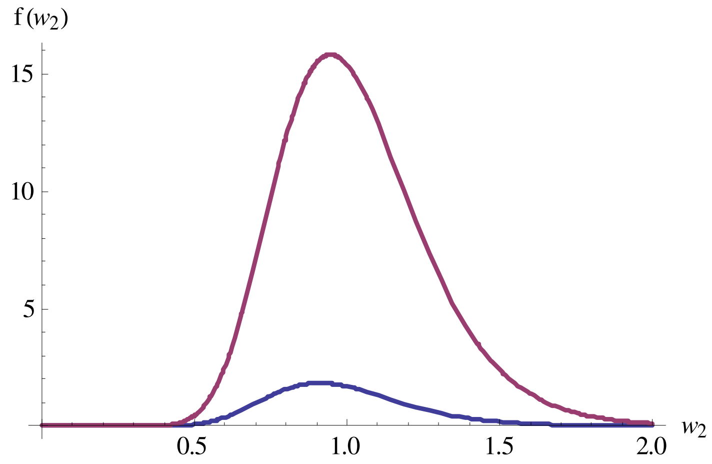

Figure 4.3 Estimated distribution of climate division 1 and 9 for W_2 - (3.21) (blue) and (3.15) (purple).

It is observed that the main difference between these two figures is the peakedness of the pdfs for both the generalized- and compound extended cases. In both cases, the model fit under the compound extended distribution is preferred (blue line), as a result from the AIC comparison.

Consider now, under the compound extended model (i.e. the preferred model, (3.21)) and for the comparison between division 1 and division 9, the value of $P(W_2 > 1) = 1 - P(W_2 < 1) = 1 - P\left(\frac{Y_1}{Y_2} < 1\right)$. This value is calculated, using the package Mathematica, as $P(W_2 > 1) = 1 - P(W_2 < 1) = 1 - 0.468634 = 0.531366$. This translates to the probability that division 1's drought duration will be more / longer, than that of division 9 - which seems reasonable considering that division 1 falls within the arid climate of the state, whereas division 9 does not. This corresponds with the geographical outlay of the divisions as described in section 4.1.1.

In the second considered case (drought versus nondrought duration), the parameter estimates are given below in Tables 4.3 and 4.4. By substituting these estimates into equations (3.15) and (3.21) respectively, the corresponding pdfs were drawn and is given (Figures 4.4 to 4.6). Note again that the parameters n_1 and n_2 is assumed known as it corresponds to the respective sample sizes, and was not estimated. In addition, it is noted that Nadarajah (2008) used a pooled sample of the data after a test of homogeneity of the estimated parameters across all climate divisions - whilst such a test could be conducted here, it falls outside the scope of this dissertation, and the data will be considered within its respective climate divisions, and examined as such.

	Climate division		
Parameter estimates	1	2	8
n_1	84	67	76
n_2	84	67	76
$\hat{m}$	20.00713	20.057935	19.999673
$\hat{r}$	2.880700	4.0048911	9.2246327
$\hat{\xi}$	-8.314×10^{-8}	1.7786×10^{-8}	-2.44×10^{-8}
$\hat{\theta}_1$	3.518×10^{-17}	4.73×10^{-17}	2.949×10^{-17}
$\hat{\theta}_2$	2.7391218	2.2691616	4.4891381
AIC	3562.96	2317.6171	3071.0484

Table 4.3 Parameter estimates of W_2, (3.15) for case 2

	Climate division		
Parameter estimates	1	2	8
n_1	84	67	76
n_2	84	67	76
$\hat{m}$	19.75478	20.023679	19.995659
$\hat{\gamma}$	-9.095×10^{-8}	-1.924×10^{-8}	-1.336×10^{-7}
$\hat{\theta}_1$	1.681×10^{-18}	-6.66×10^{-17}	4.304×10^{-17}
$\hat{\theta}_2$	3.581955	2.856712	5.6915757
AIC	3561.63	2315.9189	3054.5989

Table 4.4 Parameter estimates of W_2, (3.21) for case 2

It is seen that when W_2 is fitted to this data, the distribution under the bivariate noncentral compound extended F distribution yields a marginal better fit in terms of the AIC. Since these values of the AIC are so similar one might be tempted to infer that either distribution of W_2, under bivariate noncentral generalized F distribution or under bivariate noncentral compound extended F distribution is suitable for modelling the data. This, in essence, is true - but when considering a parsimonious approach, the model with less parameters is preferable. In this case, it is W_2 under the bivariate noncentral compound extended F distribution - also having a lower AIC. The estimated pdfs of these distributions are given below. Again, the graph of the distribution of ((3.21) - blue) neglects to sufficiently highlight the long tails of this distribution.

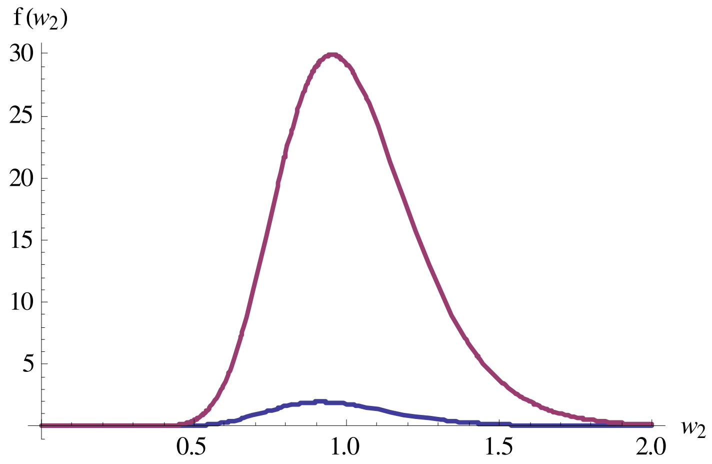

Figure 4.4 Estimated distribution of climate division 1 for W_2 - (3.21) (blue) and (3.15) (purple).

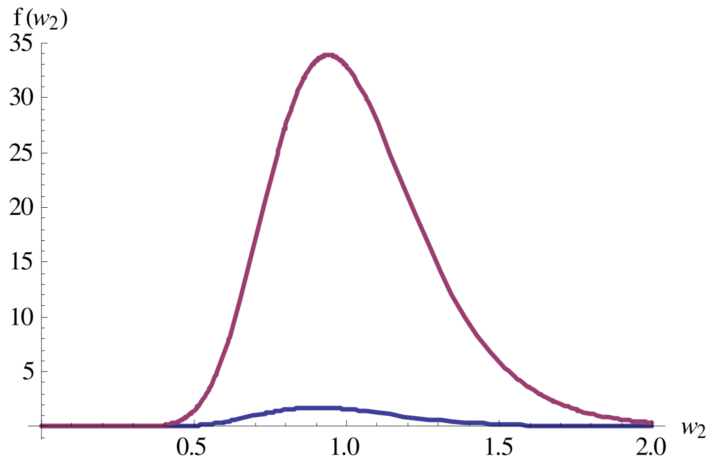

Figure 4.5 Estimated distribution of climate division 2 for W_2 - (3.21) (blue) and (3.15) (purple).

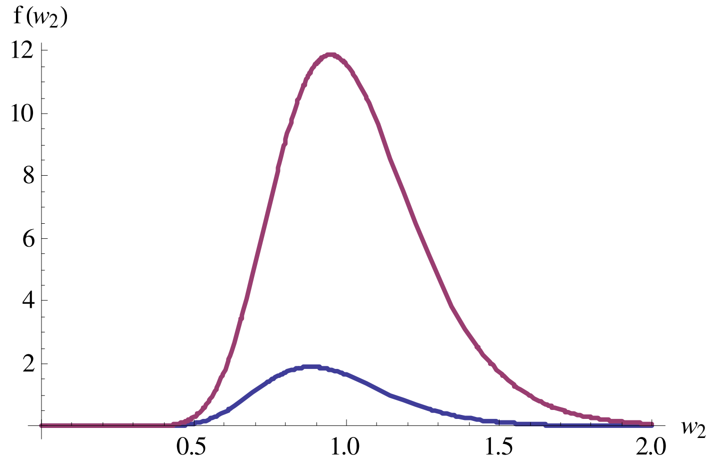

Figure 4.6 Estimated distribution of climate division 8 for W_2 - (3.21) (blue) and (3.15) (purple).

As before, the most notable change in the form of the distributions for these different climate divisions, is the peakedness of the distributions. As illustrated in section 3.5.1.2 and section 3.5.2.2, the observed distributions of this composite in both the generalized- and compound extended case is skew.

Again, consider the compound extended model (i.e. the preferred model, (3.21)) for division 8, the value of $P(W_2 > 1) = 1 - P(W_2 < 1) = 1 - P\left(\frac{Y_1}{Y_2} < 1\right)$. This value is calculated, using the package Mathematica, as $P(W_2 > 1) = 1 - P(W_2 < 1) = 1 - 0.508195 = 0.491805$. By considering the construction of the variables for case 2, this translates to the probability that division 8's drought duration will be less / shorter, than that of its nondrought duration. This seems to be a reasonable finding since division 8 lies more to the east of the state - i.e. lies more toward the humid / continental climatic side of the state, therefore resulting in less drought-ridden months than otherwise.

Chapter 5

Conclusions

5.1 Overview

In this dissertation, a well-known method was used to obtain bivariate *noncentral* distributions, namely the compounding method. This method has been used to a limited extent in literature to achieve a noncentral distributional goal, but to the author's knowledge, this study is the first to structure the literature in a compounding way (thus, by defining some conditional distribution which is subsequently unconditioned) which provides valuable insight to the construction of bivariate noncentral distributions. Thereby great value has been added to the distribution theory literature by not only presenting some alternative representations of existing bivariate *noncentral* distributions (such as the bivariate noncentral generalized chi-square distribution), but also with the addition of some new bivariate noncentral distributions (such as the bivariate noncentral compound extended chi-square distribution and their corresponding bivariate noncentral F distributions). Furthermore, important distributions of composites have been derived and graphically presented.

Chapter 1 presented a motivation for this study where a gap in the literature was identified in terms of identifying elegant constructions of bivariate noncentral distributions via the compounding method. A brief overview was given of currently relevant literature in terms of bivariate- and noncentral distributions.

In Chapter 2, the bivariate noncentral distribution took form by introducing noncentrality in Poisson probability form. An overview of two univariate noncentral chi-square distributions was presented, where the systematic structuring initially identified as an aim was achieved by rephrasing known distributions in terms of conditional (compound) central distributions, and thereafter unconditioning with Poisson probabilities as the compounding factors to obtain the corresponding noncentral distribution (see (1.1)). This was extended to each of the respective chi-square distributions' bivariate cases, by first identifying the central cases, then defining the central case as a conditional distribution, and finally unconditioning this distribution to obtain the bivariate noncentral distribution with Poisson probabilities as the compounding factors (see Description 2.10). Each of the considered bivariate noncentral distributions' mgf was determined which can be used in future to obtain the moments required to investigate the correlation structure. Subsequently the univariate distributions of the composites as described in Chapter 1 were derived for each of the two bivariate noncentral chi-square distributions. The obtained bivariate noncentral pdfs, together with their respective composite pdfs were visually illustrated for different parameter values.

Next, in Chapter 3, a brief reminder was given of the univariate noncentral F distribution, and also showed that its construction is of a compounding nature. Subsequently corresponding bivariate F distributions were derived from their bivariate chi-square counterparts in Chapter 2. A similar approach as in Chapter 2 followed where these newly obtained bivariate central F distributions were defined as conditional distributions, and then the corresponding unconditional distribution was obtained with Poisson probabilities as compounding factors which is equivalent to its bivariate noncentral F distribution (see Description 3.3). Expressions for the product moments for both obtained bivariate noncentral distributions were derived, and can be used in future to obtain the required moments to investigate the correlation structure. Again, the univariate distributions of the composites as described in Chapter 1 were derived for both bivariate noncentral F distributions and the pdfs of the bivariate noncentral F distributions and its corresponding composites were visually illustrated for different parameter values.

To highlight the use of some the newly obtained results, Chapter 4 saw the application of certain composite distributions from Chapter 3 to real data.

In final conclusion, this study contributed to the field of bivariate noncentral distributions using the compounding method.

5.2 Future directions

Future research in this field can be undertaken in a variety of ways. One of these is to possibly extend the bivariate noncentral distributions to multivariate noncentral distributions via the compounding method. Other bivariate models (such as the bivariate F distribution as considered by Nadarajah (2008), or other bivariate gamma models) could also be investigated and extended to their noncentral counterparts via this compounding method. Futhermore, the statistical properties (expected values, variances, regression equations, correlation structures, information criteria such as Rényi and Shannon entropy) of the obtained results in this study could also be investigated.

APPENDICES

A. Lists of figures and tables

Tables

Figures

B. Abbreviations and notation

pmf	probability mass function
pdf	probability density function
mgf	moment generating function
x_i	observation from random variable X_i
n_i	size of sample i
θ_i	noncentrality parameter of variable X_i
μ_i	mean of variable U_i
r	additional parameter
ξ	correlation component
γ	extension parameter
$\equiv$	identical distributions
$g_K(k)$; $h_I(i)$	pmf of random variables K and I respectively
$f_X(x)$	pdf of random variable X
$M_X(t)$	mgf of random variable X
$f_{X_1,X_2}(x_1,x_2)$	joint pdf of random variables X_1 and X_2
$M_{X_1,X_2}(t_1,t_2)$	joint mgf of random variables X_1 and X_2
$\Gamma(\cdot)$	Gamma function
$B(\cdot,\cdot)$	Beta function
$(\alpha)_j$	Pochhammer coefficient
${}_rF_s$	Hypergeometric function with parameters r and s
$K_\tau(\cdot)$	Modified Bessel function of the second kind
$L_j^n(\cdot)$	Laguerre polynomial of degree j

C. Special functions and results

Result 1 *Transformation result for composite* $W_3 = \frac{X_1}{X_1+X_2}$

If $W_3 = \frac{X_1}{X_1+X_2}$, and $W_2 = \frac{X_1}{X_2}$, then $W_2 = \frac{W_3}{1-W_3}$. Furthermore, the Jacobian of the transformation is given by $\frac{dw_2}{dw_3} = \frac{1}{(1-w_3)^2}$, and it follows that

$$f_{W_3}(w_3) = f_{W_2}\left(\frac{w_3}{1-w_3}\right)\frac{dw_2}{dw_3}. \tag{C.1}$$

Result 2 *(Bain & Engelhardt (1992), p. 206)*

For a transformation of k variables $y = \mathbf{u}(\mathbf{x})$ with a unique solution $x = (x_1, x_2, ..., x_k)$, the Jacobian is the determinant of the $k \times k$ matrix of partial derivatives:

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} & \cdots & \frac{\partial x_1}{\partial y_k} \\ \frac{\partial x_2}{\partial y_1} & \ddots & & \vdots \\ \vdots & & & \\ \frac{\partial x_k}{\partial y_1} & \cdots & & \frac{\partial x_k}{\partial y_k} \end{vmatrix}.$$

Result 3 *(Gradshteyn & Ryzhik (2007), p.892; p.897, eq. 8.310.1)*

The gamma function, denoted $\Gamma(\alpha)$, is defined as

$$\Gamma(\alpha) = \int_0^\infty e^{-x}x^{\alpha-1}dx, \tag{C.2}$$

where $\mathrm{Re}(\alpha) > 0$.

Result 4 *(Gradshteyn & Ryzhik (2007), p.908; p.909, eq. 8.380.1)*

The beta function, denoted $B(\alpha, \beta)$, is defined as

$$B(\alpha,\beta) = \int_0^\infty x^{\alpha-1}(1+x)^{-(\alpha+\beta)}dx \tag{C.3}$$

$$B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \tag{C.4}$$

where $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 0$.

Result 5 *(Mathai (1993), p.96)*

The Pochhammer coefficient is defined as

$$(\alpha)_j = \alpha(\alpha+1)\dots(\alpha+j-1) = \frac{\Gamma(\alpha+j)}{\Gamma(\alpha)}, \tag{C.5}$$

where $j = 1, 2, \ldots$, $(\alpha)_0 = 1$, $\alpha \neq 0$, $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\alpha+j) > 0$ and $\Gamma(\cdot)$ is the gamma function (see Result C.2).

Result 6 *(Mathai (1993), p.96)*

The hypergeometric series with r upper parameters and s lower parameters is defined as

$${}_rF_s(\alpha_1, \ldots, \alpha_r; \beta_1, \ldots, \beta_s; x) = \sum_{j=0}^{\infty} \frac{(\alpha_1)_j \cdots (\alpha_r)_j}{(\beta_1)_j \cdots (\beta_s)_j} \frac{x^j}{j!}, \tag{C.6}$$

where $(\alpha)_j$ is the Pochhammer symbol (and where $B(\cdot,\cdot)$ is the beta function (see Result C.5).

The following holds for the series:

(i) if any α_i, $i = 1, \ldots r$, is a negative integer or zero the series terminates and ${}_rF_s$ becomes a polynomial in x provided none of β_k, $k = 1, \ldots, s$, is zero or a negative integer;

(ii) if any β_k, $k = 1, \ldots s$, is zero or a negative integer then the series is not defined unless there is an α_i, $i = 1, \ldots r$, such that $(\alpha_i)_j$ becomes zero first. That is, suppose α_i and β_k are two negative integers such that $(\alpha_i)_\ell = 0$ for $\ell \geq j$ and $(\beta_k)_\ell = 0$ for $\ell \geq n$. Then in order for ${}_rF_s$ to be defined j must be less than n;

(iii) the series converges for all x if $r \leq s$ and for $|x| < 1$ if $r = s + 1$;

(iv) the series diverges for all x, $x \neq 0$ for $r > s + 1$;

(v) if $r = s + 1$ and $|x| = 1$, the series is absolutely convergent if $\mathrm{Re}(\gamma) < 0$ where $\gamma = \sum_{j=1}^{r} \alpha_j - \sum_{j=1}^{s} \beta_j$; divergent if $\mathrm{Re}(\gamma) \geq 1$; and if $r = s+1$ and $|x| = 1$, $x \neq 1$, the series is conditionally convergent if $0 \leq \mathrm{Re}(\gamma) < 1$.

- A special case of this function used in this dissertation is the Gauss hypergeometric function:

$${}_2F_1(\alpha, \beta; \varsigma; x) = \sum_{j=0}^{\infty} \frac{(\alpha)_j (\beta)_j}{(\varsigma)_j} \frac{x^j}{j!} \quad , \ |x| < 1. \tag{C.7}$$

- The series expansion of e, as well as the binomial series, can be written as a special case of Result C.6:

$$_0F_0\left(;x\right)=\sum_{j=0}^{\infty}\frac{x^j}{j!}=e^x, \tag{C.8}$$

and

$$_1F_0\left(;x\right)=\sum_{j=0}^{\infty}\frac{(\alpha)_j}{j!}x^j=(1-x)^{-\alpha}\,,\quad |x|<1. \tag{C.9}$$

Result 7 *(Gradshteyn & Ryzhik (2007), p. 917, eq. 8.432.3)*

The modified Bessel function of the second kind is defined by

$$K_\tau(z)=\frac{\left(\frac{z}{2}\right)^\tau\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\tau+\frac{1}{2}\right)}\int_1^\infty e^{-zx}(x^2-1)^{\tau-\frac{1}{2}}dx \tag{C.10}$$

for $\mathrm{Re}(\tau+\frac{1}{2})>0$, $|\arg z|<\frac{\pi}{2}$; or $\mathrm{Re}(z)=0$ & $\mathrm{Re}\,(\tau)=0$.

Result 8 *(Gradshteyn & Ryzhik (2007), p. 368, eq. 3.471)*

$$\int_0^\infty x^{\tau-1}e^{-\frac{\beta}{x}-\gamma x}dx=2\left(\frac{\beta}{\gamma}\right)^{\frac{v}{2}}K_\tau\left(2\sqrt{\beta\gamma}\right) \tag{C.11}$$

for $\mathrm{Re}(\beta)>0, \mathrm{Re}(\gamma)>0$.

Result 9 *(Gradshteyn & Ryzhik (2007), p. 346, eq. 3.381.4)*

$$\int_0^\infty x^{\beta-1}e^{-\alpha x}dx=\frac{1}{\alpha^\beta}\Gamma(\beta) \tag{C.12}$$

for $\alpha,\beta>0$.

Result 10 *(Kotz et. al. (2006a))*

The j^{th} generalized Laguerre polynomial is given by

$$L_j^n\left(x\right)=\frac{1}{j!}\sum_{l=0}^{j}(-1)^l\binom{j}{l}\frac{\Gamma(n+j+1)}{\Gamma(n+l+1)}x^l \tag{C.13}$$

such that $n>-1$.

Result 11 *(Prudnikov et. al. (1986a), p. 309, eq. 2.2.9.7)*

$$\int_0^\infty \frac{x^{\alpha-1}}{(ax^2+2bx+c)^\rho}dx = a^{-\frac{\alpha}{2}}c^{\frac{\alpha}{2}-\rho}\ {}_2F_1\left(\frac{\alpha}{2}, \rho-\frac{\alpha}{2}; \rho+\frac{1}{2}; 1-\frac{b^2}{ac}\right) \tag{C.14}$$

where $a > 0$, $b^2 < ac$, and $0 < \mathrm{Re}(\alpha) < 2\,\mathrm{Re}(\rho)$.

Result 12 *(Prudnikov et. al. (1986b), p. 462, eq. 2.19.3.3)*

$$\int_0^\infty x^\lambda e^{-px} L_n^\lambda(cx)\,dx = \frac{\Gamma(\lambda+n+1)\,(p-c)^n}{n!p^{\lambda+n+1}} \tag{C.15}$$

where $\mathrm{Re}(p) > 0$ and $\mathrm{Re}(\lambda) > -1$.

Result 13 *(1. Prudnikov et. al. (1986a), p. 298, eq. 2.2.4.24; 2. Gradshteyn & Ryzhik (2007), p. 315, eq. 3.194.3)*

(1) provides the following integral relation:

$$\int_0^\infty \frac{x^{\alpha-1}}{(z+x)^\rho}dx = z^{\alpha-\rho}B(\alpha, \rho-\alpha) \tag{C.16}$$

where $|\arg z| < \pi$ and $\mathrm{Re}(\rho) > \mathrm{Re}(\alpha) > 0$, and where $B(\cdot,\cdot)$ is the beta function (see Result C.4). A similar result is found in (2):

$$\int_0^\infty \frac{x^{\mu-1}}{(1+\beta x)^\kappa}dx = \beta^{-\mu}B(\mu, \kappa-\mu) \tag{C.17}$$

where $|\arg\beta| < \pi$ and $\mathrm{Re}(\mu) > \mathrm{Re}(\kappa) > 0$.

Result 14 *(Bain & Engelhardt (1992), p. 268)*

A random variable X is said to follow a chi–square distribution with n degrees of freedom, denoted by $X \sim \chi^2(n)$, if X has pdf

$$f_X(x) = \frac{x^{\frac{n}{2}-1}e^{-\frac{1}{2}x}}{2^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)}, \quad x > 0 \tag{C.18}$$

where $n > 0$, and where $B(\cdot,\cdot)$ is the beta function (see Result C.4). The mgf of X is given by

$$M_X(t) = (1-2t)^{-\frac{n}{2}}. \tag{C.19}$$

Result 15 *(Bain & Engelhardt (1992), p. 276)*

A random variable Y is said to follow an F distribution, denoted by $Y \sim F(n_1, n_2)$, if Y has pdf

$$f_Y(y) = \frac{\Gamma\left(\frac{n_1+n_2}{2}\right)}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)} \left(\frac{n_1}{n_2}\right)^{\frac{n_1}{n_2}} y^{\frac{n_1}{2}-1} \left(1 + \frac{n_1}{n_2} y\right)^{-\frac{n_1+n_2}{2}}, \quad y > 0 \tag{C.20}$$

where $n_1, n_2 > 0$, and where $B(\cdot,\cdot)$ is the beta function (see Result C.4).

Result 16 *(Bain & Engelhardt (1992), p. 101)*

A random variable I is said to follow a negative binomial distribution, denoted by $I \sim NB(\gamma^2, n)$, if I has pmf

$$h_I(i) = \frac{\Gamma\left(\frac{n}{2}+i\right)}{\Gamma\left(\frac{n}{2}\right) i!} \left(\gamma^2\right)^i \left(1-\gamma^2\right)^{\frac{n}{2}}, \quad i = 0, 1, 2, 3... \tag{C.21}$$

where $\gamma^2 > 0, n > 0$.

Result 17 *(Bain & Engelhardt (1992), p. 104)*

A random variable K is said to follow a Poisson distribution, denoted by $K \sim Poi(\theta)$, if K has pmf

$$g_K(k) = \frac{e^{-\theta}\theta^k}{k!}, \quad k = 0, 1, 2, 3... \tag{C.22}$$

where $\theta > 0$. The expected value of this distribution is such that $E(K) = \theta$, and the terms "expected value" and "parameter" are used interchangeably.

D. Computer programs

Maximum likelihood estimation in SAS/IML

For ratio of type II (eq. 3.15)

```
options nodate pageno=1 ps=5000 ls=96; footnote; title;
   title 'MLE Estimation for W2 - Generalized F composite - Div.  1 & 9';
   proc iml; **EQUATION 3.13;
   use sasuser.div1;
   read all into data1;
   data1 = data1[3:nrow(data1),ncol(data1)-2:ncol(data1)];
   use sasuser.div7;
   read all into data2;
   data2 = data2[3:nrow(data2),ncol(data2)-2:ncol(data2)];
   x1 = data1[,3]; * drought for first division;
   x2 = data2[,3]; * drought for second division;
   w2 = x1/x2;
   n1 = nrow(w2);
   n2 = n1;
   thres = 30;
   *Define likelihood function;
   start logl(par) global(w2,thres,n1,n2);
    eksi   = par[1];
    theta1 = par[2];
    theta2 = par[3];
    r      = par[4];
    m      = par[5];
    dens   = j(n1,1,0);
    poiss1 = exp(-theta1/2) * exp(-theta2/2) ;
    do index = 1 to n1;
    val = 0;
    w2val = w2[index,1];
    do i = 0 to thres;
     do k1 = 0 to thres;
      do k2 = 0 to thres;
       do l1 = 0 to i;
        do l2 = 0 to i;
          const = eksi**(2*i) * comb(i,l1) * comb(i,l2) * (-1)**(l1 + l2)
                  * gamma(r/2 + i)/gamma(r/2) / fact(i)
                  *gamma( (n1 + n2 + m)/2 + k1 + k2 + l1 + l2)
                      / (gamma(n1/2 + k1 + l1)* gamma(n2/2 + k2 + l2) * gamma(m/2))
                  *(n1)**(n1/2 + k1 + l1) * (n2)**(n2/2 + k2 + l2);
          poiss2 = (theta1/2)**k1 / fact(k1) * (theta2/2)**k2 / fact(k2);
          betav  = beta(n1/2 + n2/2 + l1 + l2 + k1 + k2,m/2);
          variab = w2val**(n1/2 + k1 + l1 - 1)
```

```
                 * (w2val*n1 + n2)**(-1*(n1/2 + n2/2 + k1 + k2 + l1 + l2));
          val    = val + const*betav*variab*poiss1*poiss2;
        end; *l2;
       end; *l1;
      end; *k2;
     end; *k1;
    end; *i;
    dens[index,1] = val;
    end; *index;
    lg = log(dens);
    v  = lg[+];
    return(v);
   finish logl;
   *Calculate ML estimates;
   par  = {0.1 2 2 4 20}; *initial values;
   optn = {1 1}; *options for optimization procedure;
   con  = {-1 0 0 0 0,
            1 .  .  .  .};
   call nlpnra(rc,xr,"logl",par,optn,con);
   xr  = n1 // n2 // xr`;
   row = {"n1" "n2" "eksi hat" "theta1 hat" "theta2 hat" "r hat" "m hat"};
   if rc>0 then print xr[r=row label='ML Estimates for model:'];
           else print 'Algorithm does not converge';
   akaike = 2*(nrow(xr)-2) - 2*nllg;
   print akaike[label='AIC for this model, given this data:'];
   run;
   quit;
```

For ratio of type II (eq. 3.21)

```
options nodate pageno=1 ps=5000 ls=96; footnote; title;
   title 'MLE Estimation for W2 - Compound Ext F composite - Div.  1 & 7';
   proc iml;
   use sasuser.div1;
   read all into data1;
   data1 = data1[3:nrow(data1),ncol(data1)-2:ncol(data1)];
   use sasuser.div7;
   read all into data2;
   data2 = data2[3:nrow(data2),ncol(data2)-2:ncol(data2)];
   x1 = data1[,3]; * drought for first division;
   x2 = data2[,3]; * drought for second division;
   w2 = x1/x2;
   n1 = nrow(w2);
   n2 = n1;
   thres = 30;
```

```
*Define likelihood function;
start logl(par) global(w2,thres,n1,n2);
 gam    = par[1];
 theta1 = par[2];
 theta2 = par[3];
 m      = par[4];
 dens   = j(n1,1,0);
 gamsq  = gam**2;
 poiss1 = exp(-theta1/2) * exp(-theta2/2) ;

 do index = 1 to n1;
 val = 0;
 w2val = w2[index,1];
 do i1 = 0 to thres;
  do i2 = 0 to thres;
   do k1 = 0 to thres;
    do k2 = 0 to thres;
      const = gamma(n1/2 + i1) * gamma(n2/2 + i2)
                  / (gamma(n1/2) * fact(i1) * gamma(n2/2) * fact(i2))
              * gamsq**(i1+i2) * (1-gamsq)**(-1*(i1+i2+k1+k2))
              *gamma( (n1 + n2 + m)/2 + k1 + k2 + i1 + i2)
                 / (gamma(n1/2 + k1 + i1) * gamma(n2/2 + k2 + i2)*gamma(m/2))
              *(n1)**(n1/2 + k1 + i1) * (n2)**(n2/2 + k2 + i2);
      poiss2 = (theta1/2)**k1 / fact(k1) * (theta2/2)**k2 / fact(k2);
      betav  = beta(n1/2 + n2/2 + i1 + i2 + k1 + k2,m/2);
      variab = w2val**(n1/2 + k1 + i1 - 1)
               * ((w2val*n2 + n1)/(1-gamsq))**(-1*(n1/2+n2/2+k1+k2+i1+i2));
      val = val + const*betav*variab*poiss1*poiss2;
     end; *k2;
    end; *k1;
   end; *i2;
  end; *i1;
 dens[index,1] = val;
 end; *index;
 lg = log(dens);
 v  = lg[+];
 return(v);
finish logl;
*Calculate ML estimates;
par  = {0.1 1 2 20}; *initial values;
optn = {1 1}; *options for optimization procedure;
con  = {-1 0 0 0,
         1 .  .  .};
call nlpnra(rc,xr,"logl",par,optn,con);
xr  = n1 // n2 // xr‘;
row = {"n1" "n2" "gamma hat" "theta1 hat" "theta2 hat" "m hat"};
```

```
if rc>0 then print xr[r=row label='ML Estimates for model:'];
        else print 'Algorithm does not converge';
akaike = 2*(nrow(xr)-2) - 2*nllg;
print akaike[label='AIC for this model, given this data:'];
run;
quit;
```

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