Construction and analysis of some nonstandard finite difference methods for the FitzHugh-Nagumo equation

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Abstract

In this work, we construct four versions of nonstandard finite difference schemes in order to solve the FitzHugh-Nagumo equation with specified initial and boundary conditions under three different regimes giving rise to three cases. The properties of the methods such as positivity and boundedness are studied. The numerical experiment chosen is quite challenging due to shock-like profiles. The performance of the four methods is compared by computing L_1, L_∞ errors, rate of convergence with respect to time and CPU time at given time, $T = 0.5$. Error estimates have also been studied for the most efficient scheme.

Keywords: FitzHugh-Nagumo, nonstandard finite difference scheme, positivity, boundedness, different regimes, error estimates.

1 Introduction

The study of nonlinear partial differential equations of phenomena (dispersion, dissipation, diffusion, convection) arising in inhomogeneous system is of huge concern from mathematical, physical and biological point of view. The theoretical design based on nonlinear partial differential equations with varying or non-varying coefficient can precisely portray for instance the wave dynamics of pulses circulating in inhomogeneous systems [1]. Remarkably, nonlinear partial differential equations with variable coefficients describe well in diverse physical or material circumstances compared to their constant coefficients counterparts. The key standing is to find closed form solutions for nonlinear partial differential equations of physical or practical suitability. This could be a difficult task and sometimes impossible due to the fact that for many practical problems, the resulting nonlinear partial differential equations of interest are non-integrable. The integrability part of nonlinear partial differential equations is of considerable concern due to its link with the understanding of the physical and dynamical phenomena in nonlinear systems [2]. For instance the Kuramoto-Sivashinsky equation, the Ginzburg-Landau equation, the Korteweg-de Vries-Burgers equation, the Fisher's equation, the Burgers-Huxley equation and the Fitzhugh-Nagumo equation are practically well-known equations of this sort [3]. An exception takes place when a non-integrable nonlinear partial differential equations becomes integrable for some given values of the parameters involved into the equation. In this instance, the exact solutions can be written explicitly. Therefore, looking for some exact meaningful solutions is a hot topic because of the wide applications of nonlinear partial differential equations.

Various efficient approaches for getting exact solutions of nonlinear partial differential equation have been introduced. We can mention the subsidiary ordinary differential equation method [4, 5, 6], solitary wave ansatz method [7, 8], sine-cosine method [9], Hirota bilinear method [10], F-expansion method [11] and the Jacobi elliptic functions method [12] amongst others.

The FitzHugh-Nagumo equation is one important nonlinear partial differential equation in biology and is given by

$$u_t - u_{xx} = \beta u(1-u)(u-\gamma). \tag{1}$$

We note that $\gamma \in (0,1)$ monitors the overall dynamics of the equation [13] and is regarded as threshold of Allee effect [14]. $u(x,t)$ is the unknown function depending on the temporal variable t, and the spatial variable, $x \in \Omega$ (bounded domain), β stands for an intrinsic growth rate [15]. The FitzHugh-Nagumo equation has diverse applications in the field of flame propagation, logistic population growth, neurophysiology, autocatalytic chemical reaction and nuclear theory [16, 17]. It is worth noting that the Fitzhugh-Nagumo equation for the case $\beta = 1$ is called the classical or the standard Fitzhugh-Nagumo equation.

2 Organisation of the paper

The paper is organised as follows. In section 3, we describe the physical behaviour of FitzHugh-Nagumo equation and provide some literature on Allee effect and intrinsic growth rate [15]. In section 4, we list some methods used previously for solving FitzHugh-Nagumo equation. In section 5, we describe the numerical experiment chosen. In section 6, we give some information on basic dynamical behaviour of FitzHugh-Nagumo equation. In sections 7 to 10, we present four versions of nonstandard finite difference schemes, study some of their properties and present some numericals results. In section 11, we obtain error estimate for NSFD3 scheme. Section 12 shows the relationship between physical behaviour and numerical solutions. Section 13 highlights the salient features of the paper. Tables and figures are presented in sections 14 and 15 respectively. A short version of this work has been published in [18].

3 Physical behaviour and description of parameters in FitzHugh-Nagumo equation

3.1 Physical behaviour

FitzHugh-Nagumo equation was first established by Hodgkin and Huxley in the early 1950's by performing an experiment on a squid giant axon (voltage clamp) and they changed their studies of transmembrane potential, currents and conductance into a circuit-like model. The conclusion obtained from the experiment was a system of four ordinary differential equations that precisely portrayed remarkable propagation alongside an axon. Although Hodgkin-Huxley's experiment has shown to be a good model to describe a signal propagation along a nerve, it is difficult to be analysed. FitzHugh [19] and Nagumo [20] tackled this problem a decade later when they restrained the original system of four variables down to a simpler model of only two variables. Their simple model is simpler to be analysed and furthermore still depicts the essential phenomena of the dynamics (physical behavior) which are:

(1) a sufficiently large stimulus will set off a considerable response, and

(2) after such a stimulus and response, the medium needs a period of recovery before it can be impulsed again.

These two properties indicated above are described as excitable ($\gamma < 0$: the nerve is in excitable form) and refractory ($\gamma > 0$: the nerve is in refractory form and does not respond to external stimulation). Excitation occurs quickly while recovery happens slowly.

3.2 Biological description of the parameter γ and β

Allee effect was introduced by the pioneer W.C. Allee [21] by demonstrating that goldfish grew faster in water which had previously contained other goldfish than in water which had not. The threshold of Allee effect is denoted by γ in Eq. (1). Moreover, Allee [21] carried out an experiment with a range of species and concluded that larger groups may spur reproduction, extend survival in adverse conditions and also improve protection [21, 22]. Despite the fact that the Allee effect is properly well known, the notion has a range of significance not all of which are allowed by contemporary use. Furthermore, Allee did not give a definition but he plainly considered "certain aspects of survival [23]" rather than total fitness and the following definitions follow:

(1) Allee effect is a positive relationship between any component of individual fitness and either numbers or density of conspecifics [24].

(2) The Allee effect induces minimum viable population sizes or a threshold value in the critical spatial lengths of the initial distributions below which the population dies out. It also induces the critical value of the spatial length of an initial nucleus (problem of critical aggregation as in [25]).

The remark made from the original idea of Allee and its observations, is, the above definition demands that some measurable component of the fitness of an organism for instance probability of dying or reproducing is significant in a wide population (components of mean fitness meant to provide an overall increase or decrease with increasing abundance relying on the relative strength of negative density dependence). Moreover, it is worthy to differentiate between component Allee effect which is manifested by a component of fitness and demographic Allee effect which manifest at the level of total fitness [24]. It is shown in [26, 27] that a strong Allee effect assigns to the population that has a negative growth when the size of the population is below certain threshold value while a weak Allee effect means that growth is positive and increasing. Overall, the Allee effect induces a rich variety of spatio-temporal dynamics in the considered epidemological model.
Furthermore, some researchers used logistic growth (in the form of travelling infection waves) and growth with a strong Allee effect in the modelling of biological or ecological phenomena. Those waves, are waves of extinction, which occur when the proposed disease is introduced in the wake of the invading host population [28]. Moreover, the Allee effect leads to bistability in the local transmission dynamics [28]. In combination with the minimum viable population size, this has serious implications for eventual control methods, since they do not necessarily help in reducing the basic reproductive ratio anymore. If the disease's infectious is considerable in comparison with the demographic reproduction, the Allee effect becomes less important due to the fact that the population dynamics is dominantly driven by the disease to extinction. Recent results show that the Allee effect produces possible limit cycle oscillations with mass action transmission [28] in the vital dynamics of the model. Furthermore, the vital dynamics are generally governed by a strong Allee effect. This can be caused by difficulties in finding mating partners at small densities, genetic inbreeding, demographic stochasticity or a reduction in cooperative interactions [24, 29, 30]. It should be noted, moreover, that the study of Allee effect dynamics is justified in its own rights, because this is largely lacking in the epidemiological literature [28]. Though the model of FitzHugh-Nagumo is the one of the straightforward models, it displays complex dynamics that have not been fully studied.

The intrinsic growth rate denoted by β, in the presence of migration, is an accurate measure of how quickly a population would ultimately grow if for instance current age-specific rates of fertility, mortality and migration were sustained indefinitely in contrast to the actual growth rate of a population (equal weight to all migrants in regard to age). For example, migrants are level-headed in their expected future number of births at the age when they come or go (will not be feigned by migration that happens beyond the end of childbearing [15]).

4 Some methods for solving FitzHugh-Nagumo equation

There are various outcomes about local and global solutions of FitzHugh-Nagumo equations that develop existence, uniqueness, smoothness, stability, approximation or numerical solutions but none has been qualified to the exact solution formulae. Furthermore, it is worth underlining that, lately semi-analytic methods have been suggested to construct exact solutions of Fitzhugh-Nagumo equation [31]. The study also unveiled an heterozygote inferiority when $\gamma \in (0,1)$ [32] but if $\gamma = -1$, Eq. (1) is known as Newell-Whitehead equation representing dynamical behaviour close to the bifurcation point. Furthermore, it has also been shown that the exact solution of Eq. (1) demonstrates the fusion of two travelling fronts of the same sense and change to a front joining two stable constant states. Additionally, Jackson [33] harnessed Galerkin's approximations to solve Eq. (1). Bell et al. [34] searched closely the singular perturbation of N-front travelling waves while Gao et al. [35] looked into the existence of wavefronts and impulses. Krupa et al. [36] examined fast and slow waves of Eq. (1). Moreover, Shonbek [37] explored the higher-order derivatives of solutions of Eq. (1) whereas, Chou et al. [38] investigated exotic dynamic behaviour of Eq. (1).

Various methods for Eq. (1) has been investigated such as: Adomian Decomposition Method (ADM) [39], Homotopy Pertubation Method (HPM) [40, 41, 42, 43], Variational Iteration Method (VIM) [44, 45, 46], Differential Transform Method (DTM) [47].
Recently, for β depending on time, many authors like Triki and Wazwaz [1], considered a generalized Fitzhugh-Nagumo equation given by

$$u_t + \alpha(t)u_x - \Gamma(t)u_{xx} + \beta(t)u(1-u)(\gamma - u) = 0, \tag{2}$$

where $\alpha(t)$, $\Gamma(t)$, $\beta(t)$ are arbitrary functions of t, exhibits time-changing coefficients and linear dispersion term. They showed the existence and uniqueness of soliton solutions and used ansatz and tanh method with specific solitary wave. Jacobi-Gauss-Lobatto collocation method was used by Bhrawy [17] to solve the generalized Fitzhugh-Nagumo equation. Polynomial differential quadrature method (PDQM) for numerical solutions of the generalized Fitzhugh-Nagumo equation with time-dependent coefficients was also utilized by Ram et al [49].
Our focus in this work is for the case $\alpha = 0$, $\Gamma(t) = 1$ and β independent of t.
Mickens [48] introduced the Nonstandard Finite Difference (NSFD) method to obtain solutions of various partial differential equations. The derivations are mostly based on the idea of dynamical consistency which are positivity, boundedness, monotonicity of the solutions, stability of all fixed points [50]. After generalizing these results, Mickens stated the following three key rules in constructing NSFD schemes:

(1) The order of discrete derivatives should be equal to the order of corresponding derivatives arising in the differential equation.

(2) Discrete approximation for derivatives, in general have non trivial denominator functions, for example

$$u_t \approx \frac{u_j^{n+1} - u_j^n}{\phi(\Delta t, \lambda)} \tag{3}$$

where

$$\phi(\Delta t, \lambda) = \Delta t + O(\Delta t^2). \tag{4}$$

(3) Nonlinear terms must be represented by nonlocal discrete discretization. For example

$$u_j^2 \approx u_j u_{j+1},\ u_j^2 \approx \left(\frac{u_{j-1}+u_j+u_{j-1}}{3}\right) u_j, \tag{5}$$

and

$$u^3 \approx 2u_j^3 - u_j^2 u_{j+1},\ u_j^3 \approx u_{j-1}u_j u_{j+1}. \tag{6}$$

5 Numerical experiment

We solve Eq. (1) where $u(x,t)$ is the unknown function which depends on the spatial variable, $x \in (-10, 10) \subset \Omega = \mathbb{R}$ and temporal variable, t.
The initial condition is $u(x,0) = u_0(x) = \frac{1}{2} - \frac{1}{2}\tanh\left(\frac{\sqrt{\beta}}{2\sqrt{2}}x\right)$ [51] and the boundary conditions are

$$u(-10,t) = \frac{1}{2} - \frac{1}{2}\tanh\left[\frac{\sqrt{\beta}}{2\sqrt{2}}(-10-ct)\right],\ u(10,t) = \frac{1}{2} - \frac{1}{2}\tanh\left[\frac{\sqrt{\beta}}{2\sqrt{2}}(10-ct)\right].$$

We note that the initial condition is non-negative i.e $u(x,0) \geq 0$.
The exact solution is $u(x,t) = \frac{1}{2} - \frac{1}{2}\tanh\left[\frac{\sqrt{\beta}}{2\sqrt{2}}(x-ct)\right]$, where $\beta > 0$ and $c = -\sqrt{\frac{\beta}{2}}(2\gamma-1)$ with $\gamma \in (0,1)$. In this work, we consider three cases:

Case 1 : $\beta = 0.5\ (0 < \beta < 1)$, $\gamma = 0.2$.

Case 2 : $\beta = 1,\ \gamma = 0.2$.

Case 3 : $\beta > 1\,(\beta = 2),\ \gamma = 0.2$.

We construct four versions of NSFD schemes in order to discretise

$$u_t - u_{xx} = \beta u(1-u)(u-\gamma).$$

In all the four methods, we use the same discretisation for u_t and u_{xx}. We approximate u_t by $\frac{u_j^{n+1}-u_j^n}{\phi_2(\Delta t)}$ where $\phi_2(\Delta t) = \phi_2(k) = \frac{e^{\beta k}-1}{\beta}$ and u_{xx} by $\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\psi_1(\Delta x)\psi_2(\Delta x)}$ where $\psi_1(\Delta x) = \psi_1(h) = \frac{1-e^{-\beta h}}{\beta}$ and $\psi_2(\Delta x) = \psi_2(h) = \frac{e^{\beta h}-1}{\beta}$. We expect the theoretical rate of convergence in time to be equal to one. Lax-Equivalence theorem cannot be used to study convergence as we are working with a non-linear partial differential equation.
We test the performance of the schemes over some different values of β over the domain,
$x \in [-10, 10]$ at time, $T = 0.5$. The spatial step size h is chosen as 0.1. We use some different values for the temporal step size k which must satisfy the condition for positivity and boundedness.

6 Basic dynamical behaviour and a priori bound of Eq. (1)

In this section, we present a theorem on the existence and uniqueness of the solution to the dynamical behaviour of Eq. (1) and a priori bound of the solution. We recall some results from [14].

Theorem 1 ([14])
Suppose that $0 < \gamma < 1$ and $\Omega \subset \mathbb{R}^n$ is bounded domain with smooth boundary:

(a) If the initial condition $u_0(x) = u(x,0)$ is positive $(u_0(x) \geq 0)$ then Eq. (1) has unique solution $u(x,t)$ such that $u(x,t)$ positive $(u(x,t) \geq 0)$ for $t \in (0,\infty)$ and $\overline{\Omega}$;

(b) If $u_0(x) \leq 1$ then $u(x,t)$ tends to 0 uniformly as $t \to \infty$;

(c) For any solution $u(x,t)$ of Eq. (1), $\limsup_{t\to\infty} u(x,t) \leq 1$.

Proof The full proof is from [14]. Let us define

$$F(u) = \beta\, u(1-u)(u-\gamma).$$

(a) From [52, 53], by letting $\underline{u}(x,t) = 0$ and $\overline{u}(x,t) = u^*(t)$, where u^* is the unique solution to

$$\frac{du}{dt} = F(u),\ u(0) = u^*,\ u^* = \sup_{x\in\overline{\Omega}} u_0(x).$$

Then $\underline{u}(x) = 0$ and $\overline{u}(x,t) = u^*(t)$, are the lower-solution and upper-solution to Eq. (1), respectively. Furthermore since

$$\overline{u}_t(x,t) - \overline{u}_{xx}(x,t) - F(\overline{u}(x,t)) = 0 \geq 0 = \underline{u}(x,t) - \underline{u}_{xx}(x,t) - F(\underline{u}(x,t))$$

the boundary condition is satisfied and $0 \leq u_0(x) \leq u^*$ by using lower/upper-solution Definition 8.1.2 in [52] or Definition 5.3.1 in [53]. It follows using Theorem 8.3.3 in [52], that Eq. (1) has unique globally defined solution $u(x,t)$ which satisfies $0 \leq u(x,t) \leq u^*(t)$, $t \geq 0$. Moreover, by using the strong maximum principle, $u(x,t) > 0$, for $t \geq 0$ and $\forall\, x \in \overline{\Omega}$.

(b) We assume that $u_0(x) \leq u^* < 1$, then $u^*(t) \to 0$ as $t \to \infty$.

(c) It is straightforward from (b) that $\limsup_{t\to\infty} u(x,t) \leq 1$.

□

In the following section, we construct four numerical methods to in order to solve Eq. (1).

7 NSFD1 scheme

We note that the right hand side of Eq. (1) is $\beta\,(-u^3 + (1+\gamma)u^2 - \gamma u)$. We use the following discrete approximations for the right hand side of Eq. (1) as used by Namjoo and Zibaei [54]:

$$-\beta\,\gamma u(x_j,t_n) \approx -\beta\,\gamma u_j^{n+1},\ -\beta\,(u(x_j,t_n))^3 \approx \beta\left(-\frac{3}{2}\left(u_{j-1}^n\right)^2 u_j^{n+1} + \frac{1}{2}\left(u_{j-1}^n\right)^3\right), \tag{7}$$

$$\beta\,(1+\gamma)\,(u(x_j,t_n))^2 \approx \beta\,(1+\gamma)\,\left(u_{j-1}^n\right)^2. \tag{8}$$

The following scheme is proposed:

$$\frac{u_j^{n+1} - u_j^n}{\phi_2(k)} - \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\psi_1(h)\,\psi_2(h)} = \beta\left(-\frac{3}{2}\left(u_{j-1}^n\right)^2 u_j^{n+1} + \frac{1}{2}\left(u_{j-1}^n\right)^3\right) + \beta\,(1+\gamma)\,\left(u_{j-1}^n\right)^2 - \beta\,\gamma\,u_j^{n+1}. \tag{9}$$

A single expression for the scheme is

$$u_j^{n+1} = \frac{(1-2R)u_j^n + R(u_{j+1}^n + u_{j-1}^n) + \beta\phi_2(k)\left((1+\gamma)\left(u_{j-1}^n\right)^2 + \frac{1}{2}\left(u_{j-1}^n\right)^3\right)}{1 + \beta\,\gamma\,\phi_2(k) + \frac{3}{2}\beta\,\phi_2(k)\left(u_{j-1}^n\right)^2}, \tag{10}$$

where $R = \frac{\phi_2(k)}{\psi_1(h)\,\psi_2(h)}$.

As discussed in the introduction, the theory of nonstandard finite difference required dynamical consistency (positivity, boundedness, preservation of fixed points) which helps to avoid numerical instabilities.
The fixed points of Eq. (1) are $u_1^* = 0$, $u_2^* = 1$ (which are stable) and $u_3^* = \gamma$ which is unstable. Furthermore, Roger and Mickens [55] showed preservation of local stabilities of all fixed points.

Theorem 2 (Dynamical consistency)
If $1 - 2R \geq 0$, the numerical solution of Eq. (10) satisfies

$$0 \leq u_j^n \leq 1 \Longrightarrow 0 \leq u_j^{n+1} \leq 1,$$

and the dynamical consistency (positivity and boundedness) holds for all relevant values of n and j.

Proof

(1) If $1 - 2R \geq 0$, then

$$R = \left(\frac{e^{\beta k} - 1}{\beta}\right)\left(\frac{\beta}{1 - e^{-\beta h}}\right)\left(\frac{\beta}{e^{\beta h} - 1}\right) \leq \frac{1}{2}. \tag{11}$$

Since $u_j^n \geq 0$ and $1 + \beta\,\gamma\,\phi_2(k) + \frac{3}{2}\beta\,\phi_2(k)\,\left(u_{j-1}^n\right)^2 > 0$, NSFD1 is positive definite under the condition

$$k \leq \frac{1}{\beta}\ln\left[1 + \frac{1}{2\beta}\frac{\left(e^{\beta h} - 1\right)^2}{e^{\beta h}}\right]. \tag{12}$$

(2) We assume that $0 \leq u_j^n \leq 1$. If the scheme is bounded, we need to prove that $0 \leq u_j^{n+1} \leq 1$. Consider

$$\begin{aligned}(u_j^{n+1} - 1)\left(1 + \beta\gamma\phi_2(k) + \frac{3}{2}\beta\phi_2(k)\left(u_{j-1}^n\right)^2\right) &= (1 - 2R)u_j^n + R(u_{j+1}^n + u_{j-1}^n)\\ &\quad + \beta\phi_2(k)\left((1+\gamma)\left(u_{j-1}^n\right)^2 + \frac{1}{2}\left(u_{j-1}^n\right)^3\right)\\ &\quad - 1 - \beta\phi_2(k)\gamma - \frac{3}{2}\beta\phi_2(k)\left(u_{j-1}^n\right)^2.\end{aligned} \tag{13}$$

It follows $\left(u_{j-1}^n\right)^3 = u_{j-1}^n\left(u_{j-1}^n\right)^2 \leq \left(u_{j-1}^n\right)^2$ since $0 \leq u_j^n \leq 1$ for all values of n and j. Therefore,

$$\begin{aligned}(u_j^{n+1} - 1)\left(1 + \beta\gamma\phi_2(k) + \frac{3}{2}\beta\phi_2(k)\left(u_{j-1}^n\right)^2\right) &\leq 1 - 2R + 2R + \beta\gamma\phi_2(k)\left(u_{j-1}^n\right)^2\\ &\quad + \beta\phi_2(k)\left(\left(u_{j-1}^n\right)^2 + \frac{1}{2}\left(u_{j-1}^n\right)^2\right)\\ &\quad - 1 - \beta\,\gamma\,\phi_2(k) - \frac{3}{2}\beta\phi_2(k)\left(u_{j-1}^n\right)^2\\ &= 0.\end{aligned} \tag{14}$$

Hence $u_j^{n+1} - 1 \leq 0$. Thus, we conclude that NSFD1 scheme is bounded.

□

Using $h = 0.1$, for positivity we have from (12),

(a) $k \leq 4.9948 \times 10^{-3}$ for $\beta = 0.5$.

(b) $k \leq 4.9917 \times 10^{-3}$ for $\beta = 1.0$.

(c) $k \leq 4.9917 \times 10^{-3}$ for $\beta = 2.0$.

We tabulate L_1 and L_∞ errors, rate of convergence in time (with respect to L_∞ error) and CPU time at some different values of k using $\gamma = 0.2$, $h = 0.1$ at time, $T = 0.5$ for three cases namely; $\beta = 0.5$, 1.0, 2.0 using NSFD1 scheme in Tables 1, 2 and 3.

The scheme does not give satisfactory results especially in regard to the rate of convergence as the theoretical rate of convergence with respect to time is one. This could be due to the approximation of $-\beta\,(u(x_j,t_n))^3$ by $\beta\left(-\frac{3}{2}\left(u^n_{j-1}\right)^2 u^{n+1}_j + \frac{1}{2}\left(u^n_{j-1}\right)^3\right)$ and approximation of $\beta\,(1+\gamma)\,(u(x_j,t_n))^2$ by $\beta\,(1+\gamma)\,\left(u^n_{j-1}\right)^2$ where u^n_{j-1} and u^{n+1}_j are both non-local approximation.

Plots of u against x for the three cases using NSFD1 scheme are shown in Fig (1). Corresponding plot of errors against x are shown in Fig (2).
We observe that as we increase the values of β, the profile becomes more stiff and the problem becomes more challenging for the numerical scheme.

8 NSFD2 scheme

In this section, we make use of the following approximations [54] for the right hand of Eq. (1):

$$-\beta\,\gamma\,u(x_j,t_n) \approx -\beta\,\gamma\,u^{n+1}_j,\;\; \beta\,(1+\gamma)\,(u(x_j,t_n))^2 \approx \beta(1+\gamma)\,u^{n+1}_j u^n_j, \tag{15}$$

$$-\beta\,(u(x_j,t_n))^3 \approx -\beta\,u^{n+1}_j\left(u^n_j\right)^2. \tag{16}$$

This gives the following scheme:

$$\frac{u^{n+1}_j - u^n_j}{\phi_2(k)} - \frac{u^n_{j+1} - 2u^n_j + u^n_{j-1}}{\psi_1(h)\,\psi_2(h)} = \beta\left(-u^{n+1}_j\left(u^n_j\right)^2 + (1+\gamma)u^{n+1}_j u^n_j - \gamma\,u^{n+1}_j\right), \tag{17}$$

which can be rewritten as

$$u^{n+1}_j = \frac{(1-2R)u^n_j + R(u^n_{j+1} + u^n_{j-1})}{1 + \beta\,\phi_2(k)\,\gamma - \beta\,(1+\gamma)\,\phi_2(k)u^n_j + \beta\,\phi_2(k)\left(u^n_j\right)^2},\text{ where } R = \frac{\phi_2(k)}{\psi_1(h)\,\psi_2(h)}. \tag{18}$$

Theorem 3 (Dynamical consistency)
If $1 - 2R \geq 0$ and $1 - \beta\phi_2(k) \geq 0$, the numerical solution of Eq. (18) satisfies

$$0 \leq u^n_j \leq 1 \Longrightarrow 0 \leq u^{n+1}_j \leq 1,$$

and the dynamical consistency (positivity and boundedness) holds for all relevant values of n and j.

Proof

(1) If $1-2R \geq 0$ and $1-\beta\phi_2(k) \geq 0$, then NSFD2 is positive definite. Indeed we require $1+\beta\,\phi_2(k)\,\gamma - \beta\,(1+\gamma)\,\phi_2(k)u_j^n + \beta\,\phi_2(k)\left(u_j^n\right)^2 > 0$. Since $0 \leq u_j^n \leq 1$,

$$\begin{aligned} 1-\left[\beta(1+\gamma)\phi_2(k)u_j^n - \beta\phi_2(k)\gamma\right] + \beta\phi_2(k)\left(u_j^n\right)^2 &\geq 1-\beta\phi_2(k) + \beta\phi_2(k)\left(u_j^n\right)^2 \\ &\geq 1-\beta\phi_2(k). \end{aligned} \tag{19}$$

NSFD2 is positive definite under the conditions

$$k \leq \begin{cases} \frac{1}{\beta}\ln(2)\,, \\ \frac{1}{\beta}\ln\left(1+\frac{1}{2\beta}\frac{\left(e^{\beta h}-1\right)^2}{e^{\beta h}}\right). \end{cases} \tag{20}$$

(2) We note that $0 \leq u_j^n \leq 1$. We need to check if NSFD2 is bounded. Consider

$$\begin{aligned} &(u_j^{n+1}-1)\left(1+\beta\phi_2(k)\gamma - \beta(1+\gamma)\phi_2(k)u_j^n + \beta\phi_2(k)\left(u_j^n\right)^2\right) = (1-2R)u_j^n \\ &+ R(u_{j+1}^n + u_{j-1}^n) - 1 - \beta\gamma\phi_2(k) + \beta(1+\gamma)\phi_2(k)u_j^n - \beta\phi_2(k)\left(u_j^n\right)^2. \end{aligned} \tag{21}$$

Since $0 \leq u_j^n \leq 1$ for all values for n and j,

$$\begin{aligned} &(u_j^{n+1}-1)\left(1+\beta\phi_2(k)\gamma - \beta(1+\gamma)\phi_2(k)u_j^n + \beta\phi_2(k)\left(u_j^n\right)^2\right) \leq 1-2R+2R \\ &-1-\beta\gamma\phi_2(k) + \beta(1+\gamma)\phi_2(k)u_j^n - \beta\phi_2(k)\left(u_j^n\right)^2 = \\ &-\beta\phi_2(k)\left[\left(u_j^n\right)^2 - \gamma u_j^n - u_j^n + \gamma\right] = -\beta\phi_2(k)(u_j^n-1)(u_j^n-\gamma) \leq 0. \end{aligned} \tag{22}$$

Hence $0 \leq u_j^{n+1} \leq 1$ and therefore NSFD2 satisfies the boundedness properties.

□

Using $h=0.1$, for positivity we have from Eq. (20)

(a) $k \leq 1.3863$ and $k \leq 4.9948 \times 10^{-3}$ for $\beta = 0.5$.

(b) $k \leq 6.9315 \times 10^{-1}$ and $k \leq 4.9917 \times 10^{-3}$ for $\beta = 1.0$.

(c) $k \leq 3.4657 \times 10^{-1}$ and $k \leq 4.9917 \times 10^{-3}$ for $\beta = 2.0$.

We tabulate L_1 and L_∞ errors, rate of convergence in time (with respect to L_∞ error) and CPU time at some different values of k using $\gamma = 0.2$, $h = 0.1$ at time, $T = 0.5$ for three cases namely; $\beta = 0.5$, 1.0, 2.0, using NSFD2 scheme in Tables 4, 5 and 6.

We observe that L_1 error, L_∞ errors are quite small for all the three cases. The rate of convergence with respect to time is approximatively one for the case $\beta = 0.5$. The rate of convergence for $\beta = 1$ is approximately one if k is properly chosen as 0.005. For $\beta = 2$, the rate of convergence deviate from ideal value of one quite significantly.

Plots of u against x for the three cases using NSFD2 scheme are shown in Figs 3 and 4.

9 NSFD3 scheme

We consider the ordinary differential equation

$$\epsilon \frac{du}{dt} = f(u), \tag{23}$$

where $f(u) = u(1-u)(u-\gamma)$, and $\epsilon > 0$ is a real parameter, $\gamma \in (0,1)$. Roger and Mickens [55] showed that the following NSFD scheme

$$\epsilon \frac{u^{n+1} - u^n}{\phi_2(\Delta t)} = -(2u^{n+1} - u^n)(u^n)^2 + (1+\gamma)(u^n)^2 - \gamma u^{n+1}. \tag{24}$$

where $\phi_2(\Delta t) = \frac{e^{\frac{\Delta t}{\epsilon}} - 1}{\frac{1}{\epsilon}}$, preserves positivity and local stabilities of all fixed points. We construct a Nonstandard finite difference scheme using idea from Roger and Mickens [55]. We propose the following scheme for Eq. (1):

$$\frac{u_j^{n+1} - u_j^n}{\phi_2(k)} - \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\psi_1(h)\,\psi_2(h)} = \beta\left(-\left(2u_j^{n+1} - u_j^n\right)\left(u_j^n\right)^2 + (1+\gamma)\left(u_j^n\right)^2 - \gamma u_j^{n+1}\right). \tag{25}$$

A single expression for NSFD3 scheme is

$$u_j^{n+1} = \frac{(1-2R)u_j^n + R(u_{j+1}^n + u_{j-1}^n) + \beta\,\phi_2(k)\left(\left(u_j^n\right)^3 + (1+\gamma)\left(u_j^n\right)^2\right)}{1 + \beta\,\phi_2(k)\,\gamma + 2\beta\,\phi_2(k)\left(u_j^n\right)^2}. \tag{26}$$

Theorem 4 (Dynamical consistency)
If $1 - 2R \geq 0$, the numerical solution of Eq. (26) satisfies

$$0 \leq u_j^n \leq 1 \Longrightarrow 0 \leq u_j^{n+1} \leq 1,$$

and the dynamical consistency (positivity and boundedness) holds for all relevant values of n and j.

Proof

(1) NSFD3 is positive definite if $1-2R \geq 0$. The following condition must be satisfied namely,

$$k \leq \frac{1}{\beta} \ln\left(1 + \frac{1}{2\beta}\frac{\left(e^{\beta h} - 1\right)^2}{e^{\beta h}}\right). \tag{27}$$

(2) We next check if the NSFD3 is bounded:

$$\begin{aligned}(u_j^{n+1} - 1)\left(1 + \beta\,\phi_2(k)\gamma + 2\beta\phi_2(k)\left(u_j^n\right)^2\right) &= (1-2R)u_j^n \\ &+ R(u_{j+1}^n + u_{j-1}^n) + \beta\phi_2(k)\left(\left(u_j^n\right)^3 + (1+\gamma)\left(u_j^n\right)^2\right) \\ &- 1 - \beta\,\gamma\phi_2(k) - 2\beta\,\phi_2(k)\left(u_j^n\right)^2.\end{aligned} \tag{28}$$

We note that $0 \leq u_j^n \leq 1$ for all values of n and j. Therefore

$$\begin{aligned}(u_j^{n+1} - 1)\left(1 + \beta\gamma\phi_2(k) + 2\beta\phi_2(k)\left(u_{j-1}^n\right)^2\right) &\leq 1 - 2R + 2R + \beta\phi_2(k)\left(\gamma\left(u_j^n\right)^2 + \left(u_j^n\right)^2\right) \\ &+ \beta\phi_2(k) - 1 - \beta\,\gamma\phi_2(k) - 2\beta\phi_2(k)\left(u_j^n\right)^2 = 0.\end{aligned} \tag{29}$$

Hence $0 \leq u_j^{n+1} \leq 1$ and therefore NSFD3 satisfies the boundedness properties.

□

We tabulate L_1 and L_∞ errors, rate of convergence in time and CPU time at some different values of k using $\gamma = 0.2$, $h = 0.1$ at time, $T = 0.5$ for three cases namely; $\beta = 0.5$, 1.0, 2.0 using NSFD3 scheme in Tables 7, 8 and 9.
The scheme is very effective for $\beta = 0.5$, 1.0 in respect to L_1 and L_∞ errors, rate of convergence in time. The scheme is effective for $\beta = 2$ in regard to rate of convergence if the time step size is carefully chosen.
Plots of u against x for the three cases using NSFD3 scheme are shown in Fig 5 and corresponding plots of errors against x are displayed in Fig 6.

10 NSFD4 scheme

We observe that NSFD1 has convergence issues for all the three values of β used. NSFD4 scheme is a modification of NSFD1 scheme: We approximate $-\beta\,(u(x_j,t_n))^3$ by $\beta\left(-\frac{3}{2}\left(u_j^n\right)^2 u_j^{n+1} + \frac{1}{2}\left(u_j^n\right)^3\right)$ instead of $\beta\left(-\frac{3}{2}\left(u_{j-1}^n\right)^2 u_j^{n+1} + \frac{1}{2}\left(u_{j-1}^n\right)^3\right)$. We also approximate $\beta\,(1+\gamma)\,(u(x_j,t_n))^2$ by $\beta\,(1+\gamma)\,\left(u_j^n\right)^2$ instead of $\beta\,(1+\gamma)\,\left(u_{j-1}^n\right)^2$.
We propose

$$\frac{u_j^{n+1}-u_j^n}{\phi_2(k)} - \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\psi_1(h)\,\psi_2(h)} = \beta\left(-\frac{3}{2}\,(u_j^n)^2\,u_j^{n+1} + \frac{1}{2}\,(u_j^n)^3\right) + \beta\,(1+\gamma)\,(u_j^n)^2 - \beta\,\gamma\,u_j^{n+1}. \tag{30}$$

A single expression for the scheme is

$$u_j^{n+1} = \frac{(1-2R)u_j^n + R(u_{j+1}^n + u_{j-1}^n) + \beta\phi_2(k)\left((1+\gamma)\left(u_j^n\right)^2 + \frac{1}{2}\left(u_j^n\right)^3\right)}{1 + \beta\,\gamma\,\phi_2(k) + \frac{3}{2}\beta\,\phi_2(k)\left(u_j^n\right)^2}, \tag{31}$$

where $R = \frac{\phi_2(k)}{\psi_1(h)\,\psi_2(h)}$.

Theorem 5 (Dynamical consistency)
If $1-2R \geq 0$, the numerical solution of Eq. (31) satisfies

$$0 \leq u_j^n \leq 1 \Longrightarrow 0 \leq u_j^{n+1} \leq 1,$$

and the dynamical consistency (positivity and boundedness) holds for all relevant values of n and j.

Proof

(1) NSFD4 is positive definite if $1-2R \geq 0$. We have $1 + \beta\,\gamma\,\phi_2(k) + \frac{3}{2}\beta\,\phi_2(k)\left(u_j^n\right)^2 > 0$ since $u_j^n \geq 0$. It follows that

$$k \leq \frac{1}{\beta}\ln\left[1 + \frac{1}{2\beta}\frac{\left(e^{\beta h}-1\right)^2}{e^{\beta h}}\right]. \tag{32}$$

(2) By assumption, $0 \leq u_j^n \leq 1$. We have

$$(u_j^{n+1} - 1)\left(1 + \beta\,\gamma\,\phi_2(k) + \frac{3}{2}\beta\,\phi_2(k)\left(u_j^n\right)^2\right) = (1-2R)u_j^n + R(u_{j+1}^n + u_{j-1}^n) + \beta\,\phi_2(k)\left((1+\gamma)\left(u_j^n\right)^2 + \frac{1}{2}\left(u_j^n\right)^3\right) - 1 - \beta\,\phi_2(k)\gamma - \frac{3}{2}\beta\,\phi_2(k)\left(u_j^n\right)^2. \quad (33)$$

It follows that $\left(u_j^n\right)^3 = u_j^n\left(u_j^n\right)^2 \leq \left(u_j^n\right)^2$ since $0 \leq u_j^n \leq 1$ for all values of n and j. Therefore,

$$\begin{aligned}(u_j^{n+1} - 1)\left(1 + \beta\gamma\phi_2(k) + \frac{3}{2}\beta\phi_2(k)\left(u_j^n\right)^2\right) &\leq 1 - 2R + 2R + \beta\gamma\phi_2(k)\left(u_j^n\right)^2 \\ &+ \beta\phi_2(k)\left(\left(u_j^n\right)^2 + \frac{1}{2}\left(u_j^n\right)^2\right) \\ &- 1 - \beta\,\gamma\,\phi_2(k) - \frac{3}{2}\beta\phi_2(k)\left(u_j^n\right)^2 \\ &= \beta\,\gamma\,\phi_2(k)\left(\left(u_j^n\right)^2 - 1\right) \leq 0. \end{aligned} \quad (34)$$

Hence $u_j^{n+1} - 1 \leq 0$ and the therefore NSFD4 is bounded.

□

Using $h = 0.1$, for positivity we have from Eq. (32),

(a) $k \leq 4.9948 \times 10^{-3}$ for $\beta = 0.5$.

(b) $k \leq 4.9917 \times 10^{-3}$ for $\beta = 1.0$.

(c) $k \leq 4.9917 \times 10^{-3}$ for $\beta = 2.0$.

We tabulate L_1, L_∞ errors, rate of convergence and CPU time at some different values of k using $\gamma = 0.2$, $h = 0.1$ at time, $T = 0.5$ for three cases namely; $\beta = 0.5$, 1.0, 2.0, using NSFD4 scheme in Tables 10, 11 and 12.

The scheme gives good results for $\beta = 0.5$, 1.0 if k is properly chosen and is an improvement over NSFD1. The scheme is not effective for $\beta = 2.0$ as the numerical rate of convergence deviates from one.
Plots of u against x for the three cases using NSFD4 scheme are shown in Fig 7 and plot of errors against x are displayed in Fig 8.

Remark
If β is chosen much larger than the coefficient of diffusion (say $\beta = 5$ or 10) for the numerical experiment considered in this work, the profile becomes stiff and the problem becomes difficult to solve.
In Agbavon et al. [56], the numerical solution of Fisher's equation with coefficient of diffusion term much smaller than reaction was obtained for an initial condition consisting of an exponential function. NSFD methods were used and range of values of k was quite restricted. To obtain accurate results very small values of k had to be used. Some modification was made to the NSFD methods.

11 Error estimates for NSFD3

In this section, we study the error estimate for NSFD3. Before any further discussion, we recall some of results from [57, 58]. We consider the system of partial differential equations:

$$V_t = D(x,t,V)\Delta V + \sum_{j=1}^{n} M_j(x,t,V)\frac{\partial V}{x^j} + F(x,t,V),\ (x,t) \in \Omega \times (0,\infty), \tag{35}$$

$$V(x,0) = V_0(x)\ \ x \in \Omega, \tag{36}$$

$$P(x)[V(x,t) - W(x)] + Q(x)\frac{\partial V}{\partial n} = 0\ \ (x,t) \in \partial\Omega \times (0,\infty) \tag{37}$$

Here $\Omega \subset \mathbb{R}^\mu$ is a connected, bounded open set with piecewise smooth boundary and $V(x,t) \in \mathbb{R}^\nu$. Let S be the parallelepiped $S = \prod_{i=1}^{\mu}[a_i, b_i]$ in $V-$space. We make the following assumptions:

(a) D, M_j, and P and Q are diagonal matrices.

(b) There is a constant $K_1 > 0$ such that

$$d_i(x,t,V) - \frac{K_1}{2}|m_{ji}(x,t,V)| \geq 0,\quad (x,t,V) \in \overline{\Omega} \times [0,\infty) \times S,\ \ \forall i,\ \ j.$$

d_i and m_{ji} are bounded and smooth. Furthermore they are i th diagonal entries of D and M respectively.

(c) F is smooth function and verifies $F(x,t,V) \cdot N_s(V) \leq 0),\ \forall\ (x,t,V) \in \overline{\Omega} \times [0,\infty) \times \partial S$. N is outer normal on S. Also We assume there is constant K_2 such that $\left|\frac{\partial F^i}{\partial V^i}(x,t,V)\right| \leq K_2,\ \forall\ (x,t,V) \in \overline{\Omega} \times [0,\infty) \times S$ and for all i.

(d) $V_0(x)$ and $W(x)$ are in S for relevant x.

(e) $\sup_{i,x,\alpha}|D_x^\alpha V^i(x,t)| < \infty$ for $1 \leq i \leq \mu$, $x \in \Omega$ and $|\alpha| = \sum_{j=1}^{\nu}|\alpha_j| \leq 4$. $\|V\|_p = \sup_{x,\alpha}|D_x^\alpha V^i(x,t)|_\infty$, where p is non-negative integer.

We define the finite difference equation

$$\frac{V_j^{n+1} - V_j^n}{\Delta t} = \sum_{j=1}^{\mu}\left\{\left[D\frac{\Delta_j^2}{(\Delta x^j)^2} + M_j\frac{\Delta_j}{(2\,\Delta x^j)}\right](z_1\,V^{n+1} + z_2\,V^n)\right\}_j + F, \tag{38}$$

where D, M_j and F are evaluated at (x_j, t_n, V_j^n). Here z_1 and z_2 are nonnegative such that $z_1 + z_2 = 1$. We also define the error in the numerical method as

$$e_j^n = v_j^n - V_j^n \tag{39}$$

where v is the exact solution of (35)- (37) and V is the finite-difference approximation. We state the following theorem from [57] and simple case is treated in [58] using Dirichlet boundary condition.

Theorem 6 (HOFF [57])
Assume that the solution v of (35)- (37) is smooth in the sense of assumption (e) and assume that the difference scheme (38) is consistent and stable. Then

$$\|e^n\|_\infty \leq \frac{e^{-\sigma\,t_n}}{l + e^{-(\sigma+p)\,t_n}}[\|e^0\|_\infty + p\,l] + \frac{l}{l + e^{-(\sigma+p)\,t_n}}\,diam\,S, \tag{40}$$

where $p = p(t_n)$ is a positive function of t_n which depends upon the parameters appearing in (35)- (37) and upon $\max_{0\leq t\leq t_n}\|V(\cdot,t)\|_4$; $l = \Delta t + \sum_{j=1}^{\mu}(\Delta x^j)^2$; and $\sigma \geq 0$ is arbitrary.

Remark
Note that the Theorem 6 is a general case of the following theorem in which Dirichlet boundary condition is used:

Theorem 7 (Sanz-Serna and Stuart [58])
Under the above assumptions $(a)-(e)$ and under the conditions that as $t \to \infty$, v approaches an equilibrium and asymptotically stable and the grids are refined in such a way that $\left(\Delta t/(\Delta x)^2\right) \leq \epsilon \leq \frac{1}{2}$ then there exist constants l_0 and C depending upon only f, v and ϵ, such that for $\Delta x < l_0$, the numerical solution V^n exists for all positive integers n and satisfies the error bound. Then

$$\|e^n\|_2 \leq C\, l, \;\; l = \Delta t + (\Delta x)^2. \tag{41}$$

Remark (ANDÔ [59])
Let $p \geq 1$ be a real number. The p-norm (also called l_p-norm of L_p space) of vector

$x = (x_1, \;\; \cdots, x_n)$ is

$$\| x \|_p = \left(\sum_1^n |x_i|^p \right)^{\frac{1}{p}}.$$

We apply the above theorems to our problem. We take $S = [0,1]$; $diamS = 1$; $F = f(u) = \beta\, u(1-u)(u-\gamma)$. $D = 1$; $M = 0$; $P(x) = 0$; $Q(x) = 1$. $\nu = 1$; $\mu = 1$. The fixed points of Eq. (1) are $u_1^* = 0$, $u_2^* = 1$ and they are asymptotically stable. Also $u_3^* = \gamma$ is fixed point but it is unstable [55]. From assumption of Theorem 1 we have $u_0(x) \in [0,1]$, $\forall\, x$. Furthermore $u(x,t) \in [0,1]$, $\forall\, x,\ t$. $f(u) = \beta\, u(1-u)(u-\gamma)$, is smooth in u. $\partial s = \{0, 1, \gamma\}$. Hence $0 = f(x,t,0) \cdot N_s(u) = 0$; $0 = f(x,t,1) \cdot N_s(u) = 0$; $0 = f(x,t,\gamma) \cdot N_s(u) = 0$. $f_u = \beta\{(1-u)(u-\gamma) - u(u\gamma) + u(1-u)\}$. $|f_u|_\infty = \max|f_u| \leq 2\,\gamma + 1 = K_2$. $u_x = -\frac{\sqrt{\beta}}{4\sqrt{2}} + \frac{\sqrt{\beta}}{4\sqrt{2}} \tanh(\frac{\sqrt{\beta}}{2\sqrt{2}}(x - ct))$. $|u_x| < \infty$ since tanh is bounded on $[-10, 10]$.

We can conclude since the NSFD3 scheme is dynamical consistent and stable under the condition $k \leq \frac{1}{\beta} \ln \left(1 + \frac{1}{2\beta} \frac{\left(e^{\beta h} - 1\right)^2}{e^{\beta h}} \right)$ and $l = \phi_2(k) + \psi_1(h)\, \psi_2(h)$. The error bound (error estimate) is

$$\|e^n\|_\infty \leq \frac{e^{-\sigma\, t_n}}{l + e^{-(\sigma+p)\, t_n}} [\|e^0\|_\infty + p\, l] + \frac{l}{l + e^{-(\sigma+p)\, t_n}}.$$

As $t_n \to \infty$, we have

$$\|e^n\|_\infty \leq 1.$$

12 Relationship between physical behaviour and numerical solution

In section 3.1, we mentioned that the model of FitzHugh-Nagumo equation which gives signal propagation (in form of wave) along a nerve has two properties which are excitable ($\gamma < 0$: the nerve is in excitable form) and refractory ($\gamma > 0$: the nerve is in refractory form and does not respond to external stimulation) [19, 20]. Excitation occurs quickly while recovery happens slowly. We also mentioned that FitzHugh-Nagumo equation has a lot of applications in biology [16, 17]. It was proved in [13] that the solution of the model of FitzHugh-Nagumo equation has biological meaning only if $\gamma \in (0,1)$. That makes sense since if $\gamma < 0$, the nerve is in excitable form and depends on external simulation. Therefore the solution of the model can be out of bound. In biological point of view, the solutions of the model should remain positive and bounded as time progresses if $\gamma \in (0,1)$ (please refer to section 6 of basic dynamical behaviour and a priori bound). In our work, we are dealing with the case where the solution biologically makes sense i.e. $\gamma \in (0,1)$. We have constructed four schemes namely; NSFD1, NSFD2, NSFD3, NSFD4 which are dynamically consistent (please refer to theorems 2, 3, 4, 5). Dynamically consistency means preservation of the properties of the original model. Our four schemes preserve the positivity, boundedness and fixed points of the model.

13 Conclusion

In this work, we have constructed four nonstandard finite difference schemes namely; NSFD1, NSFD2, NSFD3, NSFD4 in order to solve FitzHugh-Nagumo equation under three different regimes. The first order time derivative and the second order spatial derivative are approximated in the same manner for all of the methods and it is only the nonlinear polynomial in the partial differential equation which is discretised differently. We derive conditions under which the schemes are positive definite and bounded.
NSFD1 is not effective and gives issues in regard to its rate of convergence in time for all the three values of β used. NSFD4 is a major improvement over NSFD1. NSFD4 is effective for $\beta = 0.5, 1.0$. NSFD2 is effective for $\beta = 0.5$. NSFD2 is quite effective in regard to rate of convergence for $\beta = 1.0, 2.0$ provided k is carefully chosen. NSFD3 seems the best scheme followed by NSFD2 and NSFD4 when we check performance of the methods based on L_1, L_∞ errors and rate of convergence in time (with respect to L_1 error). We studied the error estimate for NSFD3 and found that it cannot go beyond one as we progress in time.

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Competing interests
None of the authors have competing interests in the manuscript.

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14 Tables

Time step (k)	L_1 error	L_∞ error	Rate of convergence	CPU (s)
0.005	1.7339×10^{-2}	2.6693×10^{-3}	-	1.007
0.0025	1.7383×10^{-2}	2.6742×10^{-3}	-3.656×10^{-3}	1.229
0.00125	1.7405×10^{-2}	2.6767×10^{-3}	-1.824×10^{-3}	1.752

Table 1: Computation of L_1, L_∞ errors, rate of convergence and CPU time using NSFD1 for $-10 \le x \le 10$, $\gamma = 0.2$, $h = 0.1$, $\beta = 0.5$ at time, $T = 0.5$.

Time step (k)	L_1 error	L_∞ error	Rate of convergence	CPU (s)
0.005	3.4919×10^{-2}	7.4411×10^{-3}	-	1.032
0.0025	3.5051×10^{-2}	7.4613×10^{-3}	-5.443×10^{-3}	1.280
0.00125	3.5117×10^{-2}	7.4715×10^{-3}	-2.713×10^{-3}	1.815

Table 2: Computation of L_1, L_∞ errors, rate of convergence and CPU time using NSFD1 for $-10 \le x \le 10$, $\gamma = 0.2$, $h = 0.1$, $\beta = 1$ at time, $T = 0.5$.

Time step (k)	L_1 error	L_∞ error	Rate of convergence	CPU (s)
0.005	6.9598×10^{-2}	2.0534×10^{-2}	-	0.608
0.0025	6.9987×10^{-2}	2.0625×10^{-2}	-8.041×10^{-3}	0.804
0.00125	7.0182×10^{-2}	2.0671×10^{-2}	-1.205×10^{-2}	1.277

Table 3: Computation of L_1, L_∞ errors, rate of convergence and CPU time using NSFD1 for $-10 \le x \le 10$, $\gamma = 0.2$, $h = 0.1$, $\beta = 2$ at time, $T = 0.5$.

Time step (k)	L_1 error	L_∞ error	Rate of convergence	CPU (s)
0.005	2.0909×10^{-4}	3.3657×10^{-5}	-	0.502
0.0025	1.0409×10^{-4}	1.8821×10^{-5}	1.006	0.750
0.00125	5.8981×10^{-5}	1.1664×10^{-5}	8.195×10^{-1}	1.107

Table 4: Computation of L_1, L_∞ errors, rate of convergence and CPU time using NSFD2 for $-10 \le x \le 10$, $\gamma = 0.2$, $h = 0.1$, $\beta = 0.5$ at time, $T = 0.5$.

Time step (k)	L_1 error	L_∞ error	Rate of convergence	CPU (s)
0.005	5.9109×10^{-4}	1.3556×10^{-4}	-	0.939
0.0025	3.1703×10^{-4}	7.8179×10^{-5}	8.987×10^{-1}	1.119
0.00125	2.1886×10^{-4}	5.0919×10^{-5}	5.346×10^{-1}	1.535

Table 5: Computation of L_1, L_∞ errors, rate of convergence and CPU time using NSFD2 for $-10 \leq x \leq 10$, $\gamma = 0.2$, $h = 0.1$, $\beta = 1$ at time, $T = 0.5$.

Time step (k)	L_1 error	L_∞ error	Rate of convergence	CPU (s)
0.005	1.7319×10^{-3}	5.4629×10^{-4}	-	0.607
0.0025	1.1015×10^{-3}	3.3108×10^{-4}	6.528×10^{-1}	1.293
0.00125	8.8733×10^{-4}	2.3146×10^{-4}	3.119×10^{-1}	1.245

Table 6: Computation of L_1, L_∞ errors, rate of convergence and CPU time using NSFD2 for $-10 \leq x \leq 10$, $\gamma = 0.2$, $h = 0.1$, $\beta = 2.0$ at time, $T = 0.5$.

Time step (k)	L_1 error	L_∞ error	Rate of convergence	CPU (s)
0.005	1.3466×10^{-4}	2.0390×10^{-5}	-	0.532
0.0025	6.5704×10^{-5}	8.1762×10^{-6}	1.035	0.862
0.00125	3.2908×10^{-5}	4.6063×10^{-6}	9.975×10^{-1}	1.231

Table 7: Computation of L_1 and L_∞ errors, rate of convergence and CPU time using NSFD3 for $-10 \leq x \leq 10$, $\gamma = 0.2$, $h = 0.1$, $\beta = 0.5$ at time, $T = 0.5$.

Time step (k)	L_1 error	L_∞ error	Rate of convergence	CPU (s)
0.005	3.6478×10^{-4}	6.7996×10^{-5}	-	1.063
0.0025	1.8672×10^{-4}	2.4832×10^{-5}	9.661×10^{-1}	1.207
0.00125	1.0268×10^{-4}	2.5881×10^{-5}	8.627×10^{-1}	1.709

Table 8: Computation of L_1 and L_∞ errors, rate of convergence and CPU time using NSFD3 for $-10 \leq x \leq 10$, $\gamma = 0.2$, $h = 0.1$, $\beta = 1$ at time, $T = 0.5$.

Time step (k)	L_1 error	L_∞ error	Rate of convergence	CPU (s)
0.005	1.0211×10^{-3}	1.8991×10^{-4}	-	1.013
0.0025	5.5377×10^{-4}	1.5864×10^{-4}	8.827×10^{-1}	1.443
0.00125	5.4617×10^{-4}	1.6408×10^{-4}	1.993×10^{-2}	2.673

Table 9: Computation of L_1 and L_∞ errors, rate of convergence and CPU time using NSFD3 for $-10 \leq x \leq 10$, $\gamma = 0.2$, $h = 0.1$, $\beta = 2$ at time, $T = 0.5$.

Time step (k)	L_1 error	L_∞ error	Rate of convergence	CPU (s)
0.005	7.5948×10^{-5}	1.1653×10^{-5}	-	0.987
0.0025	3.5996×10^{-5}	4.2640×10^{-6}	1.077	1.274
0.00125	2.2770×10^{-5}	4.1322×10^{-6}	6.607×10^{-1}	1.659

Table 10: Computation of L_1, L_∞ errors, rate of convergence and CPU time using NSFD4 for $-10 \leq x \leq 10$, $\gamma = 0.2$, $h = 0.1$, $\beta = 0.5$ at time, $T = 0.5$.

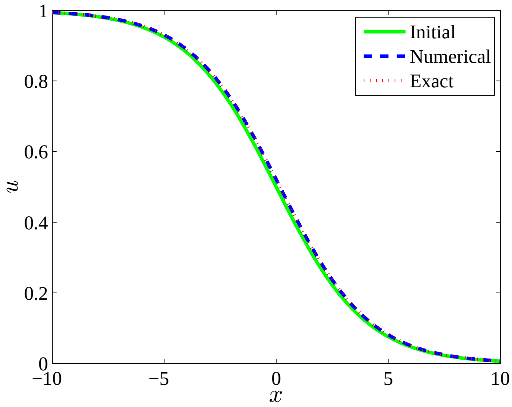
Time step (k)	L_1 error	L_∞ error	Rate of convergence	CPU (s)
0.005	1.9783 $\times 10^{-4}$	3.4931 $\times 10^{-5}$	-	0.479
0.0025	1.0298 $\times 10^{-4}$	2.0513 $\times 10^{-5}$	9.418 $\times 10^{-1}$	1.190
0.00125	1.0260 $\times 10^{-4}$	2.4227 $\times 10^{-5}$	5.333 $\times 10^{-2}$	1.652

Table 11: Computation of L_1, L_∞ errors, rate of convergence and CPU time using NSFD4 for $-10 \leq x \leq 10$, $\gamma = 0.2$, $h = 0.1$, $\beta = 1$ at time, $T = 0.5$.

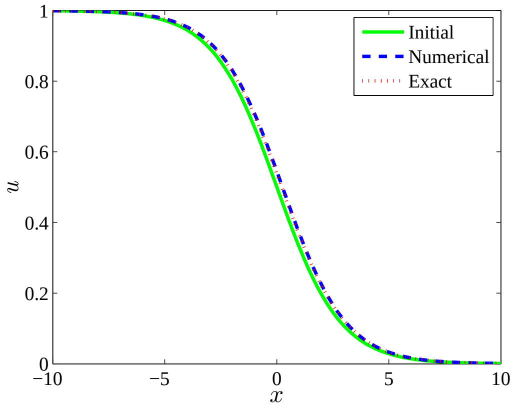
Time step (k)	L_1 error	L_∞ error	Rate of convergence	CPU (s)
0.005	5.7010 $\times 10^{-4}$	1.1881 $\times 10^{-4}$	-	0.551
0.0025	4.4661 $\times 10^{-4}$	1.4439 $\times 10^{-4}$	3.521$\times 10^{-1}$	0.798
0.00125	6.0359 $\times 10^{-4}$	1.5759 $\times 10^{-4}$	-4.345$\times 10^{-1}$	1.237

Table 12: Computation of L_1, L_∞ errors, rate convergence and CPU time using NSFD4 for $-10 \leq x \leq 10$, $\gamma = 0.2$, $h = 0.1$, $\beta = 2$ at time, $T = 0.5$.

15 Figures



(a) $h = 0.1$, $\beta = 0.5$, $k = 0.005$.



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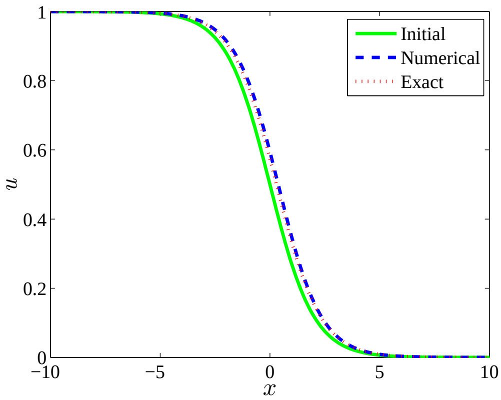
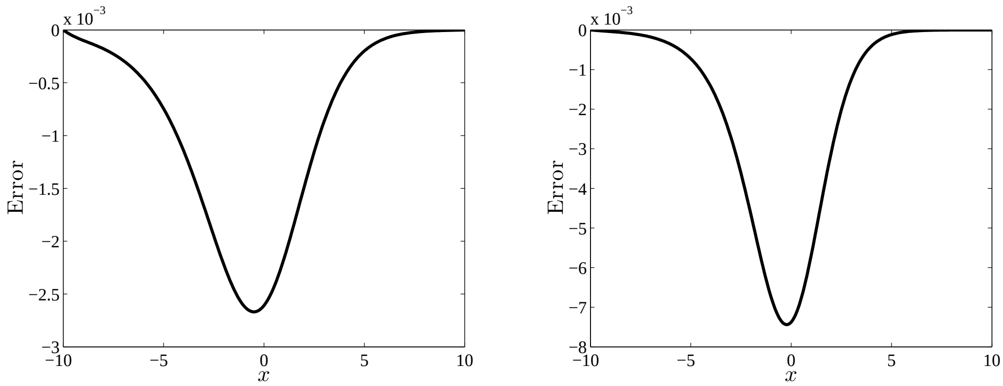


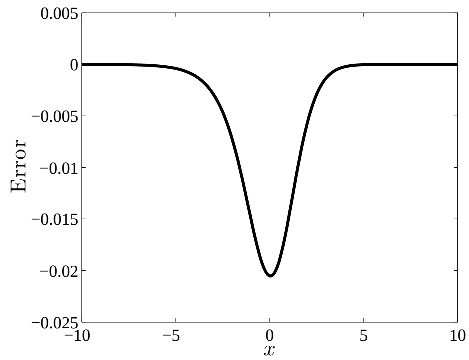
(c) $h = 0.1$, $\beta = 2$, $k = 0.005$.

Figure 1: Plot of u vs x using NSFD1 scheme at time $T = 0.5$, where $x \in [-10, 10]$ for different values of β namely; $0.5\,,\ 1.0\,,\ 2.0$.



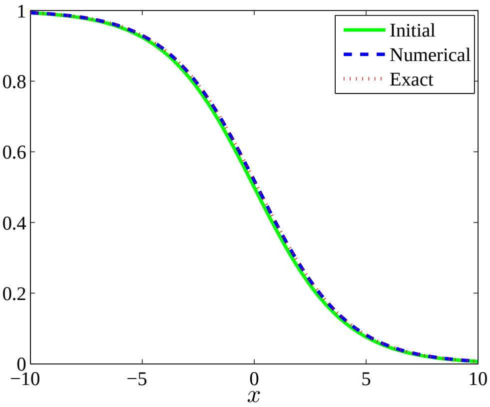
(a) $h = 0.1$, $\beta = 0.5$, $k = 0.005$.

(b) $h = 0.1$, $\beta = 1$, $k = 0.005$.

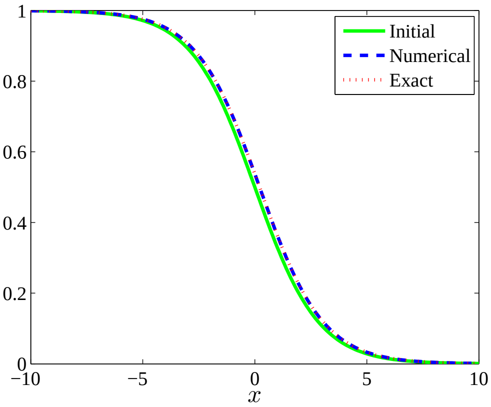


(c) $h = 0.1$, $\beta = 2$, $k = 0.005$.

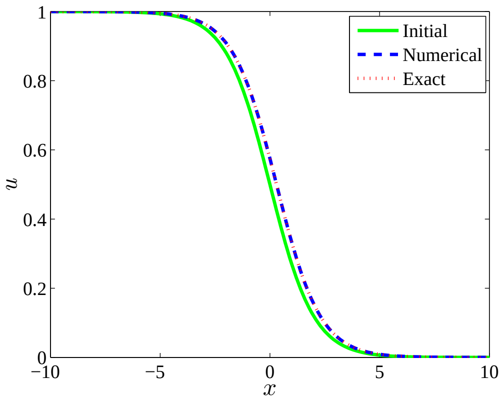
Figure 2: Plot of error vs x using NSFD1 scheme at time $T = 0.5$, where $x \in [-10, 10]$ for different values of β respectively 0.5 , 1.0 , 2.0.



(a) $h = 0.1$, $\beta = 0.5$, $k = 0.005$.

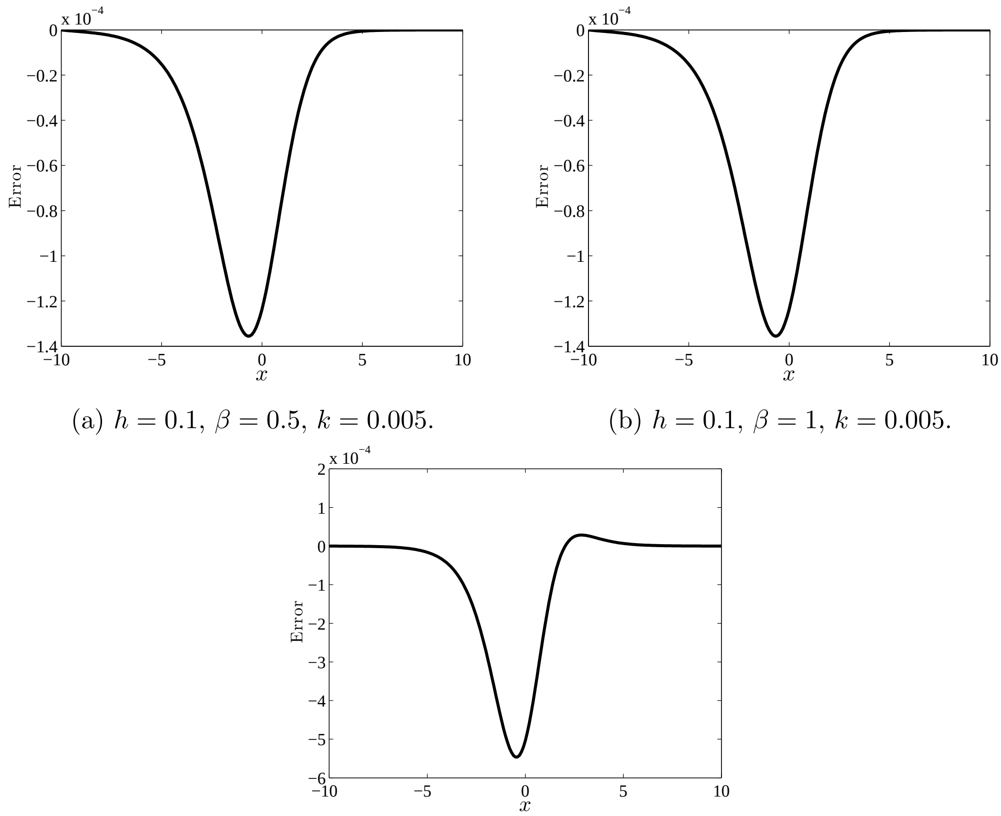


(b) $h = 0.1$, $\beta = 1$, $k = 0.005$.



(c) $h = 0.1$, $\beta = 2$, $k = 0.005$.

Figure 3: Plot of u vs x using NSFD2 scheme at time $T = 0.5$, where $x \in [-10, 10]$ for different values of β namely; $0.5\,,\ 1.0\,,\ 2.0$.

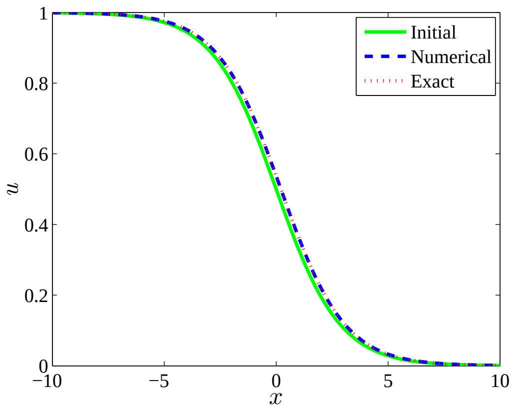


(a) $h = 0.1$, $\beta = 0.5$, $k = 0.005$.

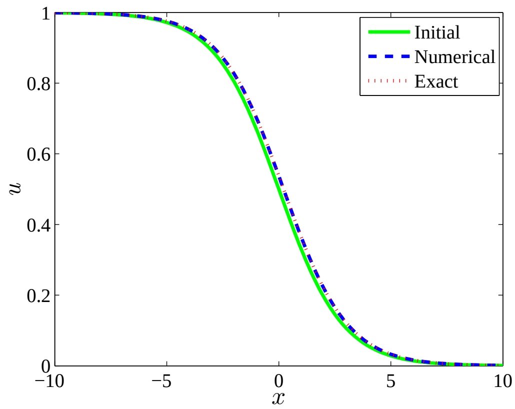
(b) $h = 0.1$, $\beta = 1$, $k = 0.005$.

(c) $h = 0.1$, $\beta = 2$, $k = 0.005$.

Figure 4: Plot of error vs x using NSFD2 scheme at time $T = 0.5$, where $x \in [-10, 10]$ for different values of β respectively 0.5 , 1.0 , 2.0.



(a) $h = 0.1$, $\beta = 0.5$, $k = 0.005$.



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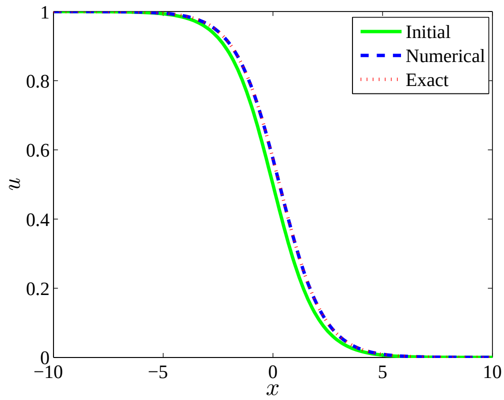
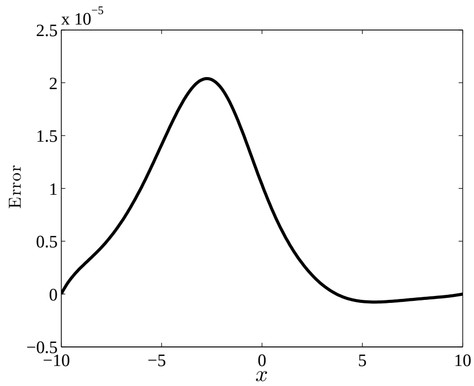
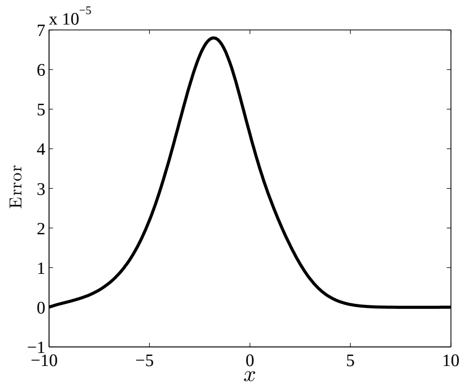


(c) $h = 0.1$, $\beta = 2$, $k = 0.005$.

Figure 5: Plot of u vs x using NSFD3 scheme at time $T = 0.5$, where $x \in [-10, 10]$ for different values of β namely; 0.5 , 1.0 , 2.0.



(a) $h = 0.1$, $\beta = 0.5$, $k = 0.005$.



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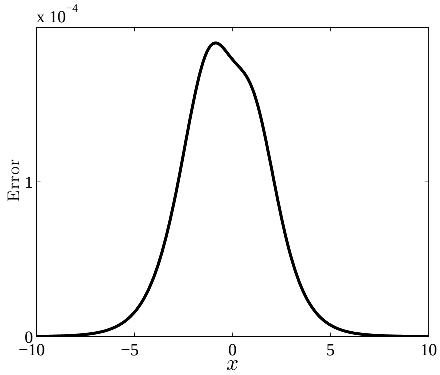
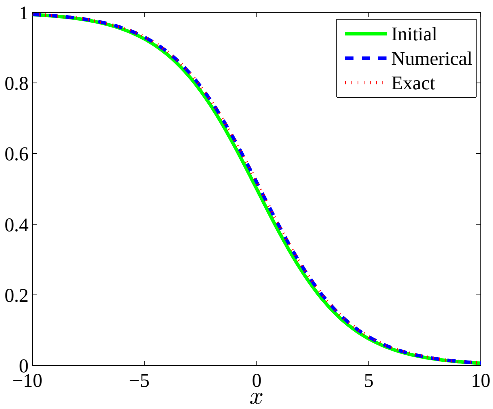
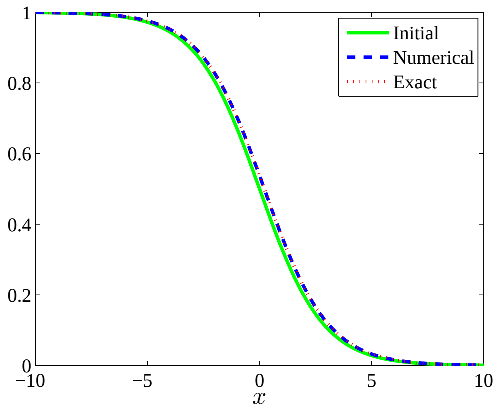


(c) $h = 0.1$, $\beta = 2$, $k = 0.005$.

Figure 6: Plot of error vs x using NSFD3 scheme at time $T = 0.5$, where $x \in [-10, 10]$ for different values of β respectively $0.5\,,\, 1.0\,,\, 2.0$.



(a) $h = 0.1$, $\beta = 0.5$, $k = 0.005$.



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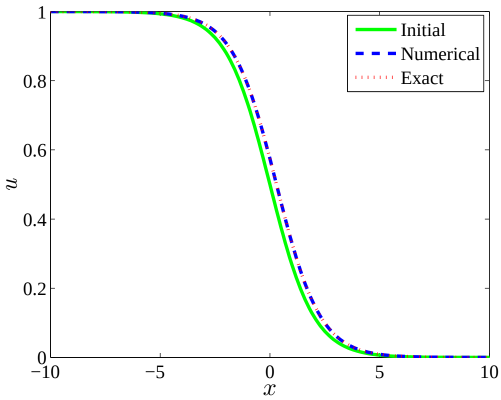
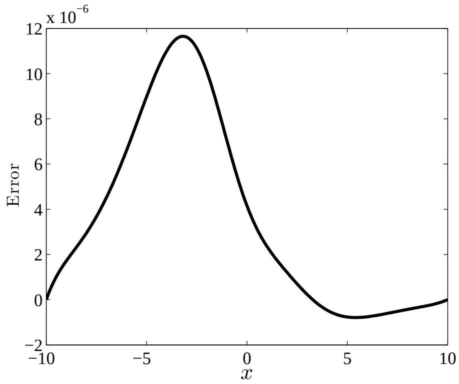
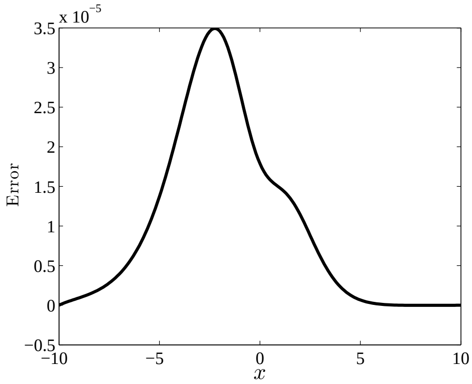


(c) $h = 0.1$, $\beta = 2$, $k = 0.005$.

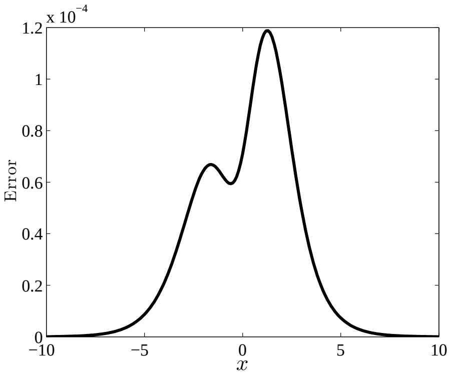
Figure 7: Plot of u vs x using NSFD4 scheme at time $T = 0.5$, where $x \in [-10, 10]$ for different values of β namely; $0.5\,,\ 1.0\,,\ 2.0$.



(a) $h = 0.1$, $\beta = 0.5$, $k = 0.005$.



(b) $h = 0.1$, $\beta = 1$, $k = 0.005$.



(c) $h = 0.1$, $\beta = 2$, $k = 0.005$.

Figure 8: Plot of error vs x using NSFD4 scheme at time $T = 0.5$, where $x \in [-10, 10]$ for different values of β respectively 0.5 , 1.0 , 2.0.