Characterization of reflexive Banach spaces

by

S. P. Mbambo (student no. 19377712)
Supervisor: Dr. M. D. Mabula

Submitted in fulfilment of the requirements for the degree

Master of Science (Mathematics)

In the Faculty of Natural and Agricultural Sciences

University of Pretoria

Pretoria

23 November 2020

ABSTRACT

A cone K in a vector space X is a subset which is closed under addition, positive scalar multiplication and the only element with additive inverse is zero. The pair (X, K) is called an ordered vector space. In this study, we consider the characterizations of Reflexive Banach spaces. This is done by considering cones with bounded and unbounded bases and the second characterization is by reflexive cones. The relationship between cones with bounded and unbounded bases, and reflexive cones is also considered. We provide an example to show distinction between such cones.

Acknowledgements

I am very grateful to my supervisor Dr. Mokhwetha Mabula for the effort he has out into this dissertation, and for all the encouragement and wisdom which he has imparted unto me throughout the year and for having faith in me. I take this opportunity to thank Ntombifuthi Khoza and Sabelo for reading the entire dissertation and making numerious suggestions and corrections.

I also thank Dr Mabula for funding some of my visits to University of Pretoria. I acknowledges with many thanks the financial support recieved from the UFS - Mathematics department and UFS Staff study support grant for funding part of my tution fees, conferences (University of North West and Malawi, where I presented part of this work and attending positivity conference in University of Pretoria) and Supervision visits to University of Pretoria and for their support without which this essay would not have been written. I also thank the UCDP for financing Malawi conference and my visitations to my supervisor.

I particularly thank Prof J. Meyer, our mathematics HOD, for his support and encouragement throughout this project. I also thank all my friends and colleagues for the positivity, advice and encouragement that they had given me.

Declaration

I, ... declare that the Dissertation, which I hereby submit for the degree Master of Science at the University of Pretoria, is my own work and has not previously been submitted by me for a degree at this or any other tertiary institution.
SIGNED: ....
DATE: ....

Table of Contents

INTRODUCTION

A subset K of a vector space X is called a cone if it is closed under addition, positive scalar multiplication and the only element of K with additive inverse is zero. Cones play very important role in pure and applied mathematics, in particular, the theory of ordered vector spaces, Riesz spaces, economic and optimization are based on properties of cones. The notion of cones also bring together the order structure and topology. This is achieved when one consider a convex subset of K, called a base. In this settings we can study the relations between order and topology.

In particular, one define an order on a vector space using a base of a cone (considered subcone of a cone), such order induced there so called Bronstead order which is mainly used to find a maximal element of a convex subset of a space X. This can also be used to study the relationship between Caristi fixed pint theory and Ekeland's variation principle. The order generated by a base of cone is not necessarily a lattice order. Therefore, the ordered vector with order induced by a base of a cone is more general that Riesz spaces.

The theory of reflexive Banach spaces were studied from 1970s, mainly by James. In this study we consider the characterizations of reflexive Banach spaces using the notion of cones. We consider, characterizations, the first one is using cones with bounded and unbounded base, the second one is using reflexive cones. The cone with a bounded and unbounded bases is called a mixed based cone.

We consider the relationship between reflexive cones and properties of their bases. In particular, we observe that the existence of a basic sequence in a reflexive cone depend on the existence of bounded and/or unbounded bases. The notion of reflexive cone allows us to obtain some important results on spaces. For example, if X is an ordered Banach space ordered by reflexive cone K and K normal, then X is Dedekind complete (order complete). Therefore, we can immediately, deduce that the positive cone $C^+[0,1]$ of $C[0,1]$, the space of all continuous functions on $[0,1]$ is not reflexive, since $C^+[0,1]$ is normal and $C[0,1]$ is not Dedekend complete. It is known that if a Banach X is reflexive, then its dual space X^* is also reflexive. However, this is not necessarily true on cones. That is, we can find a reflexive cone and its dual not reflexive.

The theory of reflexive Banach spaces plays a big role in Mathematical

Analysis, economics, finance and fixed point theory [1, 3, 5, 6, 11, 14, 15, 29]. For instance, in economic, the Leontief model is a model to find, an element e in X^+, where X is ordered Banach space, such that the map

$$T : X^+ \to X^+, Tx = x - e \text{ or } x = Tx + e$$

has a solution, depend on the reflexivity of the space X. That is, the exists $x^{**} \in (X^+)^{**}$ such that $x^{**} = T^{**}x^{**} + e$ and also there exists a sequence $(x_n) \subset X^*$ such that $x_n - Tx_n \to e$. Therefore, the solution $Tx = x - e$ exists in a reflexive Banach space.

We organize our work as follows;

In chapter 1, we recall definitions of partial ordering, cones, ordered vector spaces and Riesz spaces. We consider the relationship between different types of cones, namely, normal, lattice, and generating cones, as well as the relationship between Riesz space and ordered vectors space. We consider some examples to illustrate the difference between the notions, whenever is necessary.

In chapter 2, we consider cones with bases and provide examples of a cone with no base, mixed base cones, cone with only unbounded bases. We also consider the relationship between a cone and dual cone.

In chapter 3, we consider the characterization of Banach space in terms of a cones with bounded and unbounded base. We study also the example of cone with both bounded and unbounded bases in a reflexive space.

In chapter 4, we consider the characterization of reflexive Banach spaces using reflexive cones. We also study the relationship between reflexive cones and the properties.

1 Preliminaries.

In this chapter we recall definitions of partially ordered sets, ordered vector spaces and Riesz spaces. We also give some results regarding cones and their properties as well as some examples of these cones. We also provide examples to illustrate the difference between these cones. Definitions of ordered Banach spaces and Banach lattices are also recalled as well as special type of Banach space called Reflexive Banach space. Examples of these spaces are provided. We study operators acting between Banach lattices, in particular the relationship between the regular and order bounded operators. We also recall definitions of order isomorphism and order isometry and give examples.

1.1 Partially Ordered sets.

Definition 1.1. ([30, Definition 1.1]) The relation $\leq$ on a set X is called a partial ordering if,

(i) $x \leq x$ for every $x \in X$, (reflexivity).

(ii) $x \leq y$ and $y \leq z$ implies $x \leq z$ for every $x, y, z \in X$, (transitivity).

(iii) $x \leq y$ and $y \leq x$ implies $x = y$ for every $x, y \in X$. (antisymmetry).

The pair $(X, \leq)$ is called a partially ordered set. If x and y are points of X such that $x \leq y$ or $y \leq x$, we say that x and y are comparable.

Definition 1.2. Let $(X, \leq)$ be a partially ordered set, Y is a non - empty subset of X and $x_0 \in X$. The point x_0 is called,

(i) an upper bound if $y \leq x_0$ for all $y \in Y$. If $x_0 \leq z$ for any other upper bound z of Y, then x_0 is called a supremum of Y. We write $x_0 = \sup\{y : y \in Y\}$ for the supremum of Y.

(ii) a lower bound if $x_0 \leq y$ for all $y \in Y$. If $z \leq x_0$ for any other lower bound z of Y, then x_0 is called an infimum of Y. We write $x_0 = \inf\{y : y \in Y\}$ for the infimum of Y.

(iii) a maximal element of X if it follows from $x_0 \leq x \in X$ that $x_0 = x$.

(iv) a largest element of X if $x_0 \geq x$ for all $x \in X$.

(v) a minimum element of X if it follows from $x_0 \geq x \in X$ that $x_0 = x$.

(vi) a smallest element of X if $x_0 \leq x$ for all $x \in X$.

Proposition 1.3. Let $(X, \leq)$ be a partially ordered space. If $x_0 \in X$ is the largest element, then x_0 is a maximal element.

Proof. Assume x_0 is the largest element of X, that is, $x_0 \geq x$ for all $x \in X$. This means no $x \in X$ is strictly greater that x_0. Therefore x_0 is the maximal element in X. □

We show that the converse of the above results is not neccesarily true.

Example 1.4. (cf [30, Example 1.2]) Consider partially ordered space $(\mathbb{R}^2, \leq)$, where $\leq$ is a pointwise ordering, that is,

$$(x_1, x_2) \leq (y_1, y_2) \text{ if and only if } x_1 \leq y_1 \text{ and } x_2 \leq y_2$$

and the closed unit disc

$$A = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\} \subseteq \mathbb{R}^2.$$

The points (x, y) such that

$$x \geq 0, y \geq 0, x^2 + y^2 = 1$$

are maximal elements. However, they are not the largest, since $(0, 1)$ is not comparable with any of other maximal elements.

Definition 1.5. ([30, Definition 1.3]) A partially ordered set X is called;

(i) Dedekind complete if every non - empty subset of X that is bounded above (bounded below) has a supremum (infimum).

(ii) Dedekind σ- complete if every non - empty finite or countable subset of X that is bounded above (bounded below) has a supremum (infimum).

(iii) a lattice if every subset consisting of two points, x, y, has a supremum denoted by $x \vee y$ and an infimum denoted by $x \wedge y$.

1.2 Ordered vector space and Riesz spaces.

Definition 1.6. ([30, Definition 4.1]) A real vector space X is called an ordered vector space, denoted by $(X, \leq)$, if X is partially ordered such that the vector space structure and the order structure are compatible, that is,

(i) $x \leq y$ implies $x + z \leq y + z$ for every $z \in X$.

(ii) $\overline{0} \leq x$ implies $\overline{0} \leq \alpha x$ for every $x \in X$ and $0 \leq \alpha$.

Definition 1.7. Let X be an ordered vector space and $x, y \in X$. The set

$$[x, y] = \{z \in X : x \leq z \leq y\}$$

is called an order interval of X.

Definition 1.8. Let C be a subset of an ordered vector space X. The set C is called

(i) order convex if $[x, y] \subset C$ for every $x, y \in C$.

(ii) order bounded if $C \subseteq [x, y]$ for some $x, y \in X$.

Definition 1.9. ([2, Definition 1.14]) An ordered vector space X is a Riesz space if every pair of vectors $x, y \in X$ has a supremum and an infimum in X.

Definition 1.10. Let X be a Riesz space and $x \in X$. We recall the following notations,

(i) $x^+ = x \vee \overline{0}$,

(ii) $x^- = (-x) \vee \overline{0}$,

(iii) $|x| = x \vee (-x)$.

Theorem 1.11. ([30, Theorem 5.1]) Let X be a Riesz space, $x, y, z \in X$, then

(i) x^+, x^- are elements of X^+; $(-x)^+ = x^-$, and $|-x| = |x|$.

(ii) $x = x^+ - x^-, x^+ \wedge x^- = \overline{0}$ and $|x| = x^+ + x^-$. Hence $|x| \in X^+$.

(iii) $\overline{0} \leq x^+ \leq |x|$ and $\overline{0} \leq x^- \leq |x|$.

(iv) $x \leq y$ if and only if $x^+ \leq y^+$ and $x^- \leq y^-$.

(v) $x + y \vee z = (x + y) \vee (x + z)$.

(vi) $x - (y \wedge z) = (x - y) \vee (x - z)$.

Remark 1.12. The elements x^+, x^- and $|x|$ are called the positive part, negative part and absolute value of x, respectively.

We recall the following characterization of Riesz space.

Lemma 1.13. [2, Lemma 1.15] An ordered vector space X is a Riesz space if and only if for every pair of vectors $x, y \in X$ their supremum $x \vee y$ exists in X.

Proposition 1.14. (cf [2, Exercise 1.3.2]) An ordered vector space X is a Riesz space if and only if for $x \in X$ the supremum $x^+ = x \vee \overline{0}$ exists in X.

Proof. Let X be a Riesz space. Then for $x, \overline{0} \in X$, we have $x^+ = x \vee \overline{0} \in X$ by definition of X.
Conversely, let $x^+ = x \vee \overline{0}$ be in an ordered vector space X. Then for $x, y \in X$ we have $x - y \in X$ since X is a vector space and $(x-y)^+ = (x-y) \vee \overline{0} \in X$ by the assumption. Therefore

$$\begin{aligned} x \vee y &= [(x-y)+y] \vee (\overline{0}+y) && \\ &= (x-y) \vee \overline{0} + y && \text{by Theorem 1.11 (v)} \\ &= (x-y)^+ + y && \text{by the assumption} \\ &\in X && \text{by definition of X} \end{aligned}$$

By Lemma 1.13, an ordered vector space X is a Riesz space. □

We give an example of a Riesz space.

Example 1.15. Let $X = C([0,1])$ be the set of all real continuous functions on [0,1]. Then X is a Riesz space.

Proof. We define the ordering $f \leq g$ whenever $f(x) \leq g(x)$ for all $x \in [0,1]$. The set X is a partially ordered vector space. To see, let $f, g, h \in X$. We have that $f(x) = f(x)$ which implies $f(x) \leq f(x)$ for all $x \in [0,1]$, hence $\leq$ is reflexive. If we assume $f \leq g$ and $g \leq h$, we get $f(x) \leq g(x)$ and $g(x) \leq h(x)$ for all $x \in [0,1]$. Since we have real valued functions, then $f(x) \leq h(x)$ for all $x \in [0,1]$ and so $f \leq h$. Hence we have transitivity. Assuming $f \leq g$ and $g \leq f$, we get $f(x) \leq g(x)$ and $g(x) \leq f(x)$ for all $x \in [0,1]$. Thus $f(x) = g(x)$ for all $x \in [0,1]$, and $f = g$. Hence we have anti - symmetry. As for ordering structure, assume $f \leq g$, which implies $f(x) \leq g(x)$ for all $x \in [0,1]$. Thus we have that $f(x) + h(x) \leq g(x) + h(x)$ for all $x \in [0,1]$ and $h \in X$. Hence we have that $f + h \leq g + h$ for any $h \in X$. Similarly we assume $\overline{0} \leq f$ which gives $0 \leq f(x)$ for all $x \in X$. Now $0 \leq \alpha f(x)$ for all $x \in X$ provided $\alpha \geq 0$. Therefore $(X, \leq)$ is an ordered vector space. We now show it is a lattice. Let $f, g \in X$, then

$$f \vee g = \tfrac{1}{2}((f+g) + |f-g|)$$

and

$$f \wedge g = \tfrac{1}{2}((f+g) - |f-g|)$$

are continuous functions on [0,1] by properties of continuity. Thus

$$f \vee g, f \wedge g \in X.$$

Since f, g are arbitrary then X is a Riesz space. $\square$

We now give example of ordered vector space which is not a Riesz space.

Example 1.16. (cf [2, Excercise 1.3.4]) Let $X = C^1[0,1]$ be set of all continuously differentiable functions on [0,1]. Then X is an ordered space which is not a Riesz space.

Proof. We define the ordering $f \leq g$ if and only if $f(x) \leq g(x)$ for all $x \in [0,1]$. The set X is a partially ordered vector space. The proof is similar to the one given in Example 1.15. Secondly, we show X is not a Riesz space. To see, consider the functions $f(x) = x$ and $g(x) = 1-x$. Clearly, they are continuous and differentiable on [0,1], that is, $f, g \in X$. Now

$$f(x) \vee g(x) = \sup(f,g) = |x - \tfrac{1}{2}| + \tfrac{1}{2} = \begin{cases} x, & \text{if } x \geq \frac{1}{2}; \\ 1-x, & \text{if } x < \frac{1}{2}. \end{cases}$$

is continuous but not differentiable (at $x = \frac{1}{2}$) on [0,1]. Since

$$\tfrac{d(f(x) \vee g(x))}{dx} = \begin{cases} 1, & \text{if } x > \frac{1}{2}; \\ -1, & \text{if } x < \frac{1}{2}. \end{cases}$$

Thus $f(x) \vee g(x) \notin X$. Therefore $(X, \leq)$ is an ordered vector space but not a Riesz space. $\square$

1.3 Cones and their properties.

Definition 1.17. Let X be a vector space and K non-empty subset of X. The set K is called a cone if:

(i) $f \in K, g \in K$ then $f + g \in K$.

(ii) $f \in K$ implies $\alpha f \in K$ for any real number $\alpha \geq 0$.

(iii) $f \in K, f \in -K$ implies $f = 0$.

Definition 1.18. A subset K of a vector space X is said to be convex if $x, y \in K$ implies $tx + (1-t)y \in K$ for all $t \in [0, 1]$.

Remark 1.19. Cones are convex sets, since $t, 1-t \geq 0$, and $tx, (1-t)y$ are in K by (i) and (ii).

We now give an example of a cone.

Example 1.20. Consider vector space $\mathbb{R}^2$. The subset

$$K = \{(x, y) \in \mathbb{R}^2 : x > 0 \text{ or } (x = 0 \text{ and } y \geq 0)\}$$

is a cone in $\mathbb{R}^2$ and it is called lexicographic cone.

Proof. K is not empty, since $(0, 0) \in K$. To show (i) in Definition 1.17, let $(x_1, y_1), (x_2, y_2) \in K$ and consider the following 3 cases.

a) If $x_1, x_2 > 0$, then $x_1 + x_2 > 0$ and therefore $(x_1, y_1) + (x_2, y_2) \in K$.

b) If $x_1 > 0$ and $x_2 = 0$ and $y_2 \geq 0$ then $x_1 + x_2 > 0$ and therefore $(x_1, y_1) + (x_2, y_2) \in K$.

c) If $x_1 = 0, x_2 = 0$ and $y_1 \geq 0, y_2 \geq 0$ then $x_1 + x_2 = 0$ and $y_1 + y_2 \geq 0$ and therefore $(x_1, y_1) + (x_2, y_2) \in K$.

To show (ii) in Definition 1.17, let $(x_1, y_1) \in K, \alpha \in \mathbb{R}$ and $\alpha \geq 0$ and consider the following 2 cases.

a) If $x_1 = 0$ and $y_1 \geq 0$ then $\alpha x_1 = 0$ and $\alpha y_1 \geq 0$, therefore $\alpha(x_1, y_1) = (\alpha x_1, \alpha y_1) \in K$.

b) If $x_1 > 0$ then

   (1) $\alpha x_1 > 0$ for $\alpha > 0$,

   (2) $\alpha x_1 = 0$ and $\alpha y_1 = 0$ if $\alpha = 0$ for any y_1.

   Therefore $\alpha(x_1, y_1) = (\alpha x_1, \alpha y_1) \in K$.

Lastly, we show that $K \bigcap (-K) = \{\overline{0}\}$, where

$$-K = \{(x, y) \in \mathbb{R}^2 : x < 0 \text{ or } x = 0 \text{ and } y \leq 0\}.$$

To see, first we have

$$\begin{aligned} K \bigcap (-K) &= \{(x, y) \in \mathbb{R}^2 : x > 0 \text{ or } (x = 0 \text{ and } y \geq 0)\} \cap \{(x, y) \in \mathbb{R}^2 : x < 0 \text{ or } x = 0 \text{ and } y \leq 0\} \\ &= \{(x, y) \in \mathbb{R}^2 : x > 0 \text{ or } (x = 0 \text{ and } y \geq 0) \text{ and } x < 0 \text{ or } (x = 0 \text{ and } y \leq 0)\}. \end{aligned}$$

and for $(x, y) \in K \bigcap (-K)$ we consider the following cases:

(a) $x > 0$ and $x < 0$. Not possible.

(b) $x > 0$ and ($x = 0$ or $y \leq 0$). Not possible.

(c) ($x = 0$ or $y \geq 0$) and $x < 0$. Not possible.

(d) ($x = 0$ or $y \geq 0$) and ($x = 0$ or $y \leq 0$) implying that $x = 0, y = 0$.

Since (x, y) is arbitrary then $K \bigcap (-K) = \{\overline{0}\}$. Hence K is a cone. □

Next we show that an arbitrary cone K of a vector space X defines a vector ordering on X.

Proposition 1.21. Let X be a vector space and K a cone in X. The relation " $\leq$ " defined by $x \leq y$ if and only if $y - x \in K$ for all $x, y \in X$, is a partial ordering on X.

Proof. Let $x, y, z \in X$. Since $\overline{0} \in K$ we see that $x \leq x$ for all $x \in X$, meaning that X is reflexive. Secondly, assume that $x \leq y$ and $y \leq x$. Then

$$y - x \in K \text{ and } x - y \in K.$$

But

$$-(y - x) = (x - y) \in K.$$

Then by definition, $x - y = \overline{0}$. Hence $x = y$. Therefore X is anti - symmetric. Finally, assume that $x \leq y$ and $y \leq z$. Then

$$y - x \in K \text{ and } z - y \in K.$$

By definition

$$z - x = (y - x) + (z - y) \in K.$$

Thus, $x \leq z$ and therefore X is transitive. □

Proposition 1.22. Let X be an ordered vector space. The set

$$X^+ = \{x \in X : x \geq \overline{0}\}$$

is a cone.

Proof. If $x, y \in X^+$, then $x + y \geq \overline{0} + y = y \geq \overline{0}$. Thus $x + y \in X^+$. If $x \in X^+$ and $\alpha \geq 0$. Then $x \geq \overline{0}, \alpha x \geq \alpha \overline{0} = \overline{0}$. Thus $\alpha x \in X^+$. Lastly, $-X^+ = \{-x : x \in X^+\} = \{-x : x \geq \overline{0}\} = \{x : x \leq \overline{0}\}$. So, $X^+ \cap (-X^+) = \{x : x \leq \overline{0} \text{ and } x \geq \overline{0}\} = \{\overline{0}\}$. Thus X^+ is a cone. □

The set X^+ is called positive cone and its elements are called positive elements.

Definition 1.23. A non empty subset D of an ordered vector space X is said to be directed upwards if for every pair $x, y \in D$ there exists $z \in D$ such that $x \leq z$ and $y \leq z$.

Definition 1.24. A Riesz space X is Archimedean if $\overline{0} \leq nx \leq y$ for all $n \in \mathbb{N}$ and some $y \in X^+$ implies that $x = \overline{0}$.

Next we show that not all Riesz spaces are Archimedean.

Example 1.25. Let $X = \mathbb{R}^2$ and $\leq$ the lexicographic ordering, that is, $(x_1, x_2) \leq (y_1, y_2)$, if and only if

$$x_1 < y_1 \text{ or } (x_1 = y_1 \text{ and } x_2 \leq y_2).$$

The space $(X, \leq)$ is a Riesz space but not Archimedean.

Proof. X is a Riesz space, since any two arbitrary vectors $u, v \in X$ are comparable. To see it is not Archimedean, consider $(0, 1)$ and $(1, 1)$ in $\mathbb{R}^2$ with $(1, 1) \in (\mathbb{R}^2)^+$. Then $(0, 1) \leq (1, 1)$ since $0 < 1$ and for all $n \in \mathbb{N}$, we have $\overline{0} \leq n(0, 1) \leq (1, 1)$ but $(0, 1) \neq (0, 0) = \overline{0}$. Therefore $(X, \leq)$ is not Archimedean. □

Definition 1.26. A cone K of an ordered vector space X is called

(i) generating if $X = K - K$.

(ii) Archimidean if the order induced by K on X makes X an Archimedean space.

(iii) lattice if X is a Riesz space.

The following is a well known result about Riesz space.

Proposition 1.27. Let X be a Riesz space and K be a cone in X. Then K is generating.

Proof. Let K be a cone of a Riesz space X. So, for any $x \in X$ we have that

$$x = x^+ - x^- \text{ and } x^+, x^- \in K.$$

Hence K is generating. □

Theorem 1.28. (cf [30, Theorem 6.4]) (Riesz decomposition property) Let $u, z_1, z_2 \in X^+$ satisfy $u \leq z_1 + z_2$. Then there exist $u_1, u_2 \in X^+$ such that $u_1 \leq z_1, u_2 \leq z_2$ and $u = u_1 + u_2$.

Proof. Let $u_1 = u \wedge z_1$ and $u_2 = u - u_1$. Then $\overline{0} \leq u_1$, because $u \geq \overline{0}$ and $z_1 \geq \overline{0}$. So $u_1 \in X^+$ and $u_1 \leq z_1$. Since $u_1 \leq u$, we have $u_2 = u - u_1 \geq \overline{0}$, thus $u_2 \in X^+$. We now show that $u_2 \leq z_2$.

$$\begin{aligned} u_2 &= u - u_1 \\ &= u - (u \wedge z_1) \\ &= (u - u) \vee (u - z_1) && \text{by Theorem 1.11 (vi)} \\ &= \overline{0} \vee (u - z_1) \\ &\leq z_2 && \text{since } \overline{0} \leq z_2 \text{ and } u - z_1 \leq z_2 \end{aligned}$$

Hence the result. □

Next we give characterization of an ordered vector space with Riesz decomposition property.

Theorem 1.29. (cf [2, Lemma 1.51]) An ordered vector space X has the Riesz decomposition property if and only if

$$[\overline{0}, x] + [\overline{0}, y] = [\overline{0}, x + y]$$

holds for all $x, y \in X^+$.

Proof. Assume that X has a Riesz decomposition property. Let $x \in X^+$ be an arbitrary vector such that $x \leq u + v$ for $u, v \in X^+$ where $\overline{0} \leq x_1 \leq u$ and $\overline{0} \leq x_2 \leq v$ and $x = x_1 + x_2$. Then $x \in [\overline{0}, u + v], x_1 \in [\overline{0}, u]$ and $x_2 \in [\overline{0}, v]$. This also means $x = x_1 + x_2 \in [\overline{0}, u] + [\overline{0}, v]$. Hence the result. Conversely, assume that

$$[\overline{0}, x] + [\overline{0}, y] = [\overline{0}, x + y]$$

holds for all $x, y \in X^+$. Take any $u \in X^+$ such that $u \in [\overline{0}, x] + [\overline{0}, y] = [\overline{0}, x + y]$. Then $\overline{0} \leq u \leq x + y$ and there exist $x_1, y_1 \in X^+$ such that $\overline{0} \leq x_1 \leq x, \overline{0} \leq y_1 \leq y$ and $u = x_1 + x_2$. Hence the result. □

Corollary 1.30. (cf [2, Corollary 1.55]) Every Riesz space has the Riesz decomposition property.

Proof. Let X be a Riesz space and X^+ be a lattice cone. Take $u, v \in X^+$ such that $0 \leq x \leq u + v$ for any $x \in X$. Clearly

$$A = \{\overline{0}, x - u\} \leq \{v, x\} = B.$$

Let $x_1 = \sup A$, it follows that $\overline{0} \leq x_1 \leq v$ and if we let $x_2 = x - x_1$, then from $x - u \leq x_1 \leq x$ we get $\overline{0} \leq x_2 = x - x_1 \leq u$. Clearly, $x_1 + x_2 = x$ and so X has the Riesz decomposition. □

1.4 Ordered Banach spaces and Banach lattices.

Definition 1.31. ([19, Definition 3.2]) Let X be an ordered vector space over $\mathbb{R}$. A real - valued function $\| \bullet \| : X \to \mathbb{R}$ is said to be a norm on X if

(i) $\|\alpha x\| = |\alpha| \|x\|$,

(ii) $\|x + y\| \leq \|x\| + \|y\|$,

(iii) $\|x\| \geq 0$ and $\|x\| = 0$ if and only if $x = \overline{0}$.

for all $\alpha \in \mathbb{R}$ and $x, y \in X$. The pair $(X, \| \bullet \|)$ is said to be an ordered normed space.

The following normed vector space and subspaces will occur in several cases later.

(i) Sequence spaces: For $1 \leq p < \infty$, consider the set

$$\ell_p = \{(x_1, x_2, ..., x_i, ...) : x_i \in \mathbb{R} \text{ and } \sum_{i=1}^{\infty} |x_i|^p < \infty\}.$$

For $x = (x_1, x_2, ..., x_i, ...) \in \ell_p$, we define a norm as follows

$$\|x\|_p = \left(\sum_{i=1}^{\infty} |x_i|^p \right)^{\frac{1}{p}}.$$

(ii) Set of bounded sequences in $\mathbb{R}$:

$$\ell_\infty = \{(x_1, x_2, ..., x_i, ...) : x_i \in \mathbb{R} \text{ and } \sup_{i \in \mathbb{N}} |x_i| < \infty\}.$$

For $x = (x_1, x_2, ..., x_i, ...) \in \ell_\infty$, we define a norm as follows

$$\|x\|_\infty = \sup_{i \in \mathbb{N}} \{|x_i| : i = 1, 2, 3, ...\}$$

(iii) $c = \{x \in \ell_\infty : x_i \text{ converges in } \mathbb{R} \text{ as } i \to \infty\}$

(iv) $c_0 = \{x \in c : x_i \to 0 \text{ as } i \to \infty\}$

(v) $c_{00} = \{x \in \ell_p$: all but infinitely many x_i's are equal to 0.$\}$

Also, $c_{00} \subset \ell_p \subset c_0 \subset c \subset \ell_\infty$, for $1 \le p < \infty$.
Now we show that converse does not hold by only showing the following; $c_0 \nsubseteq \ell_p$ and $\ell_p \nsubseteq c_{00}$.

Example 1.32. Let $X = c_0$ and $Y = \ell_p, 1 \le p < \infty$. The space $X \nsubseteq Y$ in general. To this end, take $x_n = \dfrac{1}{n^{\frac{1}{p}}}$. Then $(x_n) \subset c_0$, since $\lim\limits_{n\to\infty} \dfrac{1}{n^{\frac{1}{p}}} = 0$ for $1 \le p < \infty$. But $\left(\frac{1}{n^{\frac{1}{p}}}\right) \nsubseteq \ell_p$, since

$$\sum_{n=1}^{\infty} \left|\frac{1}{n^{\frac{1}{p}}}\right|^p = \sum_{n=1}^{\infty} \left(\left|\frac{1}{n}\right|^{\frac{1}{p}}\right)^p \qquad \text{since } \frac{1}{n} > 0 \text{ for all } n$$
$$= \sum_{n=1}^{\infty} \frac{1}{n}$$

is a p - series with $p = 1$ and therefore diverges.

Next we show that $\ell_p \nsubseteq c_{00}$.

Example 1.33. Let $X = \ell_p, 1 \le p < \infty$ and $Y = c_{00}$. The space $X \nsubseteq Y$ in general. To this end, take $x_n = \dfrac{1}{n^{\frac{2}{p}}}$. Then $(x_n) \subset \ell_p$ since for $1 \le p < \infty$, the series

$$\sum_{n=1}^{\infty} \left|\frac{1}{n^{\frac{2}{p}}}\right|^p = \sum_{n=1}^{\infty} \frac{1}{n^2} \qquad \text{since } \frac{1}{n^2} > 0 \text{ for all } n$$

is a p - series with $p > 1$. But $(x_n) \nsubseteq c_{00}$ because (x_n) has infinitely many nonzero terms.

Definition 1.34. Let $(X, \|\bullet\|)$ be a normed space, $x \in X$ and $r > 0$. The set

$$B(x, r) = \{y \in X : \|x - y\| < r\}$$

is open ball and $B[x, r] = \{x \in X : \|x\| \le 1\}$ a closed ball in X with radius $r > 0$. We now consider special cases of balls in X.

(i) $B[0, 1] = \{x \in X : \|x\| \le 1\}$

(ii) $B(0, 1) = \{x \in X : \|x\| < 1\}$

(iii) $S(0, 1) = \{x \in X : \|x\| = 1\}$

are called a closed, open unit balls, and unit sphere respectively. A closure of set A in X is denoted by clA and intA in X denotes interior of A.

We now prove the following property.

Proposition 1.35. Let $(X, \|\bullet\|)$ be normed space. Then norm topology is Hausdorff.

Proof. Take $x, y \in (X, \|\bullet\|)$ such that $x \neq y$. The $\|x-y\| > 0$. Now let ϵ be in the set $(0, \frac{\|x-y\|}{2})$. Then $B(x,\epsilon) \cap B(y,\epsilon) = \emptyset$. We prove this by contradiction. Suppose $B(x,\epsilon) \cap B(y,\epsilon) \neq \emptyset$. Then there is

$$z \in B(x,\epsilon) \cap B(y,\epsilon).$$

This imply that $\|z-y\| < \epsilon$. So

$$\begin{aligned} 0 &\leq \|x-y\| \\ &\leq \|x-z\| + \|z-y\| \\ &< \epsilon + \epsilon \\ &< \frac{\|x-y\|}{2} + \frac{\|x-y\|}{2} \\ &= \|x-y\|, \end{aligned}$$

which is a contradiction, since $\|x-y\| \not< \|x-y\|$. □

Definition 1.36. Let $(X, \|\bullet\|)$ be a normed space and C a subset of X.

(i) The set C is said to be closed, if for any sequence $(x_n) \subset C$ with $x_n \to x$ in X imply $x \in C$.

(ii) A sequence $(x_n) \subset X$ is said to be Cauchy, if for every $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that whenever $m > N$ and $n > N$ then $\|x_n - x_m\| < \epsilon$.

We can now show that not all cones are closed.

Example 1.37. Consider ordered normed vector space $\mathbb{R}^2$. The lexicographic cone in $\mathbb{R}^2$ with the following ordering

$$(x_2, y_2) \geq (x_1, y_1) \text{ if } x_2 > x_1 \text{ or } x_2 = x_1 \text{ and } y_2 \geq y_1$$

is not closed. To see this, take the sequence $(\frac{1}{n}, -1) \in K$ for all $n \in \mathbb{N}$ since $x = \frac{1}{n} > 0$ for all n and converges to $(0,-1) \notin K$ since $x = 0$ and $y \ngeq 0$.

Definition 1.38. An ordered normed space X is said to be an ordered Banach space if every Cauchy sequence in X converges to a limit in X. That is, if X is complete.

Definition 1.39. ([18, Definition 1.6.7]) Let $\| \bullet \|$ and $\| \bullet \|_0$ be two norms on a normed vector space X. We say $\| \bullet \|$ is equivalent to $\| \bullet \|_0$ if there are two real numbers $\alpha, \beta > 0$ such that

$$\alpha\|x\| \leq \|x\|_0 \leq \beta\|x\|$$

for all $x \in X$.

Theorem 1.40. Let $\| \bullet \|$ and $\| \bullet \|_0$ be two norms on a vector space X. Then $\| \bullet \|$ and $\| \bullet \|_0$ are equivalent if and only if ($x_n \overset{\|\bullet\|}{\to} x$ if and only if $x_n \overset{\|\bullet\|_0}{\to} x$)

We state the following useful result without proof.

Theorem 1.41. Let K be a subset of Banach space X. The following statements are equivalent.

(ii) K is closed.

(ii) K is complete.

Example 1.42. We show that $C^1[0,1]$, the space of all continuously differentiable functions, is not a closed subspace of Banach space

$$(C[0,1], \| \bullet \|_\infty).$$

To this end, consider a function f, defined by $f(x) = \left|x - \frac{1}{2}\right|$. Then $f \in C[0,1]$. We show that $f \notin C^1[0,1]$. Now,

$$f'(x) = \begin{cases} 1, & \text{if } x > \frac{1}{2}; \\ -1, & \text{if } x < \frac{1}{2}. \end{cases}$$

Then $f \notin C^1[0,1]$ since

$$\lim_{x \to (\frac{1}{2})^-} \frac{f(x) - f(\frac{1}{2})}{x - \frac{1}{2}} = -1 \neq 1 = \lim_{x \to (\frac{1}{2})^+} \frac{f(x) - f(\frac{1}{2})}{x - \frac{1}{2}}.$$

We next show that there is a sequence of functions in $(C[0,1], \| \bullet \|_\infty)$ that converges to f. Take

$$f_n(x) = \sqrt{\left(x - \frac{1}{2}\right)^2 + \frac{1}{n}} \qquad \text{for all } x \in [0,1].$$

Now,

$$|f_n(x) - f(x)| = \left| \sqrt{\left(x - \frac{1}{2}\right)^2 + \frac{1}{n}} - \sqrt{\left(x - \frac{1}{2}\right)^2} \right|$$

$$= \frac{(x-\frac{1}{2})^2 + \frac{1}{n} - (x-\frac{1}{2})^2}{\sqrt{\left(x-\frac{1}{2}\right)^2 + \frac{1}{n}} + \sqrt{\left(x-\frac{1}{2}\right)^2}},$$

$$= \frac{\frac{1}{n}}{\sqrt{\left(x-\frac{1}{2}\right)^2 + \frac{1}{n}} + \sqrt{\left(x-\frac{1}{2}\right)^2}}.$$

Since

$$\sqrt{\left(x - \frac{1}{2}\right)^2 + \frac{1}{n}} + \sqrt{\left(x - \frac{1}{2}\right)^2} \geq \frac{1}{\sqrt{n}},$$

it follows

$$|f_n(x) - f(x)| \leq \tfrac{1}{\sqrt{n}},$$

for all $x \in [0, 1]$, so that

$$\|f_n - f\|_\infty = \tfrac{1}{\sqrt{n}} \to 0, \text{ as } n \to \infty.$$

So (f_n) is uniformly convergent to f but $f \notin C^1[0, 1]$. This implies that $C^1[0, 1]$ is not a closed subspace of $(C[0, 1], \| \bullet \|_\infty)$. Hence by Theorem 1.41, $C^1[0, 1]$ is not complete with respect to the norm $\| \bullet \|_\infty$.

We show that $C^1[0, 1]$ is complete with a different norm but first we show that the function f_n defined above does not converge with respect to this norm.

Example 1.43. We show that a sequence (f_n) defined above is not convergent with respect to the norm

$$\|f\| = \|f\|_\infty + \|f'\|_\infty$$

for $f \in X$.

Proof. Let f_n be defined as in above example and consider the following function,

$$f(x) = \left|x - \tfrac{1}{2}\right|$$

and its derivative,

$$f'(x) = \begin{cases} 1, & \text{if } x > \frac{1}{2}; \\ -1, & \text{if } x < \frac{1}{2}. \end{cases}$$

Claim: $f_n \overset{\|\bullet\|}{\nrightarrow} f$.
To see this, note that

$$\begin{aligned}
\|f_n - f\| &= \sup_{x\in[0,1]} |f_n(x) - f(x)| + \sup_{x\in[0,1]} |f_n'(x) - f'(x)| \\
&= \sup_{x\in[0,1]} \left| \sqrt{\left(x-\frac{1}{2}\right)^2 + \frac{1}{n}} - \sqrt{\left(x-\frac{1}{2}\right)^2} \right| + \sup_{x\in[0,1]} \left| \frac{x-\frac{1}{2}}{\left[\left(x-\frac{1}{2}\right)^2 + \frac{1}{n}\right]^{\frac{1}{2}}} - f'(x) \right| \\
&= 0 + \sup_{x\in[0,1]} \left| \frac{x-\frac{1}{2}}{\left[\left(x-\frac{1}{2}\right)^2\right]^{\frac{1}{2}}} - f'(x) \right|,
\end{aligned}$$

as $n \to \infty$.

If $n \to \infty$, then $\|f_n - f\| = \sup_{x\in[0,1]} \left| \frac{x-\frac{1}{2}}{\left|x-\frac{1}{2}\right|} - f'(x) \right| \to 2 \neq 0$. Since

$$\frac{x-\frac{1}{2}}{\left|x-\frac{1}{2}\right|} = \begin{cases} 1, & \text{if } x > \frac{1}{2}; \\ -1, & \text{if } x < \frac{1}{2}. \end{cases}$$

So $f_n \overset{\|\bullet\|}{\nrightarrow} f$. □

Remark 1.44. By Theorem 1.40, the norms $\|\bullet\|_\infty$ and $\|\bullet\|$ are not equivalent, since $f_n \overset{\|\bullet\|_\infty}{\rightarrow} f$ but $f_n \overset{\|\bullet\|}{\nrightarrow} f$.

Now we recall the following lemma.

Lemma 1.45. ([19, Lemma 6.6]) Let (f_n) be a sequence of continuously differentiable functions on [0,1] such that

(i) $f_n \to f$ uniformly, and

(ii) $f_n' \to g$ uniformly.

Then f is differentiable and $f' = g$.

We next show that the space $(C^1[0,1], \|\bullet\|)$ is complete.

Proposition 1.46. The space $(C^1[0,1], \|\bullet\|)$ is complete where

$$\|f\| = \|f\|_\infty + \|f'\|_\infty.$$

To see, let (f_n) be a Cauchy sequence in $C^1[0,1]$, that is, $f_n, f_n' \in C[0,1]$ so that, by the completeness of $C[0,1]$, there exists $f \in C[0,1]$ with

$$\|f_n - f\|_\infty \to 0.$$

Similarly, (f_n') is also Cauchy sequence in $C[0,1]$ so that, again by the completeness of $C[0,1]$, there exists $g \in C[0,1]$ with $\|f_n' - g\|_\infty \to 0$. By Lemma 1.45, the function f is differentiable with $f' = g$, so that $f_n \to f$ in $C^1[0,1]$ with respect to the norm of $C^1[0,1]$.

Definition 1.47. Let X be a Riesz space. A norm $\| \bullet \|$ on X is called a lattice norm if $|x| \leq |y|$ implies $\|x\| \leq \|y\|$ for all $x, y \in X$.

Definition 1.48. ([26, Definition 5.1]) If $\| \bullet \|$ is a lattice norm on ordered vector space X, the pair $(X, \| \bullet \|)$ is a normed Riesz space; if in addition, $(X, \| \bullet \|)$ is Banach space, it is called a Banach lattice.

We now give 2 examples of Banach lattices which are used more often in this project.

Example 1.49. Consider the ordered Banach space $X = C[0,1]$ with the norm

$$\|f\|_\infty = \sup_{t \in [0,1]} |f(t)|.$$

We show that X is a Banach lattice.

Proof. By definition we only show that the norm is a lattice norm. Let $|f| \leq |g|$, that is, $|f(t)| \leq |g(t)|$ for each $t \in [0,1]$. Then

$$\|f\|_\infty = \sup_{t \in [0,1]} |f(t)| \leq \sup_{t \in [0,1]} |g(t)| = \|g\|_\infty.$$

Thus we have a lattice norm. Therefore $(X, \| \bullet \|_\infty)$ is a Banach lattice. □

Example 1.50. The ordered Banach space $(\ell_p, \| \bullet \|_p)$ is a Banach lattice.

Proof. We show that $\| \bullet \|_p$ is a lattice norm. We define a partial ordering as follows, if $x, y \in \ell_p$ then we say $x \leq y$ if and only if $x_i \leq y_i$ for all i. Let $x = (x_n), y = (y_n) \in \ell_p$ with $|x| \leq |y|$, that is $|x_i| \leq |y_i|$ for all i. Then

$$\|x\|_p = \left(\sum_{i=1}^{\infty} |x_i|^p \right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^{\infty} |y_i|^p \right)^{\frac{1}{p}} = \|y\|_p.$$

Thus, $(\ell_p, \| \bullet \|_p)$ is Banach lattice. $\square$

Next we show that lattice cones are closed.

Theorem 1.51. Let $(X, \| \bullet \|)$ be a Banach lattice and let $K = X^+$ be the positive cone of X. Then K is closed.

Proof. Let (x_n) be a sequence in K such that $x_n \to x \in X$ as $n \to \infty$. Therefore, for every $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that $\|x_n - x\| < \epsilon$ for all $n \geq N$. Since

$$|x_n - x^+| \leq |x_n - x|$$

$(x^+ \geq x)$ and X is Banach lattice, it follows that for all $n \geq N$

$$\||x_n - x^+|\| = \|x_n - x^+\| \leq \|x_n - x\| < \epsilon.$$

Thus $x_n \to x^+$ as $n \to \infty$. Since the norm topology is Hausdsorff, we obtain $x^+ = x \in K$. $\square$

We now revisit the notion of cone and define some cone that depends on a norm.

Definition 1.52. A cone K in a normed ordered vector space $(X, \| \bullet \|)$ is called

(i) normal if for every $x, y \in K$ such that $x \leq y$ we have that $\|x\| \leq M\|y\|$, for all $M \geq 1$.

(ii) solid if its interior is non - empty. That is, if there exists $r > 0$ such that $B(a, r) \subset K$ for some a in K and $r > 0$.

Proposition 1.53. If K is a solid cone in a normed vector space $(X, \| \bullet \|)$. Then K is a generating cone.

Proof. Since cone K is solid then $B(a, r) \subset K$ for some $a \in int(K)$ and $r > 0$. Let $x \in X, x \neq \bar{0}$. Then there exists $\alpha > 0$ such that $\alpha x \in B(0, r)$. Hence

$$a + \alpha x \in B(a, r) \subset K,$$

that is,

$$\alpha x \in K - a \subset K - K.$$

It follows that

$$x \in \alpha^{-1}(K-K) = \alpha^{-1}K - \alpha^{-1}K = K - K.$$

By definition of a cone, $\{\overline{0}\} = K \cap (-K)$. Therefore $\overline{0} \in K - K$, and thus $X = K - K$. □

The next example shows that the converse of Proposition 1.53 is not necessarily true.

Example 1.54. Consider the Riesz space ℓ_1 with norm $\|\bullet\|_1$ and its positive cone ℓ_1^+. Then ℓ_1^+ is generating but $\text{int}(\ell_1^+) = \emptyset$.

Proof. By Proposition 1.27, ℓ_1^+ is generating, since ℓ_1 is the Riesz space. To see that $\text{int}(\ell_1^+) = \emptyset$, suppose on the contrary that there is $y \in \text{int}(\ell_1^+)$. Therefore there exists $\epsilon > 0$ such that $B(y, \epsilon) \subset \ell_1^+$. Since $y \in \ell_1^+$, there exists $N \in \mathbb{N}$ such that $\overline{0} \leq y_n < \frac{\epsilon}{2}$ for all $n \geq N$. Now, define $z \in \ell_1$ by

$$z_n = \begin{cases} y_n, & \text{if } n \neq N; \\ -\frac{\epsilon}{2}, & \text{if } n = N. \end{cases}$$

Then $z \in B(y, \epsilon)$, since

$$\begin{aligned} \|y - z\|_1 &= \sum_{n=1}^{\infty} |y_n - z_n| \\ &= \left| y_N + \frac{\epsilon}{2} \right| && \text{since all other terms are zero} \\ &\leq |y_N| + \left| \frac{\epsilon}{2} \right| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

But $z \notin \ell_1^+$, since one of the term, $z_N = -\frac{\epsilon}{2} < 0$. Since y and ϵ are arbitrary we have that $\text{int}(\ell_1^+) = \emptyset$. □

The following relationship between lattice cones and normal cones follows from the definition of a lattice norm with $M = 1$.

Theorem 1.55. Let $(X, \|\bullet\|)$ be an ordered normed space and K be a cone on X. If K is a lattice, then K is normal.

The following relates Archimedean cones to normal cones.

Proposition 1.56. Let $(X, \|\bullet\|)$ be an ordered normed space and K be a cone on X. If K is normal, then K is Archimedean.

Proof. Suppose that K is normal cone of an ordered normed space X. Assume on the contrary that $y \in K$ but $y \neq \overline{0}$. Now, if $ny \leq x$ for all n, then for all n we have that,

$$\begin{aligned}
|ny| &\leq |x| \\
\|ny\| &\leq M\|x\| && \text{(for } M \geq 1\text{, since } K \text{ is normal)} \\
|n|\|y\| &\leq M\|x\| \\
\frac{n}{M}\|y\| &\leq \|x\| \\
\frac{n}{M} &\leq \frac{\|x\|}{\|y\|}.
\end{aligned}$$

This contradicts the Postulate of Archimedes, thus y cannot be non - null. Therefore we have that $y \in -K$. Hence K is Archimedean. □

Remark 1.57. Theorem 1.55 and Proposition 1.56 imply that lattice cones are archimedean.

The next example shows that the converse of the Proposition 1.56 is not necessarily true.

Example 1.58. Consider the ordered normed space $X = C^1[0,1]$ and the cone

$$K = \{f \in X : f \geq \overline{0}\}.$$

Furthermore, consider the norm in this space defined by

$$\|f\| = \|f\|_\infty + \|f'\|_\infty,$$

where $\|f\|_\infty = \sup_{t\in[0,1]} |f(t)|$. Then K is Archimedean but not normal.

Proof. We first show that K is Archimedean. To see, let $f - ng \in K$ for all n, where $g \in X$. This means that,

$$f(t) - ng(t) \geq \overline{0} \text{ for all } t \in [0,1].$$

Now this implies that for all n and for some $t_0 \in [0,1]$ we have that,

$$ng(t_0) \leq f(t_0).$$

Since $f(t_0) \in \mathbb{R}^+, g(t_0) \in \mathbb{R}$ and $\mathbb{R}$ is an Archimedean space, it follows that $g(t_0) \leq \overline{0}$. Since t_0 is arbitrary chosen in $[0,1]$, we have that $g \leq \overline{0}$. Hence the cone makes the space X an Archimedean space, that is, K is an Archimedean cone. Finally, we show that K is not normal. To this end, take $g(x) = x$ and $f(x) = x^{2M}$, where $M \geq 1$. Then

$$\|f\| = \|f\|_\infty + \|f'\|_\infty = 1 + 2M.$$

Using this norm, we get that $\|g\| = 2$. From normality inequality

$$\|f\| \leq M\|g\|,$$

we have $1 + 2M \leq 2M$ which leads to the contradiction $1 \leq 0$. $\square$

We now show an example of a cone which is not Archimedean and therefore, it is not lattice (see Remark 1.57) but has Riesz decomposition property.

Example 1.59. Let $X = \mathbb{R}^2$ and consider the cone

$$K = \{(x, y) \in \mathbb{R}^2 : x, y > \overline{0}\} \cup \{(0, 0)\}.$$

Then K is not a lattice cone and has Riesz decomposition property.

Proof. We first show that K is not a lattice cone by showing that is not Archimedean. To see, we need to show that there exists some $y \in \mathbb{R}^2$ such that for all $x \in K, ny \leq x$ but $y \in K$. This result follows immediately when we choose $y = (0, 0)$.
Next we show that K has a Reisz decomposition property. To see, suppose that X has a coordinatewise ordering, that is, for $(x_1, x_2) \leq (y_1, y_2)$ we have $x_1 \leq y_1$ and $x_2 \leq y_2$. Now we take $a = (a_1, a_2), b = (b_1, b_2)$ and $c = (c_1, c_2)$ in K such that

$$\overline{0} \leq a \leq b + c.$$

This implies

$$0 \leq a_1 \leq b_1 + c_1 \text{ and } 0 \leq a_2 \leq b_2 + c_2.$$

We need to show that there exist $u = (u_1, u_2), v = (v_1, v_2) \geq \overline{0}$ such that $u \leq b$ and $v \leq c$ and $a = u+v$. We only need to show for $0 \leq a_1 \leq b_1+c_1$. To see, take $u_1 = \dfrac{a_1 b_1}{b_1 + c_1}$ and $v_1 = \dfrac{a_1 c_1}{b_1 + c_1}$, where $b_1+c_1 \neq \overline{0}$, then $u_1+v_1 = a_1$ and

$$\begin{aligned} u_1 &= \frac{a_1 b_1}{b_1 + c_1} \\ &\leq b_1 && \text{since } \frac{a_1}{b_1 + c_1} \leq 1 \end{aligned}$$

and

$$\begin{aligned} v_1 &= \frac{a_1 c_1}{b_1 + c_1} \\ &\leq c_1 && \text{since } \frac{a_1}{b_1 + c_1} \leq 1 \end{aligned}$$

if $b_1 + c_1 = \overline{0}$ that is, when $b_1 = \overline{0}$ and $c_1 = \overline{0}$, let $u_1 = v_1 = \overline{0}$, then result hold trivially. Similarly, $u_2 \leq b_2, v_2 \leq c_2$ and $a_2 = u_2 + v_2$ and thus the result. Therefore K has a Riesz decomposition property. □

1.5 Operators on Ordered Vector Space.

Definition 1.60. ([16, Definition 2.3]) Let X and Y be two vector space. The mapping $T : X \to Y$ is said to be linear operator if, for all $x, y \in X$ and $\alpha, \beta \in \mathbb{R}$ we have,

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y).$$

T is a linear functional if $Y = \mathbb{R}$.

Definition 1.61. ([19, Definition 4.2]) A linear operator T from a normed space $(X, \| \bullet \|_X)$ into a normed space $(Y, \| \bullet \|_Y)$ is said to be continuous at the point $x_0 \in X$ if for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $x \in X$ we have

$$\|x - x_0\|_X < \delta \text{ implies } \|T(x) - T(x_0)\|_Y < \epsilon.$$

An operator T is said to be continuous if it is continuous at every point in X.

Definition 1.62. ([19, Definition 4.3]) Let X and Y be normed spaces with norms $\| \bullet \|_X$ and $\| \bullet \|_Y$ respectively and $T : X \to Y$ a linear operator. The operator T is said to be bounded if there exists $C > 0$ such that for all $x \in X$ we have

$$\|T(x)\|_Y \leq C\|x\|_X.$$

Here is an example of an unbounded operator.

Example 1.63. Let $X = P[0, 1]$ be the set of polynomial on [0,1]. Since $f \in P[0, 1]$ is bounded, we define the norm

$$\|f\|_\infty = \sup_{t \in [0,1]} |f(t)|$$

for $f \in X$. A differentiation operator $T : X \to X$ defined by

$$T(f(t)) = \tfrac{d}{dt} f(t), \text{ for } f(t) \in X$$

is linear but unbounded operator.

Proof. We first show the linearity. To see, let $f, g \in X$ and $\alpha, \beta \in \mathbb{R}$. Since the polynomial functions are differantiable, then by properties of differentiation we have,

$$\begin{aligned} T(\alpha f + \beta g)(t) &= \frac{d}{dt}(\alpha f + \beta g)(t) \\ &= \frac{d}{dt}(\alpha f(t) + \beta g(t)) \\ &= \alpha \frac{d}{dt} f(t) + \beta \frac{d}{dt} g(t) \\ &= \alpha T(f(t)) + \beta T(g(t)). \end{aligned}$$

So T is a linear operator. Lastly we show that T is unbounded. Consider the polynomial function $f_n(t) = t^n, n > 1$. Then $(T(f_n))(t) = nt^{n-1}$ for all $t \in [0, 1]$. Clearly $\|f_n\| = 1$ and $\|T(f_n)\| = n$. It follows that T is an unbounded operator. $\square$

The following theorem show that the concepts of continuity and boundedness are closely related.

Theorem 1.64. ([19, Theorem 4.4]) Let $X, \|\bullet\|_X)$ and $(Y, \|\bullet\|_Y)$ be normed spaces. The linear operator $T : X \to Y$ is bounded if and only if it is continuous.

Proof. If T is bounded, take any $y \in X$, then by Definition 1.62, there exists $C > 0$ such that for all $x \in X$, we have

$$\|T(x) - T(y)\|_Y = \|T(x - y)\|_Y \leq C\|x - y\|_X.$$

For a given $\epsilon > 0$, we set $\delta = \frac{\epsilon}{C}$. Then for $x, y \in X$,

$$\|x - y\|_X < \delta \text{ implies } \|T(x) - T(y)\|_Y \leq C\|x - y\|_X < \epsilon$$

Since $y \in X$ is arbitrary, it follows that T is continuous. Conversely, if T is continuous at any $y \in X$, then given $\epsilon > 0$ there exists a $\delta > 0$ such that $\|x - y\|_X < \delta$ implies $\|T(x) - T(y)\|_Y \leq \epsilon$. Let $z \neq \overline{0}$ in X and let $x = y + \frac{\delta}{\|z\|_X} z$. Since T is linear we have,

$$\|T(x) - T(y)\|_Y = \|T(x - y)\|_Y = \left\| T\left(\frac{\delta}{\|z\|_X} z\right)\right\|_Y = \frac{\delta}{\|z\|_X}\|T(z)\|_Y < \epsilon.$$

Thus $\|T(z)\|_Y < \frac{\epsilon}{\delta}\|z\|_X$. Picking $C = \frac{\epsilon}{\delta}$, we get the result T is bounded. $\square$

Definition 1.65. Let $T : X \to Y$ be a linear operator, where X and Y are ordered vector spaces. Then T is said to be

(i) positive if $T(x) \geq \overline{0}$ for all $x \geq \overline{0}$.

(ii) regular if T can be written as a difference of two positive operators.

(iii) order bounded if T carries order bounded subsets of X to bounded subsets of Y.

Proposition 1.66. Let X and Y be Banach lattices. Every linear operator $T : X \to Y$ is regular.

Proof. Suppose that $T : X \to Y$ is linear operator. Since for any $x \in X$, we have $Tx \in Y$, which is a Riesz space, then by Theorem 1.11 (ii)

$$Tx = (Tx)^+ - (Tx)^- = T^+(x) - T^-(x),$$

where $T^+(x) = (Tx) \vee \overline{0}$ and $T^-(x) = (-Tx) \vee \overline{0}$. Therefore $T^+(x) \geq \overline{0}$ and $T^-(x) \geq \overline{0}$. It follows that T is the difference between to positive operators thus T is a regular operator. □

Proposition 1.67. Let X and Y be ordered vector spaces. Every positive operator $T : X \to Y$ is order bounded.

Proof. Consider an order interval on X, say $[a, b]$. For $x \in [a, b]$ it is true that $x - a \in X^+$ and $b - x \in X^+$. Now since T is positive, it follows that

$$T(x - a) \geq \overline{0} \text{ implies } Tx \geq Ta$$

and similarly,

$$Tx \leq Tb.$$

Hence for any $x \in [a, b]$ we have that $Tx \in [Ta, Tb]$, thus the operator is order bounded. □

Proposition 1.68. Let X be a Banach lattice and $[a, b]$ and $[c, d]$ be order bounded intervals on X. The following statements are true;

(i) The intervals $\alpha[a, b]$ is an order interval in X for $\alpha > 0$.

(ii) The interval $[a, b] + [c, d]$ is contained in an order interval $[a + c, b + d]$ on X.

Proposition 1.69. (cf [30, Theorem 18.3]) Let X and Y be ordered vector spaces. Every regular operator $T : X \to Y$ is order bounded.

Proof. Consider an order interval on X, say $[a, b]$. Since T is regular, there exists positive operators T_1 and T_2 such that

$$T = T_1 - T_2.$$

Now by Proposition 1.67, if $x \in [a, b]$ we have that $T_1 x \in [T_1 a, T_1 b]$ and $T_2 x \in [T_2 a, T_2 b]$. By Proposition 1.68 we have that $[T_1 a, T_1 b] - [T_2 a, T_2 b]$ is contained in an order interval. Therefore T is order bounded. □

Definition 1.70. ([18, Definition 1.4.13]) Let $(X, \|\bullet\|_X)$ and $(Y, \|\bullet\|_Y)$ be normed spaces. A linear operator $T : X \to Y$ is said to be an isomorphism if T is a bijection and continuous and its inverse operator $T^{-1} : Y \to X$ is also continuous.

Theorem 1.71. ([18, Proposition 1.4.14]) Let $(X, \|\bullet\|_X)$ and $(Y, \|\bullet\|_Y)$ be normed spaces. A surjective linear operator $T : X \to Y$ is an isomorphism if and only if there are positive constants s and t such that

$$s\|x\|_X \leq \|Tx\|_Y \leq t\|x\|_X$$

whenever $x \in X$.

Definition 1.72. ([2, Definition 19]) Let $(X, \leq_X)$ and $(Y, \leq_Y)$ be two ordered normed spaces. A linear operator $T : X \to Y$ is called an order isomorphism if for all $x, y \in X$ we have

(i) T surjective and

(ii) $x \leq_X y$ if and only if $T(x) \leq_Y T(y)$.

If there exists an order isomorphism from X to Y, then X and Y are called order isomorphic ordered vector spaces. An order isomorphism from a partially ordered set to itself is called an order automorphism.

Proposition 1.73. Let X, Y be two ordered vector spaces and $T : X \to Y$ be an order isomorphism. Then

(i) T is injective.

(ii) T is surjective.

Proof. (i) We show that T is injective. To this end, suppose $Tx = Ty$. This is true if and only if

$$Tx \leq_Y Ty \text{ and } Ty \leq_Y Tx \text{ (since } Y \text{ is an ordered vector space).}$$

Since T is order isomorphism then above inequalities imply that

$$x \leq_X y \text{ and } y \leq_X x.$$

And it is true if and only if $x = y$.
(ii) T is surjective by definition of order isomorphism. $\square$

Next we give example to show that the converse is not true, that is, not every bijective map between ordered sets is an order isomorphism.

Example 1.74. Consider the ordered vector space, $(\mathbb{R}, \leq)$ where $\leq$ denotes the usual order. Then map $T : \mathbb{R} \to \mathbb{R}$ defined by $T(x) = -x$ is bijective but not order preserving.
To see, if $x \leq y$ then $T(x) = -x \geq -y = T(y)$.

Next we show a map that is neither bijective nor order preserving.

Example 1.75. Let $X = \mathbb{N} = Y$ with usual the order $<$. Define $T : X \to Y$, by

$$Tx = \begin{cases} 0, & \text{if } x \text{ is even} \\ 1, & \text{if } x \text{ is odd.} \end{cases}$$

Then T is neither injective nor order isomorphism. To this end, T is not order preserving since $1 < 2$ but $T1 = 1 > 0 = T2$ and is also not injective since $1 \neq 3$ but $T(1) = 1 = T(3)$.

Example 1.76. Let $(X, \leq)$ be a partially ordered set. Then the identity operator on X is an order automorphism.

Definition 1.77. Let $(X, \|\bullet\|_X)$ and $(Y, \|\bullet\|_Y)$ be ordered normed spaces. A linear operator T is an order isometry if $\|Tx\| = \|x\|$ whenever $x \in X$. The normed space X is order isometrically embedded in the normed space Y if there is an order isometric from X onto Y. Then the normed spaces X and Y are called order isometric.

2 Cones with unbounded and bounded bases.

In this chapter, we consider cones with bounded, unbounded and compact bases. These concepts were used to characterize the reflexivity of a Banach space.

2.1 Properties of a Cone with a base.

We first consider some definitions we will use in this chapter.

Definition 2.1. Let X be a normed space and K a positive cone in X. The set

$$X^* = \{f \in C(X) : f : X \mapsto \mathbb{R}\},$$

of all continuous linear functional on X, is called dual of X, the set

$$K^* = \{f \in X^* : f(k) \geq 0, \text{ for all } k \in K\},$$

all continuous linear functionals that are positive on K, is a natural dual wedge of K and the set

$$K^{*s} = \{f \in X^* : f(k) > 0, \text{ for all } k \in K \setminus \{\overline{0}\}\}$$

denote all strictly positive linear functionals on K.

We recall the following two important Hahn - Banach Theorems.

Theorem 2.2. (Hahn - Banach separation theorem) Let X be a normed space and E_1, E_2 be a nonempty disjoints convex subsets of X, where E_1 is open in X. Then for some $f \in X^*$ and $t \in \mathbb{R}$, we have $f(x_1) < t \leq f(x_2)$ for all $x_1 \in E_1$ and $x_2 \in E_2$.

Theorem 2.3. (Hahn - Banach extension theorem) Let X be a normed space, Y be a subspace of X and $g \in Y^*$. Then there is some $f \in X^*$ such that $f_{|Y} = g$ and $\|f\| = \|g\|$.

The next proposition provides the conditions for a set K^* to be a cone.

Proposition 2.4. Let X be a normed space and K be a cone in X. The set K^* is a cone if and only if $\text{cl}(K - K) = X$.

Proof. Suppose on the contrary, that $\text{cl}(K-K) \neq X$. Then by Theorem 2.2, there exists a non-zero $f \in X^*$ which is zero on $K - K$, that is, $f(x) = 0$ for $x \in K - K$. This means $x = x_1 - x_2$, where $x_1, x_2 \in K$, we have

$f(x_1 - x_2) = f(x_1) - f(x_2) = 0$. Hence $f(x_1) = f(x_2)$. By Theorem 2.2, $x_1 = x_2$. Thus, $x = \overline{0} \in K \cap (-K)$, since K is a cone. This means $f \in K^* \cap (-K^*)$. Contradiction, since K^* is a cone and $f \neq \overline{0}$. Conversely, Suppose that $\mathrm{cl}(K - K) = X$. To show that K^* is a cone, we need only to show that $K^* \cap (-K^*) = \{\overline{0}\}$. To see, by assumption, for each $x \in X$, we particularly have

$$B[x, 1] \cap (K - K) \neq \emptyset.$$

This implies there is $\overline{0} \neq x_0 \in B[x, 1]$ and for some $x_1, x_2 \in K$ such that $x_0 = x_1 - x_2$. Hence $f(x_0) = f(x_1 - x_2) = f(x_1) - f(x_2) \neq 0$. This implies $f(x_1) \neq f(x_2)$, that is f is one - to - one. By Theorem 2.3, f can be extended to X and thus $f(x_1) \neq f(-x_1) = -f(x_1)$ since $x_1 \neq -x_1$ if $x_1 \neq \overline{0}$ and by linearity of f. This is true for all $f \in K^*$ and $x_1 \in K$. Hence the result. □

Let K be a cone in a normed space X. We show that the set K^{*s} can be empty.

Example 2.5. Consider $(\mathcal{B}([0, 1]), \|\bullet\|_\infty)$, a normed space of bounded functions on [0,1]. Let

$$\mathcal{B}^+ = \{f \in \mathcal{B}[0, 1] : f(t) \geq 0 \text{ for all } t \in [0, 1]\}$$

be a positive cone of $\mathcal{B}[0, 1]$. The set of strictly positive functional $(\mathcal{B}^+)^{*s} = \emptyset$.

Proof. Suppose that ϕ is a strictly positive linear functional on $\mathcal{B}[0, 1]$. For $t \in [0, 1]$, let $e_t \in \mathcal{B}[0, 1]$ be given by

$$e_t(w) = \chi_{[0,1]}(w) = \begin{cases} 1, & \text{if } w = t; \\ 0 & \text{if } w \in [0, 1] \setminus \{t\}. \end{cases}$$

Let

$$A_n = \left\{t \in [0, 1] : \phi(e_t) > \tfrac{1}{n}\right\}.$$

Then $[0, 1] = \bigcup_{n=1}^{\infty} A_n$. Since the set $[0, 1]$ is uncountable, atleast one of the subsets A_n must be infinite. Suppose that A_n is infinite and let $t_1, t_2, ...$ be a countable subset of A_n. Let $e \in \mathcal{B}[0, 1]$ be defined by $e(w) = 1$ for all $w \in [0, 1]$. Then, for every $m \in \mathbb{N}$,

$$e_m := \sum_{k=1}^{m} e_{t_k} = \chi_{\{t_1, \ldots, t_m\}} \leq e,$$

so that

$$\tfrac{m}{n} < \sum_{k=1}^{m} \phi(e_{t_k}) = \phi(e_m) \leq \phi(e).$$

Letting $m \to \infty$, one obtains the contradition, $\phi(e) = \infty$. $\square$

In a vector space, there are different types of cones as we discussed in the previous section, we now consider another type of cone on a vector space. Let U be convex subset of X. The set $Cone(U) = \{\lambda u : u \in U \text{ and } \lambda \geq 0\}$ is a cone. We prove this in the proposition.

Proposition 2.6. Let U be a convex set in a vector space X. The set $\text{Cone}(U)$ is a cone in X.

Proof. The set $\text{Cone}(U)$ is not an empty set, since $\overline{0} = 0u \in \text{Cone}(U)$ for all $u \in U$. We now show that $\text{Cone}(U)$ is closed under addition and scalar multiplication. Let $x, y \in \text{Cone}(U)$, then there exist $u_1, u_2 \in U$ and $\lambda_1 \geq 0, \lambda_2 \geq 0$ such that $x = \lambda_1 u_1$ and $y = \lambda_2 u_2$. Then for $\lambda \geq 0$, we have $\lambda x = \lambda\lambda_1 u_1 \in \text{Cone}(U)$ since $\lambda\lambda_1 \geq 0$ and $u_1 \in U$.

$$\begin{aligned} x + y &= \lambda_1 u_1 + \lambda_2 u_2 \\ &= 1(ru_1 + (1-r)u_2) \qquad \text{if } \lambda_1 = r, \lambda_2 = 1-r \text{ and } 0 < r < 1 \end{aligned}$$

Thus $x + y \in \text{Cone}(U)$ since $1 > 0$ and $ru_1 + (1-r)u_2 \in U$ (U is convex). Lastly we show that $\text{Cone}(U) \cap (-\text{Cone}(U)) = \{\overline{0}\}$.
Note

$$\begin{aligned} -\text{Cone}(U) &= \{\lambda u : u \in U, -\lambda \geq 0\} \\ &= \{\lambda u : u \in U, \lambda \leq 0\}, \end{aligned}$$

and then

$$\begin{aligned} \text{Cone}(U) \cap (-\text{Cone}(U)) &= \{\lambda u : u \in U, \lambda \geq 0\} \cap \{\lambda u : u \in U, \lambda \leq 0\} \\ &= \{\lambda u : u \in U, \lambda \leq 0 \text{ and } \lambda \geq 0\} \\ &= \{\lambda u : u \in U, \lambda = 0\} \\ &= \{0u : u \in U\} \\ &= \{\overline{0}\}. \end{aligned}$$

$\square$

Example 2.5 shows that K^{*s} can be empty. Now we consider the conditions so that $K^{*s} \neq \emptyset$.

Theorem 2.7. Let $(X, \|\bullet\|)$ be a normed space and K a cone in X. Then $K^{*s} \neq \emptyset$ if and only if there exists an open convex set U in X such that

(i) $\overline{0} \notin U$,

(ii) $K \subseteq cone(U)$.

Proof. Assume $K^{*s} \neq \emptyset$. Take any $f \in K^{*s}$ and let $U = \{y \in X : f(y) = 1\}$. Firstly, $U = f^{-1}(1)$ is open since f is continuous. Secondly, we show that U is convex. Let $x, y \in U$ and $r \in (0,1)$, then $f(x) = 1$ and $f(y) = 1$. We claim that

$$rx + (1-r)y \in U.$$

Now,

$$\begin{aligned} f(rx + (1-r)y) &= f(rx) + f((1-r)y) && \text{since } f \text{ is linear} \\ &= rf(x) + (1-r)f(y) && \text{since } f \text{ is linear} \\ &= r + 1 - r && \text{assumption} \\ &= 1 \end{aligned}$$

Thus,

$$rx + (1-r)y \in U.$$

Also, $\overline{0} \notin U$ since

$$\begin{aligned} f(\overline{0}) &= 0 && \text{since } f \text{ is linear} \\ &\neq 1. \end{aligned}$$

Lastly, we show that $K \subseteq Cone(U)$. Let $x \in K$. If $x = \overline{0}$, then $\overline{0} \in Cone(U)$ since $\overline{0} = 0u$ for all $u \in U$ and $\lambda = 0$. If $x > 0$, then we have either $x \in U \subset cone(U)$ or $f(x) \neq 1$. For the latter, there exists $\sigma > 0$ such that $f(\sigma x) = \sigma f(x) = 1$ and $\sigma x \in K$. This means $\sigma x \in U \subset cone(U)$, that is, $x \in \frac{1}{\sigma} cone(U) = cone(U)$. Hence the result. Conversely, suppose U is an open convex set such that (i) and (ii) holds. Since $\overline{0} \notin U$, then by Theorem 2.2, there exists $f \in X^*$ such that $f(\overline{0}) < f(u)$ for all $u \in U$. Thus

$$f(\overline{0}) = 0 < f(u)$$

for all $u \in U$. Clearly, $K \setminus \{\overline{0}\} \subseteq U \subseteq \text{Cone}(U)$. Then $f(k) > 0$ for all $k \in K \setminus \{\overline{0}\}$. Thus, $f \in K^{*s}$. Hence the result. □

Definition 2.8. ([2, Definition 1.46]) Let a set K be a cone in a normed space X. A nonempty convex subset B of $K \setminus \{\overline{0}\}$ is said to be a **base** for the cone K if for each $x \in K \setminus \{\overline{0}\}$ there exist $\lambda > 0$ and $b \in B$ both uniquely determined such that $x = \lambda b$.

We show an example of a cone with a base.

Example 2.9. Let $X = \mathbb{R}^3$ and

$$\begin{aligned} K &= \{(x, y, z) \in \mathbb{R}^3 : z \geq \sqrt{x^2 + y^2}\} \\ &= \{\lambda(x, y, 1) \in \mathbb{R}^3 : \lambda \geq 0 \text{ and } x^2 + y^2 \leq 1\}, \end{aligned}$$

the ice cream cone. The cone K has a base,

$$B = \{(x, y, 1) : x^2 + y^2 \leq 1\}.$$

We consider the characterization of a cone with a base.

Proposition 2.10. (cf (compare [2, Theorem 1.47])) Let $(X, \|\bullet\|)$ be a normed space and $K \subseteq X$ a cone. The cone K has a base if and only if $K^{*s} \neq \emptyset$.

Proof. Let $f \in K^{*s}$ and $A = \{y \in X : f(y) = 1\}$. Let $B = A \cap K$, clearly $B \subset K$. We claim that B is a base for K. To this end, first note that $\overline{0} \notin B$. This is true since $f(\overline{0}) = 0 \neq 1$ (f is linear). Secondly we show that $B \neq \emptyset$. Let $\overline{0} \neq x \in K$ then $\frac{x}{\|x\|} \in K$ (since K is a cone). Now

$$\begin{aligned} f\left(\frac{x}{\|x\|}\right) &= \frac{1}{\|x\|} f(x) && \text{f is linear} \\ &= \frac{1}{\|x\|} \|x\| && \text{by Theorem 2.2} \\ &= 1. \end{aligned}$$

Therefore, $b = \dfrac{x}{\|x\|} \in B$ for all $x \in K$. (1)

Lastly we show that B is convex. Let $x, z \in B$ and $0 < \lambda < 1$. Then $\lambda x + (1 - \lambda) z \in K$(K is a convex cone) and

$$\begin{aligned} f(\lambda x + (1 - \lambda) z) &= \lambda f(x) + (1 - \lambda) f(z) && f \text{ is linear} \\ &= \lambda + 1 - \lambda && x, z \in B \\ &= 1. \end{aligned}$$

By (1), $x = \|x\|b = f(x)b \in K$ for all $x \in K, b \in B$ and $\lambda = f(x) \geq 0$. This choice is uniquely determined, to see, let $x = \lambda_1 b_1 = \lambda_2 b_2$ with $\lambda_i > 0$ and $b_i \in B$ for each i. Then we have

$$\begin{aligned}
\lambda_1 &= \lambda_1 f(b_1) && b_1 \in B \\
&= f(\lambda_1 b_1) && f \text{ is linear} \\
&= f(\lambda_2 b_2) && \\
&= \lambda_2 f(b_2) && f \text{ is linear} \\
&= \lambda_2 && b_2 \in B
\end{aligned}$$

And from this we get $b_1 = b_2$. Therefore B is base by definition of a base. Conversely, suppose that B is a base for K. Since $\overline{0} \notin clB$, there exists open convex neighborhood V of $\overline{0}$ such that $V \cap clB = \emptyset$. Note: $\overline{0} = \overline{0} + \overline{0} \notin B + V$, since $\overline{0} \notin B$. $B + V = \bigcup_{x \in B} x + V$ is open as the union of translates of V. $B + V$ is convex as a sum of convex sets.

$$\begin{aligned}
\text{Cone}(B + V) &= \{\lambda b + \lambda v : v \in V, b \in B, \lambda \geq 0\} \\
&= \{\lambda b : b \in B, \lambda \geq 0\} + \{\lambda v : v \in V, \lambda \geq 0\} \\
&= K + \{\lambda v : v \in V, \lambda \geq 0\} && \text{B is a base of } K.
\end{aligned}$$

This shows that $K \subseteq \text{Cone}(B + V)$. And therefore $K^{*s} \neq \emptyset$ by theorem 2.7. □

Remark 2.11. The Proposition 2.10 shows that each base B for the cone K is associated with a base $B_f = \{x \in K : f(x) = 1\}$ defined by $f \in K^{*s}$.

The following example shows that not all closed cones has a base.

Example 2.12. Consider the space, $X = (\mathcal{B}[0,1], \| \bullet \|_\infty)$ and X^+ as in Example 2.5. The cone X^+ is closed.

Proof. To this end, let $(f_n) \subset X^+$ such that

$$f_n \to f \in X,$$

that is $f_n(t) \to f(t) \in \mathbb{R}$ for all $t \in$ [0,1]. Since $0 \leq f_n(t) \leq M$ for each n and for all $t \in$ [0,1]. Then $0 \leq f(t) \leq M < \infty$ for all $t \in$ [0,1]. This means $f \in X^+$ and therefore X^+ closed. □

2.2 Cones with bounded and unbounded bases.

Now we consider cones with mixed bases, the cones with bounded and unbounded bases.

Definition 2.13. A base B of a cone K in a normed space $(X, \| \bullet \|)$ is called a **bounded** base if there exists $\alpha \in \mathbb{R}$ such that $\|b\| \leq \alpha$ for all $b \in B$.

Below we show an example of a cone with bounded base.

Example 2.14. Let $X = \mathbb{R}$ and K ice cream cone in X. The base

$$\begin{aligned} B &= \{(x, y, 1) : x^2 + y^2 \leq 1\} \\ &= (0, 0, 1) + \{(x, y, 0) : x^2 + y^2 \leq 1\} \end{aligned}$$

of K is bounded. To this end, note B translates to the $\{(x, y, 0) : x^2+y^2 \leq 1\}$ which is bounded.

Definition 2.15. ([18, Definition 2.5.1]) Let X be a normed space. The weak topology on X is the smallest topology such that every member of the dual space X^* is continuous with respect to that topology.

We next consider the characteristics of a cone with unbounded base.

Theorem 2.16. Let $(X, \| \bullet \|)$ be a normed space and K a cone in X. Suppose that K contains a sequence (y_n) such that,

(i) there exists an $m > 0$ such that $\|y_n\| \geq m$ for all n,

(ii) (y_n) converges weakly to $\overline{0}$.

Then K has unbounded base.

Proof. Suppose on contrary that B is a bounded base for K. Then there exists $M > 0$ such that $\|b\| \leq M$ for all $b \in B$. Also, by definition of a base, for each n there exists $\alpha_n > 0, \overline{b}_n \in B$ such that $y_n = \alpha_n \overline{b}_n$. Thus

$$\begin{aligned} M &\geq \|\overline{b}_n\| && \text{(since B is bounded)} \\ &= \frac{\|y_n\|}{\alpha_n} \\ &\geq \frac{m}{\alpha_n}. \end{aligned}$$

Thus,

$$\frac{1}{\alpha_n} \leq \frac{M}{m}. \tag{1}$$

Since (y_n) converges weakly to $\overline{0}$, that is, $f(y_n) \to 0$ for all $f \in X^*$. Then for each $f \in X^*$ and each $\epsilon > 0$, there exists $n \in \mathbb{N}$ such that

$|f(y_n) - f(\overline{0})| < \epsilon$ for all $n \geq N$ if and if $|f(y_n) - 0| < \epsilon$ for all $n \geq N$.

This means $|f(y_n)| < \epsilon$ for all $n \geq N$ and

$$\begin{aligned} f(y_n) &= f(\alpha_n \bar{b}_n) \\ &= \alpha_n f(\bar{b}_n) \qquad \text{since } f \text{ linear.} \qquad (2) \end{aligned}$$

Note, since $\alpha_n > 0$ for each $n \in \mathbb{N}$, then $0 < \frac{1}{\alpha_n} \leq \frac{M}{m}$. Now we show that $|\frac{1}{\alpha_n} f(y_n)| < \epsilon$ also for all $n \in \mathbb{N}$. Let $n \geq N$ be such that

$$|f(y_n)| < \frac{\epsilon m}{2(M+1)},$$

then

$$\begin{aligned} \left|\frac{1}{\alpha_n} f(y_n)\right| &= \left|\frac{1}{\alpha_n}\right| |f(y_n)| \\ &< \frac{\epsilon}{2} \\ &< \epsilon. \end{aligned}$$

This implies $\lim_{n\to\infty} \frac{1}{\alpha_n} f(y_n) = 0$. Thus by (2) $\lim_{n\to\infty} f(b_n) = 0$. Therefore, (b_n) converges weakly to 0. Hence $\overline{0} \in clB^w$(weak closure). Now we show that $clB^w = clB$.

case 1 $clB^w \subset clB$. Let $x_0 \in X$, and $x_0 \notin clB$. Then by Theorem 2.2, there exists $f \in X^*$ and $\alpha \in \mathbb{R}$ such that for all $x \in clB$, $f(x_0) < \alpha < f(x)$, and hence the set $\{x \in X : f(x) < \alpha\}$ is a weak neighborhood of x_0 such that

$$\{x \in X : f(x) < \alpha\} \cap B = \emptyset,$$

since the weak topology is Hausdorff. Thus $x_0 \notin clB^w$. Hence the results.

case 2 $clB \subset clB^w$. Since clB^w is weakly closed and weak topology is coarser than a norm topology, it follows that $clB \subset clB^w$. Therefore,

$$clB = clB^w.$$

Hence $\overline{0} \in clB$. This contradict the fact that B is a base.

Thus K have unbounded base only. □

We consider a space with only unbounded base.

Corollary 2.17. Let $X = \ell_p, 1 < p < \infty$ and $K = \ell_p^+$. The cone K has only unbounded base.

Proof. We prove for $n = 2$. That is, ℓ_2. Take

$$B_f = \{x \in \ell_p^+ : \|x\|_2 \leq 1 = f(x)\}$$

and $f \in (\ell_2^*)^+$. Now take $e_i = (0, ..., 0, \overset{i^{th}}{1}, 0, 0, ...) \in \ell_2^+$. Then $\|e_i\|_2 = 1 > 0$ for each i. Finally (e_i) converges weakly to $\overline{0}$. To see, let $f \in (\ell_2^*)^+$, then there is some $y = (y_i) \in \ell_2^+$ such that

$$f(e_i) = \sum_{i=1}^{\infty} e_i y_i = y_i.$$

Claim: $f(e_i) = y_i \to 0$ as $i \to \infty$. To see, since $(y_i) \in \ell_2$ then $\sum_{i=1}^{\infty} |y_i|^2 < \epsilon$. From calculus this means that $\lim_{i \to \infty} |y_i|^2 = 0$, that is, for every $\epsilon > 0$ there exist $N \in \mathbb{N}$ such that for all $i \geq N$ we have

$$||y_i|^2 - 0| < \epsilon.$$

So $|y_i|^2 < \epsilon$, that is, $|y_i| < \epsilon^{\frac{1}{2}}$. Therefore $y_i \to 0$ as $i \to \infty$. Hence by Theorem 2.16, ℓ_2^+ admits unbounded base only. □

Next we give an example of a mixed based cone. We start first by showing that cone admits an unbounded base. The prove for a positive cone of ℓ_1 works differently, so we give a proof separately.

Corollary 2.18. Let $X = \ell_1$ and $K = X^+$. The cone K admits unbounded base.

Proof. To see, consider a base $B_f = \{x \in \ell_1^+ : \|x\|_1 \leq 1 = f(x)\}$ defined by $f \in (\ell_1^+)^*$. Take $(y_i) = (\overbrace{0, 0, ..., 0}^{i-1}, 1, \frac{1}{2}, \frac{1}{4}, ...) \in c_0^+ \subset \ell_\infty^+$. Consider $e_i = \{0, 0, ..., 0, \overset{i^{th}}{1}, 0, 0, ...\}$ in ℓ_1^+. Then $\frac{e_i}{f(e_i)} \in B_f$ and

$$\|\tfrac{e_i}{f(e_i)}\|_1 = \frac{1}{f(e_i)} = \frac{1}{\sum_{k=1}^{i} e_k y_k} = \frac{1}{\frac{1}{i}} \to \infty \text{ as } i \to \infty.$$

Therefore B_f is unbounded. □

We now consider cones with closed and bounded base.

Proposition 2.19. If a cone K in a normed space $(X, \|\bullet\|)$ admits a closed bounded base, then K is closed.

Proof. Let $(x_n)_{n\in\mathbb{N}} \in K$ and $x_n \to x \in X$ as $n \to \infty$. We show $x \in K$. Since base B of K is closed, then for each n there exists $\lambda > 0, b_n$ such that $x_n = \lambda b_n \to \lambda b$ (since $b_n \to b \in B$). Again

$$\lambda b = x \tag{1}$$

since the norm topology is Hausdorff. Thus $b = \frac{x}{\lambda} \in B$. Since B is bounded then

$$\|b\| = \|\tfrac{x}{\lambda}\| \leq M, \text{ for some } M > 0.$$

That is,

$$\|x\| \leq \lambda M < \infty. \tag{2}$$

Hence by (1) and (2), $x \in K$. □

The following example shows that the converse of the above result is not true in general.

Example 2.20. The positive cone ℓ_2^+ is closed by Example 1.51 and does not admit a bounded base by Corollary 2.17.

We consider example of a closed cone that only admits bounded base.

Examples 2.21. Let $(X, \| \bullet \|)$ be a normed space. If $x \in S(0,1)$, then $K = \text{cone}(x + \frac{1}{2}B[0,1])$, is a closed cone with a bounded base.

Proof. Firstly, note that K is a cone by Proposition 2.6, since the set

$$B = x + \tfrac{1}{2}B[0,1]$$

is convex. Therefore B is a base of cone K by definition, since

$$\overline{0} \notin x + \tfrac{1}{2}B[0,1]$$

and it is also bounded (since it is a translation of bounded set $B[0,1]$). Therefore K is closed, since it admits a bounded base. □

The following corollary shows that converse of Proposition 2.19 is true for a finite dimensional normed space. We will give the conditions for an infinite dimensional later.

Corollary 2.22. Let X be a finite dimensional normed and K a closed cone with a base B in X. Then B is bounded.

Proof. A closed unit ball $B[0,1]$ is closed and bounded in X since X is a finite normed space. A set $B^+[0,1] = B[0,1] \cap K$ is also closed and bounded since K is also closed. Then a set $B = \{x \in B^+[0,1] : c \leq x, c > \overline{0}\}$ is closed and bounded in $B^+[0,1]$ and it is a base of a closed cone $K = cone(B)$. $\square$

We consider now some characterization of bounded base. We first give the following definition.

Definition 2.23. A linear functional f on a cone K of a normed space $(X, \|\bullet\|)$ is called uniformly monotonic on K if $f(x) > a\|x\|$ for all $x \in K$, and $a > 0$.

Theorem 2.24. (cf, [20, Proposition 2]) Let B_f be a base for a cone K in a normed space $(X, \|\bullet\|)$ defined by strickly positive linear functional f. Then B_f is bounded if and only if f is uniformly monotonic on K.

Proof. Let $B_f = \{x \in K : f(x) = 1\}$ be a base for a cone K defined by $f \in K^{*s}$. Firstly, assume that B_f is bounded, that is, there exists $M > 0$ such that

$$\|x\| \leq M < M + 1 \text{ for all } x \in B_f.$$

Thus $\dfrac{1}{M+1}\|x\| < 1 = f(x)$ in B_f and by Hahn - Banach theorem,

$$\frac{1}{M+1}\|x\| < f(x) \text{ for all } x \in K.$$

Hence f is uniformly monotonic on K.
Conversely, assume that there exists $a > 0$ such $a\|x\| < f(x)$ for all $x \in K$. By definition of a base, $f(x) = 1$ in B_f and thus $a\|x\| < 1$, that is, $\|x\| < \dfrac{1}{a}$ for all $x \in B_f$. Therefore B_f is bounded. $\square$

Example 2.25. Take normed space $(\ell_1, \|\bullet\|_1)$ and let B_f denote base of the cone ℓ_1^+ defined by linear functional $f \in (\ell_1^+)^*$. The linear functional is uniformly monotonic and therefore the cone has a bounded base. To see, let

$x \in \ell_1^+$, and $\|x\|_1 = \sum_{i=1}^{\infty} x_i < \infty$. If $y \in \ell_\infty^+$ and $f \in \ell_1^*$, then

$$\begin{aligned}
\|x\|_1 &= \sum_{i=1}^{\infty} x_i && (x_i \geq 0)\\
&\leq \sum_{i=1}^{\infty} x_i y_i && (y \geq 0)\\
&= \sum_{i=1}^{\infty} |x_i y_i| && (x, y \geq 0)\\
&= f(x)
\end{aligned}$$

Note, $\|x\|_1 < (1+b)\sum_{i=1}^{\infty} x_i y_i$ if $(b > 0)$, then $\frac{1}{b+1}\|x\|_1 < \sum_{i=1}^{\infty} x_i y_i = f(x)$. Hence the result.

Remark 2.26. Cone in a normed space admitting both bounded and unbounded bases is called mixed based cone. By Corollary 2.18 and Example 2.25, ℓ_1^+ is a mixed based cone.

Definition 2.27. (compare [20, Proposition 2.3]) Let $(X, \|\bullet\|)$ be a normed space, $(X^*, \|\bullet\|_{X^*})$ be normed dual of X and K a closed cone in X. K in X is said to have an angle property if there exists $f \in X^* \setminus \{\bar{0}\}$ and $\epsilon \in (0, 1]$ such

$$K \subseteq \{x \in X : f(x) \geq \epsilon \|f\|_{X^*}\|x\|\}.$$

We give an example of a cone that satisfying angle property.

Example 2.28. Consider an ice cream cone K in a normed space $(\mathbb{R}^3, \|\bullet\|_2)$ defined as in Example 2.9. We show that K satisfies an angle property. To see, take a uniformly monotonic functional

$$f = e_3 = (0, 0, 1) \in (\mathbb{R}^3)^* \setminus \{\bar{0}\},$$

with the norm $\|\bullet\|_2$ on $(\mathbb{R}^3)^* = \mathbb{R}^3$. This is possible since K has a bounded base by Example 2.14 and by Theorem 2.24 such f exists. Now we have

$$\begin{aligned}
e_3 \bullet (x, y, z) = (0, 0, 1) \bullet (x, y, z) &= z && \text{where } x, y, z \in K\\
&\geq 1\sqrt{x^2 + y^2} && \text{from Example 2.9}\\
&= \epsilon \|e_3\|_2 \|(x, y, 0\|_2
\end{aligned}$$

where $\epsilon = 1, \|e_3\|_2 = 1$. Hence the result.

The following main theorem gives a characterization of a closed cone with a bounded base in terms of a solid cone K^*. We first give the Lemma that will be useful in the main Theorem.

Lemma 2.29. Let K be a cone in a normed space X. If K admits a base B then $K - K = X$.

Proof. Suppose on the contrary, that $K - K \neq X$. Then there exists $x \in X$ such that $x \neq x_1 - x_2$ for all $x_1, x_2 \in K$. Since K has a base, then $x \neq \lambda_1 b_1 - \lambda_2 b_2$ where $b_1, b_2 \in B$. Which is contradiction. Hence the result. □

Theorem 2.30. (cf [22, Theorem A]) Let $(X, \|\bullet\|)$ be a normed space, $(X^*, \|\bullet\|_{X^*})$ be normed dual of X and K a closed cone in X. The following statements are equivalent.

(i) K has the angle property.

(ii) K admits a bounded base.

(iii) $K^* = \{f \in X^* : f(x) \geq 0, \text{ for all } x \in K\}$ is a solid cone.

Proof. *(i)* $\Longrightarrow$ *(ii)* Assume that a cone K has a property (a). Let $x \in K$ and $x > \bar{0}$, then $\|x\| > 0$ and by assumption $f(x) > 0$. Therefore $f \in K^{*s}$, implying K has a base by Proposition 2.10. By Theorem 2.24 with $a = \epsilon\|f\|$, a base of K is bounded.
(ii) $\Longrightarrow$ *(iii)* Assume that a cone K has a bounded base. By Proposition 2.4 and lemma 2.29, K^* is a cone in X^*. Lastly, we show that K^* is solid. Since K has a bounded base, by Proposition 2.10 and Theorem 2.24, there exists a $f \in K^{*s} \subset K^* \subset X^* \setminus \{\bar{0}\}$ such that

$$f(x) \geq a\|x\| \text{ for all } x \in K \text{ and } a > 0.$$

Let $g(x) = a\|x\|$, need to show that $g \in B_r(f) \subset K^{*s} \subset K^*$. We get the result if we choose $a = \frac{1}{\|x\|}\left(\frac{r}{2} + f(x)\right)$, since

$$\begin{aligned}\|f(x) - g(x)\|_\infty &= \left\|f(x) - f(x) - \frac{r}{2}\right\|_\infty \\ &= \frac{r}{2} < r\end{aligned}$$

Which shows that that K^* is a solid cone and $\text{int}K^{*s} = \text{int}K^* \neq \emptyset$.
(iii) $\Longrightarrow$ *(i)* Assume K^* is a solid cone. Then there exists $g \in B_r(f) \subset K^*$ for $f \in K^*$ and $r > 0$, then $g(x) \geq 0$. Assume that $f(x) - g(x) \geq 0$ and

$g(x) = a\|x\|$ for $x \in K$ and $a = \epsilon\|f\|_{X^*} > 0$, where $\epsilon \in (0, 1]$. Since f can be extended to the whole of X by Banach - Hahn theorem, then

$$K = \{x \in X : x \geq 0 \text{ and } f(x) \geq \epsilon\|f\|_{X^*}\|x\|\}$$
$$\subseteq \{x \in X : f(x) \geq \epsilon\|f\|_{X^*}\|x\|\}$$

Hence the result. □

We now show by example that $K^{*s} \neq intK^*$.

Examples 2.31. Let $X = \ell_2$ and $X^+ = \ell_2^+$ be cone in X. The cone X^+ has a base this means, $(\ell_2^+)^{*s} \neq \emptyset$ by Proposition 2.10. By Corollary 2.17, ℓ_2^+ admits unbounded base only, meaning $\text{int}(\ell_2^+)^* = \emptyset$ by the above theorem. Most importantly, $int(\ell_2^+) = int(\ell_2^+)^* = \emptyset$.

Remark 2.32. We consider the following relationships between cones with bounded and unbounded bases and structures of K^* and K^{*s} based on the above results.
Let K be a closed cone of a normed space X and $K^{*s} \neq \emptyset$.

If $intK^* = intK^{*s} \neq \emptyset$. Then

(a) $K^{*s} \neq intK^{*s}$ implies a cone K has bounded base for $f \in intK^{*s}$ and unbounded base for $f \in K^{*s} \backslash intK^{*s}$. Take ℓ_1 for a example.

(b) $K^{*s} = intK^{*s}$ implies a cone K has bounded base for all $f \in K^{*s}$. An example is given by Proposition 2.21.

A closed cone in finite dimensional spaces admits only bounded base, while infinite dimensional space a closed cone can admit both bounded and unbounded bases, for instance ℓ_1 and ℓ_2.

Theorem 2.33. Let X be a normed space and let $K \subset X$ be a closed cone such that $K^{*s} \neq \emptyset$. If $K^{*s} = intK^{*s}$ then either the base B_f is bounded for every $f \in K^{*s}$ or B_f is unbounded for every $f \in K^{*s}$.

Definition 2.34. A normed space X is compact if and only if every sequence in X has a convergent subsequence.

We mention the following important Theorem without a proof, proof can be found in [16, Theorem 5.5].

Theorem 2.35. ([16, Theorem 5.5]) Let $(X, \|\bullet\|)$ be a normed space, then the subset $B[0, 1]$ of X is compact if and only if X is finite dimensional.

We now show an example of a base which is not compact.

Examples 2.36. Consider an infinite dimensional Banach space, $(C([0,1]), \| \bullet \|_\infty)$, with a positive cone defined by

$$K = \{f \in X : f(t) \geq 0 \quad \text{for all } t \in [0,1]\}$$

and the norm defined

$$\|f\|_\infty = \sup_{i \in \mathbb{N}} \{|f_i(t)| : t \in [0,1]\}.$$

A base

$$\begin{aligned} B_g &= \{f \in K : g(f) = 1\} \\ &= \{f \in K : \|f\|_\infty \leq 1\} \end{aligned}$$

where $g : X \to \mathbb{R}$ is linear functional, is not compact. To this end, let $f_n : [0,1] \mapsto \mathbb{R}$ be defined by

$$f_n(t) = \begin{cases} nt, & \text{if } 0 \leq t \leq \frac{1}{n}; \\ 1, & \text{if } \frac{1}{n} \leq t \leq 1. \end{cases}$$

Clearly, continuous function $f_n \in B_g$. Then $\|f_n - f_m\|_\infty = 1$ for all $n \neq m$. Meaning that terms are 1 unit apart, so they can't have a convergent subsequence. So B_g is not compact.

Theorem 2.37. Let $(X, \| \bullet \|)$ be a Banach space and $K = X^+$. If K has a compact base, then X is finite - dimensional.

Proof. Let B be a compact base of K. Then there exists $M > 0$ such that $\|b\| \leq M$ for all $b \in B$. Again, since $\overline{0} \notin \overline{B} = B$, there exists w such that $\|b\| \geq w$ for all $b \in B$. Let $K_M = \{k \in K : \|k\| \leq M\}$. Then K_M is closed and convex (and hence it is weak - closed). We now prove that K_M is compact. To this end, we first show that $K_M \subseteq \{ab : 0 \leq a \leq \frac{M}{w} : b \in B\}$. Let $k \in K_M$, then there exists $\alpha > 0$ and $b \in B$ such that $k = \alpha b$ and $\|\alpha b\| \leq M$. This implies $\|b\| \leq \frac{M}{\alpha}$ and $w \leq \frac{M}{\alpha}$. Therefore $0 < \alpha \leq \frac{M}{w}$ and $b \in B$. Which shows that

$$K_M = \{\alpha b : 0 < \alpha \leq \tfrac{M}{w} : b \in B\} \subseteq \{ab : 0 \leq a \leq \tfrac{M}{w} : b \in B\}.$$

Now, $[0, \frac{M}{w}] \times B$ is compact in the product topology of $\mathbb{R} \times X, \mathbb{R}$ has usual topology and X has the norm topology. Again, X is a topological space, and so multiplication by scalar is continuous from $\mathbb{R} \times X \mapsto X$. The set

$\{ab : 0 \leq a \leq \frac{M}{w} : b \in B\}$ is compact as a scalar multiple of a compact set. Therefore, K_M is closed set, and so K_M is compact as a closed subset of compact set. Now, let $X_M = \{y \in X : \|y\| \leq M\}$. Then X_M is closed and $X_M \subseteq K_M - K_M$. But $K_M - K_M$ is compact and hence X_M is compact. Therefore X is finite - dimensional by Theorem 2.35. □

Definition 2.38. $E \subset X$ is weakly compact if E is compact in a weak topology, that is, if $f(B)$ is compact in $\mathbb{R}$, where $f \in X^*$.

Definition 2.39. A base B of a cone K in a normed space X is weakly compact if and only if for sequence $(x_i) \subset B$ there exists subsequence $(x_{i_k}) \subset B$ such that $x_{i_k} \xrightarrow{w} x_0$ in B, that is, $f(x_{i_k}) \to f(x_0)$ in $\mathbb{R}$ as $k \to \infty$ for some $f \in X^*$.

We consider cones with weak compact base.

Proposition 2.40. If a cone K of a Banach space $(X, \| \bullet \|)$ has weak - compact base B, then K admits a bounded base.

Proof. Assume that B is weakly compact, then B is weakly closed and weakly bounded. This implies B is closed and bounded. □

Converse of the above result is not true in general.

Examples 2.41. The cone ℓ_1^+ admit bounded bases but not weakly compact ones. (This will be deduced from Theorem 2.44)

We now consider the relationship between bounded, unbounded, and weakly compact base.

Theorem 2.42. ([7, Theorem 3.3]) Let X be a normed space and K a weakly closed cone of X so that $B[0,1]^+ = B[0,1] \bigcap K$ is weakly compact then the base B_f is bounded for every $f \in K^{*s}$ or B_f is unbounded for every $f \in K^{*s}$.

Theorem 2.43. ([7, Lemma 3.4]) Let X be a normed space and let $K \subset X$ be a closed cone such that $K^{*s} \neq \emptyset$. If $K^{*s} = intK^{*s}$ then B_f is weakly compact for every $f \in K^{*s}$.

Using Theorem 2.33, Theorem 2.43 can be rephrased as follows in terms of boundedness of a base of a cone:

Theorem 2.44. Let X be a normed space and K a cone in X. If a base B_f of K is bounded for all $f \in K^{*s}$ or unbounded for all $f \in K^{*s}$ then B_f is weakly compact.

Example 2.45. Let $X = \ell_2, Y = \ell_1$ and $X^+ = \ell_2^+, Y^+ = \ell_1^+$ a positive cones in X and Y respectively. The cone X^+ admits a weakly compact base B_f since B_f is unbounded for all $f \in K^{*s}$ and Y^+ does not admits a weakly compact base since it mixed based.

3 Characterization of reflexive Banach spaces via cones with bounded and unbounded bases.

In this chapter we discuss two characterizations of reflexive Banach space in terms of closed cones with bounded and unbounded bases and we also look at characterization of non reflexivity of a Banach space. We discuss the notion of isometry between positive cone ℓ_1^+ and other cones. Based on the results in this section, we give a proof of Milman Theorem on non - reflexive Banach space.

3.1 Reflexive Banach space and Banach lattice.

3.1.1 Reflexive Banach space.

Definition 3.1. Let $(X, \|\bullet\|)$ be a normed space, $(X^*, \|\bullet\|_{X^*})$ be a norm dual of X and $(X^{**}, \|\bullet\|_{X^{**}})$ be a norm dual of X^*, then X is reflexive if the canonical embedding $J : X \to X^{**}$ is surjective.

Next we give a classical characterization of a reflexive Banach space.

Theorem 3.2. ([16, Theorem 16.5]) Let X be a Banach space. The following statements are equivalent.

(1) X is a reflexive.

(2) $B[0, 1]$ is weakly compact.

(3) Every bounded sequence in X has a weakly convergence subsequence.

We next consider a classical characterization of reflexive Banach space by Alaoglu.

Theorem 3.3. [Alaoglus] Let X be a Banach space. Then X is reflexive if and only if a closed cone K admits a weakly - compact base.

Next we consider characterization of reflexivity of Banach space in terms of cones with bounded bases.

Theorem 3.4. (cf [7, Theorem 3.5]) Let $(X, \|\bullet\|)$ be a Banach space. Then X is reflexive if and only if there exists a closed cone K in X such that int $K \neq \emptyset$ and K^{*s} =int K^{*s}.

Proof. Suppose X is reflexive. Let $x \in S(0, 1)$ and the cone

$$K_x = \text{cone}(x + \tfrac{1}{2}B[0,1])$$

which is closed and convex (therefore weakly closed) admits a closed and bounded base $x + \frac{1}{2}B[0,1]$. That is, $intK_x^{*s} \neq \emptyset$. Again $intK_x \neq \emptyset$ since if $\|x_0\| > 1$, we have $x_0 + \frac{1}{2}B[0,1] \subset K_x$.

$B[0,1] = B^{**}[0,1]$ is weakly compact in X by Alaoglu's Theorem since X is reflexive. And therefore $B^+[0,1] = B[0,1] \bigcap K$ is weakly compact as a weakly closed subset of $B[0,1]$ and the base $B_f \subset B^+[0,1]$ in K_x is also a weak compact, that is, B_f is bounded for all $f \in K^{*s}$. Therefore $K_x^{*s} = intK_x^{*s}$. Hence the result.

Conversely, suppose that there exists a closed cone K such that int $K \neq \emptyset$ and $K^{*s} = \text{int}K^{*s}$. Then there exists $f \in K^{*s}$ such that $B_f = \{k \in K : f(k) = 1\}$ is a weakly compact base for the cone K. Now, since int $K \neq \emptyset$, we can find $x_0 \in K$ such $\|x_0\| > 2$ and the set $G = B[x_0, 1] \subset K$. Since G is a bounded and closed convex subset (therefore weakly bounded and closed) of K, there exists a real number $\alpha > 0$ such that $0 \leq f(g) \leq \alpha$ for every $g \in G$. Hence by Theorem 2.37

$$G \subset \bigcup_{0 \leq \beta \leq \alpha} \beta B_f = \{\beta b : 0 \leq \beta \leq \alpha, b \in B_f\}$$

and the set $\bigcup_{0 \leq \beta \leq \alpha} \beta B_f$ is weakly compact set. Since G is a closed convex set, it is also weakly compact. Therefore X is reflexive since $B[0,1]$ is weakly compact. □

The above result, shows that in a reflexive Banach space, there exists a closed cone K_x with a weakly compact base. Meaning that such a closed cone admit either bounded bases or unbounded bases but not both.

Corollary 3.5. (cf [21, Theorem 1]) Let X be a Banach space. If X is reflexive then for each cone $K^* \subset X^*$ admitting a bounded base, the cone $K = \{x \in X : f(x) \geq 0 \text{ for all } f \in K^*\}$ has a nonempty interior.

Proof. If a cone K^* admits a bounded base, then $intK^{**} \neq \emptyset$ in X^{**}. Since X is reflexive, $intK \neq \emptyset$ in X. Hence the result. □

The next theorem shows that in a reflexive Banach space, every closed cone with a base admit a weakly compact base.

Theorem 3.6. (cf [7, Theorem 3.6]) Let X be a Banach space. Then X is reflexive if and only if for every closed cone K in X such that int $K^{*s} \neq \emptyset$ we have $K^{*s} = \text{int}K^{*s}$.

Proof. Let $K \subset X$ be a closed cone such that $intK^{*s} \neq \emptyset$. Then there exists a bounded base B_f in K defined by $f \in K^{*s}$. Since the space X is reflexive, $B_X = B^{**}[0,1]$ is a weakly compact in X. $B^+[0,1] = B[0,1] \bigcap K$ is weakly compact in X as a weakly closed subset of $B[0,1]$. Therefore $B_f \subset B^+[0,1]$ is bounded for all $f \in K^{*s}$. Thus $K^{*s} = intK^{*s}$.
Conversely, the cone $K = \text{cone}(x + \frac{1}{2}B[0,1])$ for all $x \in S(0,1)$ will satisfies the desired property. Because the cone K is closed with a bounded base $B = x + \frac{1}{2}B[0,1]$. That is, $intK \neq \emptyset$ and $intK^{*s} \neq \emptyset$. Since by assumption, $K^{*s} = intK^{*s}$ for each $x \in S(0,1)$, therefore B is a bounded base for all $x \in S(0,1)$. That is, B is a weakly compact base in K and therefore $B[0,1]$ is a weakly compact in X. Hence the result. □

Note that theorem 3.6 can be rewritten as follows:

Theorem 3.7. Banach space is reflexive if and only if every closed cone K is such that $K^{*s} = intK^{*s}$ or $intK^{*s} = \emptyset$.

And in terms of boundedness of a base of a cone, theorem 3.7 can be reformulated in the following way:

Theorem 3.8. A Banach space X is reflexive if and only if each closed cone K with a base in X is such that either K has a bounded base for all $f \in K^{*s}$ or K has an unbounded base for all $f \in K^{*s}$.

Corollary 3.9. $\mathbb{R}^n$ is a reflexive Banach space.

Proof. The positive cone $(\mathbb{R}^n)^+$ is closed by Theorem 1.51 since $\mathbb{R}^n$ is normed Riesz space and admits bounded base only by Theorem 2.22. Then by the above Theorem, $\mathbb{R}^n$ is reflexive. □

We next show that in a finite dimensional space, we have a mixed based cone.

Example 3.10. Let $X = \mathbb{R}^2$. Define a cone

$$\begin{aligned} C &= \{x \in \mathbb{R}^2 : x_2 > |x_1|\}, \qquad \text{where } x = (x_1, x_2) \\ &= \{x \in \mathbb{R}^2 : -x_2 < x_1 < x_2\}. \end{aligned}$$

The cone C has both bounded and unbounded bases and it is open.

Proof. We first show that cone C is open. Since $-x_2 < x_1 < x_2$, we have that $x_2 - x_1 > 0$ and $x_1 + x_2 > 0$. Now let $\epsilon_1 = x_2 - x_1$ and $\epsilon_2 = x_1 + x_2$. We need to show that

$$\begin{aligned}
\|x - y\|_\infty &= \|(x_1 - y_1), (x_2 - y_2)\|_\infty \\
&= \sup\{|x_1 - y_1|, |x_2 - y_2|\} \\
&< \epsilon.
\end{aligned}$$

So $y = (y_1, y_2) \in C$, that is, $-y_2 < y_1 < y_2$. To do that, we consider the following cases.

Case 1 $|x_1 - y_1| < x_2 - x_1$ and $|x_2 - y_2| < x_2 - x_1$.

(a)

$$\begin{aligned}
|x_1 - x_2| < x_2 - x_1 &\iff x_1 - x_2 < y_1 - x_1 < x_2 - x_1 \\
&\iff 2x_1 - x_2 < y_1 < x_2
\end{aligned}$$

(b)

$$\begin{aligned}
|x_1 - x_2| < x_2 + x_1 &\iff x_1 - x_2 < y_2 - x_2 < x_2 - x_1 \\
&\iff x_1 < y_2 < 2x_2 - x_1
\end{aligned}$$

Case 2 $|x_1 - y_1| < x_2 + x_1$ and $|x_2 - y_2| < x_2 + x_1$.

(a)

$$-x_1 - x_2 < y_1 - x_1 < x_2 + x_1 \iff -x_2 < y_1 < 2x_1 + x_2$$

and

$$|x_1| - |y_1| < x_2 + x_1 \iff -|y_1| < x_2 + x_1 - |x_1|$$

Then $-(-y_1) < x_2 + x_1 - x_1$ as one of the possibilities. Therefore $y_1 < x_2$.

(b)

$$
\begin{aligned}
|x_2| - |y_2| < |x_2 - y_2| &< x_2 + x_1 \\
-|y_2| &< x_2 - |x_2| + x_1 \\
-|y_2| &< x_2 + |x_2| + x_1 \\
-|y_2| &< 2|x_2| + x_1 \\
-|y_2| &< 2|x_2| + y_2 \\
-|y_2| &< 2|x_2| + |y_2| \\
|y_2| &> -|x_2| \\
y_2 &> -(-x_2) \qquad \text{as one of the possibilities.}
\end{aligned}
$$

Therefore $y_2 > x_2$. This implies $y_2 \geq \overline{0}$ since $x_2 \geq \overline{0}$.

Let $y_1 = \min\{x_2, x_1 + 2x_2\}$, then $y_1 < x_2 < y_2 < 2x_2 - x_1$. Since y_1 is arbitray and $y_2 \geq \overline{0}$, then $|y_1| < y_2$ and so $y \in C$.
Finally, we find bounded base and unbounded base in the cone C, namely, bounded base $B_1 = \{(x, y) \in C : y = 1\}$ and unbounded base

$$B_2 = \{(x, y) \in C : y = x^2 + 1\}.$$

□

Remark 3.11. Note that the example does not contradicts Theorem 3.8, since C is open.

3.1.2 AL - and AM - Banach lattices.

We conclude this section with a result concerning the Banach lattices. First note that an infinite dimensional Banach lattice ℓ_2 is reflexive since its positive cone ℓ_2^+ admits unbounded bases only and infinite dimensional Banach lattice ℓ_1 is not reflexive since its positive cone ℓ_1^+ admit both bounded and unbounded bases.

Definition 3.12. A Banach lattice $(X, \| \bullet \|)$ is an

(i) ([26, Definition 8.1]) AL - space if $\|x+y\| = \|x\|+\|y\|$ for all $x, y \in X^+$.

(ii) ([26, Definition 7.1]) AM - space if $\|x \vee y\| = \max\{\|x\|, \|y\|\}$ for all $x, y \in X^+$.

Next we investigate whether a Banach lattice $\mathbb{R}^n$ is AL - and / or AM - space.

Examples 3.13. Consider a Banach lattice $(\mathbb{R}^n, \| \bullet \|_1)$ with a norm

$$\|x\|_1 = \sum_{i=1} |x_i| \text{ for all } x \in \mathbb{R}^n$$

and a coordinatewise ordering. We show that $\mathbb{R}^n$ is both AL - and AM - space.

Proof. First we show that $\mathbb{R}^n$ is AL - space. Take $x, y \in (\mathbb{R}^n)^+$, then

$$\begin{aligned}
\|x + y\|_1 &= \sum_{i=1}^{n} |x_i + y_i| \\
&= \sum_{i=1}^{n} x_i + y_i, \qquad\qquad x_i + y_i \geq 0 \\
&= \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} y_i \\
&= \sum_{i=1}^{n} |x_i| + \sum_{i=1}^{n} |y_i| \\
&= \|x\|_1 + \|y\|_1.
\end{aligned}$$

Next we show that $\mathbb{R}^n$ is AM - space. Take $x, y \in (\mathbb{R}^n)^+$, then

$$\begin{aligned}
\|x \vee y\|_1 &= \sum_{i=1}^{n} |x_i \vee y_i| \\
&= \sum_{i=1}^{n} (|x_i| \vee |y_i|) \\
&= \sum_{i=1}^{n} |x_i| \vee \sum_{i=1}^{n} |y_i| \\
&= \|x\|_1 \vee \|y\|_1 \\
&= \max\{\|x\|_1, \|y\|_1\}.
\end{aligned}$$

This shows that $\mathbb{R}^n$ is both AL - and AM - space. □

Proposition 3.14. Let $(X, \| \bullet \|)$ be a Banach lattice.

(i) If a lattice cone X^+ has a bounded base, then X^* is an AM - space.

(ii) If X^* is not reflexive, then X is not reflexive.

Proof. (i) Let $\psi \in (X^+)^{*s}$ and $f \in X^*$, since a base B_ψ is bounded, there exists $\alpha \in \mathbb{R}^+$ such that $\psi(x) \leq \alpha = |f(x)|$ for all $x \in B_\psi$. Equivalently, for each $\epsilon > 0, |f(x)| < \epsilon\psi(x)$ for $x \in X \setminus B_\psi$. Note that $\psi \in K^{*s}$ is such that $\psi > 0$. Therefore the set $\{\lambda > 0 : |f| \leq \lambda\psi\}$ is not empty for all $f \in X^*$. We first show that a real-valued function $\| \bullet \|_\infty$ defined by

$$\|f\|_\infty = \inf\{\lambda > 0 : |f| \leq \lambda\psi\}$$

is a norm on X^*.
$\|f\|_\infty = \inf\{\lambda > 0 : |f| \leq \lambda\psi\} \geq 0$ since for each λ_0 in the set

$$\{\lambda > 0 : |f| \leq \lambda\psi\}$$

we get that $\lambda_0 > 0$. If $f = \overline{0}$, then

$$\begin{aligned}
\|f\|_\infty &= \inf\{\lambda > 0 : |f| \leq \lambda\psi\} \\
&= \inf\{\lambda > 0 : \overline{0} \leq \lambda\psi\} \\
&= \inf\{\mathbb{R}^+\} = 0.
\end{aligned}$$

Now if $\|f\|_\infty = 0$ then $\inf\{\lambda > 0 : |f| \leq \lambda\psi\} = 0$, which implies for all $\lambda > 0$ we have that $\|f\|_\infty \leq \lambda\psi$. So it follows that $\|f\|_\infty \leq \frac{1}{n}\psi$ for all $n \in \mathbb{N}$. This gives us $|f| = 0$ implying $f = 0$.
Now for propety (ii) of a norm,

$$\begin{aligned}
\|\alpha f\|_\infty &= \inf\{\lambda > 0 : |\alpha f| \leq \lambda\psi\} \\
&= \inf\{\lambda > 0 : |\alpha||f| \leq \lambda\psi\} \\
&= |\alpha| \inf\{\lambda > 0 : |f| \leq \lambda\psi\} \\
&= |\alpha|\|f\|_\infty
\end{aligned}$$

For property (iii), note that since $\psi > 0$ we have

$$\begin{aligned}
\|\psi\|_\infty = \inf\{\lambda > 0 : |\psi| \leq \lambda\psi\} &= \inf\{\lambda > 0 : \psi \leq \lambda\psi\} \\
&= \inf\{\lambda > 0 : 1 \leq \lambda\} \\
&= 1. \qquad (4)
\end{aligned}$$

Then if μ and γ are smallest positive real numbers such that $|f| \leq \mu\psi$ and $|g| \leq \gamma\psi$ then by properties of absolute values

$$|f + g| \leq |f| + |g| \leq (\mu + \gamma)\psi.$$

So we have that,

$$\begin{aligned}
\|f + g\|_\infty &\leq \|(\mu + \gamma)\psi\|_\infty \\
&= |\mu + \gamma|\|\psi\|_\infty \\
&= |\mu + \gamma| \qquad \text{by (4)} \\
&= \|f\|_\infty + \|g\|_\infty.
\end{aligned}$$

Hence the result. Therefore the set $(X^*, \| \bullet \|_\infty)$ is a Banach space.
To show that $\| \bullet \|_\infty$ defines a lattice norm, assume $|f| \leq |g|$. So

$$\begin{aligned} \|f\|_\infty &= \inf\{\lambda > 0 : |f| \leq \lambda\psi\} \\ &\leq \inf\{\lambda > 0 : |g| \leq \lambda\psi\} \\ &= \|g\|_\infty. \end{aligned}$$

Thus $(X^*, \| \bullet \|_\infty)$ is a Banach lattice.
Lastly, we show that $\| \bullet \|_\infty$ is an AM - norm. To this end, let $f, g \in (X^*)^+$. From $f \leq \|f\|\psi$ and $g \leq \|g\|\psi$, we have

$$f \vee g \leq \|f\|\psi \vee \|g\|\psi, \text{ so } \|f \vee g\| \leq \|f\| \vee \|g\|.$$

On the other hand, from $\|f\| \leq \|f \vee g\|$ and $\|g\| \leq \|f \vee g\|$, we have $\|f\| \vee \|g\| \leq \|f \vee g\|$. Therefore $\|f \vee g\| = \|f\| \vee \|g\|\}$ for all $f, g \in (X^*)^+$ and then $(X^*, \| \bullet \|_\infty)$ is AM - space.
2. If X^* is not reflexive, then X^* is an infinite dimensional AM - space with a lattice cone C that admits bounded base B_f. Note that X is also infinite dimensional space. We complete the proof by showing that X contains a cone that admits bounded base. From $intC^{*s} \neq \emptyset$, there exists $f \in C^{*s} \subset X^{**}$ such that $f(g) > \bar{0}$, where $g \in K^{*s} \subset X^*$. This means for $x \in K \setminus \{\bar{0}\}$ we have $0 < f(g(x)) \leq \alpha$ since B_f is bounded. This also shows that the base $B_{f(g)}$ is weakly bounded in X with $f(g) \in K^{*s}$. Therefore bounded in X. And X contains a mixed base cone, since X is infinite. Hence X is non reflexive. □

3.2 Mixed based cones, cones conically isomorphic to ℓ_1 and Non - Reflexive Banach spaces.

Theorem 3.15. (cf [7, Theorem 4.1]) A Banach space X in non-reflexive if and only if there exists a closed cone $K \subset X$ such that $intK^{*s} \neq \emptyset$ and $intK^{*s} \neq K^{*s}$.

Proof. Let $K \subset X$ such that $intK^{*s} \neq \emptyset$ and $intK^{*s} \neq K^{*s}$, that is K admits bounded and unbounded bases. By Theorem 2.33,

$$B^+[0,1] = B[0,1] \bigcap K$$

is not a weakly compact base, that is, $B[0,1]$ is not weakly compact in X. Hence a Banach space X is not reflexive. □

We now restate Theorem 3.15.

Theorem 3.16. A non - reflexive Banach space admits a mixed base cone.

Example 3.17. By the above Theorem, infinite dimensional Banach space ℓ_1 is non - reflexive since ℓ_1^+ admits both bounded and unbounded bases.

Next we show another example of a mixed based cone.

Examples 3.18. Consider the Banach space, $(c, \|\bullet\|_\infty)$ and a cone c^+. The cone c^+ is a mixed base cone.

Proof. Firstly, we show that a cone admits a bounded base. Take

$$x = (x_i) \in c^+ \subset \ell_\infty^+,$$

then there exists $y = (y_i) \in \ell_1^+$ such that

$$\begin{aligned} \|x\|_\infty &= \sup_{i \in \mathbb{N}}\{x_i\} \\ &\le \sum_{i=1}^{\infty} x_i && x_i \ge 0 \\ &\le \sum_{i=1}^{\infty} x_i y_i && y_i \ge 0 \\ &= f(x_i) = \alpha > 0 && f \in (c^+)^*. \end{aligned}$$

Hence the results. Lastly, we show that c^+ admits unbounded base as well. Let $B_f = \{x \in c^+ : f(x) = 1\}$ be a base of a cone c^+ defined by $f \in (c^+)^*$. Take the subsequence $y_{i_k} = (1, \frac{1}{2}, \frac{1}{4}, \frac{1}{6}, ...) \in \ell_1^+$ and take $e_{i_k} = (0, 0, 0, ..., \overset{i^{th}}{1}, 0, 0, ...) \in c^+$ then $y_{i_k} \to 0$ and $e_{i_k} \to 1$ as $k \to \infty$. $\left(\frac{e_{i_k}}{f(y_{i_k})}\right) \in B_f$ and $\left\|\frac{e_{i_k}}{f(y_{i_k})}\right\|_\infty = \frac{1}{\sum_{i=1}^{\infty} y_{i_k} e_{i_k}} = \frac{1}{\frac{1}{i}} \to \infty$ as $i \to \infty$. Thus, B_f is unbounded. Hence the result. This shows that c is a non - reflexive infinite dimensional Banach space. □

We now give notion of order isomorphism.

Definition 3.19. Let X and Y be normed spaces ordered by the cones P, K respectively. The cone P is said to be order isomorphic to the cone K of Y if there exists an additive, positively homogeneous, one - to - one, map T_1 of P onto K such that T_1 and T_1^{-1} are continuous in the induced topologies. Then we also say the cone P is embeddable in Y and that T_1 is a order isomorphism of P onto K.

T_1 can be extended to a linear operator on $P - P$ as follows;

$$T_1(x_1 - x_2) = T_1(x_1) - T_1(x_2) \text{ where } x_1, x_2 \in P.$$

If P is generating then T_1 can be extended to a whole space X.
We now give an example of a order isomorphism between cones. We first give the following definition.

Definition 3.20. ([21, Definition 4.1.1]) Let (x_n) be a sequence in a Banach space X. Then a sequence (x_n) in an ordered Banach space X is called a Schauder basis for X if for each $x \in X$ there is a unique sequence (α_n) of scalars such that

$$x = \sum_{n=1}^{\infty} \alpha_n x_n.$$

$$P = \left\{ \sum_{n=1}^{\infty} \alpha_n x_n : \alpha_n \geq 0, \text{for any } \ n \right\}$$

is the positive cone of a Schauder basis (x_n) of X. A sequence (x_n) of X is a positive basis of X if it is a Schauder basis of X and $X^+ = P$. The Banach space c_0 is one of the examples of space with positive bases.

Definition 3.21. ([27, Definition 3.1] Let (x_n) be a basis of a Banach space X. The sequence of linear functionals (f_n) defined by

$$f_j(x) = \alpha_j, x = \textstyle\sum_{j=1}^{\infty} \alpha_j x_j \in X, j = 1, 2, ...,$$

is called a sequence of coefficient functionals associated to the basis (x_n), a.s.c.f for short.

Examples 3.22. Consider two positive cones, ℓ_1^+ and c_0^+. Let $a = (a_i) \in \ell_1^+$ and $\{x_i\}$ be a Schauder basis for c_0^+. Consider the closed cone $Q \subseteq c_0^+$ defined as

$$Q = \left\{ x \in c_0^+ : x = \sum_{i=1}^{\infty} a_i x_i, (a_i) \subset \ell_1^+ \right\}.$$

Define the map $T : \ell_1^+ \to Q$ by

$$T(a) = \sum_{i=1}^{\infty} a_i x_i.$$

T is a conical isomorphism of ℓ_1^+ onto Q.

Proof. This map is well defined since $\{x_i\}$ is bounded by being convergent. Firstly, we show that T is additive and positively homogeneous. To this end, if $a, b \in \ell_1^+$ and $\lambda \geq 0$, then $a + b \in \ell_1^+$ and $\lambda a \in \ell_1^+$. So

$$T(a+b) = \sum_{i=1}^{\infty}(a_i + b_i)x_i = \sum_{i=1}^{\infty} a_i x_i + \sum_{i=1}^{\infty} b_i x_i = T(a) + T(b)$$

and

$$T(\lambda a) = \sum_{i=1}^{\infty}(\lambda a_i)x_i = \lambda\left(\sum_{i=1}^{\infty} a_i x_i\right) = \lambda T(a).$$

Hence the result. Secondly, we show that T is injective and surjective. T is injective, since $x = \sum_{i=1}^{\infty} a_i x_i \in Q$ is uniquely determined by $a = (a_1, a_2, ..., a_i, ...) \in \ell_1^+$. T is also surjective map. Indeed, if $x \in Q$ then the exists a sequence $(a_i) \in \ell_1^+$ such that $x = \sum_{i=1}^{+} a_i x_i$. Therefore we have $a = (a_i) \in \ell_1^+$ such that $x = T(a)$. Hence the results. Lastly, T is continuous on its domain since $\{x_i\}$ is a bounded basic sequence in c_0. The closed cones ℓ_1^+ onto Q are complete subsets of Banach spaces ℓ_1 and c_0 respectively. Since T is continuous and surjective then by Open mapping theorem, $T^{-1} : Q \to \ell_1^+$ is also continuous. Therefore T is order isomorphism of ℓ_1^+ onto Q. □

We recall the following lemma.

Lemma 3.23. ([7, Lemma 3.4]) Let X be a Banach space and let $K \subset X$ be a closed cone such that $K^{*s} \neq \emptyset$. If $K^{*s} = intK^{*s}$ then B_f is weakly compact for every $f \in K^{*s}$.

Definition 3.24. A conical hull of a non - empty set X is a set denoted by $conv(X)$ and defined as

$$conv(X) = \left\{\sum_{n=1}^{N} \alpha_n x_n : N \in \mathbb{N}, x_n \in X, \sum_{n=1}^{N} \alpha_n = 1, \alpha_n \geq 0, \text{for any } n\right\}.$$

We now consider cones that isomorphic to ℓ_1^+ and show that such cones are mixed based.

Theorem 3.25. ([7, Theorem 4.5]) Let X be a Banach space and $K \subset X$ be a closed cone order isomorphic to the cone ℓ_1^+ then K is a mixed base cone such that $\text{int}K = \emptyset$.

Proof. Let $T : \ell_1^+ \to K$ be an isomorphism and we denote again by T its continuous extension to ℓ_1 (due to Hahn-Banach Theorem). Since the ℓ_1^+ has a closed and bounded base B, the set $T(B)$ is a closed (to see, let $x_n \in B$ such that $x_n \to x \in \ell_1^+$ then $g(x_n) \to g(x) \in \mathbb{R}$, where $g \in (\ell_1^+)^{*s}$.

Then for each $n, g(x_n) = 1 \to g(x) \in \mathbb{R}$, implying that $g(x) = 1$ that is, $x \in B$.) and bounded base for K. Hence there exists $f \in X^*$ such that $B_f = \{k \in K : f(k) = 1\}$, is a bounded base for K. Therefore $f \in intK^{*s}$. Now let U be unbounded base of ℓ_1^+. Then $T(U)$ is a closed unbounded base for K. Hence all $f \in K^{*s}$ separate $T(U)$ and 0 (by Hanh - Banach Theorem). We claim that there exists $f \in K^{*s} \setminus intK^{*s}$. To see, assume $K^{*s} = intK^{*s}$. Then by Lemma 3.23 bounded base B_f is weakly compact. Consider $\{e_i\}$ of ℓ_1 given by sequence of unit vectors in ℓ_1 and write $k_n = T(e_n)$ in K. Then there exists $v > 0$, such that $\|k_n\| \leq v$ and $f(k_n) \leq v\|f\|$. Hence

$$\{k_n\} \subset \left(\bigcup_{0 \leq \alpha \leq v\|f\|} \alpha B_f \right) \bigcap K = \{\alpha b : 0 \leq \alpha \leq v\|f\|, b \in B_f\} \bigcap K.$$

where $\bigcup \alpha B_f$ is weakly compact. Therefore there exists $k_0 \in K$ such that k_n converges weakly to k_0. Hence

$$k_0 \in \bigcap_{j=1}^{\infty} cl(conv\{k_j, k_{j+1}, ...\})$$

then

$$T^{-1}(k_0) \in T^{-1} \left(\bigcap_{j=1}^{\infty} cl(conv\{k_j, k_{j+1}, ...\}) \right) = \bigcap_{j=1}^{\infty} cl(conv\{e_j, e_{j+1}, ...\}).$$

Contradiction, since $\bigcap_{j=1}^{\infty} cl(conv\{e_j, e_{j+1}, ...\}) = \emptyset$. Because ℓ_1^+ is a mixed base cone, therefore not weakly compact. Finally, we show that $\text{int}K = \emptyset$. To see, suppose on the contrary that there exists an open set $O \subset K$. Since the map T is continuous, the set $T^{-1}(O) \subset \ell_1^+$ is open, contradiction against $\text{int}\ell_1^+ = \emptyset$. □

Remark 3.26. Let $(X, \| \bullet \|)$ be a Banach space and Q be a closed cone in X order isomorphic to the cone ℓ_1^+, then there exists a order isomorphism T of ℓ_1^+ onto Q.

Lemma 3.27. Let $(X, \| \bullet \|)$ be an ordered Banach space. If T is order isomorphism of ℓ_1^+ onto a cone $Q \subset X$ then there are positive constants $\alpha, \beta > 0$ such that

$$\alpha\|x\|_1 \leq \|T(x)\| \leq \beta\|x\|_1 \text{ for each } x \in \ell_1^+.$$

Proof. Note that T and T^{-1} are non zero operators since T is an isomorphism. So we have

$$\|Tx\| \leq \|T\|_\infty \|x\|_1 \text{ and } \|x\|_1 = \|T^{-1}(T(x))\|_1 \leq \|T^{-1}\|_\infty \|Tx\|$$

for each $x \in \ell_1^+$, since both $\|T\|$ and $\|T^{-1}\|$ are continuous. By letting $\alpha = \|T^-\|_\infty^{-1}$ and $\beta = \|T\|_\infty$ we get the desired inequalities. □

Since ℓ_1 ordered by the componentwise ordering is a Banach lattice, whose lattice cone is ℓ_1^+ then the cone ℓ_1^+ is generating, i.e., $\ell_1 = \ell_1^+ - \ell_1^+$ ([1]) and $x = x^+ - x^- = \sup\{x, \overline{0}\} - \inf\{-x, \overline{0}\}$ for every $x \in \ell_1$. The order isomorphism T can be extended to a one - to - one linear operator of ℓ_1 onto $Q - Q$ by taking $T(x) = T(x^+) - T(x^-)$.
The extension of T is continuous on the whole space ℓ_1.

Proof. To this end, we have

$$\begin{aligned} \|T(x)\| &= \|T(x^+) - T(x^-)\| \\ &= \|T(x^+) + (-T(x^-))\| \\ &\leq \|T(x^+)\| + \|T(x^-)\| \\ &\leq \alpha(\|x^+\|_1 + \|x^-\|_1) \\ &= \alpha\|x\|_1 && \text{since } \ell_1 \text{ is an AL - space.} \end{aligned}$$

Hence the result. □

We now recall the following types of basic sequences in a Banach space.

Definition 3.28. Let (x_n) be a sequence in a Banach space X. Then (x_n) is a weak Cauchy sequence if it is a Cauchy sequence in X endowed with the weak topology; equivalently, if $\lim_{n\to\infty} f(x_n)$ exists for every $f \in X^*$.

Definition 3.29. Let (x_n) be a sequence in a Banach space X. A sequence (x_n) is a trivial weak Cauchy sequence if it is a weak cauchy sequence that does not weakly converge.

Definition 3.30. ([17, Definition 1.a.1.] A sequence (x_n) in a Banach space X is a basic sequence if it is a Schauder basis for a closed linear span (denoted by cl(span(x_n))).

Definition 3.31. ([17, Definition 1.a.10.]) Let (x_n) be a basic sequence of in a Banach space X. A sequence (b_n) of non - zero elements in X is a block basis of (x_n) if there exist a sequences (p_n) and (q_n) such that $1 = p_1 \leq q_1 < p_2 \leq q_2 < \ldots$, and $b_n = \sum_{i=p_n}^{q_n} a_i x_i$, a non-trivial linear combination of x_i where the sequence $(a_i) \subset \mathbb{R}$.

Definition 3.32. Let (x_n) be a basic sequence of a Banach space X. A sequence (b_n) of a non - zero elements in X is a convex block basis of the basic sequence (x_n) if the sequence a_n in the previous definition is such that $a_i \geq 0$ for all i and $\sum_{i=p_n}^{q_n} a_i = 1$ for each n.

Definition 3.33. ([24, Definition 1.1]) Let (x_n) be a basic sequence of in a Banach space X. A sequence (x_n) is a strong summing sequence if (x_n) is a weak Cauchy basic sequence such that whenever the sequence of real numbers (a_n) satisfy

$$\sup_j \left\| \sum_{n=1}^{j} a_n x_n \right\| < \infty$$

then $\sum_{n=1}^{\infty} a_n$ converges.

Proposition 3.34. ([10, Theorem 5] Let (x_n) and (y_n) be a basic sequences in a Banach spaces X and Y respectively, then (y_n) is equivalent to (x_n) if there exists an isomorphism

$$T : cl(span(y_n)) \to cl(span(x_n))$$

such that $T(y_n) = x_n$ for each n.

We now give examples of a block basic sequence and a strong summing sequence.

Example 3.35.

(i) Let a sequence (e_n) of standard unit vectors in c_0 be the Schauder basis of c_0. Then a sequence (y_n) given by $y_n = \sum_{i=1}^{n} \lambda_i e_i$ where $\lambda_i = 1$ for all i is a block basic sequence taken from (e_n).
If $\lambda_i = \frac{1}{2^i}$, then (y_n) defined above is a convex block basic sequence since $\sum_{i=1}^{\infty} \lambda_i = 1$.

(ii) A basic sequence $(e_n) \in \ell_1$ is strong summing sequence. We first show that (e_n) is weakly cauchy. To see, take any $f = (f_n) \in \ell_\infty$. Then

$$\begin{aligned} f(e_n) &= \sum_{n=1}^{\infty} |f_n e_n| \\ &= |f_n|. \end{aligned}$$

Thus

$$\begin{aligned}
\lim_{n\to\infty} f(e_n) &= \lim_{n\to\infty} |f_n| \\
&< \sup |f_n| \\
&= \|f\|_\infty \\
&< \infty && \text{since } f \in \ell_\infty.
\end{aligned}$$

Therefore (e_n) is weak cauchy basic sequence.

Next we show that $\sup_n \left\| \sum_{i=1}^{n} a_i e_i \right\| < \infty$ for particular sequence $(a_i) = \left(\frac{1}{2^i}\right)$. To this end, we have

$$\sup_n \left\| \sum_{i=1}^{n} \frac{1}{2^i} e_i \right\| = \sup_n \left| \frac{1}{2^n} \right| = \frac{1}{2} < \infty$$

Hence the results, since $\sum_{i=1}^{\infty} \frac{1}{2^i} = 1 < \infty$.

Next we recall the following two lemmas.

Lemma 3.36. (cf [7, Theorem 4.6]) Every bounded sequence in a Banach space has either a weak Cauchy subsequence or a subsequence equivalent to the standard basis of ℓ_1.

Proof. Firstly, if $(X, \| \bullet \|_\infty)$ is a reflexive Banach space, each bounded sequence in X has a weak Cauchy subsequence since weak convergence subsequence imply weak Cauchy subsequence.
Secondly, if X is a non - reflexive, then in $B[0,1]$ in X (not weakly compact) there is a bounded (weak) sequence (x_n) with nontrivial weak cauchy subsequence, say (x_n), for simplicity. Therefore, there exists a subsequence (x_{n_i}) of (x_n) that does not converges weakly, that is,

$$\|f(x_n) - f(x_m)\|_\infty = 1,$$

for $f \in X^*$ and all $n \neq m$. Again, since X is non reflexive Banach space, ℓ_1^+ is cone conically isomorphic to a closed mixed base cone $Q \subset X^+$, that is $T : \ell_1^+ \to Q$ is isomorphism. Since $Q - Q = X$, the T can be extended isomorphically to ℓ_1 onto X, using Hahn - Banach Theorem. Therefore, (x_{n_i}) in X will be equivalent to unit standard basis vector (e_n) in ℓ_1. □

Lemma 3.37. (cf [7, Theorem 4.7]) Every nontrivial weak Cauchy sequence in a Banach space has either a strongly summing subsequence or a convex basis equivalent to the summing basis.

Proof. By Lemma 3.36, nontrivial weak Cauchy sequence contain a subsequence (x_i) equivalent to the standard basis (e_i) of ℓ_1 and therefore for $(y_i) \in \ell_1^+$ with $y_i \geq 0$ we have $\sup_n \left\| \sum_{i=1}^{n} y_i x_i \right\| < \epsilon$ and $\sum_{i=1}^{\infty} y_i < \infty$. Showing that (x_i) is a strongly summing subsequence. Or a block convex basis (b_i) of (x_i) described in the above Theorem can be constructed as follows: for $(a_i) \in \ell_1^+$, we have $b_i = \sum_{k=1}^{i} a_k x_k$ with $\|b_i\|_{c_0} = \max_{0 \leq k \leq i} |a_k|$ and $\sum_{i=1}^{\infty} a_i = 1$. The basis sequence (b_n) is a our block basis sequence and it is equivalent to the summing basis $\left\{ c_n = \sum_{i=1}^{n} e_i \right\}$ of the space c_0. □

Next, we mention a converse of Theorem 3.25 with some important results.

Theorem 3.38. ([7, Theorem 4.8]) Let X be a Banach space. If there exists a closed mixed based cone $K \subset X$ then there exists a conical isomorphism of ℓ_1^+ onto a cone $Q \subseteq K$. Moreover only three cases occur:

(i) ℓ_1^+ embeds in X,

(ii) c_0 embeds in X, that is Q is conically isomorphic to

$$K_{summ}c_0 = \left\{ \sum_{k=1}^{\infty} \lambda_k b_k \in c_0 : \lambda_k \geq 0 \right\} \text{ where } b_k = \sum_{i=1}^{k} e_i,$$

(iii) $Q = \{q \in X : q = \sum_{i=1}^{\infty} \psi_i q_i, \psi_i \in \mathbb{R}, \psi_i \geq 0 \text{ for each } i\}$ where $(q_n) \subset X$ is a strong summing sequence.

Next we prove the Milman's characterization Theorem of non - reflexivity in terms of a mixed base cone by combining Theorems 3.15, 3.25 and 3.38.

Theorem 3.39. (cf [7, Theorem 4.4]) A Banach space X is non - reflexive if and only if the positive cone of ℓ_1 is embeddable in X.

Proof. Assume that X is non - reflexive, then by Theorem 3.15, there exists a closed cone $K \subset X$ such that $(\text{int}K^*)^+ \neq \emptyset$ and $\text{int}K^{*s} \neq K^{*s}$, this means that base B_f is bounded for $f \in intK^{*s}$ and base B_g unbounded for $g \in K^{*s} \setminus intK^{*s}$ by Remark 2.32. That is, K is a closed mixed based cone. By Theorem 3.38 there exists a conical isomorphism of ℓ_1^+ onto a cone K. That is ℓ_1^+ is embeddable in X.
Conversely, assume ℓ_1^+ is embeddable in X, that is there exists a closed cone $K \subset X$ isomorphic to ℓ_1^+. By Theorem 3.25, K is a mixed based cone

such int$K = \emptyset$. Therefore by Theorem 3.15, a Banach space X is non - reflexive. $\square$

Definition 3.40. ([18, Theorem 2.5.23]) A Banach space is weakly complete whenever every weak Cauchy sequence weakly converges.

We consider an example of a Banach space that is not weakly complete.

Example 3.41. The Banach space c_0 is not weakly complete. To see, consider the sequence $y = (y_n) = \sum_{i=1}^{n} e_i \in c_0$. We first show that y is weakly Cauchy, that is $\lim_{n\to\infty} (y_n)$ exists for all $f \in c_0^*$. Now, take any $f \in c_0^* = \ell_1$, then

$$\begin{aligned} f(y) &= \sum_{i=1}^{\infty} |f_i y_i| \\ &= \sum_{i=1}^{\infty} |f_i| y_i && y_i \geq 0 \\ &= \sum_{i=1}^{n} |f_i| && \text{by definition } y. \end{aligned}$$

And thus

$$\begin{aligned} \lim_{n\to\infty} f(y_n) &= \lim_{n\to\infty} \sum_{i=1}^{n} |f_i| \\ &= \sum_{i=1}^{\infty} |f_i| \\ &< \infty && \text{since } f \in \ell_1. \end{aligned}$$

Next we show that (y_n) is not weakly convergent, that is,

$$f(y_n) \nrightarrow f(y) \tag{3}$$

for $y \in c_0$. Note that $y_n \to y = (1, 1, 1, 1, ...) \notin c_0$. This means for (y_n) there is no $y \in c_0$ such that $f(y_n) \to f(y)$.

We recall the following Proposition.

Proposition 3.42. ([24, Proposition 1.4]) Let (x_n) be a summing sequence in a Banach space X, and (f_n) be its a.s.c.f in X^*. The X^* is not weakly complete.

We now show that whenever X is weakly complete Banach space, only the first situation listed in Theorem 3.38 occurs.

Corollary 3.43. (cf [7, Corollary 4.10]) If X is a weakly complete Banach space and X contains a mixed base cone K then ℓ_1 embeds in X.

Proof. Note that K is closed since it admits bounded base and X is non-reflexive since K contains both bounded B_f and unbounded B_g bases for $f \in intK^*$ and $g \in K^{*s} \setminus intK^*$. By Theorem 3.36, sequence $(x_n) \in B_f$ has

(i) either a subsequence (x_{n_j}) equivalent to (e_j) of ℓ_1

(ii) or a weak cauchy subsequence.

Using Theorem 3.38 on (i), we get the result that ℓ_1^+ embeds on X. From (ii) a weak cauchy subsequence (x_{n_j}) converges weakly since X is weakly complete. By Lemma 3.37, the following occur: (x_{n_j})

(i) has no strongly summing subsequence, that is, situation (iii) on Theorem 3.38 cannot occur.

(ii) has no convex block basis equivalent to the summing basis of c_0, that is, c_0 is not embedding in K, meaning, situation (ii) on Theorem 3.38 cannot occur .

□

Next we provide conditions on a Banach space for the first two conditions listed on Theorem 3.38 to occur.

Corollary 3.44. (cf [7, Corollary 4.11]) Let X be a Banach space such that X^* is weakly complete. If X contains a mixed based cone then X contains either c_0 or ℓ_1.

Proof. Note that X is a non - reflexive, since it contains a closed mixed based cone K. As in the above Corollary, a sequence $(x_n) \in B_f$ (bounded base of K with $f \in intK^*$) has either a subsequence (x_{n_i}) of (x_n) equivalent to (e_i) of ℓ_1 or a weakly cauchy subsequence.
If (x_{n_i}) is equivalent to (e_i) of ℓ_1 then by results of Theorem 3.38, ℓ_1^+ embeds on X.
Or (x_{n_i}) is a non - trivial weakly cauchy by the results of Theorem 3.38. By Lemma 3.37, (x_{n_i}) contains either a convex block basis $(x_{n_{i_j}})$ equivalent to the summing basis (b_j) of c_0 or a strong summing subsequence $(x_{n_{i_j}})$.
If $(x_{n_{i_j}})$ is a convex block basis then by Theorem 3.38 c_0 embeds in X. But X cannot contain a strong summing sequence because if it has it would

mean that X^* is not weakly complete by Proposition 3.42. Contradicting the assumption that X^* is weakly complete. $\square$

4 Reflexive cones.

In this chapter, we consider a notion of reflexive cones and their properties. The characterization of reflexive Banach space by reflexive cones is also discussed. We also discuss the relationship between reflexive cones and subcones of ℓ_1^+. We recall definitions of semi - interior points, open decomposition as well as some of their properties.

4.1 Reflexive cones and their properties.

Theorem 4.1. ([2, Lemma 1.6]) Given a cone K in an ordered vector space X. The cone K is generating if and only if X is directed upwards by ordering induced by K.

Proof. Suppose K is a generating cone in an ordered vector space X. That is, $X = K - K$. Then there are vectors $a, b, c, d \in K$ such that $x = a - b$ and $y = c - d$ or equivalently, $x + b = a$ and $y + d = c$. Since $b, d \geq \overline{0}$, it follows that $x \leq a$ and $y \leq c$. Note that $a + c \in X$ and that $x \leq a + c$ and $y \leq a + c$. This implies that X is directed. Conversely, let X be directed and take $x \in X$. Then there is $u \in X$ such that $u \geq x$ and $u \geq \overline{0}$ (since $\overline{0} \in X$). Now let $v = u - x$. Then $v \geq \overline{0}$. Thus $u, v \in K$ and $x = u - v$. Hence K is generating. □

Definition 4.2. Let K be a cone of a vector space X. A vector $e \in K$ is an order unit if for each $x \in X$ there exists some $\lambda > 0$ such that $x \leq \lambda e$.

Theorem 4.3. Let X be an ordered vector space. If X has order unit, then the positive cone K is generating.

Proof. Suppose X has order unit. For $x \in X$, there is $\lambda > 0$ such that $x \leq \lambda e$ or equivalently, $x - \lambda e \leq \overline{0}$. This implies that $v = x - \lambda e \in -K$. Also $u = \lambda e \in K$ since $e \in K$. Thus $x = u + v$ with $u \in K$ and $v \in -K$. Therefore, K is generating. □

Definition 4.4. A subset S of a topological vector space X is said to be absorbing if for every $x \in X$ there exists $r > 0$ such that $x \in \alpha S$ for all $|\alpha| \geq r$.

Theorem 4.5. (cf [18, Proposition 1.3.13]) An open unit ball $B(0, 1)$ in a normed space $(X, \|\bullet\|)$ is absorbing.

Proof. Take $x \in X$ and $\alpha > 0$ such that $\|x\| < \alpha$. Now,

$$\alpha^{-1}\|x\| < \alpha^{-1}\alpha = 1$$

and by defintion,

$$\alpha^{-1}\|x\| = \|\alpha^{-1}x\| < 1.$$

Thus, $\alpha^{-1}x \in B(0,1)$. And therefore $x \in \alpha B(0,1)$. Thus, the set $B(0,1)$ is absorbing. □

Definition 4.6. A subset S of a topological vector space X is said to be balanced if $\alpha S \subseteq S$ whenever $|\alpha| \leq 1$.

Theorem 4.7. A closed unit ball $B[0,1]$ in a normed space $(X, \| \bullet \|)$ is balanced.

Proof. Let $x \in B[0,1]$, then $\alpha x \in \alpha B[0,1]$. Need to show that $\alpha x \in B[0,1]$ if $|\alpha| < 1$. This is true since $\alpha x \in X$(vector space) and

$$\begin{aligned} \|\alpha x\| &= |\alpha|\|x\| \\ &\leq |\alpha| && \text{since } \|x\| \leq 1 \\ &< 1. \end{aligned}$$

□

Definition 4.8. Let K be a cone in an ordered normed space X. The point $x_0 \in K$ is an interior point if $x_0 + \alpha B[0,1] \subseteq K$.

Remark 4.9. It follows from the above definition that the point $x_0 \in K$ is an interior point if $x_0 + \alpha B^+[0,1] \subseteq K$ and $x_0 - \alpha B^+[0,1] \subseteq K$, where $B^+[0,1] = B[0,1] \cap K$. The first inclusion is always true for $x_0 \in K$ since $\alpha B^+[0,1] \in K$.

Second inclusion is not always true as the following theorem shows.

Theorem 4.10. let K be a cone in an ordered Banach space $X, y \in X$ and $-x \notin K$, then there is $\alpha > 0$ such that $y - \alpha x \notin K$

Proof. Suppose $y - \alpha x \in K$ for all $\alpha > 0$. Since K is a cone, we have $\frac{1}{\alpha}y - x \in K$ for all $\alpha > 0$. Now, if $\alpha \to \infty$, then $-x \in K$ (Contradiction). □

Definition 4.11. (I. Polyrakis) Let K be a cone in an ordered normed space and $\alpha > 0$. A point x_0 is called semi - interior point of K if

$$x_0 - \alpha B^+[0,1] \subseteq K$$

for some $\alpha > 0$.

Lemma 4.12. (cf [3, Example 2.5]) In a Riesz space $(\mathbb{R}^2, \leq)$, the function $\| \bullet \|_n$, defined by

$$\|(x,y)\|_n = \begin{cases} |x| + |y|, & \text{if } xy \geq 0, \\ \max\{|x|,|y|\} - \frac{n-1}{n}\min\{|x|,|y|\}, & \text{if } xy < 0. \end{cases}$$

and its unit ball is the polygon of $\mathbb{R}^2$ with vertices

$$(1,0), (0,1), (-n,n), (-1,0), (0,-1), (n,-n).$$

is a norm on $\mathbb{R}^2$

Proof. (1) Let $x, y \in \mathbb{R}^2$. If $xy \geq 0$. Then $\| \bullet \|_n$ is norm (1 - norm on $\mathbb{R}^2$).

(2) If $xy < 0$ and $\alpha \in \mathbb{R}$.

(i) for $(x,y) \in X_n$ we have

$$\begin{aligned} \|(\alpha x, \alpha y)\|_n &= \max\{|\alpha x|, |\alpha y|\} - \frac{n-1}{n}\min\{|\alpha x|, |\alpha y|\} \\ &= |\alpha| \max\{|x|,|y|\} - |\alpha|\frac{n-1}{n}\min\{|x|,|y|\} \\ &= |\alpha| \|(x,y)\|_n. \end{aligned}$$

(ii) Let $(x_1,y_1),(x_2,y_2) \in X_n$. Now,

$$\|(x_1,y_1) + (x_2,y_2)\|_n$$

$$\begin{aligned} &= \max\{|x_1+x_2|, |y_1+y_2|\} - \frac{n-1}{n}\min\{|x_1+x_2|, |y_1+y_2|\} \\ &\leq \max\{|x_1|+|x_2|, |y_1|+|y_2|\} - \frac{n-1}{n}\min\{|x_1|+|x_2|, |y_1|+|y_2|\} \\ &= \max\{(|x_1|,|y_1|) + (|x_2|,|y_2|)\} - \frac{n-1}{n}\min\{(|x_1|,|y_1|) + (|x_2|,|y_2|)\} \\ &= \max\{(|x_1|,|y_1|)\} - \frac{n-1}{n}\min\{(|x_1|,|y_1|)\} \\ &+ \max\{(|x_2|,|y_2|)\} - \frac{n-1}{n}\min\{(|x_2|,|y_2|)\} \\ &= \|(x_1,y_1)\|_n + \|(x_2,y_2)\|_n \end{aligned}$$

Note that if $xy < 0$, we have

$$\max\{|x|,|y|\} \geq \min\{|x|,|y|\}$$

$$\max\{|x|,|y|\} > \frac{n-1}{n}\min\{|x|,|y|\} \quad \text{since } 0 \leq \frac{n-1}{n} < 1$$

$$\max\{|x|,|y|\} - \frac{n-1}{n}\min\{|x|,|y|\} > 0$$

and therefore, $\|(x,y)\|_n \geq 0$. Thus $\|(x,y)\|_n = 0$ if and only if $|x|+|y| = 0$, that is, if and only if $x = y = \overline{0}$. Hence the result. $\square$

It is clear that an interior point of a cone K is a semi - interior point. However, the converse is not true in general.

Example 4.13. (cf [3, Example 2.5]) Consider $(\mathbb{R}^2, \leq)$ as in above lemma. Let E be a space of all pointwise bounded sequences in $\mathbb{R}^2$. Now consider the ordered triple $(E, K, \|\bullet\|_\infty)$, where $K = \{(x_n) \in E : x_n \in (\mathbb{R}^2)^+\}$ for any n and $\|x\|_\infty = \sup\limits_{n\in\mathbb{N}} \|x_n\|_n$. Let $X = K - K$ be subspace of E. The pair (X, X^+) is an ordered space. Suppose also that $X = P - P$ is the subspace of E generated by the cone P and suppose that X is ordered by the cone $X^+ = P$. We show that X^+ has a semi - interior point but no interior point.

Proof. Let $\mathbf{1}$ be the constant sequence $(1,1)$ of X for any n. Claim: $\mathbf{1}$ is not an interior point of X^+. To see, for any m we take $y = (y_n) \in X$ with $y_m = (-2, 2)$ and $y_n = (0,0)$ if $n \neq m$. Since $(-2)(2) < 0$, then

$$\|y\|_\infty = \sup\{\max\{|-2|, |2|\} - \tfrac{m-1}{m}\min\{|-2|, |2|\}\} = 2 - \tfrac{m-1}{m}(2) = \tfrac{2}{m}$$

and $\mathbf{1} + y = (-1, 3) \notin X^+$. Therefore $\mathbf{1} + \alpha B[0,1] \nsubseteq X^+$ for any $\alpha > 0$. In the same way $x \in X^+$ is not an interior point and then that X^+ has empty interior.

We next show that $\mathbf{1}$ is a semi - interior point of X^+.

First note that

$$\begin{aligned} B^+[0,1] &= B[0,1] \cap X^+ \\ &= \{x \in X : \|x\|_n \leq 1\} \cap \{x \in X : x \geq \overline{0}\} \\ &= \{x \in X^+ : \|x\|_n \leq 1\}. \end{aligned}$$

Now, for any $(x_n) \in B^+[0,1]$. Then $x_{n_1}, x_{n_2} \geq \overline{0}$. Therefore, by definition,

$$\begin{aligned} \|x_n\|_n &= |x_{n_1}| + |x_{n_2}| \\ &= x_{n_1} + x_{n_2} \\ &\leq 1. \end{aligned}$$

Thus, $\overline{0} \leq x_{n_1} + x_{n_2} \leq 1$. Hence $x_{n_1}, x_{n_2} \leq 1$. Therefore,

$$(0,0) \leq (1,1) - (x_{n_1}, x_{n_2}) \leq (1,1).$$

This implies that $\mathbf{1} - B^+[0,1] \subseteq X^+$. Hence $\mathbf{1}$ is semi - interior of X^+. $\square$

Lemma 4.14. (cf [2, Lemma 1.7]) Let K be a cone of normed vector space X. A vector $e \in K$ is an order unit if and only if it is an interior point of K.

Proof. Assume a vector e is an interior point of K. Then for each $-x \in X$, there is $\alpha > 0$ such $e + \alpha(-x) = e - \alpha x \in K$. That is, $e - \alpha x \geq \overline{0}$ or equivalently $e \geq \alpha x$. Thus $x \leq \frac{1}{\alpha}e$. Hence e is an order unit of K. Conversely, suppose e is an order unit of K and take $-x \in X$. Therefore, there is $\alpha_0 > 0$ such that $-x \leq \alpha_0 e$. Now, take $\alpha \in [0, \alpha_0]$. So, $-x \leq \alpha_0 e \leq \alpha e$. Therefore, $\overline{0} \leq \alpha e + x$ for all $\alpha \in [0, \alpha_0]$. This implies that e is an internal point of K. □

Theorem 4.15. ([3, Theorem 2.8]) Let $(X, \| \bullet \|)$ be an ordered Banach space and K be a closed and generating cone of X. Then any semi - interior point x_0 of K is an order unit of X.

Proof. Suppose x_0 is a semi - interior point of K and let $x_1 \in K$, then by definition 4.11 there exists $\alpha > 0$ such that $x_0 - \alpha\dfrac{x_1}{\|x_1\|} \geq \overline{0}$, that is,

$$kx_0 \geq \frac{x_1}{\|x_1\|}, \qquad \text{where } k = \frac{1}{\alpha} \tag{4}$$

since $\dfrac{x_1}{\|x_1\|} \in B^+[0,1]$. Now let $x \in X$. Since K is generating there exist $x_1, x_2 \in K$ such that $x = x_1 - x_2$. Then

$$\begin{aligned} x &= x_1 - x_2 \\ &\leq x_1 \\ &= \|x_1\| \frac{x_1}{\|x_1\|} \\ &\leq \|x_1\| k x_0 && \text{by (1)} \\ &\leq a k x_0 && a \geq \|x_1\|, a > 0 \text{ since } K \text{ is closed} \end{aligned}$$

and similarly

$$\begin{aligned} x &= x_1 - x_2 \\ &\geq -x_2 \\ &= \|x_2\| \frac{-x_2}{\|x_2\|} \\ &\geq -\|x_2\| k x_0 && \text{by (1)} \\ &\geq -a k x_0 && a \geq \|x_2\|, a > 0 \text{ since } K \text{ is closed.} \end{aligned}$$

Therefore $x \in [-akx_0, akx_0]$ and x_0 is an order unit of X. □

Definition 4.16. Let X^+ be a positive cone of normed space X. We say X^+ gives an open decomposition if there exists $\alpha > 0$ so that

$$\alpha B[0,1] \subseteq B^+[0,1] - B^+[0,1],$$

where $B^+[0,1] = B[0,1] \cap X^+$. That is, the convex set $B^+[0,1] - B^+[0,1]$ is a neighborhood of zero.

Next we give an examples of a cones that gives an open decomposition.

Example 4.17. Consider the lexicographic cone and unit ball $B[0,1]$ in $\mathbb{R}^2$, where $\|(x,y)\| = \sup\{|x|, |y|\}$. Then

$$\begin{aligned} B^+[0,1] &= B[0,1] \cap K \\ &= \{(x,y) : 0 < x \leq 1 \text{ and } -1 \leq y \text{ or } x = 0 \text{ and } 0 \leq y \leq 1\}. \end{aligned}$$

It can be seen that the points, $x_1, x_2 \in B^+[0,1]$ are such that $\|x_1 - x_2\| \leq 2$. For example, if we take two points at right - hand corners of $B^+[0,1]$, namely $(1,1), (1,-1) \in B^+[0,1]$, we get

$$(1,1) - (1,-1) = (0,2) \in B^+[0,1] - B^+[0,1]$$

and that $\|(0,2)\| \leq 2$. So if $0 < \alpha < 2$ then $\alpha B[0,1] \subseteq B^+[0,1] - B^+[0,1]$. Therefore K gives an open decomposition.

Theorem 4.18. [cf, (Krein - Smulian)] Let $(X, \|\bullet\|)$ be an ordered Banach space and X^+ a closed and generating positive cone, then X^+ gives an open decomposition.

Proof. Since a neighborhood $B[0,1]$ of zero is both absorbing and balanced then by Theorem 2.10 in [7], there exists some $\alpha > 0$ such that

$$\alpha B[0,1] \subseteq B^+[0,1] - B^+[0,1].$$

That is, X^+ gives an open decomposition which also means that

$$(B[0,1] \cap X^+) - (B[0,1] \cap X^+)$$

is a neighbourhood of zero. □

We consider the conditions for a semi - interior point to be an interior point.

Proposition 4.19. ([3, Proposition 2.4]) If X is a Banach space ordered by the closed and generating cone K, then any semi - interior point of K is an interior point of K.

Proof. Let x_0 be a semi - interior point of K. Then $x_0 - \alpha B^+[0,1] \subseteq K$, for some $\alpha > 0$. Again, $\alpha B^+[0,1] \subseteq K$. So by definition,

$$(x_0 - \alpha B^+[0,1]) + \alpha B^+[0,1] = x_0 + \alpha\,(B^+[0,1] - B^+[0,1]) \subseteq K.$$

By Theorem 4.18, there exists $a > 0$ such $aB[0,1] \subseteq B^+[0,1] - B^+[0,1]$, therefore we have

$$x_0 + a\alpha B[0,1] \subseteq x_0 + \alpha\,(B^+[0,1] - B^+[0,1]) \subseteq K$$

and x_0 is an interior point of K. □

We recall the following lemma.

Lemma 4.20. Let (X, τ) be an ordered Hausdorff topological vector space whose cone X^+ has a nonempty interior. Then X^+ is Archimedean if and only if it is closed.

We use Theorem 4.18 to give an example of a cone that gives an open decomposition.

Example 4.21. Let $X = C^1[0,1]$, with norm $\|f\|$, defined by

$$\|f\| = \|f'\|_\infty + \|f\|_\infty$$

for $f \in X$ and $K = X^+$, then K gives open decomposition.

Proof. We know that $(X, \|\bullet\|)$ is an ordered Banach space. We first show that $f(x) = \mathbf{1}$ for $x \in [0,1]$ is an order unit. Take $f \in X$, then f is continuous on the closed and bounded interval [0,1]. Therefore $f(x)$ attains a maximum on [0,1]. Let $a = \max\limits_{x\in[0,1]} f(x)$ and $a(x) = a$, a constant function, then $a(x) \in X$ since it is continuous and differentiable on [0,1]. Let $\mathbf{1}(x) = 1$ and $\lambda = \max\{a, 1\} > 0$, then $f \leq a(x) = a \leq \max\{a, 1\} = \lambda 1$. Further, as $\mathbf{1}(x) > \overline{0}$, we have $\mathbf{1}(x) \in X^+$. Since $f \in X$ is arbitrary, this show that 1 is an order unit of X. Thus X has order unit. Therefore $f(x) \leq \|f\|_\infty \mathbf{1} \leq \|f\|\mathbf{1}$ for all $f \in X$, where $\mathbf{1}(x)$ is constant function and order unit in X. Thus by theorem 4.3 X^+ is generating. Again by Proposition 1.35 the norm topology is Hausdorff. Therefore, by Lemma 4.20, X^+ is closed since by Proposition 1.56 and Example 1.58, X^+ is also Archimedean. Hence X has an open decomposition property. □

Definition 4.22. A cone K of a Banach space X is reflexive if the set

$$B^+[0,1] = B[0,1] \cap K$$

is weakly compact, that is, every sequence in $B^+[0,1]$ has a weakly convergent subsequence.

Now we give an example of a reflexive cone.

Example 4.23. Consider the base B_f for the positive cone ℓ_p^+ of the Banach space ℓ_p, with $1 < p < \infty$ defined by $f \in \ell_q^+$ where $\frac{1}{p} + \frac{1}{q} = 1$. Then by Corollary 2.17 B_f is closed and unbounded. Let $x_0 \in B_f, \rho > 0$ with $\rho > \|x_0\|_p$,

$$B_\rho = \{x \in B_f : \|x\|_p \leq \rho\}.$$

Then B_ρ has a non - empty interior and the set

$$K = \{\lambda x : \lambda > 0, x \in B_\rho\}$$

generated by its base B_ρ is a cone since B_ρ is convex. The

$$B^+[0,1] = B[0,1] \cap K$$

is weakly compact, that is, K is reflexive.

Proof. First note that K is closed (and thus weakly closed) by Proposition 2.19 since its base B_ρ is closed and bounded. Let (x_{n_k}) be a subsequence of (x_n) in $B^+[0,1]$ such that $x_{n_k} \stackrel{w}{\to} x$ in ℓ_p. Then

$$x_{n_k} \in B[0,1] \cap K$$

and since K is weakly closed and $B[0,1]$ is weakly compact (ℓ_p is reflexive) then $x_{n_k} \stackrel{w}{\to} x$ in $B[0,1] \cap K$. That is, $x_{n_k} \stackrel{w}{\to} x$ in $B^+[0,1]$ and thus K is reflexive. □

We consider the following properties of reflexive cone.

Proposition 4.24. (cf [8, Remark 3.1.1] Let $(X, \| \bullet \|)$ be a Banach space and K a cone in X. If K is reflexive, then it is closed.

Proof. Suppose K is a reflexive cone in a Banach space $(X, \| \bullet \|)$. Take a sequence (x_n) in K such that $x_n \to x \in X$. Therefore, (x_n) is bounded. That is, there is $m > 0$ such that $\|x_n\| \leq m$ for all $n \in \mathbb{N}$, or equivalently $x_n \in mB^+[0,1]$. Now, K reflexive implies that $mB^+[0,1]$ is weakly closed and hence closed. So $x \in mB^+[0,1] \subseteq K$. Thus K is closed. □

Corollary 4.25. (cf [8, Remark 3.1.2]) Any closed cone K of a reflexive space $(X, \| \bullet \|)$ is reflexive.

Proof. Supose X is a reflexive space and K a closed cone of X. Then by Theorem 3.2 $B[0,1]$ is weakly compact. Therefore, every sequence (x_n) in $B[0,1]$ has weakly convergent subsequence in $B[0,1]$. Now, take a set $B^+[0,1] = B[0,1] \cap K$ and let (y_n) be a sequence in $B^+[0,1]$. Then

$$(y_n) \in B[0,1] \cap K.$$

Therefore, there is a subsequence (y_{n_k}) of (y_n) such that $y_{n_k} \xrightarrow{w} y \in B[0,1]$ since $B[0,1]$ is weakly compact. But since K is closed and convex it follows, that K is weakly closed, then $y \in K$. That is $y \in B[0,1] \cap K = B^+[0,1]$. Thus $B^+[0,1]$ is weakly compact. So K is reflexive. □

We obtain the converse of the above results for generating cones.

Corollary 4.26. If a cone K of a Banach space X is reflexive and generating then X is reflexive.

Proof. Let K be a reflexive generating cone of a Banach space X. By Theorem 4.18 a cone K gives an open decomposition since K is closed (by being reflexive) and generating, that is, $\alpha B[0,1] \subseteq B^+[0,1] - B^+[0,1]$, where

$$B^+[0,1] = B[0,1] \cap K.$$

Note that $\alpha B[0,1]$ is weakly closed since it is closed and convex. Now let, $y_n \in \alpha B[0,1]$ then $y_n \in B^+[0,1] - B^+[0,1]$ that is,

$$y_n = x_n - z_n \in B^+[0,1] - B^+[0,1]$$

where $x_n, z_n \in B^+[0,1]$. Since $B^+[0,1]$ is weakly compact (K is reflexive) then there exist subsequences (x_{n_k}) and (z_{n_k}) of (x_n) and (z_n) respectively such that $x_{n_k} \xrightarrow{w} x$ and $z_{n_k} \xrightarrow{w} z$. Therefore for subsequence (y_{n_k}) of y_n, we have

$$y_{n_k} = x_{n_k} - z_{n_k} \xrightarrow{w} x - z$$

in $B^+[0,1] - B^+[0,1]$. And $x - z \in \alpha B[0,1]$ since $\alpha B[0,1]$ is weakly closed. Therefore $B[0,1]$ is weakly compact and therefore X is reflexive. □

The next theorem shows that reflexive cone K of a Banach space X coincide with their second dual cone in X^{**}, that is, $K = K^{**}$.

Theorem 4.27. ([8, Theorem 3.3]) Let X be a Banach space, we denote by $J_X : X \to X^{**}$ the natural embedding of X in X^{**}. A closed cone K of X is reflexive if and only if $J_X(K) = K^{**}$, where $K^{**} \subset X^{**}$.

Next, we mention without proof the important Theorem that characterizes a reflexive Banach space by means of a reflexive cone.

Theorem 4.28. ([8, Theorem 3.5]) A Banach space X is reflexive if and only if there exists a closed cone K of X so that the cones K and K^* are reflexive.

This Theorem implies that in every non reflexive Banach space a reflexive cone cannot have a dual cone which is reflexive.

Definition 4.29. The Rademacher functions $\{r_n\}_{n=0}^{\infty}$ on $[0,1]$ are defined by $r_n(x) = \operatorname{sign}(\sin 2^n \pi x)$, where

$$\operatorname{sign}(\sin 2^n \pi x) = \begin{cases} 1, & x \in \bigcup_{k=0}^{2^{n-1}-1} \left(\frac{2k}{2^n}, \frac{2k+1}{2^n} \right), \\ 0, & x = \frac{k}{2^n}, k = 0, ..., 2^n, \\ -1, & x \in \bigcup_{k=0}^{2^{n-1}-1} \left(\frac{2k+1}{2^n}, \frac{2k+2}{2^n} \right). \end{cases}$$

Note that $|r_n| = 1$ almost everywhere on $[0,1]$ and $r_0 = 1$ for $x \in [0,1]$.

Theorem 4.30. (Khintchine's Inequalties) There exist constants

$$A_p, B_p \text{ where } 1 \leq p < \infty$$

such that for any finite sequence of scalars $(a_i)_{i=1}^n$ and any $n \in \mathbb{N}$

$$A_p \left(\sum_{i=1}^n |a_i|^2 \right)^{\frac{1}{2}} \leq \left\| \sum_{i=1}^n a_i r_i \right\|_p \leq B_p \left(\sum_{i=1}^n |a_i|^2 \right)^{\frac{1}{2}}$$

$$\text{if } 1 \leq p < 2$$

and

$$\left(\sum_{i=1}^n |a_i|^2 \right)^{\frac{1}{2}} \leq \left\| \sum_{i=1}^n a_i r_i \right\|_p \leq B_p \left(\sum_{i=1}^n |a_i|^2 \right)^{\frac{1}{2}}$$

$$\text{if } p > 2.$$

What it says.

(i) – Khintchine's Inequalties tells us that $(r_i)_{i=1}^{\infty}$ is a basic sequence equivalent to the standard basis of ℓ_2 in every $L_p, 1 \leq p < \infty$.
  – In $L_\infty, (r_i)_{i=1}^{\infty}$ is isometrically equivalent to canonical ℓ_1 - basis.

(ii) $(r_i)_{i=1}^{\infty}$ is an orthonormal sequence in ℓ_2, such that

$$\left\| \sum_{i=1}^{n} a_i r_i \right\|_p = \left(\sum_{i=1}^{n} |a_i|^2 \right)^{\frac{1}{2}}$$

for any scalars (a_i). But $(r_i)_{i=1}^{\infty}$ is not complete.

(iii) Khintchine's Inequalties, all norms $\| \bullet \|_p, 1 \leq p < \infty$ are equivalent on the linear span of the Rademacher function on L_p.

Now we can provide an example of a reflexive cone such that its dual is not reflexive.

Example 4.31. Let $X = L_1([0,1]), Y$ is the closed subspace of X generated by the Rademacher functions (r_i), and let K be the positive cone of (r_i). Recall that (r_i) is a basic sequence in $L_1([0,1])$, equivalent to the standard basis of ℓ_2 by Theorem 4.30, therefore Y is isomorphic to ℓ_2 by Theorem 4.30 and the cone K is reflexive. By Theorem 4.28, the dual cone K^* of K in $L_\infty([0,1])$ is not reflexive.

4.2 Bases of reflexive cones.

Theorem 4.32. (cf [8, Theorem 4.3]) Any reflexive cone of the Banach space is not a mixed based cone.

Proof. Suppose that a cone K of a Banach space X is reflexive, then by definition $B^+[0,1]$ is weakly compact. By Theorem 2.43 we have either the base B_f is bounded for every $f \in K^{*s}$ or B_f is unbounded for every $f \in K^{*s}$. This means K cannot have a mixed based cone. □

The converse of the above theorem is not true in general as the following example shows.

Example 4.33. To see, we consider the cone c_0^+ in a Banach space c_0. This closed cone is not a mixed base cone since it is not conically isomorphic to ℓ_1^+, but c_0^+ is not reflexive since it contains a closed subcone isomorphic to ℓ_1^+.

The next result provides condition for a reflexive cone.

Proposition 4.34. ([8, Proposition 4.4]) Let X be a Banach space ordered by the closed cone K. If the set $K^{*s} \neq \emptyset$ and for any $f \in K^{*s}$ the base B_f for K defined by f is bounded, then the cone K is reflexive.

The converse is not true in general.

Example 4.35. The cone ℓ_2^+ is reflexive by Corollary 4.25 since ℓ_2^+ is a closed cone of a reflexive space ℓ_2 and it has a base B_f defined by unbounded bases for all $f \in K^{*s}$ by Corollary 2.17.

Definition 4.36. ([27, II Definition 10.1]) A basis (x_n) of a Banach space X is called to be of type ℓ^+ if (x_n) is bounded and there is a constant $\eta > 0$ such that for all finite sequences $\alpha_1, \alpha_2, ..., \alpha_n \geq 0$

$$\left\| \sum_{i=1}^{n} \alpha_i x_i \right\| \geq \eta \sum_{i=1}^{n} \alpha_i$$

Example 4.37. Let (e_n) be a sequence such that $e_n = (0, 0, ..., \overset{n^{th}}{1}, 0, 0, ...)$. A basis $(e_n) \in \ell_1$ is of type ℓ^+. To see, first note that (e_n) is bounded since $\|e_n\|_1 = \sum_{n=1}^{\infty} |e_n| = 1 < \infty$. Now take any finite sequence $(\alpha_i)_{i=1}^n$ such that $0 \leq \alpha_i < \infty$ for each i. Then $\sum_{i=1}^{n} \alpha_i < \infty$ and

$$\left\| \sum_{i=1}^{n} \alpha_i e_i \right\| = \alpha_i < \infty.$$

Because they are both finite real numbers, there exists $\eta > 0$ such

$$\left\| \sum_{i=1}^{n} \alpha_i e_i \right\| \geq \eta \sum_{i=1}^{n} \alpha_i.$$

We recall the following four theorems.

Theorem 4.38. ([27, II Theorem 10.2]) Let (x_n) be a bounded basis of a real Banach space X with the associated sequence of coefficient functionals and let $K_{(x_n)}$ be the cone associated to the basis (x_n). The following are equivalent

(i) (x_n) is of type ℓ^+.

(ii) $K_{(x_n)} = \left\{ \sum_{i=1}^{\infty} \alpha_i x_i : \alpha_n \geq 0, n = 1, 2, ..., \sum_{i=1}^{\infty} \alpha_i < \infty \right\}$.

(iii) $K_{(x_n)}$ has a bounded base.

Theorem 4.39. Let (x_n) be a sequence in a Banach space $(X, \| \bullet \|)$ such that x_n is not norm convergent $\overline{0}$.

(i) If $x_n \overset{w}{\to} \overline{0}$ or

(ii) If (x_n) is weakly Cauchy and not weakly convergent,

then (x_n) has a basic subsequence.

Theorem 4.40. ([Rosenthal, 7]) A Banach space X contains a subspace isomorphic to ℓ_1 if and only if it has a bounded sequence with no weakly Cauchy subsequence.

Theorem 4.41. ([27, II Theorem 10.1]) Let (x_n) be a basis of a Banach space X, with

$$\sup_{1 \leq n < \infty} \|x_n\| < \infty.$$

The following statement are equivalent:

(i) (x_n) is of type ℓ^+.

(ii) There exists an $f \in X^*$ such that $f(x_n) \geq 1$ for $n \in \mathbb{N}$.

We now provide conditions for a cone to be reflexive.

Theorem 4.42. ([8, Theorem 4.5]) A closed cone K of a Banach space X is reflexive if and only if K does not contain a closed cone isomorphic to ℓ_1^+.

Proof. Let X be a Banach space, K a reflexive cone of X and K_1 a closed subcone of K. Then K_1 is reflexive by Corollary 4.25. Suppose that K_1 is isomorphic to ℓ_1^+. Then K_1 by Theorem 3.25, is a mixed based cone with empty interior. But this contradicts Theorem 4.32. Therefore, no closed subcone of K is isomorphic to ℓ_1^+.
Conversely, suppose that K does not contain a closed cone isomorphic to ℓ_1^+ and K is not reflexive cone. Then by definition, the set $B^+[0,1] = B[0,1] \cap K$ is not weakly compact set. Hence, there is a sequence $\{x_n\}$ in $B^+[0,1]$ with no weakly convergent subsequence. So by Theorem 4.40, there exists a weakly Cauchy subsequence (x_{n_α}) of (x_n), since K does not contain a closed cone isomorphic to ℓ_1^+. Again by Theorem 4.39, $\{x_n\}$ has a basic subsequence $\{x_{n_B}\}$. The sequence $\{x_n\}$ does not have a weakly convergent subsequence, therefore $\{x_n\}$ is not weakly convergent to $\overline{0}$. Hence, there is $f \in X^*$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $f(x_{n_k}) \geq 1$ for each $k \in \mathbb{N}$. Therefore by Theorem 4.41, $\{x_{n_k}\}$ is a basic sequence of ℓ^+- type and the cone

$$P = \left\{ p \in X : p = \sum_{k=1}^{\infty} \alpha_k x_{n_k} : \alpha_k \geq 0 \text{ for every } k \in \mathbb{N} \right\} \subseteq K$$

generated by $\{x_{n_k}\}$ is isomorphic to ℓ_1^+ by Theorem 4.39. But this contradicts our assumption. $\square$

Corollary 4.43. ([8, Corollary 4.6]) If the closed cones $K \subseteq X, Q \subseteq Y$ of the Banach spaces X, Y are isomorphic we have: K is reflexive if and only if Q is reflexive.

Proof. Let T be an isomorphism of K onto Q. That is, T is an additive, positively homogeneous, one - to - one map of K onto Q such that T and T^{-1} are continuous in the induced topology. Suppose K is reflexive and that Q is nonreflexive. Then by Theorem 4.42, there is a closed subcone Q_1 of Q which is isomorphic to ℓ_1^+. Therefore $T^{-1}(Q_1)$ is a closed cone of K, since cone Q_1 is closed in Q and T is isomorphism of K onto Q, isomorphic to ℓ_1^+, because composite of two isomorphisms is an isomorphism. This leads to contradiction. $\square$

Theorem 4.44. ([8, Theorem 4.7]) Suppose that K is a reflexive cone of a Banach space X. If K has a bounded base defined by $f \in X^*$, then K does not contain a basic sequence.

Proof. Let $\{x_n\} \subseteq K$ be a basic sequence and let $y_n = \dfrac{x_n}{f(x_n)}$ for each n. Then (y_n) is a basic sequence as a scalar multiple of a basic sequence and

$$\begin{aligned} f(y_n) &= \frac{f(x_n)}{f(x_n)} && \text{since } f \text{ is linear and } f(x_n) \in \mathbb{R} \\ &= 1 && \text{for each } n. \end{aligned}$$

Now, (y_n) is a basic sequence of ℓ^+ - type since (y_n) is bounded. To this end,

$$\begin{aligned} \|y_n\| &= \left\| \frac{x_n}{f(x_n)} \right\| \\ &= \frac{1}{|f(x_n)|} \|x_n\| \\ &\leq \frac{M}{|f(x)|}, && M > 0, \text{ since } K \text{ is bounded} \end{aligned}$$

and by Theorem 4.41. Hence the cone

$$P = \left\{ p \in X : p = \sum_{k=1}^{\infty} \alpha_k y_n : \alpha_k \geq 0 \text{ for every } k \in \mathbb{N} \right\} \subseteq K$$

generated by $\{y_n\}$ is isomorphic to ℓ_1^+ which is a contradiction. $\square$

Theorem 4.45. ([Bessaga - Pelczynski Selection Principle]) Let (x_n) be a weakly null, normalized sequence in the Banach space X. Then (x_n) admits of a basic sequence.

Theorem 4.46. (cf [8, Theorem 4.8]) Suppose that K is a reflexive cone of a Banach space X. If X has an unbounded base defined by $f \in X^*$, then K contains a normalized basic sequence $\{x_n\}$ which converges weakly to zero.

Proof. Suppose K is reflexive and has unbounded base B_f. Then there is a sequence (y_n) in B_f such that $y_n \geq \overline{0}$ and $\|y_n\| \to \infty$ for each $n \in \mathbb{N}$ or equivalently $\|y_n\| > n$ for each $n \in \mathbb{N}$. Now, consider the sequence $x_n = \frac{y_n}{\|y_n\|}$. Then $\|x_n\| = 1$ and hence (x_n) is in $B^+[0,1]$. The set $B^+[0,1]$ is weakly compact, since K is reflexive, therefore (x_n) has a weakly convergent subsequence (x_n). Say $x_n \overset{w}{\to} x$. That is, $f(x_n) \to f(x), f \in X^*$. Since $B^+[0,1]$ is weakly closed, $x \in B^+[0,1] \subseteq K$. Now,

$$
\begin{aligned}
f(x) = \lim_{n\to\infty} f(x_n) &= \lim_{n\to\infty} f\left(\frac{y_n}{\|y_n\|}\right) \\
&= \lim_{n\to\infty} \frac{1}{\|y_n\|} \qquad \text{since } y_n \in B_f, \text{ that is, } f(y_n) = 1 \\
&= 0, \qquad \text{since } \lim_{n\to\infty} \|y_n\| = \infty.
\end{aligned}
$$

hence $x = 0$, since f is strictly positive on K. By Theorem 4.45 $\{x_n\}$ has a basic subsequence. □

Definition 4.47. ([18, Definition 4.2.10]) A basis (x_n) for a Banach space X is unconditional if, for every $x \in X, x = \sum_n x_n$ is unconditionally convergent, that is, if the series $\sum_n \alpha_n x_n$ converges for every choice of $(\alpha_n) \in \ell_\infty$.

Theorem 4.48. ([27, II Theorem 16.3]) Let (x_n) a sequence in a Banach space X and (f_n) be a corresponding sequence of linear functional in X^*. The following statements are equivalent:

(i) (x_n) is an unconditional basis of X.

(ii) A cone K in X is normal and generating.

Theorem 4.49. (cf [8, Theorem 4.9]) Let $(X, \|\bullet\|)$ be a Banach space with an unconditional basis. If K is reflexive with an unbounded base defined by a vector of X^*, then $\mathrm{cl}(K - K)$ contains an infinite dimensional reflexive subspace.

Proof. Suppose that $K = \{\lambda x : x \in B_f, f \in X^*\}$ is a reflexive cone in a Banach space $(X, \|\bullet\|)$ with unbounded base $B_f, f \in X^*$. By Theorem 4.46, K has a normalized basic weakly null sequence (x_n). That is,

$$K = \left\{ y \in X : y = \sum_{n=1}^{\infty} \alpha_n x_n, \alpha_n \geq 0, \|x_n\| = 1, n \in \mathbb{N} \right\}.$$

Hence by Theorem 4.45 (x_n) has an unconditional basic subsequence (x_{n_k}). That is, $\sum_{k=1}^{\infty} \alpha_k x_{n_k}$ converges for all choices of $\alpha_k \in \mathbb{R}$. Now let

$$K_{(x_{n_k})} = \left\{ x \in X : x = \sum_{k=1}^{\infty} \alpha_k x_{n_k}, \alpha_k \geq 0, k \in \mathbb{R} \right\}.$$

be a cone generated by (x_{n_k}). Then $K_{(x_{n_k})}$ is a closed subcone of K since $\{x_{n_k}\} \subseteq \{x_n\}$ and $\sum_{k=1}^{\infty} \alpha_k x_{n_k} < \infty$. Hence $K_{(x_{n_k})}$ is a reflexive subcone of K by Corollary 4.25 since K is reflexive. $K_{(x_{n_k})}$ is also generating by Theorem 4.48 because (x_{n_k}) is unconditional basis. Now let Y be a subspace of X generated by $K_{(x_{n_k})}$, that is, $Y = K_{(x_{n_k})} - K_{(x_{n_k})} \subseteq \mathrm{cl}(K - K)$, since $K_{(n_k)}$ is closed and $K_{(n_k)} \subseteq K$. Then Y is closed since $K_{(x_{n_k})}$ is closed. Then note that

$$\begin{aligned} Y &= K_{(x_{n_k})} - K_{(x_{n_k})} \\ &= \left\{ x \in X : x = \sum_{k=1}^{\infty} \alpha_k x_{n_k} - \sum_{k=1}^{\infty} \lambda_k x_{n_k}, \alpha_k, \lambda_k \geq 0, k \in \mathbb{N} \right\} \\ &= \left\{ x \in X : x = \sum_{k=1}^{\infty} (\alpha_k - \lambda_k) x_{n_k}, \alpha_k - \lambda_k \in \mathbb{R} \right\} \\ &= \mathrm{cl}(\mathrm{span}\{x_{n_k}\}) \qquad \text{since } Y \text{ is closed.} \end{aligned}$$

Again, $K_{(x_{n_k})}$ gives an open decomposition in Y by Theorem 4.7 since $K_{(x_{n_k})}$ is closed and generating. Lastly, Y is an infinite dimensional reflexive by Corollary 4.26 since Y contains a reflexive and generating cone $K_{(x_{n_k})}$ that is generated by an infinite subsequence (x_{n_k}). □

Proposition 4.50. A vector space X ordered by a reflexive cone K is Archmedean.

Proof. Suppose K is a reflexive cone and that $nx \leq y$ for $n \in \mathbb{N}, x \in X$ and $y \in K$. We claim that $x \leq \overline{0}$. Now, $nx \leq y$ if and only if $y - nx \in K$. And $\frac{1}{n}(y - nx) = \frac{y}{n} - x \in K$. If $n \to \infty$, $\frac{1}{n}y - x \to -x$. Since K is reflexive, it is closed. Hence $-x \in K$. That is $-x \geq \overline{0}$ or equivalently $x \leq \overline{0}$. Thus, X is Archimedean. □

The following two important properties of spaces ordered by reflexive cones are worth mentioning.

Theorem 4.51. ([8, Theorem 7.1]) Any Banach space X, ordered by a reflexive and normal cone K, is Dedekind complete.

Corollary 4.52. ([8, Corollary 7.5]) Any reflexive and generating cone of an infinite dimensional Banach space X with a bounded base cannot be a lattice cone.

We recall results based on the Riesz decomposition property and corollary which is due to [2], and use it in proving some lattice property.

Theorem 4.53. ([2, Theorem 2.46 (Andô)]) For an ordered Banach space X with a closed, generating and normal cone, the following statements are equivalent:

(i) X has the Riesz space decomposition property.

(ii) X^{**} is a Riesz space.

(iii) X^{*} has the Riesz decomposition property.

Corollary 4.54. ([2, Corollary 2.43]) For an ordered norm space X whose closed cone X^{+} is generating and normal we have the following:

(i) then dual cone $(X^{+})^{*}$ is generating in X^{*}.

(ii) If X is also a reflexive Banach space, then dual cone $(X^{+})^{*}$ is normal.

We complete this section by one of the important lattice property of a Banach space ordered by a reflexive cone.

Theorem 4.55. (cf [8, Theorem 7.2]) A Banach space X ordered by a normal, generating and reflexive cone K has the Riesz decomposition property if and only if X is a lattice.

Proof. Assume that X is a lattice, then X has the Riesz decomposition property by Corollary 1.30.
Conversely, suppose that X has the Riesz decomposition property. By Theorem 4.53, X^{*} is a Riesz space. Since X is also reflexive as a Banach space

of a closed, generating and reflexive cone then by Corollary 4.54 dual cone $(X^+)^*$ is generating and normal. Then by Theorem 4.53, X^{**} is a Riesz space. Since X is reflexive, then $X = X^{**}$ is also a Riesz space. Therefore X has a lattice property. □

References

[1] M.R. Alfuraidan, M.A. Khamsi, *Ekeland Variational Principle on Weighted Graphs*, Proceedings of the American Mathematical Society, 147 (2019), 5313 - 5321.

[2] C.D. Aliprantis, R. Tourky, Cones and Duality, Graduate studies in Mathematics, AMS, Volume 84, 2007.

[3] A. Basile, M.G. Graziano, M. Papadaki, I.A. Polyrakis, *Cones with Semi - interior Points and Equilibrium*, Centre for Studies in Economics and Finance, Working Paper no. 443, 2016.

[4] C. Bessaga, A. Pelczynski, *On bases and unconditional convergence series in Banach spaces*, Studia Mathematica, T, 17(1958), 152 - 164.

[5] A. Brondsted, *Fixed Points and Partial Orders*, Proceedings of the American Mathematical Society, 60(1976), 365 - 366.

[6] A. Brondsted, *Common Fixed Points and Partial Orders*, Proceedings of the American Mathematical Society, 77(1979), 365 - 368.

[7] E. Casini, E. Miglierina, *Cones with bounded and unbounded bases and reflexivity*, Nonlinear Analysis 72(2010), 2356 - 2366.

[8] E. Casini, E. Miglierina, I.A. Polyrakis, F. Xanthos, *Reflexive cones*, Positivity 17(2013), 911 - 933.

[9] G.P. Curbera, *A note on function spaces generated by Rademacher series*, Proceedings of the Edinburgh Mathematical Society, 40(1997), 119 - 126.

[10] J. Diestel, Sequences and Series in Banach Spaces, Springer - Verlag New York, inc., Berlin, Heidelberg, 1984.

[11] M. Frittelli, E.R. Gianin, *Putting Order in Risk Measures*, Journal of Banking and Finance, 26(2002), 1473 - 1486.

[12] Z.Q. Han, *Technical Note: Remarks on the Angle Property and Solid cones*, Journal of Optimization Theory and Applications, 82(1994), 149 - 157.

[13] V. Katsikis, I.A. Polyrakis, *Positive Bases in Ordered Subspaces with the Riesz Decomposition Property*, Studia Mathemetica, 174 (2006), no. 3, 233 - 253.

[14] D.G. Konstantinides, C.E. Kountzakis, *Risk measures in ordered normed linear spaces with non - empty cone - interior*, Insurance: Mathematics and Economics, 48(2011), 111 - 122.

[15] C.E. Kountzakis, *Risk Measures on Non - reflexive Banach Spaces*, J. Math. Anal. and Appl, 373(2011), 548 - 652.

[16] B.V. Limaye, Functional Analysis, New Age International Limited, Publishers, New Delhi - Bangalore - London, 1996.

[17] J. Lindenstrauss, L Tzafriri, Classical Banach Spaces I and II, Springer, Berlin, Heidelberg, 1977.

[18] R.E. Megginson, An Introduction to Banach Space Theory, Springer - Verlag New York, inc., Berlin, Heidelberg, 1998.

[19] S. Ovchinnikov, Functional Analysis, An Introductory Course, Springer, Switzerland, 2018.

[20] I. Polyrakis, *Bases for cones and Reflexivity*, Quaestiones Mathematicae, 24(2001), 165 - 173.

[21] I.A. Polyrakis, *Demand Functions and Reflexivity*, J. Math. Anal. App, 338(2008), 695 - 704.

[22] J.H. Qiu, *A Cone Characterization of Reflexive Banach Spaces*, J. Math. Anal. App, 256(2001), 39 - 44.

[23] J.H. Qiu, *Technical note: On Solidness of Polar Cones*, Journal of Optimization Theory and Applications, 109(2001), 199 - 214.

[24] H. Rosenthal, *A characterization of Banach spaces containing* c_0, J. Amer. Math. Soc., 7(1994), 707 - 748.

[25] H. Rosenthal, *A characterization of Banach spaces containing* ℓ_1, Proc. Natl. Acad. Sci. U.S.A., 71(1974), 2411 - 2413.

[26] H.H. Schaefer, Banach Lattices and Positive Operators, Springer - Verlag, Berlin - Heidelberg - New York, 1974.

[27] I Singer,Bases in Banach Spaces I, Springer - Verlag, New York, Heidelberg, Berlin, 1970.

[28] F. Sullivan, *A Characterization of Complete Metric Spaces*, Proceedings of the American Mathematical Society, 83(1981), 345 - 346.

[29] C. Swartz, *The Nikodym Boundedness Theorem for lattice - valued measures*, Arch. Math., 53(1989), 390 - 393.

[30] A.C. Zaanen, Introduction Operator Theory in Riesz Spaces, Springer, Berlin - Heidelberg - New York,1997.