Journal of Inequalities and Special Functions
ISSN: 2217-4303, URL: www.ilirias.com/jiasf
Volume 10 Issue 1(2019), Pages 111-122.

ON SOME HERMITE-HADAMARD INTEGRAL INEQUALITIES IN MULTIPLICATIVE CALCULUS

MUHAMMAD AAMIR ALI, MUJAHID ABBAS AND AZHAR ALI ZAFAR

Abstract. In this paper, we establish some new Hermite-Hadamard integral inequalities for log-ϕ-convex and ϕ-convex functions in the framework of multiplicative calculus. Furthermore, some results related to differentiable log-ϕ-invex functions are also obtained.

1. Introduction

Grossman and Katz [14] initiated the study of Non-Newtonian calculus and modified the classical calculus introduced by Newton and Leibnitz in the 17th century. On the other hands, Bashirov et al. [3] studied the concept of multiplicative calculus and presented a fundamental theorem of multiplicative calculus.

Since then a number of interesting results has been obtained in this direction. For more discussion and applications of this discipline, we refer to [28], [2, 3, 4] and [26]. Some elements of stochastic multiplicative calculus have been investigated in [17] and [13]. Bashirov and Riza [5] also studied complex multiplicative calculus.

Another popular Non- Newtonian calculus, known as bigeometric calculus is studied in [29], [15], [1], [18], [27], [6].

Recall that, multiplicative integral called *integral is denoted by $\int_a^b (f(x))^{dx}$ whereas the ordinary integral is denoted by $\int_a^b f(x)dx$. This is due to the fact that the sum of product terms in the definition of a proper Riemann integral of f on $[a,b]$ is replaced with the product of terms raised to certain powers. It is also known that [3] if f is positive and Riemann integrable on $[a,b]$, then it is *integrable on $[a,b]$ and

$$\int_a^b (f(x))^{dx} = e^{\int_a^b \ln(f(x))dx}.$$

Consistent with [3] , the following results and notations will be needed in the sequel.

(i) $\int_a^b ((f(x))^p)^{dx} = \int_a^b \left((f(x))^{dx}\right)^p$,

(ii) $\int_a^b (f(x)g(x))^{dx} = \int_a^b (f(x))^{dx} . \int_a^b (g(x))^{dx}$,

---

1991 *Mathematics Subject Classification.* 26D07, 26D15, 35A23.

Key words and phrases. Hermite-Hadamard Inequalities, Convex Set and Multiplicative Integrals.

Submitted October 11, 2018. Published December 23, 2018.
Communicated by Mikail Et.

(iii) $\int_a^b \left(\frac{f(x)}{g(x)}\right)^{dx} = \frac{\int_a^b (f(x))^{dx}}{\int_a^b (g(x))^{dx}}$,

(iv) $\int_a^b (f(x))^{dx} = \int_a^c (f(x))^{dx} . \int_c^b (f(x))^{dx}, \quad a \leq c \leq b.$

(v) $\int_a^a (f(x))^{dx} = 1$ and $\int_a^b (f(x))^{dx} = \left(\int_a^b (f(x))^{dx}\right)^{-1}$.

On the other hand, the notion of convexity plays a significant role in many disciplines such as mathematical finance, economics, engineering, management sciences,and optimization theory.

In the recent years, several extensions and generalizations of convexity have been investigated. Noor [22] extended the concept of a convex function to ϕ-convex functions. For more results in this direction, we refer to [19] and [22] .

Hermite and Hadamard showed independently that the convex functions are related to an integral inequality. Hadamard's inequality for convex functions has received much attention in recent years and a remarkable variety of refinements and generalizations have been obtained (see for example, [7, 8, 9, 10, 11, 12]).

The aim of this paper is to establish Hermite Hadamard type integral inequalities for log-ϕ-convex functions, and ϕ-convex functions in the setup of multiplicative calculus.

2. Preliminaries

Let K be a nonempty closed set in $\mathbb{R}^n$, and K° the interior of K. We denote by $\langle .,. \rangle$ and $\|.\|$ the inner product and norm on $\mathbb{R}^n$, respectively. Let f, $\phi : K \to \mathbb{R}$ be continuos mappings.

We recall the following well known results and concepts.

Definition 2.1 A set K is said to be convex, if for any $a, b \in K$,

$$(1-t)a + tb = a + t(b-a) \in K, \text{ for all } t \in [0,1]. \tag{2.1}$$

Definition 2.2 A set K is said to be ϕ-convex, if for any $a, b \in K$,

$$a + te^{i\phi}(b-a) \in K, \text{ for all } t \in [0,1]. \tag{2.2}$$

If we take $\phi = 0$, then ϕ-convex set becomes a convex set. The converse does not hold in general.

Definition 2.3 The function f on the convex set K is said to be convex, if for any $a, b \in K$, we have

$$\begin{aligned} f(a + t(b-a)) &= f((1-t)a + tb) \\ &\leq (1-t)f(a) + tf(b), \text{ for all } t \in [0,1]. \end{aligned}$$

The function f is said to be concave iff $-f$ is convex.

Definition 2.4 The function f on the ϕ-convex set K is said to be ϕ-convex with respect to ϕ, if

$$f(a + te^{i\phi}(b-a)) \leq (1-t)f(a) + tf(b), \quad \forall\, a, b \in K, \quad t \in [0,1].$$

The function f is said to be ϕ-concave iff $-f$ is ϕ-convex. Note that, every convex function is ϕ-convex but the converse does not hold in general.

Definition 2.5 The function f on the convex set K is called quasi convex, if

$$f(a + t(b-a)) \leq \max\{f(a), f(b)\}, \quad \forall\, a, b \in K, \quad t \in [0,1].$$

Definition 2.6 The function f on the ϕ-convex set K is called quasi ϕ-convex, if

$$f(a + te^{i\phi}(b-a)) \leq \max\{f(a), f(b)\}, \quad \forall\, a, b \in K, \quad t \in [0,1].$$

Definition 2.7 The function f on the convex set K is called logarithmic convex, if

$$f(a+t(b-a)) \leq (f(a))^{1-t}(f(b))^{t}. \tag{2.3}$$

Moreover, we have

$$\log f(a+t(b-a)) \leq (1-t)\log f(a) + t\log f(b) \quad \forall\, a,b \in K, \quad t \in [0,1].$$

Definition 2.8 The function f on the convex set K is called logarithmic ϕ-convex, if

$$f(a+te^{i\phi}(b-a)) \leq (f(a))^{1-t}(f(b))^{t}. \tag{2.4}$$

Definition 2.9 The function f on the ϕ-convex set K is said to be logarithmic ϕ-convex with respect to ϕ, if

$$f(a+te^{i\phi}(b-a)) \leq (f(a))^{1-t}(f(b))^{t}.$$

Moreover, we have

$$\begin{aligned}&\log f(a+te^{i\phi}(b-a))\\ &\leq (1-t)\log f(a) + t\log f(b) \quad \forall\, a,b \in K, \quad t \in [0,1].\end{aligned}$$

In view of this fact, we have the following.

Definition 2.10 The differentiable function f on the ϕ-convex set K is said to be a log-ϕ-invex function with respect to ϕ , if

$$\log f(b) - \log f(a) \geq \left\langle \frac{f'_{\phi}(a)}{f(a)}, b-a \right\rangle \quad \forall\, a,b \in K.$$

It is well known [10, 11, 24, 25] that if f is a convex function on the interval $I=[a,b]$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}, \qquad \forall\, a,b \in I, \tag{2.5}$$

which is known as the Hermite-Hadamard inequalities for the convex functions. For some results related to this classical result, we refer to [10, 11, 24, 25] and the references therein.

Dragomir and Mond [10] proved the following Hermite-Hadamard type inequalities for the log-convex functions:

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \exp\left[\frac{1}{b-a}\int_a^b \ln[f(x)]dx\right]\\ &\leq \frac{1}{b-a}\int_a^b G(f(x), f(a+b-x))dx\\ &\leq \frac{1}{b-a}\int_a^b f(x)dx\\ &\leq L(f(a), f(b)) \leq \frac{f(a)+f(b)}{2}, \end{aligned} \tag{2.6}$$

where $G(p,q)=\sqrt{pq}$ is the geometric mean and $L(p,q)=\frac{p-q}{\ln p - \ln q}(p \neq q)$ is the logarithmic mean of the positive real numbers p,q ($for\ p=q$,we put $L(p,q)=p$).

Pachpatte [24] obtained some other refinements of the Hermite-Hadamard inequality for differentiable log-convex functions.

From now onward, unless otherwise stated, we assume that $K = \left[a, a + e^{i\phi}(b-a)\right]$ and $0 \leq \phi \leq \frac{\pi}{2}$.

Note that, if $K = \left[a, a + e^{i\phi}(b-a)\right]$ is an interval, then the ϕ-convex functions can be characterized as follows:

$$\begin{vmatrix} 1 & 1 & 1 \\ a & x & a + e^{i\phi}(b-a) \\ f(a) & f(x) & f(a + e^{i\phi}(b-a)) \end{vmatrix} \geq 0,$$

where $x = a + te^{i\phi}(b-a) \in K$.

Using this definition, it can be easily shown that ϕ-convex functions satisfy the inequalities of the form:

$$f(x) \leq f(a) + \frac{f(b) - f(a)}{e^{i\phi}(b-a)}(x-a). \tag{2.7}$$

3. Main Results

Theorem 3.1. *If $f : K \to (0, \infty)$ is a ϕ-convex function on the interval of real numbers in K° and $a, b \in K^\circ$ with $a < a + e^{i\phi}(b-a)$ and $0 \leq \phi \leq \frac{\pi}{2}$, then*

$$\left(\int_a^{a+e^{i\phi}(b-a)} (f(x))^{dx}\right)^{\frac{1}{e^{i\phi}(b-a)}} \leq \frac{(f(b))^{\frac{f(b)}{f(b)-f(a)}} \cdot (f(a))^{\frac{f(a)}{f(a)-f(b)}}}{e}.$$

Proof. As f is a ϕ-convex function, we have

$$\begin{aligned}
\int_a^{a+e^{i\phi}(b-a)} (f(x))^{dx} &= e^{\int_a^{a+e^{i\phi}(b-a)} \ln(f(x))dx} \\
&= e^{\int_0^1 \ln\left(f(a+te^{i\phi}(b-a)\right)e^{i\phi}(b-a)dt} \\
&\leq e^{e^{i\phi}(b-a) \int_0^1 (\ln((1-t)f(a)+tf(b)))dt} \\
&= e^{e^{i\phi}(b-a)\left\{\ln f(b) - (f(b)-f(a)) \int_0^1 \frac{t}{f(a)+t(f(b)-f(a))} dt\right\}} \\
&= e^{e^{i\phi}(b-a)\left\{\begin{array}{c} \ln f(b) - (f(b)-f(a)) \int_0^1 [\frac{1}{f(b)-f(a)} \\ -\frac{f(a}{(f(b)-f(a))(f(a)+t(f(b)-f(a)))}]dt \end{array}\right\}} \\
&= e^{e^{i\phi}(b-a)\left\{(\ln f(b) - 1 + \frac{f(a)}{f(b)-f(a)}(\ln f(b) - \ln f(a)))\right\}} \\
&= \left[e^{\ln f(b) - 1 + \ln\left(\frac{f(b)}{f(a)}\right)^{\frac{f(a)}{f(b)-f(a)}}}\right]^{e^{i\phi}(b-a)} \\
&= \left[\frac{(f(b))^{\frac{f(b)}{f(b)-f(a)}} \cdot (f(a))^{\frac{f(a)}{f(a)-f(b)}}}{e}\right]^{e^{i\phi}(b-a)}.
\end{aligned}$$

Hence

$$\left(\int_a^{a+e^{i\phi}(b-a)} (f(x))^{dx}\right)^{\frac{1}{e^{i\phi}(b-a)}} \leq \frac{(f(b))^{\frac{f(b)}{f(b)-f(a)}} \cdot (f(a))^{\frac{f(a)}{f(a)-f(b)}}}{e}.$$

□

Corollary 3.2. *If $f : K = [a,b] \to (0,\infty)$ is a convex function on the interval of real numbers in K° and $a, b \in K^\circ$, then*

$$\left(\int_a^b (f(x))^{dx}\right)^{\frac{1}{b-a}} \le \frac{(f(b))^{\frac{f(b)}{f(b)-f(a)}} . (f(a))^{\frac{f(a)}{f(a)-f(b)}}}{e}.$$

Proof. From Theorem 1 we get this inequality for $\phi = 0$. □

Theorem 3.3. *If $f : K \to (0,\infty)$ is a log-ϕ-convex function on K, then*

$$\begin{aligned}\left(\int_a^b (f(x))^{dx}\right)^{\frac{1}{e^{i\phi}(b-a)}} &\le G(f(a), f(b)) \\ &\le L(f(a), f(b)) \le A(f(a), f(b)),\end{aligned}$$

where $G(.,.)$ $L(.,.)$, $A(.,.)$ are geometric, logarithmic and arithmetic means, respectively.

Proof. Since f is a ϕ-convex function, we have

$$\begin{aligned}\int_a^{a+e^{i\phi}(b-a)} (f(x))^{dx} &= e^{\int_a^{a+e^{i\phi}(b-a)}(\ln(f(x)))dx} \\ &= e^{e^{i\phi}(b-a)\int_0^1 \ln\left(f(a+te^{i\phi}(b-a))\right)dt} \\ &\le e^{e^{i\phi}(b-a)\int_0^1 \ln\left(f(a)^{1-t} f(b)^t\right)dt} \\ &= e^{e^{i\phi}(b-a)\int_0^1 ((1-t)\ln f(a) + t \ln f(b))dt} \\ &= e^{e^{i\phi}(b-a)\left\{\frac{\ln f(b) - \ln f(a)}{2} + \ln f(a)\right\}} \\ &= e^{e^{i\phi}(b-a)\left\{\frac{\ln f(b) + \ln f(a)}{2}\right\}} \\ &= \left(e^{\{\ln f(b) + \ln f(a)\}}\right)^{\frac{e^{i\phi}(b-a)}{2}} \\ &= \left(e^{\{\ln f(b) . f(a)\}}\right)^{\frac{e^{i\phi}(b-a)}{2}} \\ &= (f(a).f(b))^{\frac{e^{i\phi}(b-a)}{2}} \\ &\le \left(\sqrt{f(a).f(b)}\right)^{e^{i\phi}(b-a)} = (G(f(a), f(b)))^{e^{i\phi}(b-a)} \\ &\le (L(f(a), f(b)))^{e^{i\phi}(b-a)} \le \left(\frac{f(a)+f(b)}{2}\right)^{e^{i\phi}(b-a)}.\end{aligned}$$

Hence,

$$\begin{aligned}\left(\int_a^b (f(x))^{dx}\right)^{\frac{1}{e^{i\phi}(b-a)}} &\le \sqrt{f(a).f(b)} = G(f(a), f(b)) \\ &\le L(f(a), f(b)) \le \frac{f(a)+f(b)}{2} = A(f(a), f(b)).\end{aligned}$$

□

Corollary 3.4. *If $f : K = [a,b] \to (0,\infty)$ is a log convex function on the interval $[a,b]$, then*

$$\left(\int_a^b (f(x))^{dx}\right)^{\frac{1}{b-a}} \leq= G(f(a), f(b))$$
$$\leq L(f(a), f(b)) \leq A(f(a), f(b)).$$

Proof. From Theorem 3, we obtain this inequality for $\phi = 0$. □

Theorem 3.5. *Let $f, g : K \to (0,\infty)$ be log-ϕ-convex functions on the interval of real numbers in K° and $a, b \in K^\circ$. Then*

$$\left(\int_a^{a+e^{i\phi}(b-a)} (f(x)g(x))^{dx}\right)^{\frac{1}{e^{i\phi}(b-a)}} \leq \sqrt{f(a)f(b).g(a)g(b)} = G(f(a)f(b), g(a)g(b))$$
$$\leq L(f(a)f(b), g(a)g(b)) \leq \frac{f(a)f(b) + g(a)g(b)}{2}.$$

Proof. As f, g are log-ϕ-convex functions, therefore

$$\int_a^{a+e^{i\phi}(b-a)} (f(x)g(x))^{dx} = e^{\int_a^{a+e^{i\phi}(b-a)} (\ln(f(x)g(x)))dx}$$
$$= e^{e^{i\phi}(b-a)\int_0^1 \left(\ln(f(a+te^{i\phi}(b-a))g(a+te^{i\phi}(b-a))\right)dt}$$
$$\leq e^{e^{i\phi}(b-a)\int_0^1 \left(\ln([f(a)g(a)]^{1-t}[f(b)g(b)]^{t})\right)dt}$$
$$= e^{e^{i\phi}(b-a)\int_0^1 ((1-t)\ln(f(a)g(a))+t\ln(f(b)g(b)))dt}$$
$$= e^{e^{i\phi}(b-a)\left\{\frac{\ln(f(b)g(b))-\ln(f(a)g(a))}{2}+\ln(f(a)g(a))\right\}}$$
$$= e^{e^{i\phi}(b-a)\left\{\frac{\ln(f(b)g(b))+\ln(f(a)g(a))}{2}\right\}}$$
$$= e^{\frac{e^{i\phi}(b-a)}{2}\{\ln(f(b)g(b))+\ln(f(a)g(a))\}}$$
$$= e^{\frac{e^{i\phi}(b-a)}{2}\{\ln(f(b)g(b)).(f(a)g(a))\}}$$
$$= \left(e^{\{\ln(f(b)g(b)).(f(a)g(a))\}}\right)^{\frac{e^{i\phi}(b-a)}{2}}$$
$$= (f(a)f(b).g(a)g(b))^{\frac{e^{i\phi}(b-a)}{2}}$$
$$= (G(f(a)f(b), g(a)g(b)))^{e^{i\phi}(b-a)}$$
$$\leq (L(f(a)f(b).g(a)g(b)))^{e^{i\phi}(b-a)}$$
$$\leq \left(\frac{f(a)f(b) + g(a)g(b)}{2}\right)^{e^{i\phi}(b-a)}.$$

Hence

$$\begin{aligned}
&\left(\int_a^{a+e^{i\phi}(b-a)} (f(x)g(x))^{dx}\right)^{\frac{1}{e^{i\phi}(b-a)}} \\
&\leq \sqrt{f(a)f(b).g(a)g(b)} \\
&= G(f(a)f(b), g(a)g(b)) \\
&\leq L(f(a)f(b), g(a)g(b)) \\
&\leq A(f(a)f(b), g(a)g(b)).
\end{aligned}$$

□

Corollary 3.6. *If $f, g : K = [a, b] \to (0, \infty)$ is a log convex functions on the interval of real numbers in K° and $a, b \in K^\circ$, then*

$$\begin{aligned}
\left(\int_a^b (f(x)g(x))^{dx}\right)^{\frac{1}{b-a}} &\leq \sqrt{f(a)f(b).g(a)g(b)} = G(f(a)f(b), g(a)g(b)) \\
&\leq L(f(a)f(b), g(a)g(b)) \\
&\leq A(f(a)f(b), g(a)g(b)).
\end{aligned}$$

Proof. This follows from Theorem 5 by taking $\phi = 0$. □

Theorem 3.7. *If $f, g : K \to (0, \infty)$ are differentiable log-ϕ-invex functions on the interval of real numbers in K° and $a, b \in K^\circ$, then*

$$\begin{aligned}
&\int_a^{a+e^{i\phi}(b-a)} (2f(x)g(x))^{dx} \\
&\geq \int_a^{a+e^{i\phi}(b-a)} \left[\begin{array}{c} f\left(\frac{2a+e^{i\phi}(b-a)}{2}\right) g(x) \exp\left[\left\langle \frac{f_\phi'\left(\frac{2a+e^{i\phi}(b-a)}{2}\right)}{f\left(\frac{2a+e^{i\phi}(b-a)}{2}\right)}, x - \frac{2a+e^{i\phi}(b-a)}{2}\right\rangle\right] + g\left(\frac{2a+e^{i\phi}(b-a)}{2}\right) \\
\times f(x) \exp\left[\left\langle \frac{g_\phi'\left(\frac{2a+e^{i\phi}(b-a)}{2}\right)}{g\left(\frac{2a+e^{i\phi}(b-a)}{2}\right)}, x - \frac{2a+e^{i\phi}(b-a)}{2}\right\rangle\right]
\end{array}\right]^{dx}.
\end{aligned}$$

Proof. Since f, g are differentiable log-ϕ-invex functions. So, we have

$$\begin{aligned}
&\log f(x) - \log f(y) \geq \left\langle \frac{f_\phi'(y)}{f(y)}, x - y\right\rangle, \text{ and} \\
&\log g(x) - \log g(y) \geq \left\langle \frac{g_\phi'(y)}{g(y)}, x - y\right\rangle \quad \forall\ g(x), g(y) \in K,
\end{aligned}$$

which implies that

$$\log \frac{f(x)}{f(y)} \geq \left\langle \frac{f_\phi'(y)}{f(y)}, x - y\right\rangle.$$

That is,

$$f(x) \geq f(y) \exp\left[\left\langle \frac{f'(y)}{f(y)}, x - y\right\rangle\right] \tag{3.1}$$

$$g(x) \geq g(y) \exp\left[\left\langle \frac{g_\phi'(y)}{g(y)}, x - y\right\rangle\right]. \tag{3.2}$$

Multiplying on both sides of (3.1) and (3.2) by $g(x)$ and $f(x)$, respectively and then adding the resultants, we have

$$\begin{aligned}
&2f(x)g(x)\\
&\geq g(x)f(y)\\
&\times \exp\left[\left\langle \frac{f'_\phi(y)}{f(y)}, x-y\right\rangle\right] + f(x)g(y)\exp\left[\left\langle \frac{g'_\phi(y)}{g(y)}, x-y\right\rangle\right]. \qquad (3.3)
\end{aligned}$$

By taking $y = \frac{2a+e^{i\phi}(b-a)}{2}$ in (3.3), we obtain that

$$\begin{aligned}
&2f(x)g(x)\\
&\geq g(x)f\left(\frac{2a+e^{i\phi}(b-a)}{2}\right)\exp\left[\left\langle \frac{f'_\phi\left(\frac{2a+e^{i\phi}(b-a)}{2}\right)}{f\left(\frac{2a+e^{i\phi}(b-a)}{2}\right)}, x-\frac{2a+e^{i\phi}(b-a)}{2}\right\rangle\right]\\
&+f(x)g\left(\frac{2a+e^{i\phi}(b-a)}{2}\right)\exp\left[\left\langle \frac{g'_\phi\left(\frac{2a+e^{i\phi}(b-a)}{2}\right)}{g\left(\frac{2a+e^{i\phi}(b-a)}{2}\right)}, x-\frac{2a+e^{i\phi}(b-a)}{2}\right\rangle\right],
\end{aligned}$$

$$\begin{aligned}
&\int_a^{a+e^{i\phi}(b-a)} (2f(x)g(x))^{dx}\\
&\geq \int_a^{a+e^{i\phi}(b-a)} \left[\begin{array}{c} f\left(\frac{2a+e^{i\phi}(b-a)}{2}\right)g(x)\exp\left[\left\langle \frac{f'_\phi\left(\frac{2a+e^{i\phi}(b-a)}{2}\right)}{f\left(\frac{2a+e^{i\phi}(b-a)}{2}\right)}, x-\frac{2a+e^{i\phi}(b-a)}{2}\right\rangle\right] + g\left(\frac{2a+e^{i\phi}(b-a)}{2}\right)\\ \times f(x)\exp\left[\left\langle \frac{g'_\phi\left(\frac{2a+e^{i\phi}(b-a)}{2}\right)}{g\left(\frac{2a+e^{i\phi}(b-a)}{2}\right)}, x-\frac{2a+e^{i\phi}(b-a)}{2}\right\rangle\right]\end{array}\right]^{dx}.
\end{aligned}$$

□

Corollary 3.8. *If $f, g : K = [a, b] \to (0, \infty)$ are differentiable log invex functions on the interval of real numbers in K° and $a, b \in K^\circ$ with $a < b$. Then*

$$\begin{aligned}
&\int_a^b (2f(x)g(x))^{dx}\\
&\geq \int_a^b \left[f\left(\frac{a+b}{2}\right)g(x)\exp\left[\left\langle \frac{f'_\phi\left(\frac{a+b}{2}\right)}{f\left(\frac{a+b}{2}\right)}, x-\frac{a+b}{2}\right\rangle\right]\right.\\
&\left.+g\left(\frac{a+b}{2}\right)\times f(x)\exp\left[\left\langle \frac{g'_\phi\left(\frac{a+b}{2}\right)}{g\left(\frac{a+b}{2}\right)}, x-\frac{a+b}{2}\right\rangle\right]\right]^{dx}.
\end{aligned}$$

Proof. By taking $\phi = 0$ in Theorem 7, we obtain the result. □

Theorem 3.9. *If $f, g : K \to (0, \infty)$ are ϕ-convex functions on the interval of real numbers in K° and $a, b \in K^\circ$, then*

$$\left(\int_a^{a+e^{i\phi}(b-a)} (f(x)g(x))^{dx}\right)^{\frac{1}{e^{i\phi}(b-a)}} \leq \frac{(f(b)g(b))^{\frac{f(b)g(b)}{f(b)g(b)-f(a)g(a)}} . (f(a)g(a))^{\frac{f(a)g(a)}{f(a)g(a)-f(b)g(b)}}}{e}.$$

Proof. Since f,g are ϕ-convex functions, we have

$$\begin{aligned}
\int_a^{a+e^{i\phi}(b-a)}(f(x)g(x))^{dx} &= e^{\int_a^{a+e^{i\phi}(b-a)}\ln(f(x)g(x))dx}\\
&= e^{e^{i\phi}(b-a)\int_0^1 \ln\left(f(a+te^{i\phi}(b-a)\right)g(a+te^{i\phi}(b-a))dt}\\
&\le e^{e^{i\phi}(b-a)\int_0^1 \ln(((1-t)f(a)+tf(b))((1-t)g(a)+tg(b)))}\\
&= e^{e^{i\phi}(b-a)\int_0^1(\ln(f(a)g(a)+t(f(b)g(b)-f(a)g(a)))dt}\\
&= e^{e^{i\phi}(b-a)\left\{\ln f(b)g(b)-(f(b)g(b)-f(a)g(a))\int_0^1\frac{t}{f(a)g(a)+t(f(b)g(b)-f(a)g(a))}dt\right\}}\\
&= e^{e^{i\phi}(b-a)\left\{\begin{array}{c}\ln f(b)g(b)-(f(b)g(b)-f(a)g(a))\int_0^1[\frac{1}{f(b)g(b)-f(a)g(a)}\\ -\frac{f(a)g(a)}{(f(b)g(b)-f(a)g(a))(f(a)g(a)+t(f(b)g(b)-f(a)g(a)))}]dt\end{array}\right\}}\\
&= e^{e^{i\phi}(b-a)\left\{(\ln f(b)g(b)-1+\frac{f(a)g(a)}{f(b)g(b)-f(a)g(a)}(\ln f(b)g(b)-\ln f(a)g(a)))\right\}}\\
&= \left[e^{\ln f(b)g(b)-1+\ln\left(\frac{f(b)g(b)}{f(a)g(a)}\right)^{\frac{f(a)g(a)}{f(b)g(b)-f(a)g(a)}}}\right]^{e^{i\phi}(b-a)}\\
&= \left[\frac{(f(b)g(b))^{\frac{f(b)g(b)}{f(b)g(b)-f(a)g(a)}}.(f(a)g(a))^{\frac{f(a)g(a)}{f(a)g(a)-f(b)g(b)}}}{e}\right]^{e^{i\phi}(b-a)}.
\end{aligned}$$

Hence

$$\left(\int_a^{a+e^{i\phi}(b-a)}(f(x)g(x))^{dx}\right)^{\frac{1}{e^{i\phi}(b-a)}} \le \frac{(f(b)g(b))^{\frac{f(b)g(b)}{f(b)g(b)-f(a)g(a)}}.(f(a)g(a))^{\frac{f(a)g(a)}{f(a)g(a)-f(b)g(b)}}}{e}.$$

□

Corollary 3.10. *If $f,g:K=[a,b]\to(0,\infty)$ are convex functions on the interval of real numbers in K° (the interior of K) and $a,b\in K^\circ$, then*

$$\left(\int_a^b(f(x)g(x))^{dx}\right)^{\frac{1}{b-a}} \le \frac{(f(b)g(b))^{\frac{f(b)g(b)}{f(b)g(b)-f(a)g(a)}}.(f(a)g(a))^{\frac{f(a)g(a)}{f(a)g(a)-f(b)g(b)}}}{e}.$$

Proof. Take $\phi=0$ in Theorem 9. □

Theorem 3.11. *If $f,g:K\to(0,\infty)$ are ϕ-convex and log-ϕ-convex functions, respectively on the interval of real numbers K° and $a,b\in K^\circ$, then*

$$\begin{aligned}
\left(\int_a^{a+e^{i\phi}(b-a)}(f(x)g(x))^{dx}\right)^{\frac{1}{e^{i\phi}(b-a)}} &\le \frac{(f(b))^{\frac{f(b)}{f(b)-f(a)}}.(f(a))^{\frac{f(a)}{f(a)-f(b)}}.G(g(a),g(b))}{e}\\
&\le \frac{(f(b))^{\frac{f(b)}{f(b)-f(a)}}.(f(a))^{\frac{f(a)}{f(a)-f(b)}}.L(g(a),g(b))}{e}\\
&\le \frac{(f(b))^{\frac{f(b)}{f(b)-f(a)}}.(f(a))^{\frac{f(a)}{f(a)-f(b)}}.A(g(a),g(b))}{e}.
\end{aligned}$$

Proof. Let f, g be ϕ-convex and log-ϕ-convex functions, respectively. Then

$$
\begin{aligned}
\int_a^{a+e^{i\phi}(b-a)} (f(x)g(x))^{dx} &= e^{\int_a^{a+e^{i\phi}(b-a)} \ln(f(x)g(x))dx} \\
&= e^{e^{i\phi}(b-a)\int_0^1 \ln\left(f(a+te^{i\phi}(b-a)\right).g(a+te^{i\phi}(b-a)))dt} \\
&\leq e^{e^{i\phi}(b-a)\int_0^1 (\ln((1-t)f(a)+tf(b).(g(a))^{1-t}(g(b))^{t}))dt} \\
&= e^{e^{i\phi}(b-a)\int_0^1 (\ln((1-t)f(a)+tf(b))+\ln((g(a))^{1-t}(g(b))^{t}))dt} \\
&= e^{e^{i\phi}(b-a)\int_0^1 (\ln((1-t)f(a)+tf(b))+(1-t)\ln(g(a))+t\ln(g(b))))dt} \\
&= e^{e^{i\phi}(b-a)\left\{\int_0^1 (\ln((1-t)f(a)+tf(b)))dt+\int_0^1((1-t)\ln(g(a))+t\ln(g(b))))dt\right\}} \\
&= e^{e^{i\phi}(b-a)\left\{\ln f(b)-(f(b)-f(a))\int_0^1 \frac{t}{f(a)+t(f(b)-f(a))}dt+\frac{\ln g(a)+\ln g(b)}{2}\right\}} \\
&= e^{e^{i\phi}(b-a)\left\{\begin{array}{c} \ln f(b)-(f(b)-f(a))\int_0^1 \frac{1}{f(b)-f(a)} \\ -\frac{f(a}{(f(b)-f(a))(f(a)+t(f(b)-f(a)))}dt+\ln(g(a).g(b))^{\frac{1}{2}} \end{array}\right\}} \\
&= e^{e^{i\phi}(b-a)\left\{(\ln f(b)-1+\frac{f(a)}{f(b)-f(a)}(\ln f(b)-\ln f(a)))+\ln(g(a).g(b))^{\frac{1}{2}}\right\}} \\
&= \left[e^{\ln f(b)-1+\ln\left(\frac{f(b)}{f(a)}\right)^{\frac{f(a)}{f(b)-f(a)}}+\ln(g(a).g(b))^{\frac{1}{2}}}\right]^{e^{i\phi}(b-a)} \\
&= \left[\frac{(f(b))^{\frac{f(b)}{f(b)-f(a)}}.(f(a))^{\frac{f(a)}{f(a)-f(b)}}.G(g(a),g(b))}{e}\right]^{e^{i\phi}(b-a)}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\left(\int_a^{a+e^{i\phi}(b-a)} (f(x)g(x))^{dx}\right)^{\frac{1}{e^{i\phi}(b-a)}} \\
&\leq \frac{(f(b))^{\frac{f(b)}{f(b)-f(a)}}.(f(a))^{\frac{f(a)}{f(a)-f(b)}}.G(g(a),g(b))}{e} \\
&\leq \frac{(f(b))^{\frac{f(b)}{f(b)-f(a)}}.(f(a))^{\frac{f(a)}{f(a)-f(b)}}.L(g(a),g(b))}{e} \\
&\leq \frac{(f(b))^{\frac{f(b)}{f(b)-f(a)}}.(f(a))^{\frac{f(a)}{f(a)-f(b)}}.A(g(a),g(b))}{e}.
\end{aligned}
$$

$\square$

Corollary 3.12. *Let $f, g : K = [a, b] \to (0, \infty)$ are convex and log convex functions, respectively on the interval of real numbers in K° and $a, b \in K^\circ$, then*

$$\begin{aligned} &\left(\int_a^b (f(x)g(x))^{dx}\right)^{\frac{1}{b-a}} \\ &\leq \frac{(f(b))^{\frac{f(b)}{f(b)-f(a)}} . (f(a))^{\frac{f(a)}{f(a)-f(b)}} .G(g(a), g(b))}{e} \\ &\leq \frac{(f(b))^{\frac{f(b)}{f(b)-f(a)}} . (f(a))^{\frac{f(a)}{f(a)-f(b)}} .L(g(a), g(b))}{e} \\ &\leq \frac{(f(b))^{\frac{f(b)}{f(b)-f(a)}} . (f(a))^{\frac{f(a)}{f(a)-f(b)}} .A(g(a), g(b))}{e}. \end{aligned}$$

Proof. The result follows from Theorem 11, if we take $\phi = 0$. □

Acknowledgments. Authors are thankful to the reviewers for their useful comments and remarks which improved the presentation of this paper.

References

[1] D. Aniszewska, Multiplicative Runge-Kutta method, Nonlinear Dynamics, 50, (2007) 265-272.
[2] A. E. Bashirov and G. Bashirova, Dynamics of literary texts, Online Journal of Communication and Media Technologies, 1 (3), (2011) 60-82.
[3] A. E. Bashirov, E. Kurpınar and A. Özyapıcı, Multiplicative calculus and its applications, Journal of Mathematical Analysis and Applications, 337 (1), (2008) 36-48.
[4] A. E. Bashirov, E. Kurpınar, Y. Tando and A. Özyapıcı, On modeling with multiplicative differential equations, Applied Mathematics, 26(4), (2011) 425-438.
[5] A. E. Bashirov and M. Riza, On complex multiplicative differentiation, TWMS Journal of Applied Engineering Mathematics, 1(1), (2011) 75-85.
[6] F. Córdova-Lepe, The multiplicative derivative as a measure of elasticity in economics, TEMAT-Theaeteto Atheniensi Mathematica, 2(3), (2006) online.
[7] S. S. Dragomir, Two mappings in connection to Hadamrad inequality, J. Math. Anal. Appl. 167, (1992) 49-56.
[8] S. S. Dragomir, On Hadamard inequalities for the convex functions, Mat. Balkancia 6, (1992) 215-222.
[9] S. S. Dragomir and C. Buse, Refinment of Hadamard inequality for multiple integrals, United Math. 47, (1995) 193-198.
[10] S.S Dragomir and B.Mond, Integral inequalities of Hadamard type for log-convex functions, Demonst. Math., 31, (1998) 354-364.
[11] S. S. Dragomir and C. E. M. Pearce, Slected topics on Hermite-Hadamard type inequalities, RGMIA Monograph, Victorial University, Australia (2000).
[12] S. S. Dragomir, J. E. Pecaric and L. E. Person, Some inequalities of Hadamard type, Soochow J. Math. 21, (1995) 335-341.
[13] Y. L. Daletskii and N. I. Teterina, Multiplicative stochastic integrals, Uspekhi Matematicheskikh Nauk, 27(2:164), (1972) 167-168.
[14] M. Grossman and R. Katz, Non-Newtonian Calculus, Lee Press, Pigeon Cove, MA (1972).
[15] M. Grossman, Bigeometric Calculus: A System with a Scale-Free Derivative, Archimedes Foundation, Rockport, MA (1983).
[16] Y. Gurefe, U. Kadak, E. Misirli and A. Kurdi, A new look at the classical sequence spaces by using multiplicative calculus, U. P. B. Sci. Bull. Series A78, (2016) 9-20.
[17] R. L. Karandikar, Multiplicative decomposition of non-singular matrix valued continuous semimartingales, The Annals of Probability, 10(4), (1982) 1088-1091.
[18] W. Kasprzak, B. Lysik and M. Rybaczuk, Dimensions, Invariants Models and Fractals, Ukrainian Society on Fracture Mechanics, SPOLOM, Wroclaw-Lviv, Poland (2004).

[19] K. Inayat Noor and M. Aslam Noor, Relaxed strongly nonconvex functions, Appl. Math. E-Notes,6, (2006) 259-267.
[20] U. Kadak and H. Efe, The construction of Hilbert spaces over the Non-Newtonian field, International Journal of Analysis (2014) 1-10.
[21] U. Kadak and Y. Gurefe, A generalization on weighted means and convex functions with respect to the Non-Newtonian calculus, International Journal of Analysis (2016) 1-9.
[22] M. Aslam Noor, Some new classes of nonconvex functions, Nonl. Funct. Anal. Appl., 11, (2006) 165-171.
[23] A. Özyapici and E. Misirli, Exponential approximation on multiplicative calculus, 6th ISAAC Congress, (2007) p. 471.
[24] B.G. Pachpatte, A note on integral inequalitis involving two log convex functions, Math. Inequal. Appl., 7, (2004) 511-515.
[25] J. E. Pecaric, F Proschan and Y.L. Tong, Convex functions, Partial Orderings and Statistical Applications, Acedamic Press, New York (1992).
[26] M. Riza, A. Özyapıcı and E. Kurpınar, Multiplicative finite difference methods, Quarterly of Applied Mathematics, 67 (4), (2009) 745-754.
[27] M. Rybaczuk, A. Kedzia and W. Zielinski, The concepts of physical and fractional dimensions II. The differential calculus in dimensional spaces, Chaos Solutions Fractals, 12, (2001) 2537-2552.
[28] D. Stanley, A multiplicative calculus, Primus, IX(4), (1999) 310-326.
[29] V. Volterra and B. Hostinsky, Operations Infinitesimales Lineares, Herman, Paris (1938).

Muhammad Aamir Ali, Department of Mathematics, Government College University Faislabad, Sahiwal Campus, Pakistan, Mujahid Abbas, Department of Mathematics, Government College University, Lahore 54000, Pakistan and Department of Mathematics and Applied Mathematics, University of Pretoria, Lynnwood road, Pretoria 0002, South Africa, Azhar Ali Zafar, Department of Mathematics, Government College University, Lahore 54000

E-mail address: `mahr.muhammad.aamir@gmail.com`