



Research Article

An Enriched $\alpha - \mu$ Model as Fading Candidate

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This paper introduces an enriched $\alpha - \mu$ distribution which may act as fading model with its origins via the scale mixture construction. The distribution's characteristics are visited and its feasibility as a fading candidate in wireless communications systems is investigated. The analysis of the system reliability and some performance measures of wireless communications systems over this enriched $\alpha - \mu$ fading candidate are illustrated. Computable representations of the Laplace transform for this scale mixture construction are also provided. The derived expressions are explored via numerical investigations. Tractable results are computed in terms of the Meijer G-function. This unified scale mixture approach allows access to previously unconsidered underlying models that may yield improved fits to experimental data in practice.

1. Introduction

In the field of communication systems, fading channels are characterized with statistical distributions to describe the signal degradation from the transmitter to receiver of wireless signals. Commonly used fading models such as the Rayleigh distribution assume received signals can be found by the addition of vector sums representing scattering, diffraction, and reflection from different objects. These models have a drawback in that they can only model the fading accurately when the scattering is homogeneous and require a large number of interfering signals to apply the central limit theorem (CLT) and so are built upon an underlying assumption of normality [1]. The $\alpha - \mu$ distribution was introduced by Yacoub [2] such that the parameters have physical meaning. Fading is modelled in nonlinear environments where there exists a spatial correlation between the surfaces which causes the diffusion and scattering and introduces the $\alpha - \mu$ model with the normal assumption for the underlying process behaviour. The $\alpha - \mu$ model is statistically complex in its explanation of fading while also being diverse and contains certain well-known statistical distributions which are of interest within the statistics and wireless communications paradigm. This fading candidate has been found valuable in

the modelling of on-body communications networks as well as vehicle communication (see [3]).

The underlying assumption of normality has received some criticism. Increasing evidence in real applications has illustrated that the normal distribution is not always an appropriate choice, as many experimental measurements show a lepto- or platykurtic shape. Ollila et al. [4] noted that a more general assumption than that of the normal may not be far from reality, with an underlying t assumption being deployed within communication systems to account for severe fading by Choi et al. [5]. The t distribution is useful and well known in statistics, but there is a lack of results in communication systems pertaining to real life modelling when compared to its normal counterpart. This raises the challenge of modelling data which emanate from inhomogeneous environments. Indeed, He et al. [6] and Qiu [7] explicitly asked what the consequences of analyses are when underlying normality does not seem plausible. The contribution of the work in this paper aims to assist answering this question. A more pliable approach is to model these data by taking into account a stochastic element which is embedded in the variance component of the underlying model; this leads to the consideration of scale mixtures of normal (SMN) distribution alternatives. For the interested reader, the SMN

approach and the value thereof can be consulted in, for example, Andrews and Mallows [8]; Choy and Chan [9]; and Lachos Dávila et al. [10].

The continuous demand for reliable communications systems has caused to the resurgence of research in various channel modelling approaches in system design and evaluation to improve model performance. Therefore, in this paper, we propose a SMN alternative of the well-known $\alpha - \mu$ distribution. This paper generalizes Yacoub's $\alpha - \mu$ model to one that emanates from an underlying SMN assumption for the process behaviour. In this way, an enriched $\alpha - \mu$ distribution, a powerful umbrella model, is derived which allows the practitioner to amend the underlying distribution in the case that observed experimental data exhibit greater heterogeneity than the usual $\alpha - \mu$ distribution would accommodate. To the authors' knowledge, this representation of the $\alpha - \mu$ model within communications systems has not yet been considered. Some performance measures of a fading channel subject to this enriched $\alpha - \mu$ model are comparatively investigated against that of the well-studied normal (for recent contributions in this domain, see [11–13]).

The SMN class uses the Laplace transform technique to illustrate that a variable X with location μ and scale σ has a SMN representation if it can be expressed as

$$f_X(x|\mu, \sigma) = \int_0^{\infty} N(x|\mu, t^{-1}\sigma^2)W(t; \theta)dt, \quad (1)$$

where $N(x|\cdot, \cdot)$ is the normal density function and $W(t; \theta)$ is a density function of arbitrary variable T defined on \mathcal{R}^+ in which θ is a scalar or vector parameter indexing the distribution of T (see [9]). In this case, $W(t; \theta)$ is called the mixing density of this SMN representation, and the distribution of X is denoted by $\text{SMN}(\mu, \sigma^2, W(\cdot))$. In general, $W(t; \theta)$ may be posed as an arbitrary random variable. The advantage of the representation in (1) is gaining access to class of previously unexplored distributions to accommodate experimental data better in practice.

In this paper, two mixing densities are of particular interest. The usual $\alpha - \mu$ model is obtained when the random variable T is degenerate at 0; that is, $W(t; \theta)$ is given by

$$W(t; \theta) = \delta(t - 1), \quad (2)$$

where $\delta(\cdot)$ denotes the Dirac delta function. In the case where an enriched $\alpha - \mu$ model emanates from an underlying t distribution, the mixing density is given by (see [14])

$$W(t; \theta) = \frac{\nu(\nu t/2)^{(\nu/2)-1}}{2\Gamma(\nu/2)\exp(\nu t/2)}, \quad (3)$$

where $\theta = \nu > 0$ degrees of freedom and $\Gamma(\cdot)$ denotes the gamma function ([15], equation 8.310.1). Note that $W(t; \theta) \equiv \text{Gam}((\nu/2), (\nu/2))$, where $\text{Gam}((\nu/2), (\nu/2))$ denotes the density of a gamma random variable with both parameters equal to $\nu/2$. In the literature, these two mixing densities, respectively, lead to what is called the "normal" and the " t " case.

This paper's contribution can be summarised as follows:

- (1) A SMN ($\alpha - \mu$) model is systematically developed with genesis from the communication system

platform and particular statistical characteristics of this model are derived

- (2) Expressions for system reliability are derived and investigated
- (3) Computable representations of the Laplace transform of special cases of this SMN ($\alpha - \mu$) are given
- (4) Speculative comparison between the different considered mixing densities in terms of outage probability is included

The departure point of this paper is the systematic construction of an enriched $\alpha - \mu$ distribution with genesis as a fading model in communication systems; this is given in Section 2. Subsequently, in Section 3, the system reliability is analyzed via the signal to noise ratio (SNR), amount of fading (AoF), and the Laplace transform which is of value in practice for evaluation the average bit-error-rate (ABER). The direct applications which may arise from the proposed model are comparatively investigated via the outage probability in Section 4 with final thoughts given in Section 5.

2. System Model and Statistical Characteristics

In this section, some theoretical properties such as the density of the SMN ($\alpha - \mu$) distribution will be proposed and derived.

2.1. System Model. The following theorem introduces the $\text{SMN}(\alpha - \mu)$ model and derives the corresponding density function.

Theorem 1. *Let X_i and Y_i be mutually independent SMN processes with $E(X_i) = E(Y_i) = 0$ and $\text{var}(X_i) = \text{var}(Y_i) = (\hat{r}^\alpha/2\mu)$; hence, $X_i, Y_i \sim \text{SMN}(0, (\hat{r}^\alpha/2\mu), W(t; \theta))$. The α -power envelope (amplitude of the process stemming from X_i and Y_i) emanating from the SMN construction is defined by (see [2])*

$$R^\alpha = \sum_{i=1}^{\mu} (X_i^2 + Y_i^2), \quad (4)$$

where α and μ are positive integers. The density of the envelope R is given by

$$f_R(r) = \frac{\alpha r^{\alpha\mu-1} \mu^\mu}{\Gamma(\mu) \hat{r}^{\alpha\mu}} \int_0^{\infty} t^\mu \exp\left[-\mu t \left(\frac{r}{\hat{r}}\right)^\alpha\right] W(t; \theta) dt, \quad r > 0, \quad (5)$$

where $\hat{r} = \sqrt[\alpha]{E(R^\alpha)}$. The distribution with density (5) is called a SMN ($\alpha - \mu$) distribution.

Proof. Note that $(\nu(t))^2 = (\hat{r}^\alpha/2\mu) (\chi^2(\nu))$ denotes a chi-square distributed random variable with $\nu > 0$ degrees of freedom; then $(X_i/\nu(t))|t \sim N(0, 1)$ and $(Y_i/\nu(t))|t \sim N(0, 1)$; therefore, $\sum_{i=1}^{\mu} X_i^2 + Y_i^2/(\nu(t))^2|t = (R^\alpha/(\nu(t))^2)|t \sim \chi^2(2\mu)$ since $\sum_{i=1}^{\mu} X_i^2/(\nu(t))^2|t \sim \chi^2(\mu)$ and $\sum_{i=1}^{\mu} Y_i^2/(\nu(t))^2|t \sim \chi^2(\mu)$. Let $K(t) = R^\alpha/(\nu(t))^2|t$; thus, $K(t) \sim \chi^2(2\mu)$ with the conditional density of the α -power envelope as

$$f_{R^\alpha|t}(r^\alpha) = \frac{1}{2^\mu \Gamma(\mu) (\nu(t))^2} \left(\frac{r^\alpha}{(\nu(t))^2} \right)^{\mu-1} \exp\left(-\frac{r^\alpha}{2(\nu(t))^2} \right). \quad (6)$$

Subsequently, the conditional density of the envelope, R , is

$$f_{R|t}(r) = \left(\frac{\alpha}{2^\mu \Gamma(\mu) (\nu(t))^{2\mu}} \right) r^{\alpha\mu-1} \exp\left(-\frac{r^\alpha}{2(\nu(t))^2} \right), \quad (7)$$

with the unconditional distribution given by (5).

Consider $W(t; \theta)$ as given by (2), and then (5) simplifies to the well-known $\alpha - \mu$ model [16]:

$$f_{\text{normal}}(r) = \frac{\alpha r^{\alpha\mu-1} \mu^\mu}{\Gamma(\mu) \hat{r}^{\alpha\mu}} \exp\left[-\mu \left(\frac{r}{\hat{r}} \right)^\alpha \right], \quad r > 0, \quad (8)$$

and if $W(t; \theta)$ as given by (3), then

$$\begin{aligned} f_t(r) &= \frac{\alpha r^{\alpha\mu-1} \mu^\mu}{\Gamma(\mu) \hat{r}^{\alpha\mu}} \int_0^\infty t^\mu \exp\left[-\mu t \left(\frac{r}{\hat{r}} \right)^\alpha \right] \frac{\nu(\nu t/2)^{(\nu/2)-1}}{2\Gamma(\nu/2) \exp(\nu t/2)} dt \\ &= \frac{\alpha r^{\alpha\mu-1} \mu^\mu \nu^{1/2}}{\Gamma(\mu) \hat{r}^{\alpha\mu} \Gamma(\nu/2) 2^{\nu/2}} \frac{\Gamma(\mu + \nu/2)}{(\mu(r/\hat{r})^\alpha + (\nu/2))^{\mu+(\nu/2)}} \end{aligned} \quad (9)$$

with $\mu + (\nu/2) > 0$ and $\mu(r/\hat{r})^\alpha + (\nu/2) > 0$. \square

2.2. Statistical Characteristics. In this section, some statistical characteristics necessary for the remainder of the paper of the SMN ($\alpha - \mu$) distribution are derived. The results are presented without proof.

Theorem 2. From (5), the j^{th} moment of the SMN ($\alpha - \mu$) type model is

$$E(R^j) = \frac{\alpha \mu^\mu}{\Gamma(\mu) \hat{r}^{\alpha\mu}} \int_0^\infty r^{\alpha\mu+j-1} \int_0^\infty t^\mu \exp\left[-\mu t \left(\frac{r}{\hat{r}} \right)^\alpha \right] W(t; \theta) dt dr. \quad (10)$$

For $W(t; \theta)$ given by (2), it follows from (10) and Gradshteyn and Ryzhik ([15]; equation 3.326-2) that

$$E_{\text{normal}}(R^j) = \frac{\hat{r}^j \Gamma(\mu + (j/\alpha))}{\mu^{j/\alpha} \Gamma(\mu)}. \quad (11)$$

This will hold for $\alpha > 0$, $\hat{r}^\alpha > 0$, and $\alpha\mu + j > 1$.

For $W(t; \theta)$ given by (3), it follows from (10) and Gradshteyn and Ryzhik ([15]; equation 3.241-4) that

$$E_t(R^j) = \frac{\Gamma(\mu + (j/\alpha)) \Gamma((\nu/2) - (j/\alpha))}{\Gamma(\mu) \Gamma((\nu/2))} \left(\frac{\nu \hat{r}^\alpha}{2\mu} \right)^{j/\alpha}, \quad \nu > \frac{2j}{\alpha}. \quad (12)$$

Figure 1 illustrates the value which the SMN structure provides the practitioner with. This graph illustrates the CDF, $F(r) = \int_0^r f(t) dt$, using (5) for the two considered mixing densities for certain arbitrary values of the parameters and highlights this potential heavier tail characteristic. The proposed SMN platform in this paper allows theoretical and resultant practical access to

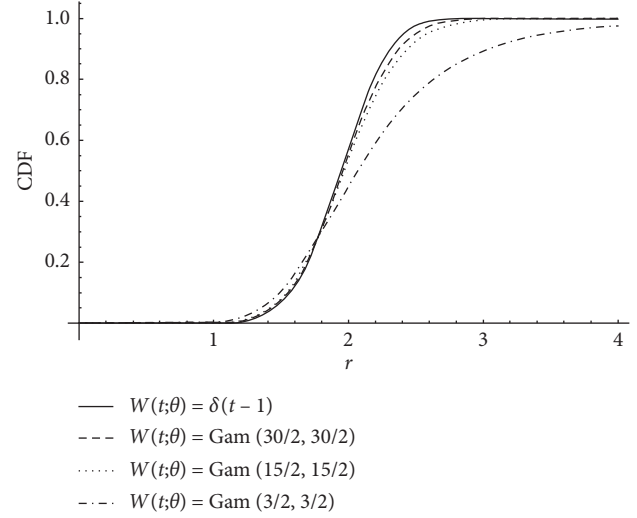


FIGURE 1: Illustration of CDFs of the SMN ($\alpha - \mu$) for different mixing densities from (5).

previously unconsidered models, providing flexibility for modelling that may yield improved fits to experimental data in practice when these data exhibit potential heavier tail behaviour [2, 13]

3. Reliability Analysis over an Enriched Candidate

In this section, the objective is to derive expressions for different performance metrics under SMN ($\alpha - \mu$) model as proposed in Section 2. Tractable expressions for the SNR, AoF, and the Laplace transform are derived and analyzed for the SMN ($\alpha - \mu$).

3.1. SNR. To measure and investigate the performance of the SMN ($\alpha - \mu$) type model, the instantaneous SNR is needed. The instantaneous SNR expressed in terms of the channel envelope is $\gamma = \hat{\gamma}(R/\hat{r})^2 = \hat{\gamma}R^2(1/E(R^2))$, where $\hat{\gamma} = E(\hat{r}^2)$ (E_b/N_0), and E_b is the energy per bit and N_0 is the noise spectral density [17]. The SNR is the building block to calculate the outage probability, AoF and ABER [18]. In this section, the density, CDF, AoF, and Laplace transform of the SNR are derived.

From (5), the density of the SNR for the SMN ($\alpha - \mu$) model is

$$f(\gamma) = \frac{\alpha \mu^\mu}{2\Gamma(\mu)} \left(\frac{\sqrt{\gamma}}{\sqrt{\hat{\gamma}}} \right)^{\alpha\mu} \gamma^{-1} \int_0^\infty t^\mu \exp\left[-\mu t \left(\frac{\sqrt{\gamma}}{\sqrt{\hat{\gamma}}} \right)^\alpha \right] W(t; \theta) dt. \quad (13)$$

For the $W(t; \theta)$ (2) and (3), (13) simplifies to

$$f_{\text{normal}}(\gamma) = \frac{\alpha \mu^\mu}{2\Gamma(\mu)} \gamma^{(\alpha\mu/2)-1} \hat{\gamma}^{-(\alpha\mu/2)} \exp\left[-\mu \left(\frac{\sqrt{\gamma}}{\sqrt{\hat{\gamma}}} \right)^\alpha \right], \quad \gamma > 0, \quad (14)$$

and

$$f_t(\gamma) = \frac{\alpha\mu^\mu \gamma^{\nu/2} \Gamma(\mu + (\nu/2))}{\Gamma(\mu)\Gamma(\nu/2)2^{(\nu/2)+1}} \gamma^{(\alpha\mu/2)-1} \hat{\gamma}^{-(\alpha\mu/2)} \frac{1}{(\mu(\sqrt{\gamma/\hat{\gamma}})^\alpha + (\nu/2))^{(\mu+(\nu/2))}}, \quad \gamma > 0, \quad (15)$$

respectively. The CDF of the SNR (13):

$$\begin{aligned} F_\gamma(\gamma_{OP}) &= \int_0^{\gamma_{OP}} f_\gamma(\gamma) d\gamma \\ &= \frac{\alpha\mu^\mu}{2\Gamma(\mu)\hat{\gamma}^{\alpha\mu/2}} \int_0^{\gamma_{OP}} \gamma^{(\alpha\mu/2)-1} \int_0^\infty t^\mu \exp\left[-\mu t \left(\frac{\sqrt{\gamma}}{\sqrt{\hat{\gamma}}}\right)^\alpha\right] W(t; \theta) dt d\gamma, \end{aligned} \quad (16)$$

with threshold $\gamma_{OP} > 0$. The CDF of the SNR's for the normal and t case will be obtained next.

- (i) From (2), (14), and (16) and using Gradshteyn and Ryzhik ([15]; equation (3.381)-(8)), it follows

$$F_{normal}(\gamma_{OP}) = \frac{\gamma(\mu, \mu(\gamma_{OP}/\hat{\gamma})^{\alpha/2})}{\Gamma(\mu)}, \quad (17)$$

where $\gamma(\cdot)$ denotes the incomplete gamma function ([15]; equation 8.350-1).

- (ii) Similarly from (3), (15), (16), and Gradshteyn and Ryzhik ([15]; equation 3.194-1), it follows:

$$F_t(\gamma_{OP}) = \int_0^{\gamma_{OP}} \frac{\alpha\mu^\mu \Gamma(\mu + (\nu/2)) \gamma^{\nu/2}}{\Gamma(\mu)\Gamma(\nu/2)2^{(\nu/2)+1} (\mu(\sqrt{\gamma/\hat{\gamma}})^\alpha + (\nu/2))^{(\mu+(\nu/2))}} \gamma^{-1} \left(\frac{\sqrt{\gamma}}{\sqrt{\hat{\gamma}}}\right)^\alpha d\gamma. \quad (18)$$

Let $z = \gamma^{\alpha/2}$; then, the latter equals

$$F_t(\gamma_{OP}) = {}_2F_1\left(\mu + \frac{\nu}{2}, \mu; 1 + \mu, -\frac{2}{\nu} \mu \hat{\gamma}^{-(\alpha/2)} \gamma_{OP}^{\alpha/2}\right) \frac{\Gamma(\mu + (\nu/2)) 2^\mu \mu^{\mu-1} \gamma_{OP}^{\alpha\mu/2}}{\Gamma(\mu)\Gamma(\nu/2) \nu \mu \hat{\gamma}^{\alpha\mu/2}}, \quad (19)$$

where ${}_2F_1(\cdot)$ denotes the Gauss hypergeometric function ([15]; equation 9.14-1).

3.2. *AoF*. The AoF is a quantitative measure giving an indication of the severity or level of fading that is present in the model [18, 19]. The AoF is calculated as

$$\text{AoF} = \frac{\text{var}(\gamma)}{E(\gamma)^2} = \frac{E(\gamma^2) - E(\gamma)^2}{E(\gamma)^2}, \quad (20)$$

and when $\text{AoF} > 1$, the fading is considered to be severe. The AoF for the normal case is given as

$$\text{AoF}_{normal} = \frac{\Gamma(\mu + (4/\alpha)) - [\Gamma(\mu + (2/\alpha))]^2 [\Gamma(\mu)]^{-1}}{[\Gamma(\mu + (2/\alpha))]^2 \mu^{-(1/\alpha)} [\Gamma(\mu)]^{-1}}, \quad (21)$$

since $E_{normal}(\gamma) = \hat{\gamma} \Gamma(\mu + (2/\alpha)) / \mu^{2/\alpha} \Gamma(\mu)$ and $E_{normal}(\gamma^2) = \hat{\gamma}^2 \Gamma(\mu + (4/\alpha)) / \mu^{4/\alpha} \Gamma(\mu)$. The t case follows similarly as

$$\text{AoF}_t = \frac{\Gamma(\mu + (4/\alpha)) \Gamma((\nu/2) - 4) - ([\Gamma(\mu + (2/\alpha)) \Gamma((\nu/2) - (\alpha/2))]^2 / (\Gamma(\mu) \Gamma(\nu/2)))}{[\Gamma(\mu + (2/\alpha)) \Gamma(\nu/2 - (\alpha/2))]^2 / (\Gamma(\mu) \Gamma(\nu/2))}, \quad (22)$$

since

TABLE 1: AoF (22) for different values of ν and μ , when $\alpha = 2$.

ν	$\mu = 4$	$\mu = 5$	$\mu = 6$
5	2.75	2.6	2.5
6	1.5	1.4	1.3333
7	1.083333	1	0.94444
8	0.875	0.8	0.75
9	0.75	0.68	0.633333
10	0.666667	0.6	0.555556
11	0.607143	0.542857	0.5
12	0.5625	0.5	0.458333
13	0.527778	0.466667	0.425926

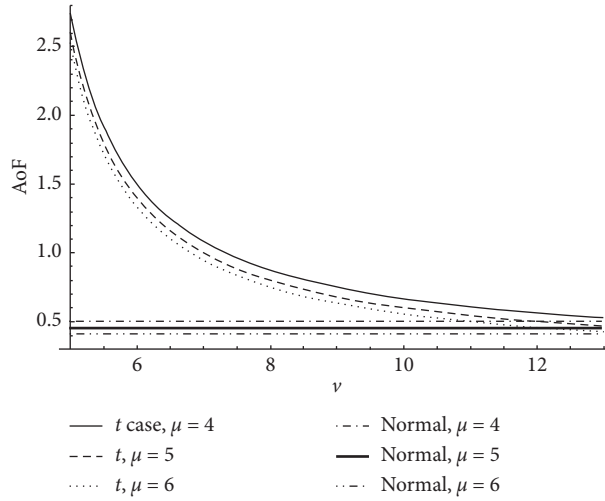


FIGURE 2: AoF (21) and AoF (22) for different values of ν and μ , when $\alpha = 2$.

$$\begin{aligned}
 E_t(\gamma) &= \frac{\Gamma(\mu + (2/\alpha))\Gamma((\nu/2) - (2/\alpha))}{\Gamma(\mu)\Gamma(\nu/2)} \frac{\gamma^{2/\alpha}\tilde{\gamma}}{(2\mu)^{2/\alpha}} \\
 E_t(\gamma^2) &= \frac{\Gamma(\mu + (4/\alpha))\Gamma((\nu/2) - (4/\alpha))}{\Gamma(\mu)\Gamma(\nu/2)} \frac{\gamma^{4/\alpha}\tilde{\gamma}}{(2\mu)^{4/\alpha}}
 \end{aligned}
 \tag{23}$$

From Table 1 and Figure 2, we observe that the AoF is > 1 for small values of ν , corresponding to the observations of Choi et al. [5] that the underlying t assumption is suitable for cases of severe fading. The AoF for the underlying normal case is equal to 0.5, 0.447214, and 0.408248 (see (22)) for the corresponding three different μ values with $\alpha = 2$; it is further noted that as ν increases, the AoF decreases for the underlying t model.

3.3. Laplace Transform. The Laplace transform is useful as an integral transform in different performance metrics. Relying on the Laplace-based approach, the ABER can be evaluated as $P_{\text{aver}} = (a_m/\pi) \int_0^{\pi/2} L(b_m^2/2 \sin^2(\theta))d\theta$, where $a_m = 1$ and $b_m^2 = 2(E_b/N_0)$ (see [20]). Analytical evaluation of ABER for fading channel distributions has been of interest lately [21]. Therefore, alternative expressions for the Laplace transform are presented here, in order to enable the researcher to

obtain numerical results for the ABER for this proposed model (see (5)). From (13), the Laplace transform is

$$\begin{aligned}
 L(c) = E[\exp(-c\gamma)] &= \frac{\alpha\mu^t}{2\Gamma(\mu)\tilde{\gamma}^{\alpha\mu/2}} \sum_{k=0}^{\infty} \sum_{u=0}^{\infty} \frac{(-c)^k}{k!} \frac{(-\mu)^u}{u!\tilde{\gamma}^{\alpha u/2}} \\
 &\int_0^{\infty} \gamma^{(\alpha(\mu+u)/2)+k-1} \int_0^{\infty} t^{\mu+u} W(t; \theta) dt d\gamma.
 \end{aligned}
 \tag{24}$$

An alternative expression, using an approach similar to that of Magableh and Matalgah [17], follows as

$$\begin{aligned}
 L(c) &= \frac{\alpha\mu^t}{2\Gamma(\mu)\tilde{\gamma}^{\alpha\mu/2}} \int_0^{\infty} \int_0^{\infty} G_{0,1}^{1,0}(c\gamma | \bar{\cdot}) \gamma^{(\alpha\mu/2)-1} \\
 &\times G_{0,1}^{1,0}\left(\mu t \gamma^{\alpha/\mu} \tilde{\gamma}^{-(\alpha/2)} \Big|_{\mu} \bar{\cdot}\right) W(t; \theta) t^{\mu} dt d\gamma,
 \end{aligned}
 \tag{25}$$

where $G(\cdot)$ denotes Meijer's G-function [22].

The expressions for the underlying normal and t case are of special interest.

- (1) Using (2), (14), and Gradshteyn and Ryzhik ([15]; equation 3.326-2), the Laplace transform for the normal case is

$$L_{\text{normal}}(c) = \int_0^\infty \frac{\alpha \mu^\mu \gamma^{(\alpha\mu/2)-1}}{2\Gamma(\mu)\hat{\gamma}^{\alpha\mu/2}} \exp\left[-\mu\left(\frac{\gamma}{\hat{\gamma}}\right)^{\alpha/2}\right] \exp(-c\gamma) d\gamma$$

$$= \frac{1}{\Gamma(\mu)} \sum_{k=0}^\infty \frac{(-c)^k}{k!} \frac{\Gamma(\mu + (2k/\alpha))}{\mu^{2k/\alpha}} \hat{\gamma}^k.$$

(26)

An alternative form is

$$L_{\text{normal}}(c) = \frac{\alpha \mu^\mu}{2\Gamma(\mu)\hat{\gamma}^{\alpha\mu/2}} \int_0^\infty \gamma^{(\alpha\mu/2)-1} G_{0,1}^{1,0}\left(\frac{\mu}{\hat{\gamma}^{\alpha/2}} \gamma^{1/k} \middle|_0^1\right) G_{0,1}^{1,0}(c\gamma|_0^1) d\gamma$$

$$= \frac{\alpha \mu^\mu}{2\Gamma(\mu)\hat{\gamma}^{\alpha\mu/2}} \frac{2^{1/2} \alpha^{\alpha\mu-1/2}}{(2\pi)^{\alpha/2} c^{\alpha\mu/2}} G_{\alpha,2}^{2,\alpha}\left(\left(\frac{\mu}{\hat{\gamma}^{\alpha/2}}\right)^2 \frac{\alpha^\alpha}{c^\alpha 2^2} \middle| \begin{matrix} \frac{2-\alpha\mu}{2\alpha}, \frac{3-\alpha\mu}{2\alpha}, \dots, \frac{\alpha-1-\alpha\mu}{2\alpha} \\ 0, \frac{1}{2} \end{matrix}\right).$$

(27)

(2) For the t case, applying Gradshteyn and Ryzhik ([15], equation 3.241-4) results in

Meijer's G -function for the integral considered can be found in ([17, 23], p. 346).

$$L_t(c) = \frac{\alpha \mu^\mu \Gamma(\mu + (\nu/2)) \nu^{\nu/2}}{\Gamma(\mu) \Gamma(\nu/2) 2^{(\nu/2)+1} \hat{\gamma}^{\alpha\mu/2}} \int_0^\infty (\mu \gamma^{\alpha/2} \hat{\gamma}^{-(\alpha/2)} + (\nu/2))^{-(\mu+(\nu/2))} \exp(-c\gamma) d\gamma$$

$$= \frac{\alpha \mu^\mu 2^{\mu-1}}{\Gamma(\mu) \Gamma(\nu/2) \hat{\gamma}^{\alpha\mu/2} \nu^\mu} \sum_{k=0}^\infty \frac{(-1)^k}{c^{\alpha(\mu+k)/2}} \frac{\Gamma(\mu + (\nu/2) + k) \Gamma(\alpha(\mu+k)/2)}{k!} \left(\frac{2\mu}{\hat{\gamma}^{\alpha/2} \nu}\right)^k,$$

(28)

where $0 < \mu + (2k/\alpha) < \mu + (\nu/2)$, $(\nu/2) \neq 0$ and $\mu \hat{\gamma}^{-(\alpha/2)} \neq 0$. An alternative expression for (29) is obtained from ([22, 24], p. 81) and (15) as

$$L_t(c) = \frac{\alpha^{1/2(\alpha\mu+1)} \mu^\mu 2^{2\mu+(\nu/2)-(1/2)(3+\alpha)}}{\Gamma(\mu) \Gamma(\nu/2) (c\hat{\gamma})^{(\alpha\mu/2)} \nu^\mu} c^{-(\alpha\mu/2)} (\pi)^{-(1/2)(1+\alpha)}$$

$$\times G_{2+\alpha,2}^{2,2+\alpha}\left(\frac{4\mu^2 \alpha^\alpha}{\nu^2 c^\alpha \hat{\gamma}^\alpha} \middle| \begin{matrix} \frac{1-\mu-(\nu/2)}{2}, \frac{2-\mu-(\nu/2)}{2}, \frac{1-(\alpha\mu/2)}{\alpha}, \frac{2-(\alpha\mu/2)}{\alpha}, \dots, \frac{1-(\alpha\mu/2)+\alpha-1}{\alpha} \\ 0, \frac{1}{2} \end{matrix}\right).$$

(29)

Particular values are evaluated for the Laplace transform for the normal and t cases with these different expressions (see Table 2). This table illustrates the accuracy of the different expressions for the Laplace transform which the practitioner may consider.

4. Analytical Illustration and Discussion

An often considered performance measure for systems operating in fading environments is the outage probability, P_{out} , defined as the probability that the output SNR falls below a given threshold γ_{OP} ; thus, $P_{\text{out}}(\gamma) = F(\gamma_{OP})$ (see (16)).

TABLE 2: Values of Laplace transform computations for some arbitrary parameter values.

Normal		
Equation	(26)	(27)
$L(0.1)$	0.797194370359114	0.797194370359114
$L(0.25)$	0.619690325863945	0.619690325863945
$L(0.5)$	0.455537937967763	0.455537937967763
$L(0.75)$	0.36041670118177	0.36041670118177
$L(1)$	0.297676390851642	0.297676390851642
t ($\nu = 3$)		
Equation	(28)	(29)
$L(0.1)$	0.624743148313397	0.624743148313397
$L(0.25)$	0.4840824983658209	0.4840824983658212
$L(0.5)$	0.373557082353699	0.373557082353697
$L(0.75)$	0.3113678073601211	0.3113678073601208
$L(1)$	0.269619410990287	0.269619410990287
t ($\nu = 15$)		
Equation	(28)	(29)
$L(0.1)$	0.753959031928999	0.753959031928984
$L(0.25)$	0.5805936227825265	0.5805936227824519
$L(0.5)$	0.431300920344367	0.431300920343910
$L(0.75)$	0.3463761840086145	0.3463761840072217
$L(1)$	0.290221789375107	0.290221789374477

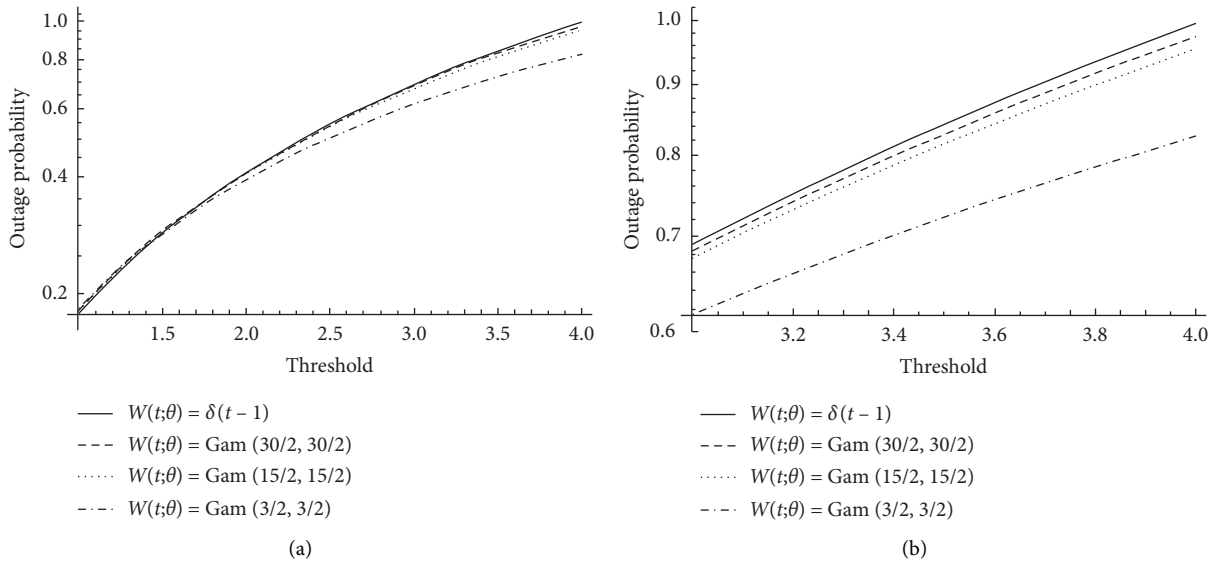


FIGURE 3: Outage probabilities of SNR of the SMN ($\alpha - \mu$) for different mixing densities from 5, for $\alpha = 2, \mu = 1, \hat{\gamma} = 3$.

By comparing the outage probability of the SMN ($\alpha - \mu$) with the mixing densities, i.e. for the underlying normal and t distributions, as depicted in Figure 3, it can be observed that the assumption of an underlying t distribution results in a lower outage probability than the $\alpha - \mu$ distribution even for large degrees of freedom (the bottom graph in Figure 3 is a magnification of the top graph; for a smaller interval on the x -axis to provide the reader with ease of interpretation). As expected, the SMN ($\alpha - \mu$) approaches the $\alpha - \mu$ distribution as the degrees of freedom of the t distribution increase, since a larger degree of freedom results in the tails being lighter and closer to what is expected with the underlying normal distribution. These results and arguments highlight the contribution the SMN ($\alpha - \mu$) model which enables the

practitioner with engineering expertise in the communications systems arena. This unified scale mixture platform allows theoretical and practical access to previously unconsidered models that may yield improved fits to experimental data.

5. Conclusion

In this paper, we proposed a SMN generalization to Yacoub's $\alpha - \mu$ distribution, to model process behaviour. This provides a flexible tool for the practitioner to model observed experimental data when the underlying normality assumption does not seem feasible. Based on the performance evaluation for the two considered mixing densities, it is clear that there

are instances in which the ABER and outage probability of the $\alpha - \mu$ distribution can be improved through the use of the proposed SMN ($\alpha - \mu$) model. This reiterates the arguments presented in Choi et al. [5] and Ollila et al. [4] that fading models which do not exhibit normality from its genesis may be of distinct practical value. This paper provides a framework within a computational reach utilising a SMN construction within the fading environment, which could now be implemented by the practitioner when field data justify it. Further, as a potential future point of departure, these scale mixture density representations situated within the fading environment may provide particular advances within Bayesian computing, especially in the case of the Gibbs sampler, where this hierarchical representation may lessen the computational strain within implementation of Bayesian statistical inference (see, for example, [25, 26]). Furthermore, the SMN ($\alpha - \mu$) model can be extended and implemented, for example, within the cascaded $\alpha - \mu$ fading environment [3].

Data Availability

The code used in this article may be obtained upon request from the corresponding author.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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