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Universality in graph properties allowing constrained growth

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Abstract

A graph property is a class of graphs which is closed under isomorphisms. Some properties are also closed under one or more specified constructions that extend any graph into a supergraph containing the original graph as an induced subgraph. We introduce and study in particular the concept that a property $\mathcal{P}$ "allows finite spiking" and show that there is a universal graph in every induced-hereditary property of finite character which allows finite spiking.

We also introduce the concept that $\mathcal{P}$ "allows isolated vertex addition" and constructively show that there is a unique graph with the so-called $\mathcal{P}$-extension property in every induced-hereditary property $\mathcal{P}$ of finite character which allows finite spiking and allows isolated vertex addition; such a graph is then universal in $\mathcal{P}$ too.

Infinitely many examples which satisfy the conditions of both these results are obtained by taking the property of K_n-free graphs for an arbitrary integer $n \geq 2$.

Keywords: Countable graph; Property of graphs; Universal graph; Finite character; Spiking

Contents

1. Introduction

In general, the notation and terminology of [1] will be used. All graphs considered are simple, undirected and unlabelled, and have countable vertex sets. The symbol $\mathcal{I}$ denotes the class of all such graphs. We highlight some of the graph-theoretical concepts needed in the sequel.

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A *graph property* is a class (here always non-empty) of graphs, closed under isomorphisms. If two graphs are isomorphic, we refer to any one of them as a *clone* of the other. To avoid potential conceptual problems with proper classes or large numbers of clones, we may select a particular subset Skel($\mathcal{I}$) (a "skeleton") of the class $\mathcal{I}$, with elements one specific graph chosen from each isomorphism class in $\mathcal{I}$. (Since clones share all their graph properties, and since we have for many purposes, in a property, no reason to distinguish between clones, this move is unproblematic.) Graph properties can therefore be considered as *sets* ($\mathcal{I}$, $\mathcal{P} \subseteq \mathcal{I}$) of graphs. In general we follow [2] for notation and terminology on properties.

A property $\mathcal{P}$ is *induced-hereditary* if, whenever $G \in \mathcal{P}$ and $H \leq G$, then $H \in \mathcal{P}$ too. Note that, for any graph G, the set of graphs $\leq G := \{H \mid H \leq G\}$ is induced-hereditary. The set of all induced-hereditary properties is a well-studied lattice (amongst others in [2]) with set inclusion as the partial ordering; we denote this lattice by $\langle \mathbb{K}_{\leq}; \subseteq \rangle$. While the very important and well-investigated induced-hereditary properties $\mathcal{P}$ preserve $\mathcal{P}$-membership "downward along $\leq$" – *all induced subgraphs* of a $\mathcal{P}$-graph are again in $\mathcal{P}$ – we shall later introduce some new types of graph property which look "upward along $\leq$": Properties $\mathcal{P}$ which *allow spiking* or *allow isolated vertex addition* bestow $\mathcal{P}$-membership on *some inducing supergraphs* of a $\mathcal{P}$-graph and allow a form of "constrained growth". For graphs not necessarily in $\mathcal{P}$ but having another property, namely the *$\mathcal{P}$-extension property*, some form of growth is allowed for induced subgraphs which happen to be in $\mathcal{P}$.

Following [3], we say that the graph property $\mathcal{P}$ is *of finite character* if, whenever we have for a graph G that for every finite $H \leq G$ the inclusion $H \in \mathcal{P}$ holds, then we have $G \in \mathcal{P}$ too. Being of finite character can also be construed as allowing another form of constrained growth—from the finite to the infinite. Note that, for an induced-hereditary property $\mathcal{P}$, we have that $\mathcal{P}$ is of finite character if and only if the equivalence "$G \in \mathcal{P}$ if and only if every finite induced subgraph of G is in $\mathcal{P}$" holds. Examples of such properties are easily obtained by forbidding finite graphs: If $\mathcal{F}$ is any set of finite graphs then the property $-\mathcal{F}$ is defined by $-\mathcal{F} := \{G \in \mathcal{I} \mid F \not\leq G \text{ for each } F \in \mathcal{F}\}$; if $\mathcal{F} = \{F\}$, then we write $-F$ for $-\mathcal{F}$. These properties are well-known as induced-hereditary properties and are easily seen to be of finite character. In the next section we shall consider cases where $-\mathcal{F}$ interdicts complete graphs, and cycles.

Let $\mathcal{P}$ be a set of countable graphs. Following [1], we define a graph U to be a *universal graph for* $\mathcal{P}$ if every graph in $\mathcal{P}$ is (a clone of) an induced subgraph of U; it is a *universal graph in* $\mathcal{P}$ if $U \in \mathcal{P}$ too. We write $\mathcal{U}(\mathcal{P})$ for the set of graphs which are universal in $\mathcal{P}$. In [4] it is shown that, in some well-defined sense, $\mathcal{U}(\mathcal{P}) = \emptyset$ for most (even induced-hereditary) properties $\mathcal{P}$.

In the next section we show that there is a universal graph in every induced-hereditary property of finite character which allows finite spiking, a case where $\mathcal{U}(\mathcal{P}) \neq \emptyset$. In Section 3 we then demonstrate that there is a unique graph with the so-called $\mathcal{P}$-extension property in every induced-hereditary property $\mathcal{P}$ of finite character which allows finite spiking and also isolated vertex addition; such a graph is then universal in $\mathcal{P}$ too. The properties of being K_n-free ($n \geq 2$) instantiate these results.

2. Spiking

In [4] it was also pointed out how difficult it is in general for an arbitrary property $\mathcal{P} \in \mathbb{K}_{\leq}$ to decide, from some description of $\mathcal{P}$ in mathematical language, whether $\mathcal{U}(\mathcal{P}) \neq \emptyset$. We shall now demonstrate that any induced-hereditary $\mathcal{P}$ of finite character which "allows finite spiking" (to be defined) does indeed contain a universal member which is constructible in a recursive manner.

We define a new operation on graphs called *spiking*. Let G be any graph, I a countable index set and $\{W_i \mid i \in I\}$ an indexed set of subsets W_i of the vertex set of G. For each $i \in I$, let w_i be an element (vertex to be) outside $V(G)$, with $w_i \neq w_j$ when $i \neq j$. We define a new graph, designated as $G\langle\{W_i\}_{i\in I}\rangle$ and called "*G spiked at the W_i*", as follows:

$V(G\langle\{W_i\}_{i\in I}\rangle) := V(G) \cup \{w_i \mid i \in I\}$, while

$E(G\langle\{W_i\}_{i\in I}\rangle)$ consists of $E(G)$ together with, for every $i \in I$, all possible edges between w_i and elements of W_i. We note that I may be empty, in which case the spiked G is just G itself; also W_i may be empty (in which case w_i is an isolated vertex of $G\langle\{W_i\}_{i\in I}\rangle$); and the W_i may overlap or even be equal for different indices. If I is a singleton so that we have a single $W \subseteq V(G)$, we write $G\langle W\rangle$ and call it *G single-spiked at W*.

As an example of the employment of spiking, we mention the *Mycielski construct* $M(G)$ of a graph G [5], resulting from G in two spiking steps. First we construct $G' := G\langle\{N(v) \mid v \in V(G)\}\rangle$ by spiking at all the neighbourhoods of vertices of G. Let $W := \{w_v \mid v \in V(G)\}$ be the set of new vertices of G' that have been added to those of G. Then $M(G) := G'\langle W\rangle$.

To define the concept that a general property $\mathcal{P}$ *allows spiking* we consider any configuration with the following components:
(i) a graph $G \in \mathcal{P}$;
(ii) a countable index set I and a set $\{w_i \mid i \in I\}$, disjoint from $V(G)$, with $w_i \neq w_j$ when $i \neq j$;
(iii) for each $i \in I$ an internal $H_i \leq G$ (i.e. $V(H_i) \subseteq V(G)$) and a set $W_i \subseteq V(H_i)$; when we write $H_i\langle W_i\rangle$, the new spiking vertex is w_i;
(iv) for each $i \in I$, $H_i\langle W_i\rangle \in \mathcal{P}$.

Now we say that $\mathcal{P}$ *allows spiking* when for every situation as just described we have that $G\langle\{W_i\}_{i\in I}\rangle \in \mathcal{P}$. We also say that $\mathcal{P}$ *allows finite spiking* when in this stipulation I and each $V(H_i)$ and hence W_i are restricted to be finite sets; and $\mathcal{P}$ *allows single-spiking* when I is a singleton (and H finite).

In our first result a star, defined in [1] to be a graph of the form $K_{1,n}$, is a graph consisting of a vertex which is adjacent to each member of a non-empty countable set of pairwise non-adjacent vertices.

Lemma 1. *Every induced-hereditary property that contains all stars and allows spiking is closed under forming Mycielski constructs.*

Proof. Let $\mathcal{P}$ be any property as described in the lemma and suppose $G \in \mathcal{P}$. Using the notation introduced above, we let $V(G)$ act as the index set I, we choose for each $v \in V(G)$ the internal $H_v \leq G$ to be $G[N(v)]$ and the set $W_v \subseteq V(H_v)$ to be $V(H_v) = N(v)$ itself. Then $H_v\langle W_v\rangle$, which is a clone of $G[N(v) \cup \{v\}]$, is in $\mathcal{P}$ for each $v \in V(G)$ since $\mathcal{P}$ is induced-hereditary. Hence the graph G' formed by spiking in this situation is in $\mathcal{P}$ too.

For the second step of the Mycielski construct we use single-spiking at the countable set W of new vertices of G' to form $G'\langle W\rangle$; this is allowed in $\mathcal{P}$ since W is an independent set of G' and hence $G'[W]\langle W\rangle$ is a star, which is in $\mathcal{P}$. Then $M(G)$, which is a clone of $G'\langle W\rangle$, is in $\mathcal{P}$ too. □

Our next result exhibits examples of induced-hereditary properties of finite character which allow spiking.

Lemma 2. *For every integer $n \geq 2$, the property $-K_n$ is induced-hereditary, of finite character, and allows spiking.*

Proof. Given the general remarks on $-\mathcal{F}$ in Section 1, only the spiking needs attention here. Consider any graph $G \in -K_n$ with a set $\{H_i\}_{i\in I}$ of internal induced subgraphs and a set $\{W_i\}_{i\in I}$ of subsets of $V(G)$ with each $W_i \subseteq V(H_i)$ such that every $H_i\langle W_i\rangle$ does not contain the complete graph K_n. Should $G\langle\{W_i\}_{i\in I}\rangle$ contain this complete graph, then it is clear that no two or more of the new w_i can be vertices of this complete graph since the new vertices form an independent set. Hence, if none of the new w_i are used, then this complete graph must lie in G and if one of the new w_i is used then this complete graph must lie in a single $H_i\langle W_i\rangle$, both of which are impossible since these graphs are in $-K_n$ by our assumptions. Hence $G\langle\{W_i\}_{i\in I}\rangle \in -K_n$. □

It is well known that, if G is a triangle-free graph, then so is the Mycielski construct $M(G)$. (This fact is used in a typical constructive proof of the existence of triangle-free graphs with arbitrary high chromatic numbers [5].) We can now record, as a consequence of Lemma 2, that the property $-K_n$, for every integer $n \geq 3$, is also closed under forming Mycielski constructs.

Corollary 1. *The property $-K_n$, for every integer $n \geq 3$, is closed under forming Mycielski constructs: If G is K_n-free, then so is $M(G)$.*

Proof. Clearly, the induced-hereditary property $-K_n$ contains all stars if $n \geq 3$ and allows spiking by Lemma 2. Hence the result follows immediately from Lemma 1. □

We now show that properties defined by forbidding cycles do not, in general, allow spiking. In this lemma, the set $\mathcal{C}$ denotes any set of cycles including at least one of order $n \geq 4$.

Lemma 3. *The property $-\mathcal{C}$ is induced-hereditary, of finite character, and does not allow (even finite) spiking.*

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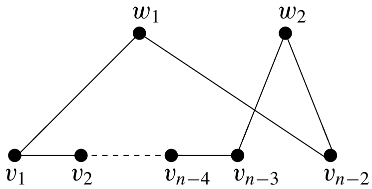
Proof. Again, only the no-spiking needs attention here. For each $n \geq 4$ we construct a graph $G_n \in -\mathcal{C}$ which by finite spiking extends to C_n, thus exemplifying that $-\mathcal{C}$ does not allow spiking, (since at least one such C_n is forbidden in $-\mathcal{C}$). Let $V(G_n) := \{v_1, v_2, \ldots, v_{n-2}\}$ and $E(G_n) := \{v_i v_{i+1} \mid 1 \leq i \leq n-4\}$ (which is empty when $n = 4$ and $V(G_4) = \{v_1, v_2\}$), so that G_n consists of an induced path (which is a single vertex if $n = 4$) on $v_1, v_2, \ldots, v_{n-3}$ and an isolated vertex v_{n-2}; hence $G_n \in -\mathcal{C}$. We now define two induced subgraphs H_i of G_n and two sets of vertices by taking each $H_i := G_n[W_i]$ with $W_1 := \{v_1, v_{n-2}\}$ and $W_2 := \{v_{n-3}, v_{n-2}\}$ (which are equal if $n = 4$, even then with $w_1 \neq w_2$) and look at the spiked $G_n\langle\{W_1, W_2\}\rangle$:



The spiked $G_n\langle\{W_1, W_2\}\rangle$

It is clear that $H_1\langle W_1\rangle$ and $H_2\langle W_2\rangle$, being just paths, are in $-\mathcal{C}$, while $G_n\langle\{W_1, W_2\}\rangle = C_n$. For at least one $n \geq 4$ this C_n is forbidden from being an induced subgraph of any member of $-\mathcal{C}$, i.e., $G_n\langle\{W_1, W_2\}\rangle \notin -\mathcal{C}$. So this G_n falsifies the allowance of spiking for the property $-\mathcal{C}$. □

We remark that the use of the singleton set $\{K_n\}$ in Lemma 2, as against the set $\mathcal{C}$ in Lemma 3 which may contain more than one graph, is dictated by the fact that a more general set of complete graphs does not give a more general situation (since, if $m \leq n$, then $K_m \not\leq G$ implies $K_n \not\leq G$). The property of triangle-free graphs, $-C_3 = -K_3$, is a special case covered by Lemma 2.

We now turn to questions around the existence of universal graphs in properties allowing some forms of constrained growth and start with

Theorem 1. *There is a universal graph in every induced-hereditary property of finite character which allows finite spiking.*

Proof. Consider any induced-hereditary property $\mathcal{P}$ of finite character which allows finite spiking. In Section 2.1 of [6] the construction of the graph $U(\mathcal{P})$, in general universal **for** $\mathcal{P}$, is explicated: $U(\mathcal{P}) = U_1 \cup U_2 \cup \ldots$, with $U_1 < U_2 < \cdots$. Here U_{n+1} results recursively from U_n by finite spiking: $U_{n+1} := U_n\langle\{W_i\}_{i\in I}\rangle$, where $\{G_i \mid i \in I\}$ constitutes a skeleton (there called $\mathcal{F}_{n+1}$) of the class of those graphs in $\mathcal{P}$ with $n+1$ vertices, with $G_i = H_i\langle W_i\rangle$ and $H_i \in \mathcal{F}_n$. U_1 is the graph consisting of a single vertex, while U_n is universal for $\mathcal{F}_n$; hence $H_i \leq U_n$ and $H_i \in \mathcal{P}$. (Unlike the situation in [6] the graphs here are simple, undirected, and unlabelled.) So $U_1 \in \mathcal{P}$ and with $\mathcal{P}$ allowing finite spiking, every $U_n \in \mathcal{P}$. The finite character of $\mathcal{P}$ guarantees that also $U(\mathcal{P}) \in \mathcal{P}$, i.e., there is a universal graph in $\mathcal{P}$ and $\mathcal{U}(\mathcal{P}) \neq \emptyset$. □

From Lemma 2 and Theorem 1, for the properties $-K_2 \subset -K_3 \subset \cdots$, we can now easily deduce

Corollary 2. *For every integer $n \geq 2$ there is a graph, say Z_n, such that*
(i) Z_n is universal in $-K_n$; and
(ii) $Z_2 < Z_3 < \cdots$, but (for all $k \geq 1$) $Z_{n+k} \not\leq Z_n \in -K_{n+k}$.

This result, in combination with Lemma 2, reconfirms the well-known fact [7] and [8] that there is a universal triangle-free (or K_3-free) graph. It is also known (see [9]) that $-C_4$ does not have a universal member; in fact it is shown in [10] that for a finite set $\mathcal{C}$ of cycles it holds that $-\mathcal{C}$ has a universal member if and only if $\mathcal{C} = \{C_3, C_5, \ldots, C_{2k+1}\}$ for some positive integer k. Using these remarks and Lemma 3 we can now easily deduce

Theorem 2. *For the cycle-free property $-C_n$ of countable graphs the following are equivalent:*
(i) $n = 3$,
(ii) $-C_n$ allows spiking, and
(iii) $-C_n$ has a universal member.

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The fact that every induced-hereditary property of the form $-\mathcal{C}$ with $\mathcal{C} = \{C_3, C_5, \ldots, C_{2k+1}\}$ and $k \geq 2$ contains a universal graph cannot be deduced from Theorem 1 since, although it is of finite character, it does not allow finite spiking by Lemma 3.

3. Constructing a $\mathcal{P}$-extensible universal member

The classical notion that a graph "has the extension property" from [11] is extended in [12] to the following for an arbitrary property $\mathcal{P}$: A countable graph G *has the $\mathcal{P}$-extension property* when the following holds: whenever U and V are two disjoint finite subsets of $V(G)$ such that $G[U \cup V] \in \mathcal{P}$, while the graph obtained by enlarging $G[U \cup V]$ with one new vertex adjacent to all and only the vertices of U is also in $\mathcal{P}$, then this fact is reflected inside G, i.e., there exists a vertex z of G, outside of $U \cup V$, which is adjacent in G to every vertex of U and to no vertex of V. (Note that G having the $\mathcal{P}$-extension property signifies the allowance of a specific type of constrained growth to induced subgraphs of G which have property $\mathcal{P}$.) In Theorem 2 of [12] it is then shown that any countable graph which has the $\mathcal{P}$-extension property is also universal for an induced-hereditary $\mathcal{P}$; Theorem 6 of [12] takes this further by showing that any such universal graph in $\mathcal{P}$ is unique, i.e., for every induced-hereditary property $\mathcal{P}$ there is up to isomorphism at most one graph in $\mathcal{P}$ which has the $\mathcal{P}$-extension property.

In order to ensure, beyond Theorem 1 of Section 2, the existence in $\mathcal{P}$ of a $\mathcal{P}$-extensible universal member an extra (though not exorbitant) price can be paid. We say that a graph property $\mathcal{P}$ *allows isolated vertex addition (i-v-a)* when the following holds: Given any $G \in \mathcal{P}$, $v \notin V(G)$, and the graph $G' = (V(G) \cup \{v\}, E(G))$, we have $G' \in \mathcal{P}$. We note that any induced-hereditary property which is *additive*, i.e., closed under disjoint unions of graphs (another form of constrained growth), allows isolated vertex addition. Any property which allows isolated vertex addition of course allows the isolated addition of finite independent vertex sets. Any induced-hereditary property (inevitably containing the graph with one vertex) which allows i-v-a has every finite edgeless graph as a member. Now consider any induced-hereditary property $\mathcal{P}$ of finite character which allows i-v-a (like, e.g., a $\mathcal{P}$ as mentioned in the next theorem). Such a $\mathcal{P}$ allows the isolated addition of any *countable* independent vertex set (although this fact is not used in the proof of the next theorem): Consider namely a $G \in \mathcal{P}$, a countable set W disjoint from $V(G)$, and $G' = (V(G) \cup W, E(G))$. To show that $G' \in \mathcal{P}$ it is sufficient, by the finite character of $\mathcal{P}$, to show that $H' \in \mathcal{P}$ for any finite $H' \leq G'$. If $V(H') \subseteq W$, then H', being a finite edgeless graph, is in $\mathcal{P}$. If not, then there exists a finite $H \leq G$, $H \in \mathcal{P}$, such that $H' = (V(H) \cup (V(H') \cap W), E(H))$. Then $H' \in \mathcal{P}$, since $\mathcal{P}$ allows the isolated addition of finite independent vertex sets. So, in this case, $\mathcal{P}$ contains the property of all countable edgeless graphs.

Theorem 3. *Let $\mathcal{P}$ be any induced-hereditary property of finite character which allows finite spiking and i-v-a. Then $\mathcal{P}$ has a member with the $\mathcal{P}$-extension property (which hence is the unique universal member of $\mathcal{P}$ with the $\mathcal{P}$-extension property).*

Proof. For any induced-hereditary property $\mathcal{P}$, [12] Section 2 describes (in notation which we shall confirm here) the recursive construction of a graph $X(\mathcal{P})$ which has the $\mathcal{P}$-extension property (and hence is the unique graph with the $\mathcal{P}$-extension property which is universal *in* $\mathcal{P}$ in case $X(\mathcal{P}) \in \mathcal{P}$). In general, however, we know only that $X(\mathcal{P})$ is universal *for* $\mathcal{P}$. By closely tracing the stages of the construction of $X(\mathcal{P})$, we shall demonstrate that when an induced-hereditary property $\mathcal{P}$ allows finite spiking and i-v-a, then each recursive step preserves membership of $\mathcal{P}$, finally yielding $X(\mathcal{P}) \in \mathcal{P}$ by $\mathcal{P}$ being of finite character.

A denumerable sequence of graphs in $\mathcal{P}$ is constructed for which the strict induced subgraph relations $X(\mathcal{P})^1 < X(\mathcal{P})^2 < \cdots$ hold, and then $X(\mathcal{P})$ is defined as the limit of this ascending sequence:

$$X(\mathcal{P}) = \bigcup \{X(\mathcal{P})^k \mid k \text{ is a positive integer}\}$$

in which $V(X(\mathcal{P}))$ is the union of the sets $V(X(\mathcal{P})^k)$, and similarly for $E(X(\mathcal{P}))$.

$X(\mathcal{P})^1 \in \mathcal{P}$: $X(\mathcal{P})^1$ has a unique single vertex, say v, and no edges. By the induced-hereditariness of $\mathcal{P}$, we surely have $X(\mathcal{P})^1 \in \mathcal{P}$.

$X(\mathcal{P})^2 \in \mathcal{P}$: $V_1^1 = \emptyset$ and $V_2^1 = \{v\}$ are the two elements of $Pow(V(X(\mathcal{P})^1))$. We add two new vertices, v_1^1 for V_1^1 and v_2^1 for V_2^1 to $V(X(\mathcal{P})^1)$ to obtain $V(X(\mathcal{P})^2) = \{v, v_1^1, v_2^1\}$. For the empty V_1^1 its corresponding new vertex v_1^1 stays isolated in $X(\mathcal{P})^2$, while for the non-empty V_2^1 we look at the graph $X(\mathcal{P})^1[V_2^1] + v_2^1$ (in this case K_2). Should this graph be in $\mathcal{P}$, then join each vertex in V_2^1 (here only v) to v_2^1 by an edge in $E(X(\mathcal{P})^2)$. If not, then not. This describes $E(X(\mathcal{P})^2)$.

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$X(\mathcal{P})^1\langle V_1^1\rangle = X(\mathcal{P})^1\langle\emptyset\rangle$ (with $\emptyset \subseteq V(H_1)$ and $H_1 = X(\mathcal{P})^1$, if you are meticulous) is $X(\mathcal{P})^1$ with the isolated vertex v_1^1 added to it; since $\mathcal{P}$ allows i-v-a, this graph is in $\mathcal{P}$. Consider $X(\mathcal{P})^1$ together with the new vertex v_2^1. Should $E(X(\mathcal{P})^2)$ contain an edge between v and v_2^1, then this is so because $X(\mathcal{P})^1[V_2^1] + v_2^1 = X(\mathcal{P})^1\langle V_2^1\rangle \in \mathcal{P}$. If not, then we can again see $X(\mathcal{P})^1$ together with the isolated new vertex v_2^1 as another clone of $X(\mathcal{P})^1\langle\emptyset\rangle \in \mathcal{P}$. So $X(\mathcal{P})^2 = X(\mathcal{P})^1\langle\{W_1, W_2\}\rangle$, with either $W_1 = \emptyset$ and $W_2 = V_2^1$, or else $W_1 = W_2 = \emptyset$, while both $X(\mathcal{P})^1\langle W_1\rangle$ and $X(\mathcal{P})^1\langle W_2\rangle$ are in $\mathcal{P}$. Since $\mathcal{P}$ allows finite spiking, $X(\mathcal{P})^2 \in \mathcal{P}$.

Now we describe the general recursive step in the construction. Suppose that the graphs $X(\mathcal{P})^1, X(\mathcal{P})^2, \ldots, X(\mathcal{P})^k \in \mathcal{P}$ have already been constructed, and consider all the subsets $V_1^k, V_2^k, \ldots, V_n^k$ of $V(X(\mathcal{P})^k)$ one by one, where $V_1^k = \emptyset$. (If $|V(X(\mathcal{P})^k)| = m$, then $n = 2^m$.) For each V_i^k $(1 \le i \le n)$, add one new vertex v_i^k to $V(X(\mathcal{P})^k)$ to obtain $V(X(\mathcal{P})^{k+1}) := V(X(\mathcal{P})^k) \cup \{v_1^k, v_2^k, \ldots, v_n^k\}$. For each i, look whether the graph $X(\mathcal{P})^k[V_i^k] + v_i^k \in \mathcal{P}$ or not. If so, add to $E(X(\mathcal{P})^k)$ all the edges wv_i^k for $w \in V_i^k$ into $E(X(\mathcal{P})^{k+1})$; if not, add no edges between v_i^k and elements of V_i^k. This determines $E(X(\mathcal{P})^{k+1})$.

Now consider the n single-spiked graphs $X(\mathcal{P})^k\langle W_i\rangle$ with vertex sets $V(X(\mathcal{P})^k) \cup \{v_i^k\}$, where $W_i = \emptyset$ when $i = 1$ or $X(\mathcal{P})^k[V_i^k] + v_i^k \notin \mathcal{P}$; and $W_i = V_i^k$ when $X(\mathcal{P})^k[V_i^k] + v_i^k \in \mathcal{P}$.

Then each $X(\mathcal{P})^k\langle W_i\rangle \in \mathcal{P}$, either because $\mathcal{P}$ allows i-v-a (when $W_i = \emptyset$) or by the definition of $E(X(\mathcal{P})^{k+1})$ and the fact that $\mathcal{P}$ allows (single) spiking. Since $\mathcal{P}$ allows finite spiking, we have $X(\mathcal{P})^{k+1} = X(\mathcal{P})^k\langle\{W_i\}_{1\le i\le n}\rangle \in \mathcal{P}$.

We note that the induced-subgraph relations $X(\mathcal{P})^1 < X(\mathcal{P})^2 < \cdots$ hold and define $X(\mathcal{P})$ (as said above) to be the limit of this ascending sequence:

$$X(\mathcal{P}) := \bigcup\{X(\mathcal{P})^k \mid k \text{ is a positive integer}\}.$$

Each $X(\mathcal{P})^k$ is universal for the property of all those members of $\mathcal{P}$ with no more than k vertices. Every finite induced subgraph of $X(\mathcal{P})$ is an induced subgraph of some $X(\mathcal{P})^k \in \mathcal{P}$ and hence (by the induced-hereditariness of $\mathcal{P}$) itself in $\mathcal{P}$. Since $\mathcal{P}$ is of finite character, we finally have $X(\mathcal{P}) \in \mathcal{P}$, with [12] Theorems 2 and 3 establishing the universality of $X(\mathcal{P})$ for (now *in*) $\mathcal{P}$. □

It is easy to see that, for every integer $n \ge 2$, the property $-K_n$ (which is of finite character and allows finite spiking by Lemma 2) also allows i-v-a and therefore satisfies all the conditions of Theorem 3. Hence we have the following strengthening of Corollary 2(i):

Corollary 3. *For every integer $n \ge 2$ there is a universal graph in the property $-K_n$; in fact, there is, up to isomorphism, only one which has the $-K_n$-extension property.*

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