Sustainable vector/pest control using the permanent Sterile Insect Technique

R. Anguelov[a], Y. Dumont[a,b,c], I.V. Yatat Djeumen[a]

[a] *University of Pretoria, Department of Mathematics and Applied Mathematics, Pretoria, South Africa*
[b] *CIRAD, UMR AMAP, F-97410 Saint-Pierre, La Réunion, France*
[c] *AMAP, Univ Montpellier, CIRAD, CNRS, INRAE, IRD, Montpellier, France*

Abstract

Vector/Pest control is essential to reduce the risk of vector-borne diseases or losses in crops. Among all biological control tools, the sterile insect technique (SIT), which consists of massive releases of sterile insects to reach elimination or to lower a vector/pest population under a certain threshold, is the most promising one. The models presented here are minimalistic with respect to the number of parameters and variables. The first model deals with the dynamics of the vector population while the second model tackles the interaction between treated males and wild female vectors. For the vector population model, equilibrium $\mathbf{0}$ is globally asymptotically stable when the basic offspring number, $\mathcal{R} \leq 1$ whereas $\mathbf{0}$ becomes unstable and one stable positive equilibrium exists, with well-determined basins of attraction, when $\mathcal{R} > 1$. For the SIT model, we obtain a threshold number of treated males above which the control of wild population is effective using massive releases. When the amount of treated males is lower than the aforementioned threshold, the SIT model experiences a weak Allee effect, i.e. $\mathbf{0}$ becomes locally asymptotically stable, while a positive equilibrium still exists. Practically, massive releases of sterile males are only possible for a short period. That is why using the Allee effect, we develop a new strategy to maintain the wild population under a certain threshold, for a permanent and sustainable low level of SIT control. We illustrate our theoretical results with numerical simulations. In particular, we study the combination of SIT with other control tools, like mechanical control and adulticide.

Keywords: Sterile Insect Technique, Vector control, Pest control, Weak Allee effect, Monotone system.

1. Introduction

In the last decades, the development of sustainable vector control methods has become one of the most challenging issues to reduce the impact of human vector borne diseases, like malaria, dengue, chikungunya or crop pests, like fruit flies.

Several control techniques have been developed or are under development. However, the process to reach field applications is long and complex. Modeling, and in particular Mathematical Modeling has become a useful tool in Human Epidemiology since the pioneering works of Sir R. Ross and his malaria model [16, 17]. Numerous models have been developed

Email addresses: roumen.anguelov@up.ac.za (R. Anguelov), yves.dumont@cirad.fr; yves.dumont@up.ac.za (Y. Dumont), ivric.yatatdjeumen@up.ac.za (I.V. Yatat Djeumen)

to understand the dynamics of diseases and pests to test "in silico" the usefulness or not of control strategies (and their combination).

In this paper, we focus on the Sterile Insect Technique (SIT). This is an old control techniques that have been used more or less successfully on the field against various kind of Pests or Vectors (see [11] for various examples). The classical SIT consists of mass releases of males sterilized by ionizing radiation. The released sterile males transfer their sterile sperms to wild females, which results in a progressive decay of the targeted population. For mosquitoes, other sterilization techniques have been developed using either genetics (the RIDL technique) or bacteria (wolbachia) [18]. For fruit flies, only ionizing radiation has been used, so far [11].

This work builds on [20], where SIT against mosquitoes only has been considered. However, it is important to notice that the results obtained in [20] can be used against crop pests too. An important assumption in [20] is that the insect population dynamics exhibit a strong Allee effect. Then, the application of SIT for an estimated finite time is sufficient to drive the population below the minimum survival density. However, for insect population the minimum survival density tends to be very close to extinction, that is an area of the domain, where deterministic modelling is not considered adequate. Hence, in this paper we do not make such assumption, but rather propose a strategy, which relies on an Allee effect generated by the SIT control. Indeed, in previous works, e.g. [2, 8, 20], it has been shown that even low levels of SIT control produce a tangible minimum survival density, below which the population is driven to elimination. To this end, we need to keep the insect population at a very low level and/or to sustain the decay, such that SIT cannot be discontinued or suppressed. Otherwise, the wild population will recover and reach its initial state. In this sense we talk about "permanent" SIT. The level of permanent SIT control is determined by the available resources (to produce sterile males). Once this level is known, higher level of releases can be used in short term in order to bring the insect population density below the minimum survival level associated with the lower, but long-term, sustainable SIT level of control. The aim of this paper is to show the feasibility of this type of SIT control strategy as well as specific methods for calculating its essential parameters.

In practice, it is well known that SIT alone is not sufficient to control a wild population. In general, it is recommended by IAEA (the International Atomic Energy Agency) to combine several control tools. We will first consider Mechanical Control (MC), which consists of removing the breeding sites, because it was showed in [6, 10] that MC can be efficient in addition to be sustainable. A second control tool is the use of massive spraying of adulticide, like Deltamethrin, the only authorized adulticide in La Réunion (France). Even if it is very efficient to reduce adult populations, it can be very detrimental to the environment, and also mosquitoes can develop resistance, if its use is too long, like in the French West Indies.

The outline of the paper is as follows. In the next section, we present a minimalistic entomological model of wild insect population and the discussion of its global dynamical properties. Section 3 deals with the study of the SIT mathematical model in the case of constant and continuous SIT releases. The key finding is the identification of a threshold number of sterile male vectors above which the control of the wild population is effective; that is, the wild population declines to extinction. Section 4 is devoted to the characterization of the minimal time necessary to reduce the size or spatial density of wild vector population under a given threshold when using SIT releases, that is, by considering the SIT model studied in section 3. Section 5 deals with the study of the SIT mathematical model in the case of periodic and impulsive SIT releases. Notably, by using suitable comparison

arguments, we provide condition of reaching elimination of wild vector population with periodic and pulse SIT releases and, characterize the minimal time necessary to lower the wild vector population under a given threshold in order to reduce the epidemiological risk. The theoretical results are discussed and supported by numerical simulations in section 6. In section 6, we simulate and discuss combinations of SIT with MC and/or adulticide. Finally, in section 7, we summarize the main outputs of this work, and their novelties compared to previous SIT works, and how it can be extended in the future.

2. A minimalistic entomological model

The model presented in this section is minimalistic in the sense that it uses smallest possible number of compartments which allows for adequate modelling of the mechanism of SIT control. We simplify the model developed in [2] by considering only three compartments: A, the aquatic stage (gathering eggs, larvae and pupae stages), F, the mature female, and M the male. Nevertheless, and we will see in the sequel, it has the same asymptotic properties as the other mentioned models. The advantages of using this simpler model are: On the one hand, while the model remains biologically meaningful, it allows a full theoretical analysis. On the other hand, it is more generic and can be applied to a variety of insect populations.

Following the compartmental diagram given in Figure 1, we derive the following system of ordinary differential equations:

$$\begin{cases} \dfrac{dA}{dt} &= \phi F - (\gamma + \mu_{A,1} + \mu_{A,2}A)A, \\ \dfrac{dM}{dt} &= (1-r)\gamma A - \mu_M M, \\ \dfrac{dF}{dt} &= r\gamma A - \mu_F F, \end{cases} \tag{1}$$

where the parameters and state variables are described in Table 1.

Symbol	Description
A	Aquatic stage (gathering eggs, larvae, nymph stages)
F	Fertilized and eggs-laying females
M	Males
ϕ	Number of viable eggs at each deposit per capita (per day)
γ	Maturation rate from larvae to adult (per day)
$\mu_{A,1}$	Density independent mortality rate of the aquatic stage (per day)
$\mu_{A,2}$	Density dependent mortality rate of the aquatic stage (per day$\times$ number)
r	Sex ratio
Λ	Number of sterile insects released per unit of time
$1/\mu_F$	Average lifespan of female (in days)
$1/\mu_M$	Average lifespan of male (in days)
$1/\mu_T$	Average lifespan of sterile male (in days)

Table 1: Description of parameters and state variables of model (1).

Contrary to [20], we assume a density-dependent mortality rate in the aquatic stage. This may correspond to an intra-specific competition between the larvae stages, for instance.

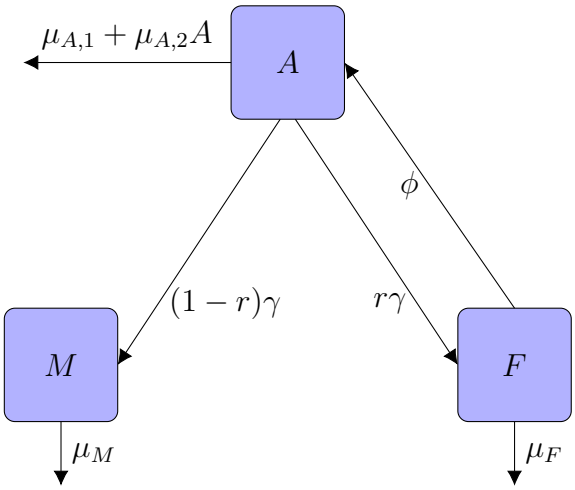


Figure 1: Flow diagram of model (1).

However, the forthcoming methodology could be applied for a system where the non-linearity stands for the birth-rate, like in [20, 8, 9].

The inequalities between vectors are considered here in their usual coordinate-wise sense, i.e., for any $x = (x_i)_{i=1,...,n},\ y = (y_i)_{i=1,...,n} \in \mathbb{R}^n,\ n \geq 1$,

- $x \leq y \Leftrightarrow x_i \leq y_i,\ i = 1, ..., n,$
- $x < y \Leftrightarrow x \leq y,\ x \neq y,$
- $x \ll y \Leftrightarrow x_i < y_i,\ \ i = 1, ..., n.$

Hence, we define the order intervals:

$$[x, y] = \{z \in \mathbb{R}^n : x \leq z \leq y\},$$

$$[x, y) = \{z \in \mathbb{R}^n : x \leq z < y\},$$

$$(x, y) = \{z \in \mathbb{R}^n : x < z < y\}.$$

We set $x = (A, M, F)'$ and $\mathcal{D} = \mathbb{R}^3_+ = \{x \in \mathbb{R}^3 : x \geq \mathbf{0}\}$. Then model (1) can be written in the form

$$\frac{dx}{dt} = f(x), \tag{2}$$

where $f : \mathbb{R}^3 \to \mathbb{R}^3$ represents the right hand side of (1). Function f is continuous and continuously differentiable on $\mathbb{R}^3$. Thus, according to [23, Theorem III.10.VI], for any initial condition a unique solution exists, at least locally. The vector field defined by f is either tangential or directed inwards on ∂D. Therefore, for any initial condition in $\mathcal{D}$ the solution of (2) remains in $\mathcal{D}$ for its maximal interval of existence [23, Theorem III.10.XVI]. In the sequel we consider the vector population model in the form (1) or in the form (2) on the domain $\mathcal{D}$. In order to obtain existence of the solutions in $\mathcal{D}$, it is sufficient to obtain a priori upper bounds. This can be done as follows.

We observe that system (1) is monotone [19, Proposition 3.1.1]. Indeed, for any $x \in \mathcal{D}$ the Jacobian

$$J(x) = \begin{pmatrix} -(\gamma + \mu_{A,1}) - 2\mu_{A,2}A & 0 & \phi \\ (1-r)\gamma & -\mu_M & 0 \\ r\gamma & 0 & -\mu_F \end{pmatrix} \tag{3}$$

is a Metzler matrix, i.e. all its off diagonal entries are non-negative. The inequality

$$r\gamma - \frac{\mu_F}{2\phi}(\gamma + \mu_{A,1} + \mu_{A,2}A) < 0 \tag{4}$$

holds for all sufficiently large A. Let $m > 0$ and let A_m be so large that in addition to (4) the following inequalities also hold:

$$\begin{array}{rcll} A_m & \geq & m, & \\ F_m & := & \dfrac{(\gamma + \mu_{A,1} + \mu_{A,2}A_m)A_m}{2\phi} & \geq m, \\ M_m & := & \dfrac{2(1-r)\gamma A_m}{\mu_M} & \geq m. \end{array} \tag{5}$$

For every $m > 0$ let

$$b_m = (A_m, M_m, F_m)' \tag{6}$$

be a vector with coordinates satisfying (4) and (5). Then

$$f(b_m) = \begin{pmatrix} -\phi F_m \\ -(1-r)\gamma A_m \\ A_m(r\gamma - \dfrac{\mu_F}{2\phi}(\gamma + \mu_{A,1} + \mu_{A,2}A_m)) \end{pmatrix} < \mathbf{0}. \tag{7}$$

Using [19, Proposition 3.2.1], the solution initiated at b_m is decreasing. Then, using again the monotonicity of the system, see [19, Proposition 3.2.1], for any solution of (1) initiated in $\mathcal{D}$ we have

$$x(t) \leq b_{||x(0)||_\infty}. \tag{8}$$

The a priori upper bound given in (8) provides for existence of the solution for all $t \geq 0$. Therefore, (1) defines a dynamical system on $\mathcal{D}$.

The stability properties of the extinction equilibrium $\mathbf{0} = (0, 0, 0)'$ are usually described in terms of the basic offspring number $\mathcal{R}$ of the population, i.e. the average self-reproduction of an individual (number of females produced by a single female) during its lifetime, assuming that the population is so small that the density dependent mortality can be ignored. The basic offspring number related to model (1) is obtained by the next generation method [22]:

$$\mathcal{R} = \frac{r\gamma\phi}{\mu_F(\gamma + \mu_{A,1})}. \tag{9}$$

The Jacobian of system (1) computed at the extinction equilibrium is

$$J(\mathbf{0}) = \begin{pmatrix} -(\gamma + \mu_{A,1}) & 0 & \phi \\ (1-r)\gamma & -\mu_M & 0 \\ r\gamma & 0 & -\mu_F \end{pmatrix}. \tag{10}$$

Its eigenvalues are $-\mu_M$ and the roots of the equation

$$\lambda^2 + (\gamma + \mu_{A,1} + \mu_F)\lambda + (\gamma + \mu_{A,1})\mu_F(1 - \mathcal{R}) = 0. \tag{11}$$

It is easy to see that if $\mathcal{R} < 1$, all eigenvalues of $J(\mathbf{0})$ are either negative or have negative real parts, that is $\mathbf{0}$ is asymptotically stable. If $\mathcal{R} > 1$, the Jacobian has two negative eigenvalues and a positive one. Hence, $\mathbf{0}$ is unstable.

The existence of an endemic equilibrium also depends on the value of $\mathcal{R}$. Setting the right hand side of (1) to zero we obtain the equilibrium $\mathbf{0}$ and the equilibrium $E^* = (A^*, M^*, F^*)'$ given by

$$\begin{cases} A^* &= \dfrac{(\gamma+\mu_{A,1})}{\mu_{A,2}}(\mathcal{R}-1), \\ M^* &= \dfrac{(1-r)\gamma A^*}{\mu_M}, \\ F^* &= \dfrac{r\gamma A^*}{\mu_F}. \end{cases} \tag{12}$$

Clearly, $E^* \in \mathcal{D}$ and $E^* \neq \mathbf{0}$ if and only if $\mathcal{R} > 1$. We summarize these results with some more details related to basins of attraction of equilibria in the following theorem.

Theorem 1. *Model (1) defines a forward dynamical system on $\mathcal{D}$. Furthermore,*

1) If $\mathcal{R} \leq 1$ then $\mathbf{0}$ is globally asymptotically stable on $\mathcal{D}$.
2) If $\mathcal{R} > 1$ then E^* is stable with basin of attraction

$$\mathcal{D} \setminus \{x = (A, M, F)' \in \mathbb{R}^3_+ : A = F = 0\},$$

and $\mathbf{0}$ is unstable with the non negative $M-$axis being a stable manifold.

Proof. As mentioned, it remains to prove the statements regarding the basins of attraction. We use an approach similar to the approach in [1] for the analysis of bi-stable monotone systems. 1) Let $\mathcal{R} \leq 1$. Let $x = x(t)$ be any solution initiated in $\mathcal{D}$. Denote by $y = y(t)$ the solution of (1) with initial condition $y(0) = b_{||x(0)||_\infty}$. It follows from the inequality (7) that the function y is decreasing and, therefore, it converges. The limit is necessarily an equilibrium (see also [19, page 35]). Considering that there is only one equilibrium in $\mathcal{D}$, we conclude that $\lim_{t\to+\infty} y(t) = \mathbf{0}$. Using that (1) is a monotone system, the inequalities $\mathbf{0} \leq x(0) \leq b_{||x(0)||_\infty}$, we have

$$\mathbf{0} \leq x(t) \leq y(t), \quad t \geq 0.$$

Therefore, $\lim_{t\to+\infty} x(t) = \mathbf{0}$, which proves the global asymptotic stability of $\mathbf{0}$ on $\mathcal{D}$.

2) To prove the stability and basin of attraction we use [19, Theorem 2.2.2]. This theorem applies to strongly monotone systems. We recall that if the Jacobian of f is a Metzler irreducible matrix for every $x \in \mathcal{D}$, then (2) is strongly monotone [19, Theorem 4.1.1]. The Jacobian (3) associated with (1) is not irreducible, since the equation for M can be decoupled. We consider the subsystem for A and F, that is,

$$\begin{cases} \dfrac{dA}{dt} &= \phi F - (\gamma + \mu_{A,1} + \mu_{A,2}A)A, \\ \dfrac{dF}{dt} &= r\gamma A - \mu_F F, \end{cases} \tag{13}$$

which defines a dynamical system on $\mathbb{R}^2_+$. The Jacobian

$$\tilde{J}(A, F) = \begin{pmatrix} -(\gamma+\mu_{A,1}) - 2\mu_{A,2}A & \phi \\ r\gamma & -\mu_F \end{pmatrix} \tag{14}$$

is clearly irreducible. We apply [19, Theorem 2.2.2] to the two dimensional interval

$$\{(A, F)' \in \mathbb{R}^2_+ : 0 \leq A \leq A^*,\ 0 \leq F \leq F^*\}.$$

It follows that, all solutions initiated in this interval, excluding the end points, converge either all to $(0,0)'$ or all to $(A^*,F^*)'$. The characteristic equation of $\tilde{J}(0,0)$ is exactly (11), which produces one positive and one negative root. Considering that $\tilde{J}(0,0)$ is a Metzler matrix, it has a strictly positive eigenvector corresponding to the positive eigenvalue. Hence, it is not possible that all solutions converge to $(0,0)'$. Therefore, they all converge to $(A^*,F^*)'$. The implication for the three dimensional system (1) is that all solutions initiated in the interval $[\mathbf{0},E^*]$, excluding the M-axis, converge to E^*.

Using similar argument as in 1), any solution initiated at a point larger than E^* converges to E^*. Since any point in $\mathcal{D}\setminus\{x=(A,M,F)'\in\mathbb{R}^3_+ : A=F=0\}$ can be placed between a point below E^*, but not on M-axis and a point above E^*, all solutions initiated in $\mathcal{D}\setminus\{x=(A,M,F)'\in\mathbb{R}^3_+ : A=F=0\}$ converge to E^*. The monotone convergence of the solutions initiated below and above E^* implies the asymptotic stability of E^* as well. The basin of attraction cannot be extended further, since the non negative M-axis is the attractive manifold corresponding to the eigenvalue $-\mu_M$ of $J(\mathbf{0})$. □

3. The SIT model in the case of constant and continuous releases

In the sequel, we assume that $\mathcal{R}>1$. We take into account the constant release of sterile male vectors M_T by adding to model (1) an equation for M_T. Altogether, the SIT model reads as

$$\begin{cases} \dfrac{dA}{dt} &= \phi F-(\gamma+\mu_{A,1}+\mu_{A,2}A)A,\\ \dfrac{dM}{dt} &= (1-r)\gamma A-\mu_M M,\\ \dfrac{dF}{dt} &= \dfrac{M}{M+M_T}r\gamma A-\mu_F F,\\ \dfrac{dM_T}{dt} &= \Lambda-\mu_T M_T. \end{cases} \tag{15}$$

In model (15), the total number of males available for mating with females is $M+M_T$. Hence, we assume that emerging immature females (from the aquatic stage) have a probability $\dfrac{M}{M+M_T}$ to mate with wild (fertile) males. Assuming t large enough, we may assume that $M_T(t)$ has reached its equilibrium value $M_T^*:=\Lambda/\mu_T$. Thus, model (15) reduces to

$$\begin{cases} \dfrac{dA}{dt} &= \phi F-(\gamma+\mu_{A,1}+\mu_{A,2}A)A,\\ \dfrac{dM}{dt} &= (1-r)\gamma A-\mu_M M,\\ \dfrac{dF}{dt} &= \dfrac{M}{M+M_T^*}r\gamma A-\mu_F F, \end{cases} \tag{16}$$

where parameters and state variables are described in Table 1.

Theorem 2. *Model (16) defines a monotone dynamical system on $\mathcal{D}$ for any value of $M_T^*\in(0,+\infty)$.*

Proof. Let us set $x=(A,M,F)'\in\mathcal{D}$ and Φ a vector-valued function such that $\Phi(M_T^*,x)=f(x)$ where f is the right hand side of system (16). In compact form, we can therefore write system (16) as follows:

$$\frac{dx}{dt}=\Phi(M_T^*,x). \tag{17}$$

Denote by $x_{M_T^*}(z,t)$ the solution of (17) satisfying $x_{M_T^*}(z,0)=z$. Consider the point b_m as given by (5). Using (7) we have

$$\Phi(M_T^*, b_m) \leq \Phi(0, b_m) = f(b_m) < \mathbf{0}. \tag{18}$$

Then the solution initiated at b_m is decreasing and, again by the monotonicity of the system, for any solution of (17) initiated in $\mathcal{D}$ we have

$$x_{M_T^*}(z,t) \leq b_{||z||_\infty}. \tag{19}$$

The a priori upper bound given in (19) provides for existence of the solution for all $t \geq 0$. Therefore, (17) defines a monotone dynamical system on $\mathcal{D}$. □

Equilibria of the SIT model (16) are obtained by solving the system

$$\begin{cases} \phi F - (\gamma + \mu_{A,1} + \mu_{A,2}A)A &= 0, \\ (1-r)\gamma A - \mu_M M &= 0, \\ \dfrac{M}{M + M_T^*} r\gamma A - \mu_F F &= 0. \end{cases} \tag{20}$$

From $(20)_1$ and $(20)_2$ we have

$$A = \frac{\mu_M}{(1-r)\gamma} M \tag{21}$$

and

$$F = \frac{(\gamma + \mu_{A,1} + \mu_{A,2}A)A}{\phi} = \frac{(\gamma + \mu_{A,1})}{\phi} \frac{\mu_M}{(1-r)\gamma} M + \frac{\mu_{A,2}}{\phi} \left(\frac{\mu_M}{(1-r)\gamma} M \right)^2. \tag{22}$$

Substituting in $(20)_3$ leads to $M = 0$ or

$$\frac{r\gamma M}{M + M_T^*} - \frac{\mu_F(\gamma + \mu_{A,1})}{\phi} - \frac{\mu_F \mu_{A,2}}{\phi} \frac{\mu_M}{(1-r)\gamma} M = 0. \tag{23}$$

Setting $\alpha = M_T^*/M$, equation (23) can be written as

$$\alpha^2 - a_1 \alpha + a_0 = 0, \tag{24}$$

where

$$\begin{aligned} a_1 &= \frac{r\gamma\phi}{\mu_F(\gamma + \mu_{A,1})} - 1 - \frac{\mu_{A,2}\mu_M}{(\gamma + \mu_{A,1})(1-r)\gamma} M_T^*, \\ a_0 &= \frac{\mu_{A,2}\mu_M}{(\gamma + \mu_{A,1})(1-r)\gamma} M_T^*. \end{aligned}$$

Setting $Q = \dfrac{\mu_{A,2}\mu_M}{(\gamma + \mu_{A,1})(1-r)\gamma}$, (24) assumes the form

$$\alpha^2 - (\mathcal{R} - 1 - QM_T^*)\alpha + QM_T^* = 0. \tag{25}$$

The discriminant of (25) is

$$\Delta(M_T^*) = ((\sqrt{\mathcal{R}} - 1)^2 - M_T^* Q)((\sqrt{\mathcal{R}} + 1)^2 - M_T^* Q).$$

The equation $\Delta(M_T^*) = 0$ has two positive solutions M_{T_1} and M_{T_2}:

$$M_{T_1} = \frac{(\sqrt{\mathcal{R}} - 1)^2}{Q}, \quad M_{T_2} = \frac{(\sqrt{\mathcal{R}} + 1)^2}{Q}. \tag{26}$$

Then, we have four possible cases:

- If $0 < M_T^* < M_{T_1}$, then $\Delta(M_T^*) > 0$ and (25) has roots α_+ and α_- given by

$$\alpha_\pm = \frac{(\mathcal{R} - 1 - QM_T^*) \pm \sqrt{\Delta(M_T^*)}}{2}. \tag{27}$$

Using that $\alpha_+ + \alpha_- = \mathcal{R} - 1 - QM_T^* > \mathcal{R} - 1 - QM_{T_1} = 2(\sqrt{\mathcal{R}} - 1) > 0$, we deduce that these roots are positive. Therefore, the system (16) has two positive equilibria $E_{1,2} = (A_{1,2}, M_{1,2}, F_{1,2})'$ with $\mathbf{0} \ll E_1 \ll E_2$ given by

$$\begin{aligned} A_{1,2} &= \frac{\mu_M}{(1-r)\gamma} M_{1,2}, \\ F_{1,2} &= \frac{(\gamma + \mu_{A,1} + \mu_{A,2} A_{1,2}) A_{1,2}}{\phi}, \\ M_1 &= \frac{M_T^*}{\alpha_+}, \\ M_2 &= \frac{M_T^*}{\alpha_-}. \end{aligned} \tag{28}$$

- If $M_T^* = M_{T_1}$ then $\Delta(M_T^*) = 0$ and (25) has only one real solution $\alpha_\dagger$, namely

$$\alpha_\dagger = \frac{\mathcal{R} - 1 - QM_T^*}{2} > 0. \tag{29}$$

Then the system (16) has one positive equilibrium $E_\dagger \gg \mathbf{0}$ given by

$$\begin{aligned} A_\dagger &= \frac{\mu_M}{(1-r)\gamma} M_\dagger, \\ F_\dagger &= \frac{(\gamma + \mu_{A,1} + \mu_{A,2} A_\dagger) A_\dagger}{\phi}, \\ M_\dagger &= \frac{M_T^*}{\alpha_\dagger}. \end{aligned} \tag{30}$$

- If $M_{T_1} < M_T^* < M_{T_2}$ then $\Delta(M_T^*) < 0$. The equation (25) has no real roots, which implies that the system (16) has no equilibria other than the origin.

- If $M_T^* \geq M_{T_2}$ then $\Delta(M_T^*) \geq 0$ and (25) has one or two real roots which are negative because $\mathcal{R} - 1 - QM_T^* \leq \mathcal{R} - 1 - QM_{T_2} = -2(1 + \sqrt{\mathcal{R}}) < 0$. Hence, as in the preceding case, the only equilibrium of (16) in its domain $\mathcal{D}$ is the origin.

Theorem 3. *For any $M_T^* > 0$ the origin $\mathbf{0}$ is an asymptotically stable equilibrium of the system (16) on $\mathcal{D}$. Furthermore, we have the following:*

(1) If $M_T^* > M_{T_1}$ then equilibrium $\mathbf{0}$ is globally asymptotically stable on $\mathcal{D}$.

(2) If $M_T^* = M_{T_1}$ then system (16) has one additional equilibrium $E_\dagger$ given by (30) such that $E_\dagger \gg \mathbf{0}$. The set $\{x \in \mathbb{R}^3 : \mathbf{0} \leq x < E_\dagger\}$ is in the basin of attraction of $\mathbf{0}$, while the set $\{x \in \mathbb{R}^3 : x \geq E_\dagger\}$ is in the basin of attraction of $E_\dagger$.

(3) If $0 < M_T^* < M_{T_1}$ then system (16) has two additional equilibria E_1 and E_2 given by (28) and such that $\mathbf{0} \ll E_1 \ll E_2$. The set $\{x \in \mathbb{R}^3 : \mathbf{0} \leq x < E_1\}$ is in the basin of attraction of $\mathbf{0}$ while the set $\{x \in \mathbb{R}^3 : x > E_1\}$ is in the basin of attraction of E_2.

Proof. Using that the eigenvalues, $\xi_1 = -(\gamma + \mu_{A,1})$, $\xi_2 = -\mu_M$, $\xi_3 = -\mu_F$, of the Jacobian matrix of the SIT model (16) at $\mathbf{0}$ are all negative, then the elimination equilibrium $\mathbf{0}$ is always asymptotically stable.

(1) Suppose that $M_T^* > M_{T_1}$. Then system (16) has only one equilibrium, namely $\mathbf{0}$. The global asymptotic stability of $\mathbf{0}$ is proved as in point 1) of Theorem 1, page 6.

(3) Assume that $0 < M_T^* < M_{T_1}$. In this case, the dynamical system (16) has three equilibria $\mathbf{0}$, E_1 and E_2. Let us consider the order interval $[\mathbf{0}, E_1]$. According to [19, Theorem 2.2.2], the solutions initiated in this interval, excluding the end points, either all converge to $\mathbf{0}$ or all converge to E_1. Since $\mathbf{0}$ is asymptotically stable, this implies that all solutions converge to $\mathbf{0}$. Moreover, straightforward computations show that the Jacobian matrix, J_{E_1}, of the SIT model (16) at E_1 is an irreducible Metzler matrix. Hence, it follows from the theory of nonnegative matrices [13, Theorems 11 and 17], [12, Proposition 3.4] that J_{E_1} has an eigenvector v with positive coordinates and associated eigenvalue ξ, which is real and has an algebraic multiplicity equal to one. Since E_1 is repelling in $[\mathbf{0}, E_1]$, then $\xi \geq 0$. Further, we have

$$\begin{aligned}
\det(J_{E_1}) &= \phi\mu_M \left(\frac{r\gamma M_1 M_T^*}{(M_1 + M_T^*)^2} + \frac{\gamma r M_1}{M_1 + M_T^*} \right) - \mu_M \mu_F (\gamma + \mu_{A,1} + 2\mu_{A,2} A_1) \\
&= \phi\mu_M \left(\frac{r\gamma M_1}{M_1 + M_T^*} - \frac{\mu_F(\gamma + \mu_{A,1})}{\phi} - \frac{\mu_F \mu_{A,2} \mu_M M_1}{\phi(1-r)\gamma} \right) \\
&\quad + \phi\mu_M M_1 \left(\frac{r\gamma M_T^*}{(M_1 + M_T^*)^2} - \frac{\mu_F \mu_{A,2} \mu_M}{\phi(1-r)\gamma} \right)
\end{aligned}$$

Taking into account that M_1 is a solution of (23), the expression in the first set of brackets is zero. Further, the expression in the second set of brackets is the derivative with respect to M of the right hand side of (23). Then, since M_1 is a simple root, this expression is not zero. Therefore, $\det(J_{E_1}) \neq 0$. Considering that $\det(J_{E_1})$ is the product of eigenvalues of J_{E_1}, we have that $\xi > 0$. Next, we consider the order interval $[E_1, E_2]$. Again following [19, Theorem 2.2.2], we deduce that the solutions initiated in this interval, excluding the end points, all converge to E_2 since E_1 is repelling in the direction of the positive vector v. Now, let $x = x(t)$ be any solution of the SIT model (16) such that $x(0) \geq E_2$. Denote by $y = y(t)$ the solution of (16) with initial data $y(0) = b_{\|x(0)\|_\infty}$. It follows from inequality (18) that the function y is decreasing and, therefore, it converges. The limit is necessarily an equilibrium greater or equal to E_2. However, there is no other equilibrium greater than E_2. Thus, the limit of $y(t)$, as t goes to infinity, is E_2. Using that model (16) is a monotone system, $E_2 \leq x(0) \leq y(0)$ implies that $E_2 \leq x(t) \leq y(t)$. Hence $\lim_{t \to +\infty} x(t) = E_2$.

The proof of point (2) is done in a similar way but by considering $E_1 := E_\dagger$ to construct the basin of attraction of the elimination equilibrium and $E_2 := E_\dagger$ to construct the basin of attraction of $E_\dagger$.

□

From Theorem 3, it is straightforward to deduce that SIT control may induce a weak Allee effect in system (16). More precisely, for a given M_T^* such that $0 < M_T^* \leq M_{T_1}$,

solutions of system (16) are either driven to elimination or persist, depending on the initial data.

Figure 2 depicts a rough illustration of the bi-stable case obtained in the last part of Theorem 3, page 9. In Figure 2, the black bullet is the wild equilibrium $E^* = (A^*, M^*, F^*)'$, the blue bullet is the positive unstable equilibrium (E_1) while the red bullet is the positive stable equilibrium (E_2). The dashed black box is the set $[\mathbf{0}, E_1)$ which is contained in the basin of attraction of $\mathbf{0}$ while the solid black box is the set $\{x \in \mathbb{R} : x > E_1\}$ which is contained in the basin of attraction E_2.

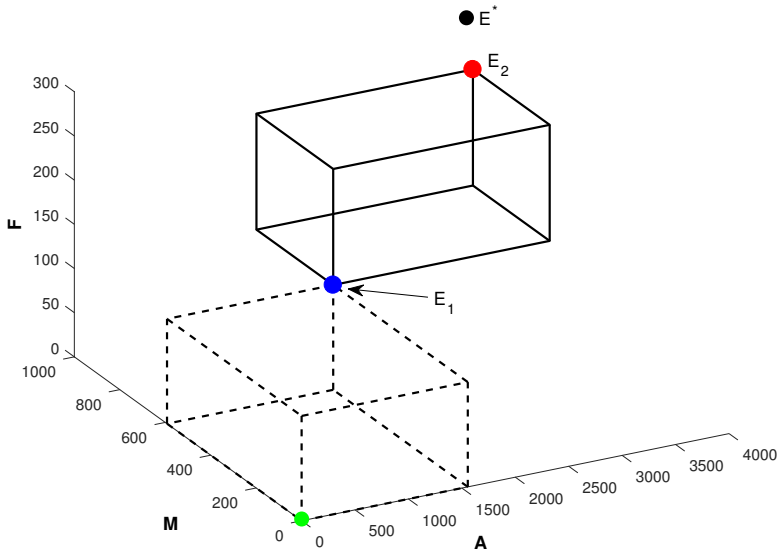


Figure 2: Rough illustration of the bi-stable case obtained in the last part of Theorem 3. The black bullet is the wild equilibrium E^*, the green bullet is the elimination equilibrium $(\mathbf{0})$, the blue bullet is the positive unstable equilibrium (E_1) while the red bullet is the positive stable equilibrium (E_2).

4. About the permanent SIT control strategy - Characterization of the $[0, E_1)$ entry-time

SIT control generally requires of massive release rate in the targeted area in order to lower the population under a certain epidemiological relevant threshold. While the possibility of elimination cannot be ruled out, it has not been observed practically. Therefore, when the SIT intervention is terminated the population recovers to its natural equalibrium. Accordingly, for $M^* = 0$ the model (2) changes to (1), for which the equilibrium E^* is globally asymptotically stable

Thus, SIT always needs to be maintained. Here we propose a practically feasible strategy consisting of massive release rate resulting in sterile insect population $\overline{M}_T^* > M_{T_1}$, followed by lower release rate which can be maintained in a long term. In order to construct this strategy we need to establish first the long-term feasible release rate. Naturally, this would depend on the available in the long-term resources. We suppose that this long-term feasible release results in equilibrium for the treated mosquitoes which we denote by $\underline{M}_T^*$, where $0 < \underline{M}_T^* < M_{T_1}$. Let $\underline{E}_1$ and $\underline{E}_2$ be the equilibria associated with treated mosquito population of $\underline{M}_T^*$ in terms of point (3) of Theorem 3. Then it follows from Theorem 3 that $[\mathbf{0}, \underline{E}_1)$ is in the basin of attraction of $\mathbf{0}$.

In this setting, the main question would be to find an estimate of the time for the solution of (2) initiated at E^* to enter the interval $[\mathbf{0}, E_1)$.

In the remainder of this section we address this question.

Let $X_{t_0}(t, a, b)$ denote the solution of system (16) with $M_T^* = b$ and satisfying $X_{t_0}(t_0, a, b) = a$, where $t_0 \geq 0$, $a, b \in \mathbb{R}$.

Theorem 4 (Existence of minimum entry time)**.** *For any $\overline{M}_T^* > M_{T_1}$ and $\underline{M}_T^* \in (0, M_{T_1})$ there exists a unique $\delta = \delta(\overline{M}_T^*, \underline{M}_T^*) > 0$ such that*

(i) for every $t \geq \delta$ we have $X_0(t, E^*, \overline{M}_T^*) < \underline{E}_1$ and

(ii) $\delta = \max\{t > 0 : X_0(t, E^*, \overline{M}_T^*) < \underline{E}_1\}$

Proof. Since $\overline{M}_T^* > M_{T_1}$, it follows from Theorem 3 that $\mathbf{0}$ is globally asymptotically stable for system (16) in $\mathcal{D}$. Hence, $X_0(t, E^*, \overline{M}_T^*)$ converges to $\mathbf{0}$. Using the notation (17) we have that

$$\Phi(\overline{M}_T^*, E^*) = -\begin{pmatrix} 0 \\ 0 \\ \dfrac{r\gamma A^* \overline{M}_T^*}{M^* + \overline{M}_T^*} \end{pmatrix} < \mathbf{0}$$

Since the system (17) is monotone, it follows from [19, Proposition 3.2.1] that the set

$$P = \{x \in \mathcal{D} : \Phi(\overline{M}_T^*, x) > \mathbf{0}\}$$

is positively invariant. As a consequence, the solution $X_0(t, E^*, \overline{M}_T^*)$ is in P for all $t \geq 0$ and, therefore, is monotone decreasing to $\mathbf{0}$. Hence, there is a unique time δ at which the solution $X_0(t, E^*, \overline{M}_T^*)$ enters the compact neighborhood $[\mathbf{0}, \underline{E}_1]$ of $\mathbf{0}$ in $\mathcal{D}$. Since $[\mathbf{0}, \underline{E}_1]$ is an order interval, once the solution $X_0(t, E^*, \overline{M}_T^*)$ enters it, it remains in it, and further decreases to $\mathbf{0}$. Therefore, it remains to prove that $X_0(\delta, E^*, \overline{M}_T^*) \neq \underline{E}_1$.

Assume the opposite, namely that $X_0(\delta, E^*, \overline{M}_T^*) = \underline{E}_1$. The positively invariant set P is bounded by the three surfaces given by the equations in (20) for $M_T^* = \overline{M}_T^*$. We note that the coordinates of $\underline{E}_1$ satisfy the first equation. The fact that the solution $X_0(t, E^*, \overline{M}_T^*)$ does not cross the surface defined by this equation implies that the graph of the solution $X_0(t, E^*, \overline{M}_T^*)$ is tangent to this surface. This implies that the gradient of the solution is orthogonal to the normal $\mathbf{n}$ of the surface at the point E_1. Simple computations show that the dot product of these two vectors is not zero. More precisely, we have

$$\Phi(\overline{M}_T^*, \underline{E}_1) \cdot \mathbf{n} = \begin{pmatrix} 0 \\ 0 \\ \dfrac{r\gamma \underline{A}_1 \underline{M}_1 (\underline{M}_T^* - \overline{M}_T^*)}{(M_1 + \overline{M}_T^*)(\underline{M}_1 + \underline{M}_T^*)} \end{pmatrix} \cdot \begin{pmatrix} -\gamma - \mu_{A,1} - 2\mu_{A,2}\underline{A}_1 \\ 0 \\ \phi \end{pmatrix} \neq 0.$$

The obtained contradiction shows that, while δ is the smallest positive such that $X_0(\delta, E^*, \overline{M}_T^*) \leq \underline{E}_1$ we have, in fact, $X_0(\delta, E^*, \overline{M}_T^*) < \underline{E}_1$. □

The value of δ established in Theorem 4 is the minimum time for the solution $X_0(t, E^*, \overline{M}^*)$ to enter the order interval $[\mathbf{0}, \underline{E}_1)$, which is in the basin of attraction of $\mathbf{0}$ when the treated population is at the low and feasible in the long term level $\underline{M}_T^*$. Hence, as a direct consequence of Theorem 3(3) we obtain the following theorem.

Theorem 5. *Let $\overline{M}_T$, $\underline{M}_T^*$, $\delta = \delta(\overline{M}_T^*, \underline{M}_T^*)$ be as given in Theorem 4 and $Y = X_0(\delta, E^*, \overline{M}_T^*)$. Then*

(i) $X_\delta(t, Y, \underline{M}_T^*) < \underline{E}_1$ *for all* $t \geq \delta$ *and*

(ii) $\lim_{t\to+\infty} X_\delta(t, Y, \underline{M}_T^*) = \mathbf{0}$.

Remark 1. *In the framework of permanent SIT control, it is important to observe that if the massive release is reduced to low release associated with treated male population of $\underline{M}_T^*$, before the prescribed period of time, δ, obtained in Theorem 4, the solution of system (16) may converge towards the positive stable equilibrium $\underline{E}_2$. Indeed, our results in Theorem 3 provide only sub-sets of the basins of attraction of $\mathbf{0}$ and $\underline{E}_2$ in the form of order intervals. Hence, if one wants the permanent SIT control strategy to be successful, one should carry out massive SIT releases during the prescribed entry-time, δ, before change to a more sustainable low level of releases. By doing so, it is ensured that solutions are inside the basin of attraction of $\mathbf{0}$, that is the population is below $\underline{E}_1$ and is driven to elimination. However, if SIT control is discontinued, i.e. $M_T^* = 0$, then the solution will converge towards the initial positive wild equilibrium, E^*.*

Remark 2. *Since systems (16) and (15) are equivalent for t sufficiently large, a difference between entry times may occur, depending if we consider $M_T(0) = 0$ or $M_T(0) = M_T^*$. When $M_T(0) = 0$, we have $M_T(t) = M_T^*(1-\exp(-\mu_T t))$, for $t \geq 0$. Thus, a straightforward release strategy to reach kM_{T_1} fast, is to first make "very massive" releases, $k'M_{T_1}$ with $k' > k$ for a few days, and then to continue with massive releases until the system reaches $[\mathbf{0}, \mathbf{E_1})$. This strategy is equivalent to look for t_k^* such that*

$$k \times M_{T_1} = k' \times M_{T_1}(1 - \exp(-\mu_T t_k^*)).$$

We find that

$$t_k^* = -\frac{1}{\mu_T} \times \ln\left(1 - \frac{k}{k'}\right). \tag{31}$$

For instance, choosing $k' = 2 \times k$ leads to $t_k^* = \dfrac{1}{\mu_T} \times \ln(2)$.

Under certain conditions we can derive an analytic approximation for the minimal time, δ, defined in Theorem 4. We deal with that issue below where we assume that

$$\mu_F < \min\{\mu_M, \gamma + \mu_{A,1}\}. \tag{32}$$

Assumption (32) is also consistent with parameter values considered, for the case of *Aedes spp.*, in [2, 5].

The following inequalities holds

$$\begin{aligned}
0 \;\leq\; A^* \;&\leq\; \frac{(\gamma+\mu_{A,1})}{\mu_{A,2}} R := A_e^0, \\
0 \;\leq\; M^* \;&\leq\; \frac{(1-r)\gamma}{\mu_M}\frac{(\gamma+\mu_{A,1})}{\mu_{A,2}} R := M_e^0, \\
0 \;\leq\; F^* \;&\leq\; \frac{r\gamma}{\mu_F}\frac{(\gamma+\mu_{A,1})}{\mu_{A,2}} R := F_e^0.
\end{aligned} \tag{33}$$

Let us consider the solution $X(t) = (A(t), M(t), F(t))'$ of system (16) with initial data E^*. In order to estimate the (minimal) time needed to drive the vector population under a given value $\underline{Y} = (\underline{A}, \underline{M}, \underline{F})' < E^*$, we will look for an analytical upper bound of $X(t)$, $X^{upper}(t)$.

According to system (16), we have

$$\left\{ \begin{array}{rcl} \dfrac{dA}{dt} & \leq & \phi F - (\gamma + \mu_{A,1})A, \\ \dfrac{dM}{dt} & = & (1-r)\gamma A - \mu_M M, \\ \dfrac{dF}{dt} & = & \dfrac{M}{M + M_T^*} r\gamma A - \mu_F F, \end{array} \right. \tag{34}$$

that is

$$\frac{dX}{dt} \leq ZX$$

where

$$Z = \begin{pmatrix} -(\gamma + \mu_{A,1}) & 0 & \phi \\ (1-r)\gamma & -\mu_M & 0 \\ r\gamma\epsilon(M_T^*) & 0 & -\mu_F \end{pmatrix}$$

and $\epsilon(M_T^*) = M^*/(M^* + M_T^*) < 1$. Let $X_e(t) = (A_e(t), M_e(t), F_e(t))'$ be the solution of

$$\frac{dX_e}{dt} = ZX_e. \tag{35}$$

Before going further, let us give the following result that is deduced from Proposition 1.4 and Corollary 1.6 in [14] thanks to the fact that systems (16) and (35) are cooperative systems.

Lemma 1. *Solutions of systems (16) and (35) with initial data such that*

$$(A^0, M^0, F^0)' \leq (A_e^0, M_e^0, F_e^0)' := X_e^0$$

satisfy

$$\forall t \geq 0, \quad X(t) \leq X_e(t).$$

We now follow the idea of [20] in our computations. The sub-matrix Z_0 of Z that reads as

$$Z_0 = \begin{pmatrix} -(\gamma + \mu_{A,1}) & \phi \\ r\gamma\epsilon(M_T^*) & -\mu_F \end{pmatrix}$$

has negative trace. Moreover, Z_0 has a positive determinant if and only if $1/R > \epsilon(M_T^*)$. Therefore, if $\epsilon(M_T^*)R < 1$ then $\mathbf{0}$ is globally asymptotically stable for system (35). In this case, its eigenvalues are real, negative and equal to $\kappa_\pm$ ($\kappa_- < \kappa_+$) associated respectively with eigenvectors $\begin{pmatrix} 1 \\ x_\pm \end{pmatrix}$ where, with assumption (32), $x_- < 0 < x_+$ and

$$\begin{array}{rcl} \kappa_\pm & = & \dfrac{-(\gamma + \mu_{A,1} + \mu_F) \pm \sqrt{(\gamma + \mu_{A,1} - \mu_F)^2 + 4\phi r\gamma\epsilon(M_T^*)}}{2}, \\ x_\pm & = & \dfrac{\gamma + \mu_{A,1} - \mu_F \pm \sqrt{(\gamma + \mu_{A,1} - \mu_F)^2 + 4\phi r\gamma\epsilon(M_T^*)}}{2\phi}. \end{array}$$

Hence, for real numbers $(a_\pm^0, b_\pm^0)' \in \mathbb{R}^4$, we have

$$\begin{pmatrix} A_e(t) \\ M_e(t) \\ F_e(t) \end{pmatrix} = \begin{pmatrix} a_+^0 e^{\kappa_+ t} + a_-^0 e^{\kappa_- t} \\ e^{-\mu_M t} M_e^0 + (1-r)\gamma \int_0^t e^{-\mu_M (t-s)} (a_+^0 e^{\kappa_+ s} + a_-^0 e^{\kappa_- s}) ds \\ b_+^0 e^{\kappa_+ t} + b_-^0 e^{\kappa_- t} \end{pmatrix}$$

where $a_\pm^0, b_\pm^0$ are computed by using the overestimation $(A_e^0, F_e^0)'$ in (33) as initial condition. In details, we found

$$\begin{cases} a_+^0 &= \dfrac{x_- A_e^0 - F_e^0}{x_- - x_+}, \qquad a_-^0 = \dfrac{-x_+ A_e^0 + F_e^0}{x_- - x_+}, \\ b_+^0 &= \dfrac{x_+ x_- A_e^0 - x_+ F_e^0}{x_- - x_+}, \qquad b_-^0 = \dfrac{-x_+ x_- A_e^0 + x_- F_e^0}{x_- - x_+}. \end{cases}$$

Note that

$$x_- - x_+ = -\frac{\sqrt{(\gamma + \mu_{A,1} - \mu_F)^2 + 4\phi r \gamma \epsilon(M_T^*)}}{\phi} < 0,$$

$a_+^0 > 0$, $b_+^0 > 0$, $a_-^0 < 0$ and $b_-^0 = x_- a_-^0 > 0$. Indeed, for $\Delta = (\gamma + \mu_{A,1} - \mu_F)^2 + 4\phi r \gamma \epsilon(M_T^*)$ we have

$$\begin{aligned} a_-^0 < 0 &\Leftrightarrow x_+ A_e^0 < F_e^0 \\ &\Leftrightarrow \frac{((\gamma + \mu_{A,1}) - \mu_F + \sqrt{\Delta})}{2\phi} \frac{(\gamma + \mu_{A,1})R}{\mu_{A,2}} < \frac{r\gamma(\gamma + \mu_{A,1})R}{\mu_F \mu_{A,2}} \\ &\Leftrightarrow (\gamma + \mu_{A,1}) - \mu_F + \sqrt{\Delta} < \frac{2\phi r \gamma}{\mu_F} \\ &\Leftrightarrow \sqrt{\Delta} < (\gamma + \mu_{A,1})(2R - 1) + \mu_F \\ &\Leftrightarrow r\gamma\phi\epsilon(M_T^*) < (\gamma + \mu_{A,1})^2 R(R-1) + r\gamma\phi \\ &\Leftrightarrow r\gamma\phi(\epsilon(M_T^*) - 1) < 0 < (\gamma + \mu_{A,1})^2 R(R-1). \end{aligned}$$

In addition, by using assumption (32) we also have

$$\kappa_+ + \mu_M = \frac{2(\mu_M - \mu_F) - (\gamma + \mu_{A,1} - \mu_F) + \sqrt{\Delta}}{2} > 0.$$

Moreover, assuming $\kappa_- \neq -\mu_M$ (which most holds generally) leads that

$$\begin{aligned} M_e(t) &= e^{-\mu_M t} M_e^0 + (1-r)\gamma \left(a_+^0 \frac{e^{\kappa_+ t} - e^{-\mu_M t}}{\mu_M + \kappa_+} + a_-^0 \frac{e^{\kappa_- t} - e^{-\mu_M t}}{\mu_M + \kappa_-} \right) \\ &= \left(M_e^0 - \frac{(1-r)\gamma a_+^0}{\mu_M + \kappa_+} - \frac{(1-r)\gamma a_-^0}{\mu_M + \kappa_-} \right) e^{-\mu_M t} + \frac{(1-r)\gamma a_+^0}{\mu_M + \kappa_+} e^{\kappa_+ t} + \frac{(1-r)\gamma a_-^0}{\mu_M + \kappa_-} e^{\kappa_- t}. \end{aligned}$$

Recall that

$$0 < \underline{Y} < E^* < X_e^0.$$

Since $a_-^0 < 0$, $A_e(t) \leq \underline{A}$ if $a_+^0 e^{\kappa_+ t} \leq \underline{A}$. That is, if

$$t \geq t_{min}^A := \frac{1}{\kappa_+} \log \left(\frac{\underline{A}}{a_+^0} \right). \tag{36}$$

By using the fact that $b_+^0 + b_-^0 = F_e^0$, we deduce that $F_e(t) \leq \underline{F}$ if $F_e^0 e^{\kappa_+ t} \leq \underline{F}$. That is, if

$$t \geq t_{min}^F := \frac{1}{\kappa_+} \log\left(\frac{\underline{F}}{F_e^0}\right). \tag{37}$$

We proved that $\kappa_+ + \mu_M > 0$ but we need to discuss the two cases $\kappa_- + \mu_M > 0$ and $\kappa_- + \mu_M < 0$.

In the case that $\kappa_- + \mu_M > 0$, with $a_-^0 < 0$ we have

$$M_e(t) \leq \left(M_e^0 - \frac{(1-r)\gamma a_-^0}{\mu_M + \kappa_-}\right) e^{-\mu_M t} + \frac{(1-r)\gamma a_+^0}{\mu_M + \kappa_+} e^{\kappa_+ t}.$$

Since $\kappa_+ > \mu_M$, we obtain

$$M_e(t) \leq \left(M_e^0 - \frac{(1-r)\gamma a_-^0}{\mu_M + \kappa_-} + \frac{(1-r)\gamma a_+^0}{\mu_M + \kappa_+}\right) e^{\kappa_+ t} := \lambda_- e^{\kappa_+ t}$$

where $\lambda_- = M_e^0 - \dfrac{(1-r)\gamma a_-^0}{\mu_M + \kappa_-} + \dfrac{(1-r)\gamma a_+^0}{\mu_M + \kappa_+} > 0$. Therefore, $M_e(t) \leq \underline{M}$ if $\lambda_- e^{\kappa_+ t} \leq \underline{M}$. That is, if

$$t \geq t_{min}^M := \frac{1}{\kappa_+} \log\left(\frac{\underline{M}}{\lambda_-}\right). \tag{38}$$

In the case that $\kappa_- + \mu_M < 0$, with $a_-^0 < 0$ we have

$$M_e(t) \leq M_e^0 e^{-\mu_M t} + \frac{(1-r)\gamma a_+^0}{\mu_M + \kappa_+} e^{\kappa_+ t} + \frac{(1-r)\gamma a_-^0}{\mu_M + \kappa_-} e^{\kappa_- t}.$$

Since $\kappa_+ > \mu_M$ and $\kappa_+ > \kappa_-$, we obtain

$$M_e(t) \leq \left(M_e^0 + \frac{(1-r)\gamma a_-^0}{\mu_M + \kappa_-} + \frac{(1-r)\gamma a_+^0}{\mu_M + \kappa_+}\right) e^{\kappa_+ t} := \lambda_+ e^{\kappa_+ t}$$

where $\lambda_+ = M_e^0 + \dfrac{(1-r)\gamma a_-^0}{\mu_M + \kappa_-} + \dfrac{(1-r)\gamma a_+^0}{\mu_M + \kappa_+} > 0$. Therefore, $M_e(t) \leq \underline{M}$ if $\lambda_+ e^{\kappa_+ t} \leq \underline{M}$ or equivalently, if

$$t \geq t_{min}^M := \frac{1}{\kappa_+} \log\left(\frac{\underline{M}}{\lambda_+}\right). \tag{39}$$

Hence, we have proved the following result.

Proposition 1. *Let $(A(t), M(t), F(t))'$ be a solution of system (16) initiated at the wild equilibrium $E^* = (A^*, M^*, F^*)'$. Assume that $\epsilon(M_T^*)R < 1$ where $\epsilon(M_T^*) = M^*/(M^* + M_T^*)$. The necessary time $\delta(M_T^*)$ to lower the vector population from E^* to $\underline{Y} = (\underline{A}, \underline{M}, \underline{F})'$, with $\underline{A} < A^*$, $\underline{M} < M^*$ and $\underline{F} < F^*$ is such that*

$$\delta(M_T^*) \geq \max(t_{min}^A, t_{min}^M, t_{min}^F)$$

where t_{min}^A is given by (36), t_{min}^F is given by (37) and t_{min}^M is given by (38) or (39).

5. SIT with periodic impulsive releases

Continuous releases, while mathematically very convenient, are not realistic. In general, in the field, releases are periodic and instantaneous. That is why, we consider the following SIT model (40) with periodic impulsive releases

$$\begin{cases} \dfrac{dA}{dt} &= \phi F - (\gamma + \mu_{A,1} + \mu_{A,2}A)A, \\ \dfrac{dM}{dt} &= (1-r)\gamma A - \mu_M M, \\ \dfrac{dF}{dt} &= \dfrac{M}{M+M_T} r\gamma A - \mu_F F, \\ \dfrac{dM_T}{dt} &= -\mu_T M_T, \\ M_T(n\tau^+) &= M_T(n\tau) + \tau\Lambda, \quad n = 1, 2, ... \end{cases} \tag{40}$$

where τ (in unit of time) is the pulse release period. The right-hand side of system (40) is locally Lipschitz continuous on $\mathbb{R}^4$. Thus, using a classic existence theorem (Theorem 1.1, p. 3 in [4]), there exists $T^* > 0$ and a unique solution defined from $(0, T^*) \to \mathbb{R}^4$. Then, using standard arguments, we show that the positive orthant $\mathbb{R}^4$ is an invariant region for system (40).

From the last two equations of system (40), we deduce that, as $t \to +\infty$, M_T converges toward the periodic solution

$$M_T^{per}(t) = \frac{\tau\Lambda}{1 - e^{-\mu_T\tau}} e^{-\mu_T(t - \lfloor t/\tau \rfloor \tau)}. \tag{41}$$

Thus, solutions of system (40) converges, in the sense of $L^\infty(0, +\infty)$ norm, to solutions of the following system

$$\begin{cases} \dfrac{dA}{dt} &= \phi F - (\gamma + \mu_{A,1} + \mu_{A,2}A)A, \\ \dfrac{dM}{dt} &= (1-r)\gamma A - \mu_M M, \\ \dfrac{dF}{dt} &= \dfrac{M}{M+M_T^{per}(t)} r\gamma A - \mu_F F. \end{cases} \tag{42}$$

System (42) is a periodic monotone dynamical system that admits one solution X_{per}. Substituting

$$\underline{M_T} := \min_{t\in[0,\tau]} M_T^{per}(t) = \frac{\tau\Lambda}{1 - e^{-\mu_T\tau}} e^{-\mu_T\tau}, \tag{43}$$

in system (42) leads to the following constant SIT model

$$\begin{cases} \dfrac{dA}{dt} &= \phi F - (\gamma + \mu_{A,1} + \mu_{A,2}A)A, \\ \dfrac{dM}{dt} &= (1-r)\gamma A - \mu_M M, \\ \dfrac{dF}{dt} &= \dfrac{M}{M+\underline{M_T}} r\gamma A - \mu_F F \end{cases} \tag{44}$$

whose solution X_M is such that $X_M \geq X_{per}$ for all time $t > 0$, using a comparison principle. Hence, applying to system (44) the results obtained in Theorems 3 and 4, we obtain conditions on the size and the periodicity of the releases to get GAS or LAS of $\mathbf{0}$. Using M_{T_1} defined in (26), we set

$$M_{T_1}^{per} = M_{T_1}\left(e^{\mu_T \tau} - 1\right). \tag{45}$$

$M_{T_1}^{per}$ is not the best release value for the periodic case. Most probably the best release value should depend on $\frac{1}{\tau}\int_0^\tau \frac{1}{M_T^{per}(t)}dt$, like in [5]. Then, following Theorem 3, page 9, we deduce

Corollary 1. *For τ and Λ given, and*

(i) Assuming

$$\tau\Lambda > M_{T_1}^{per}, \tag{46}$$

then $\mathbf{0}$ is globally asymptotically stable in (42).

(ii) Assuming

$$\tau\Lambda = M_{T_1}^{per}, \tag{47}$$

then $\mathbf{0}$ is locally asymptotically stable in (42), and $[\mathbf{0}, E_\dagger(\underline{M}_T))$ lies in its basin of attraction.

(iii) Assuming

$$0 < \tau\Lambda < M_{T_1}^{per}, \tag{48}$$

then $\boldsymbol{0}$ is locally asymptotically stable in (42), and $[\mathbf{0}, E_1(\underline{M}_T))$ lies in its basin of attraction.

Using Theorem 4, page 4, we deduce

Theorem 6. *Let $\mathbf{0} < \underline{Y} = (\underline{A}, \underline{M}, \underline{F})' < E_1(\underline{M}_T^*)$ for a given targeted release amount, $\underline{M}_T^* < M_{T_1}^{per}$. The following results hold*

- *First, assuming massive releases, with $\tau\Lambda^* > M_{T_1}^{per}$, then X_{per} converges from E^* to $\underline{Y}$ in a finite time $t^* > 0$.*
- *Second, assuming small releases, with $\tau\Lambda^* = \underline{M}_T^*$, then, for $t > t^*$, $X_{per}(t) < \underline{Y}$ and $\lim_{t\to+\infty} X_{per}(t) = \mathbf{0}$.*

Theorems 6 and 4 give us a strategy to drive, in a finite time, and keep the wild vector population under a given threshold value $\underline{Y}$, for a targeted amount of sterile male releases, namely $\underline{M}_T^*$: first, massive releases for several weeks, and then small releases according to $\underline{M}_T^*$. They are illustrated in the forthcoming section, both for constant and periodic impulsive releases. Note that it may exist a difference between entry times if we consider either $M_T(0) = 0$ or $M_T(0) = k * M_{T_1}^{per}$, with $k > 0$. When $M_T(0) = 0$, we found, at least numerically, that if we first make "very massive" periodic releases, $k' M_{T_1}^{per}$ with $k' > k$, with $k' = 2k$ or $k' = 4k$, followed by massive releases, $kM_{T_1}^{per}$, then the entry time is extended by one week (for $k' = 4k$) or two weeks (for $k' = 2k$).

6. Numerical simulations

In this part, we consider a specific application of SIT against mosquito, like aedes spp. Parameter values are given in Table 2: some of them are based on expert knowledge, others are based on values considered in previous publications [6, 3]

Symbol	ϕ	$\mu_{A,1}$	$\mu_{A,2}$	r	μ_F	μ_M	μ_T	γ
Value	10	0.05	2×10^{-4}	0.49	1/10	1/7	1/7	0.04-0.1

Table 2: *Aedes spp* entomological parameter values.

In Table 3 we provide several computations, related to the maturation rate, γ. We derive the wild (positive) equilibrium $E^* = (A^*, M^*, F^*)'$ according to (12). These wild equilibria will be used as the initial data for forthcoming simulations. In addition, we also display in Table 3 the thresholds related to the global asymptotic stability of $\mathbf{0}$ with constant release (M_{T_1}) and periodic pulse release ($M_{T_1}^{per}$).

γ	0.04	0.06	0.08	0.1
$\mathcal{R}$	21.78	26.73	30.15	32.67
A^*	9350	14150	18950	23750
M^*	1335	3031	5412	8479
F^*	1834	4160	7428	11637
M_{T_1}	863.9	2048	3745	5954
$M_{T_1}^{per}$	1484.5	3519.8	6434.3	10230

Table 3: Wild equilibrium $E^* = (A^*, M^*, F^*)'$ and Threshold values for γ with periodic treatment $\tau = 7$ days.

To illustrate Remark 2, page 13, according to (31), page 13, and using the sterile males lifespan value given in Table 2, we find that we need $t_5^* = 7 \times \ln(2) \approx 5$ days of "very massive" releases, $2k \times M_{T_1}$, to reach the size for massive releases, i.e. $k \times M_{T_1}$. Thus, for all the following entry time estimates given below, if we consider that $M_T(0) = 0$, we have to add 5 days in order to take into account that we first start with "very" massive releases during 5 days, and then we continue with massive releases until we enter $[\mathbf{0}, \mathbf{E_1})$.

In Table 4, for a given amount of sterile males to release, $\underline{M}_T^*$, we provide the values of the positive unstable equilibrium $E_1 = (A_1, M_1, F_1)'$. This is needed to define $\underline{Y} = (A_1 - \varepsilon, F_1 - \varepsilon, M_1 - \varepsilon)'$, for a given $\varepsilon > 0$, and thus to estimate the minimal time. We set $\varepsilon = 0.1$ and values of $\underline{M}_T^*$ are from expert-based knowledge.

γ \ $\underline{M}_T^*$	100	500	800
0.04	(36.59,5.2,0.36)'	(283.11,40.43,4.15)'	(878.68,125.48,23.35)'
0.06	(18.79,4.03,0.21)'	(109.67,23.49,1.45)'	(201.11,43.1,3.02)'
0.08	(12.24,3.5,0.16)'	(66.42,18.97,0.95)'	(113.54,32.4,1.7)'
0.1	(8.95,3.2,0.14)'	(47.1,16.8,0.75)'	(78.4,27.9,1.3)'

Table 4: Values of the positive (unstable) equilibrium $E_1 = (A_1, M_1, F_1)'$ that corresponds to the targeted release $\underline{M}_T^*$ and γ.

The next simulations are done using a nonstandard finite difference scheme, see e.g. [3].

6.1. Minimal time in the case of continuous and constant releases

We consider massive constant releases such that $M_T(0) = M_T^* = k \times M_{T_1}$ (see Table 3 for M_{T_1}). Using Theorem 4-(i), page 12, the minimal entry time for different values of k, γ and $\underline{M}_T^*$ are summarized in Table 5.

$k = 1.001$			
γ \ $\underline{M}_T^*$	100	500	800
0.04	6959	6889	6719
0.08	7151	7123	7112

$k = 1.01$		
100	500	800
2159	2090	1929
2224	2196	2184

$k = 1.1$		
100	500	800
656	592	479
685	658	647

$k = 1.2$			
γ \ $\underline{M}_T^*$	100	500	800
0.04	460	399	311
0.06	476	442	426
0.08	485	458	447
0.1	489	465	457

$k = 2$		
100	500	800
217	169	126
232	201	188
239	214	205
244	221	213

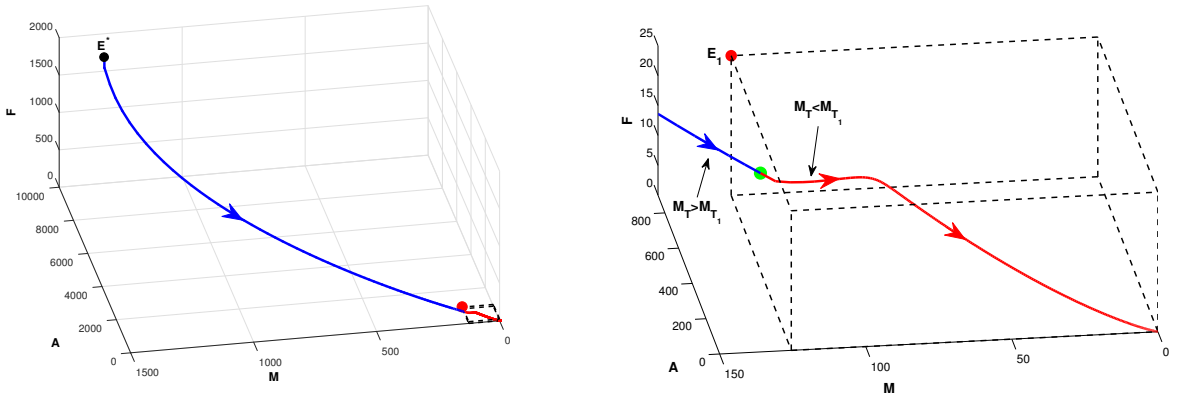
$k = 5$		
100	500	800
141	103	76
155	128	117
162	139	131
167	146	139

$k = 10$		
100	500	800
123	88	65
137	111	101
144	122	114
149	129	122

Table 5: The case of continuous and constant release. Numerical estimates of the minimal times (in days) to reach $\underline{Y}$. We set $\varepsilon = 0.1$ using massive releases, $M_T^* = k \times M_{T_1}$.

For different values of $\underline{M}_T^*$, an increase in the size of the massive releases implies a decay of the minimal time to enter $[\mathbf{0}, E_1)$. Of course lower is the value of $\underline{M}_T^*$, longer is the duration of the massive releases. However, it is interesting to notice that between $k = 5$ (where $M_T^* \in [4320, 29770]$) and $k = 10$ (where $M_T^* \in [8640, 59540]$), the gain of time is very weak if we take into account the cost and, eventually, a possible limitation in the production capacity of the sterile males. However, a cost-effectiveness analysis could be suggested in order to choose which value of k should be considered for field applications. In addition, when $\underline{M}_T^* = 100$ the impact of γ on the minimal time, is limited.

To illustrate the trajectory of the SIT system in the constant release case, we provide in a 3D-view, the trajectory related to $\gamma = 0.04$ and $k = 5$ (see Figure 3).



(a) (b)

Figure 3: The case of continuous and constant release. (a): 3D plot of the trajectory of system (16) initiated at the wild equilibrium $E^* = (9350, 1335, 1834)'$ (black dot). (b): Zoom in around the box delimited by the positive unstable equilibrium $\underline{E}_1 = (878.68, 125.48, 23.35)'$ (red dot). The green dot with coordinates (633.2,121.2,10.85)' corresponds to the start of the targeted release $\underline{M}_T^*$. $\gamma = 0.04$, $k = 5$, $M_{T_1} = 863.9$ and $\underline{M}_T^* = 800$.

Note that the red trajectory continues to decay to $\mathbf{0}$ (because of the LAS of $\mathbf{0}$), but this is very slow. However, the main objective is achieved: to maintain the wild population below E_1.

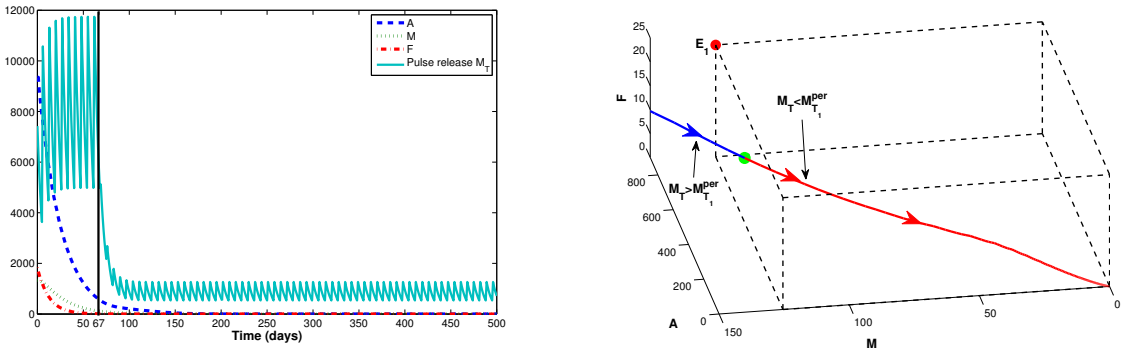
6.2. Minimal time in the case of periodic pulse releases

We consider that releases are done every week, i.e. $\tau = 7$. Thus for a given τ, we choose Λ such that $\tau\Lambda > M_{T_1}^{per}$. We also assume that, in system (40), massive periodic and pulse releases are such that $M_T(0) = k \times M_{T_1}^{per}$ (see Table 3 for $M_{T_1}^{per}$). When $M_T(0) = 0$, then we add 7 (14) days, thanks to very massive releases $k' = 4k$ ($k' = 2k$). In Table 6, we provide the results for different values of k, γ, and $\underline{M}_T^*$.

$k = 1.2$				$k = 2$			$k = 5$			$k = 10$		
γ \ $\underline{M}_T^*$	100	500	800	100	500	800	100	500	800	100	500	800
0.04	213	166	123	166	120	88	127	91	67	117	83	61
0.06	228	195	184	175	147	135	140	114	104	130	105	95
0.08	235	210	201	183	159	150	148	125	118	138	116	108
0.1	240	218	210	187	166	159	152	132	125	142	122	115

Table 6: Periodic impulsive releases are done every 7 days. Numerical estimates of the minimal times (in days) to reach $\underline{Y}$, using massive periodic impulsive releases, $M_T^* = \Lambda\tau \geq k \times M_{T_1}^{per}$

In Figure 4, we illustrate the periodic impulsive SIT control for $\gamma = 0.04$ and $k = 5$. First, with massive periodic releases, followed by small periodic releases. Again, the red trajectory indicates that the system converges (but very slowly) to $\mathbf{0}$.



(a) (b)

Figure 4: The case of periodic pulse release. Releases are done every 7 days. (a): Time-serie of the trajectory of system (16) initiated at the wild equilibrium E^*. The solid vertical black line denotes the shift from massive release to targeted release (b): Zoom in around the box delimited by the positive unstable equilibrium $\underline{E}_1 = (878.68, 125.48, 23.35)'$ (red dot). The green dot with coordinates (604.57,122.4, 9.57)' corresponds to the start of the targeted release $\underline{M}_T^*$. $\gamma = 0.04$, $k = 5$, $M_{T_1}^{per} = 1484.5$ and $\underline{M}_T^* = 800$.

Comparing the results between Table 6 and Table 5, clearly shows some similarities for large releases, while the results are far better for small periodic "massive releases", i.e. $k = 1.2$. In fact, the periodic impulsive case is strongly related to the constant release case, thanks to the fact that $< M_T^{per} >= \frac{1}{\tau}\int_{t_n}^{t_n+\tau} M_T^{per}(t)dt = \frac{\Lambda}{\mu_T} = M_T^*$. Therefore, releasing $\tau\Lambda$ sterile individuals every τ days is equivalent of releasing a constant amount, M_T^*, of sterile males over the same period. Thus, since $M_{T_1}^{per} = k \times (e^{\mu_T \tau} - 1)M_{T_1}$, as long as $k \times (e^{\mu_T \tau} - 1) > 1$, choosing Λ such that $\tau\Lambda > k(e^{\mu_T \tau} - 1)M_{T_1}$, is equivalent of choosing $M_T^* = k \times M_{T_1}$. That is why values of k smaller than 1 can be considered too. In Table 7, we provide estimates of the minimal time for $k < 1$. When $k < 0.58$, we did not observe (numerically) convergence towards $\mathbf{0}$.

However, like for the constant releases case, the larger the value of k, the lowest the time necessary to enter $[\mathbf{0}, E_1)$. Values of k chosen between 2 and 5 seem the most interesting ones.

$k = 0.58$			
γ \ $\underline{M}_T^*$	100	500	800
0.04	3073	3003	2827
0.06	3732	3696	3677
0.08	4322	4294	4282
0.1	∞	∞	∞

$k = 0.6$		
100	500	800
1065	997	852
1111	1075	1057
1137	1109	1098
1215	1191	1182

$k = 0.7$		
100	500	800
449	388	300
467	433	416
477	450	439
486	462	454

$k=0.8$				$k=0.9$			$k=1$		
γ \ $\underline{M}_T^*$	100	500	800	100	500	800	100	500	800
0.04	335	278	210	282	228	171	250	199	148
0.06	351	318	302	297	265	250	265	234	219
0.08	359	333	323	305	279	269	273	247	237
0.1	366	343	334	311	288	279	278	255	247

Table 7: Periodic impulsive releases are done every 7 days. Numerical estimates of the minimal times (in days) to reach $\underline{Y}$, using massive periodic impulsive releases, $M_T^* = \Lambda\tau \geq k \times M_{T_1}^{per}$. The symbol ∞ denotes that the result is greater than 10^6.

6.3. Mechanical control or not?

In general, using SIT alone is not efficient. It is preferable to consider other bio-control tools. Against mosquito, it has been showed that mechanical control (MC), which consists of removing the breeding sites, can be an additional efficient control tool [10, 7], and in particular coupled with SIT [8]. This is a cheap control, but it requires the support of the local population.

We now assume that the MC leads an increase of $\mu_{A,2}$, that is a decrease of the wild aquatic stage equilibrium A^* (see Table 3 for values of A^*). According to relation (12), page 6, we deduce that reducing A^* for $MC\%$ corresponds to an increase of $\mu_{A,2}$ as follows

$$\mu_{A,2,MC} \quad = \quad \frac{(\gamma+\mu_{A,1})}{(1-\frac{MC}{100})A^*}(\mathcal{R}-1). \tag{49}$$

In Table 8, we provide $\mu_{2,A,MC}$ and the wild equilibrium E^*_{MC}, for $MC = 0$, 20% and 40% in (49).

$MC=0$					$MC=20$				$MC=40$			
$\mu_{A,2,MC}$	2×10^{-4}				2.5×10^{-4}				3.3333×10^{-4}			
γ	0.04	0.06	0.08	0.1	0.04	0.06	0.08	0.1	0.04	0.06	0.08	0.1
A^*	9350	14150	18950	23750	7480	11320	15160	19000	5610	8490	11370	14250
M^*	1335	3031	5412	8479	1068	2425	4330	6783	801	1819	3247	5087
F^*	1834	4160	7428	11637	1466	3328	5943	9310	1010	2496	4457	6983

Table 8: Impact of MC on the wild equilibrium E^*_{MC}

Clearly, the impact of MC on the wild equilibrium is quite obvious. However, MC can be limited in space and time.

Since the objective of massive SIT release is to enter (rapidly) in $[\mathbf{0}, E_1)$, it is also interesting to see the impact of MC treatment on the unstable equilibrium, $E_{1,MC}$, for a given targeted amount of sterile males, $\underline{M}_T^*$. This is summarized in Tables 9 and 10. In fact, and this is a good news, we have $E_{1,MC} > E_1 = E_{1,0}$. Thus, with MC, the wild equilibrium, E^*_{MC}, decreases and the size of $[\mathbf{0}, E_1)$ increases, such that we can expect a good gain in terms of minimal time to enter in $[\mathbf{0}, E_1)$, using massive releases.

γ \ $\underline{M}_T^*$	100	500
0.04	(37.39,5.34,0.37)'	(347.57,49.56,6.14)'
0.06	(18.96,4.06,0.22)'	(115.79,24.8,1.6)'
0.08	(12.3,3.51,0.163)'	(68.24,19.49,1)'
0.1	(8.98,3.21,0.137)'	(47.88,17.09,0.78)'

Table 9: Values of $E_{1,MC}$ for different values of the targeted releases amount, $\underline{M}_T^*$, and various values of γ, when $MC = 20\%$.

γ \ $\underline{M}_T^*$	100	500
0.04	(38.82,5.54,0.4)'	(646.33,92.3,19.7)'
0.06	(19.25,4.12,0.22)'	(127.8,24.37,1.9)'
0.08	(12.4,3.54,0.166)'	(71.5,20.4,1.1)'
0.1	(9.03,3.22,0.138)'	(49.2,17.58,0.82)'

Table 10: Values of $E_{1,MC}$ for different values of the targeted releases amount, $\underline{M}_T^*$, and various values of γ, when $MC = 40\%$.

Minimal time results are given in Tables 11-12, when we consider that MC has started before SIT and goes on once SIT starts. Clearly, the gain in time is "small", indicating that MC does not drastically decay the minimal time to reach $[\mathbf{0}, E_1)$.

The case of continuous and constant release						
	$k=2$		$k=5$		$k=10$	
γ \ $\underline{M}_T^*$	100	500	100	500	100	500
0.04	213(4)	155(14)	137(4)	93(10)	120(3)	80(8)
0.06	228(4)	195(6)	152(3)	123(5)	134(3)	107(4)
0.08	236(3)	210(4)	160(2)	136(3)	141(3)	118(4)
0.1	241(3)	218(3)	164(3)	143(3)	146(3)	125(4)
The case of periodic pulse release						
	$k=2$		$k=5$		$k=10$	
γ \ $\underline{M}_T^*$	100	500	100	500	100	500
0.04	157(9)	109(11)	123(4)	82(9)	113(4)	75(8)
0.06	172(3)	142(5)	137(3)	110(4)	127(3)	101(4)
0.08	180(3)	155(4)	145(3)	122(3)	135(3)	112(4)
0.1	185(2)	162(4)	150(2)	129(3)	140(2)	119(3)

Table 11: The case when 20% of MC takes place all over the time. Numerical estimates of the minimal times (in days) to reach $\underline{Y}$, using massive releases, $M_T^* = k \times M_{T_1}$. The values in the brackets indicate the gain in days compared to SIT alone.

The case of continuous and constant release								
	$k=2$			$k=5$			$k=10$	
γ \ $\underline{M_T^*}$	100	500		100	500		100	500
0.04	206(11)	116(53)		132(9)	70(33)		114(10)	60(28)
0.06	224(8)	186(15)		147(8)	116(12)		129(8)	100(11)
0.08	232(7)	204(10)		156(6)	130(9)		138(6)	114(8)
0.1	237(7)	213(8)		161(6)	138(8)		143(6)	121(8)
The case of periodic pulse release								
	$k=2$			$k=5$			$k=10$	
γ \ $\underline{M_T^*}$	100	500		100	500		100	500
0.04	151(15)	82(38)		118(9)	62(29)		108(9)	57(26)
0.06	167(8)	134(13)		133(7)	103(11)		123(7)	94(11)
0.08	176(7)	149(10)		141(7)	117(8)		131(7)	107(9)
0.1	181(6)	158(8)		146(6)	125(7)		136(6)	115(7)

Table 12: The case when 40% of MC takes place all over the time. Numerical estimates of the minimal times (in days) to reach $\underline{Y}$, using massive releases, $M_T^* = k \times M_{T_1}$. The values in the brackets indicate the gain in days compared to SIT alone.

MC is a useful tool. However, to be really efficient, whatever the type of releases, MC needs to reduce the potential breeding site by 40%.

In fact, the combination of control strategies needs to be considered according to the location. In la Réunion, a french overseas department in the Indian Ocean where a SIT project is ongoing, there is a seasonal effect on the wild mosquito population [6], such that the best period to start SIT is between July and September, when the size of the wild mosquito population is low or reducing. In general there is a factor 10 in the population estimates between the wet season (February-March) and the dry season (July-September) (see for instance [15]). In Cali (Colombia), there is no seasonal effect, such that the wild population is more or less constant along the year. In order to use the SIT in an efficient manner in Cali, a population reduction is necessary.

One possible way, and also recommended by IAEA (the International Atomic Energy Agency) for SIT control, is to first use insecticide to reduce the population by a factor 5 or 10, and then to use SIT control. This is what we consider now: during one week, before SIT starts, we combine MC and an adulticide treatment, assuming 100% efficiency.

In Tables 13 and 14, we provide the values obtained after one week of adulticide treatment without and with MC.

Adulticide during one week				
$MC=0$				
γ	0.04	0.06	0.08	0.1
A_7	1897.9	2645.1	3387.1	4114
M_7	46.2	98.3	169.5	258.6
F_7	49.3	105.4	182.2	278.2

Table 13: Solution (A_7, M_7,F_7)' of the model after one week of adulticide treatment only.

Adulticide during one week								
	$MC = 20$				$MC = 40$			
γ	0.04	0.06	0.08	0.1	0.04	0.06	0.08	0.1
A_7	1518.4	2116	2709.7	3291.2	1138.9	1587.2	2032.5	2468.6
M_7	37	78.6	135.6	206.9	27.7	59	101.7	155.2
F_7	39.5	84.3	145.7	222.5	29.6	63.2	109.3	166.9

Table 14: Solution (A_7, M_7,F_7)' of the model after one week of adulticide treatment combined with MC.

Clearly, according to the tables above, after one weak of adulticide treatment, the size of the mosquito population has been drastically reduced, such that the SIT treatment will now start at the point $X_7 = (A_7, M_7, F_7)'$. That is why an impact on the minimal time to enter the basin $[\mathbf{0}, E_{1,MC})$ is expected.

Indeed, Table 15, page 26, clearly confirms that the gain in the entry time is rather important for the adulticide treatment only: it ranges from 35 to 95 days.

The case of continuous and constant release						
$k = 2$			$k = 5$		$k = 10$	
γ \ $\underline{M}_T^*$	100	500	100	500	100	500
0.04	122(95)	74(95)	92(49)	54(49)	85(38)	50(38)
0.06	137(95)	107(94)	106(49)	79(49)	98(39)	72(39)
0.08	146(93)	121(93)	113(49)	90(49)	105(39)	83(39)
0.1	150(94)	128(93)	118(49)	97(49)	110(39)	90(39)
The case of periodic pulse release						
$k = 2$			$k = 5$		$k = 10$	
γ \ $\underline{M}_T^*$	100	500	100	500	100	500
0.04	101(65)	60(60)	87(40)	51(40)	82(35)	48(35)
0.06	115(60)	87(60)	100(40)	74(40)	95(35)	70(35)
0.08	123(60)	99(60)	107(41)	85(40)	103(35)	81(35)
0.1	127(60)	106(60)	112(40)	91(41)	107(35)	87(38)

Table 15: Numerical estimates of the minimal times (in days) to reach $\underline{Y}$, using massive releases, $M_T^* = k \times M_{T_1}$. The values in the brackets indicate the gain in days compared to SIT alone.

In Tables 16 and 17, we present the results when MC is combined with the adulticide treatment. As, expected, the results are improved. However, the gain, compared to the adulticide treatment alone is small, such that the best combination would be "adulticide treatment for seven days, followed by permanent SIT treatment".

The case of continuous and constant release						
$k=2$			$k=5$		$k=10$	
γ \ M_T^*	100	500	100	500	100	500
0.04	116(101)	68(101)	89(52)	50(53)	82(41)	47(41)
0.06	131(101)	101(100)	102(53)	75(53)	95(42)	70(41)
0.08	139(100)	114(100)	110(52)	87(52)	103(41)	80(42)
0.1	144(100)	122(99)	115(52)	94(52)	107(42)	87(42)
The case of periodic pulse release						
$k=2$			$k=5$		$k=10$	
γ \ M_T^*	100	500	100	500	100	500
0.04	97(69)	56(64)	84(43)	48(43)	80(37)	46(37)
0.06	111(64)	83(64)	97(43)	71(43)	93(37)	68(37)
0.08	119(64)	95(64)	104(44)	82(43)	100(38)	78(38)
0.1	124(63)	102(64)	109(43)	89(43)	105(37)	85(37)

Table 16: Combination of adulticide and 20% of MC, followed by SIT. Numerical estimates of the minimal times (in days) to reach $\underline{Y}$, using massive releases, $M_T^* = k \times M_{T_1}$. The values in the brackets indicate the gain in days compared to SIT alone.

The case of continuous and constant release						
$k=2$			$k=5$		$k=10$	
γ \ M_T^*	100	500	100	500	100	500
0.04	107(110)	59(110)	85(56)	46(57)	79(44)	43(45)
0.06	123(109)	93(108)	98(57)	71(57)	92(45)	66(45)
0.08	132(107)	107(107)	106(56)	83(56)	99(45)	77(45)
0.1	137(107)	114(107)	110(57)	90(56)	104(45)	84(45)
The case of periodic pulse release						
$k=2$			$k=5$		$k=10$	
γ \ M_T^*	100	500	100	500	100	500
0.04	92(74)	50(70)	80(47)	44(47)	77(40)	42(41)
0.06	106(69)	78(69)	93(47)	67(47)	90(40)	64(41)
0.08	114(69)	90(69)	101(47)	78(47)	97(41)	75(41)
0.1	118(69)	97(69)	105(47)	85(47)	102(40)	81(41)

Table 17: Combination of adulticide and 40% of MC, followed by SIT. Numerical estimates of the minimal times (in days) to reach $\underline{Y}$, using massive releases, $M_T^* = k \times M_{T_1}$. The values in the brackets indicate the gain in days compared to SIT alone.

7. Conclusion

Generally speaking, most of the papers related to SIT (see [5, 20] and references therein) focus on "finite time" applications of SIT. This is possible, when the wild population has a so-called Allee effect. If not, then, if the SIT control stops, the system recovers (even if

the population is low). Based on previous works done by some of the authors, we study SIT control when a wild pest/vector population does not have any Allee effect. In fact, a low level of SIT can induce a weak Allee effect (if the population is sufficiently small, it can be driven to extinction), and we use this particular property to derive a realistic strategy. Indeed, using a mathematical analysis, we show that a strategy mixing massive and small releases can be used to drive and maintain a wild population at a (very) low level. In addition, the combination of SIT with other control tools, including MC and adulticide, can help to reduce the duration of the massive releases and eventually their size. To the best of our knowledge, this is the first time that such a "massive-small" releases strategy is derived for SIT. Since this work is done within the framework of a mosquito and a fruit fly SIT programs, we do hope that our strategy proposal will be considered in forthcoming field trials.

Several extensions of this work are possible. For instance, take into account the epidemiological states in order to derive the threshold value that needs to be reached by the mosquito population, using SIT, to lower the epidemiological risk, like in [8]; take into account the spatial component [6], the human behavior [9, 21], and, also, to compare all possible control treatments from an economical point of view.

Acknowledgments

This study is part of the "SIT feasibility project against *Aedes albopictus* in Reunion Island", TIS 2B (2020-2021), jointly funded by the French Ministry of Health and the European Regional Development Fund (ERDF). All authors were (partially) supported by the DST/NRF SARChI Chair in Mathematical Models and Methods in Biosciences and Bioengineering at the University of Pretoria (grant 82770). YD is also partially supported by the GEMDOTIS project, funded by the call ECOPHYTO 2018 (Action 27).

The authors thank the anonymous reviewers for their fruitful comments that greatly improved the initial manuscript.

References

[1] R. Anguelov, R. Bekker, and Y. Dumont. Bi-stable dynamics of a host-pathogen model. *Biomath*, 8:1–17, 2019.

[2] R. Anguelov, Y. Dumont, and J.M. Lubuma. Mathematical modeling of sterile insect technology for control of anopheles mosquito. *Comput. Math. Appl.*, 64:374–389, 2012.

[3] R. Anguelov, Y. Dumont, and J.M. Lubuma. On nonstandard finite difference schemes in biosciences. *AIP Conf. Proc.*, 1487:212–223, 2012.

[4] D. Bainov and P. Simeonov. *Impulsive Differential Equations: Periodic Solutions and Applications*, volume 66. CRC Press, 1993.

[5] Pierre-Alexandre Bliman, Daiver Cardona-Salgado, Yves Dumont, and Olga Vasilieva. Implementation of control strategies for sterile insect techniques. *Mathematical Biosciences*, 314:43 – 60, 2019.

[6] C. Dufourd and Y. Dumont. Impact of environmental factors on mosquito dispersal in the prospect of sterile insect technique control. *Comput. Math. Appl.*, 66(9):1695–1715, 2013.

[7] Y. Dumont, F. Chiroleu, and C. Domerg. On a temporal model for the chikungunya disease: Modeling, theory and numerics. *Mathematical Biosciences*, 213(1):80 – 91, 2008.

[8] Y. Dumont and J. M. Tchuenche. mathematical studies on the sterile insect technique for the chikungunya disease and aedes albopictus. *J. Math. Biol.*, 65 (5):809–854, 2012.

[9] Y. Dumont and J. Thuilliez. Human behaviors: A threat to mosquito control? *Mathematical Biosciences*, 281(Supplement C):9 – 23, 2016.

[10] Yves Dumont and Frederic Chiroleu. Vector control for the chikungunya disease. *Mathematical Biosciences & Engineering*, 7:313–330, 2010.

[11] V. A. Dyck, J. Hendrichs, and A. S. Robinson. *The Sterile Insect Technique, Principles and Practice in Area-Wide Integrated Pest Management*. Springer, Dordrecht, 2006.

[12] C. Guiver, D. Hodgson, and S. Townley. A note on the eigenvectors of perturbed matrices with applications to linear positive systems. *Linear Algebra and its Applications*, 509:143–167, 2016.

[13] W.M. Haddad, V. Chellaboina, and Q. Hui. *Nonnegative and Compartmental Dynamical Systems*. Princeton Uninersity Press, 2010.

[14] Markus Kirkilionis and Sebastian Walcher. On comparison systems for ordinary differential equations. *Journal of Mathematical Analysis and Applications*, 299(1):157 – 173, 2004.

[15] Gilbert Le Goff, David Damiens, Abdoul-Hamid Ruttee, Laurent Payet, Cyrille Lebon, Jean-Sébastien Dehecq, and Louis-Clément Gouagna. Field evaluation of seasonal trends in relative population sizes and dispersal pattern of aedes albopictus males in support of the design of a sterile male release strategy. *Parasites & Vectors*, 12(1):81, 2019.

[16] R. Ross. *The Prevention of Malaria, 1st edn.* John Murray, London, 1910.

[17] R. Ross. *The Prevention of Malaria, 2nd edn.* John Murray, London, 1911.

[18] Steven P Sinkins. Wolbachia and cytoplasmic incompatibility in mosquitoes. *Insect Biochemistry and Molecular Biology*, 34(7):723 – 729, 2004. Molecular and population biology of mosquitoes.

[19] H. Smith. *Monotone dynamical systems: An introduction to the theory of competitive and cooperative systems*, volume 4. American Mathematical Society, 2008.

[20] Martin Strugarek, Hervé Bossin, and Yves Dumont. On the use of the sterile insect release technique to reduce or eliminate mosquito populations. *Applied Mathematical Modelling*, 68:443 – 470, 2019.

[21] J. Thuilliez and Y. Dumont. Public mosquito abatement: A cluster randomized experiment. *The World Bank Economic Review*, 33(2):479 – 497, 2019.

[22] P. van den Driessche and James Watmough. Reproduction numbers and sub-threshold endemic equilibria for compartmental models of disease transmission. *Mathematical Biosciences*, 180(1):29 – 48, 2002.

[23] W. Walter. *Ordinary Differential Equations*. Springer, 1998.