

# Considering the use of an equal-weighted index as a benchmark for South African equity investors

By BH Taljaard and E Maré

*Submission date 29 March 2019*

*Acceptance date 28 October 2019*

## ABSTRACT

We analyse and discuss the use of an equal-weighted index as an alternative to the market capitalisation weighted (cap-weighted) index as a benchmark for active equity portfolios in the South African equity market. Our findings indicate that equal-weighted portfolios are, in general, more efficient than cap-weighted portfolios and that random active portfolios tend to display significantly improved risk-return characteristics when using an equal-weighted index as a benchmark. We find our results are robust to transaction costs involved with rebalancing.

## KEYWORDS

Benchmark; risk; capitalisation weight; equal weight; efficiency; rebalancing; costs; returns

## CONTACT DETAILS

Eben Maré, Department of Mathematics and Applied Mathematics, University of Pretoria, Private Bag X20, Hatfield, 0028; Tel: +27 (0)12 420 2835; Email: [eben.mare@up.ac.za](mailto:eben.mare@up.ac.za)

## 1. INTRODUCTION

1.1 The use of equity benchmarks has become a cornerstone of modern portfolio management. Benchmark indices are typically used by investors and asset consultants for strategic asset allocation purposes as well as performance evaluation and comparison of investment portfolios. Lloyd & Manium (2004) suggest that a good benchmark index is typically considered on the basis of investor relevance, market representativeness, the

investor's ability to invest in the benchmark as well as fully replicate it, independence as well as high data quality, and transparency with regard to the index constituents.

1.2 The market capitalisation weighted (cap-weighted) index remains the most widely used equity benchmark. This is largely due to the Capital Asset Pricing Model (CAPM) developed by Sharpe (1964), which assumes that the cap-weighted index will be mean-variance efficient.

1.3 A number of studies have, however, emerged challenging the efficiency of the cap-weighted index and consequently its use as an equity benchmark. Haugen & Baker (1991) discuss the conditions necessary for cap-weighted portfolios to be efficient. Among the conditions investigated are the assumptions that all investors agree about the risk and expected return for all securities, all investors can short-sell all securities, all investors have the same investable universe, and no investor is subject to taxation. In the absence of these assumptions, Haugen & Baker (1991) argue that the cap-weighted portfolio is inefficient. Furthermore, they show that an *ex-ante* efficient portfolio consistently outperforms the Wilshire 5000 index with lower volatility. See also, the work by Hsu (2006), Perold (2007), and Amenc et al. (2012) that questions the efficiency of cap-weighted indices.

1.4 Grinold (1992) applies a statistical test developed by Gibbons et al. (1989), which tests for the efficiency of cap-weighted benchmark portfolios. The test considers the residual return of the *ex-post* minimum variance portfolio (i.e., the return that is not correlated with the benchmark return) and considers the statistical significance of the difference between the benchmark's Sharpe ratio and that of the minimum variance portfolio—in essence, the portfolio with the highest Sharpe ratio (*ex-post*). Grinold (1992) used this test to show that the cap-weighted benchmarks in four out of five countries tested (Australia, US, UK, Germany and Japan) are not efficient. The only country that showed an efficient cap-weighted benchmark is Japan.

1.5 Hsu (2006) shows that a cap-weighted portfolio suffers from a 'return drag', which is directly proportional to the volatility of the individual stocks. Therefore, the higher the level of price inefficiency (volatility) in the market, the higher the return drag. This assumption is equivalent to an assumption of negative autocorrelation in returns. See also, DeMiguel et al. (2009), Martellini (2009), as well as Amenc et al. (2012).

1.6 Numerous studies have shown equal-weighted portfolios to be more efficient than cap-weighted portfolios. Dash & Loggie (2008) provide empirical evidence of an equal-weighted S&P500 index outperforming the cap-weighted index. Dash & Zeng (2010) confirm similar results for the S&P International 700 index. Bolognesi et al. (2013) examined the outperformance of an equal-weighted Dow Jones Eurostoxx 50 index compared with a cap-weighted index.

1.7 In terms of the efficiency of equal-weighted portfolios, DeMiguel et al. (2009) evaluated the performance of optimal asset allocation against a naïve equal-weighted portfolio. They found that none of the 14 models (extensions of the sample-based mean-variance optimisation) performed better out of sample than the equal-weighted portfolio in terms of the Sharpe ratio, certainty-equivalent return or turnover.

1.8 Under the assumption of a lognormal model of stock prices and a diversified market, Fernholz (2002) sets out a framework that allows us to compare expected returns of portfolios generated by some function against the expected returns of the cap-weighted portfolio. Under this framework, it can be shown that an equal-weighted portfolio outperforms a cap-weighted portfolio in the long run.

1.9 This framework is referred to as the Stochastic Portfolio Theory (SPT) and represents a comprehensive approach, which can be used to analyse the long-term behaviour of various portfolios to the cap-weighted portfolio under certain assumptions. As such, we provide a brief overview of this approach, the resulting equations, and some of the necessary assumptions in Appendix A.

1.10 Plyakha et al. (2015) also show, empirically, that equal-weighted portfolios (rebalanced monthly) outperform the cap-weighted portfolio in terms of total return, alpha (relative to a four-factor model), and the Sharpe ratio. Furthermore, they show that the equal-weighted portfolio derives a systematic return by being overweight small stocks, value stocks, and stocks with high idiosyncratic volatility. However, Plyakha et al. (2014) find that the higher alpha for the equal-weighted portfolio is derived from the monthly rebalancing required to maintain fixed weights.

1.11 In Malladi & Fabozzi (2017), the authors provide a theoretical framework for the relative performance between cap-weighted and equal-weighted portfolios. They confirm that a significant portion of the excess return over the cap-weighted portfolio is derived from the rebalancing process.

1.12 The majority of prior studies have focused on stocks in the United States. In Table 2, we demonstrate outperformance of an equal-weighted portfolio of the Top 40 stocks in the South African market compared with a cap-weighted portfolio of the same stocks—our methodology is highlighted in Section 3. (See also Taljaard & Maré, 2014.)

1.13 While prior literature has shown that cap-weighted portfolios are inefficient, in South Africa the further complication with cap-weighted portfolios is the issue of concentration. Kruger & Van Rensburg (2008) highlighted the concentration of a market cap index specifically in the South African context (FTSE/JSE All Share index) and compared the liquidity constraints and concentration levels of constructed capped market cap indices (either at the individual stock level or at a sector level) to address the practicalities of alternative

benchmarks. In Figure 1b, we detail the concentration levels of the FTSE/JSE All-Share index. Although the index contains over 150 shares, the top 10 stocks account for 55–60% of the index on average. This indicates high levels of idiosyncratic risk in the index.

1.14 In this paper we specifically address the potential use of an equal-weighted index as an alternative to the cap-weighted index in the South African equity market. In this paper we hope to address the practicalities of maintaining an equal-weighted portfolio by highlighting:

- a) the measurement of active bets under an equal and cap-weighted index; and
- b) why performance matters in an equity benchmark.

1.15 The outline of the article is as follows: in Section 2 we provide a discussion on active bets within a portfolio management context. We detail the key differences between managing a portfolio using an equal-weighted benchmark vs a cap-weighted benchmark. In Section 3 we discuss data and our historical methodology as well as empirical performance of equal-weighted portfolios vs cap-weighted portfolios. Section 4 is devoted to implementation issues related to equal-weighted benchmarks. We conclude in Section 5.

## 2. ACTIVE BETS AND THE EQUAL-WEIGHTED PORTFOLIO

2.1 An important reason for the use of a benchmark in active fund management is to measure the active bets of a fund manager, both in relative weight and relative return (generally referred to as alpha). In this section we address the relative weights of an active portfolio, contrasting the implications for an equal-weighted and cap-weighted benchmark given rebalancing requirements.

2.2 The key difference between an equal-weighted and a cap-weighted benchmark is rebalancing. A market cap portfolio would require almost no rebalancing as the effective re-weighting feeds through from the change in prices of constituent shares. In general, only corporate actions and changes in constituents would require manual rebalancing on the part of the portfolio manager.

2.3 In contrast, an equal-weighted portfolio requires frequent rebalancing to bring share weights back to an equal weight. The effect of this is that the fund manager would have to reallocate capital to maintain his/her active weight relative to the equal-weighted benchmark far more frequently than for a cap-weighted benchmark. The active position in this sense becomes a frequently reviewed decision on the part of the fund manager and leads to a higher turnover in the portfolio and increased transaction costs (which we address later in this article).

2.4 However, in this section, we argue that the rebalancing decision is far more intentional (and measurable) under an equal-weighted benchmark. When a share's price experiences a significant increase (or decline), the equal-weighted benchmark forces the manager to re-evaluate his/her growing active position. Leaving the weight as is would mean the manager

extends his/her relative active position over the equal-weighted benchmark. In contrast, the cap-weighted portfolio manager would maintain his/her relative active position by doing nothing.

2.5 If we view this difference in the context of an investor paying a fund manager to make active decisions, we can see how an equal-weighted benchmark would force the manager to make intentional decisions. The use of an equal-weighted benchmark also makes it very clear when a fund manager takes an active bet.

2.6 Consider an important share example in the South African market—Naspers Ltd (NPN). We highlight the equal weight of NPN under two scenarios:

- a) starting with an equal weight and rebalancing every quarter; and
- b) starting with an equal weight until 2014 and then never rebalancing.

2.7 A fund manager (under an equal-weighted benchmark) might always rebalance when he/she has no view on the stock. However, at any point the manager can cease rebalancing in a specific stock if, for example, he/she believes the counter's recent momentum is likely to persist.

2.8 For example, consider NPN between 2012 and 2014. The market cap weight starts rapidly increasing as the stock starts outperforming the FTSE/JSE Top 40. We highlight in Figure 1a if a manager were to have a long bias in the stock from 2014, he/she could simply stop rebalancing and allow the weight to float (or even increase the allocation).

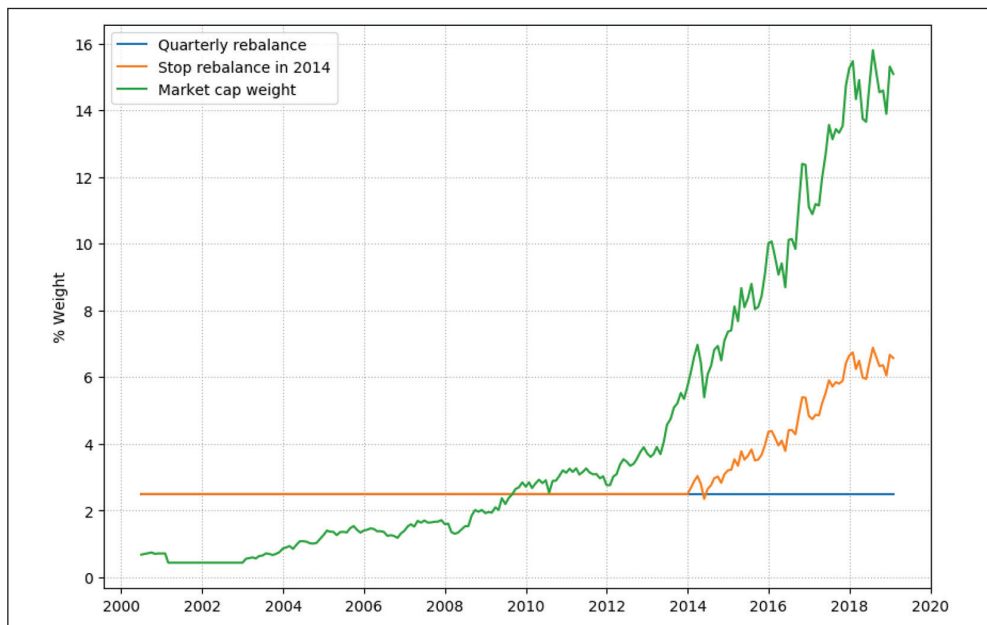


FIGURE 1A. NPN equal weight with and without rebalancing

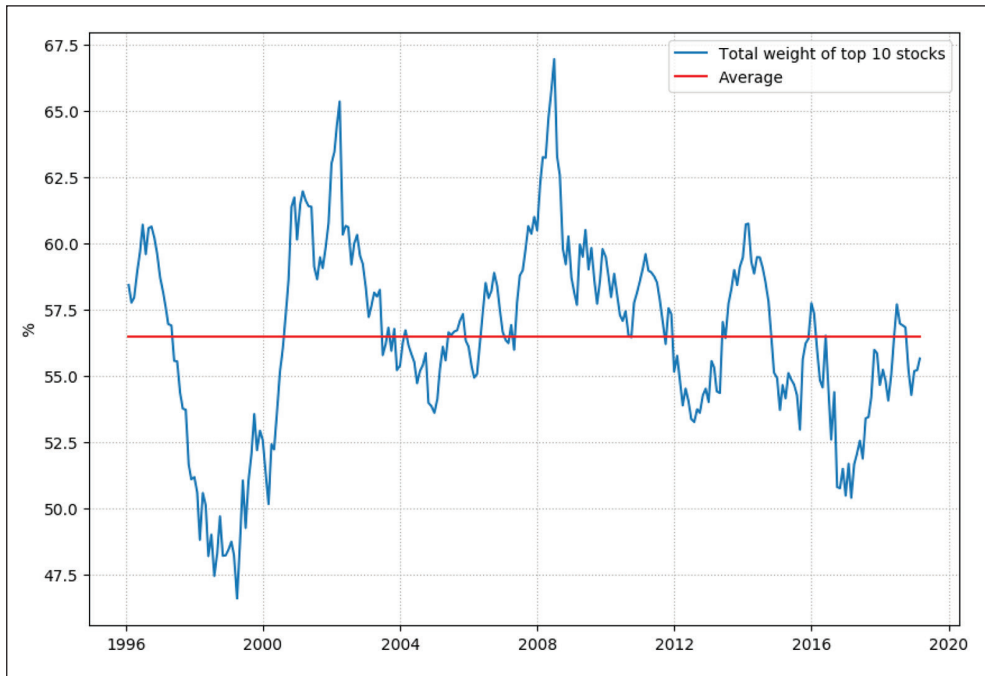


FIGURE 1B. Aggregate weight of top ten shares in the FTSE/JSE All-Share index

2.9 Secondly, the active weight as a multiple of the benchmark (equal weight) is easily calculated and highlights the fund's exposure to the specific risks in NPN. On the other hand, a fund manager with a market cap benchmark and, for example, a 2 times market cap weight exposure in NPN would show a constant multiple even as the fund's exposure to NPNs increases.

### 3. HISTORICAL PERFORMANCE AND WHY IT MATTERS

3.1 In this section we compare the historical performance of the market cap and equal-weighted portfolio for the Top 40 stocks since 2000. All portfolios are rebalanced monthly, unless otherwise specified.

#### 3.2 Data and Methodology

3.2.1 Daily stock prices, market capitalisation, and total return indices from 1 January 2000 to 31 January 2019 are sourced from Bloomberg for all South African stocks listed on the JSE (past and present). Our data also include those stocks that have been delisted in the past so as to limit the level of survivorship bias.

3.2.2 Total returns for the indices considered here are empirically tracked by constructing hypothetical portfolios. We consider the Top 40 shares by market capitalisation as at the end of each month. Portfolios are assumed to be rebalanced monthly, unless

otherwise specified. Dividends are assumed to be paid in cash and remain in the portfolio as cash until the following rebalancing dates. We assume no interest is earned on excess cash. Furthermore, we do not consider transaction costs in our analyses, unless otherwise specified.

### 3.3 Empirical Performance of the Equal Weight and Market Cap-Weighted Portfolios

3.3.1 It is well documented that the equal-weighted portfolio outperforms the market cap-weighted portfolio within US stocks (see 1.6 above). Our results show that the outperformance of the equal-weighted portfolio over the market cap-weighted portfolio appears to hold for South African stocks as well.

3.3.2 Consider Table 1, where we highlight the average rolling Sharpe ratio for the Top 40 stocks on the Johannesburg Stock Exchange using cap-weighted and equal-weighted portfolio returns (assuming monthly rebalancing). We observe that the equal-weighted portfolio outperforms the cap-weighted portfolio on a risk-adjusted basis across all rolling periods.

TABLE 1. Rolling Sharpe ratios

Average rolling 1, 3- and 5-year Sharpe ratios (2000–2019)		
	Cap-weighted	Equal-weighted
1 year	1.02	1.10
3 year	0.96	1.05
5 year	0.96	1.05

3.3.3 Figure 2 and Table 2 highlight the overall relative performance of the market cap and equal-weighted portfolios. We note that the equal-weighted portfolio outperforms the market cap portfolio and that this outperformance holds for both absolute returns and risk-adjusted measures such as the Sharpe ratio.

TABLE 2. Return statistics

Return statistics (Jan 2000–Jan 2019)					
	CAGR	Volatility	Downside deviation	Sharpe ratio	Sortino ratio
Cap-weighted	14.2%	19.7%	13.8%	0.77	1.10
Equal-weighted	14.7%	18.3%	12.9%	0.84	1.19

3.3.4 We notice, however, that there are periods where the market cap-weighted portfolio significantly outperforms the equal-weighted portfolio (2006 to 2008 and 2014 to 2018, for example). This is briefly discussed from an SPT point of view in Appendix A, and in circumstances where concentration in the cap-weighted portfolio increase and/or the correlations are high among stocks (or volatilities low), the equal-weighted portfolio is likely to underperform in the short term.

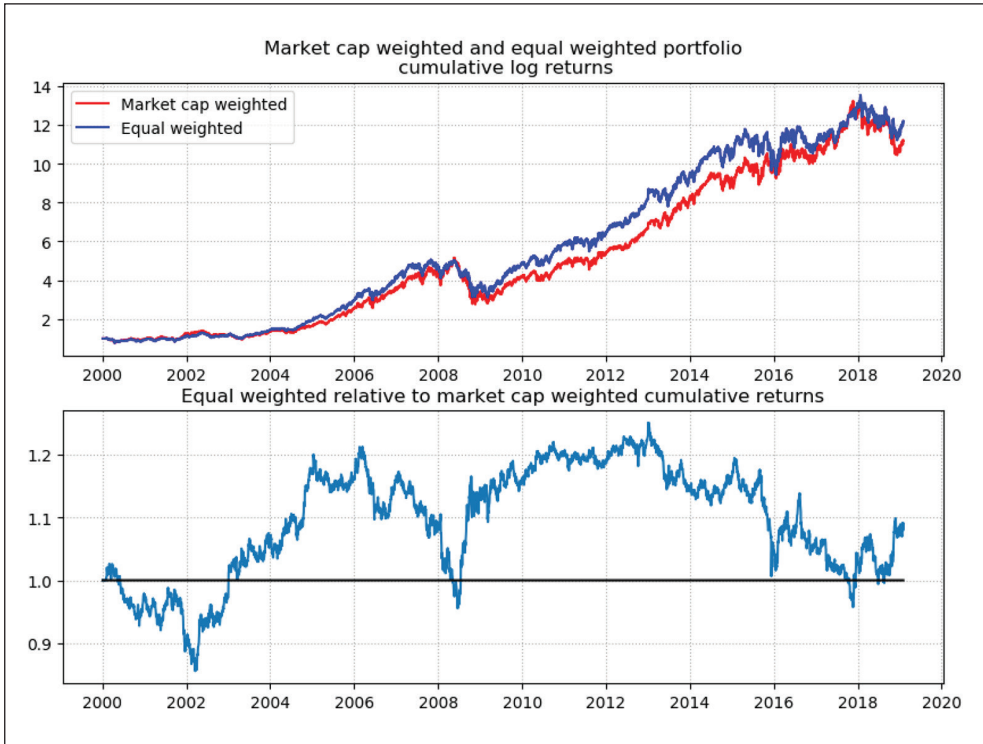


FIGURE 2. Log cumulative returns of portfolios

3.3.5 As an aside, one possible approach to improving relative performance is to model the interaction between the levels of concentration and the diversification benefit from an SPT perspective with the aim of optimising the equal-weighted portfolio in some manner. Creating a dynamic rebalancing frequency depending on levels of diversification and concentration as one example. We hope to address this research question in a future article.

3.3.6 Hallerbach (2014) also touches on this from the viewpoint of identifying the drivers of the rebalancing return as the difference between the volatility return and the dispersion discount. The volatility return is effectively the return generated from ‘buying low and selling high’, which obviously benefits from heightened volatility. This is essentially the return generated from ‘volatility harvesting’. Hallerbach (2014) also shows that as volatility in general tends to zero so does the volatility return.

3.3.7 There are occasions, however, where there is a large, sustained dispersion in the cross-sectional returns of individual stocks in the portfolio. This can lead to periods where a few stocks have outsized returns and, as such, lift the return of the market cap-weighted portfolio above the average return across all stocks for a period. During this time the equal-weighted portfolio would be diversifying away from these outperforming stocks at each rebalance by selling a portion and buying into the underperforming stocks. The dispersion discount, in other words, reduces the rebalancing return. We provide an overview



of the approach in Hallerbach (2014) in Appendix B with specific reference to how it aligns with findings in SPT.

3.3.8 These effects can lead to the underperformance in the equal-weighted portfolio, relative to the market cap-weighted portfolio. Although beyond the scope of this paper, one way to address these periods is to dynamically adjust the rebalance period via, for example, some state detection algorithm fitted on the covariance matrix (given its direct impact on rebalancing).

### 3.4 Benchmark Performance Matters

3.4.1 At a high level, the raw performance of a benchmark may not be of much concern given that it is only one aspect of many when considering a benchmark. However, the unintended consequence of a benchmark is that a manager might attempt to keep his portfolio weights very close to that of his/her benchmark. In an extreme example, a manager may replicate his/her benchmark completely (generally referred to as closet index tracking).

3.4.2 Petajisto (2013) compares the tracking error and active share for US all-equity funds from 1990–2009. They classify mutual funds as active stock pickers, factor betting funds, and closet indexers using active share and tracking error. They find that 16% of all so-called active funds could be classified as closet indexers, and a further 48% of funds could only be classified as moderately active (with an average tracking error of less than 6%). See also Cremers & Petajisto (2009).

3.4.3 We highlight the historical tracking error of South African equity unit trusts from 2005 to 2015 in Figure 3. Nearly half of these unit trusts have a tracking error of less than 5.5% w.r.t. the FTSE/JSE All Share Index, while a quarter of the unit trusts have a tracking error of less than 4.5%. Only 23% of funds have a tracking error of more than 8%.

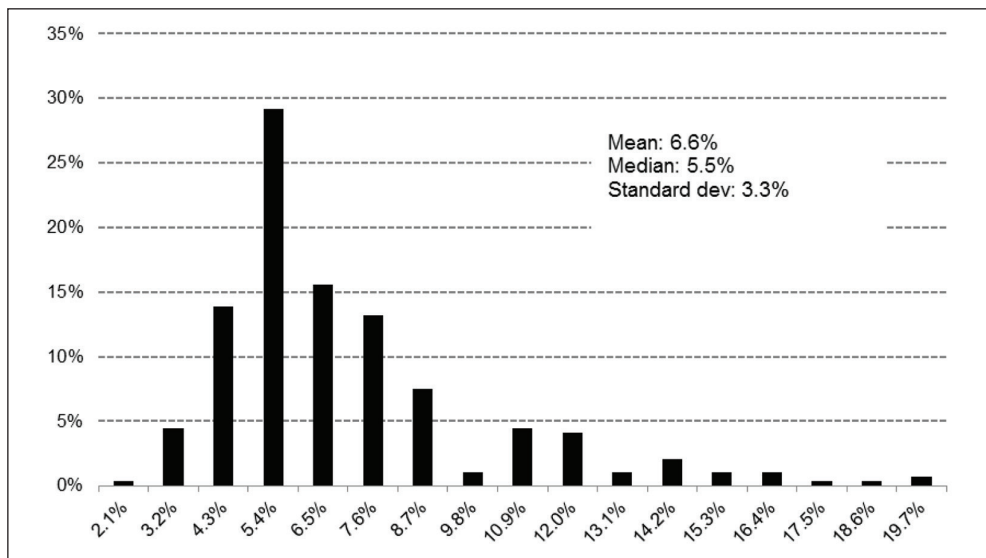


FIGURE 3. Tracking errors for South African all-equity mutual funds from 2005 to 2015

3.4.4 Although this is insufficient evidence to suggest that South African active fund managers may or may not be closet index trackers, it does suggest that South African equity fund managers do not stray too far from the benchmark (or cap-weighted index).

3.4.5 Now returns, and risk-adjusted returns, become important since the universe of fund managers' returns is likely to be centred on the market cap-weighted portfolio's return. We highlight this in the analysis below.

3.4.6 Suppose we had two fund managers, one with a market cap-weighted index as a benchmark and another with the equal-weighted index as a benchmark. Each fund manager chooses his active weights as a multiple of the benchmark weight. For example, if stock A has a market cap weight of 6% and an equal weight of 3% in the respective indices, the managers may take an active position of 2 times the benchmark. In this case, the portfolio weights in stock A would be 12% and 6% for the market cap and equal-weighted benchmarked fund managers, respectively.

3.4.7 To analyse the difference, we construct two sets of 1000 hypothetical portfolios, under both an equal and market cap-weighted benchmark. Every month the weight invested in each stock is randomly generated as a multiple between 0.5 and 2 times the respective benchmark weight.

3.4.8 Figure 4 shows the distribution of annualised returns for both sets of portfolios (market cap and equal weight benchmarks). As expected, the equal weight benchmark portfolios show a higher return. However, what is striking is the stark difference

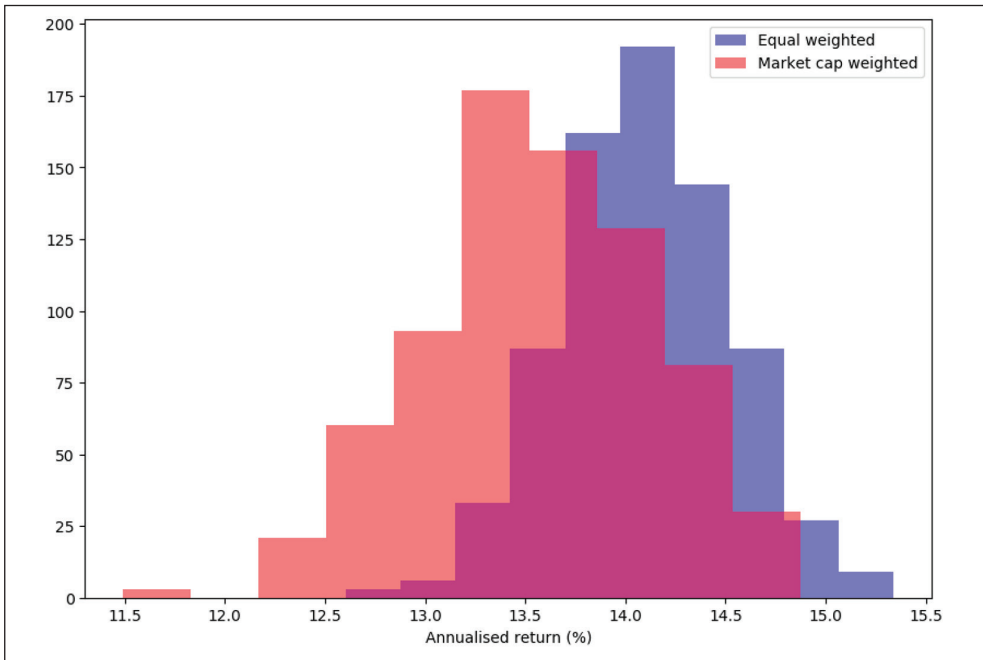


FIGURE 4. Histograms of annualised return for random portfolios under a market-cap and equal-weight benchmark

in the distributions. In fact, the average equal-weighted active portfolio would provide the same annualised return as the 80th percentile of the market cap-weighted active portfolios.

3.4.9 This difference is further exacerbated when we include volatility by measuring the Sharpe ratio (assuming a risk-free rate of zero). Here, the portfolio with a Sharpe ratio appearing at the 99th percentile (0.79) under a market cap benchmark is only equivalent to a Sharpe ratio at the 25th percentile under an equal weight benchmark. That is, 99% of the portfolios under a market cap benchmark underperform 75% of the portfolios under an equal weight benchmark.

3.4.10 We include the t-test statistics in Table 3, which highlight the statistical significance of the differences in the average CAGR and Sharpe ratios.

TABLE 3. T-test statistics for differences in CAGR and Sharpe ratios for random active portfolios

Comparison of average CAGR and Sharpe ratios for random active portfolios		
	CAGR	Sharpe ratio
Market cap-weighted	13.6%	0.74
Equal-weighted	14.1%	0.81
T-test statistic	-20.204	-56.865
P-value*	4.65x10 <sup>-80</sup>	0.0

\*One sided t-test for equal-weighted portfolio metric greater than market cap-weighted portfolio metric

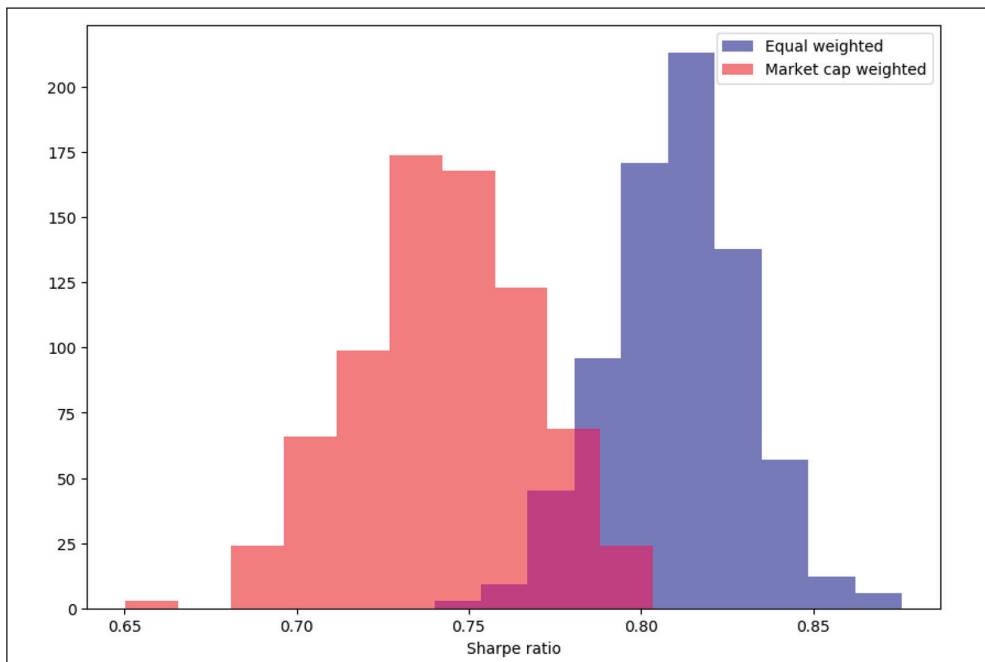


FIGURE 5. Sharpe ratio (risk-free rate=0) for random portfolios under a market-cap and equal-weight benchmark

## 4. IMPLEMENTATION: REBALANCING AND TRADING COSTS

4.1 In practice we have trading costs. Market cap-weighted portfolios have the benefit of being ‘static’ from a rebalancing perspective whereas an equal-weighted portfolio requires continuous rebalancing. Continuous rebalancing is obviously not feasible and one would, in practice, choose some appropriate rebalancing period, perhaps related to a fund manager’s view on the stocks in the portfolio as well as future realised volatility.

4.2 For the purpose of our analysis we again assume monthly rebalancing of all portfolios (there is an outstanding research question here that pertains to the optimal rebalancing period). We use realistic transaction costs of 30bp for all trading adjustments and rebalancing throughout. We replicate the results obtained in Table 2 (without transaction costs) in Table 4 but now accounting for transaction costs.

TABLE 4. Return statistics (trading cost inclusive)

Return statistics with 30bps costs (Jan 2000–Jan 2019)					
	CAGR	Volatility	Downside deviation	Sharpe ratio	Sortino ratio
Market cap-weighted	14.0%	19.7%	13.8%	0.77	1.09
Equal-weighted	14.3%	18.3%	13.0%	0.82	1.16

4.3 We see that with monthly rebalancing (trading cost inclusive) the equal-weighted portfolio’s CAGR declines by about 40bps per annum but still outperforms the cap-weighted portfolio on a return and risk-adjusted basis.

## 5. CONCLUSION

5.1 A plethora of research has shown equal-weighted portfolios to be more efficient than market cap-weighted portfolios, both theoretically in some instances and empirically in the US, for example.

5.2 We have demonstrated that a similar result holds for the South African equity market, namely, that equal-weighted portfolios outperform cap-weighted portfolios on a risk-adjusted basis. Furthermore, our results indicate that equal-weighted portfolios, even adjusted for transaction costs, would outperform cap-weighted portfolios. Secondly, we have shown that were active managers given an equal-weighted benchmark, the resulting active portfolios would have superior return and risk-adjusted return characteristics than would be the case with a market cap-weighted portfolio.

5.3 We conclude that equal-weighted portfolio benchmarks could serve as appropriate replacements for cap-weighted benchmarks in the South African equity market specifically for the active equity components of a portfolio.

## ACKNOWLEDGEMENTS

The authors would like to express their appreciation towards comments received from two anonymous reviewers which aided in the final exposition and quality of the article. The SAAJ editorial staff are thanked for their efficiency.

## REFERENCES

- Amenc, N, Goltz, F, Lodh, A & Martellini, L (2012). Diversifying the diversifiers and tracking the tracking error: Outperforming cap-weighted indices with limited risk of outperformance. *Journal of Portfolio Management* **38**(3), 72–88
- Bolognesi, E, Torluccio, G & Zuccheri, A (2013). A comparison between capitalization-weighted and equally weighted indexes in the European equity market. *Journal of Asset Management* **14**(1), 14–26
- Cremers, KJ & Petajisto, A (2009). How active is your fund manager? A new measure that predicts performance. *Review of Financial Studies* **22**(9), 3329–65
- Dash, S & Loggie, K (2008). Equal weight indexing five years later. S&P's Research Report
- Dash, S & Zeng, L (2010). Equal weight indexing seven years later. S&P's Research Report
- DeMiguel, V, Garlappi, L & Uppal, R (2009). Optimal versus naive diversification: How inefficient is the 1/N portfolio strategy? *Review of Financial Studies* **22**(5), 1915–53
- Fernholz, ER (2002). *Stochastic portfolio theory*. Springer New York
- Gibbons, MR, Ross, SA & Shanken, J (1989). A test of the efficiency of a given portfolio. *Econometrica* **57**(5), 1121–52
- Grinold, RC (1992). Are benchmark portfolios efficient? *The Journal of Portfolio Management* **19**(1), 34–40
- Hallerbach, W (2014). Disentangling rebalancing return. *Journal of Asset Management*, 301–16
- Haugen, RA & Baker, NL (1991). The efficient market inefficiency of capitalization-weighted portfolios. *The Journal of Portfolio Management* **17**(3), 35–40
- Hsu, CJ (2006). Cap-weighted portfolios are sub-optimal portfolios. *Journal of Investment Management* **4**(3), 1–10
- Kruger, R & Van Rensburg, P (2008). Evaluating and constructing equity benchmarks in the South African portfolio management context. *Investment Analysts Journal* **37**(67), 5–17
- Lloyd, WT & Manium, BK (2004). Portfolio strategies for outperforming a benchmark. *The Handbook of European Fixed Income Securities*. Vol. 108. John Wiley & Sons, 2004. Eds. Fabozzi, FJ & Choudhry, M
- Malladi, R & Fabozzi, FJ (2017). Equal-weighted strategy: Why it outperforms value-weighted strategies? Theory and evidence. *Journal of Asset Management*, **18**(3), 188–208
- Martellini, L (2009). Towards the design of better equity benchmarks. EDHEC Working Paper
- Perold, AF (2007). Fundamentally flawed indexing. *Financial Analysts Journal* **63**(6), 31–7
- Petajisto, A (2013). Active share and mutual fund performance. *Financial Analysts Journal* **69**(4), 73–93
- Plyakha, Y, Uppal, R & Vilkov, G (2014). Equal or Value Weighting? Implications for Asset-Pricing Tests

- Plyakha, Y, Uppal, R & Vilkov, G (2015). Why do equal-weighted portfolios outperform value-weighted portfolios? In Working paper, SSRN 2724535
- Sharpe, WF (1964). Capital asset prices: A theory of market equilibrium under conditions of risk. *The Journal of Finance* **19**(3), 425–42
- Taljaard, BH & Maré, E (2014). A comparison of benchmark indices for South African equity investors. Technical report UPWT 2014/15

## APPENDIX A

### Stochastic Portfolio Theory

We provide a very brief introduction to the Stochastic Portfolio Theory. For a more detailed overview we direct readers to Fernholz (2002).

Consider a stock  $X_i$  that follows the logarithmic model for continuous-time stock process:

$$d \log X_i(t) = \gamma_i(t)dt + \sigma_i(t)dW_i(t), \quad t \in [0, \infty) \quad (A1)$$

where  $\sigma_i$  and  $\gamma_i$  are the volatility and geometric growth rates, respectively, and  $dW_i$  are Brownian motions related to stock  $X_i$ .

Now consider a portfolio,  $Z_\pi$  with weights given by  $\pi(t) = (\pi_1(t), \dots, \pi_n(t))$  such that,

$$\sum_{i=1}^n \pi_i(t) = 1. \quad (A2)$$

Then it can be shown that the price process  $Z_\pi$  has the following form,

$$d \log Z_\pi(t) = \gamma_\pi(t)dt + \sum_{i=1}^n \pi_i(t)\sigma_i(t)dW_i(t), \quad (A3)$$

where

$$\gamma_\pi(t) = \sum_{i=1}^n \pi_i(t)\gamma_i(t) + \frac{1}{2} \left( \sum_{i=1}^n \pi_i(t)\sigma_{ii}(t) - \sum_{i,j=1}^n \pi_i(t)\pi_j(t)\sigma_{ij}(t) \right) \quad (A4)$$

and  $\sigma_{ij}$  represents the covariance between stock  $i$  and  $j$ .

Equation (A3) can be rewritten as,

$$d \log Z_\pi(t) = \sum_{i=1}^n \pi_i(t)d \log X_i(t) + \gamma_\pi^*(t)dt, \quad (A5)$$

where

$$\gamma_\pi^*(t) = \frac{1}{2} \left( \sum_{i=1}^n \pi_i(t)\sigma_{ii}(t) - \sum_{i,j=1}^n \pi_i(t)\pi_j(t)\sigma_{ij}(t) \right). \quad (A6)$$

Equations (A3), (A4) and (A5) highlight that the price process of a portfolio of stocks is not only a combination of their weighted growth rates but also the term,  $\gamma_\pi^*(t)$ , in Equation (A6), referred to as the excess growth rate. This term can be thought of as the benefit of diversification and its size depends on the correlations and volatilities of the individual stocks. For example, when correlations are high the benefit of diversification will be lower since the reduction in the individual stock volatilities within the portfolio with weights  $\pi(t)$  will be lower.

Now consider the market cap-weighted portfolio with weights  $\mu(t) = (\mu_1(t), \dots, \mu_n(t))$ , where,

$$\mu_{i(t)} = \frac{X_i(t)}{\sum_{i=1}^n X_i(t)}, \tag{A7}$$

and with portfolio price process  $Z_\mu$ .

Now suppose we would like to generate a portfolio with weights  $\pi$  and study its performance relative to the cap-weighted portfolio. To do this we define a portfolio-generating function  $S(\mu(t))$  and say that  $S$  generates  $\pi$  if there exists a measurable process of bounded variation,  $\Theta$ , such that the relative performance can be given by

$$\log\left(\frac{Z_\pi}{Z_\mu}\right) = \log S(\mu(t)) + \Theta(t), \quad t \in [0, T]. \tag{A8}$$

We call  $\Theta(t)$  the portfolio drift process. If we assume that the market is coherent, that is, that the time-average difference in the growth rates of two stocks is zero, then the long-term behaviour of the relative performance is determined solely by the portfolio drift term. See Definition 2.1.1 and Proposition 2.1.2 in Fernholz (2002) for further details.

We can also show that given a portfolio generating function that the portfolio weights can be derived as follows (Theorem 3.1.5 in Fernholz (2002)):

$$\pi_i(t) = \left( D_i S(\mu(t)) + 1 - \sum_{j=1}^n \mu_j(t) D_j \log S(\mu(t)) \right) \mu_i(t) \tag{A9}$$

and the drift term is given by

$$d\Theta(t) = \frac{-1}{2S(\mu(t))} \sum_{i,j=1}^n D_{ij} S(\mu(t)) \mu_i(t) \mu_j(t) \tau_{ij}(t) dt, \quad t \in [0, T] \tag{A10}$$

where  $D_i$  represent partial derivatives with respect to  $\mu_i$  and  $\tau_{ij}$  represents the relative covariance of stocks  $X_i$  and  $X_j$ , to the cap-weighted portfolio  $Z_\mu$ .

We are interested in the equal weight portfolio, which is given by the portfolio generating function,

$$S(\mu(t)) = (\mu_1(t) \cdots \mu_n(t))^{1/n}. \tag{A11}$$

This leads to the portfolio weights  $\pi_i = 1/n$  and a drift process given by

$$d\Theta(t) = \gamma_\pi^*(t) dt.$$

In other words, the drift process of the equal weight portfolio relative to the cap-weighted portfolio is given by the excess growth rate as defined in Equation (6). Since this can never be less than zero, and given the assumption of a coherent market, in the long-term the relative performance of the equal-weighted portfolio will be positive. That is, the equal-weighted portfolio will outperform the cap-weighted portfolio in the long-term.



The form of  $\log S(\mu(t))$  in Equation (11) reflects a level of concentration in the cap-weighted portfolio and given that  $0 \leq \mu_i(t) \leq 1$ , this term will be negative. However, the changes in this term are important for the relative performance over a period (the change in Equation (8)).

Of course, the market cap-weighted portfolio or index can increase in concentration for long periods of time, something the South African equity market has experienced on a few occasions. Secondly, the excess growth rate itself can vary (depending on correlations and volatilities of the individual stocks) and so, while in the long-term the equal-weighted portfolio will outperform (under the given assumptions), in the short-term the relative return will depend on the interaction of the concentration of the cap-weighted portfolio and the excess growth rate of the equal-weighted portfolio.

## APPENDIX B

### An overview of the approach to the decomposition of rebalancing return (Hallerbach, 2014)

Consider the return of a portfolio with weights  $w_{it}$  in stock  $i$  at the start of time period  $t$ . The return of the portfolio over the time period  $t$  is given by

$$r_{pt} = \sum_{i=1}^n w_{it} r_{it}$$

where

$$\sum_{i=1}^n w_{it} = 1 \text{ and } w_{it} \geq 0, \forall i, t.$$

Consider a portfolio  $p^*$  that is constantly rebalanced. Then the arithmetic mean of this portfolio is given by

$$\bar{r}_p^* = \sum_{i=1}^n w_{i0} \bar{r}_i. \quad (\text{B1})$$

The geometric growth rate can be approximated as

$$g_p^* = \bar{r}_p^* - \frac{1}{2} \sigma_p^{2*} \quad (\text{B2})$$

for the portfolio, and similarly for each stock,

$$g_i = \bar{r}_i - \frac{1}{2} \sigma_i^2. \quad (\text{B3})$$

Combining Equations (B1), (B2) and (B3) we find that

$$g_p^* = \sum_{i=1}^n w_{i0} g_i + \frac{1}{2} \left( \sum_{i=1}^n w_{i0} \sigma_i^2 - \sigma_p^{2*} \right). \quad (\text{B4})$$

Equation (B4) is analogous to Equation (A3) in Appendix A, where the second term on the right of Equation (B4) is analogous to the excess growth rate in SPT and Hallerbach (2014) refers to this as the volatility return. Similarly to SPT, the contribution of this term is always positive but its size is dependent both on the individual stock volatilities and the pairwise correlations.

A low volatility environment where stocks are highly correlated is likely to lead to a lower contribution to the equal-weighted portfolio's return. SPT takes this a step further to show that it is this volatility return (excess growth rate) that determines the long-term behaviour of the equal-weighted portfolio relative to the cap-weighted portfolio under certain assumptions.