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# Operator algebras and quantum information

by

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A dissertation submitted in partial fulfillment for the degree of Magister Scientae

in the

Faculty of Natural and Agricultural Sciences
Department of Physics

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# Declaration of Authorship

I, Kyle Oerder, declare that this dissertation titled, 'Operator algebras and quantum information' and the work presented in it are my own. I confirm that:

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- Where any part of this dissertation has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated.
- Where I have consulted the published work of others, this is always clearly attributed.
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# *Abstract*

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The C*-algebra representation of a physical system provides an ideal backdrop for the study of bipartite entanglement, as a natural definition of separability emerges as a direct consequence of the non-abelian nature of quantum systems under this formulation. The focus of this dissertation is the quantification of entanglement for infinite dimensional systems. The use of Choquet's theory of boundary integrals allows for an integral representation of the states on a C*-algebra and subsequent adaptation of the Convex Roof Measures to infinite dimensional systems. Another measure of entanglement, known as the Quantum Correlation Coefficient, is also shown to be a valid measure of entanglement in infinite dimensions, by making use of the intimate connection between separability and positive maps.

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# Symbols

| | |
|---|---|
| $\mathfrak{A}$ | a C$^*$-algebra |
| $\mathfrak{M}$ | a von Neumann Algebra |
| $\mathscr{H}$ | a Hilbert Space |
| $B(\mathscr{H})$ | the set of all bounded linear operators on the Hilbert space $\mathscr{H}$ |
| $\mathfrak{A}^*$ | the dual space of $\mathfrak{A}$ |
| $\mathfrak{A}'$ | the commutant of $\mathfrak{A}$ (set of all elements that commute with all the elements of $\mathfrak{A}$) |
| $\mathfrak{S}_{\mathfrak{A}}$ | the set of all states on the C$^*$-algebra $\mathfrak{A}$ |
| $P_{\mathfrak{A}}$ | the set of all pure states i.e. the extreme points of $\mathfrak{S}_{\mathfrak{A}}$ |
| $M_+(K)$ | the set of all Radon measures over a convex compact subset $K$ in a locally compact vector space $X$ |
| $M_1(K)$ | the subset of $M_+(K)$ with unit norm i.e. the set of all probability measures on $K$ |



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# Chapter 1

# Introduction

## 1.1 The EPR paradox: Origins of Entanglement

In 1935, Schrödinger [1] and the team of Einstein, Podolsky, and Rosen (EPR) [2] simultaneously recognized for the first time that the nonrelativistic quantum mechanics established by von Neumann implied the existence of states in a composite system that could not be recognized as the product of states acting on the individual subsystems. In general the basis vectors for a composite system $S^{AB}$ are the tensor products of the basis vectors for the individual subsystems, but a general vector of a composite system being a superposition of these vectors implies that some states achievable by the system cannot be separated into parts associated with each subsystem $S^A$ and $S^B$. These states came to be known as the entangled states.

Entanglement as a concept is now recognized as a confirmation of quantum mechanics but, ironically, it was EPR's attempt to use it to refute quantum theory in favour of a more deterministic model which lead to the former's acceptance via a thought experiment known as the EPR paradox.

EPR argued that the existence of entangled states implied that quantum mechanics was incomplete and instead proposed that, contrary to quantum theory, the values of physical quantities of an unobserved system are possessed prior to observation



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and it is not the act of measurement which gives rise to these values, thus inferring the existence of elements of reality that are not predicted by quantum theory, i.e. the local hidden variables model (LHVM) which assumes three axioms:

- "realism": that measurement results are predetermined by intrinsic independent properties of the system's constituents.
- "locality": the results of spatially separate actions are independent.
- "free will": setting of local apparatus and hidden variables are independent.

To demonstrate the EPR paradox, Bohm [3] used a pair of spin-1/2 particles prepared in the singlet state:

$$|\psi\rangle = \frac{|01\rangle - |10\rangle}{2}$$

After preparation, the particles may be separated far enough away so that any measurement performed on the first particle would only influence the second if information were allowed to travel faster than the speed of light.

A measurement of the spin along the axis corresponding to the unit vector $\hat{n}$ is given by the operator $\hat{n} \cdot \sigma = n_x\sigma_x + n_y\sigma_y + n_z\sigma_z$ where $\sigma_i$'s are the Pauli spin operators along the $i$-axis. Since any spin measurement on the first particle is commutative with a spin measurement along the second particle, then even in a quantum system the two measurements can happen simultaneously but will produce anti-correlated results for the singlet state i.e. if measurement is done along the same axis for both particles, then one particle will give the result +1 while the other will give the result -1 without fail. Under EPR's hidden variable model, then $\hat{n} \cdot \sigma$ must correspond to a physical property whose value is predetermined before the measurement took place (i.e. the assumption of realism is in place).

Once separated, each particle could undergo one of two possible spin measurements (chosen at random) and each of these measurements can produce a value of +1 or -1 depending on whether the particle's spin is aligned with the chosen axis or

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not. On the first particle, the possible measurements that could be performed are $A$: along the $z$-axis, or $B$: along the $x$-axis. On the second particle, the possible measurements are $C$: along the vector $\hat{c} = \left(-\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right)$, or $D$: along the vector $\hat{d} = \left(-\frac{1}{\sqrt{2}}, 0, +\frac{1}{\sqrt{2}}\right)$.

Bell [4] proved that the implementation of the LHVM assumptions on a bipartite system leads to a series of inequalities that govern statistical correlations on measurements (now known as the Bell inequalities) which can be violated by the entangled states under the assumptions of Quantum Mechanics. Clauser et al. [5] adopted the Bell inequalities to develop an experimentally verifiable correlation inequality known as the CHSH inequality which must be obeyed if the LHMV model is correct:

$$\langle AC\rangle + \langle BC\rangle + \langle AD\rangle - \langle BD\rangle \leq 2$$

Where $\langle XY\rangle$ represent the expectation values of simultaneously measuring $X$ on the first subsystem and $Y$ on the second. In contrast, quantum mechanics predicted that each of these expectation values is actually $\frac{1}{\sqrt{2}}$ with the exception of $\langle BD\rangle = -\frac{1}{\sqrt{2}}$ thus

$$\langle AC\rangle + \langle BC\rangle + \langle AD\rangle - \langle BD\rangle = 2\sqrt{2} > 2$$

Aspect et al. [6] experimentally verified that the singlet state violates the CHSH inequality and obeys the predictions of quantum mechanics, thus discrediting the LHVM model.

## 1.2 Entanglement as a Resource

The repeated experimental violation of the Bell inequalities by entangled states is not only a strong advocate of quantum theory but also indicates that the classical theory of statistical correlations is insufficient in quantum mechanics when dealing with composite systems. This leads to an equivalent alternative definition

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of entangled states: those which exhibit correlations that cannot be mimicked by classical theory in terms of local operations on the separate subsystems assisted by classical communication methods (the LOCC paradigm).

The presence of these additional correlations associated with entanglement elevated it from the subject of a purely philosophical debate to a resource which could be exploited. In fact, the nature of these quantum correlations allows for tasks which are wholly unachievable without them such as:

- Quantum Cryptography [7, 8]: A quantum system cannot be measured without alteration to the associated state unless the state is an eigenstate of that measurement. Thus the use of quantum systems as carriers in information sharing between two observers allows for much greater security as well as the prevention of eavesdropping which is otherwise impossible to detect in classical information sharing.

- Superdense Coding [9]: Entangled states can be used to send up to double the amount of information when in a maximally entangled state through local operations on individual subsystems and previously agreed associations with eigenstates.

- Quantum Teleportation [10]: Through the sharing of an entangled state, information can be transmitted from one observer to the next without having to move a physical particle along with it. Local measurements on a subsystem are performed and the results relayed to the holder of the second subsystem via classical communication then allowing the second subsystem to be prepared in a desired state.

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## 1.3 Motivation for the Study and Structure of the Dissertation

It is the usefulness of entanglement as a resource that is paramount in disciplines such as Quantum Information [11]. Naturally this has lead to numerous investigations into the mathematical description of entanglement aimed at characterizing, manipulating and quantifying entanglement. However these studies are largely confined to finite dimensional systems in the traditional formalism of quantum mechanics in terms of Hilbert spaces and their bounded linear operators introduced by von Neumann [12].

This dissertation will focus on the last concept of quantification of bipartite entanglement under the algebraic formalism of quantum mechanics which utilizes the more general notion of $C^*$-algebras in place of Hilbert spaces. The main justification for using the algebraic formalism is twofold in that it not only provides a mathematically rigorous theory for the systems of infinite degrees of freedom, but also transparently and naturally incorporates and highlights entanglement as a characteristic feature of quantum mechanics.

In particular, this dissertation will examine and expand on two methods of measuring entanglement and their associated correlations that were proposed by Majewski for infinite dimensions: The Convex Roof Measures (a generalized version of the Entanglement of Formation presented in [13]) and the Quantum Correlation Coefficient (introduced in [14] and refined in [15]).

## 1.4 Structure of the Dissertation

Throughout this dissertation it will be assumed that the reader is familiar with the basic theory of $C^*$-algebras (namely Chapters 1 to 4 in [16] and Chapters 1 through 13 in [17] which would be covered in a typical MSc level Analysis course) as well as standard results in measure theory, functional analysis, and topology

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(which can be found in standard introductory texts such as [18], [19], and [20]). It will be assumed that C*-algebras possess a unit throughout.

The traditional views of observables as linear operators on a Hilbert space and the states as trace class operators does not lend itself well to the mathematical description of entanglement and so this dissertation will begin (in Chapter 2) by first reworking the axioms of quantum mechanics into a more mathematically sensible form: the C*-algebra representation of a physical system. The key difference between the classical and quantum systems is the commutativity of all elements of the algebra representing the observables of the system in the former scenario. The implications of this fundamental quality is discussed and explored in order to show that entanglement is a natural occurrence embedded in the structure of the operator algebra forming the composite quantum system and the states that are allowed to act on this operator algebra.

Once the preliminaries of quantum systems have been covered and entangled states rigorously defined, a few mathematical preliminaries are discussed in Chapter 3 to establish a library of relevant tools for a novice reader. In particular, a brief recap of some less elementary aspects of topology and functional analysis are discussed as well as the more obscure concept of the integral representation of elements of a compact convex set, viz. Choquet's theory of boundary integrals, which provides an invaluable tool in the development of entanglement measures.

In Chapter 4 the nature of entanglement will be briefly discussed before an adapted variant (first introduced by [13]) of the convex roof measures is introduced based on the integral representation of states. It will be shown that these measures are suitable for use in the infinite dimensional scenarios and possess nice features such as weak* continuity and convexity which are desirable properties in the discipline of Quantum Information. A comparison between the original version introduced by Bennet et al. [21] and improved by Uhlmann [22] will be done to show that the proposed definition coincides with the original in the finite dimensional scenarios (to which the original variants are restricted). Lastly some specific examples of the convex roof measures and their properties are discussed.

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Chapter 5 is dedicated to another method of detecting entanglement known as the Quantum Correlation Coefficient that was introduced by Majewski in [14] and improved in [15]. The success of this measure of entanglement is based on the connection between positive maps and entanglement and so this connection is also discussed.

Chapter 6 is reserved for final remarks and conclusions.

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# Chapter 2

# Systems and C*-Algebras

The abstraction of quantum mechanics from the traditional formulation based on Hilbert spaces (as introduced by von Neumann) to the algebraic formulation based on C*-algebras is not only justified by the mathematical rigour that it imparts on infinite dimensional systems, but also because it clearly points out the fundamental difference between classical and quantum systems in a comparable setting. Specifically, both classical and quantum systems can be formulated in terms of C*-algebras but it is only those associated with classical systems which are abelian.

This single difference between quantum and classical systems, in their algebraic formulations, naturally leads to a characterization of entanglement that is a unique property of composite quantum systems (which is the ultimate goal of this chapter).

## 2.1 Classical Physical Systems

Before the algebraic formulation of quantum mechanics is discussed in detail, a cursory discussion of classical (Hamiltonian) mechanics is given with the aim of showing that abelian C*-algebras can be used to describe a classical system. This strengthens the motivation for adopting C*-algebras in the quantum setting so



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that classical mechanics clearly becomes a special case of the former. The primary sources used for this subsection are [23] and [24].

Any physical system (both classical and quantum mechanical in nature) requires two things:

1. A collection of possible states of the system referred to as the state space.
2. A collection of rules dictating time evolution and transformations (including observables and measurements) of one state into another within the state space.

In the traditional (Hamiltonian) classical system, the states are associated with a Hausdorff space $\Gamma$ known as the phase space. The elements $(p, q) \in \Gamma$ are the allowed momenta and positions, which themselves are vectors in a $n$-dimensional Euclidean space. For simplicity, it will be assumed that $\Gamma$ is compact. This is merely a technical simplification motivated by typical constraints on realistic systems such as a when a system is confined to a specific bounded region in space for which the energy is also bounded. How this assumption can be relaxed without altering the underlying results will be discussed briefly at the end of this section.

Measurements of these classical systems are associated with continuous stochastic variables i.e. continuous, measurable, and real-valued functions acting on the phase space $f : \Gamma \to \mathbb{R}$. These continuous stochastic variables are termed the observables of a classical system.

The set of all these classical observables form a subset of a much larger set $C(\Gamma)$, the set of all continuous complex valued functions acting on the phase space. When equipped with pointwise operations for addition, multiplication and scalar multiplication:

$$(\alpha f + \beta g)(x) = \alpha f(x) + \beta g(x)$$

$$(fg)(x) = f(x)g(x)$$

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and the sup norm:

$$\|f\| = \sup_{x\in\Gamma}\{|f(x)|\}$$

then $C(\Gamma)$ is a unital banach algebra.

An involution operation can be defined in a similar way:

$$f^*(x) = \overline{f(x)}$$

which makes $C(\Gamma)$ into a C*-algebra. In this way we can identify the set of all classical observables as the self adjoint elements of a unital C*-algebra. In particular this C*-algebra is abelian. In fact, every abelian C*-algebra is *-isomorphic to $C(K)$ for some compact Hausdorff space $K$ (see Theorem 9.4 in [17]) thus it follows that:

**Definition 2.1.** (Observables of a Classical System)
The set of *observables* are exactly the self-adjoint elements of a unital abelian C*-algebra $\mathfrak{A}$ associated with a classical system.

*Remark* 2.2. It should be noted that the above definition does not mean that any abelian C*-algebra will work for a given physical system. As a result of the Gelfand Transform, any abelian C*-algebra is *-isomorphic to $C(K)$ for some compact Hausdorff space $K$. Thus, in order for $\mathfrak{A}$ to accurately represent the observables for a classical system traditionally given by the phase space $\Gamma$, $K$ must be *-isomorphic to $\Gamma$. The choice of whether to use $\mathfrak{A}$ or $C(K)$ is then a personal choice of which space it is easier to work for a given problem. The key identifier however is the abelian nature of this algebra $\mathfrak{A}$ since without this we cannot assume this *-isomorphism exists.

To limit the states of a classical system purely to the elements of the phase space $\Gamma$ requires the assumption that the system can be measured sharply and with infinite precision, which is an absurd idealization. Thus it is far more accurate to associate the state of a classical system with a probability measures $\mu$ on $\Gamma$:

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**Definition 2.3.** (Probability Measure)

A *probability measure* $\mu$ on $\Gamma$ is any non-negative countably additive set function on the Borel sets $\mathcal{B}$ of $\Gamma$ for which $\mu(\Gamma) = 1$.

Since $\Gamma$ is a compact metric space, then any probability measure on $\Gamma$ is automatically regular in the sense that for $B \in \mathcal{B}$

$$\mu(B) = \sup\{\mu(C) : C \subseteq B, C \text{ closed}\} \tag{2.1}$$

$$= \inf\{\mu(C) : B \subseteq C, C \text{ open}\} \tag{2.2}$$

(See Chapter 2 in [25]).

The expectation value of an observable $f$ when a system is in a state $\mu$ is given by

$$\langle f \rangle = \int_{\Gamma} f \, d\mu$$

In general this scheme accommodates unsharp measurements (i.e. $\langle f^2 \rangle \neq \langle f \rangle^2$) with the (always) sharply measurable states forming a special subclass: the pure states.

**Lemma 2.4.** *For a compact Hausdorff space $K$, the Dirac point measures are the extreme points of $P(K)$ the weak* closed set of all probability measures on $K$.*

*Proof.* Firstly to show that $P(K)$ is weak* closed, let $\{\mu_n\}$ be a net in $P(K)$ that converges to $\mu$ in the weak* topology. Thus

$$\int_K f d\mu = \lim_{n\to\infty} \int_K f d\mu_n$$

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for all positive functions in $C(K)$ and in particular for the function $f(x) = 1$ which implies that

$$\begin{aligned}\mu(K) &= \int_K 1 d\mu \\ &= \lim_{n\to\infty} \int_K 1 d\mu_n \\ &= \lim_{n\to\infty} \mu_n(K) \\ &= 1 \qquad (\mu_n \in P(K) \Rightarrow \mu_n(K) = 1)\end{aligned} \tag{2.3}$$

Thus $\mu \in P(K)$ and $P(K)$ is weak* closed.

Next assume that for $x \in K$, the corresponding dirac point measure $\delta_x$ can be written as a convex combination of at least two other probability measures i.e. $\delta_x = \lambda\mu_1 + (1-\lambda)\mu_2$ where $\lambda \in (0,1)$. Then for any open set $U \subset K$ that does not contain the point $x$

$$0 = \delta_x(U) = \lambda\mu_1(U) + (1-\lambda)\mu_2(U)$$

But then $\mu_i(U) = 0$ by virtue of being a probability measure. But since this is true for any open set that does not contain $x$ then the $\mu_i = \delta_x$ everywhere on $K$ which is a contradiction to original assumption and $\delta_x$ is in fact an extreme point of $P(K)$.

Alternatively assume that $\mu \in P(K)$ is not a dirac point measure. Then there exists an open set $U \subset K$ such that $0 < \mu(U) = \alpha < 1$ and $0 < \mu(K \setminus U) = 1 - \alpha < 1$ then we can define two new probability measures on $K$ by restriction i.e. $\mu_1(V) = \frac{\mu(U\cap V)}{\alpha}$ and $\mu_2(V) = \frac{\mu((K\setminus U)\cap V)}{1-\alpha} = \frac{\mu(V\setminus U)}{1-\alpha}$. Additionally

$$\begin{aligned}\alpha\mu_1(V) + (1-\alpha)\mu_2(V) &= \mu(U\cap V) + \mu(V\setminus U) \\ &= \mu((U\cap V)\cup(V\setminus U))) \\ &= \mu(V)\end{aligned}$$

For all $V \subset K$ and hence $\mu$ cannot be an extreme point. □

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The pure states are thus the Dirac point measures while all other probability measures are the mixed states of the system. Since the Dirac point measures are (trivially) in one-to-one correspondence with the elements of $\Gamma$, it follows that if the state of a classical system is a pure state then the position and momentum of the system is precisely known. However if the state is a mixed state, then the probability measure that is associated with that state is not a Dirac measure i.e. there is a natural uncertainty with regards to the systems exact position and momentum.

**Definition 2.5.** (States of a C*-algebra )
A linear functional $\omega$ on a C*-algebra $\mathfrak{A}$ is called a *state* if it is both positive ($\omega(a^*a) \geq 0 \ \forall a \in \mathfrak{A}$) and normalized (i.e. $\omega(1) = 1$). $\mathfrak{S}_{\mathfrak{A}}$ denotes the collection of all states on $\mathfrak{A}$ and is called the *state space*. A state is called a *pure state* if it is an extreme point of $\mathfrak{S}_{\mathfrak{A}}$. The collection of all pure states on $\mathfrak{A}$ is denoted by $P_{\mathfrak{A}}$.

*Remark* 2.6. $\mathfrak{S}_{\mathfrak{A}}$ is trivially a closed subset of the closed unit ball of $\mathfrak{A}^*$, the dual space of $\mathfrak{A}$, and can thus be equipped with the weak* topology through inheritance. This can be used to show that $\mathfrak{S}_{\mathfrak{A}}$ is a compact convex Hausdorff space (see section 13.3 in [17]) and hence, by the Krein-Milman theorem, $P_{\mathfrak{A}}$ is not empty and $\mathfrak{S}_{\mathfrak{A}}$ is the convex hull of $P_{\mathfrak{A}}$.

While the definition of states for a C*-algebra initially seems at odds with the previous definition, they are in fact equivalent as a result of the following useful theorem whose proof can be found on page 40 of [20] or page 223 of [26]:

**Theorem 2.7.** *(Riesz-Markov Theorem) For a compact Hausdorff space $K$, there exists a one-to-one relationship between the positive regular Borel measures $\mu$ on $K$ and the positive linear functionals $\varphi$ on $C(K)$ given by*

$$\varphi(f) = \int_K f d\mu$$

The Riesz-Markov theorem is essential to the recasting of states of a classical system in terms of those on a C*-algebras and allows for the following:

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**Definition 2.8.** (States of a Classical System)
The state of a classical system is a normalized positive linear functional $\varphi$ on the unital abelian C*-algebra $\mathfrak{A}$ of observables.

C*-algebras can thus be naturally employed as the backbone for any classical system and the associated theory.

*Remark* 2.9. As previously noted, the assumption of compactness of $\Gamma$ is purely for convenience as it ensures that all observables (being continuous) are necessarily bounded. An alternative approach would be to consider observables as elements of the larger set of real-valued essentially bounded measurable functions on the measure space $(\Gamma, \Sigma, \mu)$, i.e. $L^\infty(\Gamma, \mu)$, which contains the spectral projections of all real measurable functions (including the unbounded ones) on $\Gamma$. The end result is still the same as $L^\infty(\Gamma, \mu)$ is a well known example of an abelian C*-algebra (see, for example, Chapter 4 in [16]).

## 2.2 Quantum Physical Systems

Classical and quantum systems seem very different when approached in the traditional scheme. This subsection seeks to address this and show that the two types of systems have a common parent structure i.e. a C*-algebra backbone with the only notable difference being that the abelian nature of a classical system falls away in a quantum one.

The traditional quantum system was built up around a set of axioms used to describe the states and observables. The original axioms were introduced by Dirac but Takhtajan's mathematical formulation appearing in [24] (and listed below) are equivalent and will be taken as the basis for a quantum system:

A1 A quantum mechanical system is associated with an infinite separable and complex Hilbert space $\mathscr{H}$.

A2 The observables of a quantum system are the self-adjoint linear operators on $\mathscr{H}$.

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A3 The states of a quantum system are the positive trace class operators $\rho$ with $\operatorname{Tr}\rho = 1$ (any operator of this form is referred to as a density matrix).

In general, a linear operator $T$ on Hilbert space $\mathscr{H}$ is not required to be defined on the entire space i.e. $D(T)$ the domain of $T$ is a linear subspace of $\mathscr{H}$. $\mathscr{L}(\mathscr{H})$ the set of all linear operators on $\mathscr{H}$ can be equipped with addition and multiplication operations defined by:

$$(T_1 + T_2)x = T_1x + T_2x \qquad \forall x \in D(T_1) \cap D(T_2)$$

$$(T_1T_2)x = T_1(T_2x) \qquad \forall x \in D(T_1T_2) = \{x \in T_2 : T_2x \in D(T_1)\}$$

The adjoint of the linear operator $T$ with $\overline{D(T)} = \mathscr{H}$ is the operator $T^*$ on the domain $D(T^*) := \{x \in \mathscr{H} : \exists x^* \in \mathscr{H}$ such that $\langle x, Ty\rangle = \langle x^*, y\rangle \quad \forall y \in \mathscr{H}\}$ defined by $T^*x = x^*$. A linear operator is self-adjoint if $D(T) = D(T^*)$ and $Tx = T^*x \quad \forall x \in D(T)$.

*Remark* 2.10. The adjoint of an operator is not defined for any element of $\mathscr{L}(\mathscr{H})$ but only for those with densely defined domains ($\overline{D(T)} = \mathscr{H}$) in order to ensure uniqueness.

The lack of a common domain for elements of $\mathscr{L}(\mathscr{H})$ presents serious problems associated with the combination of two elements. If two operators act on disjoint domains (except for the zero element which must be common to each domain) then necessarily the domain of the sum operator will then be the single value $\{0\}$ which essentially implies that the sum operator does not exist at all (i.e. it is not defined on a proper subset of $\mathscr{H}$). Additionally the sum of two self adjoint operators is not necessarily self adjoint:

**Example 2.2.1.** Consider a self-adjoint unbounded operator $T : D(T) \to \mathscr{H}$ where $D(T) \subset \mathscr{H}$ is the domain of $T$. Set $S = -T$ with domain equal to that of $T$. Then clearly $S$ is a self-adjoint unbounded operator but the sum of $S$ and $T$ is the null operator which is not self adjoint unless $D(T) = \mathscr{H}$ (or else $D((T+S)^*) = \mathscr{H}$ but $D(T+S) = D(T) \subset \mathscr{H}$).

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This issue can be avoided by requiring the operators to be bounded. An operator $T : D(T) \to \mathscr{H}$ is considered as bounded if there exists some positive real valued constant $c$ such that $\|Tx\| \leq c\|x\| \quad \forall x \in D(T)$ where the norm in this case is the standard operator norm. The set of all bounded linear operators on $\mathscr{H}$ is denoted by $B(\mathscr{H})$.

A consequence of the closed graph theorem is that $T \in B(\mathscr{H})$ if and only if $D(T) = \mathscr{H}$ and $T$ is closed [27]. Additionally $T \in B(\mathscr{H})$ if and only if $T$ is continuous at a single point of its domain (which in turn implies global continuity on its domain) and the existence of a continuous (and thus bounded) extension to the entire Hilbert space.

The argument to require boundedness is strengthened by noting that every unbounded operator can be approximated by bounded ones as a consequence of the following theorem (see Chapter 2, Theorem 1.1 of [24]):

**Theorem 2.11.** *(General Spectral Theorem of Self-Adjoint Elements)[J. von Neumann]*

*For every self-adjoint operator $T$ on the Hilbert space $\mathscr{H}$ there exists a unique projection-valued resolution of the identity $P(\lambda)$ such that for every continuous function $f$ on $\mathbb{R}$, $f(T)$ is a linear operator on $\mathscr{H}$ defined by*

$$f(T)x = \int_{-\infty}^{\infty} f(\lambda) dP(\lambda) x$$

*on the dense domain* $D(f(T)) := \left\{ x \in \mathscr{H} : \int_{-\infty}^{\infty} |f(\lambda)|^2 d(P(\lambda)x, x) < \infty \right\}$.

*Additionally:*

$$f(T)^* = \bar{f}(T)$$

$$f(T) \in B(\mathscr{H}) \Leftrightarrow f \text{ is bounded on } \sigma(T)$$

**Corollary 2.12.** *An unbounded self-adjoint operator can be approximated by bounded operators.*

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*Proof.* As a result of Theorem 2.11 $\chi_{[-n,n]}(T)$ is a bounded linear operator since the characteristic function $\chi_{[-n,n]}$ of the interval $[-n,n]$ is continuous and bounded on $\sigma(T)$ and

$$Tx = \left(\int_{-\infty}^{\infty} \lambda dP(\lambda)\right) x = \left(\lim_{n\to\infty} \int_{-n}^{n} \lambda dP(\lambda)\right) x$$
$$= \left(\lim_{n\to\infty} \int_{-\infty}^{\infty} \chi_{[-n,n]} \lambda dP(\lambda)\right) x = \left(\lim_{n\to\infty} \chi_{[-n,n]}(T)\right) x$$

□

As a consequence of this it makes sense to amend the second axiom to:

A2' The observables of a quantum system are the self-adjoint bounded linear operators on $\mathscr{H}$.

*Remark* 2.13. The alteration is not only justified on mathematical grounds to ensure that the operators are both continuous and defined on the entire domain (and thus the sum and multiplication are ensured to act on this entire domain as well), but also because any linear operator defined on a finite dimensional Hilbert space is necessarily bounded [18]. Thus the problem of unbounded self-adjoint operators is restricted to the infinite Hilbert space models, and even in these case it suffices, for any real system, to take a bounded approximation of the observable in question.

Thus, similar to the classical case, the operators of a quantum system are identifiable as the self-adjoint elements of the C*-algebra $B(\mathscr{H})$ (when equipped with the operators described previously), however unlike the classical case, $B(\mathscr{H})$ is not abelian.

To see that the traditional version of states are again linkable to the states on a C*-algebra, note that the Hilbert-Schmidt theorem links each density matrix $\rho$ to a (possibly infinite) set of orthonormal unit vectors $\{\psi_n\}$ such that

$$\rho = \sum_n \alpha_n P_{\psi_n} \qquad Tr\rho = \sum_n \alpha_n = 1$$

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where $P_{\psi_n}$ is the projection operator onto $\mathbb{C}\psi_n$ (which is a one dimensional subspace of $\mathscr{H}$) and $a_n \geq 0$ are the non-negative eigenvalues of $\rho$.

Hence it is natural to define the pure states of a quantum system as the projection operators onto one dimensional subspaces of $\mathscr{H}$. As an immediate consequence, a mixed state is always given by a convex linear combination of pure states and a state is pure if and only if it cannot be written as a non-trivial convex linear combination of states (see page 67 in [24]).

This identification of the pure states with projections onto one dimensional subspaces of $\mathscr{H}$ is equivalent to associating the pure states with unit vectors in $\mathscr{H}$, a popular alternative phrasing of the axioms of quantum mechanics (see, for example, Chapter 2 in [11]) for details).

For every unit vector, $\psi \in \mathscr{H}$, one can define a linear functional on the C*-algebra $\mathfrak{A} = B(\mathscr{H})$ by $\omega_\psi(T) := \mathrm{Tr}(P_\psi T) = (\psi, T(\psi))$. A functional defined in such a way is both positive and normalized:

$$\omega_\psi(T^*T) = (\psi, T^*T\psi) = (T(\psi), T(\psi)) = \|T\psi\|^2 \geq 0$$

By Proposition 2.3.11 in [28]:

$$\|\omega_\psi\| = \sup\{\omega_\psi(T^*T) : \|T\| = 1\} = \sup\{\|T\psi\|^2, \|T\| = 1\} \leq 1$$

since $\|T\psi\| \leq \|T\|\|\psi\| = 1$ for all $T \in B(\mathscr{H})$ such that $\|T\| \leq 1$. But then by considering the specific case of $T = P_\psi$ which (can be done since $\|P_\psi\| = 1$) then it is clear that $\|\omega_\psi\| = 1$ which is equivalent to $\omega_\psi(1) = 1$ by Corollary 13.6 in [17].

By the properties of the trace, it follows that any positive trace class operator $\rho$ with unit trace can be used to define a state on $B(\mathscr{H})$ by

$$\omega_\rho(T) = Tr(\rho T) = \sum \alpha_n(\psi_n, T\psi_n)$$

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i.e. every state of a quantum system is associated with a normalized positive linear functional on $B(\mathscr{H})$.

The converse link is not as straightforward and is a consequence of representation theory, a topic that is covered in Section 2.3.1 of [28] which is used as the main reference material for the subsequent discussion (unless stated otherwise):

**Definition 2.14.** (Representations of a C*-algebra )
A *representation* of a C*-algebra $\mathfrak{A}$ is a pair $(\mathscr{H}, \pi)$ where $\pi$ is a *-morphism of $\mathfrak{A}$ into $B(\mathscr{H})$.
A representation is called *faithful* if $\pi$ is a *-isomorphism from $\mathfrak{A}$ onto $\pi(\mathfrak{A})$.
A *cyclic representation* of $\mathfrak{A}$ is a triple $(\mathscr{H}, \pi, \Omega)$ consisting of a representation $(\mathscr{H}, \pi)$ and a vector $\Omega \in \mathscr{H}$ such that $span(\pi(a)\Omega : a \in \mathfrak{A})$ is dense in $\mathscr{H}$.

**Definition 2.15.** (Unitarily Equivalent Representations)
Two representations $(\mathscr{H}_1, \pi_1)$ and $(\mathscr{H}_2, \pi_2)$ of a C*-algebra $\mathfrak{A}$ are said to be *unitarily equivalent* if there exists a unitary operator $U : \mathscr{H}_1 \to \mathscr{H}_2$ such that:

$$\pi_1(a) = U^{-1}\pi_2(a)U \ \forall \ a \in \mathfrak{A}$$

**Definition 2.16.** (Normal States on a C*-algebra )
Let $(\mathscr{H}, \pi)$ be a representation of a C*-algebra $\mathfrak{A}$. A state is said to be $\pi$-normal if there exists a density matrix $\rho_\omega$ on $\mathscr{H}$ such that

$$\omega(a) = \mathrm{Tr}(\rho_\omega \pi(a)) \ \forall \ a \in \mathfrak{A}$$

**Theorem 2.17.** *(Canonical Cyclic Representation) For any normalised positive linear functional $\omega$ on a C*-algebra $\mathfrak{A}$, there exists a unique (up to unitary equivalence) cyclic representation $(\mathscr{H}_\omega, \pi_\omega, \Omega_\omega)$ of $\mathfrak{A}$ with $\|\Omega_\omega\| = \|\omega\| = 1$ such that*

$$\omega(a) = (\Omega_\omega, \pi_\omega(a)\Omega_\omega) \qquad \forall a \in \mathfrak{A}$$

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*Proof.* Set $I := \{a \in \mathfrak{A} : \omega(a^*a) = 0\}$ then $I$ is a left ideal in $\mathfrak{A}$ since for any $a \in \mathfrak{A}$ and $b \in I$ then:

$$0 \leq \omega((ab)^*(ab)) = \omega(b^*a^*ab) \leq \omega(\|a\|^2 b^*b) = \|a\|^2 \omega(b^*b) = 0$$

This ideal can be used to define an equivalence classes on $\mathfrak{A}$ by $[a] := \{c : c = a + b, b \in I\}$. The collection of all equivalence classes on $\mathfrak{A}$ with respect to $I$ can be endowed with a inner product defined by:

$$([a], [b]) = \omega(a^*b)$$

The completion of this collection is then a Hilbert space denoted by $\mathscr{H}_\omega$.

For each $a \in \mathfrak{A}$ a bounded linear operator $T_a$ on the equivalence classes can be defined by $T_a[b] = [ab]$.

$$T_a(\alpha[b] + \beta[c]) = T_a([\alpha b + \beta c]) = [\alpha ab + \beta ac] = \alpha[ab] + \beta[ac] = \alpha T_a([b]) + \beta T_a([c])$$

$$\|T_a([b])\|^2 = ([ab], [ab]) = \omega((ab)^*ab) = \|a\|^2 \omega(b^*b) = \|a\|\|[b]\|$$

Since the domain of this operator is dense in $\mathscr{H}_\omega$ then it extends to a bounded linear functional on $\mathscr{H}_\omega$ and without ambiguity $T_a$ will now denote this extension. The mapping $\pi_\omega : \mathfrak{A} \to B(\mathscr{H}_\omega)$ defined by $\pi_\omega(a) = T_a$ is then a $^*$-morphism:

$$\pi_\omega(\alpha a + \beta b)[c] = T_{\alpha a + \beta b}[c] = [\alpha ac + \beta bc] = \alpha T_a[c] + \beta T_b[c] = (\alpha \pi_\omega(a) + \beta \pi_\omega(b))\,[c]$$

$$\pi_\omega(ab)[c] = T_{ab}[c] = [abc] = T_a[bc] = T_a T_b[c] = \pi_\omega(a)\pi_\omega(b)[c]$$

Since $T_a$ is bounded then $T_a^*$ exists and is unique but

$$(T_{a^*}[b], [b]) = ([a^*b], b) = \omega((a^*b)^*b) = \omega(b^*ab) = ([b], [ab]) = ([b], T_a[b])$$

Hence $T_a^* = T_{a^*} \Rightarrow \pi_\omega(a^*) = (\pi_\omega(a))^*$.

Since $([1], \pi_\omega(a)[1]) = \omega(a)$ and $\pi_\omega(a)[1] = [a]$, then the set $span(\pi(a)\Omega : a \in \mathfrak{A})$

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is exactly the set of equivalence classes on $\mathfrak{A}$ with respect to $I$, which is dense in $\mathscr{H}_\omega$. By setting $\Omega_\omega := [1]$ the triple $(\mathscr{H}_\omega, \pi_\omega, \Omega_\omega)$ is a cyclic representation of $\mathfrak{A}$ with $\|\Omega_\omega\| = ([1],[1])^{1/2} = \omega(1^*1)^{1/2} = 1$.

Let a second cyclic representation $(\mathscr{H}_2, \pi_2, \Omega_2)$ exist such that

$$\omega(a) = (\Omega_2, \pi_2(a)\Omega_2) \qquad \forall a \in \mathfrak{A}$$

Then define $U : \mathscr{H}_\omega \to \mathscr{H}_2$ by $U(\pi_\omega(a)\Omega_\omega) = \pi_2(a)\Omega_2$. $U$ is well defined since it preserves the inner product on $\mathscr{H}_\omega$:

$$(U(\pi_\omega(a)\Omega_\omega), U(\pi_\omega(b)\Omega_\omega)) = (\pi_2(a)\Omega_2, \pi_2(b)\Omega_2) = \omega(a^*b) = (\pi_\omega(a)\Omega_\omega, \pi_\omega(b)\Omega_\omega)$$

Additionally since

$$(U^*U(\pi_\omega(a)\Omega_\omega), \pi_\omega(b)\Omega_\omega)) = (U(\pi_\omega(a)\Omega_\omega), U(\pi_\omega(b)\Omega_\omega)) = (\pi_\omega(a)\Omega_\omega, \pi_\omega(b)\Omega_\omega)$$

then $U^*U = I$. Similarly $UU^* = I$ hence U is unitary and $(\mathscr{H}_2, \pi_2, \Omega_2)$ is unitarily equivalent to $(\mathscr{H}_\omega, \pi_\omega, \Omega_\omega)$. □

Consider now the family of representations $\{\mathscr{H}_\omega, \pi_\omega\}_{\omega \in \mathfrak{S}_\mathfrak{A}}$ generated by the states on a C*-algebra $\mathfrak{A}$.

The direct sum of these Hilbert spaces is the space:

$$\mathscr{H} := \bigoplus_{\omega \in \mathfrak{S}_\mathfrak{A}} \mathscr{H}_\omega := \left\{ x = \{x_\omega\} \in \prod_{\omega \in \mathfrak{S}_\mathfrak{A}} \mathscr{H}_\omega : \sum_{\omega \in \mathfrak{S}_\mathfrak{A}} \|x_\omega\|^2_{\mathscr{H}_\omega} \leq \infty, x_\omega \in \mathscr{H}_\omega \right\}$$

where $\prod$ is the Cartesian product for sets and $\sum$ refers to the generalized summation.

*Remark* 2.18. The index set $\omega \in \mathfrak{S}_\mathfrak{A}$ may be countable or not but the finite subsets $F$ of $\mathfrak{S}_\mathfrak{A}$ form a directed set by inclusion in both cases. Since elements of $\mathscr{H}$ are the families of elements from all of the Hilbert spaces $\mathscr{H}_\omega$ then summation over

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$\omega \in \mathfrak{S}_{\mathfrak{A}}$ corresponds to

$$\sum_{\omega \in \mathfrak{S}_{\mathfrak{A}}} \|x_\omega\|^2_{\mathscr{H}_\omega} = \lim_F \sum_{\omega \in F} \|x_\omega\|^2_{\mathscr{H}_\omega}$$

(see page 46 in [28]).

$\mathscr{H}$ is a Hilbert space when equipped with the inner product

$$(x, y) = \sum_{\omega \in \mathfrak{S}_{\mathfrak{A}}} (x_\omega, y_\omega)_{\mathscr{H}_\omega}$$

where $(\cdot, \cdot)_{\mathscr{H}_\omega}$ refers to the inner product as defined on $\mathscr{H}_\omega$.

For every $a \in \mathfrak{A}$ define $\pi(a) : \mathscr{H} \to \mathscr{H}$ by $\pi(a)x = \{\pi_\omega(a)x_\omega\}$. Note that since $\pi(a)$ is trivially linear and setting $\|\pi(a)\| = \sup_{\omega \in \mathfrak{S}_{\mathfrak{A}}} \|\pi_\omega(a)\|$ then $\pi(a) \in B(\mathscr{H})$

Additionally the map $\pi : \mathfrak{A} \to B(\mathscr{H})$ is linear, preserves the involution ($\pi(a^*) = \pi(a)^*$), and is multiplicative ($\pi(ab) = \pi(a)\pi(b)$) and therefore a *-morphism such that

$$\begin{aligned} \pi(a) = 0 &\Rightarrow \pi_\omega(a) = 0 \ \forall \ \omega \in \mathfrak{S}_{\mathfrak{A}} \\ &\Rightarrow \omega(a^*a) = (\Omega_\omega, \pi_\omega(a^*a)\Omega_\omega) = (\pi_\omega(a)\Omega_\omega, \pi_\omega(a)\Omega_\omega) = 0 \\ &\Rightarrow a = 0 \end{aligned}$$

i.e. $\ker \pi = 0$ hence $\pi$ actually a *-isomorphism and $\{\mathscr{H}, \pi\}$ is a faithful representation of $\mathfrak{A}$.

**Theorem 2.19** (GNS construction)**.** *Every $C^*$-algebra $\mathfrak{A}$ has a faithful representation $(\mathscr{H}, \pi)$ such that the following are equivalent:*

*(1)* $\phi \in \mathfrak{S}_{\mathfrak{A}}$.

*(2)* $\exists$ *positive* $\rho \in B(\mathscr{H})$ *such that* $Tr\rho = 1$ *and* $\phi(a) = Tr[\rho\pi(a)]$.

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*Proof.* As indicated the representation $(\mathscr{H}, \pi) = \left(\bigoplus_{\omega\in\mathfrak{S}_\mathfrak{A}} \mathscr{H}_\omega, \bigoplus_{\omega\in\mathfrak{S}_\mathfrak{A}} \pi_\omega\right)$ is a faithful representation and the only thing that still needs to be proven is the equivalence relationship.

(2) $\Rightarrow$ (1) is immediate since

$$\phi_\rho(\cdot) := Tr[\rho\pi(\cdot)] = \sum_i (e_i, \rho\pi(\cdot)e_i)$$

where $\{e_i\}$ is an orthonormal basis for $\mathscr{H}$, is clearly a state on $\mathfrak{A}$.

(1) $\Rightarrow$ (2)

For each $\phi \in \mathfrak{S}_\mathfrak{A}$ (with canonical cyclic representation $\{\mathscr{H}_\phi, \pi_\phi, \Omega_\phi\}$) define the projection $P_\phi \in B(\mathscr{H}_\phi)$ by $P_\phi x = (\Omega_\phi, x)\Omega_\phi$.

Then $\phi(a) = (\Omega_\phi, \pi_\phi(a)\Omega_\phi) = Tr[P_\phi\pi_\phi(a)]$ when the trace is evaluated in an orthonormal basis containing $\Omega_\phi$. Trivially $TrP_\phi = \phi(1) = 1$ and $P_\phi \geq 0$ since it is a projection.

$P_\phi$ can be extended to a projection $\rho$ on $\mathscr{H}$ by setting $\rho$ to map the elements of the family $\{x_\omega\}_{\omega\in\mathfrak{S}_\mathfrak{A}}$ to zero except for the element from $\mathscr{H}_\phi$ upon which it acts like $P_\phi$.

Since $\pi(a)(\{x_\omega\}) = \{\pi_\omega(a)x_\omega\}$ then $(x, \rho\pi(a)x) = (x_\phi, P_\phi\pi_\phi(a)x_\phi)$. Thus by utilizing an orthonormal basis for $\mathscr{H}$ that contains $\{\Omega_\omega\}_{\omega\in\mathfrak{S}_\mathfrak{A}}$, it follows that $\phi(a) = (\Omega_\phi, \pi_\phi(a)\Omega_\phi) = Tr[\rho\pi(a)]$. Additionally $\rho$ inherits positivity as well as $Tr\rho = 1$ from $P_\phi$ as required.

□

*Remark* 2.20. The direct sum $\bigoplus_{\omega\in\mathfrak{S}_\mathfrak{A}} \mathscr{H}_\omega$ used in the GNS construction need not be taken over all of $\mathfrak{S}_\mathfrak{A}$ and can be limited to the canonical representations of the pure states over $\mathfrak{A}$ (i.e. $\bigoplus_{\omega\in P_\mathfrak{A}} \mathscr{H}_\omega$) since any mixed state is a convex combination of pure states and the operator associated with the mixed state will then be the convex combination of the operators associated with those of pure states and the convex combination operator will still obey the required properties [17].

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The GNS construction ensures that C*-algebras can naturally take over the pivotal role of Hilbert spaces in Quantum theory. While not only providing a new mathematical backdrop in which to work, this also shows explicitly that the classical and quantum realms have a common nature at their core. Moreover, it should be noted that a normal element of any C*-algebra can be used to generate an abelian C*-subalgebra (see page 41 in [16]) i.e. an abelian C*-algebra can be embedded as a C*-subalgebra in a larger non-abelian one. The converse statement is not possible: if a C*-algebra is abelian then by definition any C*-subalgebra must also be abelian. This ensures that a quantum theory based on C*-algebras will coincide with a classical one when necessary but a complete quantum theory cannot be embedded in a classical one.

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As a result of the GNS construction, the axioms used to describe a quantum system can be recast as:

**Axiom 2.21** (Axioms of Quantum Mechanics)**.**

*A1* *The observables of a quantum system are the self-adjoint elements of a non-commutative $C^*$-algebra $\mathfrak{A}$.*

*A2* *The states of a quantum system are the positive normalised linear functionals on $\mathfrak{A}$.*

*Remark* 2.22. It must be mentioned that the axioms given here are in no way intended to give a full mathematical description of a quantum theory, but instead show that the C*-algebraic approach forms a valid mathematical background to describe both quantum and classical systems. Any axioms that were not addressed (such as measurement and time evolution, see [24]) are beyond the scope of this dissertation and irrelevant to the purposes of this investigation and are thus omitted with the assumption that equivalent versions can be defined within the context of the C*-algebraic approach in the event that they are necessary.

The remainder of this section is dedicated to some useful properties of the pure states on a C*-algebra and their associated representations.

**Definition 2.23.** (Majorization)
For two positive linear functionals $\omega_1$ and $\omega_2$ on a C*-algebra $\mathfrak{A}$, $\omega_1$ *majorizes* $\omega_2$ if $\omega_1 - \omega_2$ is a positive linear functional. The set of all positive linear functionals on $\mathfrak{A}$ is a partially ordered set with respect to majorization (i.e. $\omega_1 \geq \omega_2$ if $\omega_1$ majorizes $\omega_2$)

**Lemma 2.24.** *A state $\omega$ on a $C^*$-algebra $\mathfrak{A}$ is pure if and only if the only positive linear functionals majorized by $\omega$ are scalar multiples of $\omega$*

*Proof.* Suppose $\omega$ is a pure state and let $\tau$ be a positive linear functional such that $\tau \leq \omega$. Then $0 \leq \tau(1) \leq \omega(1) = 1$.

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By Proposition 4.2.3 (ii) in [29], for any self-adjoint $a \in \mathfrak{A}$, there exists a scalar $c$ such that $-c1 \leq a \leq c1$. Hence if $\tau(1) = 0$ then $0 = \tau(-c1) \leq \tau(a) \leq \tau(c1) = 0$ for any self-adjoint $a \in \mathfrak{A}$. Since any element in $\mathfrak{A}$ is a linear combination of self-adjoint elements in $\mathfrak{A}$, then $\tau(1) = 0$ implies that $\tau = 0 = 0\omega$. Similarly if $\tau(1) = 1$ then the same argument using $\omega - \tau$ implies that $\tau = \omega$.

If $0 < \tau(1) < 1$ then set $\omega_1 = (1-b)^{-1}(\omega - \tau)$ and $\omega_2 = b^{-1}\tau$ where $b = \tau(1)$, so that

$$\omega = (1-b)\omega_1 + b\omega_2$$

But by purity of $\omega$ then $\omega = \omega_2 \Rightarrow \tau = b\omega$.

Conversely assume that the only positive linear functionals majorized by $\omega$ are scalar multiples of $\omega$. Write $\omega$ as a convex combination of two states: $\omega = (1-b)\omega_1 + b\omega_2$. Clearly $(1-b)\omega_1 \leq \omega$ and $b\omega_2 \leq \omega$ hence $(1-b)\omega_1 = a_1\omega$ and $b\omega_2 = a_2\omega$. But $\omega(1) = \omega_1(1) = \omega_2(1) = 1$ hence $a_1 = (1-b)$ and $a_2 = b$ so that $\omega = \omega_1 = \omega_2$ so $\omega$ is pure. □

*Remark* 2.25. Some books (such as [28]) exploit the if and only if nature of Lemma 2.24 to use majorization to define pure states. Throughout this dissertation, a pure state is taken to be an extreme point of $\mathfrak{S}_{\mathfrak{A}}$ as per Definition 2.5, while Lemma 2.24 is used as a convenient characteristic feature of the pure states.

The extreme point and majorization versions of identifying pure states are both based on the comparison of a particular state with the other states. An alternative is through identifying properties uniquely held by the canonical cyclic representations of the pure states.

**Definition 2.26.** (Irreducible Representations)
A representation $\{\pi, \mathscr{H}\}$ of a C*-algebra $\mathfrak{A}$ is called *irreducible* if the only subspaces of $\mathscr{H}$ that are invariant under $\pi(\mathfrak{A})$ are the trivial subspaces $\mathscr{H}$ and $\{0\}$ (i.e. $\pi(\mathfrak{A})\mathscr{K} \subseteq \mathscr{K}$ for $\mathscr{K} \subseteq \mathscr{H}$ if and only if $\mathscr{K} = \mathscr{H}$ or $\{0\}$) or equivalently if $\pi(\mathfrak{A})'$ the commutant of $\pi(\mathfrak{A})$ consists only of multiples of the identity operator (see Chapter 1 Proposition 9.20 in [30]).

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Segal recognised that irreducible representations provide another method to determine purity of a state (see Theorem 2.3.19 in [28]):

**Theorem 2.27.** *A state $\omega$ on a $C^*$-algebra $\mathfrak{A}$ is pure if and only if its associated canonical cyclic representation $\{\mathscr{H}_\omega, \pi_\omega, \Omega_\omega\}$ is irreducible.*

**Proposition 2.28.** *A $C^*$-algebra $\mathfrak{A}$ is abelian if and only if every irreducible representation of $\mathfrak{A}$ is one dimensional.*

*Proof.* If $\mathfrak{A}$ is abelian, then $\pi(\mathfrak{A}) \subseteq \pi(\mathfrak{A})'$ for every representation of $\mathfrak{A}$. Thus $\pi(\mathfrak{A})$ consists entirely of mulitples of the identity if and only if $\mathscr{H}$ is one dimensional. Conversely then, if there exists an irreducible representation $(\pi, \mathscr{H})$ of $\mathfrak{A}$ where $\mathscr{H}$ is not one dimensional, then $\mathfrak{A}$ cannot be abelian. □

## 2.3 Composite Physical Systems

### 2.3.1 Tensor Products

The focus of this subsection is to introduce the tensor product: a method of constructing new spaces by combining a collection of individual spaces, a concept which is essential when investigating composite systems. For additional details and a more complete treatment, the reader is referred to the source material for this section: Chapter 6 in [16] and Chapter 4 in [30].

**Definition 2.29.** (Algebraic Tensor Product)
Given a finite family of vector spaces $X_1, \ldots, X_n$ there exists a unique vector space, called the *algebraic tensor product* of $X_1, \ldots, X_n$ and denoted by $\bigodot_{i=1}^{n} X_i$, that is spanned by elements $\otimes_i x_i$ which are created multilinearly from each family $\{x_i \in X_i\}_{i=1}^n$ such that:

1) For each multilinear map $\varphi$ from the Cartesian product space $X_1 \times \cdots \times X_n$ to a vector space $Y$, there exists a unique linear map $\tilde{\varphi} : \bigodot_{i=1}^{n} X_i \to Y$ such that $\tilde{\varphi}(\otimes_i x_i) = \varphi(\{x_i\})$.

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2) For each partition $\bigcup_k I_k$ of $\{1,\ldots,n\}$ there exists a unique isomorphism from $\bigodot_{i=1}^{n} X_i$ onto $\bigodot_k \left( \bigodot_{i\in I_k} X_i \right)$ transforming $\otimes_i x_i$ into $\otimes_k(\otimes_{i\in I_k} x_i)$.

If the $X_i^{'s}$ are Hilbert spaces $\mathscr{H}_i$ then the linear extension of

$$(\otimes x_i, \otimes y_i) = \prod_{i=1}^{n}(x_i, y_i)$$

defines a unique inner product on $\bigodot_{i=1}^{n} \mathscr{H}_i$. The completion of $\bigodot_{i=1}^{n} \mathscr{H}_i$ under the associated norm is called the Hilbert space tensor product of the $\mathscr{H}_i^{'s}$ and is denoted by $\bigotimes_{i=1}^{n} \mathscr{H}_i$.

**Definition 2.30.** (Cross Norms)
For a family of Banach spaces $X_1,\ldots,X_n$, a norm $\gamma$ on $\bigodot_{i=1}^{n} X_i$ is called a *cross norm* if

$$\|x\|_\gamma = \prod_{i=1}^{n} \|x_i\|_i$$

for every $x = \otimes_i x_i$ in $\bigodot_{i=1}^{n} X_i$ where $\|.\|_i$ refers to the norm on $X_i$.

For a family of C*-algebras $\mathfrak{A}_i,\ldots,\mathfrak{A}_n$, the space $\bigodot_{i=1}^{n} \mathfrak{A}_i$ can uniquely be made into a *-algebra by setting:

$$(\otimes_i a_i)(\otimes_i b_i) = \otimes_i a_i b_i$$

$$(\otimes_i a_i)^* = \otimes_i a_i^*$$

**Definition 2.31.** (C*-norms and the Tensor Product of C*-algebras)
For a family $\mathfrak{A}_i,\ldots,\mathfrak{A}_n$ of C*-algebras , a norm $\beta$ on $\bigodot_{i=1}^{n} \mathfrak{A}_i$ is called a *C*-norm* if $\|ab\|_\beta \leq \|a\|_\beta \|b\|_\beta$ and $\|a^*a\|_\beta \leq \|a\|_\beta^2$. The completion of $\bigodot_{i=1}^{n} \mathfrak{A}_i$ with respect to the C*-norm $\beta$ is called the *tensor product of* $\mathfrak{A}_i,\ldots,\mathfrak{A}_n$ *with respect to* $\beta$ and is denoted by $\bigotimes_{i=1}^{n}{}_\beta \mathfrak{A}_i$.

While multiplication and involution are uniquely defined, there is (in general) more than one possible C*-norm on $\bigodot_{i=1}^{n} \mathfrak{A}_i$ which can be used to transform this *-algebra into a C*-algebra.

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**Definition 2.32.** (Projective Tensor Product)
The *projective C*-cross-norm* $\|\cdot\|_{max}$ on $\bigodot_i \mathfrak{A}_i$ is given by

$$\|a\|_{max} := \sup\{\|\pi(a)\| : \pi \text{ runs through all representations of } \bigodot_i \mathfrak{A}_i\}$$

$\bigotimes_{max} \mathfrak{A}_i$ denotes the completion of $\bigodot_i \mathfrak{A}_i$ under $\|\cdot\|_{max}$ and is called the *projective tensor product of* $\{\mathfrak{A}_i\}_{i=1}^n$.

**Definition 2.33.** (Injective Tensor Product)
The *injective C*-cross-norm* $\|\cdot\|_{min}$ on $\bigodot_i \mathfrak{A}_i$ is given by

$$\|a\|_{min} := \sup\{\|(\otimes_i \pi_i)(a)\| : \pi_i \text{ is a represntation of } \mathfrak{A}_i\}$$

Where $(\otimes_i \pi_i)(a) = \sum_{k=1}^n \otimes_i (\pi_i(a_{i,k}))$ for $a = \sum_{k=1}^n \otimes_i a_{i,k} \in \bigodot_i \mathfrak{A}_i$.

$\bigotimes_{min} \mathfrak{A}_i$ denotes the completion of $\bigodot_i \mathfrak{A}_i$ under $\|\cdot\|_{min}$ and is called the *injective C*-tensor product of* $\{\mathfrak{A}_i\}_{i=1}^n$.

The projective and injective C*-cross-norms are the extreme C*-norms in the sense that for any C*-norm $\beta$ on $\bigodot_i \mathfrak{A}_i$

$$\|a\|_{min} \leq \|a\|_\beta \leq \|a\|_{max} \qquad a \in \bigodot_i \mathfrak{A}_i$$

As a result every C*-norm is a cross norm. More importantly, if the projective and injective norms agree on $\bigodot_i \mathfrak{A}_i$, the C*-algebra completion of $\bigodot_i \mathfrak{A}_i$ is uniquely defined.

**Definition 2.34.** (Nuclear C*-algebras )
A C*-algebra $\mathfrak{A}$ is called *nuclear* if, for each C*-algebra $\mathfrak{B}$, there is only one C*-norm on $\mathfrak{A} \bigodot \mathfrak{B}$ such that its completion is a C*-algebra.

Particularly important examples of nuclear C*-algebras are the abelian C*-algebra and the finite C*-algebras.

**Definition 2.35.** (Positivity in the Algebraic Tensor Product of C*-algebras )
An element $a \in \bigodot_{i=1}^n \mathfrak{A}_i$ is *positive*, denoted by $a \geq 0$ if $a = \sum_{j=1}^m a_j^* a_j$ where $a_j \in \bigodot_{i=1}^n \mathfrak{A}_i$.

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The collection of all positive elements in $\bigodot_{i=1}^{n} \mathfrak{A}_i$ is denoted by $\left(\bigodot_{i=1}^{n} \mathfrak{A}_i\right)^+$. A linear functional $\omega$ on $\bigodot_{i=1}^{n} \mathfrak{A}_i$ is said to be *positive*, written $\omega \geq 0$, if $\omega(a) \geq 0$ for all $a \in \bigodot_{i=1}^{n} \mathfrak{A}_i$.

Under this definition $(\bigodot_i \mathfrak{A}_i)^+$ is a cone which spans $(\bigodot_i \mathfrak{A}_i)^+$ and is contained in $(\bigotimes_\beta \mathfrak{A}_i)^+$, the set of all positive elements of $\bigotimes_\beta \mathfrak{A}_i$, for any C*-norm $\beta$.

In addition, any positive linear functional $\omega$ preserves the involution and obeys the Cauchy-Schwarz inequality:

$$|\omega(a^*b)|^2 \leq \omega(a^*a)\omega(b^*b) \text{ for all } a, b \in \bigodot_{i=1}^{n} \mathfrak{A}_i$$

The dual space $\mathfrak{A}$ of any C*-algebra is a Banach space hence the algebraic tensor product of a family $\{\mathfrak{A}_i^*\}_{i=1}^n$ of dual spaces of C*-algebras is well defined. An element $\otimes_i \phi_i \in \bigodot_i \mathfrak{A}_i^*$, which is defined by

$$(\otimes_i \phi_i)(\otimes_i a_i) = \prod_{i=1}^{n} \phi_i(a_i) \text{ for all } a_i \in \mathfrak{A}_i,\ 1 \leq i \leq n$$

is a positive linear functional on $\bigodot_i \mathfrak{A}_i$ if $\phi_i$ is a positive linear functional on $\mathfrak{A}_i$.

**Definition 2.36.** (Adjoint Cross Norm)
Any C*-norm $\beta$ induces a cross-norm $\beta^*$ on $\bigodot_i \mathfrak{A}_i^*$ defined by

$$\|f\|_{\beta^*} := \sup\{|f(a)| : a \in \otimes_i \mathfrak{A}_i, \|a\| \leq 1\}$$

$\beta^*$ is called the *adjoint cross-norm* of $\beta$.

**Proposition 2.37.** *The adjoint cross-norms of the injective and projective C*-cross-norms agree on* $\bigodot_i \mathfrak{A}_i^*$.

This implies that the completion of $\bigodot_i \mathfrak{A}_i^*$ under either of these adjoint cross-norms will be the same Banach space which will be denoted by $\bigotimes_i \mathfrak{A}_i^*$. In particular, this implies that $\bigotimes_i \mathfrak{A}_i^*$ is an invariant subspace of both $(\bigotimes_{min} \mathfrak{A}_i)^*$ and $(\bigotimes_{max} \mathfrak{A}_i)^*$

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The issue of norm choice is restricted to combinations of multiple infinite non-abelian C*-algebras. Which norm is chosen depends on the properties required of the composite space. For the remainder of this dissertation, the injective C*-tensor product of algebras will be used as the primary norm choice unless otherwise stated and $\bigotimes_i \mathfrak{A}_i$ will be used as shorthand notation for $\bigotimes_{min} \mathfrak{A}_i$.

*Remark* 2.38. The adoption of the injective tensor product as the default is motivated by two important properties (see Chapter 3 in [31] for details):

- Every linear functional $\otimes_i f_i$ on $\bigodot_i \mathfrak{A}_i$ (where $f_i \in \mathfrak{A}_i^*$) can be uniquely extended to a linear functional on $\bigotimes_{min} \mathfrak{A}_i$ with the norm $\prod_i \|f_i\|$.
- $\bigodot_i \mathfrak{A}_i^*$ is weakly* dense in $(\bigotimes_{min} \mathfrak{A}_i)^*$.

### 2.3.2 Classical Composite Systems

In the traditional classical scheme, composite systems are formed from two subsystems in a trivial manner, i.e. if $\Gamma_1$ and $\Gamma_2$ represent the phase spaces of two subsystems, then $\Gamma_1$ holds no influence over $\Gamma_2$ (and vice versa). This is mathematically realized by setting the phase space of the composite system as the Cartesian product $\Gamma_1 \times \Gamma_2$ of the subsystems' phase spaces. The observables are then the self-adjoint elements of $C(\Gamma_1 \times \Gamma_2)$ as per a single classical system.

**Theorem 2.39.** *For compact Hausdorf spaces $X, Y$, $C(X) \otimes C(Y)$ is $^*$-isomorphic to $C(X \times Y)$.*

*Proof.* For every simple tensor $f \otimes g$ where $f \in C(X), g \in C(Y)$ define $g_f : Y \to C(X)$ by $g_f(y) = g(y)f$. The map $f \otimes g \to g_f$ is a $^*$-isomorphism hence $C(X) \otimes C(Y)$ is $^*$-isomorphic to $C(Y, C(X))$ the C*-algebra of all norm continuous functions from $Y$ into $C(X)$.

For $g_f \in C(Y, C(X))$ set

$$\varphi : C(Y, C(X)) \to C(X \times Y) \qquad (\varphi(g_f))(x, y) = (g_f(y))\, x = g(y)f(x)$$

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It is trivial to check that $\varphi$ is a $^*$-isomorphism from $C(Y, C(X))$ onto $C(X \times Y)$. □

Theorem 2.39 concludes that, in a C*-algebraic formulation, the observables of a classical composite system formed from two classical subsystems is given by the tensor product of $\mathfrak{A}_1 \otimes \mathfrak{A}_2$ of the abelian C*-algebras representing the observables on the individual subsystems $\mathfrak{A}_1$ and $\mathfrak{A}_2$. This also leads to a natural recovery of the independence of pure states utilized utilized in the traditional approach i.e. taking the Cartesian product of the phase spaces consequently ensured that the pure states of the composite system are given by the product of pure states of the subsystems.

*Remark* 2.40. $\mathfrak{A}_1 \otimes \mathfrak{A}_2$ is a uniquely defined abelian C*-algebra since all abelian C*-algebra are nuclear and

$$(a_1 \otimes a_2)(b_1 \otimes b_2) = a_1 b_1 \otimes a_2 b_2 = b_1 a_1 \otimes b_2 a_2 = (b_1 \otimes b_2)(a_1 \otimes a_2)$$

For all $a_1, b_1 \in \mathfrak{A}_1$ and $a_2, b_2 \in \mathfrak{A}_2$.

**Definition 2.41.** (Center of a C*-algebra )
The *Center* of a C*-algebra $\mathfrak{A}$ is the abelian C*-subalgebra[i] $Z_{\mathfrak{A}} := \{z \in \mathfrak{A} : za = az \quad \forall a \in \mathfrak{A}\}$

**Lemma 2.42.** *For any pure state $\omega$ acting on C*-algebra $\mathfrak{A}$*

$$\omega(az) = \omega(a)\omega(z) \quad \forall a \in \mathfrak{A}, z \in Z_{\mathfrak{A}}$$

*Proof.* Since any element of a C*-algebra can be written as a linear combination of at most four positive elements, it suffices to only consider the cases where $0 \leq z \leq 1$ (the latter inequality being ensured by using $z' = z/\|z\|$ in the event that $z > 1$).

Since $z \in \mathfrak{A}^+$, then $z = c^*c$ for some $c \in \mathfrak{A}$ and so for any $a = b^*b \in \mathfrak{A}^+$

$$az = b^*bz = b^*zb = b^*c^*bc = (cb)^*(cb)$$

[i](page 50 in [30])

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Thus $az \geq 0$. Since $1 - z \in \mathfrak{A}^+$, it follows that $a - az \geq 0$ by the same argument. Hence $0 \leq az \leq a$ which implies that $0 \leq \omega(az) \leq \omega(a)$ (as a consequence of Theorem 13.11(c) in [17]) for any $\omega \in P_{\mathfrak{A}}$.

Define a positive linear functional on $\mathfrak{A}$ by $\omega_0(a) = \omega(az)$. Trivially it follows that $\omega_0 \leq \omega$ and hence by Lemma 2.24 it follows that $\omega_0 = \lambda\omega$ for some $\lambda \in \mathbb{C}$ thus

$$\omega(az) = \omega_0(a) = \lambda\omega(a) = \lambda\omega(1)\omega(a) = \omega_0(1)\omega(a) = \omega(z)\omega(a)$$

□

**Corollary 2.43.** *A state $\omega$ on an abelian C*-algebra $\mathfrak{A}$ is pure if and only if it is multiplicative:*

$$\omega(ab) = \omega(a)\omega(b) \qquad \forall\ a, b \in \mathfrak{A}$$

*Proof.* When $\mathfrak{A}$ is abelian, the Center of $\mathfrak{A}$ is the entire algebra (i.e. $Z_{\mathfrak{A}} = \mathfrak{A}$) and multiplicity of pure states follows immediately from Lemma 2.42.

Conversely if a state $\omega$ on $\mathfrak{A}$ is multiplicative, let $\tau$ be a positive linear functional majorized by $\omega$.

If $a \in \ker(\omega)$ then $\omega(a^*a) = \omega(a^*)\omega(a) = 0$ and as a result $0 \leq \tau(a^*a) \leq \omega(a^*a) = 0$. By the Cauchy-Schwarz inequality $|\tau(a)|^2 \leq \tau(1)\tau(a^*a) \leq \tau(a^*a) = 0$ hence $\tau(a) = 0$ i.e. $\ker(\omega) \subseteq \ker(\tau)$. But since $0 < \tau \leq \omega$ then $\ker(\omega) = \ker(\tau)$.

If $\tau \neq 0$ then there exists an element $xin\mathfrak{A}$ such that $\tau(x) \neq 0$. Without loss of generality, it can be assumed that $\tau(x) = 1$ (by using $x/\|x\|$ if required). Then for any arbitrary $b \in \mathfrak{A}$, it follows that

$$\tau(b - \tau(b)x) = 0$$

Thus $\omega(b - \tau(b)x) = 0$ hence

$$\omega(b) = \omega(b - \tau(b)x + \tau(b)x) = \tau(b)\omega(x) = \lambda\tau(b)$$

for all $b \in \mathfrak{A}$. Thus $\omega$ is pure by Lemma 2.24. □

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**Theorem 2.44.** *If either* $\mathfrak{A}_1$ *or* $\mathfrak{A}_2$ *are abelian, then for any* $\omega \in P_{\mathfrak{A}_1 \otimes \mathfrak{A}_2}$ *there exists pure states* $\omega_1 \in P_{\mathfrak{A}_1}$ *and* $\omega_2 \in P_{\mathfrak{A}_2}$ *such that*

$$\omega = \omega_1 \otimes \omega_2$$

*Proof.* Let $\mathfrak{A}_1$ be abelian. Clearly $\mathfrak{A}_1 \otimes I_2 \subseteq Z_{\mathfrak{A}_1 \otimes \mathfrak{A}_2}$ hence Corollary 2.43 establishes that

$$\omega(a_1 \otimes a_2) = \omega(a_1 \otimes I_2)\omega(I_1 \otimes a_2)$$

Define $\omega_1(a_1) := \omega(a_1 \otimes I_2)$ and $\omega_2(a_2) := \omega(I_1 \otimes a_2)$ for all $a_1 \in \mathfrak{A}_1$ and $a_2 \in \mathfrak{A}_2$ respectively so that

$$\omega = \omega_1 \otimes \omega_2$$

Clearly $\omega_1 \in P_{\mathfrak{A}_1}$ by Corollary 2.43.

Let $\phi_1$ and $\phi_2$ be two states on $\mathfrak{A}_2$ such that

$$\omega_2 = \lambda\phi_1 + (1-\lambda)\phi_2$$

for some $\lambda \in (0,1)$. By the Cauchy-Schwarz inequality, for any $a \in \mathfrak{A}_1 \bigodot \mathfrak{A}_2$

$$\begin{aligned} |(\omega_1 \otimes \phi_1)(a)|^2 &\leq (\omega_1 \otimes \phi_1)(a^*a) && (2.4)\\ &\leq \left(\frac{1}{\lambda} + \frac{1}{1-\lambda}\right)\omega(a^*a) && (2.5)\\ &\leq \left(\frac{1}{\lambda} + \frac{1}{1-\lambda}\right)\|a\|^2_{min} && (2.6) \end{aligned}$$

Thus $\omega_1 \otimes \phi_1$ can be uniquely extended to a state $\varphi_1$ on $\mathfrak{A}_1 \bigotimes \mathfrak{A}_2$. Similarly, by repeating this argument, $\omega_1 \otimes \phi_2$ extends uniquely to a state $\varphi_2$ on $\mathfrak{A}_1 \bigotimes \mathfrak{A}_2$.

Clearly $\omega = \lambda\varphi_1 + (1-\lambda)\varphi_2$ but, by purity of $\omega$, then $\omega = \varphi_1 = \varphi_2$ which implies that $\omega_2 = \phi_1 = \phi_2$ i.e. $\omega_2$ is a pure state on $\mathfrak{A}_2$. □

Theorems 2.39 and 2.44 imply that when forming a composite (or bipartate) systems from classical subsystems, the tensor product is used for both states and observables. Additionally the resultant system is classical in nature and pure

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states of the composite system are in a one-to-one relationship with the simple tensors of pure states of the subsystems so that the independance of the subsystems present in the traditional formulation is retained (a property that will be shown to be a key difference in the quantum setting).

### 2.3.3 Quantum Composite Systems

In the traditional approach, the Hilbert space of a composite system created from two systems with associated Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$ is axiomatically defined as the tensor product of Hilbert spaces associated with the individual subsystems i.e. $\mathscr{H} = \mathscr{H}_1 \otimes \mathscr{H}_2$ [24]. The axioms for observables and states are the same as for the single system case except applied to the tensor product space $\mathscr{H}$ (i.e. the self-adjoint elements of $B(\mathscr{H}_1 \otimes \mathscr{H}_2)$ are the observables and the states are the positive trace class operators with unit trace).

The ability to represent the space of allowed observables as the tensor product of the individual observable spaces associated with each subsystem is again realizable in the traditional quantum setting i.e.

$$B(\mathscr{H}_1 \otimes \mathscr{H}_2) = B(\mathscr{H}_1) \otimes B(\mathscr{H}_2)$$

(see the special application of Proposition 1.6 on page 185 of [30]).

This concept holds true in the C*-algebra formulation as a result of the following theorem (see Theorem 4.9 on page 208 in [30]):

**Theorem 2.45.** *If two C*-algebras $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are faithfully represented by $(\mathscr{H}_1, \pi_1)$ and $(\mathscr{H}_2, \pi_2)$ respectively, then $(\mathscr{H}, \pi) := (\mathscr{H}_1 \otimes \mathscr{H}_2, \pi_1 \otimes \pi_2)$ is a faithful representation of $\mathfrak{A}_1 \otimes \mathfrak{A}_2$.*

Theorem 2.45 stipulates that the C*-algebra approach to quantum mechanics respects (and is equivalent to) the traditional approach to observables of composite quantum systems but while classical and quantum systems are treated differently

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in the traditional approach, the C*-algebra approach treats both in the same way except for the additional assumption of abelianness in the classical case. This means that any difference in the two realms is a consequence of this abelian (or nonabelian) nature of the algebra of observables and readily allows for the creation of composite systems from a combination of both classical and quantum systems (a concept which is not readily apparent in the traditional approach).

While the pure states of a C*-subalgebra can always be extended to a pure state on the parent C*-algebra (see Proposition 2.3.24 in [28]), the converse (i.e. that every restriction of a pure state to a C*-subalgebra is a pure state on the C*-subalgebra) is only true for very specific cases (see Proposition 4.2.37 in [28]).

**Proposition 2.46.** *Let $\mathfrak{B}$ and $\mathfrak{C}$ be C*-subalgebras of the C*-algebra $\mathfrak{A}$ such that every element of $\mathfrak{B}$ commutes with every element of $\mathfrak{C}$ and $\mathfrak{B} \cup \mathfrak{C}$ generates $\mathfrak{A}$ as a C*-algebra. Then $P_{\mathfrak{B}} \subseteq P_{\mathfrak{A}}|_{\mathfrak{B}}$ with equality only when $\mathfrak{C}$ is abelian.*

As a consequence of Theorem 2.44, the pure states of a composite classical system could always be written as a simple tensor of pure states associated with the subsystems. The same is not true when considering a composite quantum system.

**Theorem 2.47.** *For two C*-algebras $\mathfrak{A}_1$ and $\mathfrak{A}_2$ every pure state $\omega \in \mathfrak{S}_{\mathfrak{A}_1 \otimes \mathfrak{A}_2}$ can be written in the form $\omega = \omega_1 \otimes \omega_2$ for some pure states $\omega_1 \in \mathfrak{S}_{\mathfrak{A}_1}$ and $\omega_2 \in \mathfrak{S}_{\mathfrak{A}_2}$ if and only if either $\mathfrak{A}_1$ or $\mathfrak{A}_2$ is abelian.*

*Proof.* The forward "if" direction has already been proven in Theorem 2.44. To prove the converse, assume that $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are not abelian and let $\phi_1 \in P_{\mathfrak{A}_1}$ and $\phi_2 \in P_{\mathfrak{A}_2}$ with associated canonical cyclic representations $\{\mathscr{H}_1, \pi_1, \Omega_1\}$ and $\{\mathscr{H}_2, \pi_2, \Omega_2\}$ respectively. By Theorem 2.27, both of these representations are irreducible. Thus it can be assumed, without loss of generality, that neither $\mathscr{H}_1$ nor $\mathscr{H}_2$ are one dimensional and there exists at least two orthonormal unit vectors $\{\epsilon_1, \epsilon_2\}$ and $\{\eta_1, \eta_2\}$ in $\mathscr{H}_1$ and $\mathscr{H}_2$ respectively (Proposition 2.28). The commutation theorem establishes that $\{\mathscr{H}, \pi\} = \{\mathscr{H}_1 \otimes \mathscr{H}_2, \pi_1 \otimes \pi_2\}$ is an irreducible

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representation of $\mathfrak{A}_1 \otimes \mathfrak{A}_2$ (see Prop 11.3.2 in [32]). The state define by

$$\omega(a_1 \otimes a_2) = (\Omega, \pi_1(a_1) \otimes \pi_2(a_2)\Omega)$$

$$\Omega = \frac{1}{\sqrt{2}}(\epsilon_1 \otimes \eta_1 + \epsilon_2 \otimes \eta_2)$$

is a pure state by Theorem 2.27.

If $\omega = \omega_1 \otimes \omega_2$ for some $\omega_1 \in P_{\mathfrak{A}_1}$ and $\omega_2 \in P_{\mathfrak{A}_2}$ as required, then $\omega(a_1 \otimes a_2) = \omega(a_1 \otimes 1_{\mathfrak{A}_1})\omega(1_{\mathfrak{A}_1} \otimes a_2)$ implies

$$(\Omega, \pi_1(a_1) \otimes \pi_2(a_2)\Omega) = \left(\Omega, \pi_1(a_1) \otimes 1_{B(\mathscr{H}_2)}\Omega\right)\left(\Omega, 1_{B(\mathscr{H}_2)} \otimes \pi_2(a_2)\Omega\right)$$

Thus for any $T_1 \in B(\mathscr{H}_1)$ and $T_2 \in B(\mathscr{H}_2)$

$$(\Omega, T_1 \otimes T_2\Omega) = \left(\Omega, T_1 \otimes 1_{B(\mathscr{H}_2)}\Omega\right)\left(\Omega, 1_{B(\mathscr{H}_1)} \otimes T_2\Omega\right) \qquad (*)$$

In particular, consider the projections $P_{\epsilon_1}$ of $\mathscr{H}_1$ onto $\epsilon_1$ and $P_{\eta_2}$ of $\mathscr{H}_2$ onto $\eta_2$, for which $(P_{\epsilon_1} \otimes P_{\eta_2})\Omega = 0$. But $(*)$ implies $(\Omega, P_{\epsilon_1} \otimes P_{\eta_2}\Omega) = \frac{1}{4}$ which is a contradiction, thus $\omega$ cannot be written in the form $\omega_1 \otimes \omega_2$ □

**Definition 2.48.** (Product State)
For C*-algebras $\mathfrak{A}_1$ and $\mathfrak{A}_2$, any state on $\mathfrak{A}_1 \otimes \mathfrak{A}_2$ that can be written in the form $\omega_1 \otimes \omega_2$, where $\omega_1$ and $\omega_2$ are states on $\mathfrak{A}_1$ and $\mathfrak{A}_2$ respectively, will be called a *product state*. The set of all product states is denoted by $\mathfrak{S}_{\mathfrak{A}_1} \times \mathfrak{S}_{\mathfrak{A}_2}$

*Remark* 2.49. This is a slight abuse of notation as $V_1 \times V_2$ usually refers to the Cartesian product space consisting of pairs $(v_1, v_2)$ where $v_1 \in V_1$ and $v_2 \in V_2$ respectively. However, $V_1 \times V_2$ can be identified with the space of simple tensors by the map

$$(v_1, v_2) \to v_1 \otimes v_2$$

Theorem 2.47 establishes that the state space of the composite quantum system is far richer than the simple combination of those associated with the individual systems i.e. there exist pure states which cannot be seen as the product of states

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acting on the individual subsystems. This is far different from the classical setting where every pure state is a product state. Moreover since any state on a C*-algebra can be approximated (in the weak* topology) by a convex linear combination pure states, then any mixed state of a composite classical system (i.e. any composite C*-algebra where at least one of its constituents is abelian) can be approximated by a convex linear combination of pure product states hence motivating the following definition:

**Definition 2.50.** (Separable and Entangled States)
A state $\omega$ acting on a quantum composite system $\mathfrak{A}_1 \otimes \mathfrak{A}_2$ is called *separable* if $\omega \in \mathfrak{S}_{sep} := \overline{conv}(\mathfrak{S}_{\mathfrak{A}_1} \times \mathfrak{S}_{\mathfrak{A}_2})$ where the closure is taken with respect to the norm on $(\mathfrak{A}_1 \bigotimes \mathfrak{A}_2)^*$. A state $\varphi$ is called *entangled* if it is not separable i.e. $\varphi \in \mathfrak{S}_{\mathfrak{A}_1 \otimes \mathfrak{A}_2} \setminus \mathfrak{S}_{sep}$.

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# Chapter 3

# Mathematical Preliminary

## 3.1 On the Topological Structures of C*-algebras

When a vector space is finite dimensional, the concept of topology is relatively unimportant since the topologies of interest turn out to be the same in this case. As a consequence there is no ambiguity that arises when taking about topological properties in the finite dimensional case. In contrast the, a notion of topology becomes essential for infinite dimensional vector spaces as it is the foundation for concepts such as approximation and continuity (upon which even the notions of summation apply i.e. the axioms of vector spaces only provide for finite dimensional sums but not infinite ones).

To demonstrate this importance in the context of this dissertation explicitly, recall that the set of all states $\mathfrak{S}_{\mathfrak{A}}$ on a C*-algebra $\mathfrak{A}$ is the convex closure of its extreme points the pure states $P_{\mathfrak{A}}$. In finite dimensions this implies that every point in $\mathfrak{S}_{\mathfrak{A}}$ can be expressed as a finite convex combination of the elements in $P_{\mathfrak{A}}$ by the Carathéodory theorem. In infinite dimensions, at best a state can be viewed as the limit of a sequence of such finite combinations. However the concept of convergence requires the imposition of neighbourhoods on the vector space which is precisely the concept of defining a topology on the vector space.



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The purpose of this section is to review some elementary (but not necessarily standard) facts regarding topologies with the aim of application to the dual spaces of a C$^*$-algebra (and therefore to the state space by restriction). This subsection should thus be taken as a toolkit with the express purpose of clarifying some terminology and assumptions that vary in standard reference materials. As such this subsection should not be seen as a complete treatment of topological spaces and most concepts which do not vary greatly in standard reference materials (such as compactness and associated theorems) are omitted and will be referred to when first required. All unquoted results of this section can be found in any standard functional analysis or topology text (such as [29], [27], [19], or [33]) to which the reader is referred to for further details and a more complete treatment.

The most commonly used definition of a topology is in terms of open sets:

**Definition 3.1.** (Topology and Open Sets)
A *topology* on a set $X$ is a collection $\mathcal{T}$ of subsets of $X$ such that:

1) $\emptyset$ and $X$ are in $\mathcal{T}$.

2) The intersection of any finite collection of elements in $\mathcal{T}$ is also in $\mathcal{T}$.

3) The union of any collection (finite or infinite) of elements in $\mathcal{T}$ is also in $\mathcal{T}$.

The ordered pair $(X, \mathcal{T})$ is called a *topological space* and the elements of $\mathcal{T}$ are called the *open sets* of $X$. A subset of $X$ is called closed if its complement is open.

The notion of a topology is intimately connected to that of neighbourhoods:

**Definition 3.2.** (Neighbourhoods)
A *neighbourhood* of an element $x \in X$ of a topological space $(X, \mathcal{T})$ is a subset $\mathcal{U}$ of $X$ that contains an element of $\mathcal{T}$ which itself contains $x$ i.e. there exists an open set $U$ of $X$ such that $x \in U$ and $U \subseteq \mathcal{U}$. The collection of all neighbourhoods of $x$ is denoted by $\mathcal{N}(x)$. An open neighbourhood of $x$ is a neighbourhood of $x$ which is also an element of $\mathcal{T}$.

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The topologies of interest for this dissertation can be seen as being generated by open neighbourhoods by noting that any collection $\mathcal{B}$ of subsets in $X$ that satisfies:

- Every $x \in X$ is contained in at least one $B \in \mathcal{B}$.
- If $x \in B_1$ and $x \in B_2$ where $B_1, B_2 \in \mathcal{B}$ then there exists a $B_3 \in \mathcal{B}$ such that $x \in B_3$ and $B_3 \subseteq B_1 \cap B_2$.

can be used to generate a topology $\mathcal{T}$ on $X$ by specifying the elements of $\mathcal{T}$ to be those subsets $U$ of $X$ for which every $x \in U$ is contained in at least one $B \in \mathcal{B}$ that is itself contained in $U$ i.e. $x \in U$ implies that there exists a $B \in \mathcal{B}$ such that $x \in B$ and $B \subseteq U$ (see [19] page 78). This motivates the following definition:

**Definition 3.3.** (Neighbourhood Basis) For a topological space $(X, \mathcal{T})$ a *neighbourhood basis* of an element $x \in X$ is a family of open neighbourhoods $\mathcal{B}(x)$ of $x$ such that for every $U \in \mathcal{T}$ that contains $x$, there exists a $\mathcal{U} \in \mathcal{B}(x)$ such that $\mathcal{U} \subseteq U$.

Immediately it then follows that a topology is completely specified if a nonempty neighbourhood basis can be specified for each point (c.f.Lemma 13.2 in [19]).

**Definition 3.4.** (Norm Topology)
If a norm $\|\cdot\|$ on $X$ exists, then the *norm topology* is defined on $X$ by setting the neighbourhood basis of each $x \in X$ to consist of sets of the form:

$$\mathcal{U}(x; \epsilon) := \{y \in X : \|x - y\| < \epsilon\}$$

where $\epsilon > 0$.

The set of all topologies on a set $X$ can be partially ordered by inclusion and therefore allows comparison:

**Definition 3.5.** (Strength of Topologies)
For two topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ on $X$, $\mathcal{T}_2$ is called *weaker* or *coarser* than $\mathcal{T}_1$ (or equivalently $\mathcal{T}_1$ is called *stronger* or *finer* than $\mathcal{T}_2$) if $\mathcal{T}_2 \subseteq \mathcal{T}_1$. If either $\mathcal{T}_1 \subseteq \mathcal{T}_2$ or $\mathcal{T}_1 \supseteq \mathcal{T}_2$ then $\mathcal{T}_1$ and $\mathcal{T}_2$ are said to be *comparable*.

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**Definition 3.6.** (Continuity)
A function $f : X \to Y$, where $X$ and $Y$ are topological spaces, is said to be *continuous at the point* $x \in X$ if for every neighbourhood $\mathcal{V}$ of $f(x)$ there exists a neighbourhood $\mathcal{U} \in \mathcal{N}(x)$ such that $f(\mathcal{U}) \subseteq \mathcal{V}$. A function is called *continuous* if it is continuous at all points of $X$.

*Remark* 3.7. This definition explicitly establishes the role of topologies in continuity. Continuity is not a universal property in the sense that changing the topology on either $X$ or $Y$ may lead to the function no longer being continuous. Thus, in situations where it is possible to define multiple topologies, continuity should be specified with respect to the particular topologies used.

The following theorem provides useful alternative characterizations of continuity (c.f. Theorem 18.1 in [19]):

**Theorem 3.8.** *Let $X$ and $Y$ be topological spaces and $f : X \to Y$. Then the following are equivalent:*

*1) $f$ is continuous*

*2) For every open subset $V$ in $Y$, the set $f^{-1}(V)$ is open in $X$*

*3) For every closed subset $K$ in $Y$, the set $f^{-1}(K)$ is closed in $X$*

If a function on a topological space is real-valued, then continuity can be subdivided into two weaker conditions:

**Definition 3.9.** (Semi-Continuity)
A function $f : X \to \mathbb{R}$ on a topological space $X$ is called *lower semi-continuous (l.s.c)* if for every real number $r$, the set $\{x : f(x) \leq r\}$ is closed. $f$ is called *upper semi-continuous (u.s.c)* if for every real number $r$, the set $\{x : f(x) \geq r\}$ is closed.

Note that, by Theorem 3.8, a function $f : X \to \mathbb{R}$ is continuous if and only if $\{x : f(x) \leq r\} \cup \{y : f(y) \geq s\}$ is closed for all $r, s \in \mathbb{R}$. Hence:

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**Proposition 3.10.** *A function $f : X \to \mathbb{R}$ on a topological space $X$ is continuous if and only if it is both upper and lower semi-continuous.*

Recall that addition and scalar multiplication in a vector space $V$ are, by definition, $V \times V \to V$ and $V \times \mathbb{C} \to V$ maps respectively. Hence in order for either operation to be considered continuous, a well defined notion of topology on both $V \times V$ and $V \times \mathbb{C}$ is needed.

**Definition 3.11.** (Product Topology)
For two topological spaces $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ let $\mathcal{B}(x)$ and $\mathcal{B}(y)$ denote neighbourhood bases for any $x \in X$ and $y \in Y$ respectively. The *product topology* on $X \times Y$ is topology generated by setting the neighbourhood basis of each $(x, y) \in X \times Y$ to be the collection $\mathcal{B}(x, y)$ consisting of all sets of the form

$$\{B \times C : B \in \mathcal{B}(x), C \in \mathcal{B}(y)\}$$

**Definition 3.12.** (Topological Vector Space)
A *topological vector space* $(V, \mathcal{T})$ is a vector space $V$ equipped with a topology $\mathcal{T}$ for which the operations of vector addition and scalar multiplication are continuous with respect to the product topology on $V \times V$ and $V \times \mathbb{C}$ respectively.

**Proposition 3.13.** *Let $X, Y$ be topological vector spaces and $f : X \to Y$ a continuous function with respect to the topologies on $X$ and $Y$. $f$ will remain continuous if the topology on $Y$ becomes coarser or the topology on $X$ becomes finer.*

For any function $f : X \to Y$, where $X$ is a set and $Y$ is a topological space, it is always possible to define at least one topology on $X$ making $f$ continuous namely by defining the open sets of $X$ to be generated by sets of the form

$$\{x \in X : f(x) \in V\}$$

where $V$ is an open set in $Y$.

Let $f_1, \ldots, f_n$ be a family of functions from a set $X$ into a topological space $Y$ and $\mathcal{T}_1, \ldots, \mathcal{T}_n$ a family of topologies on $X$ such that $f_i$ is continuous with respect

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to $\mathcal{T}_i$ for all $i = 1, \ldots, n$. The intersection of $\mathcal{T}_1, \ldots, \mathcal{T}_n$ is the collection of all the open sets which appear in all of the topologies i.e. $U \in \bigcap_{i=1}^{n} \mathcal{T}_i$ if $U \in \mathcal{T}_i$ for all $i = 1, \ldots, n$. Clearly $\bigcap_{i=1}^{n} \mathcal{T}_i$ is a topology on $X$ [34]. Moreover every function in the family $f_1, \ldots, f_n$ is clearly continuous with respect to $\bigcap_{i=1}^{n} \mathcal{T}_i$ thus there exists at least one topology on $X$ for which every member of $f_1, \ldots, f_n$ is continuous. This concept, when combined with Proposition 3.13, means that the following is well defined:

**Definition 3.14.** (Weak Topology)
For a set $X$ and a family of functions $\mathcal{F}$ from $X$ into a topological space $Y$, the finest topology on $X$ such that every function in $\mathcal{F}$ is continuous is referred to as the *weak topology on $X$ induced by $\mathcal{F}$*.

In particular, if $X$ is a Banach space, then the weak topology on $X$ refers to the finest topology $X$ induced by the bounded linear functionals on $X$.

The dual space $X^*$ of any Banach space is itself a Banach space when equipped with the operator norm. Hence there exists a related norm topology on $X^*$ defined by setting the neighbourhood basis of $f \in X^*$ as per Definition 3.4.

The elements $x \in X$ define a family of bounded functionals $\hat{x}$ on $X^*$ by

$$\hat{x}(f) := f(x) \text{ for all } f \in X^*$$

where boundedness of $\hat{x}$ follows from $\|\hat{x}(f)\| = \|f(x)\| \leq \|x\|\|f\|$ so $\|\hat{x}\| \leq \|x\|$. The weakest topology on $X^*$ such that this family of functionals is continuous is referred to as the weak$^*$ topology on $X^*$.

*Remark* 3.15. The weak$^*$ topology and the weak topology on $X^*$ are in general different with the weak$^*$ topology being weaker than the weak topology by definition.

The following theorem emphasises the importance of the weak$^*$ topology (see Theorem 1.6.5(i) in [29] for details and proof):

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**Theorem 3.16** (Banach-Alaoglu Theorem)**.** *Let $X_1^*$ denote the closed unit ball in the dual space $X$ of a Banach space $X$. $X_1^*$ is compact in the weak$^*$ topology.*

The state space of $\mathfrak{S}_{\mathfrak{A}}$ of a C$^*$-algebra is a subset of the unit ball of $\mathfrak{A}$ and so, respecting the previous discussion, the following definitions are adopted (see page 53 in [28]):

**Definition 3.17.** (Topologies on $\mathfrak{S}_{\mathfrak{A}}$)
Given a C$^*$-algebra $\mathfrak{A}$, the *uniform topology* (otherwise known as the *norm topology*) on $\mathfrak{S}_{\mathfrak{A}}$ is given by setting the neighbourhood basis of $\omega \in \mathfrak{S}_{\mathfrak{A}}$ to be the collection of sets of the form

$$\mathcal{U}(\omega;\epsilon) := \{\varphi \in \mathfrak{A}^* : \|\omega - \varphi\| < \epsilon\}$$

where $\epsilon > 0$.
The *weak$^*$ topology* on $\mathfrak{S}_{\mathfrak{A}}$ is determined by defining the neighbourhood basis of $\omega \in \mathfrak{S}_{\mathfrak{A}}$ to consist of sets of the form

$$\mathcal{U}(\omega; a_1, \ldots, a_n, \epsilon) := \{\varphi \in \mathfrak{A}^* : |\omega(a_i) - \varphi(a_i)| < \epsilon \ \forall \ i = 1, \ldots, n\}$$

where $\epsilon > 0$ and $a_1, \ldots, a_n$ is any finite set of elements in $\mathfrak{A}$.

**Definition 3.18.** (Hausdorff space)
A topological space $(X, \mathcal{T})$ is called *Hausdorff* if for any two distinct points $x, y \in X$ there exist neighbourhoods $\mathcal{U} \in \mathcal{N}(x)$ and $\mathcal{V} \in \mathcal{N}(y)$ such that $\mathcal{U} \cap \mathcal{V} = \emptyset$.

*Remark* 3.19. Clearly, from their definitions, the uniform and weak$^*$ topologies on $\mathfrak{S}_{\mathfrak{A}}$ are Hausdorff.

Since $\mathfrak{A}^*$ is a Banach space, an application of the following theorem (appearing as Corollary 2.6.3 in [35]) establishes why the topology on $\mathfrak{S}_{\mathfrak{A}}$ is generally not discussed when $\mathfrak{A}$ is finite dimensional but needs to be specified for infinite dimensions:

**Theorem 3.20.** *Let $X$ be a normed space. The weak$^*$ and uniform topologies of $X^*$ coincide if and only if $X$ is finite dimensional.*

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In order to discuss convergence in topological spaces, a generalization of sequences must be introduced:

**Definition 3.21.** (Nets)
A *directed set* is a set $I$ together with a binary relationship $\leq$ on $I$ that is reflexive, transitive and every pair of elements in $I$ has an upper bound in $I$ (i.e. if $i, j \in I$ then there exists a $k \in I$ such that $i \leq k$ and $j \leq k$).
A *net* in a set $X$ is a function $f$ from a directed set $I$ into $X$ denoted by $\{x_i\}_{i\in I}$ where $x_i = f(i)$.

*Remark* 3.22. Nets generalize the concept of a sequence in the sense that a sequence is a net where the index set is the natural numbers with the normal order relationship.

**Definition 3.23.** (Convergence and Limits)
Let $X$ be a topological space and $\{x_i\}_{i\in I}$ be a net in $X$. $\{x_i\}_{i\in I}$ is said to *converge* to an element $x \in X$ (written $x_i \to x$), if for every neighbourhood $\mathcal{U} \in \mathcal{N}(x)$ there exists an $i_0 \in I$ such that $x_i \in \mathcal{U}$ for all $i \geq i_0$. The element $x$ is then called the *limit* of the net $\{x_i\}_{i\in I}$.

Since the concept of convergence is defined through the use of a neighbourhood, it naturally becomes a topological property in the sense that a net may converge with respect to one topology but not another.

While the question of whether a net is convergent then depends on the topology specified, the uniqueness of the limit (if it exists) is assured for all topological spaces of interest to this dissertation by the following theorem (see Theorem 13.7 in [33]):

**Theorem 3.24.** *Let $X$ be a Hausdorff topological space. The limit of any convergent net in $X$ is unique.*

A consequence of Theorem 3.16 is the following (see Proposition 13.8 in [17]):

**Proposition 3.25.** *The state space $\mathfrak{S}_{\mathfrak{A}}$ of a unital $C^*$-algebra $\mathfrak{A}$ is convex and compact in the weak$^*$-topology. Moreover, in the weak$^*$ topology, $\mathfrak{S}_{\mathfrak{A}}$ is the convex closure of the pure states $P_{\mathfrak{A}}$.*

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This section is closed with an extension of the Lebesque integral to a vector valued variant known as the weak integral (the reader is referred to Chapter 3 in [27] for the theory of such integrals):

**Definition 3.26.** (Weak Integration)
Let $(X, \Sigma, \mu)$ be a measure space and $V$ a topological vector space whose dual $V^*$ separates the points of $V$. A function $f : X \to V$ is called *weakly integrable* if:

1) $\varphi \circ f \in L^1(X, \Sigma, \mu)$ for all $\varphi \in V^*$.

2) There exists an element $v \in V$ such that

$$\varphi(v) = \int_X \varphi(f(x))d\mu(x) \text{ for all } \varphi \in V^*$$

The element $v$ is called the *weak integral of* $f$ and is denoted by $\int_X f(x)d\mu(x)$.

## 3.2 On the Integral Representation of States

The state space $\mathfrak{S}_{\mathfrak{A}}$ is an example of a convex weak$^*$ compact subset of a locally convex topological vector space, namely the dual space $\mathfrak{A}^*$ (see Proposition 3.25). In the case when $\mathfrak{S}_{\mathfrak{A}}$ is finite dimensional, the Carathéodory theorem establishes that any point in $\mathfrak{S}_{\mathfrak{A}}$ can be written as a finite convex combination of the extreme points of $\mathfrak{S}_{\mathfrak{A}}$ i.e. the pure states. The purpose of this section is to elaborate on the generalization of this theorem to the infinite dimensional case through the integral version of the Krein-Millman theorem which associates to every point in $\mathfrak{S}_{\mathfrak{A}}$ a measure that is concentrated on the pure states.

The Choquet theory of boundary integrals is essential to this version of the Krein-Millman theorem and will be discussed first for the most general context of a generic convex compact subset $K$ of a locally compact vector space $X$. Throughout this dissertation, $S(K)$ will denote the subset of $C(K)$ consisting of the real continuous convex functions over $K$ while $A(K)$ will denote the subset of $C(K)$

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consisting of all real continuous affine functions over $K$ i.e.

$$S(K) := \{f \in C(K) : f(\lambda x+(1-\lambda)y) \leq \lambda f(x)+(1-\lambda)f(y), x, y \in K, 0 \leq \lambda \leq 1\}$$

$$A(K) := \{f \in C(K) : f(\lambda x+(1-\lambda)y) = \lambda f(x)+(1-\lambda)f(y), x, y \in K, 0 \leq \lambda \leq 1\}$$

A Radon measure will be understood as any element of $C(K)^*$.

*Remark* 3.27. Note that each element $x \in K$ can be mapped to an element $\hat{x}$ of $A(K)^*$ where $\hat{x}$ is defined by

$$\hat{x}(f) = f(x) \text{ for all } f \in A(K)$$

This mapping $x \to \hat{x}$ is clearly affine, linear and weak$^*$ continuous ($A(K)$ being a Banach space when equipped with the sup norm and pointwise addition) thus $K$ can be viewed as a convex weak$^*$ compact subset of $A(K)^*$ when the latter is equipped with the weak$^*$ topology.

A Borel measure on $K$ is any $\sigma$-additive set function on the $\sigma$-algebra $\mathcal{B}$ over $K$ generated by the open or closed subsets of $K$. A Baire measure is any $\sigma$-additive set function on the $\sigma$-algebra $\mathcal{B}_0$ over $K$ generated by the compact $G_\delta$-sets (i.e. the compact subsets of $K$ that are the countable intersection of open sets). The Riesz Representation theorem establishes that the regular Borel measures and Radon measures are in a one-to-one relationship. Since any Baire measure is automatically regular and can be uniquely extended to a regular Borel measure (Theorem 54.D in [36]) and, conversely, any Borel measure can be used to define a Baire measure through restriction (page 230 in [36]), there is a one-to-one relationship between the regular Borel measures and the Baire measures. Throughout the remainder of this section a measure will refer to any of these three types of measure (regularity of Borel measures being assumed unless otherwise stated) and context will dictate the exact nature.

$M_+(K)$ will denote the set of all non-negative real Radon measures over $K$ and $M_1(K)$ the subset of $M_+(K)$ with unit norm (alternatively known as the set of all probability measures on $K$).

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**Definition 3.28.** (Total Variation and Norm of a Measure)
The *total variation* $|\mu|$ of a regular Borel measure $\mu$ over $K$ is defined as the measure

$$|\mu| = \mu^+ + \mu^-$$

Where $\mu^+(C) := max(\mu(C), 0)$ and $\mu^-(C) := max(-\mu(C), 0)$ for all $C \in \mathcal{B}$. The *norm* $\|\mu\|$ of a regular Borel measure $\mu$ over $K$ is given by

$$\|\mu\| = |\mu|(K)$$

*Remark* 3.29. Note that if $\mu \in M_+(K)$, then $|\mu| = \mu$ and $\|\mu\| = \mu(K)$.

**Definition 3.30.** (Support of a Measure)
A measure $\mu \in M_+(K)$ is said to be *supported* by a Borel set $C \subseteq K$ if it is the smallest Borel set such that $\mu(C) = \mu(K)$. $\mu$ is said to be *pseudosupported* by an arbitrary set $A \subseteq K$ if $\mu(B) = 0$ for all Baire sets $B$ in the complement of $A$ (i.e. $B \cap A = \emptyset$).

The support of a measure intuitively represents the only important points in $K$ which need to be considered in the sense that the support defines the region over which integration needs to be taken (since the evaluation of the measure over an measurable set in the complement of the support is zero) i.e.

$$\int_K f(x)d\mu(x) = \int_C f(x)d\mu(x)$$

where $C$ is the support of $\mu$ and $f$ is any $\mu$-measurable function.

*Remark* 3.31. The Dirac measures $\delta_x$ are the measures that are supported by a single element $x \in K$. Any measure with finite support $\{x_i\}_{i=1}^n$ is a linear combination of the associated Dirac measures $\{\delta_{x_i}\}_{i=1}^n$.

**Definition 3.32.** (Barycenter)
For a compact convex set $K$ of a locally compact topological vector space $X$, a point $x \in K$ is called the *barycenter* of a measure $\mu \in M_1(K)$ (or represented by the measure $\mu$) if

$$\mu(f) = f(x) \qquad \forall\ f \in X^*$$

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While it is not a priori clear that the barycenter of a measure exists, if it does then it must be unique (as will be shown in Proposition 3.35) thus justifying the use of the definite article i.e. "the barycenter" as opposed to "a barycenter". However since any point $x \in K$ is trivially the barycenter of at least one probability measure (namely the Dirac measure $\delta_x$), a single point may potentially act as the barycenter of multiple measures hence the following definition:

**Definition 3.33.** The nonempty subset of all probability measures that represent $x$ is denoted by $M_x(K)$.

*Remark* 3.34. By adopting the standard notation $\mu(f) = \int_K f d\mu = \int_K f(x) d\mu(x)$ the barycenter of the measure $\mu$ can be seen as the point $x = \int_K x' d\mu(x')$ where the integral is understood in the weak sense.

**Proposition 3.35.** *If $\mu \in M_1(K)$, then the barycenter of $\mu$ exists and is unique.*

*Proof.* For each $f \in X^*$ define $K_f := \{y \in K : \mu(f) = f(y)\} = f^{-1}(\mu(f)) \neq \emptyset$ which is clearly a closed subset of $K$ Since $K$ is compact, if $\bigcap_{i=1}^{n} K_{f_i} \neq \emptyset$ for an arbitrary finite collection $f_1, f_2, \ldots, f_n \in X^*$, then $\bigcap_{f \in X^*} K_f \neq \emptyset$ by Proposition 4.2.1 in [37].

Define $T : K \to \mathbb{R}^n$ by $T(x) = (f_1(x), f_2(x), \ldots, f_n(x))$ for any $f_1, \ldots, f_n \in X^*$. Clearly $T$ is linear and continuous thus $T(K)$ is a compact convex subset of $\mathbb{R}^n$. Set $z = (\mu(f_1), \mu(f_2), \ldots, \mu(f_n)) \in \mathbb{R}^n$ and assume $z \notin T(K)$ so that, by the Hahn-Banach separation theorem (see Corollary IV.3.10 in [38] combined with Theorem 5.25 in [37]), there exists a point $a \in \mathbb{R}^n$ and an $\alpha \in \mathbb{R}$ such that

$$\langle a, z \rangle > \alpha > \langle a, T(x) \rangle \qquad \forall x \in K$$

But $\langle a, z \rangle = \sum a_i \mu(f_i) = \mu(\sum a_i f_i) = \mu(g)$ and $\langle a, T(x) \rangle = \sum a_i f_i(x) = g(x)$ thus

$$\mu(g) > \alpha = \int_K \alpha d\mu(x) > \int_K g(x) d\mu(x) = \mu(g)$$

Clearly this is impossible hence $z \in T(K)$ implying $\bigcap_{i=1}^{n} K_{f_i} \neq \emptyset$ and a barycenter of $\mu$ exists.

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Let $x, y \in K$ be barycenters of $\mu \in M_1(K)$. Then $f(x-y) = f(x) - f(y) = \mu(f) - \mu(f) = 0$ for all $f \in X^*$. By the Hahn-Banach separation theorem, $X^*$ separates the points of X thus $x = y$ i.e. the barycenter of $\mu$ is unique. □

Proposition 4.1.1 in [28] provides an alternative proof to the existence of barycenters for general measures and also establishes the following:

**Proposition 3.36.** *The collection of finitely supported measures in $M_x(K)$ is dense in $M_x(K)$.*

As a result, any measure in $M_1(K)$ can be seen as the weak$^*$ limit of a net of finitely supported measures in $M_1(K)$ with the same barycenter (i.e. any probability measure can be approximated in the weak$^*$-topology by finitely supported probability measures which represent the same point).

As a reminder, the aim is to replace the generic convex set $K$ with the state space $\mathfrak{S}_{\mathfrak{A}}$ which is further motivated by recognizing the self-adjoint elements of $\mathfrak{A}$ as the space of real valued affine continuous functions on $\mathfrak{S}_{\mathfrak{A}}$ (see Chapter III.6 in [30]). As this stage, it is possible to associate each probability measure to a unique state and each state to a collection of probability measures (each of which is can be approximated by a probability measure with finite support). If a probability measure has finite support, then it is clear that its barycenter is the corresponding convex linear combination of the supporting elements (see Chapter 2 in [39]). Thus each measure in $M_\omega(\mathfrak{S}_{\mathfrak{A}})$ can be viewed as a decomposition of $\omega$ into the states forming the support of that measure. Of greatest interest (for this dissertation) will be decompositions in terms of pure states i.e. the extreme points of $\mathfrak{S}_{\mathfrak{A}}$ and so attention will now be turned to identifying elements of $M_x(K)$ which are supported by the extreme points of $K$ (those points of $K$ which cannot be non-trivially expressed as a convex linear combination of other elements of $K$).

In order to find a measure that represents a particular point $x \in K$ whose support is contained in the extreme points of $K$, the measures in $M_x(K)$ need to be compared i.e. a suitable ordering on $M_+(K)$ is needed that determines if the support of a measure is concentrated closer to the boundary of $K$.

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**Definition 3.37.** (Maximal Measures)
A partial ordering "$\succ$" is defined on $M_+(K)$ by $\mu \succ \nu$ if $\mu(f) \geq \nu(f)$ for all $f \in S(K)$. $\mu \in M_+(K)$ is called a maximal measure if, for any $\nu \in M_+(K)$ that is not a scalar multiple of $\mu$, $\nu \succ \mu$ implies $\mu = \nu$.

*Remark* 3.38. $\succ$ is trivially reflexive and transitive. Antisymmetry follows as a consequence of the set $\{f - g : f, g \in S(K)\}$ being uniformly dense in C(K)(see Lemma 6.1 in [30] or Lemma 4.1.4 in [28].

Proposition 4.1.3 in [28] emphasises the inductive properties of the relationship "$\succ$" and the consequential existence of maximal representing measures:

**Proposition 3.39.**

*(1)* $\mu \succ \nu$ *and* $\nu \in M_x(K) \Rightarrow \mu \in M_x(K)$.

*(2)* $\mu \succ \delta_x \Leftrightarrow \mu \in M_x(K)$.

*(3) Every* $x \in K$ *is represented by a measure* $\mu \in M_1(K)$ *which is maximal for the order* $\succ$.

Statement (1) can be interpreted as saying that the only "comparable" probability measures are those that represent the same points, in which case the measures $\mu$ and $\nu$ can be viewed as distributions about their common barycenter and $\mu \succ \nu$ means that $\mu$ is concentrated further away from the common barycenter so its support is therefore "closer" to the extreme points of $K$. Simultaneously, (3) establishes that at least one element in $M_x(K)$ is a maximal measure but does not ensure uniqueness of the maximal representing measure of $x$. Conditions which ensures that a maximal $\mu \in M_x(K)$ is unique will be addressed after investigating the relationship between maximality and the set $\mathcal{E}(K)$ of extreme points of $K$ viz. the conditions that ensure a maximal measure is supported by a subset of $\mathcal{E}(K)$.

**Definition 3.40.** (Upper Envelope and Boundary Sets)
The *upper envelope* $\bar{f}$ of any $f \in C(K)$ is the concave, upper semi-continuous

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function defined by

$$\bar{f}(x) = \inf\{g(x) : -g \in S(K), g \geq f\}$$

The associated *boundary set* of $f \in S(K)$ is the $G_\delta$-set (and thus also a Borel set)

$$B_f := \{x \in K : f(x) = \bar{f}(x)\} = \bigcap_{n=1}^{\infty} \left\{x \in K : \bar{f}(x) - f(x) < \frac{1}{n}\right\}$$

*Remark* 3.41. $f$ is continuous and $\bar{f}$ is upper semi-continuous implying $f - \bar{f}$ is upper semi-continuous and thus $\left\{x \in K : \bar{f}(x) - f(x) < \frac{1}{n}\right\}$ is open for all $n \in [1, \infty)$. Thus equality of the two sets given in the definition of $B_f$ follows since $\bar{f} \geq f$ by definition.

**Lemma 3.42.** *For any $\mu \in M_1(K)$ and $f \in C(K)$, there exists a $\nu \in M_1(K)$ such that $\mu \prec \nu$ and $\nu(f) = \mu(\bar{f})$.*

*Proof.* This statement follows directly from the Hahn-Banach extension theorem. For details see lemma IV 6.7 in [30]. □

**Lemma 3.43.** $x \in \mathcal{E}(K) \Leftrightarrow M_x(K) = \{\delta_x\}$.

*Proof.* By Proposition 3.36, it suffices to only consider cases when $\mu \in M_x(K)$ has finite support, i.e.

$$\mu = \sum_i \lambda_i \delta_{x_i}$$

But then $\mu$ represents the point $\sum_i \lambda_i x_i$ and by uniqueness of the barycenter $x = \sum_i \lambda_i x_i$.

Clearly then $x \in \mathcal{E}(K) \Leftrightarrow \mu = \delta_x \ \forall \ \mu \in M_x(K)$. □

The introduction of the upper envelopes and boundary sets is motivated by their relationship to the maximal measures as characterized by the following:

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**Theorem 3.44.** *A measure $\mu \in M_1(K)$ is maximal if and only if $\mu$ is supported by every boundary set i.e.*

$$\mu(B_f) = \|\mu\| \text{ for all } f \in S(K)$$

*Proof.* See either Theorem 4.1.7 in [28] or Theorem IV 6.11 in [30]. □

By definition, $\mu \in M_1(K)$ being supported by the boundary sets is equivalent to the requirement that $\mu(f) = \mu(\bar{f})$ for every $f \in S(K)$. Hence Theorem 3.44 can intuitively be seen as concluding that the maximal probability measures are supported by the sets where the convex functions achieve their maxima (i.e. sets where $f = \bar{f}$).

In addition, the extreme points of $K$ are those points that are common to all the boundary sets by the following proposition:

**Proposition 3.45.** $\mathcal{E}(K) = \bigcap_{f \in S(K)} B_f$

*Proof.* Let $x \in \mathcal{E}(K)$ and $f \in S(K)$, then by Lemma 3.42 there exists a $\nu \in M_1(K)$ such that $\nu \succ \delta_x$ and $\nu(f) = \delta_x(\bar{f}) = \bar{f}(x)$. But $\nu \succ \delta_x \Rightarrow \nu \in M_x(K)$ by (2) in Proposition 3.39 and thus $\nu = \delta_x$ by Lemma 3.43. Consequently $f(x) = \bar{f}(x)$ for all $f \in S(K)$ i.e. $x \in \bigcap_{f \in S(K)} B_f$.

Conversely if $x \notin \mathcal{E}(K)$ then there exists distinct points $x_1, x_2 \in K$ such that $x = (x_1 + x_2)/2$. By the Hahn-Banach separation theorem, there exists an $a \in A(K)$ such that $a(x_1) \neq a(x_2)$. In order to show that $x \notin \bigcap_{f \in S(K)} B_f$, it is enough to show that $x \notin B_{a^2}$. To this end consider a function $g$ such that $-g \in S(K)$ and $g \geq a^2$. Then

$$a^2(x) = \left(\frac{a(x_1) + a(x_2)}{2}\right)^2 < \frac{a^2(x_1)}{2} + \frac{a^2(x_2)}{2} \leq \frac{g(x_1)}{2} + \frac{g(x_2)}{2} \leq g(x)$$

Thus $a^2(x)$ is strictly less than $g(x)$ for all $-g(x) \in S(K)$ such that $a^2 \leq g$ which means that (by definition) $\overline{a^2}(x) > a^2(x)$ i.e. $\overline{a^2}(x) \neq a^2(x)$ hence $x \notin B_{a^2}$. □

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*Remark* 3.46. Each boundary set $B_f$ is a Borel $G_\delta$-set by definition but the extreme points $\mathcal{E}(K)$ is not necessarily Borel since it is the intersection of a nondenumerable collection of sets.

As a result of Theorem 3.44 and Proposition 3.45, it is clear that if a measure $\mu \in M_1(K)$ is supported by a Borel subset of the extreme points $\mathcal{E}(K)$, then $\mu$ is maximal. The converse is not necessarily true, i.e. it is possible to construct a scenario in which $\mu$ is maximal, $\mathcal{E}$ is Borel and $\mu(\mathcal{E}(K)) = 0$ and thus $\mu$ is cannot be supported by any Borel subset of $\mathcal{E}(K)$ [40]. It is thus essential to identify realistic (physically realizable) conditions under which a maximal $\mu \in M_x(K)$ is supported by a subset of $\mathcal{E}(K)$. A development of a general theory in this regard is beyond the scope of this dissertation, hence for the remainder of this section, the set $K$ will be replaced by the state space $\mathfrak{S}_\mathfrak{A}$ of a unital C$^*$-algebra $\mathfrak{A}$ so that $\mu \in M_1(\mathfrak{S}_\mathfrak{A})$, and the aim is to establish conditions where a maximal $\mu \in M_\omega(\mathfrak{S}_\mathfrak{A})$ is supported by a subset of the pure states.

**Definition 3.47.** (Ruelle Separability Condition)
If $\mathfrak{A}$ is a unital C$^*$-algebra, a subset $F \subseteq \mathfrak{S}_\mathfrak{A}$ is called Ruelle separable if there exists a sequence $\{\mathfrak{A}_n\}$ of C$^*$-subalgebras of $\mathfrak{A}$ whose union is dense in $\mathfrak{A}$ and each $\mathfrak{A}_n$ contains a closed, separable two-sided ideal $\mathcal{I}_n$ such that

$$F = \{\omega \in \mathfrak{S}_\mathfrak{A} : \|\omega|_{\mathcal{I}_n}\| = 1 \ \forall \ n \geq 1\}$$

Proposition 4.1.34 in [28] shows that the Ruelle separable subsets of $\mathfrak{S}_\mathfrak{A}$ possess the following desirable properties:

**Proposition 3.48.** *If $F \subseteq \mathfrak{S}_\mathfrak{A}$ is Ruelle separable, then*

*1) $F$ is a stable face of $\mathfrak{S}_\mathfrak{A}$ i.e.*

$$\omega \in F \text{ and } \mu \geq \delta_\omega \Rightarrow \mu \text{ is supported by } F$$

*2) $F$ is a Baire set.*

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*3)* $\mathcal{E}(F)$ *is a Baire set and there exists a continuous convex function* $f$ *over* $\mathfrak{S}_{\mathfrak{A}}$ *such that*

$$\mathcal{E}(F) = B_f \cap F$$

**Theorem 3.49.** *Let* $\mathfrak{A}$ *be a unital* $C^*$*-algebra and* $F \subseteq \mathfrak{S}_{\mathfrak{A}}$ *be Ruelle separable. If* $\omega \in F$ *and* $\mu \in M_\omega(\mathfrak{S}_{\mathfrak{A}})$*, then* $\mu$ *is maximal if and only if* $\mu(\mathcal{E}(F)) = 1$*.*

*Proof.* Since $\mathcal{E}(F)$ is a Baire set, then it is clearly Borel and since $\mathcal{E}(F) = \mathcal{E}(\mathfrak{S}_{\mathfrak{A}}) \cap F$ then $\mu(\mathcal{E}(F)) = 1$ implies $\mu$ is maximal (see discussion after Proposition 3.45).

Conversely assume $\mu$ is maximal. By property (2) in Proposition 3.48 and De Morgan's Law

$$\mathcal{E}(F)^c = (B_f \cap F)^c = B_f^c \cup F^c$$

But $\mu(B_f^c) = 0$ by Theorem 3.44.

By property (1) in Proposition 3.48 and property (2) in Proposition 3.39 then $\mu(F^c) = 0$.

Define $C = B_f^c \setminus B_f^c \cap F^c$, $D = F^c \setminus B_f^c \cap F^c$, and $E = B_f^c \cap F^c$. Clearly these three sets are pairwise disjoint and are subsets of sets with zero measure and thus themselves have zero measure by regularity of $\mu$. By $\sigma$-additivity of $\mu$ it follows that

$$\mu(\mathcal{E}(F)^c) = \mu(B_f^c \cup F^c) = \mu(C) + \mu(D) + \mu(E) = 0$$

Hence $\mu(\mathcal{E}(F)) = 1$. □

Consequently, any state that is contained in a Ruelle separable subset of the state space can be represented by a maximal measure supported by the pure states while simultaneously avoiding undesirable measure theoretic properties. Many states associated with physical systems are contained in Ruelle separable subsets as illustrated by the following examples:

**Example 3.1.**

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*1) When $\mathfrak{A}$ is a separable $C^*$-algebra then by setting $\mathcal{I}_n = \mathfrak{A}$ for all $n \geq 1$, clearly the entire state space $\mathfrak{S}_{\mathfrak{A}}$ is Ruelle separable.*

*2) Consider $\mathfrak{A} = B(\mathscr{H})$ for an infinite dimensional Hilbert space $\mathscr{H}$ and $\mathfrak{S}_{normal}$ the subset of $\mathfrak{S}_{\mathfrak{A}}$ consisting of the normal states on $\mathfrak{A}$ (i.e. those states $\omega$ for which there exists a density matrix $\rho$ such that $\omega(a) = Tr(\rho a)$ for all $a \in \mathfrak{A}$). $K(\mathscr{H})$, the set of all compact algebras on $\mathscr{H}$ is a closed, two-sided ideal in $B(\mathscr{H})$ and (see Proposition 2.6.14 in [28])*

$$\mathfrak{S}_{normal} = \left\{\omega \in \mathfrak{S}_{\mathfrak{A}} : \left\|\omega|_{K(\mathscr{H})}\right\| = 1\right\}$$

*Thus by setting $\mathfrak{A}_n = B(\mathscr{H})$ and $\mathcal{I}_n = K(\mathscr{H})$, $\mathfrak{S}_{normal}$ is Ruelle separable.*

*3) A quasi-local algebra is a $C^*$-algebra $\mathfrak{A}$ with a net $\{\mathfrak{A}_\alpha\}_{\alpha \in I}$ of $C^*$-subalgebras with a common identity (known as the local algebras) such that*

*i. $\mathfrak{A}_\alpha \subseteq \mathfrak{A}_\beta$ whenever $\alpha \leq \beta$*

*ii. $\overline{\cup \mathfrak{A}_\alpha} = \mathfrak{A}$*

*iii. There exists an orthogonality relationship $\perp$ on the index set $I$ and an automorphism $\sigma : \mathfrak{A} \to \mathfrak{A}$ satisfying $\sigma(\sigma(a)) = a \ \forall a \in \mathfrak{A}$, $\sigma(\mathfrak{A}_\alpha) = \mathfrak{A}_\alpha$, and whenever $\alpha \perp \beta$ then*

$$[a_\alpha^+, a_\beta^+] = [a_\alpha^+, a_\beta^-] = \{a_\alpha^-, a_\beta^-\} = 0 \ \forall a_\alpha \in \mathfrak{A}_\alpha, a_\beta \in \mathfrak{A}_\beta$$

*where $a^\pm = \frac{a \pm \sigma(a)}{2}$, $[a, b] = ab - ba$, and $\{a, b\} = ab + ba$*

*If each $\mathfrak{A}_\alpha$ is isomorphic to a subalgebra $\pi(\mathfrak{A}_\alpha)$ of $B(\mathscr{H}_\alpha)$ such that $\pi(\mathfrak{A}_\alpha)$ contains $K(\mathscr{H}_\alpha)$ as a subalgebra, then the locally normal states $\mathfrak{S}_{\mathfrak{A}}^l$ (i.e. any state $\omega \in \mathfrak{S}_{\mathfrak{A}}$ for which $\omega|_{\mathfrak{A}_\alpha}$ is normal for all $\alpha \in I$) clearly forms a Ruelle separable set of states (see [28] for details).*

The separable C$^*$-algebra $M_n(\mathbb{C})$, of $n \times n$ matrices with complex entries, is of fundamental importance to quantum information theory and, by Example 3.1 (1), its entire state space is Ruelle separable. Example 3.1 (2) is the Dirac formalism

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of Quantum Mechanics and the locally normal states of the quasi-local algebras in Example 3.1 (3) are important descriptive tools for systems in branches of mathematical physics such as quantum statistical mechanics and quantum field theory. Thus the states of most physically realizable systems are elements of a Ruelle separable subset.

As previously mentioned, there may be many elements in $M_\omega(\mathfrak{S}_\mathfrak{A})$ which are maximal for the order "$\succ$" and in actual fact the physical systems which permit multiple multiple maximal measures in $M_\omega(\mathfrak{S}_\mathfrak{A})$ are the quantum ones:

**Proposition 3.50.** *Let $\mathfrak{A}$ be a $C^*$-algebra, then each $\omega \in \mathfrak{S}_\mathfrak{A}$ is represented by a unique maximal measure if and only if $\mathfrak{A}$ is abelian.*

*Proof.* Combine Theorem 4.1.15 and Example 4.2.6 in [28] □

As a result, a state of a quantum system may be represented by multiple maximal measures, each of which must be supported by a subset of the pure states (assuming the state is contained in a Ruelle separable subset of $\mathfrak{S}_\mathfrak{A}$). These subsets need not be identical for any two measures and thus the same state may be approximated by a convex combinations of pure states in a multitude of inequivalent ways. It is thus necessary to consider all possible maximal representing measures of a particular state when considering a quantum system.

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# Chapter 4

# Measuring Bipartate Entanglement and Correlations

The existence of entangled states in composite quantum systems is definitive proof of quantum theory, but the current distinction between entangled and separable states, presented in Definition 2.50, is limiting both in terms of detection as well as comparison. The prior definition first requires that a state be decomposed into a linear combination of pure states but, in a truly quantum system, many such decompositions may exist (as shown by Proposition 3.50) and it is thus necessary to consider all possible decompositions of a given state. The inability to identify a decomposition in terms of product states could either be because there is none and the state is entangled or all possible decompositions have not been considered. This chapter will focus on alternative means to distinguish the entangled states from the separable ones by first exposing the properties which differentiate these states.



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## 4.1 Classical and Quantum Correlations

To each element $a \in \mathfrak{A}$ of a unital C$^*$-algebra, a continuous affine functional $\hat{a}$ can be defined on the state space $\mathfrak{S}_{\mathfrak{A}}$ by

$$\hat{a}(\omega) = \omega(a)$$

A probabilistic interpretation of a state follows from the observation that (c.f. Definition 3.32):

$$\hat{a}(\omega) = \mu(\hat{a}) = \int_{\mathfrak{S}_{\mathfrak{A}}} \hat{a}(\omega^{'}) d\mu(\omega^{'}) = \langle \hat{a} \rangle$$

where $\mu \in M_{\omega}(\mathfrak{S}_{\mathfrak{A}})$ and $\langle f \rangle$ denotes the expectation value of $f \in C(\mathfrak{S}_{\mathfrak{A}})$. Thus $\omega(a)$ can be interpreted as the expectation value for the observable $a \in \mathfrak{A}$.

**Definition 4.1.** (Correlated and Uncorrelated)
Let $a_1$ and $a_2$ be distinct elements of a C$^*$-algebra $\mathfrak{A}$ and $\omega \in \mathfrak{S}_{\mathfrak{A}}$. The elements $a_1$ and $a_2$ are said to be *uncorrelated* with respect to $\omega$ if

$$\omega(a_1 a_2) = \omega(a_1)\omega(a_2)$$

Otherwise they are said to be *correlated* with respect to $\omega$ By extension, a state $\omega \in \mathfrak{S}_{\mathfrak{A}}$ is said to be *uncorrelated* if every pair of elements in $\mathfrak{A}$ is uncorrelated with respect to it.

**Definition 4.2.** (Local Uncorrelated States)
A state $\omega \in \mathfrak{S}_{\mathfrak{A}_1 \otimes \mathfrak{A}_2}$ is called locally uncorrelated if for every pair $(a_1, a_2) \in \mathfrak{A}_1 \times \mathfrak{A}_2$

$$\omega(a_1 \otimes a_2) = \omega(a_1 \otimes I_2)\omega(I_1 \otimes a_2)$$

This notion of correlation is extended from the conventional notation based on random variables and is linked to entangled states through Theorem 2.47 i.e. the locally uncorrelated states are exactly the pure product states. Yet the nature of correlations present in a general separable state and an entangled state must be

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fundamentally different since entangled states only exist in truly quantum composite systems and can be either pure or mixed while locally correlated separable states are necessarily mixed states. A system in a product state $\omega_1 \otimes \omega_2$ reflects two unrelated and independent subsystems, whereas a general separable state is identified as a statistical mixing of such aforementioned systems.This latter statement implies that such scenarios can be replicated by independent preparation of subsystems through local operations coordinated by classical communication (the so called LOCC paradigm). By contrast, the pure entangled states permit local correlations between observables and so cannot be prepared by independently acting parties on subsystems. As a consequence the correlations present in any entangled states cannot be replicated by LOCC alone [41]. The correlations arising from LOCC equivalent sources are branded as classical correlations whereas the additional correlations innate to entangled states are denoted as quantum correlations.

The existence of quantum correlations, which are exclusive to entangled states, elevates entanglement from merely confirming quantum theory to an exploitable resource. In particular, some tasks are easier to complete by utilizing quantum correlations and allows for tasks which are impossible without these correlations [42].

The LOCC paradigm naturally relates to the quantification of entanglement at an operational level i.e. if a particular state can (with certainty) be transformed into another through LOCC alone, then any task performable by the latter state and LOCC must be performable by the former and LOCC. In this sense the utility of states can only be decreased through LOCC and the initial state is at least as entangled as the resultant state from LOCC.

The local unitaries hold a particularly important position in the LOCC paradigm as they represent reversible operations (consequential from unitary implying the existence of a unitary inverse). Combined with the noted degradation of entanglement under LOCC, then local unitaries must preserve entanglement so that any two states related by local unitaries must therefore have equal entanglement.

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As a result, any suitable measure of entanglement must be monotone decreasing, on average, under LOCC transformations. Since any separable state is accurately convertible to any other separable state via unitary LOCC, then any such measure automatically will achieve a shared minimum value across all separable states. Without loss of generality, this minimal value may be assumed to be zero. Moreover, the maximum of an entanglement measure will occur for a state which can be transformed into any other state via LOCC alone (such states are said to be maximally entangled). The existence of such states is not a priori (see for example [43]) but are known to exist for bipartite systems created from finite $d$-dimensional subsystems[11, 44] and will be denoted by $\sigma_d$. The existence of maximally entangled states forms the basis for the two primary bipartite operational measures: Entanglement Cost $E_C$ and Distillable Entanglement $E_D$.

$E_C$ quantifies the rate at which batches of a maximally entangled state on a $2 \times 2$-dimensional bipartite system can be converted through LOCC to an arbitrarily accurate approximation (for a large enough batch) of many copies of a given state:

$$E_C(\omega) = \inf \left\{ r : \lim_{n\to\infty} \left[ \inf_T D\left(\omega^{\otimes n}, T(\sigma_2^{\otimes rn})\right) \right] \right\}$$

where $T$ is a general LOCC process, $D$ is a suitable distance function, and $\sigma_2$ is the maximally entangled singlet state (which will be formally defined in Example 4.1) on a $2 \times 2$-dimensional system. $E_C$ is associated with quantum channel capacities as it represents how many $\sigma_2$ states are required to produce a copy of a given state through LOCC alone.

$E_D$ is associated with the reverse process of distillation wherein many copies of a given state are transformed through an LOCC process to produce output states that approximate batches of the singlet state $\sigma_2$ with arbitrary accuracy for a large enough number of copies. The efficacy with which distillation can be carried out is quantified by $E_D$ which is mathematically defined (analogously to $E_C$) as

$$E_D(\omega) = \sup \left\{ r : \lim_{n\to\infty} \left[ \inf_T D\left(T(\omega^{\otimes n}), \sigma_2^{\otimes rn}\right) \right] = 0 \right\}$$

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$E_D$ is tied to communication through quantum teleportation. The maximally entangled states are the only states that can be faithfully teleported. $E_D$ indicates the rate at which states can be converted into the reliably usable singlet state $\sigma_2$

Equality of $E_C$ and $E_D$ would require that the asymptotic limit of entanglement transformations underpinning these measures be reversible. While this is true for pure states [41], it is not true in general resulting in $E_C \leq E_D$ [45].

Calculation of the operational measures for finite dimensions is generally complicated. In addition, for infinite and multipartite systems, no single state from which all others can be created through LOCC exists i.e. there is no unique maximally entangled state[43, 46]. The development of entanglement measures from an operational point of view is thus incredibly convoluted. An axiomatic approach can be adopted to overcome some of this complexity, by considering real-valued functions of the state space that possess necessary properties to be considered as entanglement measures.

As previously discussed, the principle property any entanglement measure must possess is monotonicity under LOCC. Vidal [47] advocated that this is the only necessary postulate required arguing that any other essential properties (such as minimality on separable states) follows directly or, in the case that they do not, should be treated as convenient but optional. As such, the following notion is adopted:

**Definition 4.3.** (Entanglement Monotones and Measures)
For $\mathfrak{A} = \mathfrak{A}_1 \otimes \mathfrak{A}_2$ a function $E : \mathfrak{S}_A \to \mathbb{R}$ is called an *entanglement monotone* if

$$E(T(\omega)) \leq E(\omega)$$

for any local operation (to be defined in detail below) $T : \mathfrak{S}_{\mathfrak{A}} \to \mathfrak{S}_{\mathfrak{A}}$.

If, in addition, $\omega$ is a separable state implies that $E(\omega) = 0$, then it is called an *entanglement measure*.

*Remark* 4.4. The requirement that $E(\omega) = 0$ for separable states automatically implies that any entanglement measure is non-negative.

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To clarify the nature of the map $T$ in the above definition, in order to represent a physical quantum operation it clearly must be positive (to map states to states) and linear (so that any statistical mixture of input states is mapped to a statistical mixture of output states with conserved probability). In addition, the trace of any normal state cannot increase under the action of $T$ or else the probability of the quantum operation occurring will exceed 1 (conservation of the trace will not be imposed as it is only valid for deterministic operations, which are given the special title of quantum channel. A trace of less than 1 implies that the normalization of statistical ensembles may not be preserved which is perfectly valid for any process which is, from a probability standpoint, not memoryless. In this way any projection will still constitute a valid quantum operation). Finally, when an arbitrary finite system $M_n(\mathbb{C})$ is coupled to the system as an ancillary $\mathfrak{A} \otimes M_n(\mathbb{C})$, the trivial extension of $T$ must remain positive. Physically this corresponds to the requirement that if only one subsystem of the composite system undergoes a quantum operation, the result will still be a state on the composite system. This accommodates situations where the observed system is open and actually a subsystem of a much larger system. Since $\mathfrak{A} \otimes M_n(\mathbb{C})$ is isomorphic to $M_n(\mathfrak{A})$, the C$^*$-algebra of $n \times n$ matrices $a = [a_{ij}]$ with entries $a_{ij} \in \mathfrak{A}$ and $M_n(\mathfrak{A})^* = M_n(\mathfrak{A}^*)$ (Lemma 3.6.1 in [31]), then $T$ must be completely positive:

**Definition 4.5.** (Completely Positive Maps)
For a subspace $E$ of a C$^*$-algebra $\mathfrak{A}$ (or its dual space $\mathfrak{A}^*$), a subspace $F$ of a C$^*$-algebra $\mathfrak{B}$ (or $\mathfrak{B}^*$ respectively), and a linear map $T : E \to F$ define $T_n : M_n(E) \to M_n(F)$ by

$$T_n([e_{ij}]) = [T(e_{ij})]$$

$T$ is called *$n$-positive* if $T_n$ is positive and *completely positive* if it is $n$-positive for all $n \in \mathbb{N}$.

**Definition 4.6.** (Local Operation)
A *local operation* on a composite system $\mathfrak{A} = \mathfrak{A}_1 \otimes \mathfrak{A}_2$ is a map $T : \mathfrak{S}_\mathfrak{A} \to \mathfrak{S}_\mathfrak{A}$ of the form $T_1 \otimes I_2$ or $I_1 \otimes T_2$ where $T_i$ is a trace non-increasing completely positive map of $\mathfrak{S}_{\mathfrak{A}_i}$ into $\mathfrak{S}_{\mathfrak{A}_i}$ and $I_j$ is the identity map on $\mathfrak{S}_{\mathfrak{A}_j}$.

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*Remark* 4.7. For C$^*$-algebras, $\mathfrak{A}$ and $\mathfrak{B}$, and a bounded linear map $T : \mathfrak{A} \to \mathfrak{B}$ the transpose map ${}^tT : \mathfrak{B}^* \to \mathfrak{A}^*$ is defined by ${}^tT(\varphi) = \varphi \circ T$. It is clear, from its definition, that ${}^tT$ is positive if and only if $T$ is positive but it is also true that ${}^tT$ is completely positive if and only if T is completely positive since ${}^t(T_n) = ({}^tT)_n$ and the composition of any two completely positive maps is also completely positive. As a result, it is clear that the Schrödinger and Heisenberg formulation of dynamics are equivalent in the C$^*$-algebra setting.

The inclusion of classical communication allows parties to coordinate their actions and perform local operations in response to the outcomes of previous ones, particularly when said prior operations allow for multiple outcomes within certain probabilities (e.g. during a measurement). A complete mathematical description of an LOCC transformation is in general complex and beyond the scope of this dissertation (an interested reader is referred to Donald et al. [48] for a detailed description), but it is noted that the set of LOCC transformations is generally larger than that of local operations alone. The conceptual simplicity of LOCC transformations is offset by this mathematical complexity of representing such transformations. Two potential means of overcoming this complexity are to either consider a larger set of transformations that include the LOCC ones as a proper subset (e.g. the separable operations or positive partial transpose (PPT) operations) or alternatively consider a stronger monotonicity requirement on the entanglement measures. An example of the latter is the requirement that the entanglement measure does not increase on average under LOCC transfomations:

$$E(\omega) \geq \sum_i p_i E(\omega_i)$$

where $\{p_i, \omega_i\}$ is any ensemble of possible states $\omega_i$ and associated probabilities of occurrence $p_i$ that are obtained from $\omega$ by the action of an LOCC transformation.

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## 4.2 Convex Roof Measures

Pure state entanglement is comparatively simple next to mixed state entanglement. Moreover since any state can be represented as the barycenter of a measure supported by the pure states, intuitively an entanglement measure that is defined on the pure states can lead to an entanglement measure on a generic state. The methodology relies on first differentiating the pure product states from the entangled ones, a task which can be simplified by the following maps:

**Definition 4.8.** (Restriction Maps)
For a quantum composite system $\mathfrak{A} := \mathfrak{A}_1 \otimes \mathfrak{A}_2$ with state space $\mathfrak{S}_{\mathfrak{A}}$, the *restriction maps* $r_i : \mathfrak{S}_{\mathfrak{A}} \rightarrow \mathfrak{S}_{\mathfrak{A}_i}$ to the $i^{\text{th}}$ subsystem are defined by:

$$(r_1\omega)(a_1) = \omega(a_1 \otimes I_2) \qquad \forall a_1 \in \mathfrak{A}_1$$

$$(r_2\omega)(a_2) = \omega(I_1 \otimes a_2) \qquad \forall a_2 \in \mathfrak{A}_2$$

The first important property of the restriction maps is found in Proposition 4.1.37 in [28]:

**Proposition 4.9.** *$r_i$ is weak$^*$-continuous with $\mathscr{E}(\mathfrak{S}_{\mathfrak{A}_i}) \subseteq r_i\mathscr{E}(\mathfrak{S}_{\mathfrak{A}})$.*

**Proposition 4.10.** *For $\omega \in \mathfrak{S}_{\mathfrak{A}_1 \otimes \mathfrak{A}_2}$ if $r_i\omega$ is a pure state on $\mathfrak{A}_i$ for either $i = 1$ or $i = 2$, then $\omega = \omega_1 \otimes \omega_2$ where $\omega_i \in \mathfrak{S}_{\mathfrak{A}_i}$.*

*Proof.* Let $r_1\omega$ be a pure state and let $a_2 \in \mathfrak{A}_2$. Since $\mathfrak{A}_2$ is spanned by its positive part then without loss of generality, assume that $0 \leq a_2 \leq I_2$ hence $0 \leq \omega(I_1 \otimes a_2) \leq 1$ since $I_2 - a_2 \geq 0$.

If $\omega(I_1 \otimes a_2) = 0$ then the Cauchy-Schwarz inequality implies that:

$$\begin{aligned} |\omega(a_1 \otimes a_2)|^2 &\leq \omega\left(\left(a_1 \otimes a_2^{1/2}\right)\left(a_1 \otimes a_2^{1/2}\right)^*\right) \\ &= \omega(a_1 a_1^* \otimes a_2)\omega(I_1 \otimes a_2) \\ &= 0 \end{aligned}$$

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Thus $\omega(a_1 \otimes a_2) = 0 = \omega(I_1 \otimes a_2)\omega(a_1 \otimes I_2)$. If $\omega(I_1 \otimes a_2) = 1$ then the same argument using $I - I_1 \otimes a_2$ also yields $\omega(a_{1\otimes}a_2) = \omega(I_1 \otimes a_2)\omega(a_1 \otimes I_2)$.

Hence assume that $0 < \omega(a_1 \otimes a_2) < 1$. Then

$$(r_1\omega)(a_1) = \omega(a_1 \otimes I_2) = \omega(I_1 \otimes a_2)\frac{\omega(a_1 \otimes a_2)}{\omega(I_1 \otimes a_2)} + (1 - \omega(I_1 \otimes a_2))\frac{\omega(a_1 \otimes I_2) - \omega(a_1 \otimes a_2)}{1 - \omega(I_1 \otimes a_2)}$$
$$= \omega(a_1 \otimes 1_2)\varphi_1(a_1) + (I - \omega(a_1 \otimes 1_2))\varphi_2(a_1)$$

Where $\varphi_1$ and $\varphi_2$ are defined by:

$$\varphi_1(a_1) := \frac{\omega(a_1 \otimes a_2)}{\omega(a_1 \otimes a_2)} \qquad \text{and} \qquad \varphi_2(a_1) := \frac{\omega(a_1 \otimes I_2) - \omega(a_1 \otimes a_2)}{1 - \omega(I_1 \otimes a_2)}$$

. But $\varphi_1 = \varphi_2$ by purity of $r_1\omega$ i.e

$$\frac{\omega(a_1 \otimes a_2)}{\omega(1_1 \otimes a_2)} = \frac{\omega(a_1 \otimes I_2) - \omega(a_1 \otimes a_2)}{1 - \omega(1_1 \otimes a_2)}$$

$$\Rightarrow \omega(a_1 \otimes a_2) = \omega(a_1 \otimes 1_2)\omega(1_1 \otimes a_2) = (\omega_1 \otimes \omega_2)(a_1 \otimes a_2)$$

□

If $\omega$ is a pure state, then it is either a product state or entangled, hence the above propositions imply that the restriction maps can be employed as a reliable means of determining which of these two possibilities is applicable. The restriction maps thus has the potential to become an entanglement measure when combined with any non-negative real valued function that is defined on the state space of either subsystem and which vanishes on the pure states, and only on the pure states, of that subsystem. The focus is then to find the smallest set of additional properties that must be possessed by these functions in order for them to extend to entanglement measures on the entire set of states.

This extension from the relatively simple scenario of pure states to the mixed states is facilitated by the integral representation of states i.e. by recalling that the set $M_\omega(\mathfrak{S})$, which includes all the barycentric decompositions of the state $\omega$, contains at least one maximal element supported by the pure states. When

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the aforementioned function is continuous, then it can be applied to the mixed states through any barycentric decomposition i.e. $f(\omega) = \mu(f) = \int_{\mathfrak{S}_\mathfrak{A}} f(\omega')d\mu(\omega')$ for any $\mu \in M_\omega(\mathfrak{S}_\mathfrak{A})$ and $f \in \mathfrak{S}^*_\mathfrak{A}$. Intuitively, since there are many possible decompositions of a given state (in particular the only scenario for which the maximal element of $M_\omega(\mathfrak{S})$ is unique is the classical case where $\mathfrak{A}$ is abelian), in a truly quantum system all possible decompositions should be considered and the least expected entanglement selected.

Thus a prototype entanglement measure on a composite quantum system $\mathfrak{A} = \mathfrak{A}_1 \otimes \mathfrak{A}_2$ is

$$E(\omega) = \inf_{\mu \in M_\omega(\mathfrak{S}_\mathfrak{A})} \int_{\mathfrak{S}_\mathfrak{A}} f(r_i\varphi)d\mu(\varphi) \tag{4.1}$$

where $f : \mathfrak{S}_{\mathfrak{A}_i} \to \mathbb{R}^+$ is a continuous, non-negative, real-valued function that vanishes on the pure states of $\mathfrak{A}_i$. If $\{\mu_\alpha\}$ is a net in $M_\omega(\mathfrak{S}_\mathfrak{A})$ which converges to a point $\mu$ then clearly $\int_{\mathfrak{S}_\mathfrak{A}} \varphi d\mu(\varphi) = \lim \int_{\mathfrak{S}_\mathfrak{A}} \varphi d\mu_\alpha(\varphi) = \omega$ i.e. $\mu \in M_\omega(\mathfrak{S}_\mathfrak{A})$ thus $M_\omega(\mathfrak{S}_\mathfrak{A})$ is a closed subset of the compact set $M_1(\mathfrak{S}_\mathfrak{A})$ and so is itself compact. Since $M_\omega(\mathfrak{S}_\mathfrak{A})$ is a compact set and the integral taken over a continuous function, then the infimum is attained within that set (see page 89 in [49]). Therefore this prototype is well defined in the sense that for every $\omega \in \mathfrak{S}_\mathfrak{A}$ there exists a $\mu_0 \in M_\omega(\mathfrak{S}_\mathfrak{A})$ such that $\omega = \int_{\mathfrak{S}_\mathfrak{A}} \varphi d\mu_0(\varphi)$ and $E(\omega) = \int_{\mathfrak{S}_\mathfrak{A}} f(r_1\varphi)d\mu_0(\varphi)$. It remains to be shown that it distinguishes separable from entangled states.

Let $\omega$ be separable i.e. there exists a net $\{\omega^N\}$ that converges to $\omega$ where

$$\omega^N = \sum_i \lambda_i^N \omega_{1,i}^N \otimes \omega_{2,i}^N = \sum_i \lambda_i^N \omega_i^N \qquad \text{with } \sum_i \lambda_i = 1$$

Consider a state $\omega = \omega_1 \otimes \omega_2$ where $\omega_1$ and $\omega_2$ are not pure. By Definition 2.5, both $\omega_1$ and $\omega_2$ can be approximated as linear combinations of pure states i.e. $\omega_1 = \sum_j \alpha_j \varphi_{1,j}$ and $\omega_2 = \sum_k \beta_k \varphi_{2,k}$ where $\{\varphi_{1,j}\}$ and $\{\varphi_{2,k}\}$ are pure states on $\mathfrak{A}_1$ and $\mathfrak{A}_2$ respectively and $\sum_j \alpha_j = 1$ and $\sum_k \beta_k = 1$. By the distributivity of the tensor product (which follows directly from multilineality in Definition 2.29)

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it follows that

$$\omega = \left(\sum_j \alpha_j \varphi_{1,j}\right) \otimes \left(\sum_k \beta_k \varphi_{2,k}\right) = \sum_{j,k} \alpha_j \beta_k (\varphi_{1,j} \otimes \varphi_{2,k})$$

where $\sum_{j,k} \alpha_j \beta_k = (\sum_j \alpha_j)(\sum_k \beta_k) = 1$.

Therefore, without loss of generality, it can be assumed that each $\omega_i^N$ is a pure product state.

Define $\mu^N = \sum \lambda_i^N \delta_{\omega_i^N} = \sum \lambda_i^N \delta_{\omega_{1,i}^N \otimes \omega_{2,i}^N}$. Any net in a compact topological space contains a convergent subnet (combine Theorem 11.5 and Theorem 17.4 in [33]). Since $M_1(\mathfrak{S}_\mathfrak{A})$ is weak$^*$ compact, then the net $\{\mu^N\}$ contains a weak$^*$ convergent subnet $\{\mu^\alpha\}$ whose limit $\mu_0$ satisfies

$$\int_{\mathfrak{S}_\mathfrak{A}} \varphi d\mu_0(\varphi) = \lim_\alpha \int_{\mathfrak{S}_\mathfrak{A}} \varphi d\mu^\alpha(\varphi) = \lim \omega^\alpha = \omega$$

i.e. $\mu_0$ represents $\omega$.

Since $f$ and $r_i$ are continuous, then (by Theorem 11.8 in [33]) it follows that

$$\int_{\mathfrak{S}_\mathfrak{A}} f(r_i\varphi) d\mu_0(\varphi) = \lim_\alpha \int_{\mathfrak{S}_\mathfrak{A}} f(r_i\varphi) d\mu^\alpha(\varphi)$$

Moreover $\mu^\alpha$ is, by definition, supported on pure product states for which $f \circ r_i$ vanishes thus implying

$$\int_{\mathfrak{S}_\mathfrak{A}} f(r_i\varphi) d\mu_0(\varphi) = 0$$

i.e. there exists a $\mu_0 \in M_\omega(\mathfrak{S}_\mathfrak{A})$ such that $E(\omega) = 0$.

If $E(\omega) = 0$ then there exists a measure $\mu_0 \in M_\omega(\mathfrak{S}_\mathfrak{A})$ such that

$$\int_{\mathfrak{S}_\mathfrak{A}} f(r_i\varphi) d\mu_0(\varphi) = 0$$

Since $f$ is non-negative then $f(r_i\varphi) = 0 \ \forall \ \varphi$ in the support of $\mu_0$. But then, by the definition of $f$, $r_i\varphi$ is pure for every $\varphi$ in the support of $\mu_0$ which means that,

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by Proposition 4.10, that $\varphi = \varphi_1 \otimes \varphi_2$. Hence $\mu_0 \in M_\omega(\mathfrak{S}_\mathfrak{A})$ is supported by the product states from which it can be concluded that $\omega$ is separable.

In order for $E$ to be a true entanglement measure, it is enough that $E$ satisfy the monotonicity requirement of not increasing, on average, under LOCC transformations. This will occur whenever $f$ is concave so that

$$f \circ r_i(\omega) \geq \int_{\mathfrak{S}_\mathfrak{A}} f(r_i(\varphi)) d\mu(\varphi)$$

for any $\mu \in M_\omega(\mathfrak{S}_\mathfrak{A})$. Indeed Vidal [47] showed that this condition of concavity is the critical property required by any entanglement monotone that satisfies this stronger monotonicity requirement. Hence it is now possible to define:

**Definition 4.11.** (Convex Roof Measure)
For a composite system $\mathfrak{A} = \mathfrak{A}_1 \otimes \mathfrak{A}_2$, $F \subseteq \mathfrak{S}_\mathfrak{A}$ a Ruelle separable subset (as given in Definition 3.47), and a continuous concave function $f : \mathfrak{A}_i \to \mathbb{R}^+$ (where $i \in \{1, 2\}$) which vanishes on the pure states and only on the pure states of $\mathfrak{A}_i$, the function $E_f : F \to \mathbb{R}^+$ defined by

$$E(\omega) = \inf_{\mu \in M_\omega(\mathfrak{S}_\mathfrak{A})} \int_{\mathfrak{S}_\mathfrak{A}} f(r_i \varphi) d\mu(\varphi)$$

is called the *convex roof measure associated with* $f$.

*Remark* 4.12. Ruelle separability of $F$ is assumed in order to prevent undesriable properties in elements of $M_\omega(\mathfrak{S}_\mathfrak{A})$ as explained following Remark 3.46 and ensure the existence of a measure supported by the pure states. As discussed, this assumption is satisfied for most physical systems (Example 3.1 and subsequent remarks).

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### 4.2.1 Properties of the Convex Roof Measures

#### 4.2.1.1 Relationship to Bennet and Uhlmann's Definitions

The first convex roof measure was introduced by Bennet et al. [21] with a generalization refined by Uhlmann [22]. These measures gained their name from the latter's method of construction where a function $g : P_{\mathfrak{A}} \to \mathbb{R}^+$ defined exclusively on the pure states can be extended to the mixed states by

$$g^{\cup}(\omega) := \inf \left\{ \sum_{i=1}^{n} \lambda_i g(\omega_i) : \omega = \sum_{i=1}^{n<\infty} \lambda_i \omega_i \ , \ \omega_i \in P_{\mathfrak{A}} \ , \ \sum_{i=1}^{n<\infty} \lambda_i = 1 \right\}$$

where the infimum runs over all finite convex decompositions of $\omega$ in terms of pure states. In order for $g^{\cup}$ to make sense as an entanglement measure, the infimum cannot be taken over an empty set. So there must exist at least one optimal decomposition i.e. a finite convex decomposition of $\omega$ into pure states $\varphi_i$ so that the infimum is evaluated over a non-empty set i.e.

$$g^{\cup}(\omega) = \sum_{i=1}^{n} \lambda_i g(\varphi_i)$$

The existence of optimal decompositions is equivalent to the existence of a measure $\mu_0 \in M_\omega(\mathfrak{S})$ that is finitely supported on the pure states thus

$$\mu_0 = \sum_{i=1}^{n} \lambda_i \delta_{\varphi_i}$$

By Theorem 3.49 it is known that such a measure would be maximal with respect to the order "$\succ$" (if it exists). Thus in order to find an optimal decomposition it is only necessary to look at the maximal measures that are supported by the pure states and represent $\omega$. In order to ensure that at least one measure in $M_\omega(\mathfrak{S})$ is maximal and supported on the pure states it is enough that $\omega$ belongs to a Ruelle separable subset of $\mathfrak{S}_{\mathfrak{A}}$ (Theroem 3.49 and Proposition 3.39). However this measure need not necessarily be finitely supported.

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The current goal is to outline sufficient criteria required for a measure to be associated with an optimal decomposition so as to show compataility between the presented and historical versions. To this end, note that (see Proposition 1.3.1 in [39]):

**Proposition 4.13.** *Let $K$ be a convex compact subset of a locally compact vector space $X$ and $\mu, \nu \in M_+(K)$.*

$$\mu \succ \nu \Leftrightarrow \nu(f) \leq \mu(\bar{f}) \ \forall f \in C(K)$$

*where $\bar{f}$ is given by Definition 3.40.*

As a direct consequence, $\mu \in M_\omega(\mathfrak{S}_\mathfrak{A})$ if and only if $\mu(f) \leq \delta_\omega(\bar{f})$ for all $f \in C(\mathfrak{S}_\mathfrak{A})$. However, by Proposition 3.39 and Lemma 3.42, for any $\omega \in \mathfrak{S}_\mathfrak{A}$ and $f \in C(\mathfrak{S}_\mathfrak{A})$ there exists a $\nu \in M_\omega(\mathfrak{S}_\mathfrak{A})$ such that $\nu(f) = \delta_\omega(\bar{f}) = \bar{f}(\omega)$. Thus $\mu(f) \leq \delta_\omega(\bar{f}) = \bar{f}(\omega)$ which implies that

$$\bar{f}(\omega) = \sup_{\mu \in M_\omega(\mathfrak{S}_\mathfrak{A})} \mu(f) \tag{4.2}$$

Since the supremum will be reached on some element of $M_\omega(\mathfrak{S}_\mathfrak{A})$ then the set

$$F_f := \left\{\mu \in M_\omega(\mathfrak{S}_\mathfrak{A}) : \mu(f) = \bar{f}(\omega)\right\}$$

is not empty.

If $\mu_1, \mu_2 \in M_\omega(\mathfrak{S}_\mathfrak{A})$ then, by Equation 4.2, both $\mu_1(f) \leq \bar{f}$ and $\mu_2(f) \leq \bar{f}$. If $\mu \in F_f$ such that $\mu = \lambda\mu_1 + (1-\lambda)\mu_2$ for $\lambda \in (0,1)$ then $\bar{f}(\omega) = \lambda\mu_1(f) + (1-\lambda)\mu_2(f)$ which is only possible if $\mu_1(f) = \bar{f}$ and $\mu_2(f) = \bar{f}$ i.e. both $\mu_1$ and $\mu_2$ are elements of $F_f$ as well hence $F_f$ is a face. Moreover $F_f$ is clearly closed $\left(F_f \ni \{\mu_\alpha\} \to \mu \Rightarrow \mu(f) = \bar{f}(\omega)\right)$ and hereditary upwards ($\mu \in F_f$, $\nu \in M_\omega(\mathfrak{S}_\mathfrak{A})$ such that $\mu \succ \nu$ requires $\nu(f) = \bar{f}(\omega)$ so that $\nu \in F_f$) and so the following proposition (see Proposition 1.6.4 in [39]) applies to $F_f$:

**Proposition 4.14.** *The intersection $F \cap \mathcal{E}(Z)$ between any closed and hereditary upwards face $F$ of an ordered compact convex $K$ and the extreme boundary of the*

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*set $Z$ of all maximal (with respect to the order on $K$) elements in $K$ is not empty. In particular $\mathcal{E}(Z) \neq \emptyset$.*

As a result, the set $\{\mu(-f \circ r_i) : \mu \in M_\omega(\mathfrak{S}_\mathfrak{A})\}$ (where $f$ is a continuous concave function) will obtain its maximum value on an extreme point of $M_\omega(\mathfrak{S}_\mathfrak{A})$ which is maximal for the order "$\succ$".

**Definition 4.15.** (Simplicial Measures)
The extreme points of $M_\omega(\mathfrak{S}_\mathfrak{A})$ are known as the *simplicial measures*.

To be clear, maximality with respect to "$\succ$" is defined in terms of convex functions while Definition 4.11 utilizes a concave function $f \circ r_i$ in order to be a valid measure of entanglement. This implies that the measure for which the convex roof measures of Definition 4.11 will attain their infimum on will be the measure for which $-f \circ r_i$ obtains its maximum value i.e.:

**Proposition 4.16.** *Let $E_f$ be a convex roof measure as given in Definition 4.11. The infimum for this entanglement measure is attained on a maximal simplicial measure in $M_\omega(\mathfrak{S}_\mathfrak{A})$.*

Both Bennet et al. and Uhlmann restrict themselves to finite dimensional systems i.e. when $\mathfrak{A}$ admits a representation $\{\pi, \mathscr{H}\}$ where $\mathscr{H}$ has finite dimension $n$. In this case $\mathfrak{S}_\mathfrak{A}$ can be viewed as a compact convex subset of $\mathbb{R}^{(2n)^2}$ (under the density matrix formulation) and so any simplicial measure has finite support by Proposition 1.6.11 in [39]:

**Proposition 4.17.** *Any $\mu \in M_1^+(K)$, where $K$ is a compact convex set in $\mathbb{R}^n$, is simplicial if and only if $\mu$ is supported by an affinely independent set of (at most) $n+1$ points of $\mathbb{R}^n$.*

*Remark* 4.18. A set of points $\{x_1, \ldots, x_n\}$ in a vector space is called affinely independent if $\sum \lambda_i x_i = 0$ with $\sum \lambda_i = 0$ implies $\lambda_1 = \cdots = \lambda_n = 0$. Otherwise $\{x_1, \ldots, x_n\}$ are called affinely dependent.

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Moreover, since the entire state space of a finite system is Ruelle Separable, then the support of a maximal simplicial measure is a finite subset of the pure states so that an optimal decomposition of a state always exists. Hence Definition 4.11 is equivalent to that of Bennet et al. and Uhlmann's in finite dimensional cases.

#### 4.2.1.2 Convexity

The convexity of an entanglement measure, while not essential, is a valued property that represents information loss during a statistical mixing process.

By Proposition 3.39 $\mu \in M_\omega(\mathfrak{S}_\mathfrak{A})$ if and only if $\mu(f) \geq f(\omega)$ for all continuous, real-valued, convex functions $f$.

Let $\omega_1, \omega_2$ be two fixed elements of a Ruelle separable subset of $\mathfrak{S}_\mathfrak{A} = \mathfrak{S}_{\mathfrak{A}_1 \otimes \mathfrak{A}_2}$ and $\lambda \in (0,1)$. Consider the set

$$M(\omega_1, \omega_2, \lambda) := \{\lambda\mu_1 + (1-\lambda)\mu_2 : \mu_1 \in M_{\omega_1}(\mathfrak{S}_\mathfrak{A}), \mu_2 \in M_{\omega_2}(\mathfrak{S}_\mathfrak{A})\}$$

If $f$ is a continuous, real-valued, convex functions, then for $\nu \in M(\omega_1, \omega_2, \lambda)$ it follows that

$$\nu(f) = \lambda\nu_1(f) + (1-\lambda)\nu_2(f) \geq \lambda f(\omega_1) + (1-\lambda)f(\omega_2) \geq f(\lambda\omega_1 + (1-\lambda)\omega_2)$$

where $\nu_1 \in M_{\omega_1}(\mathfrak{S}_\mathfrak{A})$ and $\nu_2 \in M_{\omega_2}(\mathfrak{S}_\mathfrak{A})$.

Hence $M(\omega_1, \omega_2, \lambda) \subseteq M_{\lambda\omega_1 + (1-\lambda)\omega_2}(\mathfrak{S}_\mathfrak{A})$.

If $E_g$ is the convex roof measure associated with the function $g : \mathfrak{S}_{\mathfrak{A}_i} \to \mathbb{R}^+$ as given in Definition 4.11, then clearly

$$E_g(\lambda\omega_1 + (1-\lambda)\omega_2) = \inf_{\mu \in M_{\lambda\omega_1 + (1-\lambda)\omega_2}(\mathfrak{S})} \int g(r_i\varphi) d\mu(\varphi)$$

$$\leq \lambda \inf_{\mu \in M_{\omega_1}(\mathfrak{S})} \int g(r_i\varphi) d\mu(\varphi) + (1-\lambda) \inf_{\mu \in M_{\omega_2}(\mathfrak{S})} \int g(r_i\varphi) d\mu(\varphi)$$

$$= \lambda E(\omega_1) + (1-\lambda)E(\omega_2)$$

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follows from the aforementioned set inclusion. Hence the function $E_g : \mathfrak{S}_{\mathfrak{A}_i} \to \mathbb{R}^+$ is convex.

#### 4.2.1.3 Continuity

In order to show that the convex roof measures are continuous, the following preliminaries (from [50]) will be required.

**Definition 4.19.** (Hemicontinuity)
A set-valued function $f$ from a topological space $X$ that maps points $x \in X$ to a closed set of points $Y_x$ is called *lower hemicontinuous* if for every closed $K \subset X$, the set $\{x : f(x) \subset K\}$ is closed. $f$ is called *upper hemicontinuous* if for every closed $K \subset X$, the set $\{x : f(x) \cap K \neq \emptyset\}$ is closed.

**Proposition 4.20.** *A set valued function $f$ on a topological space $X$ is lower hemicontinuous if and only if for every open set $U \subset X$ the set $\{y : f(y) \cap U \neq \emptyset\}$ is open. $f$ is upper hemicontinuous if an only if for every open set $U \subset X$ the set $\{y : f(y) \subset U\}$ is open.*

*Proof.* This is immediate from the observation that

$$\{x : f(x) \subset K\}^c = \{y : f(y) \cap K^c \neq \emptyset\}$$

where $K^c$ denotes the complement of the set $K$. □

**Proposition 4.21.** *Consider $\mathfrak{A} = \mathfrak{A}_1 \otimes \mathfrak{A}_2$ (where $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are $C^*$-algebras), a continuous concave function $f : \mathfrak{S}_{\mathfrak{A}_i} \to \mathbb{R}^+$ (where $i$ is either 1 or 2) such that $f(\omega) = 0$ if and only if $\omega \in P_{\mathfrak{A}}$, and $E_f$ the convex roof measure associated with $f$ as given in Definition 4.11. $E_f$ is continuous with respect to the weak topology on $\mathfrak{S}_{\mathfrak{A}}$.*

*Proof.* To prove lower semi-continuity let $\{\omega_\alpha\}$ be a net of states in $\mathfrak{S}_{\mathfrak{A}}$ with a limit $\omega \in \mathfrak{S}_{\mathfrak{A}}$ such that $E_f(\omega_\alpha) \leq s$ for some $s \in \mathbb{R}$. Since the infimum in the definition of $E_f(\omega)$ is attained on some element in $M_\omega(\mathfrak{S}_{\mathfrak{A}})$ then it follows that

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there must exist at least one $\mu \in M_\omega(\mathfrak{S}_\mathfrak{A})$ such that $\mu(f \circ r_i) \leq s$ if $E_f(\omega) \leq s$. Hence for $\epsilon \geq 0$ choose $\mu_\alpha \in M_{\omega_\alpha}(\mathfrak{S}_\mathfrak{A})$ such that $\mu_\alpha(f \circ r_i) < s + \epsilon$ to produce a net $\{\mu_\alpha\}$ in $M_1(\mathfrak{S}_\mathfrak{A})$. Since $M_1(\mathfrak{S}_\mathfrak{A})$ is a compact set in the weak$^*$ topology, there exists a weak$^*$ convergent subnet $\{\mu_\beta\}$ with limit $\mu_0$.

For any affine, real-valued function $g$ on $\mathfrak{S}_\mathfrak{A}$ (i.e. $g \in A(\mathfrak{S}_\mathfrak{A})$)

$$g(\omega_\beta) = \mu_\beta(g) \to \mu_0(g)$$

but also

$$g(\omega_\beta) \to g(\omega)$$

Hence by the uniqueness of the limit $\mu_0(g) = g(\omega)$ for all $g \in A(\mathfrak{S}_\mathfrak{A})$ which implies that $\mu_0 \in M_\omega(\mathfrak{S})$ (because every $a \in \mathfrak{A}$ defines an affine linear functional $\hat{a}$ on $\mathfrak{A}^*$ by $\hat{a}(\omega) = \omega(a)$ and every linear functional that is continuous on $\mathfrak{A}^*$ has this form (see Chapter IV Theorem 1.3 in [38]). Thus for all $\epsilon > 0$

$$E_f(\omega) \leq \mu_0(f \circ r_i) = \lim_{\beta \to \infty} \mu_\beta(f \circ r_i) \leq s + \epsilon$$

This implies that $E_f(\omega) \leq s$ i.e. the convex roof measures are lower semi-continuous.

To prove upper semi-continuity, let $\{\omega_\alpha\}$ be a net with limit point $\omega_0$ such that $E_f(\omega_\alpha) \geq s$ for some $s \in \mathbb{R}$. For any $\mu_\alpha \in M_{\omega_\alpha}(\mathfrak{S}_\mathfrak{A})$, it follows from the definition of $E_f$ that $\mu_\alpha(f \circ r_i) \geq E_f(\omega_\alpha) \geq s$.

Any net $\{\mu_\alpha\}$(where $\mu_\alpha \in M_{\omega_\alpha}(\mathfrak{S}_\mathfrak{A})$) is convergent (by passing to a subnet if necessary) with a limit $\mu_0$. By the same argument as before, $\mu_0 \in M_{\omega_0}(\mathfrak{S}_\mathfrak{A})$. Additionally

$$\mu_0(f \circ r_i) = \lim_\alpha \mu_\alpha(f \circ r_i) \geq s$$

If $\mu(f \circ r_i) \geq s$ for every $\mu \in M_{\omega_0}(\mathfrak{S}_\mathfrak{A})$ then $E_f \geq s$ by definition. Therefore, to show $E_f \geq s$, it is enough to show that every $\mu \in M_{\omega_0}(\mathfrak{S}_\mathfrak{A})$ is the limit of a net $\{\mu'_\alpha\}$ where $\mu'_\alpha \in M_{\omega_\alpha}(\mathfrak{S}_\mathfrak{A})$.

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Denote by $q$ the set valued map defined by

$$q(\omega) := M_\omega(\mathfrak{S}_\mathfrak{A})$$

First it will be shown that $q$ is upper hemicontinuous. Consider sets of the form $O_K = \{\omega : q(\omega) \cap K \neq \emptyset\}$ where $K \subseteq M_1(\mathfrak{S}_\mathfrak{A})$ is closed (hence it is also compact).

Let $\{\omega_i\}$ be a net in $O_K$ with limit $\omega$. From any net $\{\mu_i \in q(\omega_i) \cap K\}$ select a convergent subnet $\{\mu_j\}$ with limit $\mu \in K$. But $\mu \in M_\omega(\mathfrak{S}_\mathfrak{A})$ by

$$\omega = \lim_j \omega_j = \lim_j \int \varphi d\mu_j(\varphi) = \int \varphi d\mu(\varphi)$$

Thus $O_K$ is closed meaning that $q$ is upper hemicontinuous.

It follows from Proposition 4.20 that for any open set $U \subseteq M_1(\mathfrak{S}_\mathfrak{A})$ the set $\{\omega : q(\omega) \subseteq U\}$ is open. In the weak$^*$ topology on $M_1(\mathfrak{S}_\mathfrak{A})$, a neighbourhood of $\nu$ has the form

$$\mathcal{U}(\nu; f_1, \ldots, f_n, \epsilon) := \left\{\nu' \in C(\mathfrak{S}_\mathfrak{A})^* : \left|\nu(f_i) - \nu'(f_i)\right| \leq \epsilon \ \forall \ i = 1, \ldots, n\right\}$$

where $\epsilon > 0$ and $f_1, \ldots, f_n$ is any set of elements in $C(\mathfrak{S}_\mathfrak{A})$. If $\nu_1, \nu_2 \in M_\varphi(\mathfrak{S}_\mathfrak{A})$ then $\nu_1(f) = f(\varphi) = \nu_2(f)$ for all $f \in C(\mathfrak{S}_\mathfrak{A})$ thus

$$\nu_1 \in \mathcal{U}(\nu; f_1, \ldots, f_n, \epsilon) \iff \nu_2 \in \mathcal{U}(\nu; f_1, \ldots, f_n, \epsilon)$$

It then follows that if $\mathcal{U}_\mu$ is a neighbourhood of $\mu \in M_{\omega_0}(\mathfrak{S})$ then $M_{\omega_0}(\mathfrak{S}_\mathfrak{A}) \subseteq \mathcal{U}_\mu$ which in turn implies $\{\omega : q(\omega) \subseteq \mathcal{U}_\mu\}$ is a neighbourhood of $\omega_0$ and since if $\omega_\alpha \to \omega_0$ then $\{\omega : q(\omega) \subseteq \mathcal{U}_\mu\}$ must contain at least one $\omega_\alpha$ so that $M_{\omega_\alpha}(\mathfrak{S}_\mathfrak{A}) = q(\omega_\alpha) \subseteq \mathcal{U}_\mu$ i.e. every neighbourhood $\mathcal{U}_\mu$ of $\mu$ contains at least one element $\mu'_\alpha \in M_{\omega_\alpha}(\mathfrak{S}_\mathfrak{A})$ so that every $\mu \in M_{\omega_0}(\mathfrak{S}_\mathfrak{A})$ is the limit point of some net $\{\mu'_\alpha\}$ as required. □

An immediate application of the continuity of the convex roof measures is to establish the existence of maximally entangled states via the Bauer maximality principle (see Proposition 4.1.12 in [28]):

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**Lemma 4.22.** *If $K$ is a non-empty compact convex set in a Hausdorff locally convex topological vector space and $f : K \to \mathbb{R}$ is a convex and upper semi-continuous function, then $f$ attains its maximum value on some extreme point of $K$.*

In particular, by taking $K = \mathfrak{S}_{\mathfrak{A}}$:

**Corollary 4.23.** *A convex roof measure on $\mathfrak{S}_{\mathfrak{A}_1 \otimes \mathfrak{A}_2}$ will attain its maximum value on a pure state.*

## 4.2.2 Examples and Applications

### 4.2.2.1 Entanglement of Formation

Bennet et al. [21] pioneered the convex roof measures with the introduction of the Entanglement of Formation. The original form was

$$E_F(\rho) = \min\left\{\sum p_i S(\mathrm{Tr}_1(\rho_i))\right\}$$

Where $\rho$ is the density matrix of a finite dimensional composite system ($\mathscr{H}_1 \otimes \mathscr{H}_2$ where $\dim \mathscr{H}_1 < \infty$ and $\dim \mathscr{H}_2 < \infty$), $\mathrm{Tr}_1$ is the partial trace over the first subsystem, $S(\rho) = -\mathrm{Tr}(\rho \ln(\rho))$ is the von Neumann entrophy, and the minimum is taken over finite pure state decompositions of $\rho$ (i.e. $\rho = \sum p_i \rho_i$ where $\rho_i = |\Psi_i\rangle \langle\Psi_i|$ for some unit vector $|\Psi_i\rangle \in \mathscr{H}_1 \otimes \mathscr{H}_2$).

Majewski[13] noted that there was no reason why the minimum in this definition was justified nor why both subsystems had to be finite and so improved upon the original definition (which is adopted as the standard definition for the remainder of this dissertation):

**Definition 4.24.** (Entanglement of Formation)
For any normal state $\omega(\cdot) = \mathrm{Tr}(\rho_\omega(\cdot))$ acting on a composite system $\mathfrak{A} = \mathfrak{A}_1 \otimes \mathfrak{A}_2$ with $\mathfrak{A}_1 = B(\mathscr{H})$ where $\dim(\mathscr{H}) = n < \infty$, the *Entanglement of Formation* is

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given by

$$E_F(\omega) := \inf_{\mu \in M_\omega(\mathfrak{S}_\mathfrak{A})} \int S(r_1\varphi) d\mu(\varphi)$$

where $S(\phi) = -\operatorname{Tr}(\rho_\phi \ln(\rho_\phi))$ is the Von Neumann entropy.

*Remark* 4.25. The assumption of $\mathfrak{A}_1$ being finite is solely to ensure that the von Neumann entropy satisfies the conditions required by Definition 4.11. This situation is still very general and corresponds to the realistic scenario of a "small" system entangled with a "larger" heat bath such as the environment. Additionally, since the collection of all normal states is a Ruelle separable subset of $\mathfrak{S}_\mathfrak{A}$ (Example 3.1) then it would be redundant to mention it in Definition 4.24.

**Example 4.1.** *For* $\mathfrak{A} = B(\mathscr{H}_1) \otimes B(\mathscr{H}_2) = B(\mathscr{H}_1 \otimes \mathscr{H}_2)$ *with* $\dim \mathscr{H}_1 = \dim \mathscr{H}_2 = d$. *If* $\{\epsilon_i\}$ *denotes a basis for* $\mathscr{H}_i$, *then* $\{\epsilon_{ij} = \epsilon_i \otimes \epsilon_j\}$ *is a basis for* $\mathscr{H}_1 \otimes \mathscr{H}_2$ *known as the standard basis. Consider the vector* $\Omega := \frac{1}{\sqrt{d}} \sum_{i=1}^{d} \epsilon_{ii} \in \mathscr{H}_1 \otimes \mathscr{H}_2$ *to which the state* $\sigma_d^+(\cdot) = \operatorname{Tr}(\rho_\Omega(\cdot))$ *is associated (where* $\rho_\Omega$ *is the density matrix* $|\Omega\rangle\langle\Omega|$*). It follows that* $E_F(\sigma_d^+) = \ln(d)$ *since* $r_1\sigma_d^+(\cdot) = \operatorname{Tr}\left[\frac{1}{d} I_1(\cdot)\right]$.

The state $\sigma_d^+$, known as the singlet state, has important historical significance as it is a known maximally entangled state for finite dimensional systems. Thus, since the Von Neumann entropy is bounded by $S(\rho) \leq \ln(d)$ for any density matrix $\rho \in B(\mathscr{H})$ where $\dim \mathscr{H} = d$, the Entanglement of Formation is capable of detecting maximally entangled states.

While all convex roof measures are inherently continuous, Nielsen [51] established that the Entanglement of Formation also satisfies the stronger notion of asymptotic continuity:

**Definition 4.26.** (Asymptotic Continuity)
Let $\{\omega_n\}$ and $\{\varphi_n\}$ be sequences of normal states on $\mathfrak{A}_n = B(\mathscr{H}_n) = B(\mathscr{H}_n^1 \otimes \mathscr{H}_n^2)$ where $\dim \mathscr{H}_n = d_n < \infty$. An entanglement measure $E : \mathfrak{S}_\mathfrak{A} \to \mathbb{R}^+$ is called asymptotically continuous if $\|\omega_n - \varphi_n\| \to 0$ as $n \to 0$ implies

$$\frac{E(\omega_n) - E(\varphi_n)}{1 + \ln(d_n)} \to 0$$

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Asymptotic continuity is sometimes imposed as a necessary criteria for entanglement measures because of its association with both lockability and extremal measures.

An entanglement measure is lockable if the removal (through restriction) of a single qubit (two dimensional subsystem) from a given system causes the entanglement present to be reduced by an arbitrarily large amount. Any convex entanglement measure that is not asymptotically continuous is lockable. However, the converse is not true and the Entanglement of Formation is lockable. Indeed a number of important entanglement measures, including the entanglement cost, are known to be lockable [52].

Arguably more important is that if an entanglement measure $E$ is asymptotically continuous, satisfies $E(\sigma_d^+) = \ln(d)$, and is regularizable in the sense that

$$E^\infty(\omega) := \lim_{n\to\infty} \frac{1}{n} E(\omega^{\otimes n})$$

is well defined and the limit exists, then this regularization obeys [48]:

$$E_D \leq E^\infty \leq E_C$$

where $E_D$ and $E_C$ refer to the Distillable Entanglement and Entanglement Cost respectively (as defined in Section 4.1). In other words the operational entanglement measures can be viewed as extreme bounds under these additional postulates which automatically implies a unique for the pure states, namely the Von Neumann entropy [48]. The Entanglement of Formation fits this scheme as it is regularizable and asymptotically continuous.

**Proposition 4.27.** *The entanglement of formation is subadditive i.e.*

$$E_F(\omega_1 \otimes \omega_2) \leq E_F(\omega_1) + E_F(\omega_2)$$

*where* $\omega_1, \omega_2 \in \mathfrak{S}_{\mathfrak{A}_1 \otimes \mathfrak{A}_2} = \mathfrak{S}_{\mathfrak{A}}$

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*Proof.* Denote by $\mathfrak{S}_T$ the set of all states on $(\mathfrak{A}_1 \otimes \mathfrak{A}_2) \otimes (\mathfrak{A}_1 \otimes \mathfrak{A}_2)$

$$E_F(\omega_1 \otimes \omega_2) = \inf_{\mu \in M_{\omega_1 \otimes \omega_2}(\mathfrak{S}_T)} \int S(r\varphi) d\mu$$

$$\leq \inf_{\mu_1 \times \mu_2 \in M_{\omega_1}(\mathfrak{S}_\mathfrak{A}) \times M_{\omega_2}(\mathfrak{S}_\mathfrak{A})} \int \int S\left(r \circ (\varphi_1 \otimes \varphi_2)\right) d\mu_2 d\mu_1$$

$$\leq \inf_{\mu_1 \times \mu_2 \in M_{\omega_1}(\mathfrak{S}_\mathfrak{A}) \times M_{\omega_2}(\mathfrak{S}_\mathfrak{A})} \int \int \left(S(r\varphi_1) + S(r\varphi_2)\right) d\mu_2 d\mu_1$$

$$= \inf_{\mu_1 \in M_{\omega_1}(\mathfrak{S}_\mathfrak{A})} \int S(r\varphi) d\mu_1 + \inf_{\mu_2 \in M_{\omega_2}(\mathfrak{S}_\mathfrak{A})} \int S(r\varphi) d\mu_2$$

$$= E_F(\omega_1) + E_F(\omega_2)$$

Where the first inequality follows from the set inclusion $M_{\omega_1}(\mathfrak{S}_\mathfrak{A}) \times M_{\omega_2}(\mathfrak{S}_\mathfrak{A}) \subseteq M_{\omega_1 \otimes \omega_2}(\mathfrak{S}_T)$ and the second is a result of the subadditivity of the Von Neumann entropy itself. □

As a consequence of this subadditivity, a system created from many bipartite subsystems cannot have more entanglement than the sum of its parts. This is of particular interest when expanding systems by adding an ancillary system or when considering blocks of the same system. Applying subadditivity to the latter case shows that the Entanglement of Formation is an upper bound for the Distillable Entanglement:

$$E_D \leq E_F^\infty(\omega) = \lim_{n \to \infty} \frac{1}{n} E_F(\omega^{\otimes n}) \leq \lim_{n \to \infty} \frac{1}{n} n E_F(\omega) = E_F(\omega)$$

Hayden et al. [45] proved that the Entanglement Cost is equal to the regularized Entanglement of Formation. Hence if $E_F$ is also superadditive (i.e. $E_F(\omega_1 \otimes \omega_2) \geq E_F(\omega_1) + E_F(\omega_2)$), it would be fully additive and then it would follow that

$$E_F = E_F^\infty = E_C$$

thus greatly simplifying the calculation of Entanglement Cost.

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Shor [53] established that the additivity of $E_F$ is equivalent to additivity of the Holveo Capacity and additivity of the Minimum Output Entropy, both of which would have important implications for quantum information theory if true:

- The Holveo Capacity $\chi$ of a quantum channel $\Lambda$ is defined by

$$\chi(\Lambda) := \max_{\{p_i,\rho_i\}} \left\{ S\left(\sum p_i \Lambda(\rho_i)\right) - \sum p_i S(\Lambda(\rho_i)) \right\}$$

  where the maximum is taken over all ensembles of input signal states $\rho_i$ occurring with probability $p_i$. It quantifies the maximum rate at which classical information can be sent reliably through the quantum channel. Additivity of the Holveo Capacity would imply that the use of entanglement cannot increase the capacity of a quantum channel to transmit quantum information.

- The Minimum Output Entropy of a quantum channel $\Lambda$, which is defined as

$$S^{min}(\Lambda) = \min_{\{\Psi\in\mathscr{H}, \|\Psi\|=1\}} S(\Lambda(|\Psi\rangle\langle\Psi|))$$

  measures the extent to which the output of a channel deviates from the pure input state i.e. $S^{min}$ measures the decoherence or output purity of the quantum channel which, in turn, indicates the level of noise present in said channel. Hastings [54] demonstrated that the use of entangled input states states in a particular joint channel produced a lower Minimum Output Entropy hence simultaneously disproving the additivity of the Minimum Output Entropy and showing that an entangled input state can be more resilient to noise than separable inputs.

As a result, none of these additivity conjectures are true in general. Furthermore, the weaker additivity variations (in which identical arguments are considered) are then also not valid [55] thus the Entanglement Cost is not identical to the Entanglement of Formation.

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#### 4.2.2.2 Concurrence and Tangle

The analytical complexity of the Entanglement of Formation, even in simple low dimensional systems, inspired Hill and Wooters [56] to introduce the entanglement measure known as Concurrence. The original version was limited to a pair of qubits and based on the spin-flip operator which maps a pure state $\omega \in \mathscr{H}_1 \otimes \mathscr{H}_2$ to the state

$$\tilde{\omega} = (\sigma_y \otimes \sigma_y)\omega^*$$

where $\sigma_y$ is the Pauli operator $\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$.

The Concurrence $C$ of the pure state $\omega$ is defined as

$$C(\omega) = |(\omega, \tilde{\omega})| \tag{4.3}$$

When written in the standard basis, $\omega = a\epsilon_{00} + b\epsilon_{01} + c\epsilon_{10} + d\epsilon_{11}$, it follows that $C(\omega) = 2|ad - bc|$. When combined with the observation that

$$E_F(\omega) = H\left(\frac{1 + \sqrt{1 - C^2(\omega)}}{2}\right) \tag{4.4}$$

where $H(x) = -x \ln x - (1 - x) \ln x$, then a calculation of the Entanglement of Formation becomes far simpler.

Uhlmann [22] attempted to generalize the concurrence to any $d \times d$-dimensional system based on the spin-flip being an example of a conjugation: a self-adjoint antiunitary (i.e. an antilinear and unitary) operator. For any conjugation $\Theta$, the associated $\Theta$-concurrence is defined for any pure state $\omega$ to be

$$C_\Theta(\omega) = |(\omega, \Theta\omega)|$$

This approach is complicated by the fact that a general conjugation does not commute with all unitaries (except in two dimensions) and thus a single conjugation-based concurrence will not be capable of measuring entanglement leading to the

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concept of a Concurrence vector $\{C_\alpha(\omega)\}$ [57]:

**Definition 4.28.** (Concurrence Vector)
Consider a pure state $\omega = \sum_{i=1}^{d_1}\sum_{j=1}^{d_2} a_{ij}\epsilon_{ij} \in \mathscr{H}_{d_1} \otimes \mathscr{H}_{d_2}$ written in the standard basis. Denote by $\alpha$ the ordered set $(k, l, k', l')$ where $k < k'$ label two states in the standard basis for the first subsystem and $l < l'$ likewise for the second system. Define the self-adjoint antilinear operator $\Theta_\alpha$ by

$$\Theta_\alpha(\omega) := -a^*_{kl}\epsilon_{k'l'} - a^*_{k'l'}\epsilon_{kl} + a^*_{kl'}\epsilon_{k'l} + a^*_{k'l}\epsilon_{kl'}$$

and the $\Theta_\alpha$-Concurrence of $\omega$ as

$$C_\alpha(\omega) := |(\omega, \Theta_\alpha\omega)|$$

The ordered set of values $\{C_\alpha(\omega)\}$ is called the *Concurrence Vector of* $\omega$.

*Remark* 4.29. $\Theta_\alpha$ represents the projection of $\mathscr{H}_{d_1} \otimes \mathscr{H}_{d_2}$ onto a $2 \times 2$-dimensional subspace and then performing a spin-flip on that subspace. Since $\Theta_\alpha$ is not unitary, it is not a conjugation but Uhlmann demonstrates it still possesses the key properties in order to have a well defined concurrence i.e. it is self-adjoint and antilinear. In particular, any self-adjoint antilinear operator $T$ has a polar decomposition $T = |T|\Theta = \Theta|T|$ where $|T| = (T^*T)^{1/2} = (T^2)^{1/2}$ and $\Theta$ is a conjugation.

**Theorem 4.30.** *A pure state $\omega \in \mathscr{H}_{d_1} \otimes \mathscr{H}_{d_2}$ is a product state if and only if*

$$\sum_\alpha C^2_\alpha(\omega) = 0$$

*Moreover $\sum_\alpha C^2_\alpha(\omega)$ is invariant under the action of local unitaries*

**Definition 4.31.** (Superoperator)// A *superoperator* refers to any linear operator acting on a vector space of linear operators.

Rungta et al. [58] proposed an alternative generalization based on extending the spin-flip operation itself to higher dimensions i.e. by looking for a superoperator $S_d$ which maps the collection of all the density matrices $\rho$ on a $d$-dimensional Hilbert space $\mathscr{H}_d$ into itself and satisfies:

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(1) $S_d$ preserves self-adjoints

(2) $S_d$ commutes with all unitary operators

(3) For a pure state $\omega \in \mathscr{H}_{d_1} \otimes \mathscr{H}_{d_2}$

$$(\omega, (S_{d_1} \otimes S_{d_2})(\rho_\omega)\omega) \geq 0 \text{ where } \rho_\omega = |\omega\rangle \langle\omega|$$

with equality if and only if $\omega$ is a product state

*Remark* 4.32. Here $S_{d_1} \otimes S_{d_2}$ referes to the unique operator on the density matrices in $B(\mathscr{H}_1 \otimes \mathscr{H}_2) = B(\mathscr{H}_1) \overline{\otimes} B(\mathscr{H}_2)$ such that

$$(S_{d_1} \otimes S_{d_2})(\rho_1 \otimes \rho_2) = S_{d_1}(\rho_1) \otimes S_{d_2}(\rho_2)$$

where $\rho_1$ and $\rho_2$ are density matrices in $B(\mathscr{H}_1)$ and $B(\mathscr{H}_2)$ respectively.

The third property follows from the extension of the spin-flip operator on pure states $\omega \in \mathscr{H}_2$ to a superoperator $S_2$ on a general density matrix $\rho \in B(\mathscr{H}_2)$

$$S_2(\rho) = \sigma_y \rho^* \sigma_y$$

which allows Equation 4.3 to be rewritten as

$$C(\omega) = \sqrt{(\omega, (S_{d_1} \otimes S_{d_2})(\rho_\omega)\omega)} \tag{4.5}$$

Hence Property (3) will ensure that a higher dimensional generalization will be well defined and capable of distinguishing entangled and separable pure states.

Property (1) ensures that the higher dimensional generalization of Equation (4.5) will be real-valued while Property (2) guarantees that it will be a viable entanglement measure by being unaltered by local unitaries.

Rungta et al. show that there exists only one (up to a real constant multiple) superoperator for a given $d$-dimensional system that satisfies these three properties:

$$S_d(\rho) = I - \rho$$

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which is dubbed the universal inverter.

The associated generalized pure state Concurrence, known as the I-concurrence, is defined for a pure state $\omega \in \mathscr{H}_{d_1} \otimes \mathscr{H}_{d_2}$ as

$$C_I(\omega) = \sqrt{(\omega, (S_{d_1} \otimes S_{d_2})(\rho_\omega)\omega)} = \sqrt{2\left[1 - \mathrm{Tr}((r_1\rho_\omega)^2)\right]}$$

Hence it is possible to define another convex roof measure:

**Definition 4.33.** (Concurrence)
For any normal state $\omega(\cdot) = \mathrm{Tr}(\rho_\omega(\cdot))$ acting on a composite system $\mathfrak{A} = \mathfrak{A}_1 \otimes \mathfrak{A}_2$, the *Concurrence* is given by

$$E_I(\omega) := \inf_{\mu \in M_\omega(\mathfrak{S}_\mathfrak{A})} \int C(\varphi) d\mu(\varphi)$$

where $C(\phi) = \sqrt{2\left[1 - \mathrm{Tr}((r_1\rho_\phi)^2)\right]}$ is the I-concurrence.

Despite the difference in starting point, Rungta et al's I-concurrence and the Concurrence Vector are closely related on the pure states:

$$C_I^2(\omega) = \sum_\alpha C_\alpha^2(\omega)$$

This relationship justifies the introduction of another convex roof measure:

**Definition 4.34.** (Tangle)
For any normal state $\omega(\cdot) = \mathrm{Tr}(\rho_\omega(\cdot))$ acting on a composite system $\mathfrak{A} = \mathfrak{A}_1 \otimes \mathfrak{A}_2$, the *Tangle* is given by

$$E_\tau(\omega) := \inf_{\mu \in M_\omega(\mathfrak{S}_\mathfrak{A})} \int \tau(\varphi) d\mu(\varphi)$$

where $\tau(\phi) = 2\left[1 - \mathrm{Tr}((r_1\rho_\phi)^2)\right]$.

*Remark* 4.35. There is equality between the $\tau$ and the squared I-concurrence on the pure states so $\tau$ inherits (from the I-concurrence) all the necessary properties for the Tangle to be a well defined convex roof measure. However the equality

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on the pure states but this does not extend to the mixed states, hence the tangle is an entanglement measure in its own right and not a simple variation of the concurrence.

While Equation 4.4 allowed Hill and Wooters to utilize the Concurrence to easily calculate the Entanglement of Formation for two qubits, the generalizations cannot be exploited in the same manner as they order pure states differently [59].

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# Chapter 5

# Quantum Correlations as a Measure of Entanglement

The focus of the operational entanglement measures is the exploitation of the correlations present in entangled states in order to better perform certain tasks. Since these correlations cannot be created by LOCC operations alone, the basis for all axiomatic measures is monotonicity under such operations. The convex roof measures exploit the simplicity of pure state entanglement and its measurement to build up to the more complicated mixed state entanglement through pure state decompositions and taking the lowest average entanglement over such decompositions in order to achieve a good indication of the level of entanglement present.

An alternative strategy is developed by returning to the original definition of entanglement i.e. separable states exhibit only classical correlations while entangled states contain some additional (quantum) correlations. It is thus intuitive to consider the separable states as a set of reference states and then measure entanglement of a general state through a suitable "distance" or relative proximity indicator to this collection of special states.



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*Remark* 5.1. This is in contrast to the operational measures which identify the maximally entangled states as the reference states and thus only make sense in the context where such states exist.

Towards the aim of defining a measure of quantum correlations, first some preliminaries concepts surrounding maps of C$^*$-algebras and their relationship to entanglement will be introduced. Notably the link between positive maps and separability will be utilized to justify this definition.

## 5.1 On Jordan Morphisms

**Definition 5.2.** (Mormphisms and Homomorphisms)[28]
Let $\mathfrak{A}, \mathfrak{B}$ be C$^*$-algebras and $\phi : \mathfrak{A} \to \mathfrak{B}$ a linear map.

1) $\phi$ is a *-morphism if $\phi(a^*) = \phi(a)^*$ and $\phi(ab) = \phi(a)\phi(b)$.

2) $\phi$ is a *-antimorphism if $\phi(a^*) = \phi(a)^*$ and $\phi(ab) = \phi(b)\phi(a)$.

3) $\phi$ is a C$^*$-homomorphism if $\phi(a^*) = \phi(a)^*$ and $\phi(h^2) = \phi(h)^2$ whenever $h = h^*$.

4) $\phi$ is a Jordan homomorphism if $\phi(a^*) = \phi(a)^*$ and $\phi(\{a, b\}) = \{\phi(a), \phi(b)\}$ where $\{x, y\} = xy + yx$.

**Lemma 5.3.** *A linear map $\phi : \mathfrak{A} \to \mathfrak{B}$ is a Jordan homomorphism if and only if $\phi(a^2) = \phi(a)^2 \ \forall \ a \in \mathfrak{A}$.*

*Proof.* Let $\phi(a^2) = \phi(a)^2 \ \forall \ a \in \mathfrak{A}$. By noting that $ab + ba = (a + b)^2 - a^2 - b^2$ a simple calculation clearly shows that $\phi(\{a, b\}) = \{\phi(a), \phi(b)\}$.

Conversely if $\phi$ is a Jordan homomorphism then $\phi(\{a, a\}) = \{\phi(a), \phi(a)\}$ hence $\phi(a^2) = \phi(a)^2$. □

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**Lemma 5.4.** *A linear map $\phi : \mathfrak{A} \to \mathfrak{B}$ is a Jordan homomorphism if and only if $\phi(a^n) = \phi(a)^n \ \forall \ a \in \mathfrak{A}, n \in \mathbb{N}$.*

*Proof.* Clearly, for any Jordan homomorphism $\phi$, $\phi(a^1) = \phi(a)^1$. Thus assume that $\phi(a^n) = \phi(a)^n$ for all $a \in \mathfrak{A}$ and some $n \in \mathbb{N}$. Then

$$\phi(a^{n+1}) = \frac{1}{2}\phi(\{a^n, a\}) = \frac{1}{2}\{\phi(a^n), \phi(a)\} = \frac{1}{2}\{\phi(a)^n, \phi(a)\} = \phi(a)^{n+1}$$

thus $\phi(a^n) = \phi(a)^n \ \forall \ a \in \mathfrak{A}, n \in \mathbb{N}$ follows from induction.

The converse follows trivially from Lemma 5.3. If $\phi(a^n) = \phi(a)^n \ \forall \ a \in \mathfrak{A}, n \in \mathbb{N}$ then in particular $\phi(a^2) = \phi(a)^2$ and so $\phi$ is a Jordan homomorphism. □

**Lemma 5.5.** *If $\phi : \mathfrak{A} \to \mathfrak{B}$ is a Jordan homomorphism then*

*(i)* $\phi(aba) = \phi(a)\phi(b)\phi(a)$.

*(ii)* $\phi(abc + cba) = \phi(a)\phi(b)\phi(c) + \phi(c)\phi(b)\phi(a)$.

*(iii)* $\phi([a, b]^2) = [\phi(a), \phi(b)]^2$ *where* $[x, y] := xy - yx$.

*(iv)* $\phi(ab) = \phi(a)\phi(b) = \phi(b)\phi(a)$ *whenever* $ab = ba$.

*Proof.* (i) $2\phi(aba) = \phi((ab + ba)a + a(ab + ba)) - \phi(ba^2 + a^2b) = 2\phi(a)\phi(b)\phi(a)$.

(ii) Note that $abc + cba = (a + c)b(a + c) - aba - cbc$ thus by (ii) it follows that $\phi(abc + cba) = \phi(a)\phi(b)\phi(c) + \phi(c)\phi(b)\phi(a)$.

(iii) Since $[a, b]^2 = (ab - ba)^2 = a(bab) + (bab)a - ab^2a - ba^2b$ then (iii) follows from the definition of a Jordan homomorphism, (i), and Lemma 5.3.

(iv) If $ab = ba$ then $[a, b] = 0$ hence by (iii) it follows that $\phi(a)\phi(b) = \phi(b)\phi(a)$. Thus

$$\phi(ab) = \frac{1}{2}\phi(\{a, b\}) = \frac{1}{2}\{\phi(a), \phi(b)\} = \phi(a)\phi(b)$$

□

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**Definition 5.6.** (Matrix Units)
A set $\{a_{ij} : 1 \leq i, j \leq n\}$ of elements in a C$^*$-algebra $\mathfrak{A}$ is called a *system of matrix units* if:

i) $a_{ij}^* = a_{ji}$

ii) $a_{ij}a_{kl} = \delta_{jk}a_{il}$

iii) $\sum_{i\in I} a_{ii} = 1$ in the norm topology.

If a C$^*$-algebra $\mathfrak{A}$ contains a C$^*$-subalgebra $\mathfrak{B}$ and a system of matrix units $U := \{u_{ij} : 1 \leq i, j \leq n\}$ where $n \geq 2$ such that:

i) $\mathfrak{A}$ is isomorphic to $\mathfrak{B} \otimes M_n$.

ii) $u_{ij}b = bu_{ij}$ for all $b \in \mathfrak{B}$ and $u_{ij} \in U$.

then $\mathfrak{A}$ is called a $n \times n$ *matrix algebra over* $\mathfrak{B}$.

It follows that (for such algebras) every $a \in \mathfrak{A}$ can be written uniquely as

$$a = \sum_{i,j=1}^{n} b_{ij}u_{ij} \text{ where } b_{ij} \in \mathscr{B}$$

The concept of matrix algebras is particularly useful in the context of von Neumann algebras due to the the work of Kaplansky [60]:

**Definition 5.7.** (Projections in a von Neumann Algebras)
Consider a von Neumann algebra $\mathfrak{M}$ and non-zero projections $e$ and $f$ in $\mathfrak{M}$:

- $e \geq f$ means $ef = f$.
- $e$ and $f$ are *equivalent* $(e \sim f)$ if there exists an $x$ s.t. $xx^* = e$ and $x^*x = f$.
- $e$ is called *finite* if $e \sim f$ and $e \geq f$ implies $e = f$. Otherwise $e$ is called *infinite*.

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- $e$ is called *abelian* if $e\mathfrak{M}e$ is abelian.

$\mathfrak{M}$ is called *finite* if its unit element 1 is finite (and hence all projections are finite). $\mathfrak{M}$ *purely infinite* if all nonzero central projections are infinite.

**Definition 5.8.** (Classification of von Neumann Algebras)
A von Neumann algebra $\mathfrak{M}$ is *type I* if, whenever $\mathfrak{M}$ is decomposed as a finite direct sum, every direct summand contains an abelian projection.

$\mathfrak{M}$ is *type II* if it has no abelian projections and every direct summand contains a finite projection.

$\mathfrak{M}$ is *type III* if all projections are infinite.

*Remark* 5.9. Note that all three types of von Neumann algebra are mutually exclusive.

Kaplansky established that every von Neumann algebra can be decomposed as the direct sum of two von Neumann subalgebras: a finite one and a purely infinite one. When combined with Kaplansky's other observation that any von Neumann algebra can be uniquely expressed as the direct sum of one of each type of von Neumann algebra, then every von Neumann algebra can be expressed as the sum of two type I (one finite, one infinite), two type II (one finite, one infinite) and one type III (necessarily infinite) von Neumann subalgebras.

Kaplansky ascertained that for any von Neumann algebra $\mathfrak{M}$ that does not contain an abelain projection or is purely infinite, the identity can be halved in the sense that there exists two orthogonal equivalent projections $e = xx^*$ and $f = x^*x$ in $\mathfrak{M}$ such that $e + f = 1$. In such cases $\mathfrak{M}$ can be viewed as a $2 \times 2$ matrix algebra by recognizing that $u_{11} = e, u_{12} = x, u_{21} = x^*$ and $u_{22} = f$ constitute a system of matrix units. It was also shown that any finite type I von Neumann algebra is the direct sum of a finite number of matrix algebras over a commutative algebra with identity. Hence any von Neumann algebra is expressible as direct sum of finite matrix algebras which will be useful in the course of proving the following theorem (which is sketched in [28] and left as an exercise in [30]):

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**Theorem 5.10.** *For any Jordan homomorphism $\phi$ of a $C^*$-algebra $\mathfrak{A}$ into $B(\mathscr{H})$ there exists a projection $E \in \mathfrak{B}' \cap \mathfrak{B}''$, where $\mathfrak{B}$ is the $C^*$-algebra generated by $\phi(\mathfrak{A})$, such that $\phi_1(\cdot) := \phi(\cdot)E$ is a *-morphism, and $\phi_2(\cdot) := \phi(\cdot)(1-E)$ is a *-antimorphism*

By noting that $\mathfrak{A}$ is a weak$^*$ dense subset of its double dual space $\mathfrak{A}^{**}$ and $\mathfrak{A}^{**}$ is a von Neumann algebra then this theorem follows directly from following two lemmas:

**Lemma 5.11.** *[61] Any Jordan homomorphism $\phi$ of a $C^*$-algebra $\mathfrak{A}$ into a von Neumann algebra $\mathfrak{M}$ can be extended to an weak$^*$ continuous function $\phi^{**}$ from $\mathfrak{A}^{**}$ into $\mathfrak{M}$ which is itself a Jordan homomorphism.*

*Proof.* $\phi$ is continuous when $\mathfrak{A}$ and $\mathfrak{M}$ are both given the weak$^*$ topology (with the former through inheritance from $\mathfrak{A}^{**}$). Thus the map

$$\phi^* : \rho \in \mathfrak{M}_* \to \rho \circ \phi \in \mathfrak{A}^*$$

is continuous. By noting that $(\mathfrak{M}_*)^* = \mathfrak{M}$ the map $\phi^{**}$ from $\mathfrak{A}^{**}$ into $\mathfrak{M}$ induced by $\phi^*$ extending $\phi$ is weak$^*$ continuous.

Since $\mathfrak{A}$ is weak$^*$ dense in $\mathfrak{A}^{**}$ then for every $x$ and $y$ in $\mathfrak{A}^{**}$ there exist nets $\{a_n\}$ and $\{b_n\}$ in $\mathfrak{A}$ such that $a_n \to x$ and $b_m \to y$ ultra weakly.

$$\begin{aligned}
\phi^{**}(\{x,y\}) &= \phi^{**}\left(\left\{\lim_n a_n, \lim_m b_m\right\}\right) \\
&= \lim_{n,m} \phi^{**}\left(\{a_n, b_m\}\right) \\
&= \lim_{n,m} \phi\left(\{a_n, b_m\}\right) \\
&= \lim_{n,m} \{\phi(a_n), \phi(b_m)\} \\
&= \{\phi^{**}(x), \phi^{**}(y)\}
\end{aligned}$$

□

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**Lemma 5.12.** *For any Jordan homomorphism $\phi$ of a von Neumann algebra $\mathfrak{M}$ into $B(\mathscr{H})$ there exists central projections $e$ and $f$ in $\mathfrak{B}$ (the $C^*$-algebra generated by $\phi(\mathfrak{M})$), such that $\phi_1(\cdot) := \phi(\cdot)e$ is a $^*$-morphism, $\phi_2(\cdot) := \phi(\cdot)f$ is a $^*$-antimorphism, and $\phi = \phi_1 + \phi_2$ as linear maps*

*Proof.* By Kaplansky's results, it is sufficient to prove the case when $\mathfrak{M}$ is assumed to be a $n \times n$ matrix algebra over $\mathfrak{A} \subset \mathfrak{M}$ (since the restriction of $\phi$ to each summand is decomposable as required, then clearly then $\phi$ is decomposable in the same way over the entire direct sum). Denote by $\{u_{ij}, 1 \geq i, j \geq n\}$ the associated matrix units in $\mathfrak{M}$. Note that by 5.5 (iv) it is clear that for any $x \in \mathfrak{M}$

$$\phi(x) = \sum_{i,j=1}^{n} \phi(a_{ij})\phi(u_{ij})$$

where $a_{ij} \in \mathfrak{A}$.

Towards introducing two sets of matrix units to $\phi(\mathfrak{M})$, note that if $i \neq j$ then $u_{ij} = u_{ii}u_{ij}u_{jj} + u_{jj}u_{ij}u_{ii}$. Thus by applying Lemma 5.5 (ii) as well as by defining $v_{ij} := \phi(u_{ii})\phi(u_{ij})\phi(u_{ji})$ and $w_{ij} := \phi(u_{ii})\phi(u_{ji})\phi(u_{jj})$ for $i \neq j$ that

$$\phi(u_{ij}) = v_{ij} + w_{ji} \; \forall \; i \neq j$$

Since $u_{ii}^2 = u_{ii}$ then by Lemma 5.3 $\phi(u_{ii})^2 = \phi(u_{ii})$. Additionally Lemma 5.5 (iv) and $u_{ii}u_{jj} = u_{jj}u_{ii} = 0$ whenever $i \neq j$ implies that $\phi(u_{ii})\phi(u_{jj}) = 0$ whenever $i \neq j$. Thus $\{\phi(u_{ii})\}$ is a set of orthogonal projections in $\phi(\mathfrak{M})$.

It follows immediately that $v_{ij}\phi(u_{jj}) = \phi(u_{ii})v_{ij} = v_{ij}$ and $v_{ij}\phi(u_{ii}) = \phi(u_{jj})v_{ij} = 0$ thus

$$v_{ij} = \phi(u_{ii})\phi(u_{ij}) = \phi(u_{ij})\phi(u_{jj})$$

The same argument replacing $v_{ij}$ with $w_{ij}$ shows that

$$w_{ij} = \phi(u_{ii})\phi(u_{ji}) = \phi(u_{ji})\phi(u_{jj})$$

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Immediately if $j \neq k$ then $v_{ij}v_{kl} = \phi(u_{ij})\phi(u_{jj})v_{kl} = 0$ and $w_{ij}w_{kl} = 0$.

If $i \neq j$, $j \neq k$ and $k \neq i$ then $u_{kk}u_{ij} = u_{ij}u_{kk}$ thus $\phi(u_{ij})\phi(u_{kk}) = 0$. As a consequence

$$\begin{aligned} v_{ij}v_{jk} &= \phi(u_{ij})\phi(u_{jj})v_{jk} = \phi(u_{ij})v_{jk} = \phi(u_{ij})\phi(u_{jk})\phi(u_{kk}) \\ &= (\phi(u_{ij})\phi(u_{jk}) + \phi(u_{jk})\phi(u_{ij}))\phi(u_{kk}) = \{\phi(u_{ij}), \phi(u_{jk})\}\phi(u_{kk}) \\ &= \phi(u_{ij}u_{jk} + u_{jk}u_{ij})\phi(u_{kk}) \\ &= \phi(u_{ik})\phi(u_{kk}) \\ &= v_{ik} \end{aligned}$$

An identical argument shows that $w_{ij}w_{jk} = w_{ik}$.

This enable the following definition: for each $i$ choose $j \neq i$ so that

$$v_{ii} := v_{ij}v_{ji} \text{ and } w_{ii} := w_{ij}w_{ji}$$

are well defined. Note that $v_{ij}v_{ji} = v_{ij}v_{jk}v_{ki} = v_{ik}v_{ki}$ establishes that $v_{ii}$ (and likewise $w_{ii}$) is independent of the choice of $j \neq i$. Moreover $v_{ii}v_{ii} = v_{ij}v_{ji}v_{ik}v_{ki} = v_{ii}$ thus $v_{ii}$ and $w_{ii}$ are projections.

Since and Jordan morphism preserves adjoints then $v_{ij}^* = \phi(u_{jj}^*)\phi(u_{ij}^*)\phi(u_{ii}^*) = \phi(u_{jj})\phi(u_{ji})\phi(u_{ii}) = v_{ji}$ for $i \neq j$ and $v_{ii}^* = v_{ji}^*v_{ij}^* = v_{ij}v_{ji} = v_{ii}$.

In summary, for any choice of indices $i, j, k, l$ the following hold:

$$v_{ij}v_{kl} = \delta_{jk}v_{il}$$

$$v_{ij}^* = v_{ji}$$

$$w_{ij}w_{kl} = \delta_{jk}w_{il}$$

$$w_{ij}^* = w_{ji}$$

Assuming $i \neq j$ and $k \neq l$ then $v_{ij}w_{kl} = \phi(u_{ij})\phi(u_{jj})\phi(u_{kk})\phi(u_{lk})$. Thus

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- If $j \neq k$ then $\phi(u_{jj})\phi(u_{kk}) = 0 \Rightarrow v_{ij}w_{kl} = 0$.
- If $j = k$ then

$$\begin{aligned} v_{ij}w_{jl} = v_{ij}w_{ji}w_{il} &= \phi(u_{ij})\phi(u_{jj})\phi(u_{jj})\phi(u_{ji})w_{il} \\ &= \phi(u_{ij})\phi(u_{jj})\phi(u_{ij})w_{il} = \phi(u_{ij}u_{jj}u_{ij})w_{il} = 0 \end{aligned}$$

It directly follows that $v_{ii}w_{ll} = v_{ij}v_{ji}w_{lk}w_{kl} = 0$. Thus $v_{ij}w_{kl} = 0$ for all choices of indices i.e. $\{v_{ij} : 1 \leq i, j \leq n\}$ and $\{w_{ij} : 1 \leq i, j \leq n\}$ are orthogonal.

Define $e := \sum_{i=1}^{n} v_{ii}$ and $f := \sum_{i=1}^{n} w_{ii}$. Clearly $e^2 = e$ and $f^2 = f$ (i.e. $e$ and $f$ are projections) follows immediately from $v_{ij}v_{kl} = \delta_{jk}v_{il}$ and $w_{ij}w_{kl} = \delta_{jk}w_{il}$ respectively.

Note that

$$v_{ii} = v_{ij}v_{ji} = \phi(u_{ij})\phi(u_{jj})\phi(u_{ji})\phi(u_{ii}) = \phi(u_{ij})v_{ji} = \phi(u_{ij})\phi(u_{ji})\phi(u_{ii})$$

And similarly

$$w_{ii} = \phi(u_{ji})\phi(u_{ij})\phi(u_{ii})$$

Hence

$$v_{ii} + w_{ii} = \{\phi(u_{ij}), \phi(u_{ji})\}\phi(u_{ii}) = \phi(\{u_{ij}, u_{ji}\})\phi(u_{ii}) = \phi(u_{ii})$$

It then follows that

$$\phi(1) = \phi(\sum_{i=1}^{n} u_{ii}) = \sum_{i=1}^{n} \phi(u_{ii}) = \sum_{i=1}^{n} v_{ii} + \sum_{i=1}^{n} w_{ii} = e + f$$

Thus $e\phi(\mathfrak{M})$ is an $n \times n$ matrix algebra over $\phi(A)$ with matrix units $\{v_{ij} : 1 \leq i, j \leq n\}$ and $f\phi(\mathfrak{M})$ is an $n \times n$ matrix algebra over $\phi(A)$ with matrix units $\{w_{ij} : 1 \leq i, j \leq n\}$.

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Since $\phi(\mathfrak{M})$ is generated by elements of the form $\phi(a_{ij})\phi(u_{ij})$ where $a_{ij} \in \mathfrak{A}$ thus to show that $e$ and $f$ are central in $\phi(\mathfrak{M})$, it is sufficient to demonstrate that they commute with any $\phi(a)$ (where $a \in \mathfrak{A}$ ) and $\phi(u_{ij})$.

The former is immediate since Lemma 5.5 (iv) implies $\phi(u_{ij})$ and $\phi(a)$ commute for all $a \in \mathfrak{A}$ thus both $v_{ij}$ and $w_{ij}$ commute with all $\phi(a)$ where $a \in \mathfrak{A}$.

The latter is established from

$$e\phi(u_{ij}) = \sum_{k=1}^{n} v_{kk}v_{ij} + \sum_{k=1}^{n} v_{kk}w_{ji} = \sum_{k=1}^{n} \delta_{ki}v_{kj} = v_{ij}$$

$$\phi(u_{ij})e = \sum_{k=1}^{n} v_{ij}v_{kk} + \sum_{k=1}^{n} w_{ji}v_{kk} = \sum_{k=1}^{n} \delta_{jk}v_{ik} = v_{ij}$$

i.e. $e\phi(u_{ij}) = \phi(u_{ij})e = v_{ij}$ thus $e$ commutes with all $\phi(u_{ij})$.

Similarly $f\phi(u_{ij}) = \phi(u_{ij})f = w_{ji} = \phi(u_{ij})f = \phi(u_{ij})f$ implies $f$ is central.

Clearly then $\phi_1$ and $\phi_2$ preserve involution:

$$\phi_1(x^*) = \phi(x^*)e = (e^*\phi(x))^* = (\phi(x)e)^* = \phi_1(x)^*$$

Additionally $\phi_1(x) = \phi(\sum x_{ij}u_{ij})e = \sum(\phi(x_{ij})\phi(u_{ij})e) = \sum(\phi(x_{ij})v_{ij})$.

To establish that $\phi_1$ is a *-morphism, consider $x, y \in \mathfrak{A}$ and $i \neq j$. Then

$$\begin{aligned}
\phi(xy)\phi(u_{ii}) + \phi(yx)\phi(u_{jj}) &= \phi(xyu_{ii} + yxu_{jj}) \\
&= \phi((xu_{ij} + yu_{ji})^2) \\
&= (\phi(x)\phi(u_{ij}) + \phi(y)\phi(u_{ji}))^2 \\
&= (\phi(x)(v_{ij} + w_{ji}) + \phi(y)(v_{ji} + w_{ij}))^2 \\
&= \phi(x)\phi(y)(v_{ii} + w_{jj}) + \phi(y)\phi(x)(v_{jj} + w_{ii})
\end{aligned}$$

By multiplying both sides by $v_{ij}$ (from the right) it then follows that

$$\phi(xy)v_{ij} = \phi(x)\phi(y)v_{ij}$$

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Thus it follows that $\phi_1(xy) = \phi_1(x)\phi_1(y)$ as required.

Similarly $\phi_2(x) = \sum(\phi(x_{ij})w_{ji})$ and $\phi(xy)w_{ij} = \phi(x)\phi(y)w_{ij}$ establish that $\phi_2(xy) = \phi_2(y)\phi_2(x)$ concluding that $\phi_2$ is a *-antimorphism. □

## 5.2 Positive Maps and Separability

Peres [62] established that a necessary condition for separability of a density matrix $\rho \in B(\mathscr{H}_1 \otimes \mathscr{H}_2)$ is positivity under the action of the partial transpose $I_1 \otimes T_2$ where $T_2$ represents the transposition map acting on the subsystem $B(\mathscr{H}_2)$. This Positive Partial Transpose (PPT) criteron for separable states was found to also be a sufficient criteria for separability whenever $\dim(\mathscr{H}_1 \otimes \mathscr{H}_2) \leq 6$ but not when $\dim(\mathscr{H}_1 \otimes \mathscr{H}_2) > 6$ where it was found that there exist entangled states with positive partial transpose [63, 64].

The success of the PPT criteron was recognized as a result of the positivity, but not complete positivity, of the transposition map in Horodecki et al. [63]. Thus it was established that any unital positive, but not completely positive, linear map $\Lambda$ on $B(\mathscr{H}_2)$ could be used in place of the transposition map leading to the Peres-Horodecki criterion, a broader variant of the PPT criterion which overcomes its deficits in higher dimensions:

A state $\rho \in B(\mathscr{H}_1 \otimes \mathscr{H}_2)$, where $\dim \mathscr{H}_i < \infty$ for $i \in \{1, 2\}$, is separable if and only if

$$[I_1 \otimes \Lambda](\rho) \geq 0$$

for all unital positive, but not completely positive, maps $\Lambda : B(\mathscr{H}_2) \to B(\mathscr{H}_1)$.

While a significant improvement over the PPT criterion, the Peres-Horodecki condition is still limited to finite dimensions. An extension to infinite dimensions was found by Størmer [65]:

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Consider the C$^*$-algebra $\mathfrak{A} = B(\mathscr{H}_1) \otimes B(\mathscr{H}_2)$ a normal state $\omega$ on $\mathfrak{A}$ defined by a density matrix $\rho$ is separable if and only if

$$[I_1 \otimes \Lambda]\rho \geq 0$$

for all positive maps $\Lambda : B(\mathscr{H}_2) \to B(\mathscr{H}_1)$ that are weakly continuous on all bounded sets.

It is immediate then that a sufficient criterion for separability in a more abstract setting is:

**Proposition 5.13.** *If $\omega$ is a normal state on the C$^*$-algebra $\mathfrak{A} = \mathfrak{A}_1 \otimes \mathfrak{A}_2$ such that*

$$\omega \circ (I_1 \otimes \Lambda) \geq 0$$

*For all unital positive linear maps $\Lambda : \mathfrak{A}_2 \to \mathfrak{A}_1$, then $\omega \in \mathfrak{S}_{sep}$.*

The intimate connection between positive maps and separability prompts investigation into the former. In particular, the remainder of this subsection is dedicated to a generalization of Stormer's [66] work on locally decomposable maps which will prove necessary in subsequent sections.

**Definition 5.14.** (Local Decomposable Maps)
Let $\Lambda$ be a positive linear map of a C$^*$-algebra $\mathfrak{A}$ into $B(\mathscr{H})$. $\Lambda$ is called *locally decomposable* if there exists an $M \geq 0$ such that given any normal state $\omega$ on $B(\mathscr{H})$, defined by the density matrix $\rho$, there exists:

- A Hilbert space $\mathscr{H}_\rho$.
- A bounded linear map $V_\rho$ of $\mathscr{H}_\rho$ into the closure of $B(\mathscr{H})\rho^{1/2}$ such that $\|V_\rho\| \leq M$.
- A C$^*$-homomorphism $\pi_\rho$ of $\mathfrak{A}$ into $B(\mathscr{H}_\rho)$ such that

$$V_\rho \pi_\rho(a) V_\rho^* \rho^{1/2} = \Lambda(a)\rho^{1/2}$$

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**Lemma 5.15.** *If $\Lambda : \mathfrak{A} \to B(\mathscr{H})$ is a positive linear map on a $C^*$-algebra $\mathfrak{A}$ such that $\|\Lambda\| \leq 1$ then:*

*i)* $\Lambda(a^*) = \Lambda(a)^*$

*ii)* $\Lambda(a^*a + aa^*) \geq \Lambda(a^*)\Lambda(a) + \Lambda(a)\Lambda(a^*)$

*Proof.* To prove (i), note that each self-adjoint element is the difference between two positive elements thus $\Lambda$ preserves self-adjoint elements. But then since $a = a_1 + ia_2$ for self-adjoint $a_1, a_2 \in \mathfrak{A}$ then $\Lambda(a^*) = \Lambda(a_1) - i\Lambda(a_2) = \Lambda(a)^*$.

The proof of (ii) is a consequence of the generalized Schwarz inequality [67] which states that $\Lambda(a^2) \geq \Lambda(a)^2$ for all self-adjoint $a \in \mathfrak{A}$. Since $a + a^*$ and $i(a - a^*)$ are self adjoint, then it follows that

$$\Lambda((a + a^*)^2) + \Lambda((i(a - a^*))^2) \geq \Lambda(a + a^*)^2 + \Lambda(i(a - a^*))^2$$

from which (ii) follows from a simple expansion of both sides. □

**Theorem 5.16.** *Every unital positive linear map $\Lambda$ of a $C^*$-algebra $\mathfrak{A}$ into $B(\mathscr{H})$ is locally decomposable.*

*Proof.* For a given density matrix $\rho$ which defines the normal state $\omega$ on $B(\mathscr{H})$, define the state $\varphi \in \mathfrak{S}_{\mathfrak{A}}$ by

$$\varphi(a) = \mathrm{Tr}(\rho\Lambda(a))$$

Denote by $\{\pi_\varphi, \mathscr{H}_\varphi, \Omega_\varphi\}$ the Canonical Representation of $\mathfrak{A}$ induced by $\varphi$ such that $\varphi(a) = (\Omega_\varphi, \pi_\varphi(a)\Omega_\varphi)$.

Let $I_\varphi := \{a \in \mathfrak{A} : \varphi(aa^*) = 0\}$ from which the Hilbert space $\mathscr{H}'_\varphi := \{(a + I_\varphi) : a \in \mathfrak{A}\}$ is created. Define a $^*$-anti-homomorphism $\pi'_\varphi$ of $\mathfrak{A}$ on $\mathscr{H}'_\varphi$ by

$$\pi'_\varphi(c)(a + I_\varphi) = ac + I_\varphi$$

Let $\Omega'_\varphi$ denote the vector in $\mathscr{H}'_\varphi$ such that $\varphi(a) = (\Omega'_\varphi, \pi'_\varphi(a)\Omega'_\varphi)$.

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Set $\tilde{\pi}_\varphi := \pi_\varphi \oplus \pi'_\varphi$, $\widetilde{\mathscr{H}_\varphi} := \mathscr{H}_\varphi \oplus \mathscr{H}'_\varphi$, and $\tilde{\Omega}_\varphi := \Omega_\varphi \oplus \Omega,'_\varphi$.

$\widetilde{\mathscr{H}_\varphi}$ is a Hilbert space when equipped with the inner product

$$(x \oplus x', y \oplus y') = \frac{1}{2}\left((x,y) + (x',y')\right) \ \forall x, y \in \mathscr{H}_\varphi \text{ and } x', y' \in \mathscr{H}'_\varphi$$

Note that $\tilde{\pi}_\varphi$ preserves the adjoint:

$$\tilde{\pi}_\varphi(a^*) = \pi_\varphi(a^*) \oplus \pi'_\varphi(a^*) = \pi_\varphi(a)^* \oplus \pi'_\varphi(a)^* = \tilde{\pi}_\varphi(a)^*$$

And because $\tilde{\pi}_\varphi(a^*a) = \pi_\varphi(a^*a) \oplus \pi'_\varphi(a^*a) = \pi_\varphi(a^*)\pi_\varphi(a) \oplus \pi'_\varphi(a)\pi'_\varphi(a^*)$ then

$$\tilde{\pi}_\varphi(a^2) = \tilde{\pi}_\varphi(a)^2 \text{ for all self adjoint } a \in \mathfrak{A}$$

Thus $\tilde{\pi}_\varphi$ is a c$^*$-homomorphism of $\mathfrak{A}$ into $B(\widetilde{\mathscr{H}_\varphi})$. In particular $\tilde{\pi}_\varphi$ is also Jordan morphism since

$$\begin{aligned}\tilde{\pi}_\varphi(a)\tilde{\pi}_\varphi(b) + \tilde{\pi}_\varphi(b)\tilde{\pi}_\varphi(a) &= \pi_\varphi(ab) \oplus \pi'_\varphi(ba) + \pi_\varphi(ba) \oplus \pi'_\varphi(ab) \\ &= \pi_\varphi(ab+ba) \oplus \pi'_\varphi(ba+ab) \\ &= \tilde{\pi}_\varphi(ab+ba)\end{aligned}$$

Define the map $V_\varphi$ from $\tilde{\pi}_\varphi(\mathfrak{A})\tilde{\Omega} \subset \widetilde{\mathscr{H}_\varphi}$ into $\overline{\Lambda(\mathfrak{A})\rho^{1/2}}$ by

$$V_\varphi(\tilde{\pi}_\varphi(a)\tilde{\Omega}_\varphi) = \Lambda(a)\rho^{1/2}$$

If $\tilde{\pi}_\varphi(a)\tilde{\Omega} = 0$ then both $\pi_\varphi(a)\Omega_\varphi = 0$ and $\pi'_\varphi(a)\Omega'_\varphi = 0$ hence

$$\varphi(aa^*) = (\Omega'_\varphi, \pi'_\varphi(aa^*)\Omega'_\varphi) = (\Omega'_\varphi, \pi'_\varphi(a^*)\pi'_\varphi(a)\Omega'_\varphi) = 0$$

$$\varphi(a^*a) = (\Omega_\varphi, \pi_\varphi(a^*a)\Omega_\varphi) = (\Omega_\varphi, \pi_\varphi(a^*)\pi_\varphi(a)\Omega_\varphi) = 0$$

Thus by Lemma 5.15 and the fact that $(T_1, T_2) = \mathrm{Tr}(T_1^*T_2)$ for all $T_1, T_2 \in B(\mathscr{H})$:

$$0 = \varphi(a^*a+aa^*) = \left((\Lambda(a^*a+aa^*))\rho^{1/2}, \rho^{1/2}\right) \geq \left((\Lambda(a^*)\Lambda(a) + \Lambda(a)\Lambda(a^*))\rho^{1/2}, \rho^{1/2}\right) \geq 0$$

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Thus $\Lambda(a)\rho^{1/2} = 0$ so that $V_\varphi$ is well defined and linear.

By definition

$$\begin{aligned}\|V_\varphi\| &= \sup_{\left\|\tilde{\pi}_\varphi(a)\tilde{\Omega}_\varphi\right\|=1} \left\|\Lambda(a)\rho^{1/2}\right\| \\ &= \sup_{\left\|\pi_\varphi(a)\Omega_\varphi \oplus \pi'_\varphi(a)\Omega'_\varphi\right\|^2=1} \left\|\Lambda(a)\rho^{1/2}\right\|\end{aligned}$$

But

$$\begin{aligned}\left\|\pi_\varphi(a)\Omega_\varphi \oplus \pi_\varphi^{'}(a)\Omega_\varphi^{'}\right\|^2 = 1 &\iff \left(\pi_\varphi(a)\Omega_\varphi \oplus \pi_\varphi^{'}(a)\Omega_\varphi^{'}, \pi_\varphi(a)\Omega_\varphi \oplus \pi_\varphi^{'}(a)\Omega_\varphi^{'}\right) = 1 \\ &\iff \frac{1}{2}\left[(\pi_\varphi(a)\Omega_\varphi, \pi_\varphi(a)\Omega_\varphi) + \left(\pi_\varphi^{'}(a)\Omega_\varphi^{'}, \pi_\varphi^{'}(a)\Omega_\varphi^{'}\right)\right] = 1 \\ &\iff (\Omega, \pi_\varphi(a^*a)\Omega_\varphi) + \left(\Omega^{'}, \pi_\varphi^{'}(aa*)\Omega_\varphi^{'}\right) = 2 \\ &\iff \varphi(a^*a + aa^*) = 2 \\ &\iff \left((\Lambda(a^*a + aa^*))\rho^{1/2}, \rho^{1/2}\right) = 2\end{aligned}$$

But then by Lemma 5.15 it follows that

$$\begin{aligned}\left\|\tilde{\pi}_\varphi(a)\tilde{\Omega}_\varphi\right\| = 1 &\Rightarrow \left(\left(\Lambda(a^*)\Lambda(a) + \Lambda(a)\Lambda(a^*)\right)\rho^{1/2}, \rho^{1/2}\right) \leq 2 \\ &\Rightarrow \left\|\Lambda(a)\rho^{1/2}\right\|^2 \leq 2 \\ &\Rightarrow \|V_\varphi\| \leq 2^{1/2}\end{aligned}$$

Moreover $V_\varphi\tilde{\pi_\varphi}(1)\tilde{\Omega} = V_\varphi\tilde{\Omega} = \Lambda(1)\rho^{1/2}$ so that

$$\begin{aligned}\left(V_\varphi^*\rho^{1/2}, \tilde{\pi}_\varphi(a)\tilde{\Omega}\right) &= \left(\rho^{1/2}, V_\varphi\tilde{\pi}_\varphi(a)\tilde{\Omega}\right) \\ &= \left(\rho^{1/2}, \Lambda(a)\rho^{1/2}\right) \\ &= \varphi(a) \\ &= \left(\tilde{\Omega}, \tilde{\pi}_\varphi(a)\tilde{\Omega}\right)\end{aligned}$$

Thus $V_\varphi^*\rho^{1/2} = \tilde{\Omega} \Rightarrow V_\varphi\pi_\varphi(a)V_\varphi^*\rho^{1/2} = \Lambda(a)\rho^{1/2}$.

The extension of $V_\varphi$ by continuity to the entire subspace $\overline{\tilde{\pi}_\varphi(\mathfrak{A})\tilde{\Omega}_\varphi}$ will again be

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denoted by $V_\varphi$ and used to define the map $\tilde{V}_\varphi$ of $\widetilde{\mathscr{H}_\varphi}$ into $\overline{\Lambda(\mathfrak{A})\rho^{1/2}}$ by setting $\tilde{V}_\varphi$ equal to $V_\varphi$ on $\overline{\tilde{\pi}_\varphi(\mathfrak{A})\tilde{\Omega}_\varphi}$ and zero on the orthocomplement of $\overline{\tilde{\pi}_\varphi(\mathfrak{A})\tilde{\Omega}_\varphi}$. Clearly then $\left\|\tilde{V}_\varphi\right\| \leq 2^{1/2}$ and $\tilde{V}_\varphi\pi_\varphi(a)\tilde{V}_\varphi^*\rho^{1/2} = \Lambda(a)\rho^{1/2}$ as required. □

## 5.3 The Quantum Correlation Coefficient

Consider a separable state $\omega$. As per previous discussions (c.f. page 68) that $\omega$ is represented by a measure $\mu$ which is the weak$^*$ limit of measures of the form

$$\mu_\alpha = \sum_{k=1}^{n} \lambda_k \delta_{\omega_k} = \sum_{k=1}^{n} \lambda_k \delta_{\omega_{1,k}\otimes\omega_{2,k}}$$

where $\omega_{1,k} \otimes \omega_{2,k}$ can, without loss of generality, be taken to be pure. Since the restriction is weak$^*$ continuous and affine, it follows that

$$\mu_\alpha = \sum_{k=1}^{n} \lambda_k \delta_{\omega_{1,k}} \times \delta_{\omega_{2,k}} \tag{5.1}$$

$\mu_\alpha$ is thus realized as a probability measure supported by points in $\mathfrak{S}_{\mathfrak{A}_1} \times \mathfrak{S}_{\mathfrak{A}_2}$ which (as per Remark 2.49) is identified with the space of all product states. Hence to every separable state it is possible to associate a representing measure that is the limit of measures of the form given in Equation (5.1).

The existence of a representing measure of this nature is a defining characteristic of separable states and can be exploited for entanglement detection such as in the convex roof measures. An alternative method of measuring entanglement is found by recognizing that the separable states can only contain classical correlations that arises from the statistical mixing of pure states to form a mixed state while the additional quantum correlation present in entangled states can be viewed as an intrinsic property of the state and its structure (since it is possible for both pure and mixed states). Intuitively this can be rectified by "removing" the classical correlations from a given state to leave those arising from entanglement itself. This can be achieved by noting that the measures of the form given by Equation

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(5.1) characterize these classical correlations completely i.e. a potential method to detect separability is produced by associating to each measure in $M_\omega(\mathfrak{S}_\mathfrak{A})$, another measure supported on $\mathfrak{S}_{sep}$ that can be approximated by measures of the form given in Equation (5.1) and then subtracting the barycenters of these two measures.

For any state $\varphi$ and measure $\mu \in M_\varphi(\mathfrak{S}_\mathfrak{A})$, it is possible to define two measures $\mu_1$ and $\mu_2$ on the Borel sets of $\mathfrak{S}_{\mathfrak{A}_1}$ and $\mathfrak{S}_{\mathfrak{A}_2}$ respectively by

$$\mu_i(F_i) = \mu(r_i^{-1}(F_i)) \text{ for Borel sets } F_i \subseteq \mathfrak{S}_{\mathfrak{A}_i}, i \in \{1,2\}$$

If $\mu$ is supported on the set $F_0$ then $\mu_1$ and $\mu_2$ are supported on $r_1F_0$ and $r_2F_0$ respectively. Moreover

$$r_i\varphi = r_i\left(\int \varphi' d\mu\right) = \int r_i\varphi' d\mu = \int \varphi_i' d(\mu \circ r_i) = \int \varphi_i' d\mu_i$$

thus $\mu_i \in M_{r_i\varphi}(\mathfrak{S}_\mathfrak{A})$

It then follows that if $\mu$ is finitely supported i.e. $\mu = \sum_{j=1}^n \lambda_j \delta_{\varphi_j}$, then $\mu_i = \sum_{j=1}^n \lambda_j \delta_{r_i\varphi_j}$ and $\mu_1(\{r_1\varphi_j\}) = \mu_2(\{r_2\varphi_j\})$ which then allows the following measure to be defined:

$$\boxtimes\mu := \sum_j \lambda_j \delta_{r_1\varphi_j} \times \delta_{r_2\varphi_j}$$

Clearly if $\mu$ is finitely supported, then $\boxtimes\mu$ is supported on $\mathfrak{S}_{\mathfrak{A}_1} \times \mathfrak{S}_{\mathfrak{A}_2}$. Recall that every $\mu \in M_\omega(\mathfrak{S}_\mathfrak{A})$ is the limit of a net of measures $\{\mu_k\}$ in $M_\omega(\mathfrak{S}_\mathfrak{A})$ with finite support (Proposition 3.36). It then follows that $\mu_{k,1}$ and $\mu_{k,2}$ can be defined akin to $\mu_1$ and $\mu_2$ respectively, in which case $\mu_{k,1} \to \mu_1$ and $\mu_{k,2} \to \mu_2$ in the weak$^*$ topology. Thus, for each $k$, $\boxtimes\mu_k$ can be defined as before which results in the production of a net $\{\boxtimes\mu_k\}$ where, by definition, each $\boxtimes\mu_k$ is a finitely supported probability measure on $\mathfrak{S}_{\mathfrak{A}_1} \times \mathfrak{S}_{\mathfrak{A}_2}$ and representing a separable state $\omega_k$. Since $M_1(\mathfrak{S}_\mathfrak{A})$ is weak$^*$ compact, then $\{\boxtimes\mu_k\}$ is weak$^*$ convergent (by considering a subnet if necessary) to a measure, which is denoted by $\boxtimes\mu$. If $\omega_k \to \omega$ ,where

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$\omega_k \in \mathfrak{S}_{sep}$, then $\omega \in \mathfrak{S}_{sep}$ by the definition of $\mathfrak{S}_{sep}$ and since

$$\mathfrak{S}_{\mathfrak{A}_1} \times \mathfrak{S}_{\mathfrak{A}_2} \subseteq \mathfrak{S}_{sep} \subseteq \mathfrak{S}_{\mathfrak{A}_1 \otimes \mathfrak{A}_2}$$

then $\boxtimes\mu$ is a probability measure on $\mathfrak{S}_{\mathfrak{A}_1 \otimes \mathfrak{A}_2}$ with support in $\mathfrak{S}_{sep}$ and $\int \varphi d \boxtimes \mu(\varphi)$ is a well defined element in $\mathfrak{S}_{sep}$ which allows for the following:

**Definition 5.17.** (Quantum Correlation Coefficient)
For $\omega \in \mathfrak{S}_{\mathfrak{A}_1 \otimes \mathfrak{A}_2}$ and $a \in \mathfrak{A} = \mathfrak{A}_1 \otimes \mathfrak{A}_2$, the *Quantum Correlation Coefficient* $d(\omega, a)$ is given by

$$d(\omega, a) = \inf_{\mu \in M_\omega(\mathfrak{S})} \left| \left( \int_{\mathfrak{S}_{\mathfrak{A}_1 \otimes \mathfrak{A}_2}} \varphi \, d\mu(\varphi) \right)(a) - \left( \int_{\mathfrak{S}_{\mathfrak{A}_1} \times \mathfrak{S}_{\mathfrak{A}_2}} \varphi \, d \boxtimes \mu(\varphi) \right)(a) \right|$$

Clearly $d(\omega, a)$ is non-negative and it has already been established that for any separable state there exists a measure such that $d(\omega, a) = 0$ for all $a \in \mathfrak{A}$.

In the converse case, let $\mu^a \in M_\omega(\mathfrak{S})$ be the measure such that $d(\omega, a) = 0$ for some given $a \in \mathfrak{A}$. It then follows that

$$\begin{aligned} \omega = \int \varphi \, d\mu^a(\varphi) = \int \varphi \, d \boxtimes \mu^a(\varphi) &= \lim_i \int \varphi \, d \left( \sum_i \lambda_i^a \delta_{r_1 \omega_i^a} \times \delta_{r_1 \omega_i^a} \right)(\varphi) \\ &= \lim_i \sum_i (r_1 \omega_i^a \otimes r_2 \omega_i^a) \end{aligned}$$

While this decomposition is promising, it is not sufficient to assume separability since it is point dependent. In order to prove global separability of a normal state $\omega$ (i.e. in order for there to exist a decomposition in terms of product states that is not point dependent), it is sufficient, by Proposition 5.13, to show that $\omega$ is invariant under the action of all maps of the form $\Lambda \otimes I_2$ where $\Lambda$ is a unital positive linear map i.e. $(\omega \circ \Lambda \otimes I_2)(a) \geq 0 \; \forall \; a \in \mathfrak{A}^+$.

Note that if $d(\omega, a) = 0$ for all $a \in \mathfrak{A}_1 \otimes \mathfrak{A}_2$, then in particular $d(\omega, a) = 0$ for all $a \in \mathfrak{A}^+$ and since $a \geq 0$ then $a$ can be approximated by elements in $\mathfrak{A}_1 \otimes \mathfrak{A}_2$ of

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the form $a = \sum_{k,l=1}^{n} x_k^* x_l \otimes y_k^* y_l$ then:

$$\omega(a) = \lim_i \sum_i \sum_{kl} \lambda_i^a (r_1 \omega_i^a (x_k^* x_l))(r_2 \omega_i^a (y_k^* y_l))$$

and

$$(\omega \circ (\Lambda \otimes I_2))(a) = \lim_i \sum_i \lambda_i^a \sum_{kl} (r_1 \omega_i^a (\Lambda(x_k^* x_l)))(r_2 \omega_i^a (y_k^* y_l))$$

Then by Theorem 5.16 it follows that

$$(\omega \circ (\Lambda \otimes I_2))(a) = \lim_i \sum_i \lambda_i^a \sum_{kl} (r_1 \omega_i^a (V_{\omega,i,a} \pi_{\omega,i,a} (x_k^* x_l) V_{\omega,i,a}^*))(r_2 \omega_i^a (y_k^* y_l))$$

where $\pi_{\omega,i,a}$ is a Jordan morphism which can be viewed as the sum of a *-morphism and a *-antimorphism by Theorem 5.10.

$\{x_k^* x_l\}_{kl}$ and $\{y_k^* y_l\}_{kl}$ can be viewed as positive matrices with entries in $\mathfrak{A}_1$ and $\mathfrak{A}_2$ respectively (see Lemma IV 3.1 in [30]). Under this scheme the aforementioned *-antimorphism can be expressed as a combination of a *-morphism and a transposition.

The pointwise application of states, *-morphisms, and transpositions to any positive matrix will produce a positive matrix with complex valued entries. The pointwise multiplication of the entries of two positive matrices is also a positive matrix, and it is thus possible to conclude that

$$(\omega \circ (\Lambda \otimes I_2))(a) \geq 0 \qquad \forall a \geq 0$$

and thus $\omega$ is separable i.e.

**Theorem 5.18.** *For a composite system $\mathfrak{A} = \mathfrak{A}_1 \otimes \mathfrak{A}_2$ and a normal state $\omega \in \mathfrak{S}_{\mathfrak{A}}$*

$$\omega \text{ is separable} \iff d(\omega, a) = 0 \quad \forall a \in \mathfrak{A}$$

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# Chapter 6

# Conclusion and Final Remarks

The ability to apply both Definition 4.11 and Definition 5.17 to infinite dimensional bipartite systems stems from two important points to note: first each state in a Ruelle separable set can be associated with a maximal measure that is supported on the pure states (c.f. Proposition 3.39 and Theorem 3.49), and second that each measure is can be approximated in the weak$^*$ topology by measures with finite support and representing the same state (Proposition 3.36). The combination of these two observations implies that the set of all measures that represent a separable state contains an element with a form that is unique to the separable states (c.f. Equation 5.1). The existence of this measure can be exploited to determine if a state is entangled or not either through the application of a suitable function (as in the case of the Convex Roof Measures) or through the direct "removal" of classical correlations from a state to leave only those quantum correlations arising from entanglement (as in the case of the Quantum Correlation Coefficient).

This dissertation has been limited to a discussion of bipartite entanglement i.e. the entanglement present in a composite system consisting of two distinct subsystems. Entanglement in the multipartite setting is more complicated than the bipartite setting.



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For example, the operational entanglement measures Entanglement Cost $E_C$ and Distillable Entanglement $E_D$ were defined through the identification of a maximally entangled pure state (i.e. the singlet state) that could be used to produce every other state via LOCC transformations alone. In the multipartite setting this cannot be replicated i.e. there exist pure states that are entangled but cannot be manipulated to reproduce each other[43, 46].

Even the definition of entanglement needs to be altered in the sense that the multipartite setting allows two different types of entanglement: partial entanglement and full entanglement:

- Full entanglement can be viewed as a direct extension of the current definition of entangled states to the multipartite scenario: a state $\omega$ on $\mathfrak{A}_1 \otimes \cdots \otimes \mathfrak{A}_n$ is said to be fully entangled if it is not an element of $\overline{conv}(\mathfrak{S}_{\mathfrak{A}_1} \times \cdots \times \mathfrak{S}_{\mathfrak{A}_n})$.
- Partial entanglement adopts the notion of splitting a multipartite system $\mathfrak{A}_1 \otimes \cdots \otimes \mathfrak{A}_n$ with respect to a partition $\{I_1, \ldots, I_k\}$ of the index set $I = \{1, \ldots, n\}$ such that $I_i \cap I_j = \emptyset$ whenever $i \neq j$ and $\bigcup_{i=1}^{k} I_i = I$. A state $\omega$ is then called separable with respect to that partition if $\omega \in \overline{conv}(\mathfrak{S}_{\mathfrak{A}_1} \times \cdots \times \mathfrak{S}_{\mathfrak{A}_k})$ where $\mathfrak{S}_{\mathfrak{A}_i}$ refers to the state space on the C$^*$-algebra formed form the C$^*$-algebras in the element $I_i$ of the partition. Interestingly there exist states which may be separable with respect to one partition but not another (c.f. [68]).

It is anticipated that the proposed entanglement measures (Definition 4.11 and Definition 5.17) can exploit their use of the restriction maps to be extended to well defined measures of entanglement in the multipartite case. For example full separability should be detected by Definition 5.17 when adapted to the utilize $\boxtimes\mu = \lim \boxtimes\mu_k = \lim \lambda_k \delta_{r_1\omega_k} \times \cdots \times \delta_{r_n\omega_k}$ however a full proof is beyond the scope of this dissertation and the extension of the concepts given in this dissertation as well as the connection between entanglement and positive maps are suspected to have potential for future research.

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