NUMERICAL SIMULATION OF MULTIDIMENSIONAL NONLINEAR FRACTIONAL GINZBURG-LANDAU EQUATIONS

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Abstract. Ginzburg-Landau equation has a rich record of success in describing a vast variety of nonlinear phenomena such as liquid crystals, superfluidity, Bose-Einstein condensation and superconductivity to mention a few. Fractional order equations provide an interesting bridge between the diffusion wave equation of mathematical physics and intuition generation, it is of interest to see if a similar generalization to fractional order can be useful here. Non-integer order partial differential equations describing the chaotic and spatiotemporal patterning of fractional Ginzburg-Landau problems, mostly defined on simple geometries like triangular domains, are considered in this paper. We realized through numerical experiments that the Ginzburg-Landau equation world is bounded between the limits where new phenomena and scenarios evolve, such as sink and source solutions (spiral patterns in 2D and filament-like structures in 3D), various core and wave instabilities, absolute instability versus nonlinear convective cases, competition and interaction between sources and chaos spatiotemporal states. For the numerical simulation of these kind of problems, spectral methods provide a fast and efficient approach.

1. Introduction. Applications of fractional differential equations are widely encountered in applied disciplines, such as biology, cybernetics, viscoelasticity mechanics and statistics. In what follows, we report in brief to capture and reflect the usefulness of fractional partial differential equations (FPDE) to most fields of applied science and engineering, many others are classified in [13, 20, 46, 47] and references therein.

Viscoelasticity mechanics is known to be one of the major areas in which the fractional differential equations are frequently and extensively utilized, with many papers appearing in literature [4,5,56]. Since most deformed objects exhibit viscous and elastic properties via simultaneous dissipation and storage of mechanical energy, any viscoelastic material can be regarded to as a linear system with the strain as the response function and the stress as excitation function.

For simple materials, Hooke's law states $u(t) = Ev(t)$ for a solid, and Newton's law states $u(t) = \eta dv(t)/dt$ for fluids, where u and v represent the stress and strain respectively. Both are regarded as mere mathematical models for ideal solid and fluid rather than universal laws. They are often useful to adequately represent the behaviour of real materials. But knowing well that the stress is proportional to the first derivative of the strain for fluids and to the zeroth derivative of the strain for solids, these little facts actually propelled Blair in the year 1947 to propose an intermediate derivative models for such materials as

$$u(t) = E_0 D_t^\alpha v(t)$$

with $\alpha \in (0,1)$ dependent of the material property. A year later, Gerasimov [18] introduced Caputo fractional derivative in the interval $0 < \alpha < 1$ to obtain the model

$$u(t) = \kappa_{-\infty} D_t^\alpha v(t),$$

where κ is a positive parameter commonly referred to as the diffusivity constant, and D_t^α is the fractional Laplacian operator. With fractional derivative concepts, one can obtain the Zener model, Voigt and Maxwell model. Both are known to be special form of general high order model

$$\sum_{j=0}^n a_j D^{\alpha_j} u(t) = \sum_{j=0}^m b_j D^{\beta_j} v(t).$$

There are many other important fractional (time-space) models in other fields beside mechanics. Other areas of intensive studies include; space-time fractional diffusion equation [12,48], fractional reaction-diffusion equation [43,51,52,56], fractional Navier-Stokes equation [54], fractional Burger's equation [9], semi-linear fractional dissipative equation [27], fractional conduction-diffusion equation [27] and fractional MHD equation [20,50,55] to mention a few.

Our aims in this paper are in folds: we first formulate the fractional complex Ginzburg-Landau equation and secondly, report via numerical simulations an overview of some of the phenomena that describe the complex Ginzburg-Landau equation in 1D, 2D, and 3D in the context of condensed matter physicists. Numerical experiment of multispecies noninteger-order system in high dimensions can be found in [7,32,36–43], and references therein. We introduce some of the useful definitions of fractional derivatives and formulation of fractional Ginzburg-Landau equations in Section 2. Numerical techniques for solving a general class of fractional diffusion problems is discussed in Section 3. The numerical techniques are utilized with two notable examples in Section 4. We finally conclude the paper with Section 5.

2. **Fractional derivatives and problem formulation.** In this section, we first give some basic and useful definitions of fractional derivatives, and later proceed to the derivation of fractional Ginzburg-Landau equation.

2.1. **Some basic definitions of fractional derivatives.** Some basic useful definitions and properties of the fractional derivatives are reported in brief here.

Definition 2.1. *The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$ of a function $f \in C_\mu$, $\mu \geq -1$ is defined as*

$$J_0^\nu f(x) = \frac{1}{\Gamma(\nu)} \int_0^x (x-t)^{\nu-1} f(t)dt, \quad \nu > 0, \quad J^0 f(x) = f(x).$$

The following also hold:
For $f \in C_\mu$, $\mu \geq -1$, $\alpha \geq 0$ and $\gamma > 1$: (i) $JJ^\alpha f(x) = J^\alpha J f(x)$, (ii) $J^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}$.

The Caputo fractional derivative has been applied widely because it incorporates both initial and boundary conditions in its problem formulation.

Definition 2.2. *In Caputo sense [3, 14], the fractional derivative is defined as*

$$D_*^\nu f(x) = J_\alpha^{m-\nu} D^m f(x) = \frac{1}{\Gamma(m-\nu)} \int_0^x (x-t)^{m-\nu-1} f^{(m)}(t)dt,$$

for $m-1 < \nu < m$, $m \in N$, $x > 0$, $f \in C_{-1}^m$.

Lemma 2.3. *If $m-1 < \alpha < m$, $m \in N$ and $f \in C_\mu^m$, $\mu \geq -1$, then*

$$D_*^\alpha J^\alpha f(x) = f(x), \quad J^\alpha D_*^\nu f(x) = f(x) - \sum_{k=0}^{m-1} f^k(0^+) \frac{x^k}{k!} x = 0.$$

Proof. See Momani and Odibat [28], Odibat and Momani [31], Davison and Essex [17]. □

Definition 2.4. *The Atangana and Baleanu fractional derivative in Caputo sense is defined as [4, 6, 21]*

$$_a^{ABC}\mathcal{D}_t^\alpha[y(t)] = \frac{M(\alpha)}{1-\alpha} \int_a^t y'(\tau) E_\alpha \left[-\alpha \frac{(t-\tau)^\alpha}{1-\alpha} \right] d\tau, \tag{1}$$

where $M(\alpha)$ has the same properties as in the case of the Caputo-Fabrizio fractional derivative.

Definition 2.5. *The Mittag-Leffler function is defined as (Podlubny [44])*

$$E_{\alpha,1}(z) = E_\alpha(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(1+\alpha k)}, \quad \alpha > 0, \; |z| < \infty$$

and

$$E_{\alpha,\beta}(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(1+\alpha k)}, \quad \alpha, \beta > 0, \; |z| < \infty.$$

For a comprehensive review, we refer our readers to classical papers and books [23,25,26,44,52,53,56] for details underlying theory as well as comprehensive review of the fractional differential equations.

2.2. **Fractional Ginzburg-Landau equation.** In literature, the complex Ginzburg-Landau equation (CGLE) is known to be one of the nonlinear equations which attracted a great deal of interest from physics community. The first case with real coefficient was earlier derived by Newell and Whiteehead [30] to describe Bénard convection, and was later applied to plane Poiseuille flow by Stewartson and Stuart [49]. It has also been used to describe a number of phenomena such as superconductivity, phase transitions, superfluidity and Bose-Einstein condensation [20] to mention a few. The generalized standard CGLE is given as

$$\partial_t u = u + (1 + ib)\Delta u - (1 + ic)|u|^2 u, \tag{2}$$

where u is a scaled complex function in the space direction x and time t. The real parameters b and c symbolize linear and nonlinear dispersion. The Laplacian operator hosts the second-order spatial derivatives that gives a damping of short wave excitations, the remaining nonlinear term provides energy flows from large to small or short scales.

A fractional Ginzburg-Landau equation can be used to describe the weakly nonlinear phenomena for substance with fractional dispersion. The standard Ginzburg-Landau equation of the form

$$\eta \Delta U = \gamma U - \omega U^3,$$

derived as the variational Euler-Lagrange equation $\delta E(U)/\delta U = 0$ for free energy function

$$E(U) = E_0 + \frac{1}{2}\int_\Omega \left[\eta(\nabla U)^2 + \gamma U^2 + \frac{\omega}{2}U^4\right] dV. \tag{3}$$

The easiest generalization of (3) takes the form of energy function

$$E(U) = E_0 + \frac{1}{2}\int_\Omega \left[\eta(\nabla U)^2 + \gamma U^2 + \frac{\omega}{2}U^4\right] dV_D, \tag{4}$$

where dV_D is $D-$dimensional volume element $dV_D = I_3(D,x)dV$. By using idea of Riesz fractional integral representation, we have

$$I_3(D,x) = \left[\frac{2^D \Gamma(3/2)|x|^{D-3}}{\Gamma(\frac{D}{2})}\right]$$

and with the Riemann-Liouville fractional integral definition, we have

$$I_3(D,x) = \left[\frac{|xyz|^{\frac{D}{3}-1}}{\Gamma^3(\frac{D}{3})}\right].$$

If we let

$$\mathcal{E}(U(x), \nabla U(x)) = \frac{1}{2}\left[\eta(\nabla U)^2 + \gamma U^2 + \frac{\omega}{2}U^4\right], \tag{5}$$

then we can obtain the Euler-Lagrange equation

$$I_3(D,x)\frac{\partial \mathcal{E}}{\partial U} - \sum_{j=1}^{3}\nabla_j\left[I_3(D,x)\frac{\partial \mathcal{E}}{\partial \nabla_j U}\right] = 0.$$

From (4), we obtain the generalized fractional Ginzburg-Landau equation

$$\eta I_3^{-1}(D,x)\nabla_j(I_3(D,x)\nabla_j U) - \gamma U - \omega U^3 = 0$$

which has the equivalent form

$$\eta \Delta U + G_j(D,x)\nabla_j U - \gamma U - \omega U^3,$$

where $G_j(D, x) = I_3^{-1}(D, x)\partial_j I_3(D, x)$. In fractional form, we can now generalize the energy function as

$$E(U) = E_0 + \int_\Omega \mathcal{E}(U(x), D^\alpha U(x)) dV_D,$$

where D^α denotes the Riesz fractional derivative and $\mathcal{E}$ is defined by

$$\mathcal{E}(U(x), D^\alpha U(x)) = \frac{1}{2}\left[\eta(\nabla U)^2 + \gamma U^2 + \frac{\omega}{2}U^4\right]. \tag{6}$$

When (6) is compared with (5), it is obvious that the Laplacian operator for standard PDEs has just been replaced with the Riesz fractional derivative. The Euler-Lagrange equation takes the form

$$I_3(D, x)\frac{\partial \mathcal{E}}{\partial U} + \sum_{j=1}^3 D_{x_j}^\alpha\left(I_3(D, x)\frac{\partial \mathcal{E}}{\partial D_{x_j}^\alpha U}\right) = 0$$

which is equivalent to

$$\eta I_3^{-1}(D, x)\sum_{j=1}^3 D_{x_j}^\alpha(I_3(D, x)D_{x_j}^\alpha U) + \gamma U + \omega U^3 = 0 \tag{7}$$

is known as the generalized fractional Ginzburg-Landau equation. For a special case of (7) in one dimension, we set $U = U(x)$ and use fractional integration by parts formulas

$$\int_{-\infty}^\infty f(x)D_x^\alpha g(x) dx = \int_{-\infty}^\infty g(x)D_x^\alpha f(x) dx \tag{8}$$

$$\int_{-\infty}^\infty f(x)\frac{d^\beta g(x)}{dx^\beta} dx = \int_{-\infty}^\infty g(x)\frac{d^\beta f(x)}{d(-x)^\beta} dx \tag{9}$$

to obtain the Euler-Lagrange equation

$$D_x^\alpha\left(I_1(D, x)\frac{\partial \mathcal{E}}{\partial D_{x_j}^\alpha U}\right) + I_1(D, x)\frac{\partial \mathcal{E}}{\partial U} = 0, \text{ where } I_1(D, x) = \frac{|x|^{D-1}}{\Gamma(D)},$$

and by using (6), we get

$$I_1^{-1}(D, x)D_x^\alpha(I_1(D, x)D_x^\alpha U) + \gamma U + \omega U^3 = 0.$$

When $D = 1$, we have $I_1 = 1$ in such that $D_x^{2\alpha}U + \gamma U + \omega U^3 = 0$ where D_x^α is known as the Riesz fractional derivative operator

$$D_x^\alpha f(x) = \frac{-1}{2\cos(\pi\alpha/2)\Gamma(n-\alpha)}\frac{\partial^n}{\partial x^n}\left(\int_{-\infty}^x \frac{f(u)du}{(x-u)^{\alpha-n+1}} + \int_x^\infty \frac{(-1)^n f(u)du}{(x-u)^{\alpha-n+1}}\right).$$

For a more specific case, we consider wave propagation in some media where the vector $\vec{j}$ satisfies $\vec{j} = \vec{j}_0 + \vec{p} = \vec{j}_0 + \vec{p}_\parallel + \vec{p}_\perp$, where $\vec{j}_0$ defines unperturbed wave vector, the direction of $\vec{j}_0$ is given by subscripts $(\parallel, \perp)$. By considering the dispersion law $m = m(j)$ for the wave propagation with $p \ll j_0$, we obtain

$$m(k) = m(|\vec{j}_0 + \vec{p}|) \approx m(j_0) + c(|\vec{j}_0 + \vec{p}| - \vec{j}_0) \approx m(j_0) + c\vec{p}_\parallel + \frac{c}{2j_0}\vec{p}_\perp^2 \tag{10}$$

where $c = \partial m/\partial j_0$, which in field U corresponds to coordinate space

$$-iU_t = icU_{x_1} + \frac{c}{2j_0}\Delta U, \tag{11}$$

where x_1 is given as the direction of $\vec{j}_0$. By generalizing the nonlinear dispersion relation, we get

$$m(j, |U|^2) \approx m(j, 0) + \omega|U|^2 = m(|\vec{j}_0 + \vec{p}|, 0) + \omega|U|^2 \qquad (12)$$

where $\omega = (\partial m(j, |U|^2))/(\partial|U|^2) = 0$. One can obtain the equation

$$-iU_t = icU_x + \frac{c}{2j_0}\Delta U - m(j_0)U - \omega|U|^2U, \qquad (13)$$

popularly known as the nonlinear Schrödinger equation (NSE). Again by generalizing the dispersion relation in equation (12) into the fractional case, we have

$$m(j, |U|^2) = m(\vec{j}_0, 0) + c\vec{p}_\| + c_\alpha(\vec{p}_\perp^2)^\alpha + \omega|U|^2, \;\; 0 < \alpha < 2,$$

where c_α, a constant. By adopting the relation $(-\Delta)^\alpha \leftrightarrow (\vec{p}_\perp^2)^\alpha$, we finally obtain

$$-iU_t = icU_x - \frac{c}{2j_0}(-\Delta)^\alpha U + m(j_0)U + \omega|U|^2U, \qquad (14)$$

which is known as the fractional nonlinear Schrödinger equation (FNSE) or the fractional Ginzburg-Landau equation (FGLE). To the right hand side of (14), the first term depicts the wave propagation in fractional regime, its fractional derivative is caused as a result of super-diffusion wave propagation or any other physical processes. The remainder terms are treated as wave motions interacting in nonlinear media. This equation has been used to represent some related fractional scenarios. For instance, the sign of ω results in two very different problems namely; the focusing case if $\omega > 0$ and the defocussing type if $\omega < 0$.

3. **Numerical techniques for solving fractional diffusion equations.** Let $T > 0$, $u_0 \in \mathcal{G}$, we seek a function $u : [0, T] \to \mathfrak{L}(A)$ satisfying the initial value problem

$$u'(t) + A(t)u(t) \;\; = \;\; F(u(t), t), \;\; 0 < t < T,$$
$$\qquad (15)$$
$$u(0) \;\; = \;\; u_0,$$

with

$$A(t) = A + ia(t)A, \qquad (16)$$

$A : \mathfrak{L}(A) \to \mathcal{G}$ a time-independent, positive definite, self-adjoint linear operator on the Hilbert space $(G, (\cdot, \cdot))$, with the domain $\mathfrak{L}(A)$ dense in $\mathcal{G}$, i denotes the imaginary unit, $a(t) : [0, T] \to$ is a continuous real-valued function, $F(\cdot, t) : \mathfrak{L}(A) \to \mathcal{G}, t \in [0, T]$ accounts possibly for the nonlinear operators. We assume the equation (15) has a smooth solution. For instance, example of a parabolic equation with linear operator written in the form (16) is the classical one-dimensional complex Ginzburg-Landau equation given in (2).

We let (α, β) be a strongly $A(0)$−stable s−step scheme and (α, γ) be an explicit s−step scheme, which is characterized by the polynomials α, β γ, that is

$$\alpha(\omega) = \sum_{i=0}^{s} \alpha_i \omega^i, \;\;\; \beta(\omega) = \sum_{i=0}^{s} \beta_i \omega^i, \;\;\; \gamma(\omega) = \sum_{i=0}^{s-1} \gamma_i \omega^i.$$

Assume $N \in \mathbb{N}$, $k = T/N$ to be a constant time-step, and $t_n = nk$, $n = 0, 1, 2, \ldots, N$ be a uniform sub-partition of the interval $[0, T]$. Since we are considering s−step schemes, it is reasonable to also assume that starting approximations, say $U_0, U_1, U_2, \ldots, U_{s-1}$ are provided. We consider the discretization of the above

initial value problem (15) by the three-parameter implicit-explicit (α, β, γ)−scheme. precisely speaking, we employ the implicit scheme (α, β) to discretize the linear part expected to be stiff, and for the nonlinear part of the equation we use the explicit scheme (α, γ), see [1, 2, 32] and references therein for details of derivation and stability analysis. A sequence of approximations U_m is thus recursively define to the nodal values $u_m = u(t_m)$ of the solution u of (15) by

$$\sum_{i=0}^{s} \left(\alpha_i \mathbf{I} + k\beta_i A\left(t_{n+i}\right)\right) U_{n+i} = k \sum_{i=0}^{s-1} \gamma_i F\left(U_{n+i}, t_{n+i}\right), \tag{17}$$

for $n = s, \ldots, N$. The unknown function U_{n+s} appearing in the above equation only linearly, since $\gamma_s = 0$. Therefore, to integrate in time with (17), we require to solve, at each time level, the linear equation which transforms into a linear system of equations, provided we discretize in space. The major setback of the above scheme is that it requires a very small time-step to function well, otherwise numerical instability occurs.

In mathematical modelling, the basic template to model a general class of parabolic partial differential equation is the heat or diffusion equation. In what follows, we shall one-dimensional fractional-in-space diffusion equation

$$\frac{\partial}{\partial t} u(x, t) = \delta \frac{\partial^\alpha}{\partial x^\alpha} u(x, t) \tag{18}$$

which is valid for $-\infty < x < \infty$ and $0 \leq t < \infty$ with $1 < \alpha \leq 2$. The parameter δ represents the diffusion coefficient and $u(x, t)$ is density distribution function at time t and position x. We define the fractional-in-space operator $\frac{\partial^\alpha}{\partial x^\alpha} u(x, t)$ on the basis of the Fourier transformation given by

$$\hat{F}(k) = \mathcal{F}\{F(x)\} = \int_{-\infty}^{\infty} F(x) e^{ikx} dx,$$

and its corresponding inverse

$$F(x) = \mathcal{F}^{-1}\{\hat{F}(k)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{F}(k) e^{-ikx} dk.$$

By applying the fractional-in-space operator on the density distribution $u(x, t)$ defined for $-\infty < x < \infty$ and $0 \leq t < \infty$ as

$$
\begin{aligned}
\frac{\partial^\alpha}{\partial x^\alpha} u(x, t) &= \mathcal{F}^{-1}\{(-ik)^\alpha \mathcal{F}[u(x, t)]\}, \\
\frac{\partial^\alpha}{\partial (-x)^\alpha} u(x, t) &= \mathcal{F}^{-1}\{(ik)^\alpha \mathcal{F}[u(x, t)]\}.
\end{aligned} \tag{19}
$$

Due to (19), the Fourier transform of (18) results to an ODE

$$\frac{\partial}{\partial t} \hat{u}(x, t) = \delta(-ik)^\alpha \hat{u}(x, t). \tag{20}$$

By choosing the initial function in the form

$$
\begin{aligned}
u(x, t = 0) &= g(x), \\
\hat{u}(x, t = 0) &= \mathcal{F}\{u(x, t = 0)\} = \mathcal{F}\{g(x)\} = 1,
\end{aligned}
$$

the solution of (20) becomes

$$u(k, t) = e^{\delta t(-ik)^\alpha} \hat{u}(k, 0) = e^{\delta t(-ik)^\alpha}.$$

With substitution $\alpha = a/b$ and $b = n$, we can rewrite the term $(-ik)^\alpha$ as

$$
\begin{aligned}
(-ik)^\alpha &= \left(|k|e^{i\arg(-ik)}\right)^\alpha = |k|^\alpha e^{i\alpha(-sign(k)\frac{\pi}{2}+2n\pi)} = |k|^\alpha e^{-i\alpha sign(k)\frac{\pi}{2}} \\
&= |k|^\alpha \cos\left(\frac{\alpha\pi}{2}\right)\left[1 - isign(k)\tan\left(\frac{\alpha\pi}{2}\right)\right],
\end{aligned}
$$

which results to

$$
\hat{u}(k,t) = \exp\left(\delta t|k|^\alpha \cos\left(\frac{\alpha\pi}{2}\right)\left[1 - isign(k)\tan\left(\frac{\alpha\pi}{2}\right)\right]\right)
$$

and inverse Fourier transform

$$
u(x,t) = \mathcal{F}^{-1}\left\{\exp\left(\delta t|k|^\alpha \cos\left(\frac{\alpha\pi}{2}\right)\left[1 - isign(k)\tan\left(\frac{\alpha\pi}{2}\right)\right]\right)\right\}.
$$

In the spirit of [43], the two dimensional representation of (18) can be formulated in Fourier space as

$$
U_t(\chi_x, \chi_y, t) = \delta\left((\chi_x^2)^{\alpha/2} + (\chi_y^2)^{\alpha/2}\right)U(\chi_x, \chi_y, t) + \mathcal{F}[f(u(x,y,t))],
$$

where U is the double Fourier transforms of the probability density distribution $u(x,y,t)$. In other words,

$$
\mathcal{F}[u(x,y,t)] = U(\chi_x, \chi_y, t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u(x,y,t)e^{-i(\chi_x x + \chi_y y)}dxdy.
$$

We let $\Omega^{\alpha/2} = \left((\chi_x^2)^{\alpha/2} + (\chi_y^2)^{\alpha/2}\right)$, and set $U = e^{\delta\Omega^{\alpha/2}t}\bar{U}$, to explicitly remove the stiffness property in the fractional-in-space partial derivative, we now have

$$
\partial_t\bar{U} = e^{\delta\Omega^{\alpha/2}t}\mathcal{F}[f(u)], \tag{21}
$$

We discretize the square domain of equispaced points N_x and N_y in the spatial directions of x and y. We employ the discrete fast Fourier transform (DFFT) [33] to transform (21) to a system of ODEs

$$
\partial_t\bar{U}_{i,j} = e^{\delta\Omega_{i,j}^{\alpha/2}t}\mathcal{F}[f(u_{i,j})], \tag{22}
$$

where $u_{i,j} = u(x_i, y_j)$ and $\Omega_{i,j}^{\alpha/2} = \left(\chi_x^2(i)\right)^{\alpha/2} + \left(\chi_y^2(j)\right)^{\alpha/2}$. Once the problem has been converted to ODEs, any explicit higher-order time stepping solver can be employed to advance in time.

For the temporal discretization, we adapt the improved fourth order exponential time differencing Runge-Kutta (ETD4RK) scheme proposed by Cox and Matthews [16, 22, 34, 35] as

$$
\begin{aligned}
\mathcal{U}_{n+1} &= e^{\mathbf{L}h}\mathcal{U}_n + h[4\varphi_3(\mathbf{L}h) - 3\varphi_2(\mathbf{L}h) + \varphi_1(\mathbf{L}h)]\mathbf{F}(u_n, v_n, t_n) \\
&\quad + 2h[\varphi_2(\mathbf{L}h) - 2\varphi_3(\mathbf{L}h)]\mathbf{F}(\mu_2, t_n + h/2) \\
&\quad + 2h[\varphi_2(\mathbf{L}h) - 2\varphi_3(\mathbf{L}h)]\mathbf{F}(\mu_3, t_n + h/2) \\
&\quad + h[\varphi_3(\mathbf{L}h) - 2\varphi_2(\mathbf{L}h)]\mathbf{F}(\mu_4, t_n + h),
\end{aligned} \tag{23}
$$

with the stages μ_i given as

$$
\begin{aligned}
\mu_2 &= e^{\mathbf{L}h/2}\mathcal{U}_n + (\mathbf{L}h/2)\varphi_1(\mathbf{L}h/2)\mathbf{F}(u_n, v_n, t_n) \\
\mu_3 &= e^{\mathbf{L}h/2}\mathcal{U}_n + (\mathbf{L}h/2)[\varphi_1(\mathbf{L}h/2) - 2\varphi_2(\mathbf{L}h/2)]\mathbf{F}(u_n, v_n, t_n) + h\varphi_2(\mathbf{L}h/2)\mathbf{F}(\mu_2, t_n + h/2) \\
\mu_4 &= e^{\mathbf{L}h}\mathcal{U}_n + h[(\varphi_1(\mathbf{L}h) - 2\varphi_2(\mathbf{L}h)]\mathbf{F}(u_n, v_n, t_n) + 2h\varphi_2(\mathbf{L}h)\mathbf{F}(\mu_3, t_n + h),
\end{aligned}
$$

where $\mathbf{L}, \mathbf{F}$ are the linear and nonlinear operators with functions $\varphi_{1,2,3}$ defined as

$$\varphi_1(z) = \frac{e^z - 1}{z}, \quad \varphi_2 = \frac{e^z - 1 - z}{z^2}, \quad \varphi_3 = \frac{e^z - 1 - z - z^2/2}{z^3}$$

which precisely coincide with the terms in the Lie group methods by Munthe-Kaas [29]. For details derivation and stability of the ETD4RK method, see [22, 33, 34].

4. **Numerical experiments.** In this section, we now apply the numerical techniques as discussed above to numerically solve a range of family of fractional-in-space Ginzburg-Landau equations. We present results from two notable equations that are still of current and recurring interest in one, two and three dimensions, to illustrate the queries and points that naturally arise. Details analysis of the complex Ginzburg-Landau on global well-posedness and long-time dynamics is well established in [45].

4.1. **The fractional-in-space complex Ginzburg-Landau equation.** A large number of nonlinear and physical systems have been described and modelled by the Ginzburg-Landau equations. Notably, the complex Ginzburg-Landau equation (CGLE) which was derived by Stewartson and Stuart [49] in the context of plane Poiseuille flow with strongly subcritical bifurcation. The CGLE is often studied in physics as an amplitude modulation equation that results to chaotic dynamics in areas of fluid dynamical systems. The major element in the long time dynamics of pattern forming systems is a family of solutions called coherent solitons or structures, which could be classified as a profile of light intensity, temperature or magnetic field [8,15]. Other phenomena include the superfluidity, superconductivity, liquid crystals, Bose-Einstein condensation, nonlinear waves and phase transitions.

In accordance with (14), we present the generalized scaled fractional complex Ginzburg-Landau equation (FCGLE) [36] formulated as

$$\partial_t u = \underbrace{(1 + ib)\Delta^\alpha u}_{\mathbf{L}} + \underbrace{u - (1 + ic)|u|^{2\alpha} u}_{\mathbf{F}} \tag{24}$$

where b is the real parameter that characterizes the strength of the linear dispersion, that is, the dependence of the wave frequency on the waves number, c is real coefficient that denotes nonlinear dispersion. The fractional Laplacian Δ^α describes a nonlocal operator bounded in the interval $0 < \alpha < 2$, except for the cases when $\alpha = 0, 1, 2$ which lead to standard CGLE. The fractional Laplacian operator of order α provides a damping of the short wavelength excitations and nonlinear term $\mathbf{F}$ evolves energy flows from large to short scales, and u is a scaled complex function in both time and space. Emerging structures and stability properties of (24) depends on the parameters b and c, see [32] for details. Also, Δ^α can be regarded as a pseudodifferential operator with $|\xi|^{2\alpha}$, which can be realized via the Fourier transform [45]

$$\widehat{\Delta}\mathcal{F}(\xi) = |\xi|^{2\alpha}\hat{\mathcal{F}}(\xi),$$

where $\hat{\mathcal{F}}$ denotes the Fourier transform of $\mathcal{F}$.

In one dimension, equation (24) is numerically solved with initial condition that takes the form of a Gaussian series pulse form:

$$u(x,0) = e^{-20(x-L/3)^2/L} - e^{-20(x-L/2)^2/L} + e^{-20(x-L)^2/L} \tag{25}$$

where L is the domain length. In this experiment, we let $b = 1$ and $c = 1.3$ to yield spatiotemporal evolution of u in Figure 1 at different instances of fractional

power α for $L = 150$ and final time $t = 40$. This kind of turbulent wave is known as defect turbulence [19] is shown for parameter values in the regime where the linear stability analysis indicates that the uniform solution is stable.
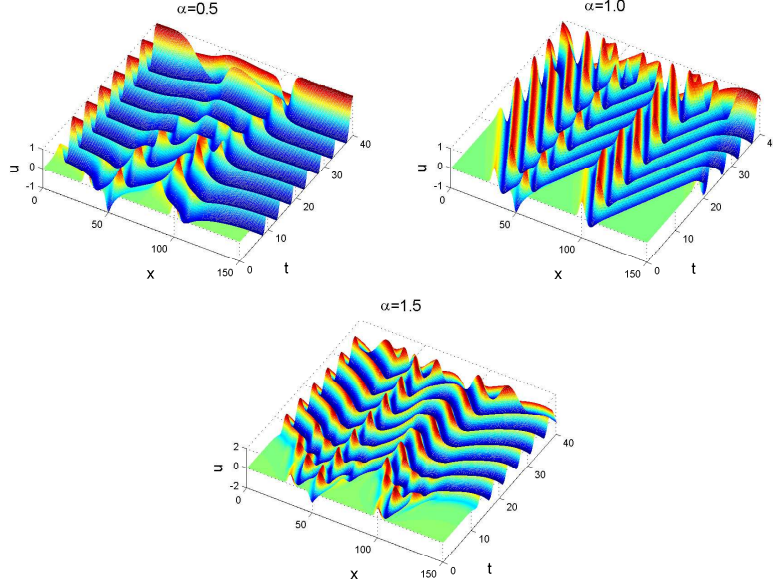

FIGURE 1. Space-time mesh results of (24) showing chaotic states in the spatiotemporal regime for parameters $(b, c) = (1, -1.3)$ at different instances of fractional index α and $t = 40$. simulation runs for $N = 200$ with step size $h = 0.1$.

Though, our primary interest in this paper is not really in one dimensional results, because they are relatively easily undertaken with either finite difference schemes or finite element collocation methods. It is in higher dimensions that the ideas given here actually become of serious value. Hence, to explore the dynamic richness of FCGLE (24), we present the numerical experiments in higher dimensions.

In 2D, we also simulate with smooth initial condition in the form of Gaussian pulse

$$
\begin{aligned}
u(x, y, 0) &= e^{-20((x-L/3)^2+(y-L/3)^2)/L} - e^{-20((x-L/2)^2+(y-L/2)^2)/L} \\
&\quad + e^{-20((x-L/2)^2+(y-L/3)^2)/L}.
\end{aligned}
\tag{26}
$$

Theoretically, we expect travelling waves to evolve from such initial condition on an infinite domain truncated at some large but finite value L. In the simulations, the domain length L is chosen large enough to give enough room for the waves pattern to propagate. The patterns that evolve in the unstable region are quite amazing and justify the status of FCGLE in studying complex behaviour.

The computation in Figure 2 utilizes 128×128 Fourier nodes on a rectangular domain of size 200×200 space units in a situation of spatiotemporal chaos obtained when any of the conditions
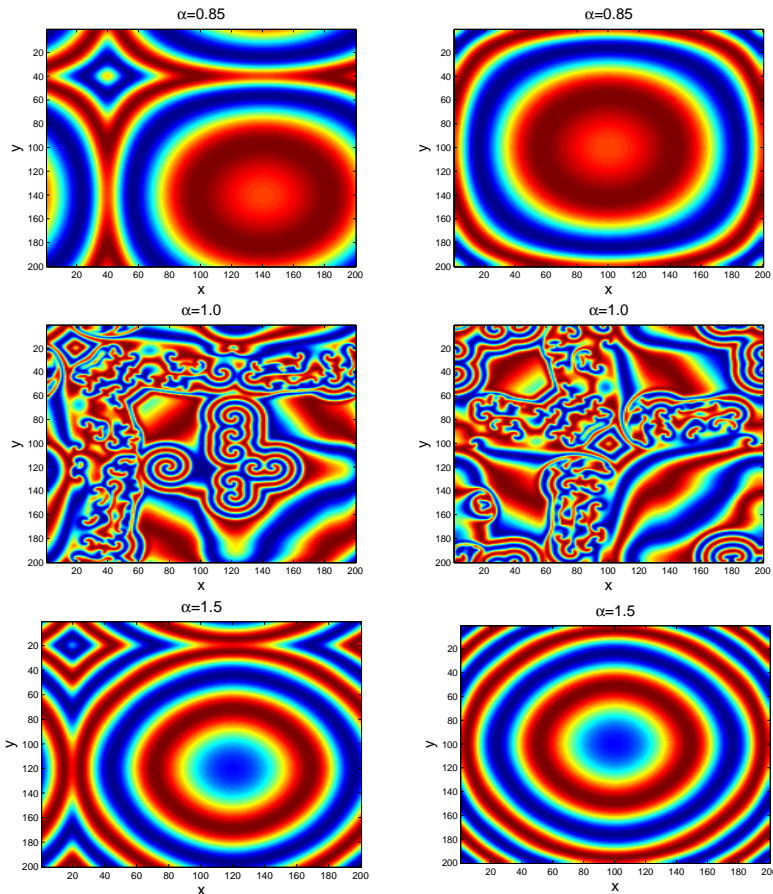
$$1 + bc < 0 \quad \text{and} \quad 1 + bc > 0$$

FIGURE 2. Solution of the 2D fractional complex Ginzburg-Landau equation on $[0, 200] \times [0, 200]$ with $b = 1$ for both the focusing case: $c = 1.3$ (first-column) and the defocussing case: $c = -1.3$ (second-column) at final time $t = 100$, $\alpha = (0.85, 1.0, 1.50)$ and $N = 200$.

is satisfied, subject to zero-flux boundary conditions. We can categorize these results into sub-diffusive $(0 < \alpha < 1)$, diffusive $(\alpha = 1)$ and super-diffusive at $(1 < \alpha < 2)$. As seen in Figure 2, one obtains stable and chaotic spatiotemporal spiral wave patterns for both the focusing $(c > 0)$ and defocussing $(c < 0)$ cases. It was observed that the distributions of probability density $u(x, y, t)$ in the sub-diffusive and super-diffusive scenarios are similar to both focusing and defocussing cases in the fractional regime. In the diffusive region, the stable patterns destabilize in the simulations to form more complex chaotic spatiotemporal spiral patterns. The spiral core is unstable in the parameter range where diffusive effects are weak when compared to dispersion.

In 3D, we experiment (24) with both random and initial conditions. The initial state is well mixed, with $u_0(x; y; z)$ taken from a random normal distribution about
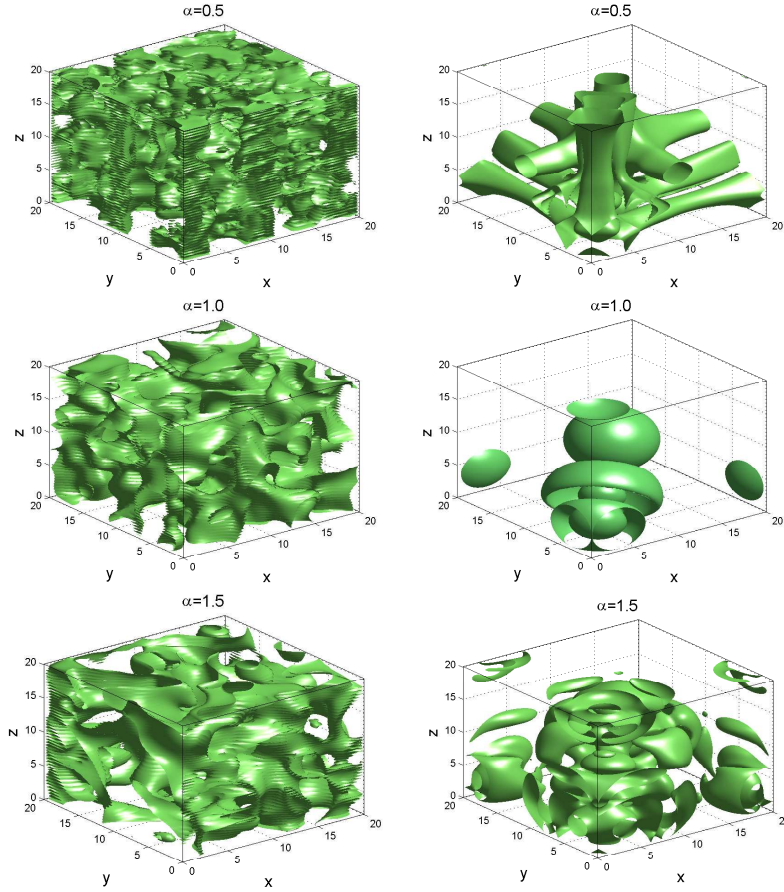
FIGURE 3. The first and second columns represent 3D results of the fractional complex Ginzburg-Landau equation on $[0, 20]^3$ obtained at instances $\alpha = (0.5, 1.0, 1.50)$ for random and initial conditions respectively. Other parameters are: $b = 1, L = 20$ and final time $t = 10$ (N=100).

0:1, while we allow the smooth initial condition to take the form

$$
\begin{aligned}
u(x, y, 0) &= e^{-20((x-L/3)^2+(y-L/3)^2)/L+(z-L/3)^2)/L} - e^{-20((x-L/2)^2+(y-L/2)^2)/L+(z-L/3)^2)/L} \\
&\quad + e^{-20((x-L)^2+(y-L)^2)/L+(z-L)^2)/L}.
\end{aligned}
\tag{27}
$$

to obtain the 3D results displayed in Figure 3. We observed that the distribution of u in space leads to series of chaotic and spatiotemporal phenomena irrespective of the initial condition used here.

4.2. **The fractional-in-space complex cubic-quintic Ginzburg-Landau equation.** The fractional complex cubic-quintic Ginzburg-Landau equation (FCCQGLE) can also be seen as another dissipative system that attracts the attentions of many researchers in the recent times. The scenarios considered in 2D bifurcate from trivial state supercritically in the parameter pair range of (b, c) where they remain bounded [32]. The scenario here has not changed qualitatively with the addition of

stabilizing higher order terms. Outside this parameter regime one requires at least the quintic terms to saturate the explosive instability provided by the cubic term. The space fractional CCQGLE is written in the form

$$\partial_t u = \epsilon u + (\phi + ib)\Delta^\alpha - (\psi - ic)|u|^2 u - (\varphi - id)|u|^4 u, \tag{28}$$

where ϵ is the linear loss constant, the parameters ϕ, ψ, φ are the angular spectral filtering, nonlinear loss or gain, and saturation of the nonlinear loss or gain respectively. Here, b represents the diffraction coefficient, c symbolizes self focusing and d stands for saturation of the nonlinear refractive index. The case when $\alpha = 0, 2$ is standard.
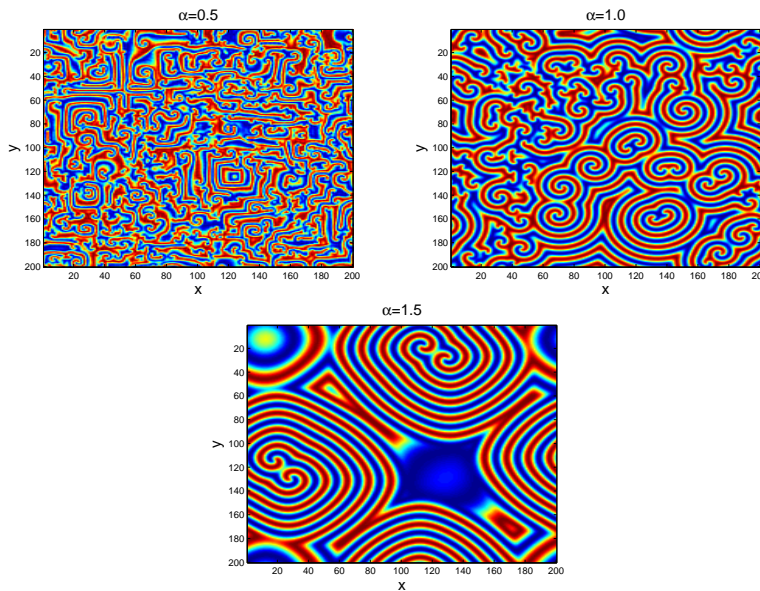

FIGURE 4. The 2D results of fractional-in-space problem (24) showing the bound state of oppositely- and like-charged spirals at some instances of fractional power α. simulation runs for $N = 200$.

We report in Figure 4 the 2D simulation results of the FCCQGLE (28) at some instances of fractional index α subject to the smooth initial condition (26) and zero flux boundary conditions clamped at the extremities of the square domain size $[0, 200] \times [0, 200]$. The parameters are taken to be $\epsilon = -0.05; b = 1.0, \phi = 1.0, c = 1.3, \psi = 1.0, d = 0.105, \varphi = 0.03, L = 20$ and final time $t = 400$. The finite-amplitude results persist stably with respect to amplitude which fluctuates below threshold $\epsilon < 0$ in a certain parameter regime, where they coexist with the linearly stable trivial solution. There exist moving fronts and stable localized spiral pulses over a finite interval of length L. Evolution of an oppositely-charged spiral and unlike-charged spiral pair from unstable into a stable antisymmetric state are evident in Figure 4 at instances of α.

Similarly, we carry-out the 3D FCCQGLE simulations in Figure 5, subject to the zero-flux boundary conditions and smooth initial condition in (27) on domain of size $[0, 20] \times [0, 20] \times [0, 20]$ with 256 Fourier modes. Other parameters are given in the figure caption. It worth mentioning that other phenomena such as Turing patterns
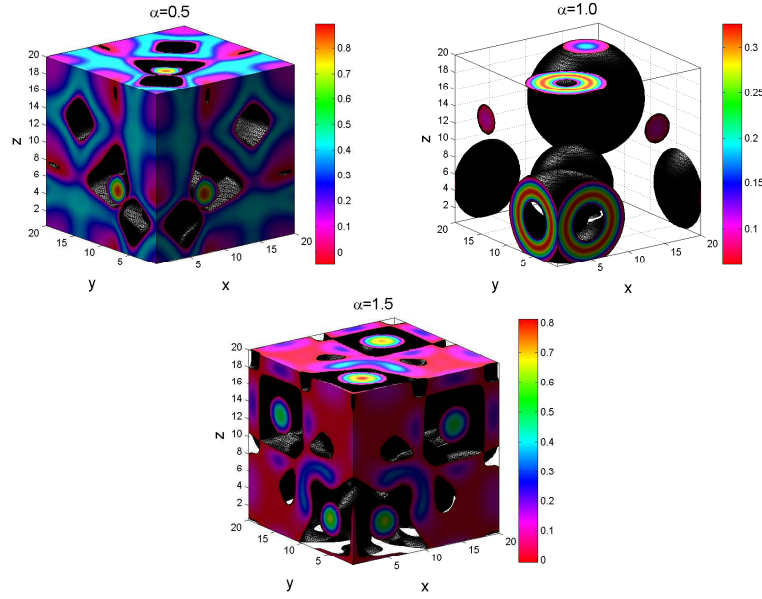
FIGURE 5. The 3D isosurfaces of $|u(x, y, z)|$ of (24) showing chaotic patterns at different instances of α for $\epsilon = -0.05; b = 1.0, \phi = 1.0, c = 1.3, \psi = 1.0, d = 0.105, \varphi = 0.03, L = 20$ and final time $t = 20$. Simulation runs for $N = 64$.

are possible, depending on the choices of the initial functions and parameters. It should also be noted in the examples considered that when $\alpha = 2$, the fractional reaction-diffusion equations correspond to the standard reaction-diffusion problems.

5. **Conclusion.** We have numerically studied equations of space-fractional reaction-diffusion with cubic nonlinearity on finite but large spatial and time domains. We have examined the temporal patterns behaviour in the steady-state solutions of two family of complex Gizburg-Landau equation when the fractional derivative index α is increasingly varied in the interval $0 < \alpha \leq 2$. Although the coherent structures discussed do reflect that the fractional complex Ginzburg-Landau and fractional complex cubic-quintic Ginzburg-Landau equations (2D spirals and 3D filaments) exhibits much richer and complicated behavior than the classical reaction-diffusion equation. We have also demonstrated in the numerical experiment that at certain value of fractional power α, the formation of patterns change from the steady-state structure to the homogeneous oscillatory structure, given that the ratio of spatial characteristics is relatively small. The study of fractional reaction-diffusion equations has yielded useful information regarding the dynamics of nonlinear phenomena. The methodology discussed in this paper can easily be extended to systems of fractional reaction-diffusion equations.

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