Optimal portfolio performance constrained by tracking error

by

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SUMMARY

OPTIMAL PORTFOLIO PERFORMANCE CONSTRAINED BY TRACKING ERROR

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Maximising investment returns is the primary goal of asset management but managing and mitigating portfolio risk also plays a significant role. Successful active investing requires outperformance of a benchmark through skilful stock selection and market timing, but these bets necessarily foster risk. Active investment managers are constrained by investment mandates such as component asset weight restrictions, prohibited investments (e.g. no fixed income instruments below investment grade) and minimum weights in certain securities (e.g. at least $x\%$ in cash or foreign equities). Such strategies' portfolio risk is measured relative to a benchmark (termed the tracking error (TE)) – usually a market index or fixed weight mix of securities – and investment mandates usually confine TEs to be lower than prescribed values to limit excessive risk taking. The locus of possible portfolio risks and returns, constrained by a TE relative to a benchmark, is an ellipse in return/risk space, and the sign and magnitude of this ellipse's main axis slope varies under different market conditions. How these variations affect portfolio performance is explored for the first time. Changes in main axis slope (magnitude and sign) acts as an early indicator of portfolio performance and could therefore be used as another risk management tool.

The mean-variance framework coupled with the Sharpe ratio identifies optimal portfolios under the passive investment style. Optimal portfolio identification under active investment approaches, where performance is measured relative to a benchmark, is less well-known. Active portfolios subject to TE constraints lie on distorted elliptical frontiers in return/risk space. Identifying optimal active portfolios, however defined, have only recently begun to be explored. The Ω ratio considers both down and upside portfolio potential. Recent work has established a technique to determine optimal Ω ratio portfolios under the passive investment approach. The identification of optimal Ω ratio portfolios is applied to the active arena (i.e. to portfolios constrained by a TE) and it is found that while passive managers should always invest in maximum Ω ratio portfolios, active managers should first establish market conditions (which determine the sign of the main axis slope of the constant TE frontier) and then invest in maximum Sharpe ratio portfolios when this slope is > 0 and maximum Ω ratios when the slope is < 0.

RESEARCH OUTPUTS

Journal articles

- Gunning, W. M. and van Vuuren, G. W. 2019. Exploring the drivers of tracking error constrained portfolio performance. *Cogent Economics*, 7(1): 1 – 15.
- Gunning, W. M. and van Vuuren, G. W. 2020. Optimal Ω-ratio portfolio performance constrained by tracking error. Submitted for publication in *Investment Management and Financial Innovations*.

Conferences

- Gunning, W. M. 2020. Features of tracking error constrained portfolios. *Actuarial Society 2020 Convention*, 7 – 8 October 2020, Cape Town International Convention Centre, Cape Town, South Africa. On behalf of the Quantitative Analytics team, EY, Johannesburg.

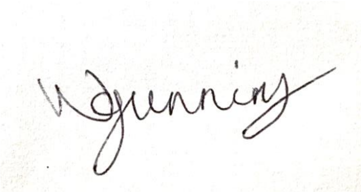
To my grandmother

Judy Hancock

for her unfailing belief in my potential
and encouragement of my trajectory
all my life

DECLARATION

I, Wade Gunning, declare that the dissertation, which I hereby submit for the degree Master of Science at the University of Pretoria, is my own work and has not previously been submitted by me for a degree at this or any other tertiary institution.

Signature: 

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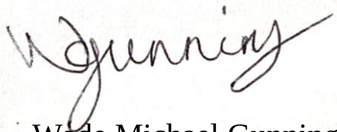
ETHICS STATEMENT

The author, whose name appears on the title page of this dissertation/thesis, has obtained, for the research described in this work, the applicable research ethics approval.

The author declares that he has observed the ethical standards required in terms of the University of Pretoria's Code of Ethics for Researchers and the Polict guidelines for responsible research.

Signature: 

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Term	Definition
Absolute risk and return	Metrics measured relative to a benchmark.
Active management	The frequent purchase and sale of securities with the intention to take advantage of changing market conditions. Actively managed portfolios are assessed relative to a benchmark usually a market index.
Alpha (α)	Gearing of portfolio returns to market or benchmark returns.
Benchmark	An assembly of constituent assets of varying weights against whose collective return a fund manager's portfolio return is assessed.
Beta (β)	The portfolio return generated when the market or benchmark return = 0%
Capital Market Line	The locus of risk-return coordinates described by the risk-free rate of return on the return axis and the tangent portfolio.
CAPM	Capital Asset Pricing Model
CML	Capital Market Line
Constant TE frontier	The locus of risk-return coordinates which trace out all possible returns for any given level of TE.
Constrained portfolios	Agent-imposed restrictions on component asset weights as usually prescribed by mandates. Examples include exclusion or inclusion of certain asset classes or the imposition of upper or lower boundaries.
Efficient frontier	The locus of coordinates in return-risk space where for every level of risk, the maximum return possible is plotted.
Excess return	The difference between the portfolio's its benchmark's return.
Expected return	Either the average of an asset's historical return, or, using expert judgement, an assessment of potential future returns given anticipated economic conditions.
GDP	Gross Domestic Product
Investor utility	A metric which incapsulates the gamut of investor preferences applicable to their portfolios.
Main axis	The locus of the risk-return coordinates joining the minimum variance on the *efficient* frontier and the benchmark.
Monotonically increasing or	In the relevant direction only.

decreasing	
MPT	Modern Portfolio Theory
Omega ratio (Ω)	A performance metric which makes no distributional assumptions of underlying portfolio returns and is particularly effective for return distributions containing extremes.
Optimal portfolios	Portfolios defined to be the "best" in some sense, for example exhibiting maximum return, minimum risk, maximal diversification, or maximum risk-adjusted return.
Passive management	The purchase and retention of securities for medium to long term investment horizons.
PST	Portfolio Selection Theory
Portfolio performance	A suite of metrics used to assess and compare portfolio behaviour.
Risk	A measure of the variability in an asset or portfolio's returns, a proxy for which is the standard deviation (volatility, denoted by σ).
Risk-adjusted return	Also known as the Sharpe ratio, this is the quotient of the difference between portfolio return and risk-free rate and the portfolio volatility.
Tangent portfolio	The portfolio which exhibits the maximum Sharpe ratio or risk adjusted return. Note that such a portfolio exists on the efficient frontier and the constant TE frontier.
TE	Tracking error
TE frontier	The locus of risk-return coordinates which trace out the maximum return possible for any given level of TE.
Tracking error	A measure of relative risk, defined as the standard deviation of the difference between portfolio and benchmark returns.
Variance	σ^2

CHAPTER 1 GENERAL INTRODUCTION

1.1 BACKGROUND AND SCOPE

1.1.1 Theme 1 – The main-axis slope of the constant TE frontier

The panic induced by the 2020 COVID-19 pandemic led to substantial sell offs of securities leading to unprecedented declines of global market indices, commodity prices, Gross Domestic Product (GDP), interest rates and consumer confidence. While the effects will be long-lasting and painful, this turbulent world offers plentiful investment opportunities (Fernandes, 2020).

There are two styles which characterise investment markets: passive and active management styles. For the passive approach, managers buy and retain securities (or portions of market indices) for relatively long periods because they believe that outperforming the market is not feasible, so the most sensible strategy is to reduce transactions (and fees) by minimising costly transactions and be content with the broad market's returns. For the latter, managers assume that through skilful selection and timing of sales and purchases, market outperformance is possible. The emphasis in this investment style is on relative performance, so skill (outperformance) is assessed relative to a prescribed, mandated benchmark. Risk is also assessed relative to the benchmark's risk (Fahling, Steurer & Sauer, 2019).

Markowitz's (1956) efficient frontier formulation has directed passive investment research for almost seven decades and the literature on associated portfolio optimisation is considerable. Sharpe (1964) introduced the concept of a maximum risk-adjusted return portfolio for 'optimal' performance, called the tangent portfolio. Active investment strategies involve more complex structures, because here, constraints restrict the investable universe. Benchmark constituents, the size of TE (portfolio risk relative to the benchmark) and asset weight floors and caps all contribute to additional complexity. Roll (1992) initiated research into TE constrained portfolio behaviour and developed the framework for describing an efficient frontier in risk/return space constrained by various levels of TE. Jorion (1992, 2003) extended this work and developed the details for the constant TE frontier: i.e. a locus of points in risk/return space which embraces the universe of risk/return combinations – relative to the benchmark's risk and return.

TE is an active risk measure (defined as the standard deviation of the difference between portfolio and benchmark returns) that reflects a portfolio manager’s decisions to deviate from the weights of a benchmark's positions with the aim of outperforming the benchmark. The inevitable risk introduced by this deviation is the TE, although it is not generally used as a risk metric to assess portfolio manager performance in isolation. Rather, other measures are used in combination with the TE for this purpose, such as Value at Risk, the Information ratio, etc. Fund managers determine the investment policy (i.e. its risk-return profile, outperformance targets, etc.) which in turn determines the TE. Thomas, Rottschafer and Zvingelis (2013) outline several causes of TE (fees, transaction costs, taxes, factor tilts, cash management and market volatility).

In mean/variance space, the universe of possible portfolios constrained by a TE is an ellipse (and in risk/return space, it is a distorted ellipse). The ellipse's orientation (designated by the sign and magnitude of the 'main axis' slope) changes through time as economic conditions change. The way the main axis slope changes under different economic conditions, is explored here for South African (SA) stocks (an emerging third world economy) for the first time. The relevant mathematics required to calculate both the sign and magnitude of the main axis slope is detailed and the way this slope changes as time evolves and market conditions change assessed. Results indicate that when the main axis slope changes sign sharply, a prolonged downturn in economic conditions inevitably follows. The effect is subtle, however. Forecasting economic conditions (and hence investment strategy) may depend on slope sign changes, but this depends on the *direction* of the change (i.e. $+ve$ to $-ve$), the magnitude of the slope before the reversal and the speed and size of the reversal. The combinations are persistent and robustly predict near economic conditions with reasonable accuracy. The way the main axis slope (and magnitude) moves through time triggers novel investment strategies for active fund managers.

1.1.2 Theme 2 – The optimal Ω ratio under a TE constraint

Investment styles follow one of two broad approaches: active and passive. Active fund managers trade frequently and engage energetically with the market. Successful active managers identify not only high-performing assets, but also time trades to extract maximal performance, buying when prices are low and selling when they are high. Skill in this space is usually measured relative to a benchmark, usually a market index or an assembly of similar securities with constraints on portfolio weights, asset quality and acceptable risk. Passive managers select and purchase desired securities and hold these for investment horizons which span periods of economic booms and busts. Such managers' proficiency is measured on an absolute basis, they minimise transaction fees and aver that "good" securities outperform in the long run.

Both styles have pros and cons, and the ebb and flow of economic activity often dictates investor style selection: passive usually in stable markets and active in volatile. Events such as the 2020 COVID-19 pandemic which severely shocked global markets, serve to emphasise the importance of agile, active investing. Managers capable and eager to quickly dispose of airline, oil or tourism-related stocks for example, avoided the worst of the downturn and significantly outperformed less-nimble investments.

Modern portfolio theory led to the design and application of the widely-used efficient frontier, which plots – in return-risk space – the locus of portfolios whose arrangement of constituent security weights generates maximal returns at each specified risk level. Sharpe's work identified the optimal portfolio on this frontier: one whose excess return (usually over the risk-free rate) per unit of risk taken to achieve that return, was maximised. This framework of asset selection is ideally suited to the passive investment style. Identifying an optimal portfolio using this construction implies the belief that markets are relatively static and that buying and holding the optimal portfolio will eventually lead to the desired risk/return characteristics.

Active investment strategies require more complex structures. Portfolios whose performance and risk are measured relative to a benchmark follow a different locus of possibilities in return/risk space. Jorion (2003) demonstrated that such portfolios occupy a distorted ellipse in this space – rather than the efficient frontier's hyperbola for absolute risk and return. The dimensions and orientation of this ellipse is governed by many factors, including the variance-covariance matrix of underlying security returns, benchmark weights in the permissible universe of investable assets, constituent portfolio weights relative to the benchmark and the size of the TE. The greater the deviation from benchmark weights, the higher the possibility for outperforming (or underperforming) that benchmark (and the higher the TE). Active managers – to limit excessive risk-taking – are often constrained to not exceed prescribed TEs. There are profound differences in the way portfolio risk and return evolve and are measured under active and passive investment styles. Standard performance metrics, in common use for passive portfolios, require complex reformulation and behave in unfamiliar ways in active space.

The Ω ratio, a performance metric which makes no distributional assumptions about asset returns, is popular amongst passive investors, but determining the asset allocation to generate an *optimal* Ω ratio portfolio eluded researchers for years. The definition of the Ω ratio imbues it with non-convex properties which do not yield to standard optimisation techniques. Recently, Kapsos, Zymler, Christofides and Rustem (2011) accomplished this feat using linear programming, but their approach has not subsequently been applied to *active* portfolios, i.e. those constrained by TEs. Maximum Ω ratio portfolios were identified on the constant TE frontier under different market conditions and these portfolios' performance were compared over time to that of universal (unconstrained) Ω ratio portfolios.

1.2 LITERATURE REVIEW

In the following sections, a condensed, specific and targeted literature review is presented for each of the pertinent fields of study. Chapter 2 provides a comprehensive review of current available literature on these topics.

1.2.1 Theme 1 – The main-axis slope of the constant TE frontier

Markowitz (1952) formulated the mean/variance framework which indicated to investors their coordinates in a return/risk plane. An efficient set of portfolios (i.e. those with a maximum return at a given absolute risk level) trace out a hyperbolic curve in this return/risk space. Sharpe (1964) established the maximum risk-adjusted portfolio return – now known as the Sharpe ratio. This ratio measures the quotient of excess return (over the risk-free rate) and portfolio risk (defined by its volatility) given as

$$SR = \frac{\mu_P - r_f}{\sigma_P}.$$

where μ_P is the portfolio annual return, r_f is the annualised risk-free rate and σ_P is the portfolio annualised volatility (or risk).

Two investment styles dominate the market: passive management (known as buy-and-hold, which is generally cheaper) and active management (strategic stock selection and timing, generally more expensive). Both passive and active fund managers are evaluated and remunerated depending on their propensity to outperform the broad market. The former accomplishes this by taking and holding small bets relative to market indices, whilst the latter attempts to generate outperformance of a prescribed benchmark (usually a market index or a – sometimes arbitrary – combination of securities) by taking a combination of bets and timing market movements. Active managers are often also constrained by a TE, which may not be exceeded under the mandated investment contract (Riccetti, 2010).

Passive fund managers generally aim for the maximum Sharpe ratio (tangent) portfolio on the efficient frontier (Markowitz, 1952) although there are several other possibilities in which passive fund managers may explore, such as the minimum variance portfolio (for highly risk-averse investors), the maximum diversification or minimum intra-correlation portfolios (for risk averse investors), etc. Active manager performance is evaluated using several criteria, one of which is the TE (Menchero, & Hu, 2006).

Roll (1992) set out the description of the maximum return portfolio, relative to a benchmark, for a given TE, a formulation that describes a hyperbolic curve, much like the efficient frontier (but shifted to the right – i.e. riskier), in risk-return space. Absolute portfolio risk is neglected in Roll's (1992) approach, so these portfolios are not optimal in a mean-variance sense and they are always riskier than the benchmark. The problem of mean-variance maximisation under a TE constraint was reconsidered by Bertrand, Prigent and Sobotka (2001) who reintroduced both absolute and relative risk (i.e. TE) aversion into their optimisation program. A range of optimisation and holding periods while ignoring transaction cost constraints was considered by Larsen & Resnick (2001). Frequent rebalancing was necessary to maintain control over total risk (though not TE risk) when portfolios are actively managed (see also Plaxco & Arnott, 2002), but this did not always lead to optimal portfolios (El-Hassan & Kofman, 2003). Jorion (2003) tackled these and other problems and established the shape of the constant TE portfolio, an ellipse in the traditional mean-variance plane, but not in mean/risk plane (where it is a distorted ellipse with no bi-axial symmetry).

When TE could vary and risk aversion was fixed (rather than only considering constant TE frontier-constrained portfolios) Bertrand (2009) found that the resulting optimal portfolios exhibited many desirable properties, such as having the same information ratio. The IR decomposition proposed by Menchero & Hu (2006) was also explored evaluated by Bertrand (2010) using risk-adjusted performance attribution previously developed by Bertrand (2005).

The literature was largely silent on *absolute* portfolio risk in the active management arena, until work by Maxwell, Daly, Thomson & van Vuuren (2018) unveiled a way to determine the asset weights to construct the TE-constrained tangent portfolio – effectively the maximum Sharpe ratio (tangent) portfolio on the constant TE frontier. This approach produced portfolios with a lower risk, but greater return than the agent's benchmark whilst satisfying the TE constraint and maximising the Sharpe ratio (Jansen & van Dijk, 2002).

The efficient frontier is shown in Figure 1.1 as well as the minimum variance and tangent portfolios (the latter at the intersection of the capital market line (CML) and the efficient frontier (in this example, $r_f = 7\%$)).

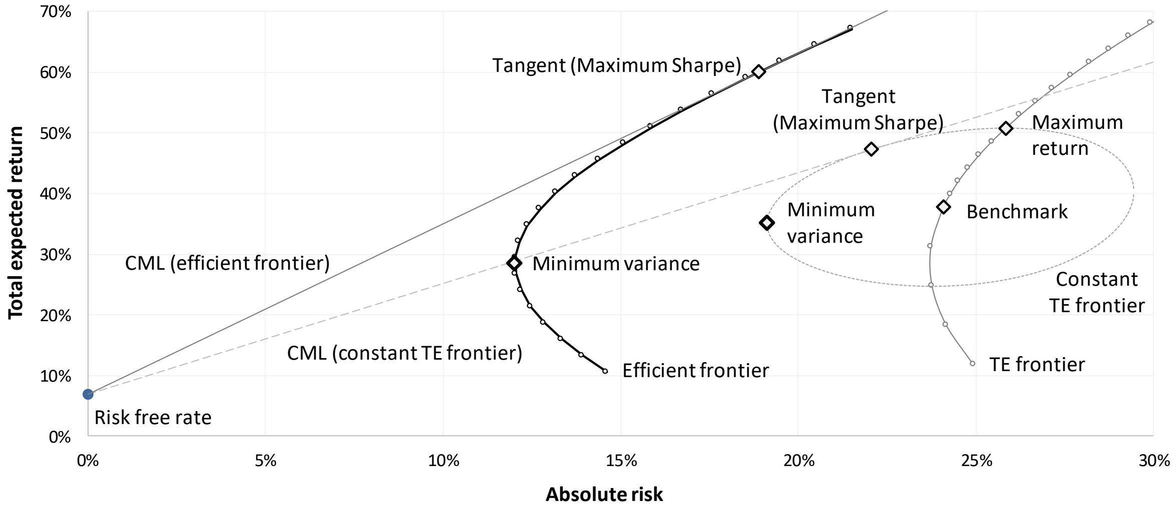

Figure 1.1. Positions of relevant frontiers and portfolios in the risk/return plane.

Source: Author calculations.

The riskier TE frontier appears to the right of the efficient frontier: this is the locus of coordinates on the risk/return plane with the highest return at increasing values of TE relative to a benchmark. Where $TE = 0\%$, the TE frontier necessarily intersects the benchmark. Finally, also shown in Figure 1.1 is the distorted ellipse of the *constant* TE frontier (here, $TE = 6\%$), with the minimum variance, maximum Sharpe and maximum return portfolios.

The tangent portfolio is at the intersection of the constant TE frontier and its CML (Maxwell, et al, 2018). The maximum return portfolio on the constant TE frontier is the maximum return coordinate on the constant TE frontier for every value of TE. Note the slightly positive slope of the constant TE frontier ellipse in this configuration.

The main axis slope, S_{MA} (defined as the slope of the line joining the *efficient frontier's* minimum variance portfolio and the *benchmark portfolio*) and its relationship with TE constrained portfolio performance is investigated here for the first time.

1.2.2 Theme 2 – The optimal Ω ratio under a TE constraint

Modern portfolio theory (MPT) is a well-established and widely implemented paradigm which asserts that investors select portfolios based upon their level of risk aversion. Set out by Markowitz (1952) the framework gives rise to a set of *efficient* portfolios – those characterised by the maximum possible return at any given risk level – which trace out a boundary in return/risk space known as the efficient frontier. The literature is replete with improvements and adaptations, augmentations, and variations of MPT. Markowitz, Schirripa & Tecotzky (1999), for example, showed how – by pooling assets – investors could collectively provide constituent members higher expected returns, for given risks, than individuals could generate alone. These results were confirmed and extended by Kwan (2003) and more recent innovations are provided by Calvo, Ivorra, & Liern (2012).

Sharpe (1966) identified an optimal portfolio on the efficient frontier, the highest *risk-adjusted* portfolio return, measured as the quotient of portfolio return in excess of the risk-free rate and the portfolio's risk (defined by its volatility), $SR = (\mu_P - r_f)/\sigma_P$ where SR is the Sharpe ratio, μ_P is the portfolio annual return, r_f is the annualised risk-free rate and σ_P is the annualised portfolio volatility (or risk). This portfolio represents the single intersection point of the capital market line (CML) hinged at the risk-free rate on the return axis and the frontier, i.e. where the CML is tangent to the frontier. Despite the many assumptions embedded in the determination of this optimised portfolio (e.g. Muralidhar, 2015), it remains a popular metric (Sharpe, 1994; Lo, 2012 and Qi, Rekkas & Wong, 2018).

MPT and the tangent portfolio reflect the passive portfolio management style in which assets are bought and held for "long" investment horizons, usually several months or years. This style enjoys the benefits of low trading costs and a through-the-cycle view of market performance resting on the assumption that superior assets will outperform the broader market even though influenced by it. The active management style (strategic stock selection and timing) while more expensive because of trading expenses, is dominated by fund managers who purchase and sell securities when prevailing conditions signal danger or opportunity (Ammann & Zimmermann, 2001). This style has been eclipsed in recent times by index (passive) investing, but this has given rise to systemic problems (Anadu, Kruttli, McCabe, Osambela, & Shin, 2018) and the alleged inferior performance of the active investment style has been challenged by Cremers, Fulkerson & Riley (2019) whose research found evidence that 'conventional wisdom' had been unfairly critical of the value of active management which continues to outperform the passive style (Berk & van Binsbergen, 2015; Pedersen, 2018 and Dolvin, Fulkerson & Krukover, 2018).

Active fund performance is assessed relative to a benchmark, commonly a market index or a selection of securities constrained by investor preferences (Clarke, de Silva & Thorley, 2002). Superior active funds should outperform the returns generated by the benchmark and simultaneously not exceed a prescribed, relative risk measure: the TE defined as the standard deviation of the differences between portfolio and fund returns (Wu & Jakshoj, 2011).

Using a mean/variance (Markowitz) framework, Roll (1992) established a description of portfolio optimisation relative to a benchmark for a given TE. These optimal, but constrained, portfolios trace out a frontier much like the efficient frontier (but shifted to the right – i.e. lower potential returns with higher risk), in return/risk space. Bertrand, Prigent & Sobotka (2001) and Larsen & Resnick (2001) reconsidered the problem of mean-variance maximisation under TE constraints, but Jorion (2003) was first to mathematically formulate the constant TE frontier, a distorted ellipse in return/risk space comprising TE-constrained portfolio return/risk coordinates. Stowe (2014) provides a recent, comprehensive treatise on the relevant mathematics governing TE constrained portfolios.

Maxwell, et al., (2018) and Maxwell & van Vuuren (2019) adapted and extended Jorion's (2003) approach to TE constrained portfolio optimisation by establishing a technique which identified the tangent portfolio on the constant TE frontier. Analogous to the tangent portfolio on the *efficient* frontier, this portfolio (where the analogous CML – also hinged at the risk-free rate on the return axis – is tangent to the constant TE frontier) represents the maximal risk-adjusted return portfolio constrained by a given TE. Daly, Maxwell & van Vuuren (2018) explored α, β and investor utility behaviour for TE constrained portfolios and Evans & van Vuuren (2019) investigated several portfolio performance metrics on the constant TE frontier. Gunning & van Vuuren (2019) surveyed the mechanisms which drive constrained portfolio performance by examining the influence of macroeconomic conditions on the shape of the TE frontier (certain market conditions alter the slope of main axis of the constant TE frontier ellipse (often from > 0 to < 0) which profoundly influences TE constrained portfolio performance).

The Ω ratio is a portfolio performance measure which captures both portfolio down and upside potential while remaining consistent with utility maximisation (Keating & Shadwick, 2002). Although now widely used, an optimal Ω ratio portfolio long eluded practitioners because it is a non-convex function, which does not lend itself to standard optimisation techniques. Kane, Bartholemew-Biggs, Cross, Dewar (2005) explored this problem empirically using simulated returns from a portfolio comprising three assets. By changing asset weights to maximise the Ω value and assuming no short selling, several local solutions were found. Extending this work to ten (real) assets and employing a global optimisation technique (not disclosed), Kane et al., (2005) identified maximal Ω portfolios and compared their performance with portfolios produced using MPT (i.e. tangent portfolios). The results showed that the allocation of weights for portfolios' constituent assets were considerably different from those based on risk minimisation. Passow (2004) and Gilli, Schumann, Di Tollo, & Cabel (2008) made laudable attempts to resolve the problem of Ω portfolio optimality, but their solutions were heuristic (which did not guarantee the accurate identification of the global optimum) and their threshold accepting methods were numerically unstable, requiring complicated fine tuning of the underlying parameters (Mausser, Saunders & Seco, 2006 and Theron & van Vuuren, 2018).

Kapsos, et al. (2011), using the Ω ratio quasi-concave property (which permits its transformation into a linear program), overcame the non-convex function problem, and established an exact formulation. This solution is a direct analogue to the mean-variance framework and its associated Sharpe ratio maximisation. Kapsos, et al's (2011) work is applicable to the passive investment style: to date, no attempts have been made to explore Ω optimal portfolios which are also subject to TE constraints.

This chapter combines several strands of related research. The work described above relating to TE constrained portfolios is described, Kapsos et al's (2011) optimal Ω ratio solutions are adapted to accommodate TE constrained portfolios and some properties of optimal, TE *constrained* Ω ratio portfolio performance are explored.

1.3 DISSERTATION RATIONALE

The formulation of the constant TE frontier in return/risk space is not novel. Jorion's (2003) work set out the mathematical description of its construction and some work has been undertaken subsequently. High active manager fees and poor performance in the 1980s led to the decline in popularity of active funds. At the same time, increased automation fuelled a boom for passive investments which had come to rely less and less on human input and more and more on machine-based pattern recognition and rapid activation times. The ebb and flow of market sentiments has recently (2018) seen the tide turn once more toward active investments: in a low-yield world, the search for relative outperformance has once again witnessed renewed interest. Maxwell et al, (2018) identified – mathematically – a maximum Sharpe ratio (tangent) portfolio which satisfied the usual active constraints (e.g. portfolio risk must be less than a prescribed TE and asset weight restrictions must be adhered to) while also generating a portfolio with the highest risk-adjusted return at a given level of risk. This precipitated much work in the area, some of it extended and developed further here.

How the orientation of the constant TE ellipse (by which is meant, the size and sign of the ellipse's main axis slope) changes with different market conditions could have consequences for the active investment approach, particularly given the market volatility experienced under increasingly common extreme events (such as the 2008 financial crisis and the 2020 COVID-19 pandemic).

As a performance metric, the popular Ω ratio makes no assumptions about the assets' return distribution. A maximum Ω ratio portfolio has been identified but its effectiveness has only been applied and tested under the passive approach. Implementing the maximal Ω ratio in a TE-constrained portfolio setting is of interest to active managers who wish to understand and exploit the behaviour of such constrained portfolios.

1.4 RESEARCH QUESTIONS

Under the active investment style, how does the orientation of the constant TE ellipse's main axis influence the behaviour of portfolios on the efficient (constrained) set? For the two optimal (maximum return and maximum Sharpe ratio) portfolios on the efficient set, how does their risk/return profile change with changing market conditions and does the size and magnitude of the long axis slope influence these profiles?

How is the unconstrained maximum Ω ratio portfolio different from the TE-constrained maximum Ω ratio portfolio? How does the TE-constrained maximum Ω ratio portfolio's return/risk profile change through time as market conditions improve and deteriorate?

1.5 DISSERTATION STRUCTURE

This dissertation is structured as follows: Chapter 2 explores how variations in the TE-constrained frontier in active return/risk space affect portfolio performance. Changes in main axis slope (magnitude and sign) are found to act as early indicators of portfolio performance and could therefore augment the existing array of management tools. Chapter 3 applies the identification of optimal Ω ratio portfolios to the active arena (i.e. to portfolios constrained by a TE) and explores the influence of market conditions on this (and other maximal portfolios) through time.

Chapter 4 summarises the findings of the entire study and suggests future research possibilities. References are included at the end of the dissertation.

1.6 GENERAL OBJECTIVES

General objectives of this research include:

- interrogate and refine the principles governing the development and implementation of the constrained TE frontier, and explore the consequences of changing market conditions on this framework both mathematically and empirically
- establish – through back-testing – whether portfolios on the TE-constrained efficient set are influenced by the ellipse's main axis slope magnitude and sign

- explore the characteristics of the unconstrained maximum Ω ratio portfolio and compare its characteristics with those of the TE-constrained maximum Ω ratio portfolio. how do these differ, how are they similar, and do they provide information on early warnings in the market to impending crises?

1.7 SPECIFIC OBJECTIVES

Specific objectives of this research are:

- using historical share price data and under the active investment style, establish whether the main axis slope of the constant TE frontier influences the performance of liquid, equity-based, TE-constrained portfolios under different market conditions and
- using historical share price data and under the active investment style, establish the principal return/risk profile differences between constrained and unconstrained maximum Ω portfolios under different market conditions.

1.8 RESEARCH DESIGN

The research design of this dissertation follows in the outline below:

Pose research problem statement and question: Portfolio optimality (in whatever form) as well as constrained portfolio optimality are complex pursuits. How may component asset weights be determined for actively managed portfolios subject to tracking-error constraints?

Critical literature review: A critical literature review is conducted by consulting and considering existing literature. Adjustments to existing risk management procedures, techniques and methodologies to solve problems are documented and highlighted in the literature studies. The existing literature for this research theme is copious. Where an entirely new approach to risk practices is required, the literature was less obliging, but this was not a constraint in this study, because popular, well-established mathematical techniques are almost always available for research endeavours and again, abundant literature exists to address and divulge these.

Theory building/adapting/testing: adaptation of existing financial tools and mathematical techniques for practical implementation enjoys rich precedent. The bulk of the results reported in this dissertation were from empirical analyses of simulated data derived using both known and innovative risk metrics.

Data collection: Data used were either simulated or from third-party, internet-based, electronic databases (e.g. McGregor BFA, Opendata and Bloomberg™ for historic index prices). Adequate data were available for all the chapters, so sample error was minimised. Data in this study comprised several published, historical time series, available from both proprietary (e.g. Bloomberg™) and non-proprietary sources (e.g. internet databases).

Conceptual development and empirical investigation: This research is intended to provide robust, but practical, solutions for use by investors and traders. As a direct result, the primary source of analytical work was Microsoft Excel™ since this tool is used by most financial institutions. These spreadsheet-based models use visual basic programming language (a flexible, functional desktop tool available to all quantitative analysts and risk managers) to develop macros to replace onerous and repetitive computing tasks. The empirical study comprises the practical implementation of the research method, using techniques and models developed in Microsoft Excel™.

The variables employed are data assembled from various historical time series. All data are available in the public domain. Some pricing data were simulated for illustration.

Illustrate and reason findings: Having analysed the data, obtained meaningful results, and displayed these appropriately, the findings were written up into article-style reports for peer review and publication. The articles have been published as detailed in Table 1.2.

Further work: To complement major findings of and ensure the continuation of much needed work not addressed in this dissertation, future work regarding the many consequences of optimal constrained portfolios is proposed for active fund managers and academics.

1.8.1 Data

Data requirements, frequency and source are shown in Table 1.1.

Table 1.1: Data requirements, frequency, and source.

#	Topic	Data required	Frequency	Sources
1	Exploring the drivers of tracking error constrained portfolio performance	Historical asset returns Variance/covariance matrices Calculated portfolio weights	Monthly	Bloomberg. S&P Capital IQ, Open-data, non-proprietary internet databases
2	Optimal Ω ratio portfolio performance constrained by tracking error			

1.8.2 Research output

Research output is shown in Table 1.2.

Table 1.2: Research output.

#	Topic	Mathematics	Research methodology
1	Gunning, W. M. and van Vuuren, G. W. 2019. Exploring the drivers of tracking error constrained portfolio performance. *Cogent Economics*, 7(1): 1 – 15.	Proprietary Microsoft Excel models Calculus (differentiation) Linear algebra and Lagrangian dynamics	Portfolio optimisation approaches Asset selection under prescribed portfolio constraints
2	Gunning, W. M. and van Vuuren, G. W. 2020. Optimal Ω-ratio portfolio performance constrained by tracking error. Submitted for publication in *Financial Innovations*.		

CHAPTER 2 LITERATURE STUDY

Successful portfolio managers are assessed based on their skill at maximising returns – a skill which underlies Modern Portfolio Theory, or Portfolio Selection Theory (PST). The main objective of PST (as well as maximising returns) is to determine how to optimally allocate assets in a portfolio, knowing an investor's risk tolerance. This 'optimal' allocation produces portfolios of chosen assets whose risk levels are commensurate with the investors level of risk aversion, thereby maximising investor utility (Ghosh & Mahanti, 2014). A trade-off between risk and return (mean/variance trade-off) is thus different for different investors, but Markowitz (1952) argued that, although this may be true, all investor preferences lie on a curve of 'efficient portfolios' called the *efficient frontier* which consists of diversified (efficient) portfolios having the lowest risk for any given level of return or, equivalently, the highest return for any given risk level,. This assembly of risk/return combinations generates the frontier.

Portfolio optimisation emerges as a single – but important – phase of the process of investment management which embraces the general procedure adopted by portfolio managers in determining 'optimal' investor portfolios. Other steps require assessing the risk preference/profile of investors and any specific investment objectives, constraints, permitted investment allocations in different asset classes or sectors, the investment strategy (value/growth, passive/active), and the performance measures and assessors to be used (Fabozzi & Markowitz, 2011). These constitute portfolio planning and optimisation models used to determine the optimal portfolios.

Markowitz's (1952) portfolio optimisation problem comprises two related criteria: expected return (mean) and risk (standard deviation) – the latter used as a proxy for the return volatility. For a single investment period formulation, Markowitz (1952) posited that investors distribute capital amongst several assets and then, over the course of the investment period, the combined return rate (assumed random) generated by the portfolio generates higher or lower capital value at the end of the period (relative to the original investment amount). Subsequent work (e.g. Mansini, Orgczak & Speranza,., 2014) successfully extended Markowitz's (1952) model to embrace multiple consecutive periods, so the concept remains the foundation upon which modern portfolio theory is based.

The portfolio selection process is divided into two main phases according to Markowitz (1952). The first phase links experience and observation to forecast the future performance of the constituent assets, and the second uses these forecasts to select the most suitable portfolio. The successful completion of the first of these stages relies on the fund manager's skill, the accuracy of the forecasting models and the formulation of estimation error. Considerable research has been conducted in this arena, but this work is beyond the scope of this dissertation. Markowitz (1952) concluded that investors should place roughly equal emphasis on return and risk in the selection of investment portfolios. In doing so, investors are considerably more likely to select utility maximising portfolios, that is, ones which are closely aligned with their preferences.

Utility maximisation represents the desired investor outcome, and represents a subset of Utility Theory, an extensively researched field of economics. Investor utility is the total satisfaction received from the consumption of capital (by whatever means). It is defined by a utility function, which assigns numeric values to all possible choices faced by the investor where the higher the numeric value of a choice, the greater the satisfaction derived from it (Fabozzi & Markowitz, 2011). PST seeks to identify – using indifference curves – the portfolio which will generate the maximum possible utility, constrained by the investor's preferences and requirements. An indifference curve collates an ensemble of choices (in this case portfolios with different risk/return combinations) which provide investors with the same utility level. Investors should, therefore, be indifferent to the allocated asset combination (and resulting portfolios). These indifference curves are usually traced out in the same (mean/variance) space as the efficient frontier, which allows portfolio managers to select optimal portfolios at the tangent point where the maximum indifference curve meets the efficient frontier (Fabozzi & Markowitz, 2011). This unique portfolio is not only efficient but is optimal for the investor in that it uniquely satisfies their risk profile/preferences (Larsen & Resnick, 2001).

Markowitz (1952) extended his 1952 work and restructured the mean variance optimisation formulation into a quadratic programming model by balancing portfolio risk and return (Markowitz, 1959). This presents portfolio managers with a quantitative tool to facilitate investment allocation decisions (Ghosh & Mahanti, 2014). This optimal allocation of holdings/investments is determined through solutions generated by this quadratic programming model (Ghosh & Mahanti, 2014). This mean variance model has been altered in various ways since its inception, namely; the single index/market model which ignores the covariance between asset returns, the CAPM (Capital Asset Pricing Model) as an extension of the single index model (considering the returns of securities to depend on the market index and not the covariance between asset pairs), and the multiple period Mean Variance model (see, for example, Clarke, de Silva & Thorley, 2002).

Using Markowitz (1952, 1959), Ghosh and Mahanti (2014) averred that an important implication of Modern Portfolio Theory was that in the asset selection process, the risk and return should not be considered in isolation but rather in conjunction with the correlation of that asset's returns with other constituents' returns. This co-movement of asset returns, if negligible or in the opposite direction, can reduce portfolio risk (volatility of returns) significantly, whilst still generating the same level of portfolio return. The process of adding uncorrelated or negatively correlated assets to reduce overall portfolio risk is known as diversification (Clarke, de Silva & Thorley, 2002).

Having selected the optimal portfolio, its performance (and hence the manager's) is measured and evaluated. This is a fundamental consideration in portfolio management and several performance measures and attribution models have been proposed. Fama's (1972) Decomposition of Total Return identifies the sources of the portfolio's return, indicating how much of the return can be attributed to the manager and how that return was earned. Other important portfolio measures include: the Treynor ratio (which measures the ratio of excess returns, above the risk-free rate, to the systematic risk, β) which indicates manager's timing skills; the Jensen index (an absolute measure which estimates manager ability to forecast returns and diversify the portfolio to protect it from excessive risk (Ghosh & Mahanti, 2014); the Information ratio (the ratio of excess return above the benchmark, to the TE of a portfolio) which estimates the manager's ability to generate excess returns (and the consistency of that return generation) and finally, the Sharpe ratio (the ratio of excess return above the risk-free rate to total portfolio risk), which indicates fund manager skill in security selection. These tools are all employed in the final phase of the investment management process and provide investors with an overall picture of portfolio manager ability, portfolio performance, and derived satisfaction levels (whether performance is aligned with investor preferences) (Fabozzi & Markowitz, 2011).

The main goal of active portfolio managers is to select and manage portfolios that outperform their benchmark. Successful portfolio management lies in the long-term mix of assets which exhibit relatively low correlations with each other. Diversification actively selects uncorrelated or low-correlated component assets to mitigate unsystematic risk associated with adversely performing investments. The rationale governing this selection process is that combinations of dissimilar portfolios will generate higher returns at lower level of risk than that of individual securities on average (Menchero & Hu, 2006). Successful asset allocation is non-trivial and represents the most important decision in portfolio construction, exceeding even that of individual security selection. Ghosh and Mahanti (2014) suggest, however, that inter-asset correlations must be considered in the individual asset selection process as the co-movement between assets reduces portfolio variance while maintaining portfolio returns.

Retaining the optimal portfolio mix involves constantly rebalancing portfolio weights. This entails repeatedly reweighting or substituting over-valued securities for undervalued ones. Fund managers are assessed and remunerated on their ability to exceed benchmark returns tantamount to a positive expected TE (Riccetti, 2010). Reducing TE, in contrast, reduces relative portfolio risk. Modern portfolio theory assumes that investors are risk averse, meaning that given two portfolios of equal expected returns, investors will always favour the less risky of the two. The trade-off associated with the risk/return portfolio is defined by a hyperbolic curve known as the efficient frontier, which categorises the highest expected return possible for any risk level. A TE (standard deviation between a portfolio's return and its benchmark) quantifies a portfolio's consistency and performance relative to its benchmark (Plaxco & Arnott, 2002).

A two-dimensional performance approach is the ambit of traditional portfolio management. Portfolio managers are evaluated on their ability to exceed benchmark returns synonymous with a positive expected TE, and reducing TE is equivalent to reducing relative portfolio variance. Roll (1992) identified that investors who aim for the highest possible excess expected return while maintaining a minimum TE were naively disregarding absolute portfolio risk. The construction of the TE frontier (Figure 2.1) illustrates the maximum expected return away from a benchmark subject to a TE constraint. It can be analytically demonstrated that the TE curve may be derived independent of benchmark returns and unless the index lies directly on the efficient frontier, these portfolios are always inefficient (Jansen & van Dijk, 2002). Furthermore, if active portfolio managers were exclusively incentivised to maximise excess returns then portfolios located on the upper half of the efficient frontier would be preferable. Expected returns are – in practice – replete with noise and unpredictable. Instead fund managers are constrained by a TE that prevents excessive amounts of absolute risk being taken in the search for superior relative returns. The consequences of not adhering to this mandate can result in punitive action from investors and regulatory authorities (El-Hassan & Kofman, 2003).

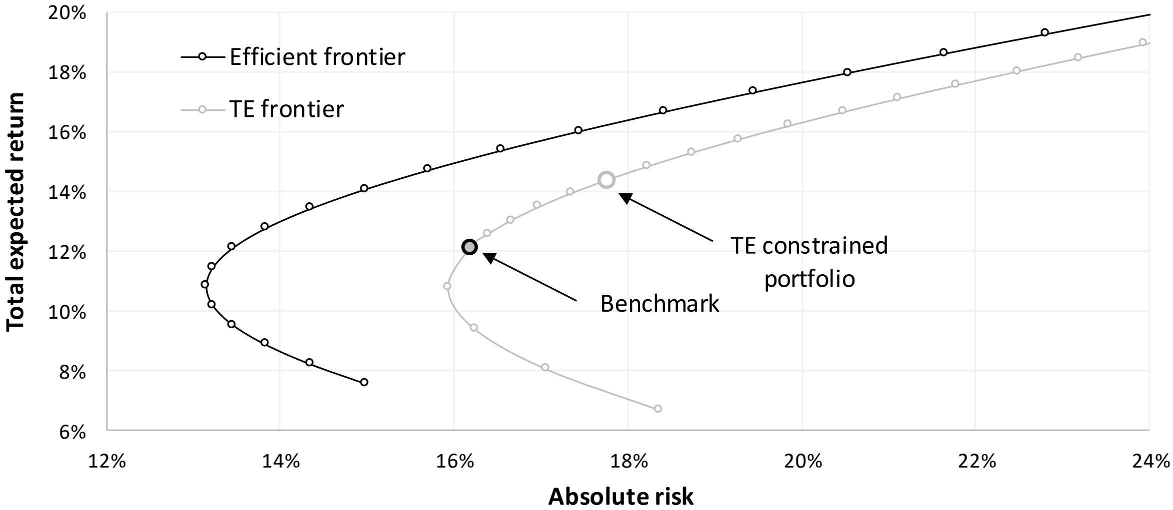

Figure 2.1: TE frontier and TE-constrained portfolio. In this example, $TE = 5\%$ and the TE constrained portfolio position shows the maximal return allowable for that level of TE.

Source: Roll (1992) and own calculations.

Each point on the TE frontier represents the maximum total expected excess return possible for a given TE. Markers indicate intervals of 1% TE deviation away from the benchmark i.e. the enlarged TE-constrained portfolio marker shown above describes the maximum excess return possible above a benchmark for a $TE = 5\%$.

Jorion (2003) investigated whether the naïve characteristic of active portfolios taking on systematically higher risk than that of the benchmark could be solved while maintaining a TE constraint. An alternative investment decision was proposed (Jorion, 2003) based on portfolio selection with the same benchmark risk but situated on the constant TE frontier as shown in Figure 2.2. Jorion (2003) showed that because of the 'flatness' (the low angle between the ellipse's main axis and the risk axis) of the ellipse, the addition of a total portfolio volatility constraint significantly improved portfolio performance. This effect was more pronounced with and lower TEs and less efficient benchmarks. Jorion (2003) demonstrated these effects by constraining the portfolio volatility to that of the benchmark, targeting portfolios with higher relative returns (but lower absolute returns).

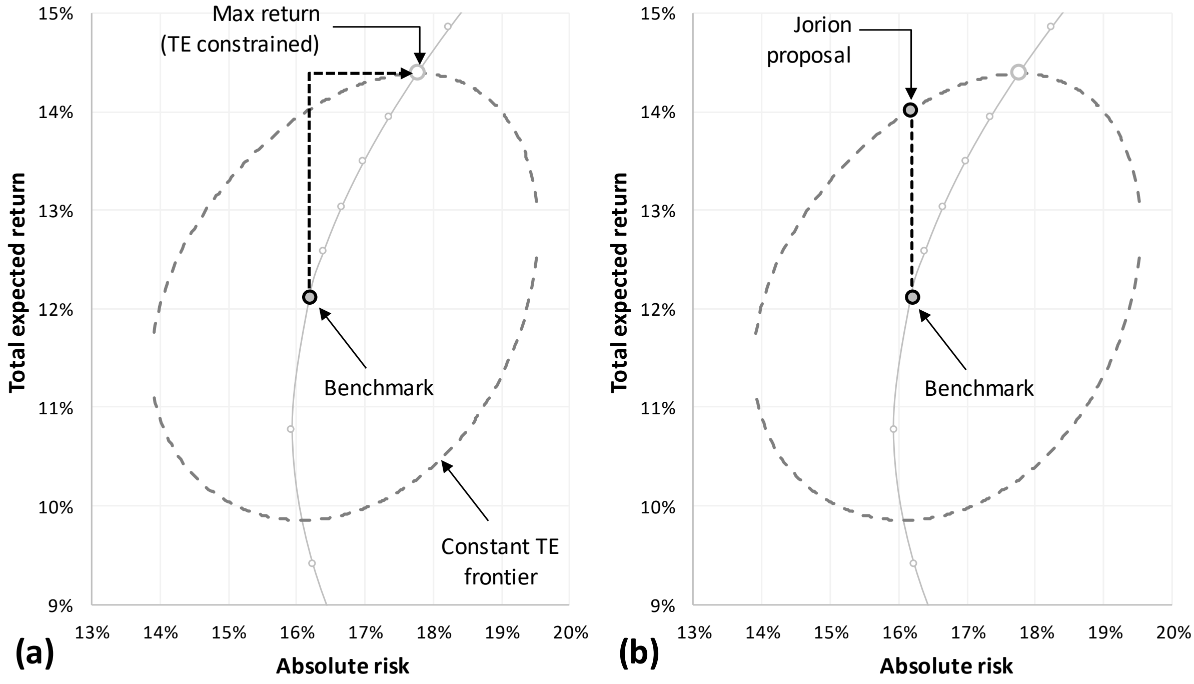

Figure 2.2: TE frontier, TE-constrained portfolio and constant TE frontier (with TE = 5%). (a) Shows the naïve portfolio: excess return is maximised for a given TE constraint. (b) shows Jorion's (2003) suggestion: observe constraints from (a) but restrict portfolio risk to that of the benchmark.

Source: Roll (1992), Jorion (2003) and own calculations.

Jorion (2003) mapped the boundary of all possible portfolios constrained by a TE and set out the mathematical description of the constant TE frontier (an ellipse in traditional mean/variance space (Figure 2.2)). In risk/return space, this is a tilted, distorted (reduced eccentric symmetry) ellipse (usually, but not always) presented at a low angle between the ellipse's long axis and the risk axis.

Figure 2.3 illustrates the constant TE frontier (in risk/return space) for various levels of TE. The grey shadow shown in the $TE = 0\%$ plane represents the realm of feasible portfolios where the outer boundaries trace the efficient frontier. For $TE = 0\%$, the constant TE frontier exists as a single point where the absolute return and risk profile equals that of the benchmark portfolio. For $TE > 0\%$, the constant TE frontier ellipse initially expands outwards until reaching the efficient frontier, as portfolios can take on more and more relative portfolio risk. Further increases of TE (in this example, for $TE > 7\%$) pushes the constant TE frontier *away* from the efficient frontier, showing that taking on unnecessary absolute risk results in less efficient portfolios.

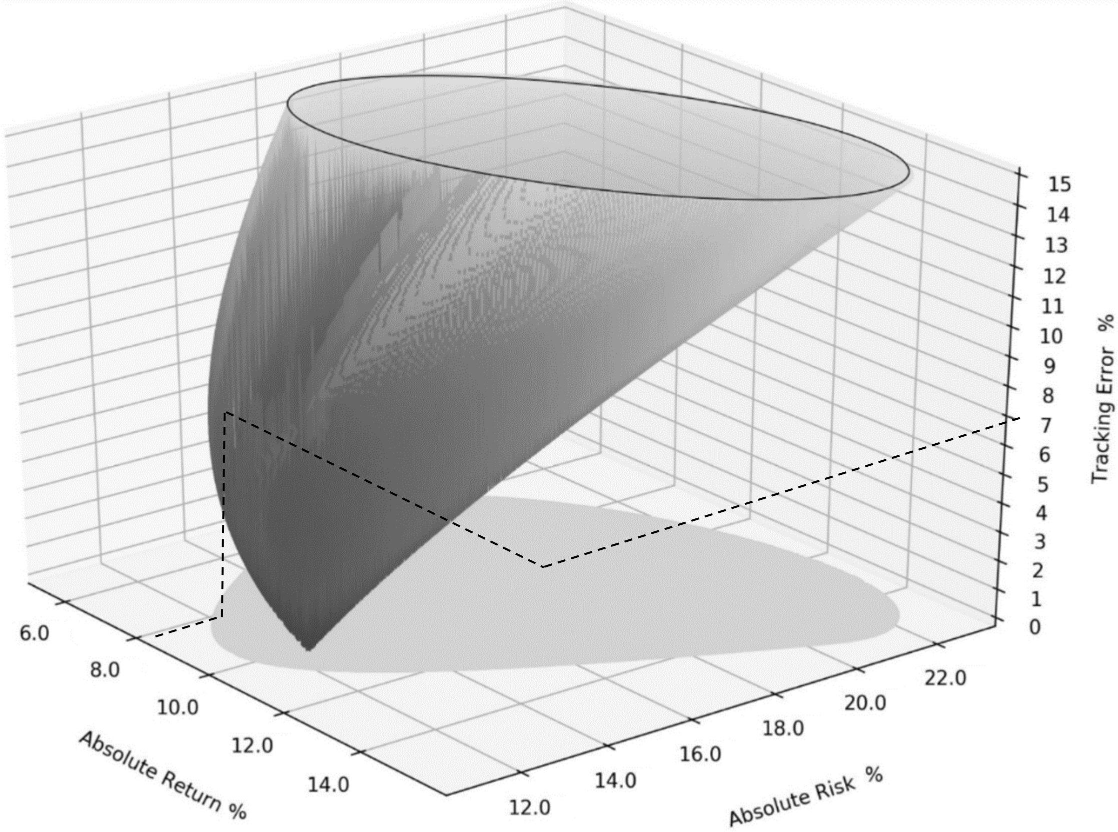

Figure 2.3: Constant TE-constrained frontier $0\% \leq TE \leq 15\%$.

Source: Daly, Maxwell, & van Vuuren (2018).

Maxwell, Daly, Thomson & van Vuuren (2018) extended this analysis by determining the portfolio with the highest risk-adjusted return. By adding an additional constraint – maximisation of the Sharpe ratio for a given TE – the authors found that these portfolios sacrifice marginal portfolio return (but exhibit considerably less absolute risk) than the maximum return portfolio (Figure 2.4). In countries such as the United Kingdom, where interest rates are low (0.25% in August 2020), the maximum Sharpe-constrained portfolio has a higher expected return and lower risk than the market return (where $\beta < 1$).

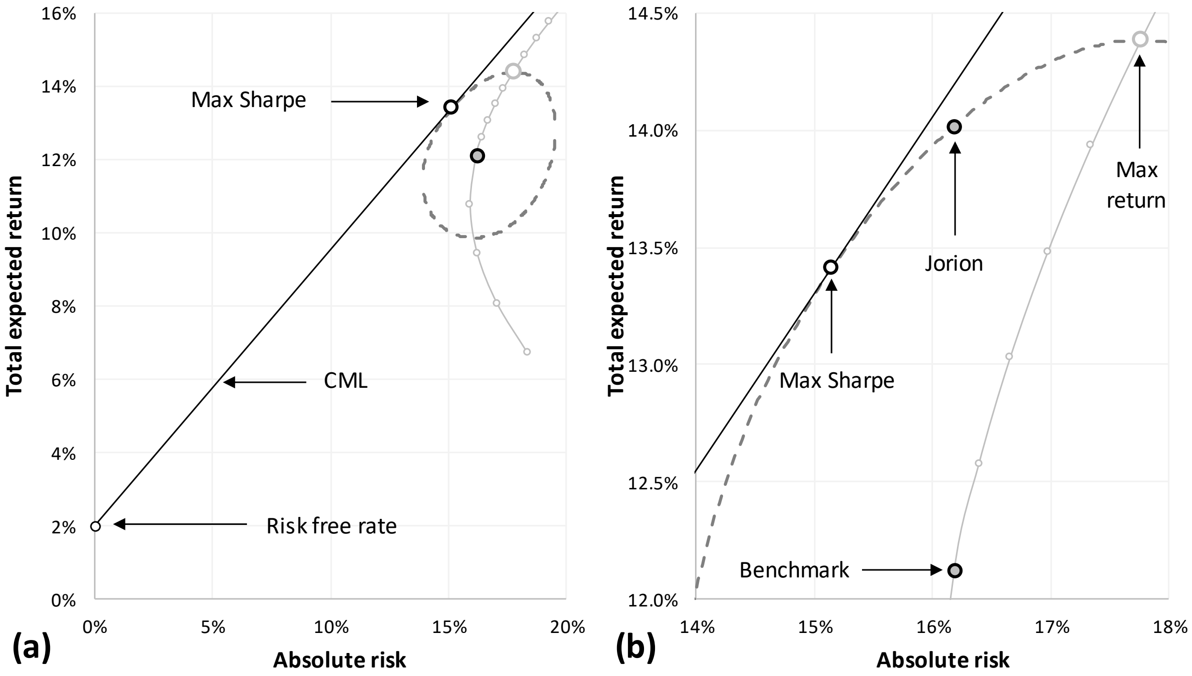

Figure 2.4: (a) TE-constrained portfolio, constant TE frontier and CML with optimal portfolio and (b) enlarged view showing all three portfolios. TE $= 5\%$ and $r_f = 2\%$.

Source: Jorion (2003) and author calculations.

Portfolio managers are sometimes required to assemble β-constrained portfolios as higher βs are theoretically indicative of higher returns. Roll (1992), however, proved this to be incorrect. Portfolios that have a lower volatility and higher expected performance relative to the benchmark have $\beta < 1$. Portfolios with higher volatility and positive benchmark outperformance (β > 1) are symptomatically inefficient and lie further to the right of the efficient set. For $\beta = 1$ (that is, not the benchmark) portfolios are located on the TE frontier but are always less efficient than the benchmark. The addition of the β constraint allowed Roll (1992) to prove the impossibility of producing β constrained portfolios that simultaneously minimises TE and outperforms the market return. The position of Jorion's (2003) proposed portfolio, and the maximum Sharpe portfolio, agrees with Roll's (1992) findings that higher performing portfolios exhibit portfolios with $\beta < 1$.

Daly, Maxwell & van Vuuren (2018) stylised the mathematics governing the β, α-TE and investor utility frontier for portfolios bound by a TE and determined that it is impossible to simultaneously satisfy more than two constraints onto the constant TE frontier (Figure 2.5). The α-TE frontier differs from other constraints in that it shows the minimum TE for various levels of *ex-ante* α. This means that for every TE constrained portfolio, a maximum α would exist at the boundary of the constant TE. A β frontier that coincides with the maximum α-TE constrained portfolio would be a maximum as defined by Roll (1992) ($\beta < 1$), although this is impractical.

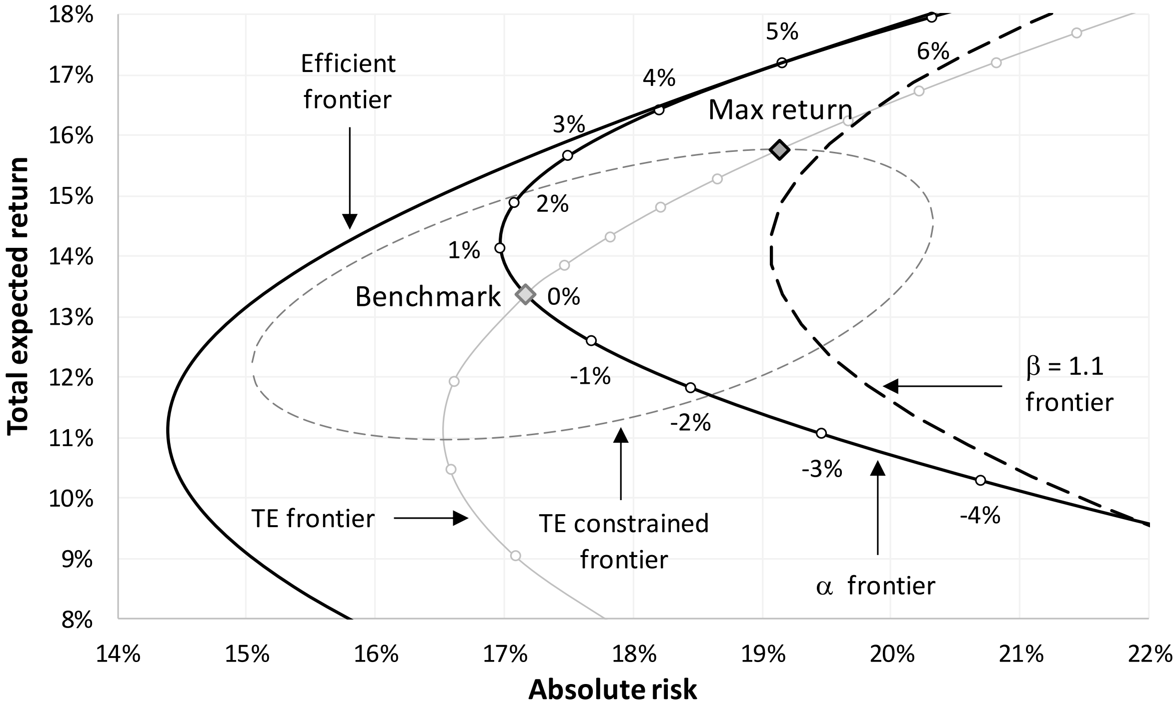

Figure 2.5: The α-TE frontier for various levels of α. Other frontiers are shown for comparison. Levels of α are indicated on the graph. TE $= 5\%, r_f = 2\%$.

Source: Author calculations.

Alexander and Baptista (2010) proposed an innovative methodology of reducing the sub optimality associated with portfolios that do not lie on the efficient set. The formulation of the α-TE frontier (Figure 2.5) is helpful to practitioners who evaluate the performance of a fund manager based on ex-post α of the investment. In addition, the α-TE frontier allows managers to identify less risky, utilitarian portfolios that are not typically selected by most active managers (Wu & Jakshoj, 2011).

Investor utility represents a quantitative investor satisfaction metric. Portfolio selection seeks to maximise utility, but this does not necessarily imply fund selection that maximise returns, minimise risk or maximise risk-adjusted returns. Rather, utility optimisation is a subjective constraint specific and unique to different investors. Indifference curves demonstrate the resulting satisfaction gained based on investment decisions, in which each point on the indifference curve represents the same level of satisfaction for different risk/return combinations (Fabozzi & Markowitz, 2011). When displayed on the mean variance plane, optimal portfolios may be chosen in which maximum indifference curves are tangential to efficient frontiers and it is at these intersections that the risk/return profiles – and investor utilities – are maximised. Daly, Maxwell & van Vuuren (2018) investigated the utility function of TE-constrained portfolios at the maximum Sharpe portfolio and found that risk aversion inreased with increasing TE, up until a point.

Maxwell & van Vuuren (2019) stylised the behaviour of alternative portfolio assemblies on the constant TE frontier (Figure 2.6). These portfolios where characterised as maximally diversified, exhibit risk parity, have minimal intra-correlation, and minimum risk for varying levels of TE. Every point along this frontier was investigated and it was discovered that such portfolios behaved adversely to mean variance efficient (unconstrained) portfolios.

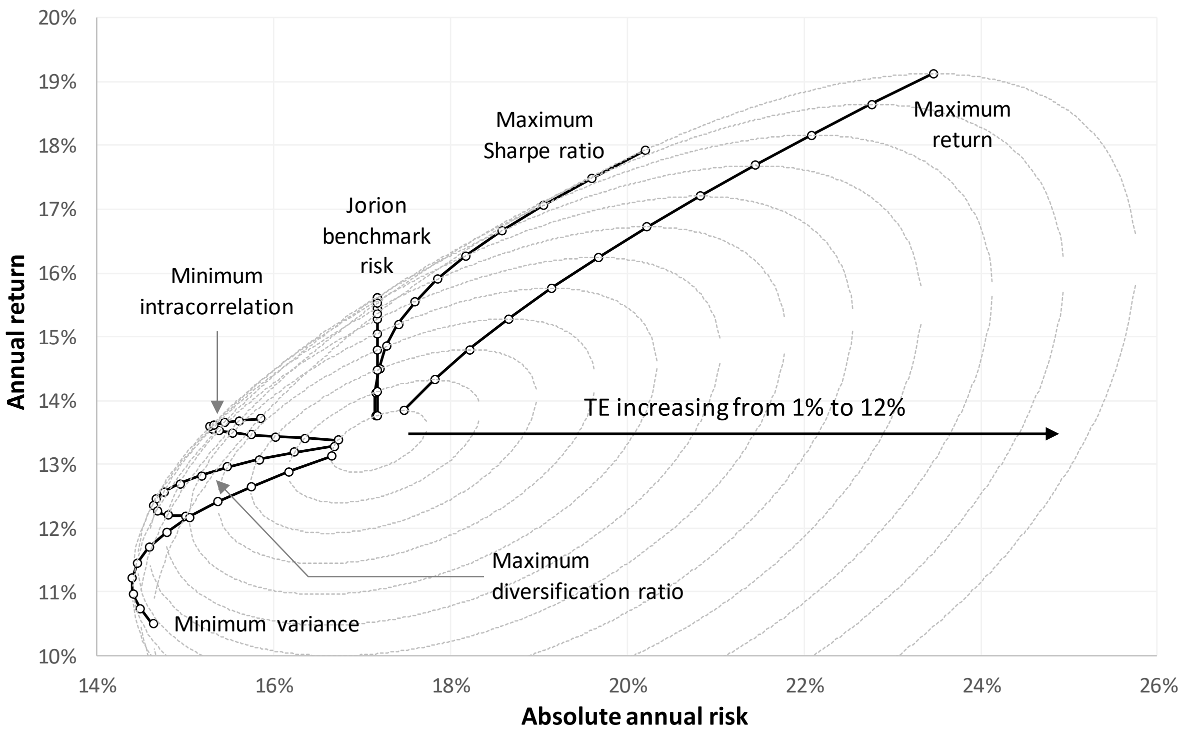

Figure 2.6: Loci of relevant portfolios in mean/risk space for $1\% \leq TE \leq 12\%$.

Source: Author calculations.

Evans & van Vuuren (2019) analysed six active TE constrained performance strategies, using numerous performance measures to assess the relative performance on the investment on the ellipse. Performance ratios reached plateaus for high TEs because of the roughly linear nature of the efficient frontier in risk/return space. Because the constant TE ellipse remains in contact with the efficient frontier for high TEs, the maximum Sharpe ratio for the former will always be approximately the same as the latter (Evans & van Vuuren, 2019).

Bertrand (2010) investigated the effect of fixing the investors level of risk aversion and allowed the TE to vary between 0 (the benchmark) and positive infinity (the minimum variance portfolio). This generated what Bertrand (2010) named 'iso-aversion frontiers' in which all optimal portfolios had the same Information ratio, allowing fund managers to make portfolio selections based on preferred investor risk. The problem with this research is that actively managed funds are generally constrained by a TE mandate, rendering Bertrand's (2010) findings partially obsolete.

Although MPT dictates that specific risk can be removed through portfolio diversification, systematic risk cannot be entirely eliminated. Movements in external factors which are beyond the control of investors or portfolio managers will always originate risks (e.g. interest rate, transactional fees, economic recessions, etc.). The capital asset pricing model (CAPM) is a theoretical tool that used to estimate an expected rate of return based on the investment's market risk. Stowe (2014) averred that the application of β and TE constraints on portfolio selection assured favourable investor utility improvements, and if implemented correctly, could produce more efficient portfolios by prudent managers.

Constructing mean-variance efficient portfolios often involve taking extreme long and short positions, hence the need for active portfolio managers to impose an asset weights constraint. Imposing constraints on portfolio weight is common in active portfolio management. Ammann and Zimmermann (2001) examined the relationship between TE and restricted asset weight deviations from the benchmark and found that large tactical asset allocation ranges implied much smaller TEs than expected. These tactical asset allocation restrictions also restricted the tactical ranges of the individual asset classes, as well as TEs of individual asset classes.

Bajeux-Besnainou, Belhaj, Maillard & Portait (2011) determined an optimal asset allocation of such agency-mandated portfolios and assessed the implications of restricting weights on fund manager performance. Bajeux-Besnainou, Belhaj, Maillard & Portait (2011) investigated the limitations associated with weight constraints on TE-constrained portfolios and found that these restrictions were mutually binding. Also, because of the weight constraint, the information ratio decreases when the fund manager deviates further from the benchmark.

The literature on constant TE frontiers is limited. Since the introduction and installation of the concept in 2003, only a handful of articles were produced before a resurgence of interest in the topic resulted in a series of connected works over the last few years (since 2017). Having established some of the background of TE-related research in this chapter, the next extends and contributes to this recent tradition by examining and assessing the drivers of TE-constrained portfolio performance, specifically the influence on portfolio performance of the orientation (angle sign) and magnitude (angle size) of the TE-ellipse's main axis with the risk (volatility) axis.

CHAPTER 3 DRIVERS OF TRACKING ERROR CONSTRAINED PORTFOLIO PERFORMANCE

3.1 ABSTRACT

Maximising returns is often the primary goal of asset management but managing and mitigating portfolio risk also plays a significant role. Successful active investing requires outperformance of a benchmark through skillful stock selection and market timing, but these bets necessarily give rise to risk. The risk, relative to the benchmark, is the TE and active managers are constrained by investment mandates including a restriction on TE. The locus of possible portfolio risks and returns, constrained by a TE is elliptical, and the main axis slope's sign and magnitude varies under different market conditions. How these variations affect portfolio performance is explored for the first time. Changes in main axis slope (magnitude and sign) are found to act as early indicators of portfolio performance and could therefore be used as another risk management tool.

3.2 INTRODUCTION

Investment styles are broadly classified as passive or active. For the former, fund managers purchase and hold securities (or fractions of market indices) for extended investment periods believing that market outperformance is impossible, so best to minimise transactions (and hence fees) and be satisfied with returns like the broad market. For the latter, managers assume that through skilful selection and timing of sales and purchases, market outperformance is possible. The emphasis in the latter investment style is on relative performance, so skill (outperformance) is assessed relative to a prescribed, mandated benchmark. Risk is also assessed relative to the benchmark's risk.

Markowitz's efficient frontier (1956) formulation has directed passive investment research for almost seven decades and the literature on associated portfolio optimisation is considerable. Sharpe (1964) introduced the concept of a maximum risk-adjusted return portfolio for 'optimal' performance, called the tangent portfolio. Active investment strategies involve more complex structures, because here, constraints restrict the investable universe. Benchmark constituents, the size of TE and asset weight floors and caps all contribute to additional complexity. Roll (1992) initiated research into TE constrained portfolio behaviour and developed the framework for describing an efficient frontier in risk/return space constrained by various levels of TE. Jorion (2003) extended this work and developed the details for the constant TE frontier: i.e. a locus of points in risk/return space which embraces the universe of risk/return combinations – relative to the benchmark's risk and return.

TE is an active risk measure (defined as the standard deviation of the difference between portfolio and benchmark returns) that reflects a portfolio manager's decisions to deviate from the weights of a benchmark's positions with the aim of outperforming the benchmark. The inevitable risk introduced by this deviation is the TE. The TE is generally not used as a risk metric to assess portfolio manager performance in isolation. Rather, other measures are used in combination with the TE to assess portfolio risk, such as Value at Risk, the information ratio, etc. Fund managers determine the investment policy (i.e. its risk-return profile, outperformance targets, etc.) which in turn determines the TE. Thomas, Rottschafer & Zvingelis (2013) outline several causes of TE (fees, transaction costs, taxes, factor tilts, cash management and market volatility).

In mean/variance space, the universe of possible portfolios constrained by a TE is an ellipse (and in risk/return space, it is a distorted ellipse). The ellipse's orientation (designated by the sign and magnitude of the 'main axis' slope) changes through time as economic conditions change. The way the main axis slope changes under different economic conditions, is explored here for South African (SA) stocks (an emerging third world economy) for the first time. The relevant mathematics required to calculate both the sign and magnitude of the main axis slope are detailed and the way this slope changes as time evolves and market conditions change is evaluated. Results indicate that when the main axis slope changes sign sharply, this presages a prolonged downturn in economic conditions. The effect is subtle, however. Forecasting economic conditions (and hence investment strategy) may depend on slope sign changes, but this depends on the *direction* of the change (i.e. $+ve$ to $-ve$), the magnitude of the slope before the reversal and the speed and size of the reversal. The combinations are persistent and robustly predict near economic conditions with reasonable accuracy. Results show that the way the main axis slope moves through time could trigger novel investment strategies for active fund managers.

3.3 MATERIALS AND METHODS

3.3.1 Materials

South Africa is an emerging economy with a history of political scandals and a volatile currency. The data for both benchmark and portfolios comprised 20 assets from the Johannesburg Stock Exchange (JSE) selected from a variety of market sectors to diversify the portfolio. The portfolio spans at least seven market sectors and seven of the largest, most liquid stocks are shown in Table 3.1. These assets are frequently traded by active managers. Monthly returns spanning 15 years from Oct-00 to Apr-19 were used. This era embraces various market conditions: the years of expansionary conditions which preceded the 2007-9 credit crisis, the credit crisis and post credit crisis turmoil.

Table 3.1 Top seven stocks (by liquidity and market capitalisation) details.

	Description	Sector
Naspers	A global internet and entertainment group and one of the largest global technology investors	Media
AVI	Food market sector embracing hot beverages, biscuits and snacks, frozen convenience foods, personal care products, cosmetics	Food producers
Shoprite	Africa's largest food retailer	Food and drug retailers
Remgro	An investment holding company with interests in banking, financial services, packaging, glass products, medical services, mining, petroleum, beverage, food and personal care products.	Financial services
MTN	Telecommunications network provider offering mobile comms, internet data bundles and contracts.	Mobile telecomms
ARM	Niche, diversified South African mining company with long-life, low-cost operating assets in key ARM mines and beneficiates iron/manganese and chrome ore, platinum group metals, copper, nickel and coal	Industrial metals and mining
Sappi	Leading global provider of sustainable wood fibre products and solutions	Forestry & Paper

Source: Bloomberg.

The benchmark was rebalanced monthly and comprised equal proportions of these highly liquid shares. The descriptive statistics of these securities are set out in Table 3.2.

Table 3.2 Descriptive statistics.

	Naspers	AVI	Shoprite	Remgro	MTN	ARM	Sappi
Mean annual return (%)	32.40	20.97	22.21	17.99	11.08	20.25	10.36
Max monthly return (%)	40.18	20.27	22.35	19.76	31.49	42.77	27.23
Min monthly return (%)	-48.57	-18.45	-16.38	-12.18	-28.09	-31.89	-43.33
Cumulative 19y return	5 217%	2 056%	2 181%	1 495%	182%	613%	114%
Annualised volatility (%)	22.07	20.39	25.80	17.47	26.34	39.54	24.54
Skewness	-0.672	0.072	0.123	0.142	0.158	0.283	-0.275
Kurtosis	4.690	0.110	-0.142	0.833	1.152	1.060	1.822

Source: Bloomberg and author calculations.

Figure 3.1 (a) illustrates all stocks' rebased cumulative share prices (rebased on Oct-00 = 100) and Figure 3.1 (b) shows the exponentially weighted moving average (EWMA) volatilities of the stocks.

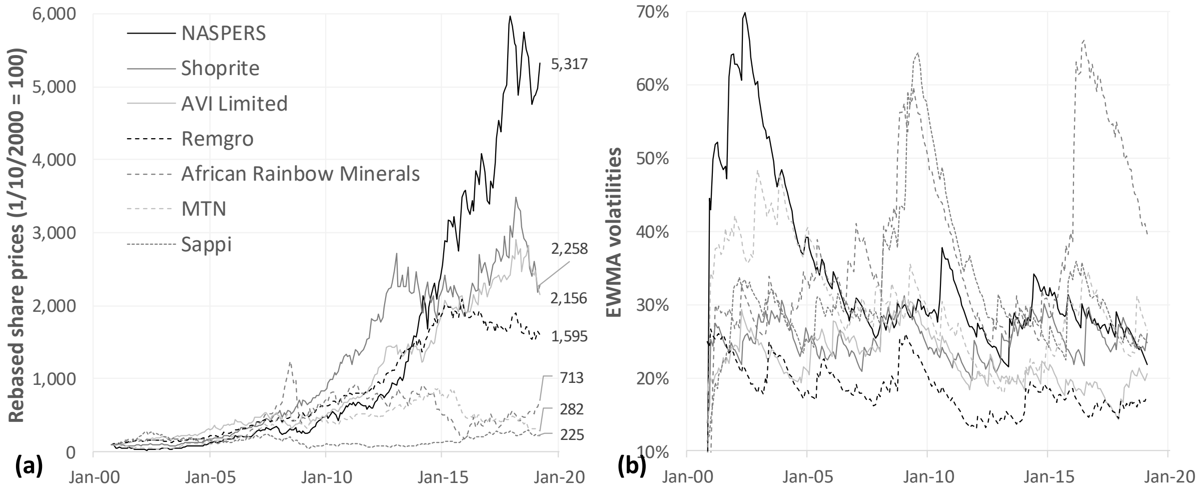

Figure 3.1. Descriptive information regarding the shares used in the analysis.

(a) Rebased cumulative share prices (Oct-00 = 100)

(b) Exponentially weighted moving average (EWMA) volatilities of constituent stocks

Source: Bloomberg and author calculations.

3.3.2 Methods

Active fund managers are assessed based on performance relative to a specified benchmark. The active investment positions they take differ from the benchmark positions according to the mandate governing the fund. For low TEs, active weights are small. For higher TEs, active weights are larger reflecting the fact that bigger bets are possible. Higher TEs, then, permit a wider range of security weights to be taken advantage of, rewarding skilled fund managers with potentially higher returns than the benchmark.

The underlying variables, matrices, and matrix notation, are defined below for a sample of N component securities:

$\boldsymbol{q}$: $1 \times N$ vector of benchmark weights

$\boldsymbol{x}$: $1 \times N$ vector of deviations from the benchmark

$\boldsymbol{q_P}$ $(= \boldsymbol{q} + \boldsymbol{x})$: $1 \times N$ vector of portfolio weights

$\boldsymbol{E}$: $1 \times N$ vector of expected returns,

$\boldsymbol{\sigma}$: $1 \times N$ vector of benchmark component volatilities

ρ: $N \times N$ benchmark correlation matrix

$\boldsymbol{V}$: $N \times N$ covariance matrix of asset returns

$\mathbf{1}$: $1 \times N$ vector of 1s and

r_f: the risk-free rate.

Net short sales *are* allowed so the total active weights $(\boldsymbol{q}_i + \boldsymbol{x}_i)$ may be < 0 for individual securities. The universe of assets may be larger than the benchmark's component set, but for Roll's (1992) methodology, no assets outside the benchmark's set may be included. Expected returns and variances are expressed in matrix notation as:

$\mu_B = \boldsymbol{qE'}$: expected benchmark return

$\sigma_B = \sqrt{\boldsymbol{qVq'}}$: volatility (risk) of benchmark return

$\mu_\varepsilon = \boldsymbol{xE'}$: expected excess return; and

$\sigma_\varepsilon = \sqrt{\boldsymbol{xVx'}}$: TE.

The active portfolio expected return and variance is given by $\mu_P = (\boldsymbol{q} + \boldsymbol{x})\boldsymbol{E}' = \mu_B + \mu_\varepsilon$ and $\sigma_P = \sqrt{(\boldsymbol{q} + \boldsymbol{x})\boldsymbol{V}(\boldsymbol{q} + \boldsymbol{x})'}$ respectively. The portfolio must be fully invested, so $(\boldsymbol{q} + \boldsymbol{x})\boldsymbol{1}' = 1$.

Merton (1972) defined $a = \boldsymbol{E}\boldsymbol{V}^{-\mathbf{1}}\boldsymbol{E}', b = \boldsymbol{E}\boldsymbol{V}^{-\mathbf{1}}\boldsymbol{1}', c = \boldsymbol{1}\boldsymbol{V}^{-\mathbf{1}}\boldsymbol{1}', d = a - \frac{b^2}{c}$ and $\Delta_1 = \mu_B - \frac{b}{c}$ where $b/c = \mu_{MV}$ and $\Delta_2 = \sigma_B^2 - \frac{1}{c}$ with $1/c = \sigma_{MV}^2$ where MV is the minimum variance portfolio.

It is useful to recall the mathematics required to generate the various frontiers.

3.3.2.1 Mean variance frontier

Minimise $\boldsymbol{q_P}\boldsymbol{V}\boldsymbol{q_P'}$ subject to $\boldsymbol{q_P}\boldsymbol{1}' = 1$ and $\boldsymbol{q_P}\boldsymbol{E}' = G$ where G is the target return. The vector of portfolio weights is $\boldsymbol{q_P} = \left(\frac{a-bG}{d}\right)\boldsymbol{q_{MV}} + \left(\frac{bG-\frac{b^2}{c}}{d}\right)\boldsymbol{q_{TG}}$ where q_{MV} is the vector of asset weights for the minimum variance portfolio given by $\boldsymbol{q_{MV}} = \boldsymbol{V}^{-1}\frac{\boldsymbol{1}}{c}$ and $\boldsymbol{q_{TG}}$ is the vector of asset weights for the tangent portfolio (with $r_f = 0$), i.e. $\boldsymbol{q_{TG}} = \boldsymbol{V}^{-1}\frac{\boldsymbol{E}}{b}$. The weights of the tangent portfolio's components, $\boldsymbol{q_{TP}}$, with $r_f \neq 0$, are:

$$q_{TP}' = \frac{\boldsymbol{V}^{-\mathbf{1}}\left(\boldsymbol{E} - r_f \cdot \boldsymbol{1}\right)'}{\boldsymbol{1} \cdot \boldsymbol{V}^{-1}\left(\boldsymbol{E} - r_f \cdot \boldsymbol{1}\right)'}$$

3.3.2.2 TE frontier

Maximise $\boldsymbol{x}\boldsymbol{E}'$ subject to $\boldsymbol{x}\boldsymbol{1}' = 0$ and $\boldsymbol{x}\boldsymbol{V}\boldsymbol{x}' = \sigma_\varepsilon^2$. The solution for the vector of deviations from the benchmark is $\boldsymbol{x}' = \pm\sqrt{\frac{\sigma_\varepsilon^2}{d}}\boldsymbol{V}^{-1}\left(\boldsymbol{E} - \frac{b}{c}\boldsymbol{1}\right)'$. The solution to this optimisation problem generates the TE frontier, a portfolio's maximal return at a given risk level and subject to a TE constraint. The benchmark may be efficient, in which case it would lie on the efficient frontier. The benchmark is often rather arbitrarily selected (a mix of stock and bonds or an imperfect market index) so it frequently is not a member of the efficient portfolio set. The TE frontier passes through the benchmark coordinates when here $TE = 0$.

3.3.2.3 Constant TE frontier

Maximise $\boldsymbol{xE'}$ subject to $\boldsymbol{x1'} = 0$, $\boldsymbol{xVx'} = \sigma_\varepsilon^2$ and $(\boldsymbol{q}+\boldsymbol{x})\boldsymbol{V}(\boldsymbol{q}+\boldsymbol{x})' = \sigma_P^2$. The vector of deviation weights from the benchmark is $\boldsymbol{x}' = -\frac{1}{\lambda_2+\lambda_3}\boldsymbol{V}^{-1}(\boldsymbol{E}'+\lambda_1+\lambda_3\boldsymbol{Vq'})$ where $\lambda_1 = -\frac{\lambda_3+b}{c}$, $\lambda_2 = \pm(-2)\sqrt{\frac{d\Delta_2-\Delta_1^2}{4\sigma_\varepsilon^2\Delta_2-y^2}} - \lambda_3$ and $\lambda_3 = -\frac{\Delta_1}{\Delta_2} \pm \frac{y}{\Delta_2}\sqrt{\frac{d\Delta_2-\Delta_1^2}{4\sigma_\varepsilon^2\Delta_2-y^2}}$. The solution for this optimisation describes an ellipse – a *constant* TE frontier – in return/risk space: the unconstrained constant TE frontier (Jorion, 2003). The north-west segment of this frontier is bounded on the west by the minimum variance portfolio (with $\sigma_P = \sqrt{\left(\sigma_B^2 + TE + 2\sqrt{TE\cdot(\sigma_B^2-\sigma_{MV}^2)}\right)}$) and in the north by the maximum return portfolio (with deviations from the benchmark weights given by $\boldsymbol{x}' = \pm\sqrt{\frac{\sigma_\varepsilon^2}{d}}\boldsymbol{V}^{-1}\left(\boldsymbol{E}-\frac{b}{c}\boldsymbol{1}\right)'$). The arc between these portfolios on the unconstrained constant TE frontier represents the efficient set of portfolios subject to a specific TE. The solution for the weights which generate the tangent portfolio (to the constant TE frontier) was recently found by Maxwell et al, (2018) to involve solving for σ_P using:

$$\frac{(r_f-\mu_B)}{\sigma_P^2} + \frac{\frac{(\Delta_1^2-d\Delta_2)\cdot(\sigma_P^2-\sigma_B^2-\sigma_\varepsilon^2)}{\sqrt{(\Delta_1^2-d\Delta_2)\left[(\sigma_P^2-\sigma_B^2-\sigma_\varepsilon^2)^2-4\Delta_2\sigma_\varepsilon^2\right]}}+\Delta_1}{\Delta_2} - \frac{\sqrt{(\Delta_1^2-d\Delta_2)\left[(\sigma_P^2-\sigma_B^2-\sigma_\varepsilon^2)^2-4\Delta_2\sigma_\varepsilon^2\right]}+\Delta_1\cdot(\sigma_P^2-\sigma_B^2-\sigma_\varepsilon^2)}{2\Delta_2\sigma_P^2}$$

then establishing μ_P on the efficient segment of the constant TE frontier and then backing out the relevant weights (see Figure 1.1).

3.3.2.4 Main axis slope, S_{MA}

The main axis slope, S_{MA} is calculated using

$$S_{MA} = \frac{\Delta_1}{\sigma_B-\sigma_{MV}} = \frac{\mu_B-\frac{b}{c}}{\sigma_B-\sigma_{MV}} = \frac{\mu_B-\mu_{MV}}{\sigma_B-\sigma_{MV}} \qquad (3.1)$$

Δ_1 determines the sign of S_{MA} because the denominator is always > 0 since $\sigma_B - \sigma_{MV} > 0$ always. A necessary and sufficient condition for $S_{MA} < 0$ is $\mu_B < \mu_{MV}$. Note that the S_{MA} is independent of TE since none of its components depend explicitly thereon. For the first time, the sign and magnitude of the S_{MA} are evaluated and how these (and constituents of the S_{MA})

change over time as market conditions evolve plus their influence on TE constrained portfolio performance are explored.

Figure 3.2 shows the relevant frontiers, portfolios, and the long axis – calculated using (3.1). The periods are (a) Sep-10 when $S_{MA} < 0$ and (b) Apr-11 when $S_{MA} > 0$.

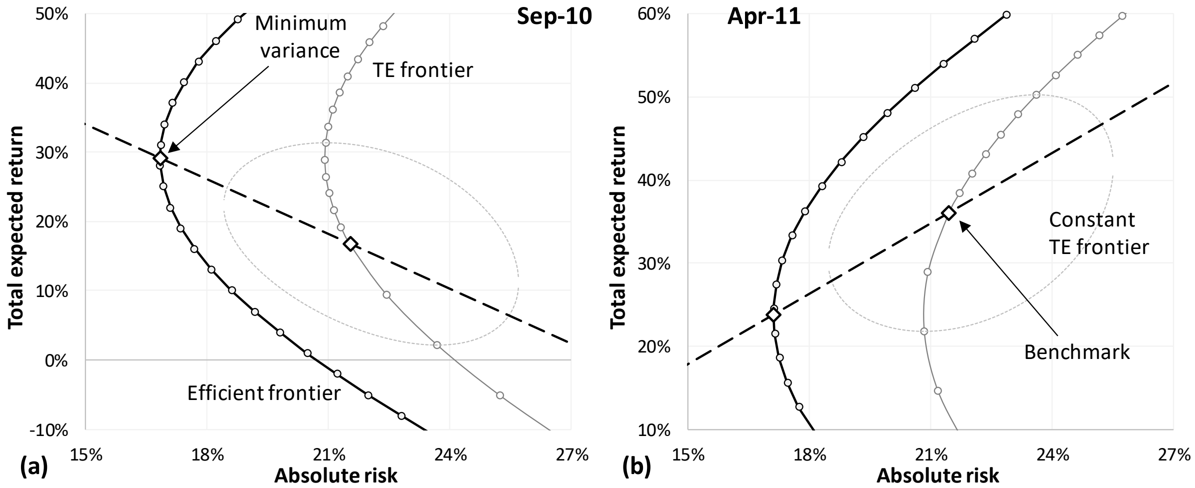

Figure 3.2. Efficient, TE and constant TE frontiers and the main axis. $TE = 6\%$ and r_f was the annualised 3-month SA treasury rate. Different levels of TE gave similar results.

(a) Sep-10 ($S_{MA} = -2.64$ [< 0]) (b) Apr-11 ($S_{MA} = +2.82$ [> 0]).

Source: Bloomberg and author calculations.

3.4 RESULTS AND DISCUSSION

Figure 3.3 shows the regression results of the S_{MA} versus the tangent portfolio's Sharpe ratios (the latter shown in Figure 1.1).

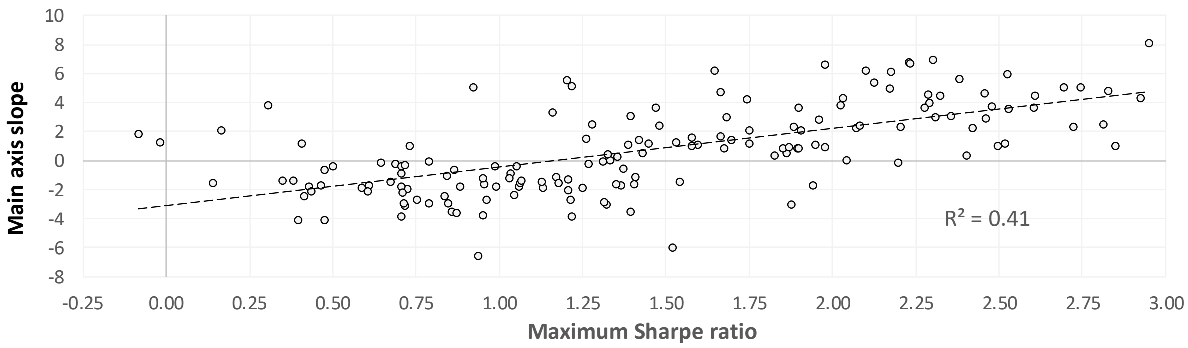

Figure 3.3. S_{MA} regressed on the tangent portfolio's maximum Sharpe ratios.

Source: Author calculations.

The maximum Sharpe ratio portfolio was compared and regressed against this portfolio (since most active managers will aim for this rather than the maximum return portfolio (Ross, 1992)), as was the portfolio with the same risk as the benchmark (Jorion, 2003) or the minimum variance portfolio on the constant TE frontier. The tangent portfolio is the optimal portfolio subject to a given TE constraint, so it lies on the efficient portfolio set, has a higher return than that of the benchmark and frequently (but not always) has a lower risk than the benchmark. Although the tangent portfolio does not generate the maximum return, it maximises the risk-adjusted return for a given TE (Maxwell, et al., 2018).

An $R^2 = 0.41$ indicates a positive relationship between S_{MA} and the maximum Sharpe ratios of the tangent portfolio. This relationship is expected: an $S_{MA} > 0$ requires that $\mu_B > \mu_{MV}$. Higher Sharpe ratios require higher risk-adjusted returns, hence the positive relationship between maximum Sharpe ratios and the S_{MA}.

Splitting the dataset into two and regressing the metrics over two time periods, (a) Nov-05 to Oct-12 and (b) Nov-12 to Apr-19 shows that these changed over time from being highly correlated to ($\rho = 0.88$) to only marginally correlated ($\rho = 0.46$) – see Figure 3.4. This is more clearly shown in Figure 3.6.

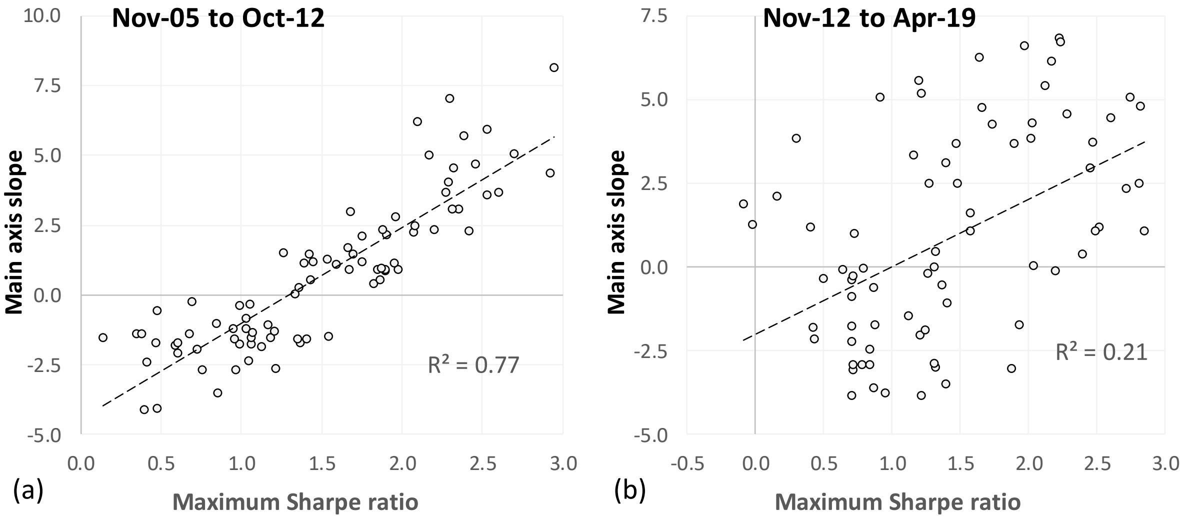

Figure 3.4. S_{MA} regressed on the tangent portfolio's maximum Sharpe ratios for the period (a) Nov-05 to Oct-12 and (b) Nov-12 to Apr-19.

Source: Author calculations.

The start of 2013 – the point at which the relationship between the S_{MA} and the maximum Sharpe ratio deteriorates – saw significant economic upheaval in South Africa. South Africa was downgraded in early 2013 by one major credit rating agency (SA National Treasury,

2013) and later that year by another (Moody's, 2013). The market became more volatile and the earnings of several corporates with international holdings became more volatile because of currency fluctuations at the time. The consequences of this are discussed later.

Why the S_{MA} and the Sharpe ratio would be correlated is better explained using Figure 3.5 for two economic milieus, (a) boom: $S_{MA} > 0$ and (b) bust: $S_{MA} < 0$. Figure 3.5(a) sets out the stylised, relative positions of frontiers and portfolios under boom conditions. Figure 3.5(b) shows the stylised position, in risk/return space, of these frontiers and portfolios in a downturn period.

In this configuration, Figure 3.5(a) shows an $S_{MA} > 0$ because $\Delta_1 > 0$.

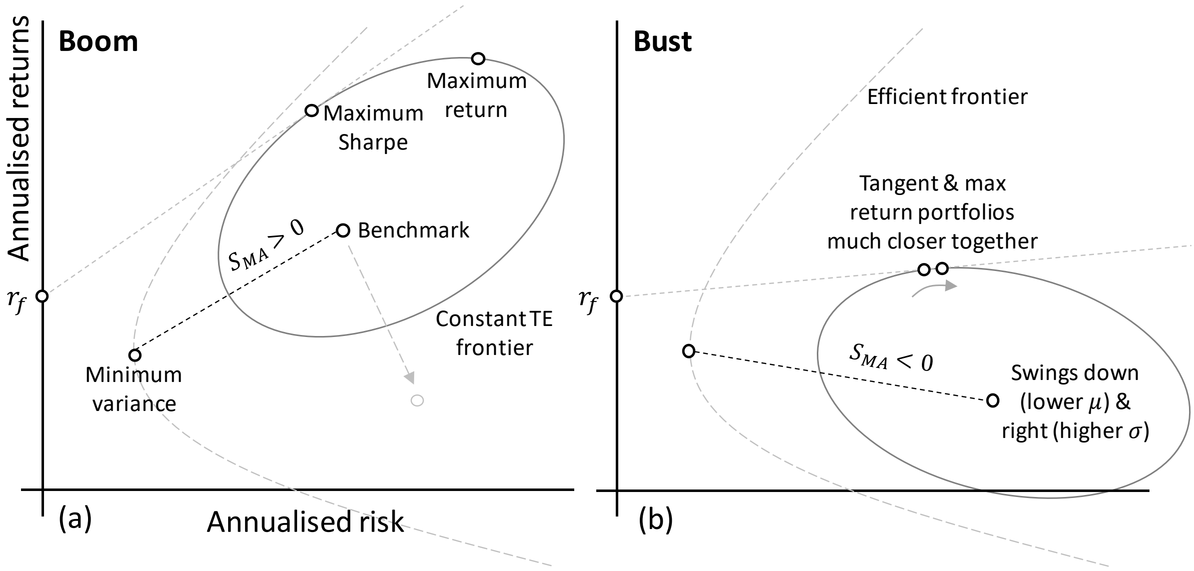

Figure 3.5. Stylised, relative positions of frontiers and portfolios under both boom and bust conditions. As a direct result, when $S_{MA} < 0$, the tangent portfolio Sharpe ratio is likely to be low, as observed (Figure 3.6).

(a) Boom conditions (b) Bust conditions

Source: Author calculations.

The efficient frontier comprises portfolios with long and short positions and is relatively stable,[1] as is the minimum variance portfolio (known to be considerably less volatile than the benchmark – see Figure 3.8) so these positions are similar in the two periods. The benchmark – being more volatile than the minimum variance portfolio – suffers a *decrease*

[1] That is, the risk/return coordinates do not alter much from month to month.

in returns and an *increase* in risk, so it shifts down and right in the risk/return plane. In sufficiently severe downturns, this can mean $S_{MA} < 0$ and the coordinates of the tangent and maximum return portfolios move much closer together, i.e. their risk and return profiles become almost indistinguishable. Because the risk of the maximum Sharpe ratio portfolio increases in market downturns and its return decreases, there should be a strong relationship between the S_{MA} and the maximum Sharpe ratio (as observed).

The relationship between S_{MA} and the maximum Sharpe ratio from the tangent portfolio over time is shown in Figure 3.6.

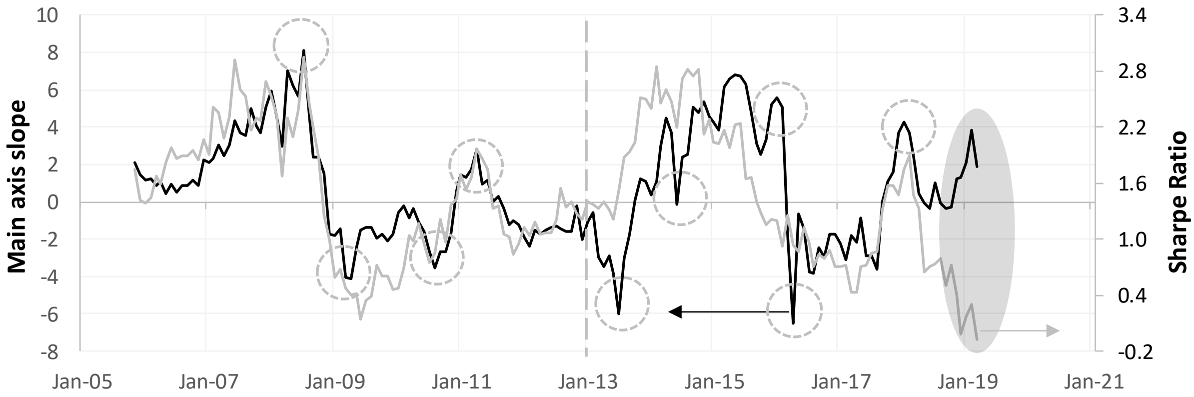

Figure 3.6. The tangent portfolio's Sharpe ratio and the S_{MA}: the grey dashed line indicates the time of multiple credit rating agency downgrades. Circles indicate turning points and the grey shaded area indicates a developing trend in which the two series diverge considerably.

Source: Author calculations.

The Sharpe ratio and the S_{MA} evolve similarly, but some interesting features are worth elucidating – particularly those which may be used to trigger investment decisions. The point at which the positive relationship deteriorates is early 2013, indicated by the grey dashed line (see Figure 3.6). In this strongly correlated phase (Figure 3.6(a)) the onset of the 2008 financial crisis marks the beginning of a sustained decline for both series (explained in Figure 6). The grey circles which occur when $S_{MA} > 0$ and are proceeded by a substantial decrease in both quantities (the S_{MA} changing from $+8$ to -4 and the maximum Sharpe decreasing by $\approx 60\%$ over a few months in both cases). The grey circles indicating turning points when $S_{MA} < 0$ are proceeded by an increase in both quantities (but considerably less for the S_{MA} compared with changes when $S_{MA} > 0$ and the maximum Sharpe increasing by $\approx 80\%$ again over a few months.

In the less correlated phase (Figure 3.6(b)) the situation is less clear. Sharp turning points when the $S_{MA} > 0$ result in corresponding small ($\approx 20\%$) changes in the maximum Sharpe ratio. Sharp changes in S_{MA} when it is < 0 sometimes results in large changes in the maximum Sharpe ratio (e.g. mid 2013, $\approx 120\%$) and sometimes not (e.g. $\approx 10\%$ in early 2014). The significant changes in the S_{MA} both decreasing and increasing in early 2016 had almost no effect on the maximum Sharpe value.

Since Feb-18, the close relationship between S_{MA} and the maximum Sharpe ratio measured has diverged even more (grey shaded region in Figure 3.7). South Africa's first technical recession in nine years was announced in Q2-18 which increased market risk and lowered annual returns (Figure 3.7). The combination of these effects contributed to the lowest measured Sharpe ratios in the entire 15-year observation period. During the same time, the S_{MA} remained positive (if only slightly) and has only begun to increase more recently. It is possible that the benchmark, comprising large, liquid stocks, has not experienced the substantial losses (and increase in risk) as those in the maximum Sharpe ratio portfolio. As a result, S_{MA} has remained positive, while the maximum Sharpe ratio has continued to decline.

These results could be used to signal exit and entry strategies in the tangent portfolio. Sharp reversals in the sign of S_{MA} indicate a prolonged period of poor performance if S_{MA} changes rapidly from > 0 to < 0 in boom conditions. The reverse is also true, S_{MA} sign reversals indicate a prolonged period of superior performance if S_{MA} changes rapidly from < 0 to > 0, again, in boom conditions. Large changes in the magnitude of S_{MA} lead directly to large changes in the maximum Sharpe ratio, and hence performance of the tangent portfolio.

Figure 3.7 shows the excess returns (over r_f) and portfolio volatility for the maximum Sharpe ratio portfolio. Note that these are the constituents of the Sharpe ratio calculation.

Figure 3.8 shows the S_{MA} and the annualised returns of the minimum variance and benchmark portfolios (averaged over three years). Note the relative volatility of returns for the two portfolios' returns.

To demonstrate the correlation between the returns of the maximum Sharpe ratio and those of the benchmark, the returns were regressed on each other and a high $R^2 = 0.92$ was found. It is not surprising that the returns of these portfolios should move together; they comprise the same constituents and the TE is tight ($TE = 6\%$), so portfolio managers will be discouraged from taking bets far from the benchmark weights.

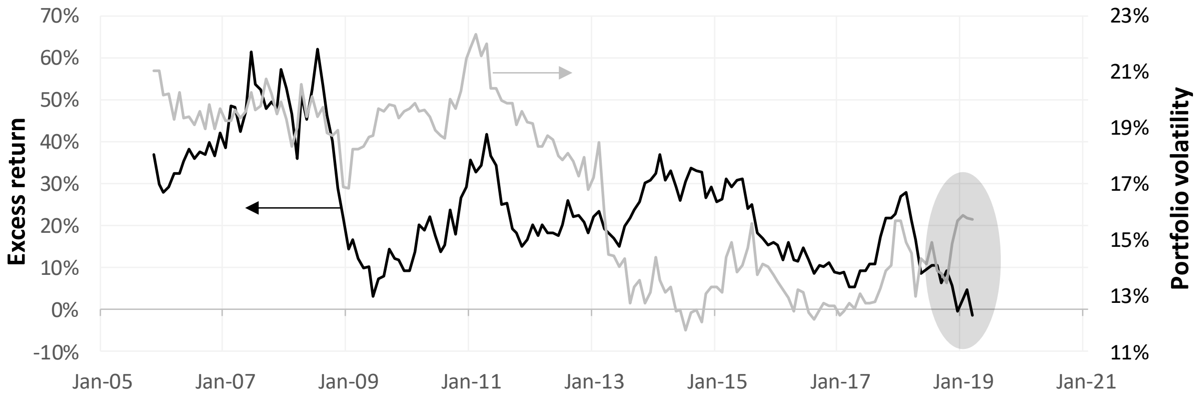

Figure 3.7. Components of the maximum Sharpe ratio portfolio (numerator: excess return (over $r_f = 6\%$) and denominator: σ_P, calculated over the same observation period for comparison. The combination of lower returns and higher risk recently (shaded area) has contributed to the decline in the maximum Sharpe ratio over the same period (shaded area in Figure 3.6).

Source: Author calculations.

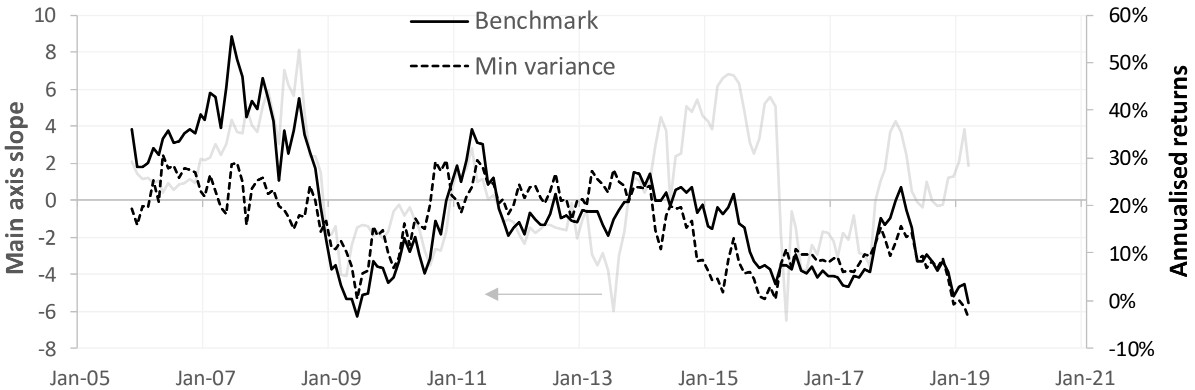

Figure 3.8. Annualised returns of the benchmark and minimum variance portfolio (averaged over three years) and the S_{MA} over the same observation period for comparison.

Source: Author calculations.

The regression (Figure 3.9) shows that by investing in the tangent portfolio, returns can be expected to be $\approx 10\%$ greater than those generated from the benchmark portfolio – for all levels of μ_B, even 0%. This reflects a highly desirable information ratio of 1.67 (relative return over the benchmark 10%, from a relative risk $TE = 6\%$).

Although a positive correlation still exists between the returns of the minimum variance portfolio and those of the benchmark, the latter influences the former considerably less (R^2 here $= 0.52$). It has already been shown that they the returns of these portfolios – although positively correlated – have substantially different volatilities (see Figure 3.9) with $\sigma_{MV} < \sigma_B$. The minimum variance portfolio generates returns which are invariably greater (by ≈ 7%) than those of the benchmark which leads to $S_{MA} > 0$ more often than $S_{MA} < 0$ (54.7% versus 45.3%).

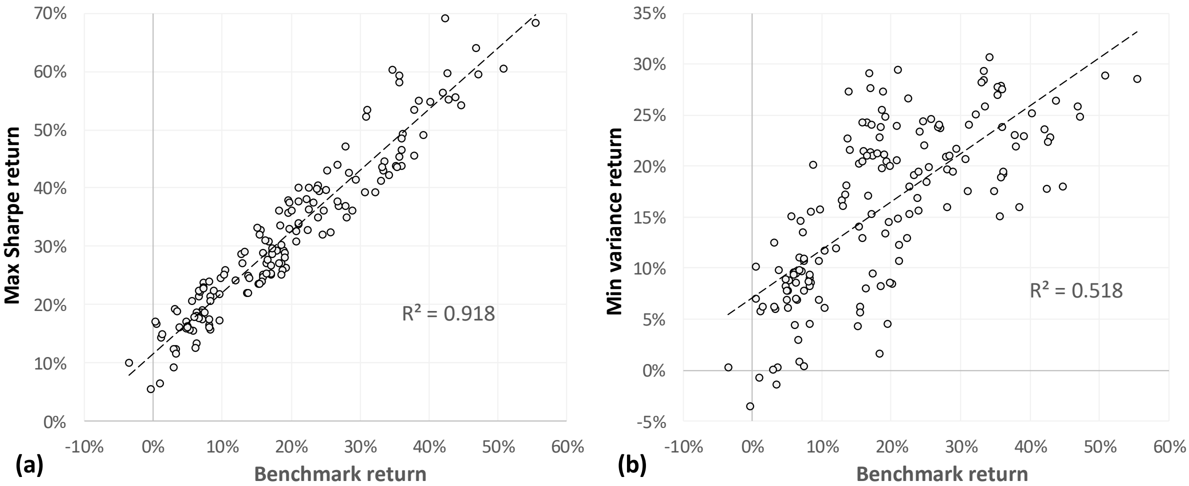

Figure 3.9. Regression of (a) monthly maximum Sharpe ratio portfolio returns and (b) monthly minimum variance portfolio returns on monthly benchmark portfolio returns.

Source: Author calculations.

The weights in the respective stocks as a function of time is shown in Figure 3.10. Three major incidents are identified: credit ratings downgrades, corruption scandals and the 'Nenegate' affair. These give rise to notable, dramatic changes in the evolving profile of security weights as well as the sharp changes in S_{MA} shown in Figure 3.6.

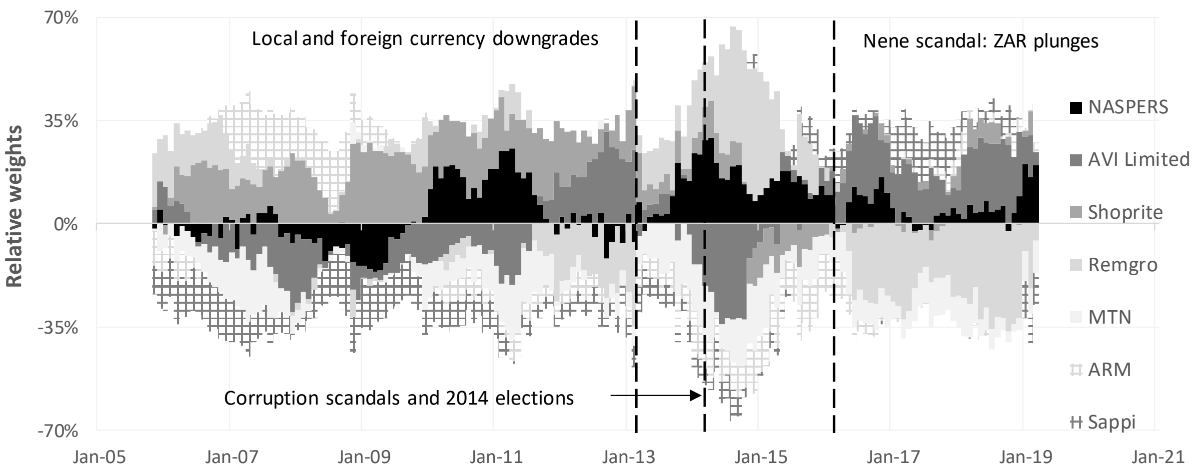

Figure 3.10. Relative portfolio constituent weights ($\boldsymbol{x_i}$) for the maximum Sharpe ratio portfolio on the constant TE frontier over the observation period. Recall that $\sum_i \boldsymbol{x_i} = 0$.

Source: Bloomberg and author calculations.

Prior to the 2013 local and foreign currency downgrade by Fitch ratings, relative holdings in Naspers fluctuates between overweight and underweight the benchmark. However, after the announcement of the downgrade (and after the 2014 corruption scandals, the uncertainty regarding the May 2014 elections and the Nenegate affair of November 2015) relative holdings in Naspers become positive and remain consistently overweight to the benchmark. Note that with monthly rebalancing note that this implies continuous increases of the weightings in Naspers. Naspers invests internationally and thus an investment in Naspers reflects considerable diversification benefits. When the ZAR depreciates it is beneficial to invest in a company which has international holdings.

Although MTN experienced good performance up until their $5.2 billion fine in 2015 it was optimal to underweight the stock relative to the benchmark weights as its performance was not as stellar as Naspers and Shoprite. Naspers and Remgro both have international holdings whereas Shoprite, MTN and African Rainbow Minerals are companies whose activities are predominantly in the developing market space. More portfolio performance research is required during political uncertainty and associated devaluation of the ZAR. More work is required before it can be definitively determined whether overweighting companies with international holdings and consecutively underweighting companies whose activities are predominantly based in the emerging market space yields better returns. While acknowledging the sometimes-disastrous consequences experienced by many South African companies who ventured into foreign markets, these results do elucidate the importance of foreign diversification.

Naspers as well as AVI Limited’s relative weightings increase significantly after Sept-18. This is due to both stocks consisting of international holdings and this period proceeds the announcement of South Africa's first technical recession in nine years (Sep-18). Again, this highlights the importance of international diversification.

CHAPTER 4 OPTIMAL OMEGA-RATIO PORTFOLIO PERFORMANCE CONSTRAINED BY TRACKING ERROR

4.1 ABSTRACT

The mean-variance framework coupled with the Sharpe ratio identifies optimal portfolios under the passive investment style. Optimal portfolio identification under active investment approaches, where performance is measured relative to a benchmark, is less well-known. Active portfolios subject to TE constraints lie on distorted elliptical frontiers in return/risk space. Identifying optimal active portfolios, however defined, have only recently begun to be explored. The Ω ratio considers both down and upside portfolio potential. Recent work has established a technique to determine optimal Ω ratio portfolios under the passive investment approach. The identification of optimal Ω ratio portfolios is applied to the active arena (i.e. to portfolios constrained by a TE) and find that while passive managers should always invest in maximum Ω ratio portfolios, active managers should first establish market conditions (which determine the sign of the main axis slope of the constant TE frontier). Maximum Sharpe ratio portfolios should be engaged when this slope is > 0 and maximum Ω ratios when < 0.

4.2 INTRODUCTION

Investment styles follow one of two broad approaches: active and passive. Active fund managers trade frequently and engage energetically with the market. Successful active managers identify not only high-performing assets, but also time trades to extract maximal performance, buying when prices are low and selling when they are high. Skill in this space is usually measured relative to a benchmark, usually a market index or an assembly of similar securities with constraints on portfolio weights, asset quality and acceptable risk. Passive managers select and purchase desired securities and hold these for investment horizons which span periods of economic booms and busts. Such managers' proficiency is measured on an absolute basis, they minimise transaction fees and aver that "good" securities outperform in the long run.

Both styles have pros and cons, and the ebb and flow of economic activity often dictates investor style selection: passive usually in stable markets and active in volatile. Events such as the 2020 COVID-19 pandemic which severely shocked global markets, serve to emphasise the importance of agile, active investing. Managers capable and eager to quickly dispose of airline, oil or tourism-related stocks for example, avoided the worst of the downturn and significantly outperformed less-nimble investments.

Modern portfolio theory led to the design and application of the widely-used efficient frontier, which plots – in return-risk space – the locus of portfolios whose arrangement of constituent security weights generates maximal returns at each specified risk level. Sharpe's work identified the optimal portfolio on this frontier: one whose excess return (usually over the risk-free rate) per unit of risk taken to achieve that return, was maximised. This framework of asset selection is ideally suited to the passive investment style. Identifying an optimal portfolio using this construction implies the belief that markets are relatively static and that buying and holding the optimal portfolio will eventually lead to the desired risk/return characteristics.

Active investment strategies require more complex structures. Portfolios whose performance and risk are measured relative to a benchmark follow a different locus of possibilities in return/risk space. Jorion (2003) demonstrated that such portfolios occupy a distorted ellipse in this space – rather than the efficient frontier's hyperbola for absolute risk and return. The dimensions and orientation of this ellipse is governed by many factors, including the variance-covariance matrix of underlying security returns, benchmark weights in the permissible universe of investable assets, constituent portfolio weights relative to the benchmark and the size of the TE. The greater the deviation from benchmark weights, the higher the possibility for outperforming (or underperforming) that benchmark (and the higher the TE). Active managers – to limit excessive risk-taking – are often constrained to not exceed prescribed TEs. There are profound differences in the way portfolio risk and return evolve and are measured under active and passive investment styles. Standard performance metrics, in common use for passive portfolios, require complex reformulation and behave in unfamiliar ways in active space.

The Ω ratio, a performance metric which makes no distributional assumptions about asset returns, is popular amongst passive investors, but determining the asset allocation to generate an optimal Ω ratio portfolio eluded researchers for years. The definition of the Ω ratio imbues it with non-convex properties which do not yield to standard optimisation techniques. Recently, Kapsos, Zymler, Christofides & Rustem (2011) accomplished this feat using linear programming, but their approach has not subsequently been applied to active portfolios, i.e. those constrained by TEs. The maximum Ω ratio portfolios are identified on the constant TE frontier under different market conditions and these portfolios' performance are compared, over time, to that of universal (unconstrained) Ω ratio portfolios.

4.3 MATERIALS AND METHODS

4.3.1 Materials

The data for both benchmark and portfolios comprised 15 stocks (from six market sectors to ensure some diversification) selected from a major emerging economy's stock exchange. These stocks are highly liquid and frequently traded by active managers; many are dual listed on international stock exchanges. Monthly returns spanning 20 years from Jan-00 to Jan-20 were used, thus covering an era characterised by different market conditions: the years of expansionary conditions which preceded the 2007-9 credit crisis, the credit crisis and post credit crisis turmoil. The currently evolving economic ramifications of the COVID-19 pandemic (May 2020) should contribute to an interesting case study.

The benchmark comprised equal proportions of these stocks and was rebalanced monthly. For the analysis that follows, five years of monthly returns were used to generate portfolio returns, volatilities and correlations. The analysis was rolled forward one month at a time (maintaining a five-year period to generate the relevant parameters) to explore the behaviour of TE constrained optimal **Ω** ratio portfolios, the impact of a sign-changing constant TE frontier main axis slope and the observed differences between security weights for different optimal portfolios constrained by TE. Although all rolling five-year periods were examined for this work (from Jan-00 to Jan-20), the analytical results which most strongly demonstrate these impacts mentioned above were selected (and presented). The two five-year periods identified were:

1. **Oct-00 – Oct-05** characterised by
    a. relatively low volatility,
    b. high returns – despite including the 9-11 US terrorist attack,
    c. a risk-free rate of 7.0%, and
    d. a main axis slope of the constant TE frontier $> \mathbf{0}$ and
2. **Oct-09 – Oct-14** characterised by
    a. high volatility,
    b. lower returns – in the aftermath of the 2007/8 global financial crisis,
    c. a risk-free rate of 5.8%, and
    d. a main axis slope of the constant TE frontier $< \mathbf{0}$.

Portfolio behaviour subject to TEs from 1% to 12% (in 1% increments) was explored. Descriptive statistics of these securities are set out in Table 4.1.

Table 4.1. Descriptive statistics for the period Oct-00 to Oct-05 and Oct-09 to Oct-14.

	Energy			**Materials**			**Retail**			**IT**		**Consumer**		**Financial**	
	A	**B**	**C**	**D**	**E**	**F**	**G**	**H**	**I**	**J**	**K**	**L**	**M**	**N**	**O**
2000 – 2005															
$\bar{\mu}$ (%)	30.0	15.3	9.3	3.9	22.5	13.8	16.0	43.7	18.9	11.5	10.4	30.2	29.6	14.0	20.1
$\mu_{\max}$ (%)	24.0	18.0	24.8	18.2	37.6	49.3	19.9	51.6	18.2	33.4	40.2	29.0	27.3	44.1	15.2
$\mu_{\min}$ (%)	-13.9	-18.2	-18.1	-20.9	-23.3	-21.9	-13.1	-23.3	-13.9	-28.6	-44.7	-11.9	-25.0	-47.1	-11.7
$\bar{\sigma}$ (%)	32.6	21.3	33.6	29.8	36.9	57.0	26.9	42.5	22.6	41.1	51.0	26.0	29.3	51.7	19.9
s	0.22	0.04	0.56	-0.16	0.75	0.67	0.50	1.13	0.04	0.13	-0.61	0.77	0.09	-0.33	0.30
κ	-0.77	1.13	0.05	-0.23	1.34	0.06	-0.30	3.23	-0.03	0.60	2.09	1.22	2.18	1.89	-0.02
2009 – 2014															
$\bar{\mu}$ (%)	14.2	-1.1	-2.0	9.0	-0.4	-26.5	20.8	41.8	17.1	14.8	39.8	37.2	12.2	37.5	7.4
$\mu_{\max}$ (%)	16.8	14.0	15.9	17.9	19.4	23.2	17.3	25.9	11.4	15.8	24.1	15.5	19.6	21.9	13.1
$\mu_{\min}$ (%)	-10.5	-14.3	-21.1	-11.8	-16.8	-28.3	-17.6	-14.1	-15.3	-9.0	-14.6	-19.0	-15.6	-11.4	-9.6
$\bar{\sigma}$ (%)	17.3	20.3	26.0	24.6	28.2	36.0	23.6	24.5	18.4	18.1	28.1	24.4	25.5	28.8	17.2

$\boldsymbol{s}$	0.52	-0.07	-0.17	0.27	0.22	-0.35	-0.38	0.54	-0.57	0.45	0.05	-0.69	0.40	0.48	0.31
$\boldsymbol{\kappa}$	0.72	0.06	0.18	-0.17	-0.69	0.50	0.31	1.90	0.56	0.38	-0.13	0.89	-0.12	-0.52	-0.04

Source: Bloomberg and author calculations.

Key: $\overline{\boldsymbol{\mu}}$ − mean *annual* return, $\boldsymbol{\mu}_{\max}$ − max *monthly* return, $\boldsymbol{\mu}_{\min}$ − min *monthly* return, $\overline{\boldsymbol{\sigma}}$ − mean annualised volatility, $\boldsymbol{s}$ − skewness and $\boldsymbol{\kappa}$ − kurtosis.

4.3.2 Methods

Active investment positions differ from benchmark positions according to the risk appetite of investors. Low TEs mean active weights must be small, while for higher TEs, active weights are larger (permitting a wider range of weights relative to the benchmark). The underlying variables, matrices and matrix notation, are defined below for a sample of N constituent securities:

$\boldsymbol{q}$: $1 \times N$ vector of benchmark weights

$\boldsymbol{x}$: $1 \times N$ vector of deviations from the benchmark

$\boldsymbol{q_P}$ $(= \boldsymbol{q} + \boldsymbol{x})$: $1 \times N$ vector of portfolio weights

$\boldsymbol{E}$: $1 \times N$ vector of expected returns,

$\boldsymbol{\sigma}$: $1 \times N$ vector of benchmark component volatilities

$\boldsymbol{\rho}$: $N \times N$ benchmark correlation matrix

$\boldsymbol{V}$: $N \times N$ covariance matrix of asset returns

$\mathbf{1}$: $1 \times N$ vector of 1s and

r_f: the risk-free rate.

Net short sales *are* allowed so the total active weights $(\boldsymbol{q}_i + \boldsymbol{x}_i)$ may be < 0 for individual securities. No assets outside the benchmark's set may be included using Roll's methodology – although in principal this is of course possible. Using matrix notation, expected returns and variances are:

$\mu_B = \boldsymbol{qE'}$: expected benchmark return

$\sigma_B = \sqrt{\boldsymbol{qVq'}}$: volatility (risk) of benchmark return

$\mu_\varepsilon = \boldsymbol{xE'}$: expected excess return; and

$\sigma_\varepsilon = \sqrt{\boldsymbol{xVx'}}$: TE.

The active portfolio expected return and variance is given by $\mu_P = (\boldsymbol{q}+\boldsymbol{x})\boldsymbol{E'} = \mu_B + \mu_\varepsilon$ and $\sigma_P = \sqrt{(\boldsymbol{q}+\boldsymbol{x})\boldsymbol{V}(\boldsymbol{q}+\boldsymbol{x})'}$ respectively. The portfolio must be fully invested, so $(\boldsymbol{q}+\boldsymbol{x})\boldsymbol{1}' = 1$.

The following definitions are also required: $a = \boldsymbol{EV^{-1}E'}, b = \boldsymbol{EV^{-1}1'}, c = \boldsymbol{1V^{-1}1'}, d = a - \frac{b^2}{c}$ and $\Delta_1 = \mu_B - \frac{b}{c}$ where $b/c = \mu_{MV}$ and $\Delta_2 = \sigma_B^2 - \frac{1}{c}$ with $1/c = \sigma_{MV}^2$ where MV is the minimum variance portfolio (Merton, 1972).

It is useful to recall the relevant mathematics which generate the various frontiers.

4.3.2.1 Mean variance frontier

Minimise $\boldsymbol{q_P V q_P'}$ subject to $\boldsymbol{q_P 1'} = 1$ and $\boldsymbol{q_P E'} = G$ where G is the target return. The vector of portfolio weights is $\boldsymbol{q_P} = \left(\frac{a-bG}{d}\right)\boldsymbol{q_{MV}} + \left(\frac{bG-\frac{b^2}{c}}{d}\right)\boldsymbol{q_{TG}}$ where q_{MV} is the vector of asset weights for the minimum variance portfolio given by $\boldsymbol{q_{MV}} = \boldsymbol{V}^{-1}\frac{\boldsymbol{1}}{c}$ and $\boldsymbol{q_{TG}}$ is the vector of asset weights for the tangent portfolio (with $r_f = 0$), i.e. $\boldsymbol{q_{TG}} = \boldsymbol{V}^{-1}\frac{\boldsymbol{E}}{b}$. The weights of the tangent portfolio's components, $\boldsymbol{q_{TP}}$, with $r_f \neq 0$, are:

$$\boldsymbol{q'_{TP}} = \frac{\boldsymbol{V^{-1}}\left(\boldsymbol{E} - r_f \cdot \boldsymbol{1}\right)'}{\boldsymbol{1} \cdot \boldsymbol{V^{-1}}\left(\boldsymbol{E} - r_f \cdot \boldsymbol{1}\right)'}$$

4.3.2.2 TE frontier

Maximise $\boldsymbol{xE'}$ subject to $\boldsymbol{x1'} = 0$ and $\boldsymbol{xVx'} = \sigma_\varepsilon^2$. The solution for the vector of deviations from the benchmark is $\boldsymbol{x'} = \pm\sqrt{\frac{\sigma_\varepsilon^2}{d}}\boldsymbol{V}^{-1}\left(\boldsymbol{E} - \frac{b}{c}\boldsymbol{1}\right)'$. The solution to this optimisation problem generates the TE frontier, a portfolio's maximal return at a given risk level and subject to a TE constraint.

4.3.2.3 Constant TE frontier

Maximise $\boldsymbol{xE'}$ subject to $\boldsymbol{x1'} = 0$, $\boldsymbol{xVx'} = \sigma_\varepsilon^2$ and $(\boldsymbol{q}+\boldsymbol{x})\boldsymbol{V}(\boldsymbol{q}+\boldsymbol{x})' = \sigma_P^2$. The vector of deviation weights from the benchmark is $\boldsymbol{x'} = -\frac{1}{\lambda_2+\lambda_3}\boldsymbol{V}^{-1}(\boldsymbol{E'} + \lambda_1 + \lambda_3\boldsymbol{Vq'})$ where $\lambda_1 =$

$-\frac{\lambda_3+b}{c}$, $\lambda_2 = \pm(-2)\sqrt{\frac{d\Delta_2-\Delta_1^2}{4\sigma_\varepsilon^2\Delta_2-y^2}} - \lambda_3$ and $\lambda_3 = -\frac{\Delta_1}{\Delta_2} \pm \frac{y}{\Delta_2}\sqrt{\frac{d\Delta_2-\Delta_1^2}{4\sigma_\varepsilon^2\Delta_2-y^2}}$. The solution for the weights which generate the tangent portfolio (to the constant TE frontier) was shown by Maxwell et al, (2018) to involve solving for σ_P using:

$$\frac{(r_f-\mu_B)}{\sigma_P^2} + \frac{\frac{(\Delta_1^2-d\Delta_2)\cdot(\sigma_P^2-\sigma_B^2-\sigma_\varepsilon^2)}{\sqrt{(\Delta_1^2-d\Delta_2)\left[(\sigma_P^2-\sigma_B^2-\sigma_\varepsilon^2)^2-4\Delta_2\sigma_\varepsilon^2\right]}}+\Delta_1}{\Delta_2} - \frac{\sqrt{(\Delta_1^2-d\Delta_2)\left[(\sigma_P^2-\sigma_B^2-\sigma_\varepsilon^2)^2-4\Delta_2\sigma_\varepsilon^2\right]}+\Delta_1\cdot(\sigma_P^2-\sigma_B^2-\sigma_\varepsilon^2)}{2\Delta_2\sigma_P^2}$$

then establishing μ_P on the efficient segment of the constant TE frontier and then backing out the relevant weights.

4.3.2.4 Constant TE frontier main axis slope, S_{MA}

The main axis slope, S_{MA} is calculated using

$$S_{MA} = \frac{\Delta_1}{\sigma_B - \sigma_{MV}} = \frac{\mu_B - \frac{b}{c}}{\sigma_B - \sigma_{MV}} = \frac{\mu_B - \mu_{MV}}{\sigma_B - \sigma_{MV}}$$

Δ_1 determines the sign of S_{MA} because the denominator is always > 0 since $\sigma_B - \sigma_{MV} > 0$ always. A necessary and sufficient condition for $S_{MA} < 0$ is $\mu_B < \mu_{MV}$. Note that the S_{MA} is independent of TE since none of its components depend explicitly thereon. For the first time, the sign and magnitude of the S_{MA} were measured and evaluated and how these (and constituents of the S_{MA}) change over time as market conditions evolve plus their influence on TE constrained portfolio performance were explored.

4.3.2.5 Optimal Ω portfolios

Consider a market with $\boldsymbol{n}$ stocks. The current time is $\boldsymbol{t = 0}$ and the end of the investment horizon is $\boldsymbol{t = T}$. A portfolio is completely characterised by a vector of weights $\boldsymbol{w} \in \mathbb{R}^{\boldsymbol{n}}$, such that $\sum_{\boldsymbol{i=1}}^{\boldsymbol{n}} \boldsymbol{w_i} = \boldsymbol{1}00\%$. The element $\boldsymbol{w_i}$ denotes the percentage of total wealth invested in the $\boldsymbol{i}$th stock at time $\boldsymbol{t = 0}$. Let $\boldsymbol{\tilde{r}_i}$ indicate the random return of asset $\boldsymbol{i}$ and with boldface the vector of return variables $\boldsymbol{\tilde{r}} \in \mathbb{R}^{\boldsymbol{n}}$. The random return of a portfolio of assets is defined as $\boldsymbol{\tilde{r}_p = w^T\tilde{r}}$.

Let $\boldsymbol{F(r_i)}$ and $\boldsymbol{f(r_i)}$ denote the cumulative density function and the probability density function, respectively. For an asset $\boldsymbol{i}$, Keating & Shadwick (2002) define the $\boldsymbol{\Omega}$ ratio as:

$$\boldsymbol{\Omega}(\tilde{\boldsymbol{r}}_i) = \frac{\int_{\tau}^{+\infty}[\mathbf{1} - \boldsymbol{F}(\boldsymbol{r}_i)]\boldsymbol{dr}_i}{\int_{-\infty}^{\tau}\boldsymbol{F}(\boldsymbol{r}_i)\boldsymbol{dr}_i} \tag{4.1}$$

Integration by parts and some algebraic transformation, the **Ω** ratio may be written:

$$\boldsymbol{\Omega}(\tilde{\boldsymbol{r}}_i) = \frac{\int_{\tau}^{+\infty}(\tilde{\boldsymbol{r}}_i - \boldsymbol{\tau})\boldsymbol{f}(\boldsymbol{r}_i)\boldsymbol{dr}_i}{\int_{-\infty}^{\tau}(\boldsymbol{\tau} - \tilde{\boldsymbol{r}}_i)\boldsymbol{f}(\boldsymbol{r}_i)\boldsymbol{dr}_i} = \frac{\mathbb{E}[(\tilde{\boldsymbol{r}}_i - \boldsymbol{\tau})^+]}{\mathbb{E}[(\boldsymbol{\tau} - \tilde{\boldsymbol{r}}_i)^+]} = \frac{\mathbb{E}(\tilde{\boldsymbol{r}}_i) - \boldsymbol{\tau}}{\mathbb{E}[(\boldsymbol{\tau} - \tilde{\boldsymbol{r}}_i)^+]} + \mathbf{1}$$

Therefore, the portfolio **Ω** ratio is:

$$\boldsymbol{\Omega}(\tilde{\boldsymbol{r}}_p) = \frac{\boldsymbol{w}^{\boldsymbol{T}}\mathbb{E}(\tilde{\boldsymbol{r}}) - \boldsymbol{\tau}}{\mathbb{E}[(\boldsymbol{\tau} - \boldsymbol{w}^{\boldsymbol{T}}\tilde{\boldsymbol{r}})^+]} + \mathbf{1} \tag{4.2}$$

Portfolio optimisation problems that aim to maximise the **Ω** ratio subjected to additional constraints on portfolio weights are explored here. The **Ω** maximisation problem can be written as

$$\max_{\omega \in \mathbb{R}^n} \frac{\boldsymbol{w}^{\boldsymbol{T}}\mathbb{E}[\tilde{\boldsymbol{r}}] - \boldsymbol{\tau}}{\mathbb{E}[(\boldsymbol{\tau} - \boldsymbol{w}^{\boldsymbol{T}}\tilde{\boldsymbol{r}})^+]} \tag{4.3}$$

s.t. $\boldsymbol{w}^{\boldsymbol{T}}\mathbf{1} = \mathbf{100}\%$ and $\underline{\boldsymbol{w}} \leq \boldsymbol{w} \leq \overline{\boldsymbol{w}}$.

The objective is to determine the allocation that gives the optimal weights ($w \in R^n$) that result in the portfolio with the maximum Ω ratio. The constraints above relate to the budget constraint and the upper and lower bound on any individual investment.

The discrete analogue for (4.2) is

$$\boldsymbol{\Omega} = \frac{\boldsymbol{w}^{\boldsymbol{T}}\overline{\boldsymbol{r}} - \boldsymbol{\tau}}{\sum_j [\boldsymbol{\tau} - \boldsymbol{w}^{\boldsymbol{T}}\boldsymbol{r}_j]^+ \boldsymbol{p}_j} \tag{4.4}$$

The optimisation problem is

$$\max_{w} \frac{\boldsymbol{w}^{\boldsymbol{T}}\overline{\boldsymbol{r}} - \boldsymbol{\tau}}{\sum_j [\boldsymbol{\tau} - \boldsymbol{w}^{\boldsymbol{T}}\boldsymbol{r}_j]^+ \boldsymbol{p}_j} \tag{4.5}$$

s.t. $\sum \boldsymbol{w}_i = \mathbf{1}$, and $\underline{\boldsymbol{w}} \leq \boldsymbol{w} \leq \overline{\boldsymbol{w}}$ and where $\boldsymbol{p}_j = \mathbf{1}/\boldsymbol{n}$.

Using the portfolio weights derived from (4.5), Ωs may be calculated, and their component numerators and denominators graphed. Figure 4.1 presents interesting similarities with the MPT's mean-variance framework and Sharpe ratio optimisation. Point above the frontier are unattainable and investors may choose better solutions for all coordinates below. Kaspos et al., (2011) named this locus of points the Ω frontier and found that its concave, non-decreasing feature arose from the optimisation problem's (4.5) convexity property. For each point on the frontier, the Ω ratio is determined by the gradient of the line passing through it and the origin, so an optimal solution (maximum Ω ratio) is that point at which the line which passes through the origin has the highest slope (i.e. tangent to the Ω frontier).

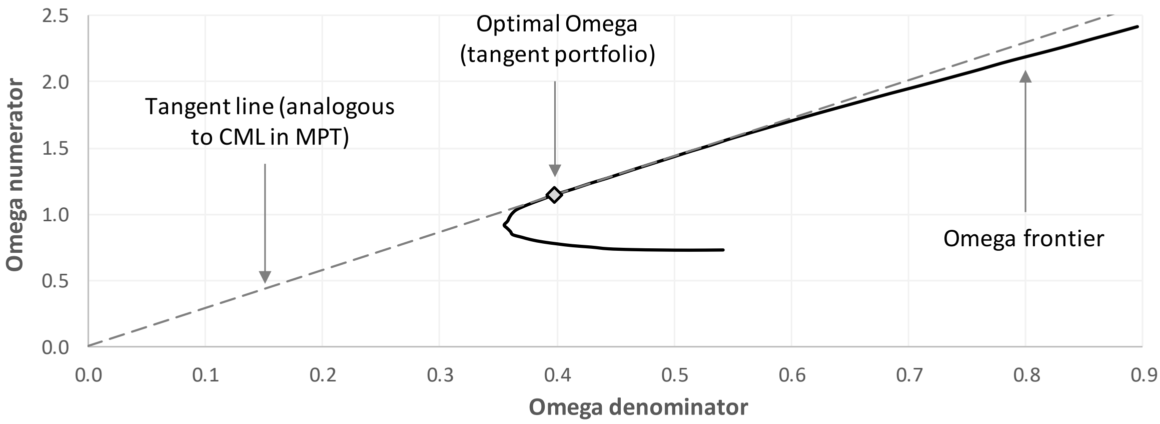

Figure 4.1: Ω frontier, analogous capital market line and location of the optimal Ω portfolio. Source: Author calculations.

4.4 RESULTS AND DISCUSSION

The aim is to generate constant TE frontiers for varying levels of TE (1% to 12% in 1% intervals) and then, for each constant TE frontier, establish the risk and return (and corresponding portfolio constituent weights) for each of the following maximal portfolios: return, Sharpe, and $\boldsymbol{\Omega}$. There are two "maximum $\boldsymbol{\Omega}$ ratio portfolios"; one which simultaneously satisfies the relevant TE constraint and maximises the return at each TE-constrained risk level and the other unconstrained by TE, i.e. a universal maximum $\boldsymbol{\Omega}$ portfolio. The former is identified by first selecting the (known) asset weights which generate the upper hemisphere of portfolios on the constant TE frontier (i.e. from minimum to maximum variance portfolios *on* the ellipse). These weights are then used to generate portfolio returns over the chosen period of interest (five years of monthly returns, rolled forward one month at a time, since Jan-00) and the associated $\boldsymbol{\Omega}$ ratio calculated for each set of 60 (5y) returns. By construction, these portfolios lie on the constant TE frontier. The $\boldsymbol{\Omega}$ ratio – measured using as threshold the benchmark return – for each portfolio is then plotted on the same $\boldsymbol{x}$-axis (risk) as the constant TE frontier (the solid black line in Figure 4.2 for $\boldsymbol{TE} = \boldsymbol{6}\%$). The unconstrained (universal) **Ω** ratio, using the same threshold, is also shown in Figure 4.2, along with the efficient frontier and the capital market line (CML) for the constant TE frontier (Maxwell, et al, 2018).

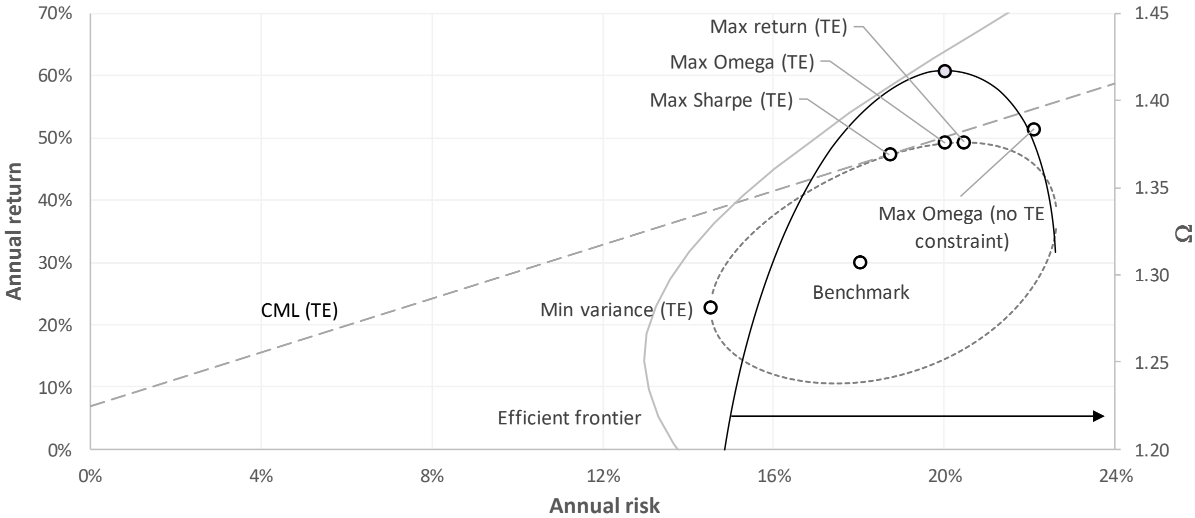

Figure 4.2. Orientation of relevant components in Oct-05. $TE = 6\%$ and $r_f = 7.0\%$. The Ω ratio as a function of risk is shown as a solid black line, tied to the right-hand axis (the maximum Ω ratio on this curve is indicated). All other elements are linked to the left-hand axis.

Source: Bloomberg and author calculations.

In Figure 4.2, the period selected was Oct-00 to Oct-05 using monthly returns. This period precedes the credit crisis of 2007-09 when markets enjoyed buoyant returns and reduced volatility, giving rise to a constant TE ellipse with a positive main axis (see Gunning & van Vuuren, 2019). The maximum Ω ratio portfolio – constrained by TE – lies between (in terms of risk and return) the maximum Sharpe ratio and maximum return portfolios, while the universal (unconstrained) Ω ratio portfolio lies outside the constant TE frontier with higher risk and higher return than all other constant TE frontier portfolios (in this example where $TE = 6\%$ and $r_f = 7.0\%$).

In Figure 4.3, the period selected was Oct-09 to Oct-14, i.e. post the worst of the turbulent market volatility instituted by the credit crisis of 2007-09. Portfolio annual returns are substantially lower than those observed in the period preceding the credit crisis and annual risk is higher: the configuration resulting in a negative main axis for the constant TE ellipse. The maximum TE-constrained Ω ratio portfolio again lies between (in terms of risk and return) the maximum Sharpe ratio and maximum return portfolios, while the universal (unconstrained) Ω ratio portfolio again lies outside the constant TE frontier with higher risk and higher return than all other constant TE frontier portfolios (in this example where $TE = 6\%$ and $r_f = 5.8\%$).

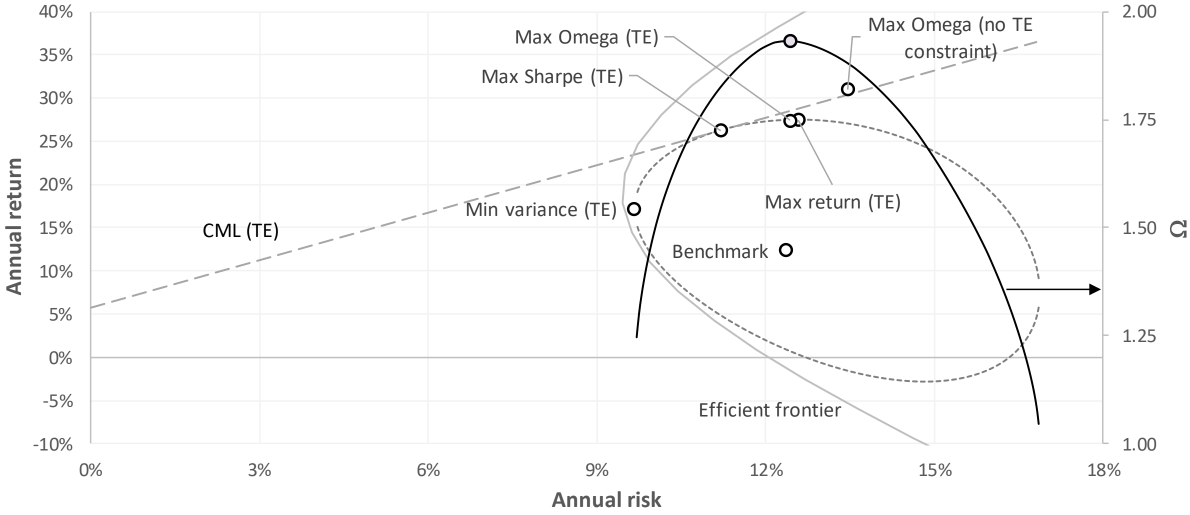

Figure 4.3. Orientation of relevant components in Oct-14. $TE = 6\%$ and $r_f = 5.8\%$.

Source: Bloomberg and author calculations.

Figure 4.4(a) shows the **Ω** frontiers for portfolios with returns selected from the two periods (Oct-00 – Oct-05 and Oct-09 – Oct-14). Figure 4.4(b) plots **Ω** ratio at each corresponding threshold. **Ω** ratios are higher in the latter period because high volatility here leads to a higher dispersion of portfolio returns. The overall increase in quantity and magnitude of returns $> \mathbf{0}\%$ in this period, combined with the greater dispersion elevates **Ω** ratios $\forall \boldsymbol{\tau} > \mathbf{0}\%$.

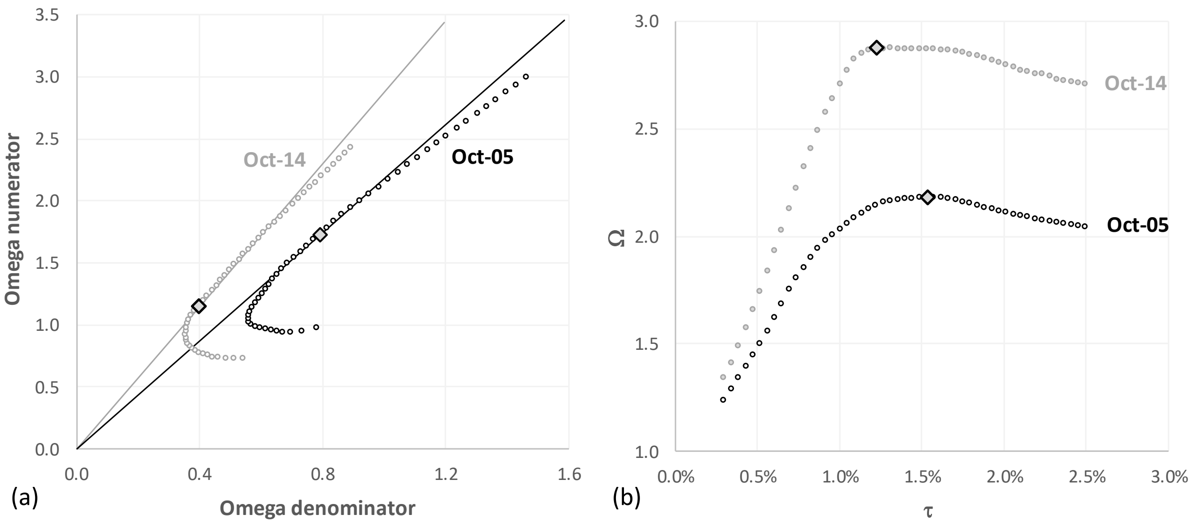

Figure 4.4. Analysis of the Ω ratio for Oct-00 – Oct-05 and Oct-09 – Oct-14.

(a) Ω frontiers (b) maximum $\Omega(\tau)$

Source: Bloomberg and author calculations.

Constituent deviations from the benchmark (i.e. x_i) in the optimal, universal, unconstrained Ω portfolios for the two periods are provided in Figure 4.5, grouped by market sector.

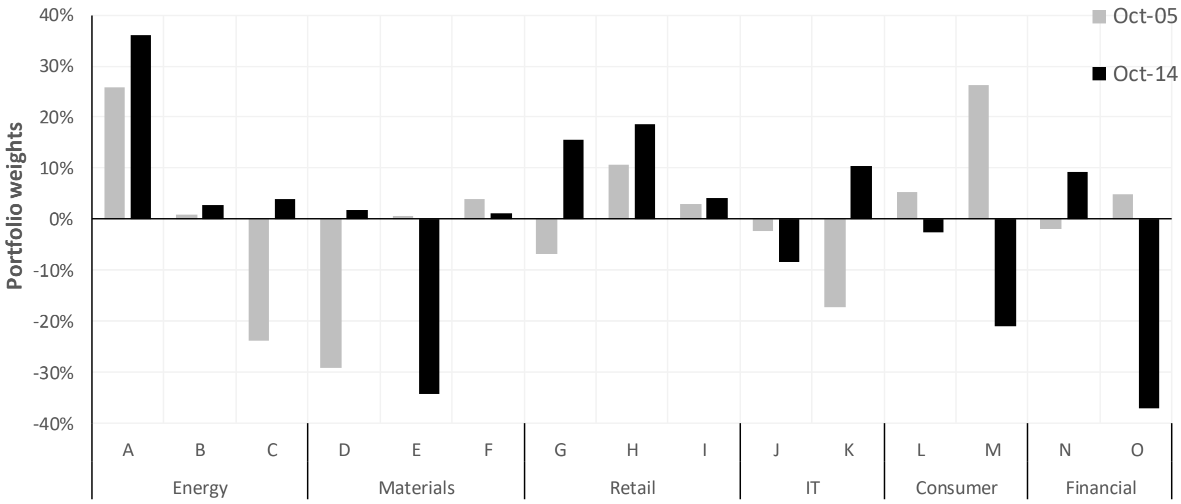

Figure 4.5. Weights in optimal, unconstrained Ω portfolios for Oct-00 – Oct-05 and Oct-09 – Oct-14.

Source: Bloomberg and author calculations.

In the former period, the optimal unconstrained **Ω** portfolio strongly overweights assets A and M while strongly underweighting C and D. Both A and M witnessed considerably growth in the low risk, high return pre-crisis period, skewing their return distributions to the right, while C and D both experienced large losses during this time, skewing both return distributions to the left.

Similar observations were noted for assets A (strong growth post the credit crisis), and E and O (large losses with widely dispersed returns) in the latter period.

The performance of the remainder of the assets was unremarkable, their return distributions characterised by low skewness and low excess kurtosis. As a result, deviations from the benchmark weights are small.

The behaviour of the maximum Sharpe ratio, maximum constrained **Ω** and maximum return portfolios as a function of TE is shown in Figure 4.6(a) for the period Oct-00 – Oct-05. The locus of the return/risk coordinates all increase monotonically as TE increases and the relative configuration is preserved for all TEs (both the risk and return of the maximum Sharpe ratio portfolio less than that of the maximum **Ω**, and in turn less than that of the maximum return portfolio). Figure 4.6(b) shows the associated Sharpe ratios for all portfolios as a function of TE – again monotonically increasing with the maximum Sharpe ratio portfolio exhibiting, as expected, the highest Sharpe ratio for all TEs. Constant TE frontiers of 3%, 7% and 12% are displayed for scale and the Sharpe ratio for the constrained optimal **Ω** portfolio is shown as a dotted line in Figure 4.6(b) for comparison. the vertical scales in Figure 4.6(a) and (b) are the same as those for Figure 4.7(a) and (b) for direct comparison.

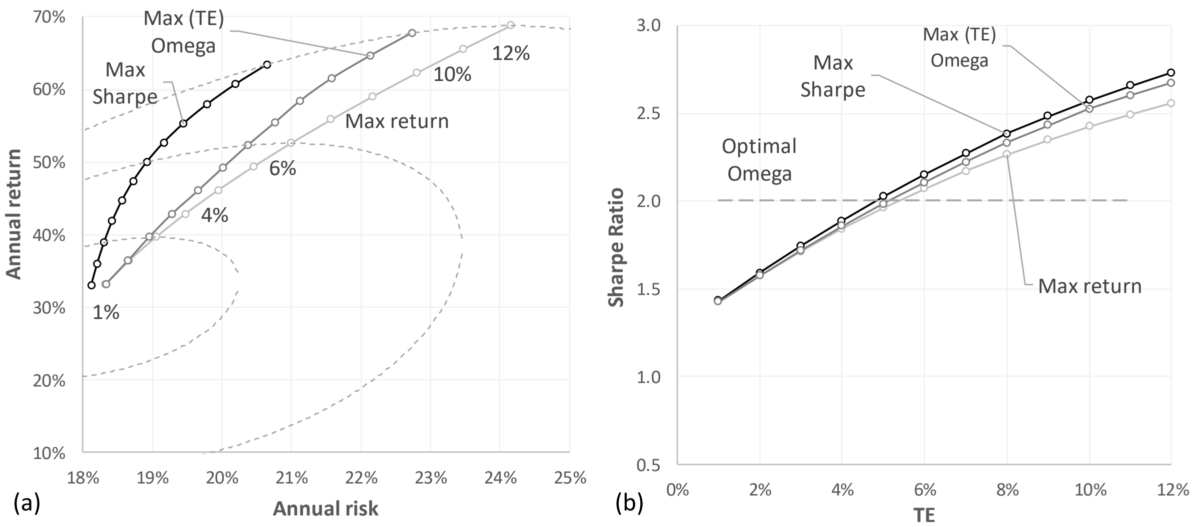

Figure 4.6. Analysis for Oct-00 – Oct-05.

(a) Return/risk profiles for relevant portfolios as a function of TE (percentages indicate TE values)

(b) Sharpe ratios versus TE for Oct-00 – Oct-05. Constant TE frontiers at 3%, 7% and 12% are shown for comparison. The optimal Ω ratio's Sharpe ratio is indicated as a dashed line.

Source: Bloomberg and author calculations.

Figure 4.7 duplicates the analysis presented in Figure 4.6, but for the period Oct-09 – Oct-14. When the main axis of the constant TE frontier is negative, returns for all maximal portfolios increase monotonically with increasing TE, while risk for these portfolios decreases then increases again as TE increases.

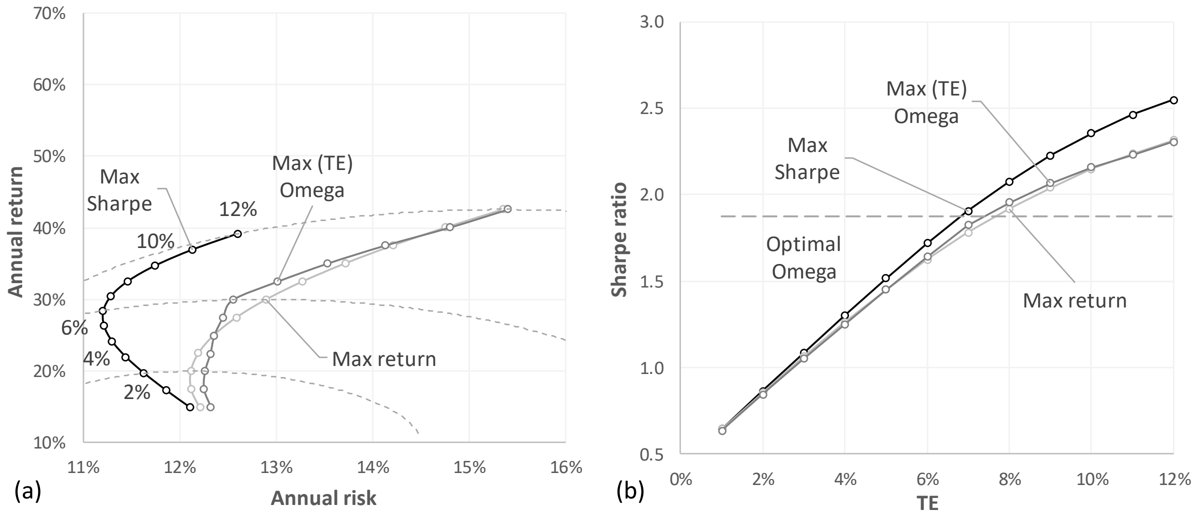

Figure 4.7. Analysis for Oct-09 – Oct-14.

(a) Return/risk profiles for relevant portfolios as a function of TE

(b) Sharpe ratios versus TE for Oct-09 – Oct-14. Constant TE frontiers at 3%, 7% and 12% are also shown for comparison.

Source: Bloomberg and author calculations.

For $\boldsymbol{TE} < \mathbf{5}\%$, the constrained maximum $\boldsymbol{\Omega}$ ratio portfolio does not lie on the efficient constrained portfolio set – it lies to the right of the maximum return portfolio, i.e. it has higher risk and lower return (recall that the efficient set spans the upper hemisphere of the ellipse from the minimum variance portfolio on the left to the maximum return portfolio on the right. Portfolios outside this region are inefficient). Because the same level of return is possible for this maximum constrained $\boldsymbol{\Omega}$ portfolio, it is inefficient. This may not, in fact, be true because the $\boldsymbol{\Omega}$ ratio makes no assumptions of return distribution normality. Instead, it uses the empirical distribution and thus may still be efficient because both the max Sharpe and max return portfolios do assume a normal distribution of returns.

Figure 4.8 compares weight deviations from the benchmark for the three portfolios over (a) the Oct-00 – Oct-05 period and (b) the Oct-09 – Oct-14 period.

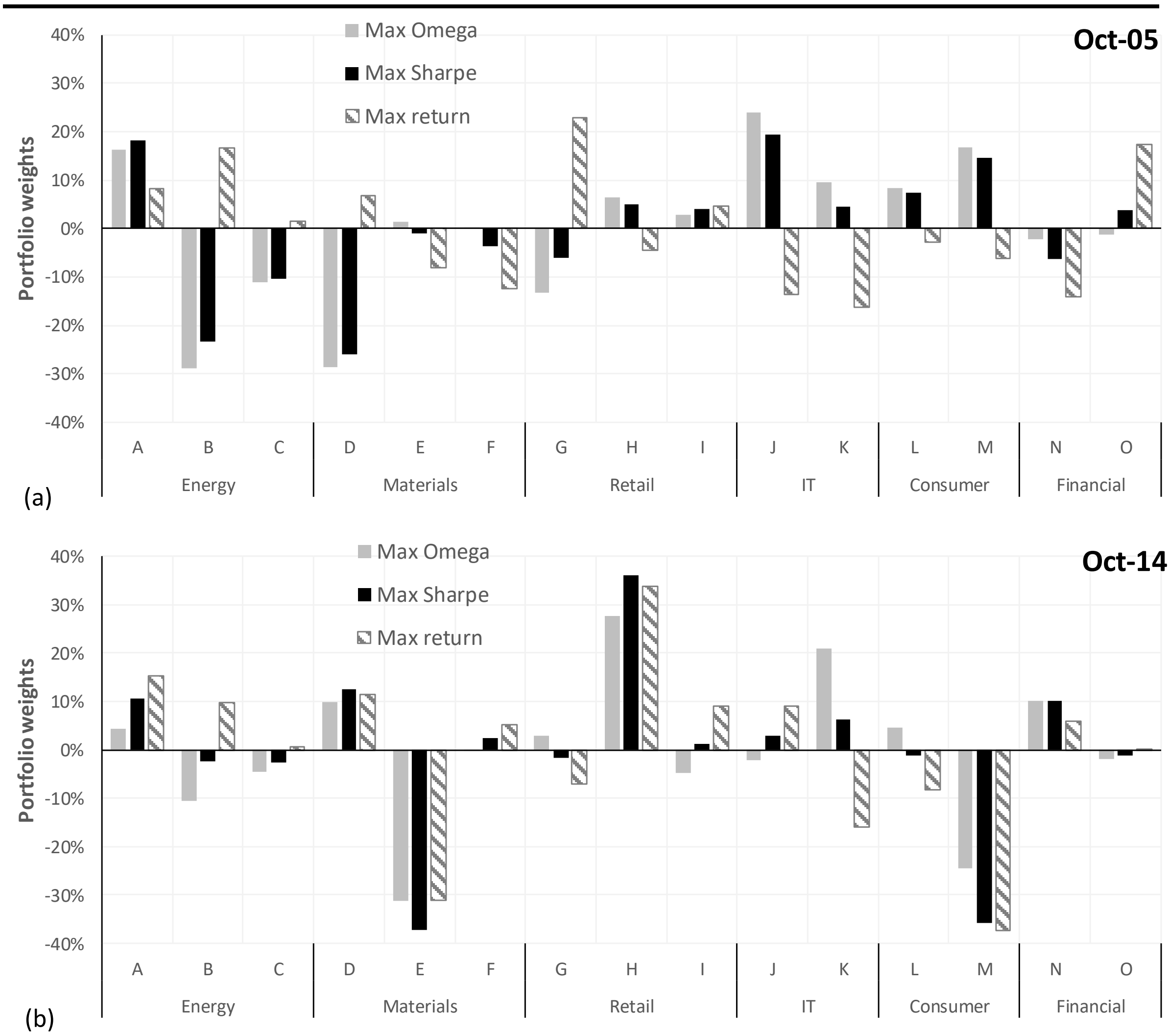

Figure 4.8: Benchmark weight deviations for relevant portfolios

(a) Oct-00 – Oct-05 (b) Oct-09 – Oct-14

Source: Bloomberg and author calculations.

Reasons for large over or underweighting remain – for either period – as discussed previously (for Figure 4.5). In Figure 4.8(a) for Oct-00 – Oct-05, the constant TE frontier's main-axis slope is $> \mathbf{0}$ (Figure 4.2) while for Figure 4.8(b) for Oct-09 – Oct-14 the main-axis slope is $< \mathbf{0}$ (Figure 4.3). Deviations of asset weights from the benchmark vary considerably over the two periods. Not only do the relative weights differ in *magnitude*, the *signs* (overweight/underweight) are also often different. The size of constituent asset deviation from the benchmark weights ($> \mathbf{0}$% or $< \mathbf{0}$%) is also greater when the main axis slope is $> \mathbf{0}$, but although the relative weights of the constituents often have different signs (like Oct-05), the magnitude of the differences are negligible. These observations are explained by the fact that when the main-axis slope is $> \mathbf{0}$, the range of risks spanned by the efficient portfolio set is greater than when the main-axis slope is $< \mathbf{0}$ (earlier period's approximately 14.5% to 20.5% (6% risk range) compared with later period's 9.5% to 12.5% (3% risk range) in this example).

Figure 4.9 presents asset K's weight deviations from the benchmark over the two periods as a function of TE. Several interesting features are apparent. For one, the profiles are broadly similar regardless of main axis slope: all increase or decrease monotonically as TE increases. This reflects the stability of benchmark deviations as TE changes – these are gradual, not abrupt. Another is that the benchmark weight deviations for the maximum Sharpe ratio and maximum return portfolios are almost identical over the two periods while those for the maximum $\boldsymbol{\Omega}$ portfolio are notably higher in the second period.

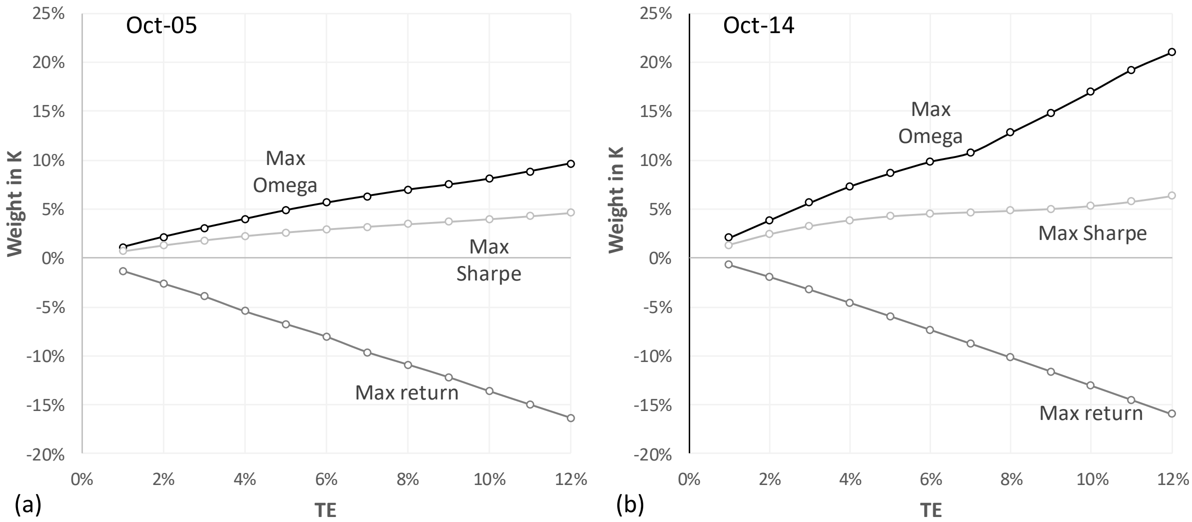

Figure 4.9: Asset K's deviation in weight from benchmark for the relevant portfolios

(a) Oct-00 – Oct-05 (b) Oct-09 – Oct-14.

Source: Bloomberg and author calculations.

Asset K is a large conglomerate, which enjoyed unprecedented success after the credit crisis. The company's profitability was buoyed by several fortuitous mergers and acquisitions, its weighting in the overall market index swelling from 1% to 10% over this period. K's return distribution becomes increasingly skewed to the right as time passes. The weights in the maximum $\boldsymbol{\Omega}$ ratio portfolio increase accordingly, as the $\boldsymbol{\Omega}$ ratio balloons. This is untrue the maximum Sharpe portfolios because this approach assumes that returns are normally distributed. As K's volatility increases over time (due to large *positive* returns), so the Sharpe ratio is penalised (as a risk-adjusted return portfolio) so weights do not change much despite a clearly strongly-performing asset. The maximum return portfolio underweights K relative to the benchmark because its performance is strongly tied to that of the benchmark (itself a representation of the broad market, being well-diversified and having equally weighted components). This leads to strong positive correlations with other asset returns which perform favourably over the periods but not spectacularly. Adding asset K increases the risk relative to the benchmark beyond that of the specified PD, so the only option is to underweight this asset at the expense of the others. This may still generate the highest return portfolio, but it is only the maximum $\boldsymbol{\Omega}$ ratio portfolio which fully exploits a strongly outperforming asset.

CHAPTER 5 CONCLUSIONS AND FUTURE DIRECTIONS

5.1 CONCLUSIONS

Markowitz's (1956) work established a solid and robust framework for efficient portfolio asset allocation and Sharpe (1964) laid out the groundwork for portfolio optimisation within Markowitz's (1956) model. Despite considerable subsequent research and the introduction of many new ideas – even investment styles – portfolio optimisation under the assumption utility maximisation seeking (higher returns preferred over lower returns) rational investors remains key to modern portfolio theory and investment approaches. Both passive and active investment styles are commonplace today (2020) and all investors make use of one (or hybrids of both) of these asset management methods. Although investor interest in the approaches wax and wane with market conditions, associated fees and costs and the inexorable search for yield, substantially more asset allocation work has focussed on passive than active investing. The gap left in the literature is now being filled by new research, invigorated by renewed interest in active investing.

Transferring well-known and successful principles from passive to active management is non-trivial: the mathematics governing the two, while similar in some respects, also differ considerably. Portfolio performance measured relative to a benchmark, it transpires, involves not only an alteration of risk/return metrics. Efficient frontiers in passive mean/variance space are hyperbolic, while in active mean/variance space, they are ellipsoidal. The features introduced by these differences unveil several interesting attributes: *maximal* variance, *maximal* return, long and short (ellipse) axes, orientation and magnitude of these axes, and a bounded mean/variance space – none of which are present in the better-known passive arena.

Confronted by the array of novel research directions, and mindful of the burst of new research in this arena, two important features of active asset allocation were chosen: what is the influence of changing market conditions on

- the orientation and magnitude of the constant TE frontier's main axis, and
- the behaviour of the maximum Ω ratio portfolio on the constant TE frontier.

5.1.1 Theme 1 – The main-axis slope of the constant TE frontier

South Africa is an emerging economy which experiences periods of financial turmoil, political scandals, and considerable currency fluctuations. Following these periods of turmoil, large rebalancing of relative asset weights is required. The results and findings reiterate the importance of diversification, for any developing economy, especially in stocks which contain a percentage of international holdings. Investors, however, shy away from uncertainty and constant rebalancing because of the high transaction costs and tax liabilities: foreign investments have decreased in South Africa over the recent past.

The significant positive relationship between the S_{MA} and the maximum Sharpe ratio confirms the link between these metrics and the possibility of a trading strategy. During boom conditions, sharp S_{MA} turning points when $S_{MA} \geq 0$ trigger roughly 12 months of improving or deteriorating Sharpe ratios, depending on whether the turning point was a local maximum or minimum. Investors may adjust portfolio holdings accordingly. In bust conditions, when the market experiences currency weakness, high market volatility or both, the S_{MA} is an unreliable indicator or future market moves and should not be used. These results, and possible investor actions, are summarised in Table 5.1.

Table 5.1. Investment strategies from sharp turning points in $\boldsymbol{S_{MA}}$.

<table>
<tr><th></th><th>Boom</th><th>Bust</th></tr>
<tr><td>$S_{MA} > 0$, downturn</td><td>Sell portfolio or decrease holdings. ≈ 1 year of declining Sharpe ratios to follow (indicating deteriorating returns and/or increasing market volatility)</td><td>No reliable signal given</td></tr>
<tr><td>$S_{MA} < 0$, upturn</td><td colspan="2">Purchase portfolio or increase holdings. ≈ 1 year of improving Sharpe ratios to follow (indicating improving returns and/or decreasing market volatility)</td></tr>
</table>

Source: Author estimates.

5.1.2 Theme 2 – The optimal $\boldsymbol{\Omega}$ ratio under a TE constraint

Identifying and characterising the behaviour of portfolios with a maximum $\boldsymbol{\Omega}$ ratio and *constrained by TEs* has been implemented and investigated here for the first time. Such portfolios differ from universal – unconstrained $\boldsymbol{\Omega}$ ratio portfolios, both in risk and return characteristics as well as constituent asset weights (and, therefore, they differ in their respective weight deviations from the benchmark). Unconstrained maximum $\boldsymbol{\Omega}$ portfolios distribute weights among components depending on both positive and negative return configurations. Portfolios which limit the magnitude of returns $< \mathbf{0}$ and encourage returns $> \mathbf{0}$ have the highest $\boldsymbol{\Omega}$ ratio and are the ones selected for optimality. TE-constrained portfolios, however, must allocate component asset weights differently. Because the relative risk level of these portfolios *must* equal the TE, constrained $\boldsymbol{\Omega}$ portfolios penalise assets whose risk profile prevents reaching relative risk equal to the TE (while favouring assets which generate portfolios with more positive returns than negative ones). This arises from the complex interplay of not only component volatilities, but also component correlations. Individual assets whose returns are strongly correlated with those of the benchmark (or "market" if the benchmark represents broad market exposure) – while appearing favourable due to high positive returns – may be penalised because their inclusion leads to relative risk different from the TE.

When the constant TE frontier's main axis is $< \mathbf{0}$ (a feature that arises only in conditions of high market turbulence, usually short-lived, lasting only a few months), the range of possible return/risk combinations for optimal portfolios (Sharpe, $\boldsymbol{\Omega}$ ratio or return) is considerably reduced. Portfolios under these conditions exhibit similar risks and returns and have similar component weights. When the constant TE frontier's main axis slope is $> \mathbf{0}$ (a longer lasting and far more prevalent feature of the constant TE frontier, arising from "normal" market conditions), the range of possible return/risk combinations is considerably greater. Variation in component asset weights is also higher when the constant TE main axis slope is $> \mathbf{0}$.

5.2 FUTURE DIRECTIONS AND RESEARCH NEEDS

5.2.1 Theme 1 – The main-axis slope of the constant TE frontier

Future work could investigate the link, if any, between the frequency of the market cycle and the cycle frequency of the changing main axis slope. The slope does not change instantaneously, but rather evolves over time, in response to changing market conditions. It is not unreasonable to assume that there is some degree of correlation between the two frequencies – or, as proxy, between the amplitude of market cycles and the size of the main axis slope. Fourier analysis (which detects underlying cycle frequencies in noisy signals) could help in this regard, as could the Kalman filter (which uses Bayesian statistics and rudimentary machine learning and could be used to assess patterns in time series data). Establishing such a connection could prove fruitful in the early detection and amelioration of financial crises.

Future work could also develop and augment the methodology, applying it to other developing and developed economies. Although there are no reasons to suspect the results will be any different (simulated runs have so far given similar results) it will be interesting to compare the impact during another economy's boom and bust periods.

Out-of-sample backtesting could also be performed on market data to determine the robustness and accuracy of the forecasts.

5.2.2 Theme 2 – The optimal Ω ratio under a TE constraint

Passive asset managers relying on absolute rather than relative performance should always allocate asset weights using the universal maximised $\boldsymbol{\Omega}$ ratio portfolio. Such portfolios have been shown to outperform other vaunted "optimal" alternatives. Active managers who require portfolios to outperform a prescribed benchmark while maintaining a prescribed level of risk relative to it subvert the mechanisms employed by optimal $\boldsymbol{\Omega}$ ratio portfolio construction, reducing – or eliminating – its effectiveness. In these cases, market conditions – which dictate the sign of the main axis slope of the constant TE frontier – should also be considered. When the main axis of the constant TE frontier is $> \mathbf{0}$, strategic active asset managers should allocate asset weights using a maximum Sharpe ratio framework.

Tactical (shorter term) active asset managers should use a maximum **Ω** ratio approach to determine asset weights. Possible future work could include an explicit, long-term, empirical investigation of the veracity of these conclusions. Current (2020) highly volatile market conditions, due to the fallout induced by the COVID-19 pandemic, could serve as an interesting case study to gauge and calibrate these effects.

The link between the family of four Johnson distributions and the Ω ratio could be examined in more detail and the strategy implemented in a constrained regime. The Ω performance measure is well-suited to the family of Johnson distributions. Passow (2004) found that Johnson-Ω portfolios, i.e. those whose higher portfolio moments were decompositions and derived to specifically include expected higher moments on fund levels gave significantly higher returns without sacrificing capital protection needs. Such a strategy, installed using an active investment strategy, could provide interesting results – especially in turbulent markets.

Further generic research could involve a Black-Litterman approach which enables tactical management of portfolios by combining information assembled from historical (or expected) returns and from some personal "view" (considered expert opinion) about asset returns. The Black & Litterman (1991, 1992) approach was introduced to facilitate effective portfolio optimisation by incorporating investor (or fund manager) views into the asset allocation process using a Bayesian method to combine the investor's views about expected asset returns with the prior information given by the vector containing the implied equilibrium returns. Posterior information is provided by a distribution whose mean is the mixed estimate of expected returns, and whose variance is a function of the covariance matrix of implied returns and of a diagonal matrix in which the confidence in the manager's views are established.

Combining a Black-Litterman framework with TE-constrained portfolio asset allocation could alleviate expected return forecasting issues and institute better allocation of funds for investments. This will form the focus of future research as this is a woefully underexplored feature of TE constrained portfolios.

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