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Varieties of De Morgan Monoids

by

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Declaration

I, Johann Joubert Wannenburg, declare that the dissertation, which I hereby submit for the degree Philosophiae Doctor at the University of Pretoria, is my own work and has not previously been submitted by me for a degree at this or any other tertiary institution. Where use has been made of the work of others, it is duly acknowledged in the text.

SIGNATURE: ....

DATE: ....

In keeping with tradition [144, 145]*, this is dedicated to my father,*
Johann Wannenburg.
Without his initial encouragement I would not be an engineer, and this thesis might have been completed years earlier. But without his continued encouragement and support it would likely not have been written at all.

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> Oh, ev'ry thought that's strung a knot in my mind
> I might go insane if it couldn't be sprung
> But it's not to stand naked under unknowin' eyes
> It's for myself and my friends my stories are sung
>
> Bob Dylan

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Abstract

Chapter 1. We review prerequisite material from universal algebra and abstract algebraic logic, as well as the connections between 'substructural logics' (defined in this chapter) and varieties of residuated lattices (defined more precisely in the next chapter). In particular, we recall the definition of the relevance logic $\mathbf{R^t}$, which is algebraized by the variety DMM of De Morgan monoids.

Chapter 2. We introduce the variety of (commutative) residuated lattices, as well as its 'involutive' variant, and we explain the basic properties of some notable subvarieties—namely, the residuated lattices whose monoid operations are square-increasing, those that are idempotent, and De Morgan monoids. A new structural characterization of the finitely subdirectly irreducible De Morgan monoids is then established. In particular, each of them consists of two chains of idempotent elements, between which a not-necessarily-idempotent subalgebra is enclosed.

Chapter 3. The four-element De Morgan monoid $\boldsymbol{C}_4$ is totally ordered and it is the only nontrivial 0-generated algebra onto which finitely subdirectly irreducible De Morgan monoids can be mapped by non-injective homomorphisms. The homomorphic *pre-images* of $\boldsymbol{C}_4$ within DMM (together with the trivial De Morgan monoids) constitute a proper subquasivariety of DMM, which is shown to have a largest subvariety U. We prove a representation theorem for the algebras in U. It exploits a construction of Slaney, which we call a 'skew reflection'.

Another representation theorem is then proved for those De Morgan monoids $\boldsymbol{A}$ that are (i) *semilinear*, i.e., a subdirect product of totally ordered algebras, and (ii) *negatively generated*, i.e., generated by the set A^- of lower bounds of the neutral element of $\boldsymbol{A}$. This set A^- can be given the structure of a Brouwerian algebra $\boldsymbol{A}^-$ (even when $\boldsymbol{A}$ is merely a square-increasing residuated lattice). Using our representation theorems, we prove that the De Morgan monoids satisfying (i) and (ii) form a variety—in fact, a locally finite variety.

Chapter 4. It is proved that there are just four minimal varieties—and

just 68 minimal quasivarieties—of De Morgan monoids. The join-irreducible covers of the four atoms in the subvariety lattice of DMM are then investigated. One of the two atoms consisting of idempotent algebras has no such cover; the other atom has just one. The remaining two atoms lack nontrivial idempotent members. They are generated, respectively, by $\boldsymbol{C}_4$ and another 4-element De Morgan monoid $\boldsymbol{D}_4$. The covers of the variety $\mathbb{V}(\boldsymbol{C}_4)$ within U are revealed. There are just ten of them (all finitely generated). In exactly six of these ten varieties, all nontrivial members have $\boldsymbol{C}_4$ as a *retract*. Beyond U, we identify infinitely many covers of $\mathbb{V}(\boldsymbol{C}_4)$ [and of $\mathbb{V}(\boldsymbol{D}_4)$] that are finitely generated, and some that are not.

These results illuminate the maximal and pre-maximal extensions of $\mathbf{R^t}$.

Chapter 5. It is known that a quasivariety K of algebras has the joint embedding property (JEP) iff it is generated by a single algebra $\boldsymbol{A}$. It is structurally complete iff the free $\aleph_0$-generated algebra in K can serve as $\boldsymbol{A}$. A consequence of this demand, called 'passive structural completeness' (PSC), is that the nontrivial members of K all satisfy the same existential positive sentences. We prove that if K is PSC then it still has the JEP, and if it has the JEP and its nontrivial members lack trivial subalgebras, then its relatively simple members all belong to the universal class generated by one of them. Under these conditions, if K is relatively semisimple then it is generated by one K-simple algebra. We also prove that a quasivariety of finite type, with a finite nontrivial member, is PSC iff its nontrivial members have a common retract. The theory is then applied to the variety of De Morgan monoids, where we isolate the sub(quasi)varieties that are PSC and those that have the JEP, while throwing fresh light on those that are structurally complete. These results further illuminate the extension lattice of $\mathbf{R^t}$.

Chapter 6. It is proved that epimorphisms are surjective in a variety K of square-increasing residuated lattices (with or without involution), provided that each finitely subdirectly irreducible algebra $\boldsymbol{A} \in \mathsf{K}$ has two properties: (1) $\boldsymbol{A}$ is negatively generated, and (2) the poset of prime filters of $\boldsymbol{A}^-$ has finite depth. Neither (1) nor (2) may be dropped. The proof adapts to the presence of bounds, and the result encompasses a range of interesting varieties of De Morgan monoids.

The surjectivity of epimorphisms is then established for certain varieties of *semilinear* algebras, not encompassed by the above theorem. In particular, epimorphisms are surjective in the variety of all semilinear idempotent residuated lattices (where (1) fails in certain irreducible members). The same applies to all varieties of negatively generated semilinear De Morgan monoids (even those with irreducible members $\boldsymbol{A}$ for which the poset of prime filters of $\boldsymbol{A}^-$ has infinite depth).

These results, and those of the following chapter, settle natural questions about Beth-style definability for a range of extensions of $\mathbf{R^t}$.

Chapter 7. It is shown that there are $2^{\aleph_0}$ varieties of Brouwerian algebras that are not structurally complete. We also show that there is a continuum of varieties of Brouwerian algebras in which epimorphisms fail to be surjective. Using these findings, we draw analogous conclusions for varieties of De Morgan monoids.

Much of this material has been accepted for publication in the form of journal articles. Most of the results in Chapters 2–4 can be found in [103, 104]. The results of Chapter 5 [resp. the first part of Chapter 6] are contained in [106] [resp. [107]], although both papers adopt a slightly more general framework. With regard to Chapter 7, the uncountability results concerning structural completeness are included in [106], while the material concerning epimorphisms is adapted from [108]. The findings of Chapters 3 and 6 concerning semilinear algebras will be written up in [147].

Contents

Introduction

De Morgan monoids are commutative monoids with a residuated distributive lattice order and a compatible antitone involution $\neg$, where $a \leqslant a^2$ for all elements a. They form a variety, DMM.

The explicit study of residuated lattices goes back to Ward and Dilworth [148], but it has older antecedents in the ideal multiplication theory of rings, and in the calculus of binary relations (see the citations in [20, 51, 65]). Much of the interest in De Morgan monoids stems, however, from their connection with relevance logic, discovered by Dunn [36] (also see his contributions to [1], as well as [101]). A key fact, for our purposes, is that the axiomatic extensions of Anderson and Belnap's logic $\mathbf{R}^{\mathbf{t}}$ and the varieties of De Morgan monoids form anti-isomorphic lattices, and the latter are susceptible to the methods of universal algebra. In Chapter 1, preliminary material regarding universal algebra and the abstract connection between algebra and logic is recounted.

Relevance logic (a.k.a. relevant logic) was originally intended as a framework in which the so-called paradoxes of material implication could be avoided. These include the weakening axiom $p \to (q \to p)$, which, when interpreted intuitively, states that if p is true then q implies p. The paradox lies in the fact that q can be any statement, even something unrelated to p. Thus, relevance logic began as a rebellion against classical logic, but it subsequently gained multiple interpretations (see for instance [126, 133, 139, 140, 142]), and it now fits under the ideology-free umbrella of substructural logics [51]. In the last part of Chapter 1, a number of substructural logics, including $\mathbf{R}^{\mathbf{t}}$, are introduced. Relative to the other logics, $\mathbf{R}^{\mathbf{t}}$ adds $\wedge, \vee$ *distributivity* and the *contraction axiom* $(p \to (p \to q)) \to (p \to q)$. This combination of additional axioms has interesting meta-logical effects; for example, $\mathbf{R}^{\mathbf{t}}$ was shown to be undecidable by Urquhart in [141], whereas the substructural logics with only one of the last-mentioned two axioms are decidable; see [22] and [100], respectively.

In 1996, Urquhart [142, p. 263] observed that "[t]he algebraic theory of relevant logics is relatively unexplored, particularly by comparison with the field of algebraic modal logic." Acquiescing in a paper of 2001, Dunn and Restall [38, Sec. 3.5] wrote: "Not as much is known about the algebraic properties of De Morgan monoids as one would like." These remarks pre-date many recent papers on residuated lattices—see the bibliography of [51], for instance. But the latter have concentrated mainly on varieties incomparable with DMM (e.g., Heyting and MV-algebras), larger than DMM (e.g., full Lambek algebras) or smaller (e.g., Sugihara monoids). This thesis, especially Chapters 2 and 3, contributes to filling that gap.

Contraction amounts to the *square-increasing law* ($x \leqslant x^2$) of De Morgan monoids. It has the effect that such an algebra is simple iff its neutral element e has just one strict lower bound. Another effect is that two such algebras with the same involution-less reduct must be equal [132]. Also, finitely generated algebras of this kind are bounded (see [136, Prop. 5] and Theorem 2.10). On the other hand, $\wedge, \vee$ distributivity—which amounts to lattice-distributivity in the algebras—causes a De Morgan monoid to be (finitely subdirectly) irreducible iff its neutral element e is join-prime. Irreducible algebras are the building blocks of any variety, so knowledge of their structural properties is valuable.

Together, contraction and distributivity give De Morgan monoids the following special feature, not shared by more general residuated lattices: an irreducible algebra $\boldsymbol{A}$ consists only of upper bounds of e and lower bounds of $\neg e$, i.e., $A = [e) \cup (\neg e]$. And, when such an algebra is bounded, it is *rigorously compact*, i.e., if $\bot \leqslant x \leqslant \top$ for all x, then $\top \cdot x = \top$, unless $x = \bot$. (Bounded residuated lattices already satisfy $\bot \cdot x = \bot$ for all x.)

A De Morgan monoid $\boldsymbol{A}$ is said to be *idempotent* or *anti-idempotent* if it satisfies $x^2 = x$ or $x \leqslant (\neg e)^2$, respectively. The idempotent De Morgan monoids are the aforementioned *Sugihara monoids*, and their structure is comparatively well understood. Anti-idempotence is equivalent to the demand that no nontrivial idempotent algebra belongs to the variety generated by $\boldsymbol{A}$ (Corollary 2.14), hence the terminology.

The main result of Chapter 2 shows that any irreducible De Morgan monoid $\boldsymbol{A}$ is either (i) a totally ordered Sugihara monoid or (ii) the union of a nontrivial interval subalgebra $[\neg a, a]$ and two chains of idempotent elements, $(\neg a]$ and $[a)$, where $a = (\neg e)^2$. In the latter case, the anti-idempotent subalgebra $([\neg a, a])$ is the e-class of a congruence θ such that $\boldsymbol{A}/\theta$ is a totally ordered Sugihara monoid in which $\neg e = e$, and all other θ-classes are singletons. The elements in the idempotent chains behave, with respect to the monoid operation $\cdot$, like the extrema of a rigorously compact algebra. We therefore turn this characterization into a representation theorem (Theorem 2.57), involving so-called 'rigorous extensions' of anti-idempotent De Morgan monoids.

Slaney [129, 130] showed that the free 0-generated De Morgan monoid is finite, and that there are only seven non-isomorphic irreducible 0-generated De Morgan monoids (see Section 3.3). No similarly comprehensive classification is available in the 1-generated case, however, where the algebras may already be infinite. Of the seven irreducible 0-generated De Morgan monoids, three are simple, namely the two element Boolean algebra $\mathbf{2}$, and two four-element algebras $\boldsymbol{C}_4$ and $\boldsymbol{D}_4$, where $\boldsymbol{C}_4$ is totally ordered (with $e < \neg e$), while e and $\neg e$ are incomparable in $\boldsymbol{D}_4$.

Slaney [130] proved that $\boldsymbol{C}_4$ is the only 0-generated nontrivial algebra onto which irreducible De Morgan monoids may be mapped by non-injective homomorphisms. We demonstrate in Chapter 3 that there is a largest variety U of De Morgan monoids consisting of homomorphic *pre-images* of $\boldsymbol{C}_4$ (along with trivial algebras), as well as a largest subvariety M of DMM such that $\boldsymbol{C}_4$ is a *retract* of every nontrivial member of M. Thus, $\mathbb{V}(\boldsymbol{C}_4) \subseteq \mathsf{M} \subseteq \mathsf{U}$. We furnish U and M with finite equational axiomatizations; each has an undecidable equational theory and uncountably many subvarieties (see Sections 3.1 and 3.4). We also provide representation theorems for the members of U and M (Section 3.2), involving a 'skew reflection' construction of Slaney [131]. This is a generalization of an older 'reflection' construction of Meyer [97] (see Section 3.4).

In Section 3.5, these representations and the main result of Chapter 2 are combined to prove a further representation theorem for De Morgan monoids that are *semilinear* (subdirect products of totally ordered algebras) and *negatively generated*, i.e., generated by lower bounds of e. It follows that these algebras form a finitely axiomatizable variety. We show that this variety is locally finite.

Apart from understanding the algebraic structure of De Morgan monoids, we aim to illuminate the structure of the subvariety lattice of DMM, because it mirrors the axiomatic extensions of $\mathbf{R^t}$. In Chapter 4 we partially describe the lower part of the subvariety lattice.

We prove that a variety of De Morgan monoids consists of Sugihara monoids iff it omits $\boldsymbol{C}_4$ and $\boldsymbol{D}_4$ (Theorem 4.1). This implies that DMM has just four minimal subvarieties (Theorem 4.2) all of which are finitely generated. They are the varieties generated by $\boldsymbol{C}_4$ and $\boldsymbol{D}_4$, and two idempotent varieties: $\mathbb{V}(\mathbf{2})$, i.e., the class BA of all Boolean algebras, and $\mathbb{V}(\boldsymbol{S}_3)$—where $\boldsymbol{S}_3$ is the three-element Sugihara monoid.

The latter part of Chapter 4 (Sections 4.2–4.5) is primarily an investigation of the covers of these four atoms within DMM. It suffices to consider the join-irreducible covers, as the subvariety lattice of DMM is distributive. We show that BA has no join-irreducible cover within DMM, and that $\mathbb{V}(\boldsymbol{S}_3)$ has just one; the situation for $\mathbb{V}(\boldsymbol{C}_4)$ and $\mathbb{V}(\boldsymbol{D}_4)$ is much more complex (see

Theorem 4.9).

With the help of the skew reflection representations from Chapter 3, we identify all of the covers of $\mathbb{V}(\boldsymbol{C}_4)$ within U. There are just ten, of which exactly six fall within M (Theorem 4.23, Corollary 4.24). All ten of these varieties are finitely generated.

Within DMM, every cover of $\mathbb{V}(\boldsymbol{D}_4)$ is semisimple—provided that it is join-irreducible. The same applies to the covers of $\mathbb{V}(\boldsymbol{C}_4)$ that are not contained in U. In both cases, we identify infinitely many such covers that are finitely generated, and some that are not even generated by their finite members (see Sections 4.4 and 4.5).

In the literature of substructural logics, subvariety lattices are more prominent than subquasivariety lattices, because they mirror the extensions of a logic by new axioms, as opposed to new inference rules. Nevertheless, some natural meta-logical problems call for a consideration of quasivarieties if they are to be approached algebraically, e.g., the identification of the structurally complete axiomatic extensions of $\mathbf{R^t}$. Chapter 5 is devoted to such problems, but already in Section 4.1 we describe the bottom of the subquasivariety lattice of DMM. Each of the four minimal varieties of De Morgan monoids is also minimal as a quasivariety, but they are not alone in this. Indeed, we prove that DMM has just 68 minimal subquasivarieties (Corollary 4.7, Remark 3.29).

Motivated by logical concerns, Chapter 5 focusses on connections between the subquasivariety and subvariety lattices of DMM. Most generally, we ask when a (quasi)variety of De Morgan algebras is 'singly generated', i.e., generated *as a quasivariety* by a single algebra. Equivalently, in logical terms, we ask when the derivable rules of an extension of $\mathbf{R^t}$ are determined by a single set of 'truth tables'. By the Łoś-Suszko Theorem 5.13, that demand amounts to a more widely meaningful variant of the relevance logician's 'variable-sharing principle' (see Definition 5.12). Maltsev [89] proved that a quasivariety K is singly generated iff it has the *joint embedding property* (JEP), i.e., any two nontrivial members of K can both be embedded into some third member.

When investigating these properties for De Morgan monoids, we found that some of our results generalized to quasivarieties whose nontrivial members lack trivial subalgebras. We call these *Kollár* quasivarieties, after [77]. The first part of Chapter 5 therefore has a universal algebraic flavour. We show that if a Kollár quasivariety K has the JEP, then its relatively simple members all belong to the universal class generated by one of them (Theorem 5.7). If, in addition, K is relatively semisimple, then it is generated (as a quasivariety) by one K-*simple* algebra.

We characterize the subvarieties of DMM that have the JEP (Theorem 5.37), but only after investigating various strengthenings of the JEP that have received attention in the literature. One such property is *structural completeness*. It has logical origins, but in algebraic terms, a quasivariety is structurally complete iff it is generated by its free $\aleph_0$-generated member. Moreover, a variety K is structurally complete iff each proper subquasivariety of K generates a proper subvariety of K. A quasivariety is said to be *hereditarily structurally complete* if each of its subquasivarieties is structurally complete. A weak variant of structural completeness, called *passive structural completeness* (PSC) asks, in effect, that any two nontrivial members of a quasivariety have the same existential positive theory. This property still implies the JEP (Theorem 5.19). We prove that a quasivariety of finite type with a finite nontrivial member is PSC iff its nontrivial members have a common retract (Theorem 5.28).

Using this fact, we describe completely the (quasi)varieties of De Morgan monoids that are PSC (Theorems 5.33 and 5.34) and conclude that, apart from the idempotent and minimal varieties, the remaining structurally complete varieties of De Morgan monoids all fall within M. We also show that, in the varietal join J of the six covers of $\mathbb{V}(\boldsymbol{C}_4)$ within M, every finite irreducible algebra is projective. It follows that every subquasivariety of J is a variety, i.e., that J is hereditarily structurally complete. (See Theorems 5.42 and 5.43.)

In Chapter 6 we investigate another algebraic property that has logical significance. In a variety of algebras, if a homomorphism is surjective, then it is an epimorphism, but the converse need not hold. Indeed, rings and distributive lattices each form varieties in which non-surjective epimorphisms arise. As it happens, this reflects the absence of unary terms defining multiplicative inverses in rings, and complements in distributive lattices, despite the uniqueness of those entities when they exist.

Such constructs are said to be *implicitly* (and not *explicitly*) *definable*. In a variety of logic, they embody implicitly definable propositional functions that cannot be explicated in the corresponding logical syntax, and Beth-style 'definability properties' *preclude* phenomena of this kind.

In particular, when a logic $\mathbf{L}$ is algebraized, in the sense of [17], by a variety K of algebras, then the *ES property* for K—i.e., the demand that *all* epimorphisms in K be surjective—amounts to the so-called *infinite Beth definability property* for $\mathbf{L}$. The most general version of this 'bridge theorem' was formulated and proved by Blok and Hoogland [13, Thms. 3.12, 3.17] (also see [105, Thm. 7.6] and the antecedents cited in both papers). In this situation, the subvarieties of K algebraize the axiomatic extensions of $\mathbf{L}$, but the ES property need not persist in subvarieties. It is therefore a well-motivated task to determine which subvarieties of K have surjective epimorphisms.

Urquhart [143] showed that epimorphisms need not be surjective in the variety of De Morgan monoids, but the goal of Chapter 6 is to locate subvarieties in which they *are* surjective. Interestingly, the results of this chapter do not depend on distributivity, nor on the presence of an involution, and they adapt to the presence of (distinguished) bounds. The context of Chapter 6 is therefore (possibly involutive) square-increasing residuated lattices (S[I]RLs). The *negative cone* of an S[I]RL $\boldsymbol{A}$, which comprises the lower bounds of e, may be given the structure of a Brouwerian or Heyting algebra $\boldsymbol{A}^-$, to which the Esakia duality of [44] applies. In particular, the *depth* of $\boldsymbol{A}$ may be defined as that of $\boldsymbol{A}^-$. In Sections 6.2–6.4 we review preliminary material regarding epimorphisms and Esakia duality.

The first main result of this chapter (Theorem 6.22) shows that in a variety K of S[I]RLs, epimorphisms will be surjective if each irreducible member of K is negatively generated and has finite depth. Neither hypothesis may be dropped. The assumptions of Theorem 6.22 persist in subvarieties and under varietal joins, so the result is labour-saving.

Beyond the scope of Theorem 6.22, a representation theorem from [59] is exploited in Sections 6.6 and 6.7 to establish the ES property for several varieties of *semilinear* residuated lattices. Success in the case of semilinear *idempotent* residuated lattices (Theorem 6.33) is particularly noteworthy, because it entails the *strong amalgamation property* for this variety. It also shows that, in a variety of SRLs with surjective epimorphisms, the irreducible algebras need not be negatively generated. On the other hand, dropping idempotence, we prove that epimorphisms are surjective in *all* varieties of negatively generated semilinear De Morgan monoids (even those with algebras of infinite depth). Here, the demand for negative generation cannot be dropped.

In Chapter 7, we supply two uncountability results. Brouwerian algebras model the positive fragment of intuitionistic propositional logic. Citkin determined the *hereditarily* structurally complete varieties of Brouwerian algebras in [32]; there are denumerably many of them. We show (in Theorem 7.4) that $2^{\aleph_0}$ varieties of Brouwerian algebras are *not* structurally complete. We then apply the reflection construction to show that a continuum of subvarieties of M are structurally *in*complete (Theorem 7.6). The cardinality of the set of structurally complete varieties of Brouwerian algebras (and of De Morgan monoids) is not known.

A result of Kreisel [78] shows (in effect) that every variety of Brouwerian algebras has a *weak* form of the ES property, whereas Maksimova established that only finitely many enjoy a certain *strong* form; see [49, 87, 88]. Uncountably many varieties of Brouwerian algebras have finite depth [79], and all of these have surjective epimorphisms; the latter claim was proved recently in [11], using Esakia duality. (Our Theorem 6.22 is a generalization of this fact.) In [11], one example of a variety K of Brouwerian algebras was exhibited in which the ES property fails. That example confirmed Blok and Hoogland's conjecture in [13] that the weak ES property really is strictly weaker than the ES property. Strengthening this finding, we show that, in $2^{\aleph_0}$ varieties of Brouwerian algebras (containing K), not all epimorphisms are surjective. We then apply the reflection construction again, and show that a continuum of varieties of De Morgan monoids having the weak ES property still lack the ES property. Moreover, these witnesses can be chosen locally finite.

Chapter 1

Substructural logics and universal algebra

As was mentioned in the introduction, the main impetus for the study of De Morgan monoids came from their connection with the relevance logic $\mathbf{R^t}$. More generally, we shall study *varieties* of De Morgan monoids, which correspond to axiomatic extensions of $\mathbf{R^t}$. The algebraic *properties* that we investigate are also motivated by the connection to logic, in that each of them is equivalent to a significant meta-logical property for the corresponding deductive system. The tools of universal algebra are well suited to such an investigation, because they often allow one to connect formal or syntactic notions with structural (purely algebraic) ones.

Another helpful feature of universal algebra is that its results often concern global properties of a class of algebras, transcending the need to reference the specific operation symbols under consideration. This level of generality will be useful, as we shall often need to consider algebras that are closely related to De Morgan monoids, but which are either more general, or have slightly different operations. All of these algebras fall under the umbrella of *residuated structures*, so this thesis is a contribution to that general field. It is therefore simultaneously a contribution to the study of *substructural logics*, because the latter are the logics modeled by residuated structures.

In the first section of this chapter we review the preliminary universal algebraic notions and notation used throughout the thesis. We also define 'Kollár quasivarieties' and discuss 'existential positive sentences', which might be less familiar to the reader.

The aim of the second section is to introduce the substructural logics corresponding to the various classes of residuated structures under investigation here, and to make this correspondence precise.

1.1 Universal algebra preliminaries

We proceed to review the basic universal algebraic definitions and constructions. Standard references on universal algebra include [7] and [24]. Our brief overview is based on [120].

Algebraic constructions

An *algebra* $\boldsymbol{A} = \langle A; F \rangle$ comprises a non-empty set A (its *universe*) and an indexed family $F = \{f^{\boldsymbol{A}} : f \in \mathcal{F}\}$ of finitary *basic operations* on A. The set $\mathcal{F}$ is called the *signature* of $\boldsymbol{A}$, and its elements are called *operation symbols*. Each $f^{\boldsymbol{A}}$ is a function from the cartesian power A^n to A, for some non-negative integer n, called the *arity* of $f^{\boldsymbol{A}}$. By *n-ary*, we mean 'with arity n'. Constants, i.e., distinguished elements of $\boldsymbol{A}$, are treated as nullary (i.e., 0-ary) basic operations. The function sending each $f \in \mathcal{F}$ to the arity of $f^{\boldsymbol{A}}$ is called the *type* of $\boldsymbol{A}$, and algebras with the same type are said to be *similar*. A *(similarity) type* can be defined without reference to particular algebras as any function from a set $\mathcal{F}$ into the set ω of non-negative integers. When $\mathcal{F}$ is a finite [resp. countable] set, we say that $\boldsymbol{A}$ (as above) has *finite* [resp. *countable*] *type*. We call $\boldsymbol{A}$ *finite* [resp. *trivial*] if A is finite [resp. $|A| = 1$].

A *subuniverse* of an algebra $\boldsymbol{A}$ is a subset of A, closed under the basic operations of $\boldsymbol{A}$. It becomes a *subalgebra* of $\boldsymbol{A}$ when equipped with the appropriate restrictions of the operations, provided it is not empty, and $\boldsymbol{A}$ is then called an *extension* of this subalgebra. Arbitrary intersections of subuniverses are again subuniverses. Given a subset X of A, we denote by $\mathrm{Sg}^{\boldsymbol{A}} X$ the subuniverse of $\boldsymbol{A}$ *generated* by X, i.e., the smallest subuniverse of $\boldsymbol{A}$ which contains X. Note that $\mathrm{Sg}^{\boldsymbol{A}} X$ is empty only when $X = \emptyset$ and $\boldsymbol{A}$ has no distinguished element. In all other cases we let $\mathbf{Sg}^{\boldsymbol{A}} X$ denote the subalgebra of $\boldsymbol{A}$ with universe $\mathrm{Sg}^{\boldsymbol{A}} X$, i.e., the subalgebra of $\boldsymbol{A}$ *generated* by X. Thus, $\boldsymbol{A}$ is *generated* by X when $\mathrm{Sg}^{\boldsymbol{A}} X = A$. In this case, if m is any cardinal $\geq |X|$, we say that $\boldsymbol{A}$ is *m-generated* (so that 'finitely generated' means 'm-generated for some finite m'). Note that an algebra is 0-generated iff it has a distinguished element and no proper subalgebra. An algebra with a distinguished element has a unique 0-generated subalgebra, which is its smallest subalgebra.

Reducts of an algebra $\boldsymbol{A}$ arise by discarding basic operations, and *subreducts* are subalgebras of indicated reducts. We call $\boldsymbol{A}$ an *expansion* of each of its reducts.

The *direct product* $\prod_{i \in I} \boldsymbol{A}_i$ of a family $\{\boldsymbol{A}_i : i \in I\}$ of similar algebras is their cartesian product, on which the appropriate basic operations are defined in terms of those of the algebras $\boldsymbol{A}_i$ in the obvious co-ordinatewise fashion. When the algebras are all the same, we may denote the resulting *direct power* by $\boldsymbol{A}^{\boldsymbol{I}}$. Products of empty families are understood to have universe $\{\emptyset\}$.

A *homomorphism* $h\colon \boldsymbol{A} \to \boldsymbol{B}$ between similar algebras is a function that preserves the basic operations $f^{\boldsymbol{A}}$ of $\boldsymbol{A}$, in the sense that

$$h(f^{\boldsymbol{A}}(\vec{a})) = f^{\boldsymbol{B}}(h(a_1), \ldots, h(a_n)) \text{ for all } \vec{a} = a_1, \ldots, a_n \in A,$$

where n is the arity of $f^{\boldsymbol{A}}$. We call h an *embedding* if it is also injective, and an *isomorphism* if it is bijective. The notation $h\colon \boldsymbol{A} \cong \boldsymbol{B}$ signifies that h is an isomorphism, and $h\colon \boldsymbol{A} \hookrightarrow \boldsymbol{B}$ that h is an embedding. The target of a surjective [resp. bijective] homomorphism is called a *homomorphic* [resp. *isomorphic*] *image* of the domain. We sometimes indicate that h is surjective with the notation $h\colon \boldsymbol{A} \twoheadrightarrow \boldsymbol{B}$. An *endomorphism* [resp. *automorphism*] of $\boldsymbol{A}$ is a homomorphism [resp. isomorphism] $h\colon \boldsymbol{A} \to \boldsymbol{A}$. If $h\colon \boldsymbol{A} \to \boldsymbol{B}$ is a homomorphism, and $\boldsymbol{C}$ and $\boldsymbol{D}$ are subalgebras of $\boldsymbol{A}$ and $\boldsymbol{B}$ respectively, then $h[C] := \{h(c) : c \in C\}$ and $h^{-1}[D] := \{a \in A : h(a) \in D\}$ are subuniverses of $\boldsymbol{B}$ and $\boldsymbol{A}$, respectively. The corresponding subalgebras are denoted by $h[\boldsymbol{C}]$ and $h^{-1}[\boldsymbol{D}]$ (if $h^{-1}[D] \neq \emptyset$). If X generates $\boldsymbol{A}$, then $h[X]$ generates $h[\boldsymbol{A}]$.

Given a map h with domain X, the *kernel* of h is the set

$$\ker h := \{\langle a, b\rangle \in X^2 : h(a) = h(b)\}.$$

A *congruence* (relation) on an algebra $\boldsymbol{A}$ is the kernel of some homomorphism with domain $\boldsymbol{A}$, i.e., it is an equivalence relation θ on A, *compatible* with each basic operation $f^{\boldsymbol{A}}$ of $\boldsymbol{A}$ in the sense that, whenever $a_k \equiv_\theta b_k$ (i.e., $\langle a_k, b_k\rangle \in \theta$) for $k = 1, \ldots, n$, then

$$f^{\boldsymbol{A}}(a_1, \ldots, a_n) \equiv_\theta f^{\boldsymbol{A}}(b_1, \ldots, b_n),$$

n being the arity of $f^{\boldsymbol{A}}$. This compatibility demand implies that the set A/θ of equivalence classes a/θ ($a \in A$) becomes a *factor* algebra $\boldsymbol{A}/\theta$ of the same type as $\boldsymbol{A}$, under the unambiguous natural definition of the basic operations: $f^{\boldsymbol{A}/\theta}(a_1/\theta, \ldots, a_n/\theta) = f^{\boldsymbol{A}}(a_1, \ldots, a_n)/\theta$. Of course, the rule $a \mapsto a/\theta$ defines a surjective homomorphism $\boldsymbol{A} \to \boldsymbol{A}/\theta$. Moreover, this leads to the first isomorphism theorem of universal algebra:

Homomorphism Theorem 1.1. *If $h\colon \boldsymbol{A} \to \boldsymbol{B}$ is a homomorphism, then $\boldsymbol{A}/\theta \cong h[\boldsymbol{A}]$, where θ is the kernel of h. The isomorphism identifies each a/θ with $h(a)$.*

For any set A and equivalence relation θ on A, the map $a \mapsto a/\theta$ is called the *canonical surjection* from A to A/θ.

Given a class K of similar algebras and an algebra $\boldsymbol{A}$ of the same type, the set $\mathrm{Con}_{\mathsf{K}}(\boldsymbol{A})$ of K-*congruences* of $\boldsymbol{A}$ (a.k.a. *relative congruences* of $\boldsymbol{A}$ when K is understood) consists of the congruences θ such that $\boldsymbol{A}/\theta \in \mathsf{K}$. The set of *all* congruences of an algebra $\boldsymbol{A}$ is denoted by $\mathrm{Con}(\boldsymbol{A})$ (i.e., $\mathrm{Con}(\boldsymbol{A}) = \mathrm{Con}_{\mathsf{K}}(\boldsymbol{A})$, where K is the class of all algebras similar to $\boldsymbol{A}$).

Second Isomorphism Theorem 1.2. *If θ and φ are congruences of an algebra $\boldsymbol{A}$, with $\theta \subseteq \varphi$, then $\varphi/\theta := \{\langle a/\theta, b/\theta\rangle \colon \langle a, b\rangle \in \varphi\}$ is a congruence of $\boldsymbol{A}/\theta$ and $(\boldsymbol{A}/\theta)/(\varphi/\theta) \cong \boldsymbol{A}/\varphi$.*

A *subdirect product* $\boldsymbol{B}$ of a family $\{\boldsymbol{A}_i \colon i \in I\}$ of similar algebras is a subalgebra of their direct product, such that each of the natural projection homomorphisms $\pi_j : \prod_{i\in I} \boldsymbol{A}_i \to \boldsymbol{A}_j$ ($j \in I$) restricts to a surjection from B to A_j (so each $\boldsymbol{A}_j$ is a homomorphic image of $\boldsymbol{B}$). An embedding $h \colon \boldsymbol{A} \to \prod_{i\in I} \boldsymbol{A}_i$ is called a *subdirect embedding* if $h[\boldsymbol{A}]$ is a subdirect product of $\{\boldsymbol{A}_i \colon i \in I\}$.

Let K be a class of similar algebras, with $\boldsymbol{A} \in \mathsf{K}$. We say that $\boldsymbol{A}$ is [*finitely*] K-*subdirectly irreducible* if the following is true for every [finite non-empty] set I and every family $\{\boldsymbol{A}_i : i \in I\}$ of *members of* K: whenever an embedding $h \colon \boldsymbol{A} \to \prod_I \boldsymbol{A}_i$ is subdirect, then $\pi_i \circ h \colon \boldsymbol{A} \cong \boldsymbol{A}_i$ for some $i \in I$. The prefix 'K-' is often replaced by the word 'relatively' when K is understood, and removed altogether when K is understood to be the class of all algebras that are similar to $\boldsymbol{A}$. We often abbreviate '[finitely] subdirectly irreducible' as *[F]SI*.

Birkhoff's Subdirect Decomposition Theorem 1.3 ([24, Thm. 8.5]). *Every algebra is isomorphic to a subdirect product of SI homomorphic images of itself.*

A *filter $\mathcal{U}$ over* a set I is a non-empty set of subsets of I, closed under taking supersets and finite intersections. It is an *ultrafilter* over I if it excludes $\emptyset$ and is not properly contained in any filter over I, except for the filter of all subsets of I. In this case, for any $J, J' \subseteq I$, if $J \cup J' \in \mathcal{U}$, then $J \in \mathcal{U}$ or $J' \in \mathcal{U}$ (in particular, just one of $J, I \setminus J$ belongs to $\mathcal{U}$). Note that there is no ultrafilter over $\emptyset$. The ultrafilter $\mathcal{U}$ is *principal* if it equals $\{J \subseteq I : x \in J\}$ for some $x \in I$. Every non-principal ultrafilter over an infinite set I contains the so-called Fréchet filter of *co-finite* subsets of I, i.e., the sets $X \subseteq I$ such that $I \setminus X$ is a finite set.

Given a non-empty family $\{\boldsymbol{A}_i \colon i \in I\}$ of similar algebras and $a \in \prod_{i\in I} A_i$, we often write a as $\langle a_i : i \in I\rangle$, so that a_i abbreviates $\pi_i(a)$. For an ultrafilter $\mathcal{U}$ over I, the relation $\theta_{\mathcal{U}}$ identifies all pairs $a, b \in \prod_{i \in I} A_i$ for which there exists $J \in \mathcal{U}$ such that

$$a_i = b_i \text{ for all } i \in J.$$

It is a congruence of $\prod_{i \in I} \boldsymbol{A}_i$. The factor algebra $\left(\prod_{i \in I} \boldsymbol{A}_i\right)/\theta_{\mathcal{U}}$, abbreviated as $\prod_{i \in I} \boldsymbol{A}_i/\mathcal{U}$, is called an *ultraproduct* of $\{\boldsymbol{A}_i \colon i \in I\}$. We abbreviate the equivalence class $a/\theta_{\mathcal{U}}$ as $a/\mathcal{U}$. We use the term *ultrapower* when the algebras $\boldsymbol{A}_i$ are all the same. If $\mathcal{U}$ is principal, then $\prod_{i \in I} \boldsymbol{A}_i/\mathcal{U} \cong \boldsymbol{A}_j$ for some $j \in I$. If $\boldsymbol{B}$ is an ultrapower $\boldsymbol{A}^I/\mathcal{U}$ of $\boldsymbol{A}$, then $\boldsymbol{A}$ is called an *ultraroot* of $\boldsymbol{B}$. Any algebra can be embedded into each of its ultrapowers by the obvious map $a \mapsto \langle a, a, a, \dots \rangle/\mathcal{U}$.

Theorem 1.4 ([24, Thm. V.2.14])**.** *Every algebra embeds into an ultraproduct of finitely generated subalgebras of itself.*

The class operator symbols

$$\mathbb{I},\ \mathbb{H},\ \mathbb{S},\ \mathbb{E},\ \mathbb{P},\ \mathbb{P}_{\mathbb{S}},\ \mathbb{P}_{\mathbb{U}} \text{ and } \mathbb{R}_{\mathbb{U}}$$

stand for closure under isomorphic and homomorphic images, subalgebras, extensions, direct and subdirect products, ultraproducts and ultraroots, respectively. For each class operator $\mathbb{O}$, we abbreviate $\mathbb{O}(\{\boldsymbol{A}_1, \dots, \boldsymbol{A}_n\})$ as $\mathbb{O}(\boldsymbol{A}_1, \dots, \boldsymbol{A}_n)$.

We now turn to some of the syntactic aspects of universal algebra.

Syntactic aspects

Recall that any *first order language* $\mathcal{L}$ includes a denumerable set *Var* of variables (which we fix throughout the thesis) and the following first order logical connectives: $\approx$, $\forall$, $\exists$, $\Longrightarrow$, & , $\sqcup$, and `not` (where $\approx$ is formal equality and $\sqcup$ denotes first order 'disjunction', and the other connectives have their usual meanings). The language $\mathcal{L}$ furthermore includes a *first order signature*, which is a pair $\langle \mathcal{F}, \mathcal{R} \rangle$ of disjoint sets, whose elements are each assigned a non-negative integer (its *arity*). The elements of $\mathcal{F}$ [resp. $\mathcal{R}$] are called *operation* [resp. *relation*] *symbols*, and relation symbols have nonzero arity. Operation symbols with zero arity are called *constant symbols*.

Let X be a set. The *terms of $\mathcal{L}$ over X* depend only on $\mathcal{F}$ and X, and are defined recursively, as follows. Define $T_0 := X \cup \{f \in \mathcal{F} \colon \rho(f) = 0\}$, where $\rho(f)$ denotes the arity of $f \in \mathcal{F}$, and for each $n \in \omega$

$$T_{n+1} := T_n \cup \{f(t_1, \dots, t_k) \colon f \in \mathcal{F},\ k = \rho(f) \text{ and } t_1, \dots, t_k \in T_n\}.$$

(Here, $f(t_1, \ldots, t_k)$ is just a string of symbols.) Then $T_\rho(X) := \bigcup_{n\in\omega} T_n$ is the set of terms of $\mathcal{L}$ over X, and $T_\rho(\mathit{Var})$ is the set of terms of $\mathcal{L}$ itself. (We write $T(X)$ for $T_\rho(X)$ when $\mathcal{L}$ is understood.)

The *complexity* $\#t$ of a term $t \in T_\rho(X)$ is the smallest n such that $t \in T_n$.

An *atomic formula* of $\mathcal{L}$ is an *equation* $s \approx t$ (with $s, t \in T_\rho(\mathit{Var})$) or an expression of the form $r(t_1, \ldots, t_k)$ where r is any relation symbol of $\mathcal{L}$ with arity k and $t_1, \ldots, t_k \in T_\rho(\mathit{Var})$.

The *first order formulas* of $\mathcal{L}$ are then built up in the usual recursive way from the atomic formulas using the first order logical connectives above. Recall that a first order formula with no free variable is called a *sentence*. For a concise account of first order logic, see [24, Sec. V.1].

Thus, algebras model first order languages in an *algebraic* signature, i.e., one with no relation symbol.

Let K be a class of similar algebras. For a set Σ of first order formulas (over the signature of K), the notation $\mathsf{K} \models \Sigma$ means that the universal closure $\forall \bar{x}\, \Phi$ of each $\Phi \in \Sigma$ is true in every algebra belonging to K.

A *quasi-equation* has the form

$$\left(\mathop{\&}_{i<n} \Phi_i \right) \implies \Phi_n,$$

where $n \in \omega$ and $\Phi_0, \ldots, \Phi_n$ are formal equations.

Definition 1.5. A *variety* [resp. *quasivariety*] is the model class of a set of equations [resp. quasi-equations].

Let $\rho : \mathcal{F} \to \omega$ be a similarity type and X a set. The elements of $T := T_\rho(X)$ are just formal strings of symbols. However, T is naturally the universe of an algebra $\boldsymbol{T} = \boldsymbol{T}_\rho(X)$, the *term algebra* over X, whose basic operations are the functions $f^{\boldsymbol{T}} : \langle t_1, \ldots, t_n \rangle \mapsto f(t_1, \ldots, t_n)$, $f \in \mathcal{F}$. Recall that nullary operation symbols of $\mathcal{F}$ are elements of T, so $\boldsymbol{T}$ exists unless $X = \emptyset$ and $\mathcal{F}$ includes no nullary symbol.

For $m \in \omega$ and $\vec{x} = x_1, \ldots, x_m \in X$, the expression '$t(\vec{x}) \in T$' signifies that $t \in T$ and that the variables occurring in t are *among* $x_1, \ldots, x_m$. Every such expression gives rise, in each algebra $\boldsymbol{A}$ of type ρ, to an m-ary *term operation* $t(\vec{x})^{\boldsymbol{A}} \colon A^m \to A$ (abbreviated as $t^{\boldsymbol{A}}$ when $\vec{x}$ is understood), which is defined recursively: if t is x_i, then $t^{\boldsymbol{A}}$ is the i-th projection $\pi_i \colon A^m \to A$; if $t_j^{\boldsymbol{A}} \colon A^m \to A$ is defined for $j = 1, \ldots, n$ and t is $f(t_1, \ldots, t_n) \in T$, where $f \in \mathcal{F}$, then $t^{\boldsymbol{A}}(\vec{a}) := f^{\boldsymbol{A}}(t_1^{\boldsymbol{A}}(\vec{a}), \ldots, t^{\boldsymbol{A}}(\vec{a}))$ for all $\vec{a} \in A^m$. The expression $t(\vec{x})$ is sometimes called an *m-ary term*, even if $x_1, \ldots, x_m$ don't all occur in t. Two algebras are *termwise equivalent* if they have the same universe and the same term operations.

The compatibility of congruences and homomorphisms with basic operations extends inductively to term operations. In any algebra $\boldsymbol{A}$, the smallest subuniverse containing a subset B consists of all $t^{\boldsymbol{A}}(\vec{b})$ such that $t^{\boldsymbol{A}}$ is a term operation of $\boldsymbol{A}$ and $\vec{b}$ a tuple of elements of B, whose length is the arity of $t^{\boldsymbol{A}}$.

There is an alternative way to view term algebras. Given a class K of algebras similar to algebra $\boldsymbol{A}$, we say that $\boldsymbol{A}$ is K-*free over* a set X if X generates $\boldsymbol{A}$ and every function from X into an algebra $\boldsymbol{B} \in \mathsf{K}$ can be extended to a homomorphism from $\boldsymbol{A}$ to $\boldsymbol{B}$. (The extension is then unique.) In this case, we call X a K-*free generating set* for $\boldsymbol{A}$.

Provided that $X \neq \emptyset$ or that some nullary symbols are available, there *is* a K-free algebra $\boldsymbol{A}$ over X, with $\boldsymbol{A} \in \mathbb{ISP}(\mathsf{K})$. Any bijection from X to a K-free generating set for another K-free algebra $\boldsymbol{C} \in \mathbb{HSP}(\mathsf{K})$ extends to a unique isomorphism from $\boldsymbol{A}$ to $\boldsymbol{C}$. We therefore denote $\boldsymbol{A}$ by $\boldsymbol{F}_{\mathsf{K}}(X)$, or by $\boldsymbol{F}_{\mathsf{K}}(m)$ if $m = |X|$.

Let K_ρ denote the class of all algebras with similarity type ρ. Then $\boldsymbol{F}_{\mathsf{K}_\rho}(X) \cong \boldsymbol{T}_\rho(X)$, for any set X, so we sometimes call $\boldsymbol{T}_\rho(X)$ the *absolutely free algebra* over X.

If we employ the elements of $\boldsymbol{B} \in \mathsf{K}$ (or those of a generating set for $\boldsymbol{B}$) as free generators for a K-free algebra $\boldsymbol{F} \in \mathbb{ISP}(\mathsf{K})$ and then map these back to themselves, we obtain a surjective homomorphism $\boldsymbol{F} \to \boldsymbol{B}$. This, and the fact that [quasi]equations are preserved by $\mathbb{I}$, $\mathbb{S}$ and $\mathbb{P}$, yields the following result.

Theorem 1.6. *Let K be a [quasi]variety. Then every algebra in K is a homomorphic image of a K-free algebra in K. In fact, for any cardinal m, every m-generated algebra in K is a homomorphic image of $\boldsymbol{F}_{\mathsf{K}}(m)$, provided that $\boldsymbol{F}_{\mathsf{K}}(m)$ exists.*

A class of algebras is *trivial* if it contains only trivial algebras. For any nontrivial class K of algebras of type ρ and any set X for which $\boldsymbol{T} = \boldsymbol{T}_\rho(X)$ exists, we define

$$\Phi_{\mathsf{K}}(X) := \{\varphi \in \mathrm{Con}(\boldsymbol{T}) : \boldsymbol{T}/\varphi \in \mathbb{IS}(\mathsf{K})\} \text{ and } \theta = \theta_{\mathsf{K}}(X) := \bigcap \Phi_{\mathsf{K}}(X).$$

The map $x \to \bar{x} := x/\theta$ ($x \in X$) is injective, as K is nontrivial. In fact, we may identify $\boldsymbol{F} = \boldsymbol{F}_{\mathsf{K}}(|X|)$ with $\boldsymbol{T}/\theta$, because $\boldsymbol{T}/\theta$ can be shown K-free over $\{\bar{x} \colon x \in X\}$ and it belongs to $\mathbb{ISP}(\mathsf{K})$ (apply the Homomorphism Theorem 1.1 to the map $t \mapsto \langle t/\varphi : \varphi \in \Phi_{\mathsf{K}}(X)\rangle$). Then, given $t(x_1, \ldots, x_m) \in T$, we write $\bar{t}$ for the element t/θ of $\boldsymbol{F}$, i.e., $\bar{t} = t^{\boldsymbol{F}}(\bar{x}_1, \ldots, \bar{x}_m)$. For $s, t \in T$, we can show that

$$\bar{s} = \bar{t} \text{ iff (every algebra in) } \mathsf{K} \text{ satisfies the equation } s \approx t, \tag{1.1}$$

using the fact that $\boldsymbol{F}$ is K-free over $\{\bar{x} : x \in X\}$.

Theorem 1.6 and (1.1) lead to the first theorem of universal algebra *not* predicted by classical algebra.

Birkhoff's Theorem 1.7. *A class of similar algebras is a variety iff it is closed under homomorphic images, subalgebras and direct products.*

There are analogues of Birkhoff's Theorem for various classes of algebras that model first order sentences more expressive than equations.

Let K be a class of similar algebras. Then K is a quasivariety iff it is closed under $\mathbb{I}$, $\mathbb{S}$, $\mathbb{P}$, and $\mathbb{P}_{\mathbb{U}}$. The smallest variety [resp. quasivariety] containing K is

$$\mathbb{V}(\mathsf{K}) := \mathbb{HSP}(\mathsf{K}) \quad [\text{resp. } \mathbb{Q}(\mathsf{K}) := \mathbb{ISPP}_{\mathbb{U}}(\mathsf{K})].$$

It is said to be *generated* by K.

By Theorem 1.6 and the fact that equations use only finitely many variables, every variety is generated as such by its free denumerably generated algebra, i.e., $\mathsf{K} = \mathbb{V}(\boldsymbol{F}_{\mathsf{K}}(\aleph_0))$ if K is a variety.[1] Two varieties K and L are *termwise equivalent* if the algebras $\boldsymbol{F}_{\mathsf{K}}(\mathit{Var})$ and $\boldsymbol{F}_{\mathsf{L}}(\mathit{Var})$ are.

Łoś' Theorem 1.8 ([24, Thm. V.2.9])**.** *Let $\boldsymbol{A} = \prod_{i \in I} \boldsymbol{A}_i/\mathcal{U}$ be an ultraproduct of similar algebras and Φ a first order sentence in the signature of $\boldsymbol{A}$. Then $\boldsymbol{A} \models \Phi$ if and only if*

$$\{i \in I : \boldsymbol{A}_i \models \Phi\} \in \mathcal{U}.$$

In particular, if $\boldsymbol{A}$ is an ultrapower of $\boldsymbol{B}$, then $\boldsymbol{A}$ satisfies exactly the same first order sentences as $\boldsymbol{B}$.

A class of similar algebras is *elementary* if it is the model class of a set of first order sentences.

Theorem 1.9. *A class of similar algebras is elementary iff it is closed under isomorphisms, ultraproducts and ultraroots.*

Łoś' Theorem is related to the *Compactness Theorem* of first order logic, which says the following (in the case of algebras):

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[1]Note that when K is a quasivariety then K need not coincide with the quasivariety $\mathbb{Q}(\boldsymbol{F}_{\mathsf{K}}(\aleph_0))$. This distinction shall be a central theme in Chapter 5 (particularly Section 5.3), where we investigate properties of quasivarieties that are generated by a single algebra.

Compactness Theorem 1.10 ([24, p. 212]). *A set Σ of first order sentences (in an algebraic signature with similarity type ρ) has a model in K_ρ provided that every finite subset of Σ does.*

The following is another interesting result concerning ultraproducts. We shall present an analogous result in the upcoming subsection concerning 'existential positive sentences'.

The Keisler-Shelah Theorem 1.11 ([29, Thm. 6.1.15]). *Two algebras $\boldsymbol{A}$ and $\boldsymbol{B}$ satisfy the same first order sentences iff they have isomorphic ultrapowers. In this case, there exists an ultrafilter $\mathcal{U}$ over a set I such that $\boldsymbol{A}^I/\mathcal{U} \cong \boldsymbol{B}^I/\mathcal{U}$.*

If K is a finite set of finite algebras, then $\mathbb{P}_{\mathbb{U}}(\mathsf{K}) \subseteq \mathbb{I}(\mathsf{K})$.

An *existential* [resp. *universal*] sentence (in an algebraic signature) is a sentence of the form $(\exists x_1)\dots(\exists x_n)\ \Phi$ [resp. $(\forall x_1)\dots(\forall x_n)\ \Phi$], where Φ is quantifier-free. Such sentences are called *positive* when Φ is a disjunction of conjunctions of equations.

The origins of the following claims are discussed in [63, Ch. 2] and [24, Sec. V.2], where proofs can also be found. Let K be a class of similar algebras. We say that K is [*positive*] *universal* if it is the model class of a set of [positive] universal first order sentences. This amounts to the demand that K be closed under [$\mathbb{H}$,] $\mathbb{S}$, $\mathbb{I}$ and $\mathbb{P}_{\mathbb{U}}$. The smallest universal [resp. positive universal] class containing K is $\mathbb{ISP}_{\mathbb{U}}(\mathsf{K})$ [resp. $\mathbb{HSP}_{\mathbb{U}}(\mathsf{K})$]. In fact, for any algebra $\boldsymbol{B}$, the class $\mathbb{ISP}_{\mathbb{U}}(\boldsymbol{B})$ is axiomatized by the *universal theory* of $\boldsymbol{B}$, i.e., the set of universal sentences satisfied by $\boldsymbol{B}$.

Partial orders and lattices

We use this opportunity to fix some definitions and notation concerning partial orders and lattices, before we introduce congruence lattices.

Recall that a structure $\langle A; \leqslant\rangle$ is a partially ordered set (briefly a *poset*) when A is non-empty and $\leqslant$ is a *partial order*, i.e., it is a reflexive, transitive, and anti-symmetric binary relation on A. Let $a, b \in A$. We define $[a) := \{c \in A : a \leqslant c\}$, i.e., the set of all upper bounds of a (including a itself), and similarly denote by $(a]$ the set of all lower bounds of a. We use $[a, b]$ to denote the set $\{c \in A : a \leqslant c \leqslant b\}$. An *interval* is a set $I \subseteq A$ that is convex, in the sense that, if $c \in [p, q]$ for some $p, q \in I$, then $c \in I$. If $a < b$ and $[a, b] = \{a, b\}$, we say that b *covers* (or is a *cover* of) a, and a is a *sub-cover* of b. An *atom* [resp. *co-atom*] is any cover [resp. sub-cover] of the least [resp. greatest] element of A (if it exists). The *(order) dual* of

$\langle A; \leqslant \rangle$ is the poset $\langle A; \geqslant \rangle$, where $\geqslant$ is $\{\langle a, b\rangle \in A^2 \colon b \leqslant a\}$. A poset $\langle A; \leqslant \rangle$ is *totally ordered* (a.k.a. a *chain*) if $a \leqslant b$ or $b \leqslant a$, for all $a, b \in A$.

In a poset $\langle A; \leqslant \rangle$, with $X \subseteq A$, an *upper bound* of X is an element $a \in A$ such that $x \leqslant a$ for all $x \in X$, and $\sup X$ denotes the least upper bound of X in $\langle A; \leqslant \rangle$, if it exists. *Lower bounds* and $\inf X$ are defined dually. Here, sup and inf abbreviate *supremum* and *infimum*, a.k.a. *join* and *meet* (respectively).

A map $h : \langle A; \leqslant^{\boldsymbol{A}} \rangle \to \langle B; \leqslant^{\boldsymbol{B}} \rangle$ between posets is *order-preserving* (a.k.a. *isotone*) if whenever $a \leqslant^{\boldsymbol{A}} a'$ then $h(a) \leqslant^{\boldsymbol{B}} h(a')$, while h is called *order-reversing* (a.k.a. *antitone*) if whenever $a \leqslant^{\boldsymbol{A}} a'$ then $h(a') \leqslant^{\boldsymbol{B}} h(a)$. The map h is called a *poset embedding* (a.k.a. an *order embedding*) if it is *order-preserving* and *order-reflecting* (i.e., if $h(a) \leqslant^{\boldsymbol{B}} h(a')$ then $a \leqslant^{\boldsymbol{A}} a'$). If, moreover, h is surjective it is called a *poset isomorphism*. Note that poset embeddings are injective. Lastly, h is an *anti-isomorphism* if it is a poset isomorphism from $\langle A; \leqslant^{\boldsymbol{A}} \rangle$ to the order dual of $\langle B; \leqslant^{\boldsymbol{B}} \rangle$.

Recall that an algebra $\boldsymbol{A} = \langle A; \wedge, \vee \rangle$ is called a *lattice* if $\wedge$ and $\vee$ are idempotent commutative associative binary operations on A and

$$a \wedge (a \vee b) = a = a \vee (a \wedge b) \text{ for all } a, b \in A.$$

The subalgebras of $\boldsymbol{A}$ are called *sublattices*.

Let $\boldsymbol{A} = \langle A; \wedge, \vee \rangle$ be any lattice, and $a, b \in A$. We associate with $\boldsymbol{A}$ its natural lattice order $\leqslant^{\boldsymbol{A}}$ (usually abbreviated $\leqslant$), i.e., the partial order on A defined by the rule

$$a \leqslant^{\boldsymbol{A}} b \text{ iff } a \wedge b = a.$$

Then $a \wedge b = \inf\{a, b\}$ and $a \vee b = \sup\{a, b\}$. Conversely, if $\langle A; \leqslant \rangle$ is a poset such that $\inf\{a, b\}$ and $\sup\{a, b\}$ exist for every $a, b \in A$, then $\langle A; \wedge, \vee \rangle$ is a lattice, where $a \wedge b := \inf\{a, b\}$ and $a \vee b := \sup\{a, b\}$ for all $a, b \in A$. For $X \subseteq A$, we sometimes use the notation $\bigwedge X := \inf X$ and $\bigvee X := \sup X$.

A lattice $\boldsymbol{A}$ is *complete* when $\inf X$ and $\sup X$ exist for every $X \subseteq A$. An element $a \in A$ is called *compact* if for every $X \subseteq A$, whenever $a \leqslant \sup X$, then there exists a finite $Z \subseteq X$, such that $a \leqslant \sup Z$. We say $\boldsymbol{A}$ is *algebraic* if it is complete and each of its elements is the join of a set of compact elements.

Any poset $\langle A; \leqslant \rangle$ for which $\inf X$ exists whenever $X \subseteq A$, forms a complete lattice, where $\sup X = \inf\{a \in A \colon a \geqslant b \text{ for all } b \in X\}$ for every $X \subseteq A$.

An element a of a lattice $\boldsymbol{A}$ is called *meet-irreducible* provided that, whenever $a = b \wedge c$ for some $b, c \in A$, then $a = b$ or $a = c$. Similarly, a is said to be *completely meet-irreducible* if, whenever $a = \inf X$ for some $X \subseteq A$, then $a \in X$. *Join-irreducible* and *completely join-irreducible* elements are defined dually.

Let K be a quasivariety with type ρ and let $\boldsymbol{A} \in \mathsf{K}$. Arbitrary intersections of K-congruences of $\boldsymbol{A}$ are again K-congruences of $\boldsymbol{A}$, so the set of K-congruences of $\boldsymbol{A}$ becomes a complete lattice $\mathbf{Con}_{\mathsf{K}}(\boldsymbol{A})$, when ordered by inclusion. The lattice $\mathbf{Con}(\boldsymbol{A}) := \mathbf{Con}_{\mathsf{K}_\rho}(\boldsymbol{A})$ is called the *congruence lattice* of $\boldsymbol{A}$. The greatest element of $\mathbf{Con}_{\mathsf{K}}(\boldsymbol{A})$ is the total relation A^2, and its least element is the identity relation $\mathrm{id}_A := \{\langle a, a\rangle : a \in A\}$.

Correspondence Theorem 1.12. *For any congruence θ of an algebra $\boldsymbol{A}$, the sublattice of $\mathbf{Con}(\boldsymbol{A})$ with universe $[\theta, A^2]$ is isomorphic to $\mathbf{Con}(\boldsymbol{A}/\theta)$ under the map $\varphi \mapsto \varphi/\theta$.*

The smallest K-congruence of $\boldsymbol{A}$ containing a set $X \subseteq A^2$ is denoted by $\Theta^{\boldsymbol{A}}_{\mathsf{K}} X$. We write $\Theta^{\boldsymbol{A}}_{\mathsf{K}}(a, b)$ for the *principal* K-congruence $\Theta^{\boldsymbol{A}}_{\mathsf{K}}\{\langle a, b\rangle\}$. (The subscript K is dropped when K is understood to be K_ρ.)

Lemma 1.13. *Let $\boldsymbol{A}$ be an algebra in a quasivariety K. Every* finitely generated K-*congruence*

$$\theta = \Theta^{\boldsymbol{A}}_{\mathsf{K}}\{\langle a_1, b_1\rangle, \dots, \langle a_n, b_n\rangle\}$$

of $\boldsymbol{A}$ is compact in $\mathbf{Con}_{\mathsf{K}}(\boldsymbol{A})$, i.e., whenever $\theta \subseteq \Theta^{\boldsymbol{A}}_{\mathsf{K}} X$, then $\theta \subseteq \Theta^{\boldsymbol{A}}_{\mathsf{K}} Y$ for some finite $Y \subseteq X$.

Conversely, it is easy to see that compact K-congruences are finitely generated, so it follows from Lemma 1.13 that $\mathbf{Con}_{\mathsf{K}}(\boldsymbol{A})$ is an algebraic lattice (and hence so is $\mathbf{Con}(\boldsymbol{A})$), for every algebra $\boldsymbol{A}$. The following generalization of (1.1) holds for quasi-equations.

Lemma 1.14. *A quasivariety K satisfies a quasi-equation*

$$(s_1(\vec{x}) \approx t_1(\vec{x}) \;\;\&\;\; \cdots \;\;\&\;\; s_m(\vec{x}) \approx t_m(\vec{x})) \implies s(\vec{x}) \approx t(\vec{x})$$

iff $\Theta^{\boldsymbol{F}}_{\mathsf{K}}(\bar{s}, \bar{t}) \subseteq \Theta^{\boldsymbol{F}}_{\mathsf{K}}\{\langle \bar{s}_i, \bar{t}_i\rangle : i = 1, \dots, m\}$, where $\boldsymbol{F} = \boldsymbol{F}_{\mathsf{K}}(X)$ is such that X includes the variables $\vec{x}$. In this case, for every $\boldsymbol{A} \in \mathsf{K}$ and $\vec{a} \in A$, we have

$$\Theta^{\boldsymbol{A}}_{\mathsf{K}}(s^{\boldsymbol{A}}(\vec{a}), t^{\boldsymbol{A}}(\vec{a})) \subseteq \Theta^{\boldsymbol{A}}_{\mathsf{K}}\{\langle s_i^{\boldsymbol{A}}(\vec{a}), t_i^{\boldsymbol{A}}(\vec{a})\rangle : i = 1, \dots, m\}.$$

Simple algebras

Let K be a quasivariety. We say that $\boldsymbol{A}$ is K-*simple* (a.k.a. *relatively simple*) when $\boldsymbol{A}$ is a nontrivial member of K and every homomorphism from $\boldsymbol{A}$

onto a nontrivial member of K is an isomorphism, i.e., id_A is a co-atom of $\mathbf{Con}_{\mathsf{K}}(\boldsymbol{A})$, or equivalently, $|\operatorname{Con}_{\mathsf{K}}(\boldsymbol{A})| = 2$. (So, $\boldsymbol{A}$ is *simple* if it is nontrivial, and every homomorphism with domain $\boldsymbol{A}$ and a nonsingleton image is injective, i.e., id_A is a co-atom of $\mathbf{Con}(\boldsymbol{A})$.)

Every nontrivial quasivariety has a relatively simple member [63, Thm. 3.1.8]; for varieties, this was proved earlier by Magari [83]. On the other hand, a finitely generated algebra need not have a simple homomorphic image [73, p. 154]. Conditions that guarantee relatively simple homomorphic images are given in the next theorem, which adapts [73, pp. 153–4] to quasivarieties.

Theorem 1.15. *Let $\boldsymbol{A}$ be a nontrivial member of a quasivariety K.*

(i) *If the total relation A^2 is compact in $\mathbf{Con}_{\mathsf{K}}(\boldsymbol{A})$, then $\boldsymbol{A}$ has a relatively simple homomorphic image in K.*

(ii) *If $\boldsymbol{A}$ is finitely generated and of finite type, then A^2 is compact in $\mathbf{Con}_{\mathsf{K}}(\boldsymbol{A})$, so $\boldsymbol{A}$ has a relatively simple homomorphic image in K.*

Proof. (i) If $\bot$ is the least element of an algebraic lattice $\boldsymbol{L}$ and $y \in L \setminus \{\bot\}$ and y is compact in $\boldsymbol{L}$, then $\{x \in L : y \nleqslant x\}$ has a maximal element, by Zorn's Lemma. Setting $\boldsymbol{L} = \mathbf{Con}_{\mathsf{K}}(\boldsymbol{A})$ and $y = A^2$, we conclude that, under the given assumptions, $\boldsymbol{A}$ has a maximal proper K-congruence θ, whence $\boldsymbol{A}/\theta \in \mathsf{K}$, so $\boldsymbol{A}/\theta$ is K-simple (by the Correspondence Theorem 1.12 and the Second Isomorphism Theorem 1.2).

(ii) Suppose $\boldsymbol{A}$ is generated by a finite subset X of A. Let Y be the union of X and the set of all $f(a_1, \ldots, a_n)$ such that $n \in \omega$, f is a basic n-ary operation of $\boldsymbol{A}$ and $a_1, \ldots, a_n \in X$. Then $A^2 = \Theta^{\boldsymbol{A}}(Y^2) \subseteq \Theta^{\boldsymbol{A}}_{\mathsf{K}}(Y^2)$. If $\boldsymbol{A}$ has finite type, then Y^2 is finite, so $A^2 = \Theta^{\boldsymbol{A}}_{\mathsf{K}}(Y^2)$ is compact in $\mathbf{Con}_{\mathsf{K}}(\boldsymbol{A})$, and the last assertion follows from (i). □

Definition 1.16. A quasivariety will be called a *Kollár quasivariety* if each of its nontrivial members has no trivial subalgebra.

Clearly, a quasivariety K is a Kollár quasivariety if its signature includes two constant symbols that take distinct values in every nontrivial member of K. This situation is common in algebraic logic, e.g., every quasivariety of Heyting algebras (see Definition 2.34) is a Kollár quasivariety. This will also be the case for many of the varieties of De Morgan monoids that we shall consider. The result below was proved first for varieties by Kollár [77], hence our nomenclature. Further characterizations of Kollár quasivarieties have been given by Campercholi and Vaggione [27, Prop. 2.3].

Theorem 1.17 (Gorbunov [62], [63, Thm. 2.3.16])**.** *A quasivariety* K *is a Kollár quasivariety iff* A^2 *is compact in* $\mathbf{Con}_{\mathsf{K}}(\boldsymbol{A})$ *for every* $\boldsymbol{A} \in \mathsf{K}$.

Corollary 1.18. *Every nontrivial member of a Kollár quasivariety has a relatively simple homomorphic image* (*in the same quasivariety*).

Proof. This follows from Theorems 1.17 and 1.15(i). □

Although the following fact is obvious, it is useful to have a reference to it when working with Kollár quasivarieties.

Fact 1.19. *If* $h\colon \boldsymbol{A} \to \boldsymbol{B}$ *is a homomorphism between members of a quasivariety, where* $\boldsymbol{A}$ *is relatively simple and* $\boldsymbol{B}$ *has no trivial subalgebra, then* h *is an embedding.*

Subdirectly irreducible algebras

Let K be a quasivariety. We denote by $\mathsf{K}_{\mathrm{RSI}}$ [resp. $\mathsf{K}_{\mathrm{RFSI}}$; K_{RS}] the class of K-subdirectly irreducible [resp. finitely K-subdirectly irreducible; K-simple] members of K. Thus, $\mathsf{K}_{\mathrm{RS}} \subseteq \mathsf{K}_{\mathrm{RSI}} \subseteq \mathsf{K}_{\mathrm{RFSI}}$, and $\mathsf{K}_{\mathrm{RSI}}$ includes no trivial algebra. An algebra $\boldsymbol{A} \in \mathsf{K}$ belongs to the class $\mathsf{K}_{\mathrm{RFSI}}$ [resp. $\mathsf{K}_{\mathrm{RSI}}$] iff the identity relation id_A is meet-irreducible [resp. completely meet-irreducible] in $\mathbf{Con}_{\mathsf{K}}(\boldsymbol{A})$. If every K-subdirectly irreducible member of K is K-simple, then K is said to be *relatively semisimple.* When K is a variety, we remove the redundant 'K-' prefixes and 'R' in the subscripts of the classes above.

Birkhoff's Subdirect Decomposition Theorem 1.3 implies that every variety K is determined by its class of subdirectly irreducible members—in fact $\mathsf{K} = \mathbb{IP}_{\mathbb{S}}(\mathsf{K}_{\mathrm{SI}})$. Therefore, to confirm that a quasi-equation holds in all members of a variety K, we need only confirm its validity in the members of K_{SI}.

An analogous result holds for every quasivariety K, i.e., $\mathsf{K} = \mathbb{IP}_{\mathbb{S}}(\mathsf{K}_{\mathrm{RSI}})$ [63, Thm. 3.1.1]. It follows that $\mathbb{Q} = \mathbb{IP}_{\mathbb{S}}\mathbb{SP}_{\mathbb{U}}$, which implies that $\mathbb{Q}(\mathsf{L})_{\mathrm{RSI}} \subseteq \mathbb{IS}(\mathsf{L})$ for any finite set L of finite algebras (in which case $\mathbb{S}(\mathsf{L})$ is again a finite set of finite algebras). For varieties we can say more.

A [quasi]variety K is said to be *finitely generated* if K is generated as a [quasi]variety by some finite set L of finite algebras. In this case, when K is a variety, L can be chosen to comprise a single finite algebra, because K is closed under $\mathbb{P}$ and $\mathbb{H}$.

An algebra is said to be *locally finite* if its finitely generated subalgebras are finite. A class K of algebras is *locally finite* if its members are. When $\mathbb{S}(\mathsf{K}) \subseteq \mathsf{K}$, it follows that K is locally finite if and only if its finitely generated members are finite.

For any algebra $\boldsymbol{A}$ and any set $X \neq \emptyset$, the algebra $\boldsymbol{F}_{\mathbb{V}(\boldsymbol{A})}(X)$ can be embedded into $\boldsymbol{A}^{(|A|^{|X|})}$ [24, p. 77]. This, with Theorem 1.6, yields:

Theorem 1.20 ([24, Thm. II.10.16])**.** *Every finitely generated variety is locally finite.*

The following fact is useful when working with locally finite varieties. (A proof can be found in [116], for instance.)

Fact 1.21. *A variety* K *of finite type is locally finite iff there is a function* $p \colon \omega \to \omega$ *such that, for each* $n \in \omega$*, every* n*-generated member of* K_{SI} *has at most* $p(n)$ *elements.*

Congruence distributive varieties

Recall that a lattice is said to be *distributive* or *modular* if it satisfies the respective law (1.2) or (1.3) below.

$$x \wedge (y \vee z) \approx (x \wedge y) \vee (x \wedge z) \tag{1.2}$$

$$y \leqslant x \implies x \wedge (y \vee z) \approx y \vee (x \wedge z) \tag{1.3}$$

We make standard use of 'Hasse diagrams' when depicting posets and lattices. Let $\boldsymbol{N}_5$ and $\boldsymbol{M}_3$ be the lattices with the following Hasse diagrams.

$\boldsymbol{N}_5$ $\boldsymbol{M}_3$

Theorem 1.22 ([7, Thms. 2.8, 2.10])**.** *Let* $\boldsymbol{A}$ *be a lattice.*

(i) $\boldsymbol{A}$ *is modular iff* $\boldsymbol{A}$ *has no sublattice isomorphic to* $\boldsymbol{N}_5$*.*

(ii) $\boldsymbol{A}$ *is distributive iff* $\boldsymbol{A}$ *has no sublattice isomorphic to* $\boldsymbol{N}_5$ *or to* $\boldsymbol{M}_3$*.*

An algebra $\boldsymbol{A}$ is called *congruence distributive* [*modular*] if $\mathbf{Con}(\boldsymbol{A})$ is a distributive [modular] lattice, and a variety is said to be *congruence distributive* [*modular*] when all of its members are. Every algebra with a lattice reduct generates a congruence distributive variety [24, Thm. II.12.3].

Jónsson's Theorem 1.23 ([72, 74])**.** *For any subclass* L *of a congruence distributive variety,* $\mathbb{V}(\mathsf{L})_{\mathrm{FSI}} \subseteq \mathbb{HSP}_{\mathbb{U}}(\mathsf{L})$*. In particular, if* L *is a finite set of finite algebras, then* $\mathbb{V}(\mathsf{L})_{\mathrm{FSI}} \subseteq \mathbb{HS}(\mathsf{L})$*.*

Let K be a [quasi]variety. A *sub[quasi]variety* of K is a subclass of K that is itself a [quasi]variety. The sub[quasi]varieties of K form a complete lattice, when ordered by inclusion (where meets are intersections, in both cases). This lattice always has at least one atom, unless K is trivial (see [7, Prop. 7.61] for the proof in the case of subvarieties). Also, every [quasi]variety of *countable type* has at most $2^{\aleph_0}$ sub[quasi]varieties (because its set of [quasi-]equations is denumerable).

It follows from Jónsson's Theorem 1.23 that, for any two subvarieties K and L of a congruence distributive variety,

$$\mathbb{V}(\mathsf{K} \cup \mathsf{L})_{\mathrm{SI}} = \mathsf{K}_{\mathrm{SI}} \cup \mathsf{L}_{\mathrm{SI}} \text{ and } \mathbb{V}(\mathsf{K} \cup \mathsf{L})_{\mathrm{FSI}} = \mathsf{K}_{\mathrm{FSI}} \cup \mathsf{L}_{\mathrm{FSI}}. \tag{1.4}$$

This can be used to show that the lattice of subvarieties of a congruence distributive variety is itself distributive [72, Cor. 4.2].

A class K of similar algebras has the *congruence extension property* (CEP) if every congruence on a subalgebra $\boldsymbol{B}$ of a member of K is the restriction $B^2 \cap \theta$ of some congruence θ on the parent algebra.

Theorem 1.24. *If K has the CEP, then $\mathbb{HS}(\mathsf{K}) \subseteq \mathbb{SH}(\mathsf{K})$ and any nontrivial subalgebra of a simple member of K is simple.*

A variety K is said to have *equationally definable principal congruences* (EDPC) if there is a finite set Σ of pairs of 4-ary terms in its signature such that, whenever $\boldsymbol{A} \in \mathsf{K}$ and $a, b, c, d \in A$, then

$$\langle c, d\rangle \in \Theta^{\boldsymbol{A}}(a, b) \text{ iff } \big(\varphi^{\boldsymbol{A}}(a, b, c, d) = \psi^{\boldsymbol{A}}(a, b, c, d) \text{ for all } \langle \varphi, \psi\rangle \in \Sigma\big).$$

Theorem 1.25 ([15])**.** *If a variety K has EDPC, then K is congruence distributive, and has the CEP, and its class of simple members is closed under ultraproducts.*

Theorem 1.26 ([15], [74, Thm. 6.6])**.** *Let K be a variety of finite type, with EDPC, and let $\boldsymbol{A} \in \mathsf{K}$ be finite and subdirectly irreducible. Then there is a largest subvariety of K that excludes $\boldsymbol{A}$. It consists of all $\boldsymbol{B} \in \mathsf{K}$ such that $\boldsymbol{A} \notin \mathbb{SH}(\boldsymbol{B})$.*

In Chapter 6, we shall make use of a theorem of Campercholi (see Theorem 6.9), which allows one to focus on FSI algebras when determining whether a variety has surjective epimorphisms. This theorem applies to any variety that is 'congruence permutable' and has 'equationally definable principal meets', so it is useful to define these conditions here.

A variety K is said to have *equationally definable principal meets* (EDPM) if it has finitely many pairs $\langle u_i(x, y, z, w), v_i(x, y, z, w)\rangle$, $i \in I$, of 4-ary terms such that for all $\boldsymbol{A} \in \mathsf{K}$ and $a, b, c, d \in A$,

$$\Theta^{\boldsymbol{A}}(a, b) \cap \Theta^{\boldsymbol{A}}(c, d) = \Theta^{\boldsymbol{A}}\{\langle u_i^{\boldsymbol{A}}(a, b, c, d), v_i^{\boldsymbol{A}}(a, b, c, d)\rangle : i \in I\}.$$

Theorem 1.27 ([16, 34])**.** *A variety* K *has EDPM iff* K *is congruence distributive and* $\mathsf{K}_{\mathrm{FSI}}$ *is a universal class.*

The *relational product* $\theta \circ \varphi$ of binary relations θ and φ on the universe of an algebra $\boldsymbol{A}$ is defined as follows. For $a, b \in A$,

$$a \equiv_{\theta \circ \varphi} b \;\text{ iff }\; a \equiv_\theta c \;\text{ and }\; c \equiv_\varphi b \;\text{ for some }\; c \in A.$$

If θ and φ are congruences, then $\theta \circ \varphi$ is a reflexive subuniverse of $\boldsymbol{A} \times \boldsymbol{A}$ and the following conditions are equivalent:

$$\theta \circ \varphi = \varphi \circ \theta, \quad \theta \circ \varphi \subseteq \varphi \circ \theta, \quad \theta \circ \varphi = \theta \vee \varphi,$$

where $\theta \vee \varphi$ is the join $\Theta^{\boldsymbol{A}}(\theta \cup \varphi)$ in the lattice $\mathbf{Con}(\boldsymbol{A})$. We say $\boldsymbol{A}$ is *congruence permutable* if these conditions hold for all $\theta, \varphi \in \mathrm{Con}(\boldsymbol{A})$. A class of algebras is *congruence permutable* if its members are.

Fleischer's Lemma 1.28 ([7, Thm. 6.2])**.** *Let* $h \colon \boldsymbol{A} \to \boldsymbol{A}_1 \times \boldsymbol{A}_2$ *be a subdirect embedding, where* $\boldsymbol{A}$ *belongs to a congruence permutable variety. Then there exist an algebra* $\boldsymbol{C}$ *and surjective homomorphisms* $h_i \colon \boldsymbol{A}_i \to \boldsymbol{C}$ $(i = 1, 2)$ *such that*

$$h[A] = \{\langle a_1, a_2 \rangle \in A_1 \times A_2 \colon h_1(a_1) = h_2(a_2)\}.$$

Existential Positive Sentences

Recall that, up to logical equivalence, an *existential positive sentence* in an algebraic signature is a first order sentence of the form $\exists x_1 \ldots \exists x_n \, \Phi$, where Φ is a (quantifier-free) disjunction of conjunctions of equations. These sentences may be variable-free (and hence quantifier-free). They have a central place in the model theory of 'positive logic' (see [114], for instance). For present purposes, their main significance derives from Theorem 1.29, which is in the spirit of the Keisler-Shelah Theorem 1.11.

Given an algebra $\boldsymbol{A} = \langle A; F \rangle$, with $S \subseteq A$, let $\boldsymbol{A}_S = \langle A; F \cup S_0 \rangle$, where S_0 consists of the elements of S, treated as new nullary operations on A. Let $\mathrm{Th}(\boldsymbol{A})$ [resp. $\mathrm{Diag}(\boldsymbol{A})$] denote the set of all first order sentences [resp. all equations not containing a variable] that are true in $\boldsymbol{A}_A$. A subalgebra $\boldsymbol{B}$ of $\boldsymbol{A}$ is called an *elementary subalgebra* (and $\boldsymbol{A}$ an *elementary extension* of $\boldsymbol{B}$) if $\boldsymbol{A}_B \models \mathrm{Th}(\boldsymbol{B})$. In this case $\boldsymbol{A}$ and $\boldsymbol{B}$ are *elementarily equivalent*, i.e., they satisfy the same first order sentences. An embedding is *elementary* if its image is an elementary subalgebra of its co-domain. Every algebra is elementarily embeddable into each of its ultrapowers, by the map defined before Theorem 1.4.

The next result appears to be folklore. It can be inferred from [64, Thm. 1.2] (also see [63, Thm. 2.3.11] and [134, Thm. 3.7]), but we provide a direct proof below.

Theorem 1.29. *Let* $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *be similar algebras. Then* $\boldsymbol{B}$ *satisfies every existential positive sentence that is true in* $\boldsymbol{A}$ *iff there is a homomorphism from* $\boldsymbol{A}$ *into an ultrapower of* $\boldsymbol{B}$.

Proof. ($\Rightarrow$) Let Σ be a finite subset of $\mathrm{Diag}(\boldsymbol{A})$. By assumption, $\Sigma \cup \mathrm{Th}(\boldsymbol{B})$ has a model that is an expansion of $\boldsymbol{B}_B$ by suitable interpretations in B of the elements of A occurring (as constant symbols) in Σ. By the Compactness Theorem 1.10, therefore, $\mathrm{Diag}(\boldsymbol{A}) \cup \mathrm{Th}(\boldsymbol{B})$ has a model, $\boldsymbol{C}$, say. Let $\boldsymbol{C}^-$ be the reduct of $\boldsymbol{C}$ in the signature of $\boldsymbol{A}, \boldsymbol{B}$. Now $\boldsymbol{C}^-$ is isomorphic to an elementary extension of $\boldsymbol{B}$, because $\boldsymbol{C}$ is a model of $\mathrm{Th}(\boldsymbol{B})$. (In particular, the negated equations in $\mathrm{Th}(\boldsymbol{B})$ separate the elements of B.) As $\boldsymbol{C}^-$ and $\boldsymbol{B}$ have the same *universal* theory, $\boldsymbol{C}^-$ embeds into an ultrapower $\boldsymbol{U}$ of $\boldsymbol{B}$. Also, there is a homomorphism from $\boldsymbol{A}$ into $\boldsymbol{C}^-$, because $\boldsymbol{C}$ is a model of $\mathrm{Diag}(\boldsymbol{A})$, so there is a homomorphism from $\boldsymbol{A}$ into $\boldsymbol{U}$.

($\Leftarrow$) Clearly, existential positive sentences persist in homomorphic images, in extensions and in ultraroots. □

Corollary 1.30. *The model class of the set of existential positive sentences satisfied by an algebra* $\boldsymbol{A}$ *is* $\mathbb{R}_{\mathbb{U}}\mathbb{E}\mathbb{H}(\boldsymbol{A})$. *(*$\mathbb{R}_{\mathbb{U}}$ *and* $\mathbb{E}$ *were defined after Theorem* 1.4 *on page* 5.)

Corollary 1.31. *The following demands on a quasivariety* K *are equivalent.*

(i) *The nontrivial members of* K *all satisfy the same existential positive sentences.*

(ii) *For any two nontrivial members of* K, *each can be mapped homomorphically into an ultrapower of the other.*

1.2 Algebraic logic preliminaries

The standard references for abstract algebraic logic include [17], [33] and [46]; also see [118] for a brief survey article. (The preliminaries that follow are based on [117] and [118].) The standard text for residuated structures and substructural logics is [51].

This section starts by making precise what it means for a logic to be ‘algebraizable’ (in the sense of Blok and Pigozzi [17]). We then introduce the ‘substructural logics’ which motivate the algebraic structures that are studied throughout the thesis.

Algebraizable logics

A *consequence relation* on a set A is a binary relation $\vdash$ from subsets of A to elements of A satisfying the two postulates below, for all $B \cup C \cup \{a\} \subseteq A$:

$$\text{if } a \in B \text{ then } B \vdash a \qquad \text{(reflexivity);}$$
$$\text{if } C \vdash b \text{ for all } b \in B \text{ and } B \vdash a \text{ then } C \vdash a \qquad \text{(transitivity).}$$

We say that $\vdash$ is *finitary* if it satisfies the next postulate as well:

$$\text{if } B \vdash a \text{ then } B' \vdash a \text{ for some finite } B' \subseteq B.$$

If $B, C \subseteq A$ then $B \vdash C$ shall mean '$B \vdash c$ for all $c \in C$', while $B \dashv\vdash C$ stands for '$B \vdash C$ and $C \vdash B$'. We shall also abbreviate $\{a_1, \ldots, a_n\} \vdash a$ as $a_1, \ldots, a_n \vdash a$ (so that $\emptyset \vdash a$ is abbreviated as $\vdash a$).

Let $\boldsymbol{F}$ be the absolutely free algebra over (our fixed set of variables) *Var* in a given algebraic signature. In logical contexts, operation symbols are often called *connectives*, and the elements of F are called *formulas* instead of terms. A *substitution* is an endomorphism of $\boldsymbol{F}$.

A (sentential) *deductive system* in this language is a consequence relation $\vdash$ on F that is *substitution-invariant* in the sense that for any $\Gamma \cup \{\alpha\} \subseteq F$,

$$\text{if } \Gamma \vdash \alpha \text{ then } h[\Gamma] \vdash h(\alpha) \text{ for all substitutions } h.$$

In this context the elements of $\vdash$ (i.e., the pairs $\langle \Gamma, \alpha \rangle$ such that $\Gamma \vdash \alpha$) are often referred to as the *derivable rules* of $\vdash$. The *theorems* of $\vdash$ are the formulas α such that $\vdash \alpha$. In our informal remarks we shall use the terms 'logic' and 'deductive system' interchangeably.

A 2-*dimensional deductive system* (briefly, a 2-*deductive system*) is defined as a substitution-invariant consequence relation on $F \times F$, where substitutions act coordinatewise, i.e., $h(\langle \alpha, \beta \rangle) := \langle h(\alpha), h(\beta) \rangle$ for every endomorphism h of $\boldsymbol{F}$ and all $\alpha, \beta \in F$.

Let K be a class of algebras in the signature under discussion. If we identify pairs $\langle \alpha, \beta \rangle \in F \times F$ with formal equations then the *equational consequence relation* $\models_{\mathsf{K}}$ becomes a 2-deductive system in our language. The meaning of $\Sigma \models_{\mathsf{K}} \alpha \approx \beta$ is: for every $\boldsymbol{A} \in \mathsf{K}$ and every assignment $\vec{a} \in A$ of values to the variables occurring in $\Sigma \cup \{\alpha \approx \beta\}$,

$$\text{if } \mu^{\boldsymbol{A}}(\vec{a}) = \nu^{\boldsymbol{A}}(\vec{a}) \text{ for all } (\mu \approx \nu) \in \Sigma, \text{ then } \alpha^{\boldsymbol{A}}(\vec{a}) = \beta^{\boldsymbol{A}}(\vec{a}).$$

If we replace K by $\mathbb{ISP}(\mathsf{K})$, this has no effect on the relation $\models_{\mathsf{K}}$. When Σ is finite then $\Sigma \models_{\mathsf{K}} \alpha \approx \beta$ has the same meaning as $\mathsf{K} \models \big(\& \, \Sigma\big) \implies \alpha \approx \beta$.

Theorem 1.32 ([46])**.** *Let* K *be a class of similar algebras that is closed under* $\mathbb{I}$*,* $\mathbb{S}$ *and* $\mathbb{P}$*. Then the following conditions are equivalent.*

(i) K *is a quasivariety.*

(ii) $\models_{\mathsf{K}}$ *is finitary.*

(iii) K *is closed under ultraproducts.*

One can naturally generalize the notion of a 2-deductive system to that of a k-deductive system, for any positive integer k [18]. A *theory* of a k-deductive system $\vdash$ is a subset T of F^k such that whenever $\Gamma \vdash \vec{\alpha}$ and $\Gamma \subseteq T$ then $\vec{\alpha} \in T$. The substitution-invariance of $\vdash$ amounts to the fact that whenever T is a theory of $\vdash$, then so is $h^{-1}[T]$, for every substitution h. Intersections of theories are theories again. So, when ordered by set inclusion, the set of all theories of $\vdash$ becomes a complete lattice that is closed under the unary operation h^{-1}, for every substitution h. We view this lattice with operators as an algebra representing $\vdash$. It is therefore natural to declare two deductive systems equivalent when their representative algebras are isomorphic:

Definition 1.33 ([14])**.** Two deductive systems with the same language but possibly different dimension are *equivalent* if there is a lattice isomorphism $\boldsymbol{\Lambda}$ between their lattices of theories such that $\boldsymbol{\Lambda}(h^{-1}[T]) = h^{-1}[\boldsymbol{\Lambda}(T)]$ for all theories T and substitutions h.

Definition 1.34. A deductive system is *algebraizable* if it is equivalent to the equational consequence relation $\models_{\mathsf{K}}$ of a class K of algebras.

It is *elementarily algebraizable* if it is equivalent to a *finitary* equational consequence relation, i.e., if we can choose K to be a quasivariety.

An elementarily algebraizable system $\vdash$ is equivalent to the equational consequence relation of a *unique* quasivariety K, called its *equivalent quasivariety* [17]. In this case we say that K *algebraizes* $\vdash$.

Let $\boldsymbol{\tau}$ be a family $\{\delta_i(x) \approx \varepsilon_i(x)\colon i \in I\}$ of unary equations. For any set $\Gamma \cup \{\alpha\}$ of formulas, we shall abbreviate

$$\{\delta_i(\alpha) \approx \varepsilon_i(\alpha)\colon i \in I\} \ \text{ as } \ \boldsymbol{\tau}(\alpha), \ \text{ and } \ \textstyle\bigcup_{\gamma\in\Gamma}\boldsymbol{\tau}(\gamma) \ \text{ as } \ \boldsymbol{\tau}[\Gamma].$$

Similarly, when $\boldsymbol{\rho}$ is a family $\{\Delta_j(x,y) : j \in J\}$ of binary formulas, then for any set $\Sigma \cup \{\alpha \approx \beta\}$ of equations, we shall abbreviate

$$\{\Delta_j(\alpha,\beta)\colon j \in J\} \ \text{ as } \ \boldsymbol{\rho}(\alpha,\beta), \ \text{ and } \ \textstyle\bigcup_{\mu\approx\nu\,\in\,\Sigma}\boldsymbol{\rho}(\mu,\nu) \ \text{ as } \ \boldsymbol{\rho}[\Sigma].$$

Theorem 1.35 ([17]; also see [14]). *A (sentential) deductive system $\vdash$ and an equational consequence relation $\models_{\mathsf{K}}$ are equivalent iff there exist a family $\boldsymbol{\tau}$ of unary equations, and a family $\boldsymbol{\rho}$ of binary formulas, such that for any set of formulas $\Gamma \cup \{\alpha\}$, we have*

$$\Gamma \vdash \alpha \;\; \textit{iff} \;\; \boldsymbol{\tau}[\Gamma] \models_{\mathsf{K}} \boldsymbol{\tau}(\alpha); \tag{1.5}$$

$$x \approx y \;=\!\mid\models_{\mathsf{K}} \boldsymbol{\tau}[\boldsymbol{\rho}(x,y)]. \tag{1.6}$$

In this case, for any set of equations $\Sigma \cup \{\alpha \approx \beta\}$, we also have

$$\Sigma \models_{\mathsf{K}} \alpha \approx \beta \;\; \text{iff} \;\; \boldsymbol{\rho}[\Sigma] \vdash \boldsymbol{\rho}(\alpha, \beta); \tag{1.7}$$

$$x \dashv\vdash \boldsymbol{\rho}[\boldsymbol{\tau}(x)]. \tag{1.8}$$

Moreover, the sets $\boldsymbol{\tau}$ and $\boldsymbol{\rho}$ are unique up to interderivability in $\models_{\mathsf{K}}$ and in $\vdash$, respectively [17, Thm. 2.15]. The set $\boldsymbol{\tau}$ is called a set of *defining equations* for $\vdash$. The existence of $\boldsymbol{\tau}$, together with (1.5), captures what it means for K to be an *algebraic semantics* for the system $\vdash$ (cf. [21]). Conditions (1.5) and (1.6) imply that $\boldsymbol{\rho}$ is a set of *equivalence formulas* for $\vdash$, i.e., that it satisfies the following conditions, for all variables $x, y, x_1, \ldots, x_n, y_1, \ldots, y_n$ and all n-ary formulas α (thus mimicking the behaviour of a biconditional):

$$\vdash \boldsymbol{\rho}(x, x)$$
$$\{x\} \cup \boldsymbol{\rho}(x, y) \vdash y$$
$$\boldsymbol{\rho}(x_1, y_1) \cup \cdots \cup \boldsymbol{\rho}(x_n, y_n) \vdash \boldsymbol{\rho}(\alpha(\vec{x}), \alpha(\vec{y}))$$

Theorem 1.36 ([14, Thms. 6.3, 6.2]). *Suppose the equivalent conditions of Theorem* 1.35 *hold for a deductive system $\vdash$ and a class K of algebras.*

(i) *When $\vdash$ is finitary then $\boldsymbol{\tau}$ can be chosen finite. Dually:*

(ii) *When $\models_{\mathsf{K}}$ is finitary (e.g., when K is a quasivariety) then $\boldsymbol{\rho}$ can be chosen finite.*

(iii) *If $\models_{\mathsf{K}}$ is finitary and $\boldsymbol{\tau}$ can be chosen finite, then $\vdash$ is finitary.*

Corollary 1.37. *An elementarily algebraizable deductive system is finitary iff its algebraization is witnessed* (*as in Theorem* 1.35) *by some finite set of defining equations.*

It can happen that a deductive system $\vdash$ is algebraized by a class K of algebras, where $\vdash$ is finitary but $\models_{\mathsf{K}}$ is not [67] (and vice versa [117]), but we shall focus only on finitary logics that are elementarily algebraizable. (These were the original 'algebraizable logics' of Blok and Pigozzi [17].)

A *(sentential) formal system* (a.k.a. a *Hilbert system*) in a given algebraic signature consists of a set of formulas (in the same language), called *axioms*, and a set of *inference rules* of the form Γ/α, where $\Gamma \cup \{\alpha\}$ is a finite set of formulas. The elements of Γ are called the *premises* of Γ/α; we call α the *conclusion*. A deductive system $\vdash$ is finitary iff it can be *axiomatized* by some formal system $\mathbf{L}$ [82]. The latter assertion means that $\vdash$ is the *deducibility relation* of $\mathbf{L}$, i.e., $\Gamma \vdash \alpha$ holds just when there is a *proof* (a.k.a. a *derivation*) *of* α *from* Γ, i.e., a finite sequence of formulas terminating with α, each item of which belongs to Γ or is a substitution instance of an axiom of $\mathbf{L}$ or of the conclusion of an inference rule of $\mathbf{L}$, where in the last case, the same substitution turns the premises of the rule into previous items in the proof.

To avoid notational clutter, we regularly attribute to a formal system $\mathbf{L}$ the significant properties of its deducibility relation $\vdash_{\mathbf{L}}$.

Given deductive systems $\vdash \subseteq \vdash'$ (in the same signature), we call $\vdash'$ an *extension* of $\vdash$. It is an *axiomatic extension* of $\vdash$ if there is a set Δ of formulas, closed under substitution, such that for any set $\Gamma \cup \{\alpha\}$ of formulas, we have

$$\Gamma \vdash' \alpha \text{ iff } \Gamma \cup \Delta \vdash \alpha.$$

If $\vdash$ is finitary, then the *finitary* extensions of $\vdash$ are produced by adjoining new axioms or new inference rules to an axiomatization $\mathbf{L}$ of $\vdash$, while the *axiomatic* extensions of $\vdash$ are produced by adjoining only new axioms to $\mathbf{L}$. Arbitrary intersections of extensions of $\vdash$ are again extensions of $\vdash$, so the extensions of $\vdash$ form a complete lattice when ordered by inclusion. When two deductive systems $\vdash_1$ and $\vdash_2$ (with possibly different dimensions) are equivalent, their extension lattices are isomorphic. In the case where $\vdash_1$ is finitary and $\vdash_2$ is the equational consequence relation $\models_{\mathsf{K}}$ of some quasivariety K, then the isomorphism maps the *finitary* extensions of $\vdash_1$ onto those of $\models_{\mathsf{K}}$, and there is an obvious lattice anti-isomorphism between the latter and the subquasivarieties of K. The composition of these maps is given explicitly in the next theorem.

Theorem 1.38 ([17])**.** *Let $\vdash$ be a finitary deductive system algebraized by quasivariety K, with a set $\boldsymbol{\tau}$ of defining equations. Then the following are mutually inverse lattice anti-isomorphisms between the lattice of finitary extensions of $\vdash$ and the subquasivariety lattice of K:*

$$\vdash' \mapsto \{\boldsymbol{A} \in \mathsf{K} \colon \boldsymbol{\tau}[\Gamma] \models_{\boldsymbol{A}} \boldsymbol{\tau}(\alpha) \text{ whenever } \Gamma \vdash' \alpha\}$$
$$\mathsf{K}' \mapsto \{\langle \Gamma, \alpha\rangle \colon \boldsymbol{\tau}[\Gamma] \models_{\mathsf{K}'} \boldsymbol{\tau}(\alpha)\}$$

If K *is a variety, then these maps restrict to mutually inverse lattice anti-isomorphisms between the lattice of axiomatic extensions of* $\vdash$ *and the subvariety lattice of* K.

The first map in Theorem 1.38 takes each finitary extension of $\vdash$ to its equivalent quasivariety [17, Cor. 4.9]. The algebraization of the extension is witnessed by same defining equations and equivalence formulas. A deductive system $\vdash$ is *consistent* if its set of formulas is not exhausted by its set of theorems, so $\vdash$ is *inconsistent* if every formula of $\vdash$ is a theorem of $\vdash$ (in which case all rules in the signature of $\vdash$ are derivable in $\vdash$). If a quasivariety K algebraizes a finitary deductive system $\vdash$, then the first map in Theorem 1.38 sends the consistent extensions of $\vdash$ to the nontrivial subquasivarieties of K (and it sends the inconsistent extension of $\vdash$ to the trivial subquasivariety of K). The maps in Theorem 1.38 also preserve the status of decision problems, for example:

Theorem 1.39. *If a variety* K *algebraizes a finitary deductive system* $\vdash$*, then* $\vdash$ *is* decidable*, i.e., its set of theorems is recursive, iff the* equational theory *of* K *(i.e., the set of equations satisfied by every member of* K*) is decidable.*

We shall not focus on decidability, but we shall note when a variety of algebras is locally finite (and therefore generated by its finite members). This is of interest because a finitely axiomatized variety of finite type that is generated by its finite members has a decidable equational theory (see for example the remarks after [120, Thm. 7.7]).

Let $\vdash$ be a deductive system, with algebraic signature $\mathcal{F}$. For any $\mathcal{F}' \subseteq \mathcal{F}$, the $\mathcal{F}'$*-fragment* of $\vdash$ is the deductive system $\vdash'$ with signature $\mathcal{F}'$ such that

$$\Gamma \vdash' \alpha \ \text{ iff } \ \Gamma \vdash \alpha,$$

for all sets $\Gamma \cup \{\alpha\}$ of formulas that involve only connectives from $\mathcal{F}'$.[2] A deductive system is said to be a *conservative expansion* of each of its fragments.

Theorem 1.40 ([17, Cor. 2.12])**.** *If a finitary deductive system* $\vdash$ *is algebraizable, then so is any* $\mathcal{F}'$*-fragment of* $\vdash$*, so long as* $\mathcal{F}'$ *contains all the connectives that occur in the defining equations and equivalence formulas witnessing the algebraization. If, moreover,* $\vdash$ *is elementarily algebraized by quasivariety* K*, then the* $\mathcal{F}'$*-fragment of* $\vdash$ *is algebraized by the quasivariety comprising the* $\mathcal{F}'$*-subreducts of members of* K*.*

[2]It is easy to see that $\vdash'$ is indeed a deductive system.

Now that the notion of algebraizability has been made precise, one can state general 'bridge theorems' that connect algebraic properties to equivalent meta-logical properties.

A deductive system $\vdash$ has a *deduction-detachment theorem* if there exists some fixed finite family $\boldsymbol{\sigma}$ of binary formulas, such that the law

$$\Gamma, \alpha \vdash \beta \text{ iff } \Gamma \vdash \boldsymbol{\sigma}(\alpha, \beta)$$

holds for all sets of formulas $\Gamma \cup \{\alpha, \beta\}$.

The following is the prototypical example of a bridge theorem.

Theorem 1.41 ([19]). *Let $\vdash$ be a finitary deductive system, that is algebraized by a* variety K. *Then $\vdash$ has a deduction-detachment theorem iff K has EDPC* (*see page* 15).

Substructural logics

The algebraic structures studied in this thesis all fall under the umbrella of *residuated structures*. They will be *lattices* with a binary operation $\cdot$ that is commutative and associative with identity e, and they have a definable *residual* function $\to$ such that the *law of residuation*

$$z \leqslant x \to y \text{ iff } x \cdot z \leqslant y$$

holds (where $\leqslant$ is the lattice order).

Roughly speaking, *substructural logics* are logics modeled by classes of residuated structures (see [51]). The ones relevant to this thesis have a set of defining equations $\boldsymbol{\tau} = \{e \leqslant x\}$, or more exactly $\boldsymbol{\tau} = \{e \wedge x \approx e\}$.[3] They all have the form $\vdash_{\mathsf{K}}$ for some class of K of residuated structures, where

$$\Gamma \vdash_{\mathsf{K}} \alpha \text{ iff } \{e \leqslant \gamma \colon \gamma \in \Gamma\} \models_{\mathsf{K}} e \leqslant \alpha,$$

for any set $\Gamma \cup \{\alpha\}$ of formulas in the signature of K (cf. condition (1.5)).

These logics are called 'substructural', because they may lack some of the *structural rules*, namely *exchange*, *contraction* and *weakening*, from Gerhard Gentzen's axiomatization of intuitionistic logic [57] by means of sequent calculi. We shall stick to an equivalent Hilbert system formulation of these logics. The *structural axioms* corresponding to the structural rules are then

$$(p \to (q \to r)) \to (q \to (p \to r)) \qquad \text{(exchange)},$$
$$(p \to (p \to q)) \to (p \to q) \qquad \text{(contraction)},$$
$$p \to (q \to p) \qquad \text{(weakening)}$$

[3] There is only one exception, namely the relevance logic **R** (defined below) in whose models e is not definable.

(where $p, q, r \in \mathit{Var}$).

In the residuated structures that model these logics, the structural axioms amount to the algebraic laws

$$x \cdot y \approx y \cdot x \qquad \text{(commutativity} \equiv \text{exchange)},$$
$$x \leqslant x \cdot x \qquad \text{(the square-increasing law} \equiv \text{contraction)},$$
$$x \leqslant e \qquad \text{(integrality} \equiv \text{weakening)}.$$

Since we always assume commutativity, all the logics with which we are concerned will have exchange. An example of a logic that adopts exchange but rejects contraction and weakening is Girard's *linear logic* [60, 138], which is motivated in part by computer science. It treats the premises of an implication as resources and is sensitive to the number of times that they are used (explaining the rejection of contraction). The following formal system axiomatizes the exponential-free fragment of linear logic with a classical negation.

Definition 1.42. Let $\mathbf{FL}_e$ be the following formal system with connectives $\wedge, \vee, \cdot, \to, \neg, \mathbf{t}$. (It abbreviates 'full Lambek calculus with exchange'; it is denoted by $\mathbf{InFL}_e$ in [51].)

Axioms of $\mathbf{FL}_e$:

A1	$p \to p$	(identity)
A2	$(p \to q) \to ((r \to p) \to (r \to q))$	(prefixing)
A3	$(p \to (q \to r)) \to (q \to (p \to r))$	(exchange)
A4	$(p \wedge q) \to p$	
A5	$(p \wedge q) \to q$	
A6	$((p \to q) \wedge (p \to r)) \to (p \to (q \wedge r))$	
A7	$p \to (p \vee q)$	
A8	$q \to (p \vee q)$	
A9	$((p \to r) \wedge (q \to r)) \to ((p \vee q) \to r)$	
A10	$\mathbf{t}$	
A11	$\mathbf{t} \to (p \to p)$	
A12	$p \to (q \to (q \cdot p))$	
A13	$(p \to (q \to r)) \to ((q \cdot p) \to r)$	
A14	$(p \to \neg q) \to (q \to \neg p)$	(contraposition)
A15	$\neg\neg p \to p$	(double negation)

Inference rules of $\mathbf{FL}_e$:

MP	$p,\ p \to q / q$	(modus ponens)
AD	$p,\ q / p \wedge q$	(adjunction)

The logic $\mathbf{FL}_e$ is algebraized by the variety of all *involutive residuated lattices*, briefly *IRLs* (see Definition 2.1). The set of defining equations and the set of equivalence formulas witnessing the algebraization are

$$\boldsymbol{\tau} := \{e \leqslant x\} \text{ and } \boldsymbol{\rho} := \{x \to y,\ y \to x\},$$

respectively. We follow the convention that the constant symbol e is replaced with the symbol $\mathbf{t}$ in a logical context.

The logics relevant to this thesis can be obtained by amending the definition of $\mathbf{FL}_e$. Consider the following axioms:

$$(p \wedge (q \vee r)) \to ((p \wedge q) \vee (p \wedge r)) \qquad \text{(distribution)}$$
$$p \to (p \to p) \qquad \text{(mingle)}$$

Classical propositional logic **CPL** can be obtained by adding all the remaining structural axioms, namely contraction and weakening, to $\mathbf{FL}_e$. The logic obtained by adding only contraction to $\mathbf{FL}_e$ shall be denoted by $\mathbf{FL}_{ec}$; it is called $\mathbf{LR}^{\mathbf{t}}$ in the relevance logic literature (and $\mathbf{InFL}_{ec}$ in [51]).

When we add distribution to $\mathbf{LR}^{\mathbf{t}}$ we get the relevance logic $\mathbf{R}^{\mathbf{t}}$. It is the main logic of interest in this thesis, because it is algebraized by the variety of De Morgan monoids. The monographs and survey articles on the subject of relevance logic include [1, 2, 23, 38, 93, 94, 122, 123, 127].

If mingle is added to $\mathbf{R}^{\mathbf{t}}$, the resulting logic is denoted by $\mathbf{RM}^{\mathbf{t}}$, and called '**R**-*mingle*'.

Given a logic **L** in the signature of $\mathbf{FL}_e$, we denote its *positive* fragment (i.e., its fragment without $\neg$) as $\mathbf{L}^+$. An axiomatization of $\mathbf{FL}_e^+$ can be obtained by deleting axioms **A14** and **A15** from Definition 1.42 and removing $\neg$ from the signature (see [69] and its references). *Positive intuitionistic propositional logic* (denoted by $\mathbf{IPL}^+$), is obtained by adding the structural axioms (weakening and contraction) to $\mathbf{FL}_e^+$. One can obtain *intuitionistic propositional logic* **IPL** by expanding the signature of $\mathbf{IPL}^+$ with a constant symbol $\bot$ and adding the axiom $\bot \to p$. The axiomatic extensions of **IPL** are called *super-intuitionistic* or *intermediate* logics.

Another fragment of interest is the fragment **R** of $\mathbf{R}^{\mathbf{t}}$ that lacks the so-called Ackermann constants (i.e., the $\mathbf{t}$-free fragment of $\mathbf{R}^{\mathbf{t}}$), which can be axiomatized like $\mathbf{FL}_e$ in Definition 1.42, by deleting axioms **A10** and **A11** and adding contraction and distribution. Although the set $\boldsymbol{\tau} = \{e \leqslant x\}$ of defining equations for $\mathbf{FL}_e$ contains the symbol $\mathbf{t}$ in the guise of e, it could be replaced by the set $\boldsymbol{\tau} = \{x \to x \leqslant x\}$, so that Theorem 1.40 still applies to systems like **R**.

If we consider $\boldsymbol{\tau}(\alpha)$ for each the axioms α above, we get the following correspondences (up to logical equivalence):

$x \wedge (y \vee z) \approx (x \vee y) \wedge (x \vee z)$	(distributivity (1.2) $\equiv$ distribution)
$x \cdot x \leqslant x$	(the *square-decreasing law* $\equiv$ mingle)

Using Theorems 1.38 and 1.40, we have the following correspondences, where the algebras mentioned will be defined in the next chapter (except for *relevant algebras*, which will be defined in Chapter 5).

Logic	**Equivalent variety**	
$\mathbf{FL}_e$	{all IRLs}	Definition 2.1
$\mathbf{FL}_e^+$	{all RLs}	Definition 2.2
$\mathbf{FL}_{ec}$	{all SIRLs}	Definition 2.8
$\mathbf{R}^{\mathbf{t}}$	{all De Morgan monoids}	Definition 2.18
$\mathbf{RM}^{\mathbf{t}}$	{all Sugihara monoids}	Definition 2.20
CPL	{all Boolean algebras}	Definition 2.30
$\mathbf{IPL}^+$	{all Brouwerian algebras}	Definition 2.32
IPL	{all Heyting algebras}	Definition 2.34
R	{all relevant algebras}	Definition 5.15

Chapter 2

The structure of De Morgan monoids

This chapter is an algebraic analysis of various classes of residuated lattices.

We start, in Section 2.1, by presenting some basic properties of (involutive) residuated lattices ([I]RLs), and progressively impose more restrictive conditions in subsequent sections. This culminates, in Section 2.5, with a new characterization of finitely subdirectly irreducible De Morgan monoids.

In Section 2.2 we discuss square-increasing [I]RLs (S[I]RLs). Whether an SIRL has an *idempotent* monoid operation is completely determined by properties of its neutral element e. This allows us to prove that an SIRL $\boldsymbol{A}$ satisfies $x \leqslant (\neg e)^2$ iff no nontrivial idempotent algebra belongs to $\mathbb{V}(\boldsymbol{A})$ (Corollary 2.14). We therefore say that such an SIRL is *anti-idempotent*. Roughly speaking, the characterization in Section 2.5 breaks up an FSI De Morgan monoid into two constructs—an anti-idempotent subalgebra and an idempotent (totally ordered) homomorphic image.

Section 2.3 establishes the universal algebraic properties of various varieties of residuated lattices. It includes well known characterizations of simple, SI and FSI [I]RLs via restrictions on the elements below e. For example, a De Morgan monoid is FSI iff the element e is join-prime.

Section 2.4 focusses on idempotent residuated lattices. The structure of idempotent De Morgan monoids, a.k.a. Sugihara monoids, is relatively well understood, in the sense that there is a transparent description of the finitely generated SI Sugihara monoids. This description and other results concerning Sugihara monoids are presented in Section 2.4.

We also take the opportunity in Section 2.4 to introduce integral SRLs (a.k.a. Brouwerian algebras), as they too are idempotent. More generally, the *negative elements* (below the neutral element e) of an S[I]RL are idempotent (and we shall see in Chapter 6 that they can be given the structure of a Brouwerian algebra). We therefore introduce *negatively generated* S[I]RLs, i.e., S[I]RLs that are generated by negative elements. We end Section 2.4 with a proof that every totally ordered negatively generated SRL is idempotent—a fact that facilitates our subsequent study of epimorphisms (in Chapter 6).

2.1 Residuated lattices

Definition 2.1. An *involutive (commutative) residuated lattice*, or briefly, an *IRL*, is an algebra $\boldsymbol{A} = \langle A; \cdot, \wedge, \vee, \neg, e\rangle$ comprising a commutative monoid $\langle A; \cdot, e\rangle$, a lattice $\langle A; \wedge, \vee\rangle$ and a function $\neg \colon A \to A$, called an *involution*, such that $\boldsymbol{A}$ satisfies the (first order) formulas $\neg\neg x \approx x$ and

$$x \cdot y \leqslant z \iff \neg z \cdot y \leqslant \neg x, \tag{2.1}$$

cf. [51].[1] Here, $\leqslant$ denotes the lattice order (i.e., $x \leqslant y$ abbreviates $x \wedge y \approx x$) and $\neg$ binds more strongly than any other operation; we refer to $\cdot$ as *fusion*.

Setting $y = e$ in (2.1), we see that $\neg$ is antitone. In fact, De Morgan's laws for $\neg, \wedge, \vee$ hold, so $\neg$ is an anti-automorphism of $\langle A; \wedge, \vee\rangle$. If we define

$$x \to y \;:=\; \neg(x \cdot \neg y) \;\text{ and }\; f \;:=\; \neg e,$$

then, as is well known, every IRL satisfies

$$x \cdot y \leqslant z \iff y \leqslant x \to z \quad \text{(the law of residuation)}, \tag{2.2}$$

$$\neg x \approx x \to f, \quad \text{hence} \quad x \cdot \neg x \leqslant f, \tag{2.3}$$

$$x \to y \approx \neg y \to \neg x \;\text{ and }\; x \cdot y \approx \neg(x \to \neg y). \tag{2.4}$$

Definition 2.2. A *(commutative) residuated lattice*—or an *RL*—is an algebra $\boldsymbol{A} = \langle A; \cdot, \to, \wedge, \vee, e\rangle$ comprising a commutative monoid $\langle A; \cdot, e\rangle$, a lattice $\langle A; \wedge, \vee\rangle$ and a binary operation $\to$, called the *residual* of $\boldsymbol{A}$, where $\boldsymbol{A}$ satisfies (2.2).

Thus, up to term equivalence, every IRL has a reduct that is an RL. Conversely, every RL can be embedded into (the RL-reduct of) an IRL; see [54] and the antecedents cited there.

Let $\boldsymbol{A}$ be an RL and let $a, b \in A$. According to the law of residuation, $a \to b$ can be characterized as the *largest* entity whose fusion with a falls

[1] The signature in [51] is slightly different, but the definable terms are not affected.

below b. The residual $\to$ of $\boldsymbol{A}$ can therefore be recovered from the $\langle A; \wedge, \vee, \cdot \rangle$ reduct of $\boldsymbol{A}$ by the rule

$$a \to b = \max\{c \in A \colon a \cdot c \leqslant b\}. \tag{2.5}$$

Every RL satisfies the following well known formulas. Here and subsequently, $x \leftrightarrow y$ abbreviates $(x \to y) \wedge (y \to x)$.

$$x \cdot (x \to y) \leqslant y \quad \text{and} \quad x \leqslant (x \to y) \to y \tag{2.6}$$
$$((x \to y) \to y) \to y \approx x \to y \tag{2.7}$$
$$x \leqslant y \to z \iff y \leqslant x \to z \tag{2.8}$$
$$(x \cdot y) \to z \approx y \to (x \to z) \approx x \to (y \to z) \tag{2.9}$$
$$(x \to y) \cdot (y \to z) \leqslant x \to z \tag{2.10}$$
$$x \cdot (y \vee z) \approx (x \cdot y) \vee (x \cdot z) \tag{2.11}$$
$$x \to (y \wedge z) \approx (x \to y) \wedge (x \to z) \tag{2.12}$$
$$(x \vee y) \to z \approx (x \to z) \wedge (y \to z) \tag{2.13}$$
$$x \leqslant y \implies \begin{cases} x \cdot z \leqslant y \cdot z \quad \text{and} \\ z \to x \leqslant z \to y \quad \text{and} \quad y \to z \leqslant x \to z \end{cases} \tag{2.14}$$
$$x \leqslant y \iff e \leqslant x \to y \tag{2.15}$$
$$x \approx y \iff e \leqslant x \leftrightarrow y \tag{2.16}$$
$$e \leqslant x \to x \quad \text{and} \quad e \to x \approx x \tag{2.17}$$
$$e \leqslant x \iff x \to x \leqslant x. \tag{2.18}$$

By (2.16), an RL $\boldsymbol{A}$ is nontrivial iff e is not its least element, iff e has a strict lower bound. Another consequence of (2.16) is that a non-injective homomorphism h between RLs must satisfy $h(c) = e$ for some $c < e$. (Choose $c = e \wedge (a \leftrightarrow b)$, where $h(a) = h(b)$ but $a \neq b$.)

In an RL, we define $x^0 := e$ and $x^{n+1} := x^n \cdot x$ for $n \in \omega$.

Lemma 2.3. *If a (possibly involutive) RL $\boldsymbol{A}$ has a least element $\bot$, then $\top := \bot \to \bot$ is its greatest element and, for all $a \in A$,*

$$a \cdot \bot = \bot = \top \to \bot \quad \textit{and} \quad \bot \to a = \top = a \to \top = \top^2.$$

In particular, $\{\bot, \top\}$ is a subalgebra of the $\cdot, \to, \wedge, \vee\, (, \neg)$ reduct of $\boldsymbol{A}$.

Proof. See [112, Prop. 5.1], for instance. (We infer $\top = \top^2$ from (2.14), as $e \leqslant \top$. The lattice anti-automorphism $\neg$, if present, clearly switches $\bot$ and $\top$.) □

If we say that $\bot, \top$ are *extrema* of an RL $\boldsymbol{A}$, we mean that $\bot \leqslant a \leqslant \top$ for all $a \in A$. An RL with extrema is said to be *bounded*. In that case, its extrema need not be *distinguished* elements, so they are not always retained in subalgebras. The next lemma is a straightforward consequence of (2.2).

Lemma 2.4. *The following conditions on a bounded IRL $\boldsymbol{A}$, with extrema $\bot, \top$, are equivalent.*

(i) $\top \cdot a = \top$ *whenever* $\bot \neq a \in A$.

(ii) $a \to \bot = \bot$ *whenever* $\bot \neq a \in A$.

(iii) $\top \to b = \bot$ *whenever* $\top \neq b \in A$.

Definition 2.5. Following Meyer [99], we say that an IRL is *rigorously compact* if it is bounded and satisfies the equivalent conditions of Lemma 2.4.

Lemma 2.6. *Let $\boldsymbol{A}$ be a rigorously compact IRL, with extrema $\bot, \top$, and let $h\colon \boldsymbol{A} \to \boldsymbol{B}$ be a homomorphism that is not a constant function. Then*

(i) $h^{-1}[\{h(\bot)\}] = \{\bot\}$ *and* $h^{-1}[\{h(\top)\}] = \{\top\}$.

(ii) *If $h(\bot)$ is meet-irreducible in $\boldsymbol{B}$, then $\bot$ is meet-irreducible in $\boldsymbol{A}$. Likewise, $\top$ is join-irreducible if $h(\top)$ is.*

(iii) *If $\boldsymbol{B}$ is totally ordered (as a lattice), then $\bot$ is meet-irreducible and $\top$ join-irreducible in $\boldsymbol{A}$.*

Proof. (i) If $\bot < a \in A$, with $h(a) = h(\bot)$, then $\top \cdot a = \top$, by rigorous compactness, so $h(\top) = h(\top) \cdot h(a) = h(\top) \cdot h(\bot) = h(\top \cdot \bot) = h(\bot)$. Similarly, if $\top > b \in A$, with $h(b) = h(\top)$, then $h(\top) = h(\bot)$, because $\top \to b = \bot$. As h is isotone, we conclude in both cases that $|h[A]| = 1$, contradicting the fact that h is not constant.

(ii) follows easily from (i), and (iii) from (ii). □

Lemma 2.7. *Let $\boldsymbol{A}$ be an IRL, with $a \in A$. Then*

$$e \leqslant a = a^2 \;\text{ iff }\; a \cdot \neg a = \neg a \;\text{ iff }\; a = a \to a.$$

Proof. The second and third conditions are equivalent, by the definition of $\to$ and involution properties. Also, $a^2 \leqslant a$ and $a \cdot \neg a \leqslant \neg a$ are equivalent, by (2.1). From $e \leqslant a$ and (2.14) we infer $\neg a = e \cdot \neg a \leqslant a \cdot \neg a$. Conversely, $a \to a \leqslant a$ and (2.17) yield $e \leqslant a$, and therefore $a \leqslant a^2$. □

The class of all RLs and that of all IRLs are finitely axiomatizable varieties [51, Thm. 2.7].

2.2 Square-increasing IRLs

Definition 2.8. An [I]RL is said to be *square-increasing*, briefly an *S[I]RL*,[2] if it satisfies

$$x \leqslant x^2 \quad (\text{the } \textit{square-increasing law}). \tag{2.19}$$

Every square-increasing RL can be embedded into a square-increasing IRL; see [97] and Section 3.4 on 'reflections', below. Moreover, Slaney [132] has shown that if two square-increasing IRLs have the same RL-reduct, then they are equal. The following formulas are valid in all square-increasing IRLs (and not in all IRLs):

$$x \wedge y \leqslant x \cdot y \tag{2.20}$$
$$(x \leqslant e \;\; \& \;\; y \leqslant e) \implies x \cdot y \approx x \wedge y \tag{2.21}$$
$$x \to (x \to y) \leqslant x \to y \tag{2.22}$$
$$e \leqslant x \vee \neg x. \tag{2.23}$$

The lemma below generalizes another result of Slaney [129, T36, p. 491] (where only the case $a = f$ was discussed, and $\boldsymbol{A}$ satisfied an extra postulate).

Lemma 2.9. *Let $\boldsymbol{A}$ be a square-increasing IRL, with $f \leqslant a \in A$. Then $a^3 = a^2$. In particular, $f^3 = f^2$.*

Proof. As $f \leqslant a$, we have $\neg a = a \to f \leqslant a \to a$, by (2.3) and (2.14), so

$$a \to \neg a \leqslant a \to (a \to a) = a^2 \to a, \tag{2.24}$$

by (2.14) and (2.9). By the square-increasing law, (2.24), (2.14) and (2.10),

$$a \to \neg a \leqslant (a \to \neg a)^2 \leqslant (a^2 \to a) \cdot (a \to \neg a) \leqslant a^2 \to \neg a.$$

Thus, $\neg(a^2 \to \neg a) \leqslant \neg(a \to \neg a)$, i.e., $a^2 \cdot a \leqslant a \cdot a$ (see (2.4)), i.e., $a^3 \leqslant a^2$. The reverse inequality follows from the square-increasing law and (2.14). □

The first assertion of the next theorem has unpublished antecedents in the work of relevance logicians. A corresponding result for 'relevant algebras' is reported in [136, Prop. 5], but the claim and proof below are simpler.

[2] In [107] the acronym 'S[I]RL' stands for '*subidempotent* [involutive] residuated lattice'. It refers to residuated lattices that satisfy the *subidempotent law* ($x \leqslant e \implies x \approx x^2$). Notice that square-increasing [I]RLs are subidempotent, owing to (2.21).

Theorem 2.10. *Every finitely generated square-increasing IRL* $\boldsymbol{A}$ *is bounded. More precisely, let* $\{a_1, \ldots, a_n\}$ *be a finite set of generators for* $\boldsymbol{A}$, *with*

$$c = e \vee f \vee \bigvee\nolimits_{i \leq n} (a_i \vee \neg a_i), \quad \textit{and} \quad b = c^2.$$

Then $\neg b \leqslant a \leqslant b$ *for all* $a \in A$.

Proof. By De Morgan's laws, every element of $\boldsymbol{A}$ has the form $\varphi^{\boldsymbol{A}}(a_1, \ldots, a_n)$ for some term $\varphi(x_1, \ldots, x_n)$ in the language $\cdot, \wedge, \neg, e$. The proof of the present theorem is by induction on the complexity $\#\varphi$ of φ. We shall write $\vec{x}$ and $\vec{a}$ for the respective sequences $x_1, \ldots, x_n$ and $a_1, \ldots, a_n$.

For the case $\#\varphi \leq 1$, note that $e, a_1, \ldots, a_n \leqslant c \leqslant b$, by the square-increasing law. Likewise, $f, \neg a_1, \ldots, \neg a_n \leqslant c \leqslant b$, so by involution properties, $\neg b \leqslant e, a_1, \ldots, a_n$. Now suppose $\#\varphi > 1$ and that $\neg b \leqslant \psi^{\boldsymbol{A}}(\vec{a}) \leqslant b$ for all terms ψ with $\#\psi < \#\varphi$. The desired result, viz.

$$\neg b \leqslant \varphi^{\boldsymbol{A}}(\vec{a}) \leqslant b,$$

follows from the induction hypothesis and basic properties of IRLs if φ has the form $\neg\psi(\vec{x})$ or $\psi_1(\vec{x}) \wedge \psi_2(\vec{x})$. We may therefore assume that φ is $\psi_1(\vec{x}) \cdot \psi_2(\vec{x})$ for some less complex terms $\psi_1(\vec{x}), \psi_2(\vec{x})$.

By the induction hypothesis and (2.14), $(\neg b)^2 \leqslant \varphi^{\boldsymbol{A}}(\vec{a}) \leqslant b^2$. As $\neg b \leqslant e$, we have $(\neg b)^2 = \neg b$, by (2.21). And since $f \leqslant c$, Lemma 2.9 gives $c^3 = c^2$, so $b^2 = c^4 = c^2 = b$. Therefore, $\neg b \leqslant \varphi^{\boldsymbol{A}}(\vec{a}) \leqslant b$, as required. □

In a square-increasing IRL, the smallest (i.e., the 0-generated) subalgebra $\boldsymbol{B}$ has top element $(e \vee f)^2 = f^2 \vee e$ (by Theorem 2.10 and (2.11)). This is a lower bound of $f \to f^2$ (by (2.2) and Lemma 2.9), so $f^2 \vee e = f \to f^2$. That the extrema of $\boldsymbol{B}$ can be expressed without using $\wedge, \vee$ is implicit in [99, p. 309]. Note also that $e \leftrightarrow f = f \wedge \neg(f^2)$ is the least element of $\boldsymbol{B}$.

An element a of an [I]RL $\boldsymbol{A}$ is said to be *idempotent* if $a^2 = a$. We say that $\boldsymbol{A}$ is *idempotent* if all of its elements are. In the next result, the key implication is (ii) ⇒ (iii). A logical analogue of (ii) ⟺ (iii) is stated without proof in [99, p. 309].

Theorem 2.11. *In a square-increasing IRL* $\boldsymbol{A}$, *the following are equivalent.*

(i) $f^2 = f$.

(ii) $f \leqslant e$.

(iii) $\boldsymbol{A}$ *is idempotent.*

Consequently, a square-increasing non-idempotent IRL has no idempotent subalgebra (and in particular, no trivial subalgebra).

Proof. In any IRL, (i) ⇒ (ii) instantiates (2.1) (as $\neg f = e$), and (iii) ⇒ (i) is trivial.

(ii) ⇒ (iii): Suppose $f \leqslant e$, and let $a \in A$. It suffices to show that $a^2 \leqslant a$, or equivalently (by (2.1)), that $a \cdot \neg a \leqslant \neg a$. Now, by the square-increasing law, (2.14), the associativity of fusion, (2.3) and (2.6),

$$a \cdot \neg a \;\leqslant\; a \cdot (\neg a)^2 \;=\; (a \cdot (a \to f)) \cdot \neg a \;\leqslant\; f \cdot \neg a \;\leqslant\; e \cdot \neg a \;=\; \neg a.$$

By (i) ⇒ (iii), $f^2 \neq f$ in each non-idempotent SIRL $\boldsymbol{A}$, so the 0-generated subalgebra of $\boldsymbol{A}$ is non-idempotent (and in particular nontrivial). □

Let $\boldsymbol{A}$ be an [I]RL. By a *filter* of $\boldsymbol{A}$, we mean a filter of the lattice $\langle A; \wedge, \vee\rangle$, i.e., a non-empty subset G of A that is upward closed and closed under the binary operation $\wedge$. A *deductive filter* of $\boldsymbol{A}$ is a filter G of $\langle A; \wedge, \vee\rangle$ that is also a *submonoid* of $\langle A; \cdot, e\rangle$, i.e., $e \in G$ and $a \cdot b \in G$ whenever $a, b \in G$. Thus, $[e)$ is the smallest deductive filter of $\boldsymbol{A}$, and whenever $b \in A$ and $a, a \to b \in G$, then $b \in G$ (as $a \cdot (a \to b) \leqslant b$, by (2.2)). The lattice $\mathbf{Fil}(\boldsymbol{A})$ of deductive filters of $\boldsymbol{A}$ and the congruence lattice $\mathbf{Con}(\boldsymbol{A})$ of $\boldsymbol{A}$ are isomorphic. The isomorphism and its inverse are given by

$$G \;\mapsto\; \boldsymbol{\Omega}^{\boldsymbol{A}} G \;:=\; \{\langle a, b\rangle \in A^2 \colon a \to b,\, b \to a \in G\};$$
$$\theta \;\mapsto\; \{a \in A \colon \langle a \wedge e, e\rangle \in \theta\}.$$

For a deductive filter G of $\boldsymbol{A}$ and $a, b \in A$, we often abbreviate $\boldsymbol{A}/\boldsymbol{\Omega}^{\boldsymbol{A}} G$ as $\boldsymbol{A}/G$, and $a/\boldsymbol{\Omega}^{\boldsymbol{A}} G$ as a/G, noting that

$$a \to b \in G \;\text{ iff }\; a/G \leqslant b/G \;\text{ in }\; \boldsymbol{A}/G. \tag{2.25}$$

Whenever $\boldsymbol{B}$ is a subalgebra of an [I]RL $\boldsymbol{A}$, and F is a deductive filter of $\boldsymbol{A}$, then $B \cap F$ is a deductive filter of $\boldsymbol{B}$ and

$$\boldsymbol{\Omega}^{\boldsymbol{B}}(B \cap F) = (\boldsymbol{\Omega}^{\boldsymbol{A}} F)|_B. \tag{2.26}$$

For any subset X of $\boldsymbol{A}$, the smallest deductive filter of $\boldsymbol{A}$ containing X is denoted by $\mathrm{Fg}^{\boldsymbol{A}} X$. When $\boldsymbol{A}$ is square-increasing, the deductive filters of $\boldsymbol{A}$ are just the lattice filters of $\langle A; \wedge, \vee\rangle$ that contain e, by (2.20). In this case $\mathrm{Fg}^{\boldsymbol{A}} X$ consists of all $a \in A$ such that

$$a \geqslant x_1 \wedge \ldots \wedge x_n \text{ for some } x_1, \ldots, x_n \in X \cup \{e\}, \text{ where } 0 < n \in \omega. \tag{2.27}$$

This yields the following lemma.

Lemma 2.12. *In a square-increasing IRL* $\boldsymbol{A}$,

(i) *if* $e \geqslant b \in A$, *then* $\mathrm{Fg}^{\boldsymbol{A}}\{b\} = [b)\,(:= \{a \in A \colon b \leqslant a\})$. *In particular,*

(ii) $[\neg(f^2))$ *is a deductive filter of* $\boldsymbol{A}$.

Here, (ii) follows from (i), because $e \geqslant \neg(f^2)$ follows from $f \leqslant f^2$.

Theorem 2.13. *Let* G *be a deductive filter of a square-increasing IRL* $\boldsymbol{A}$. *Then* $\boldsymbol{A}/G$ *is idempotent iff* $\neg(f^2) \in G$. *In particular,* $\boldsymbol{A}/[\neg(f^2))$ *is idempotent.*

Proof. $\boldsymbol{A}/G$ is idempotent iff $f/G \leqslant e/G$ (by Theorem 2.11), iff $f \to e \in G$ (by (2.25)), iff $\neg(f^2) \in G$ (as $\neg(f^2) = \neg(f \cdot \neg e) = f \to e$). □

We say that an IRL is *anti-idempotent* if it is square-increasing and satisfies $x \leqslant f^2$ (or equivalently, $\neg(f^2) \leqslant x$). This terminology is justified by the corollary below.

Corollary 2.14. *Let* K *be a variety of square-increasing IRLs. Then* K *has no nontrivial idempotent member iff it satisfies* $x \leqslant f^2$.

Proof. ($\Rightarrow$): As K is homomorphically closed but lacks nontrivial idempotent members, Theorem 2.13 shows that the deductive filter $[\neg(f^2))$ of any $\boldsymbol{A} \in \mathsf{K}$ coincides with A, i.e., K satisfies $\neg(f^2) \leqslant x$.

($\Leftarrow$): If $\boldsymbol{A} \in \mathsf{K}$ is idempotent, then $f^2 = f \leqslant e = \neg f = \neg(f^2)$, by Theorem 2.11, so by assumption, $\boldsymbol{A}$ is trivial. □

Recall from Definition 1.16 that when none of the nontrivial members of a variety K has a trivial subalgebra, then K is called a Kollár variety.

Corollary 2.15. *Every variety of anti-idempotent IRLs is a Kollár variety.*

Proof. This follows immediately from Corollary 2.14 and Theorem 2.11. □

2.3 Universal algebraic properties

Recall that an algebra $\boldsymbol{A}$ is subdirectly irreducible (SI) iff its identity relation id_A is completely meet-irreducible in its congruence lattice. Also, $\boldsymbol{A}$ is finitely subdirectly irreducible (FSI) iff id_A is meet-irreducible in $\mathbf{Con}(\boldsymbol{A})$, whereas $\boldsymbol{A}$ is simple iff $|\mathrm{Con}(\boldsymbol{A})| = 2$.

Since every variety is determined by its SI members, we need to understand these algebras in the present context. The following result is well known; see [52, Cor. 14] and [112, Thm. 2.4], for instance. Here and subsequently, an RL $\boldsymbol{A}$ is said to be *distributive* [resp. *modular*] if its reduct $\langle A; \wedge, \vee\rangle$ is a distributive [resp. modular] lattice.

Lemma 2.16. *Let $\boldsymbol{A}$ be a (possibly involutive) RL.*

(i) *$\boldsymbol{A}$ is FSI iff e is join-irreducible in $\langle A; \wedge, \vee\rangle$. Therefore, subalgebras and ultraproducts of FSI [I]RLs are FSI.*

(ii) *When $\boldsymbol{A}$ is distributive, it is FSI iff e is join-prime (i.e., whenever $a, b \in A$ with $e \leqslant a \vee b$, then $e \leqslant a$ or $e \leqslant b$).*

(iii) *If there is a largest element strictly below e, then $\boldsymbol{A}$ is SI. The converse holds if $\boldsymbol{A}$ is square-increasing.*

(iv) *If e has just one strict lower bound, then $\boldsymbol{A}$ is simple. The converse holds when $\boldsymbol{A}$ is square-increasing.*

All varieties of [I]RLs are congruence distributive, since [I]RLs have lattice reducts. This implies that Jónsson's Theorem 1.23 will always be available.

Since the join-irreducibility of e in condition (i) is expressible as a universal first order sentence, every variety of [I]RLs has equationally definable principal meets (EDPM), by Theorem 1.27.

Furthermore, all varieties of [I]RLs are congruence permutable and have the congruence extension property (CEP). These facts can be found, for instance, in [51, Sections 2.2 and 3.6].

From the CEP, it follows that $\mathbb{HS}(\mathsf{L}) = \mathbb{SH}(\mathsf{L})$ for any class L of [I]RLs; see Theorem 1.24. The CEP can easily be verified, using (2.26) and the fact that, when $\boldsymbol{A}$ is an [I]RL and $\boldsymbol{B}$ is a subalgebra of $\boldsymbol{A}$, the deductive filters of $\boldsymbol{B}$ are just the sets $B \cap F$ such that F is a deductive filter of $\boldsymbol{A}$.

Furthermore, every variety of *square-increasing* [I]RLs has equationally definable principal congruences (EDPC), as defined after Theorem 1.24 (see [51, Thm. 3.55]). In fact, in any S[I]RL $\boldsymbol{A}$, for all $a, b, c, d \in A$,

$$\langle c, d\rangle \in \Theta^{\boldsymbol{A}}(a, b) \ \text{ iff } \ (a \leftrightarrow b) \wedge e \leqslant c \leftrightarrow d.$$

Corollary 2.17. *Let K be any class of simple square-increasing [I]RLs. Then the variety $\mathbb{V}(\mathsf{K})$ is semisimple. In fact, $\mathbb{V}(\mathsf{K})_{\text{FSI}} = \mathbb{ISP}_{\mathbb{U}}(\mathsf{K})$, which consists of simple (or trivial) algebras.*

Proof. By Jónsson's Theorem 1.23, the FSI members of $\mathbb{V}(\mathsf{K})$ belong to $\mathbb{HSP}_{\mathbb{U}}(\mathsf{K})$, but the criterion for simplicity in Lemma 2.16(iv) is first order-definable and therefore persists in ultraproducts (by Łos' Theorem 1.8), while the CEP ensures that nontrivial subalgebras of simple algebras are simple (Theorem 1.24). □

Definition 2.18. A *De Morgan monoid* is a distributive square-increasing IRL. The variety of De Morgan monoids shall be denoted by DMM.

Therefore, all the items of Lemma 2.16 hold for De Morgan monoids (in particular the converses of items (iii) and (iv)). In the relevance logic literature, a De Morgan monoid is said to be *prime* if it is FSI. The reason is Lemma 2.16(ii), but we continue to use 'FSI' here, as it makes sense for arbitrary algebras.

Corollary 2.19. *In a simple anti-idempotent IRL $\boldsymbol{A}$, if $e < a \in A$, then $a \cdot f = f^2$.*

Proof. Let $e < a \in A$. By (2.14), $f = e \cdot f \leqslant a \cdot f$, but by (2.1), $a \cdot f \nleqslant f$ (since $a \cdot e \nleqslant e$), so $f < a \cdot f$. As $\boldsymbol{A}$ is simple and square-increasing, Lemma 2.16(iv) and involution properties show that f has just one strict upper bound in $\boldsymbol{A}$, which must be f^2, by anti-idempotence. Thus, $a \cdot f = f^2$. □

2.4 Idempotent varieties

The characterization of FSI De Morgan monoids that we are heading towards in the next section sorts FSI De Morgan monoids into those that are idempotent, and those are not. In this section, we focus on De Morgan monoids of the first kind (Sugihara monoids).

Sugihara monoids

Definition 2.20. A *Sugihara monoid* is an idempotent De Morgan monoid, i.e., an idempotent distributive IRL.

The variety SM of Sugihara monoids is more transparent than DMM, largely because of Dunn's contributions to [1]; see [37] also. It is locally finite, but not finitely generated. In fact, SM is the smallest variety containing the Sugihara monoid

$$\boldsymbol{Z}^* = \langle \{a : 0 \neq a \in \mathbb{Z}\}; \cdot, \wedge, \vee, -, 1 \rangle$$

on the set of all nonzero integers such that the lattice order is the usual total order, the involution $-$ is the usual additive inversion, and the monoid operation is defined by

$$a \cdot b \;=\; \begin{cases} \text{the element of } \{a, b\} \text{ with the greater absolute value, if } |a| \neq |b|\,; \\ a \wedge b \text{ if } |a| = |b| \end{cases}$$

(where $|a|$ is the natural absolute value function). In this algebra, the residual operation $\to$ is given by

$$a \to b \;=\; \begin{cases} (-a) \vee b & \text{if } a \leqslant b; \\ (-a) \wedge b & \text{if } a \nleqslant b. \end{cases}$$

Note that $e = 1$ and $f = -1$ in $\boldsymbol{Z}^*$.

Definition 2.21. An [I]RL is *semilinear* if it is isomorphic to a subdirect product of totally ordered algebras.

Total order is expressible by the universal positive sentence

$$(\forall x)(\forall y)((x \leqslant y) \sqcup (y \leqslant x)),$$

which persists under $\mathbb{H}$, $\mathbb{S}$, and $\mathbb{P}_{\mathbb{U}}$. So, because RLs are congruence distributive, Jónsson's Theorem 1.23 has the following consequence: whenever L consists of totally ordered [I]RLs, then so does $\mathbb{V}(\mathsf{L})_{\mathrm{FSI}}$, whence $\mathbb{V}(\mathsf{L})$ consists of semilinear algebras.

Lemma 2.22. *A semilinear [I]RL $\boldsymbol{A}$ is FSI iff it is totally ordered.*

Proof. ($\Rightarrow$): As $\boldsymbol{A}$ is semilinear, $\boldsymbol{A} \in \mathbb{IP}_{\mathbb{S}}(\mathsf{L})$ for some set L of totally ordered [I]RLs. Since $\boldsymbol{A}$ is FSI, $\boldsymbol{A} \in \mathbb{V}(\mathsf{L})_{\mathrm{FSI}}$, whence $\boldsymbol{A}$ is totally ordered, as above, by Jónsson's Theorem 1.23.

($\Leftarrow$): When $\boldsymbol{A}$ is totally ordered, each of its elements is join-irreducible. In particular, e is join-irreducible in $\boldsymbol{A}$, so $\boldsymbol{A}$ is FSI by Lemma 2.16(i). □

The fact that $\boldsymbol{Z}^*$ is totally ordered and generates SM yields:

Lemma 2.23. *Every FSI Sugihara monoid is totally ordered. In particular, Sugihara monoids are semilinear.*

It is shown in [66] that an [I]RL $\boldsymbol{A}$ is semilinear iff it is distributive and satisfies

$$e \leqslant (x \to y) \vee (y \to x), \tag{2.28}$$

whence the semilinear [I]RLs form a variety. The substructural logics that are algebraized by semilinear varieties of [I]RLs therefore satisfy the Gödel-Dummet axiom: $(p \to q) \vee (q \to p)$.

Definition 2.24. An IRL $\boldsymbol{A}$ is said to be *odd* if $f = e$ in $\boldsymbol{A}$.

Theorem 2.25. *Every odd De Morgan monoid is a Sugihara monoid.*

Proof. By Theorem 2.11, every odd SIRL is idempotent. □

In the Sugihara monoid $\boldsymbol{Z} = \langle \mathbb{Z}; \cdot, \wedge, \vee, -, 0 \rangle$ on the set of *all* integers, the operations are defined like those of $\boldsymbol{Z}^*$, except that 0 takes over from 1 as the neutral element for $\cdot$. Both e and f are 0 in $\boldsymbol{Z}$, so $\boldsymbol{Z}$ is odd. It follows from Theorem 2.25 and Dunn's results in [1, 37] that the variety of all odd Sugihara monoids, OSM, is the smallest quasivariety containing $\boldsymbol{Z}$, and that SM is the smallest quasivariety containing both $\boldsymbol{Z}^*$ and $\boldsymbol{Z}$.

For each positive integer n, let $\boldsymbol{S}_{2n}$ denote the subalgebra of $\boldsymbol{Z}^*$ with universe $\{-n, \dots, -1, 1, \dots, n\}$ and, for $n \in \omega$, let $\boldsymbol{S}_{2n+1}$ be the subalgebra of $\boldsymbol{Z}$ with universe $\{-n, \dots, -1, 0, 1, \dots, n\}$. The results cited above yield:

Theorem 2.26. *Up to isomorphism, the algebras $\boldsymbol{S}_n$ $(1 < n \in \omega)$ are precisely the finitely generated SI Sugihara monoids, whence the algebras $\boldsymbol{S}_{2n+1}$ $(0 < n \in \omega)$ are just the finitely generated SI odd Sugihara monoids.*

Consequently, for each $m \in \omega$, a totally ordered m-generated Sugihara monoid has at most $2m + 2$ elements. The bound reduces to $2m + 1$ in the odd case.

We cannot embed $\boldsymbol{Z}$ (nor even $\boldsymbol{S}_{2n+1}$) into $\boldsymbol{Z}^*$, owing to the involution. Nevertheless, $\boldsymbol{Z}$ is a homomorphic image of $\boldsymbol{Z}^*$, and $\boldsymbol{S}_{2n+1}$ is a homomorphic image of $\boldsymbol{S}_{2n+2}$, for all $n \in \omega$. In each case, the kernel of the homomorphism identifies -1 with 1; it identifies no other pair of distinct elements. Also, $\boldsymbol{S}_{2n-1}$ is a homomorphic image of $\boldsymbol{S}_{2n+1}$ if $n > 0$; in this case the kernel collapses $-1, 0, 1$ to a point, while isolating all other elements. Thus, $\boldsymbol{S}_3$ is a homomorphic image of $\boldsymbol{S}_n$ for all $n \geq 3$. In particular, every nontrivial variety of Sugihara monoids includes $\boldsymbol{S}_2$ or $\boldsymbol{S}_3$.

Corollary 2.27. *The lattice of varieties of odd Sugihara monoids is the following chain of order type $\omega + 1$:*

$$\mathbb{V}(\boldsymbol{S}_1) \subsetneq \mathbb{V}(\boldsymbol{S}_3) \subsetneq \mathbb{V}(\boldsymbol{S}_5) \subsetneq \dots \subsetneq \mathbb{V}(\boldsymbol{S}_{2n+1}) \subsetneq \dots \subsetneq \mathbb{V}(\boldsymbol{Z}).$$

Proof. See [1, Sec. 29.4] or [55, Fact 7.6]. □

Theorem 2.28 ([55, Thm. 7.3])**.** *Every quasivariety of odd Sugihara monoids is a variety.*

The subvariety lattice of SM is fully described in [92].

Corollary 2.29. *A quasivariety K of De Morgan monoids is a Kollár quasivariety* (*see Definition* 1.16) *iff it excludes $\boldsymbol{S}_3$.*

Proof. Clearly, if K includes $\boldsymbol{S}_3$, then it is not Kollár, because $\boldsymbol{S}_3$ is a nontrivial algebra with a trivial subalgebra.

Conversely, if $\boldsymbol{A} \in \mathsf{K}$ is nontrivial with a trivial subalgebra, then $\boldsymbol{A}$ is odd. Thus $\boldsymbol{A}$ is an odd Sugihara monoid, by Theorem 2.25. Notice that $\mathbb{Q}(\boldsymbol{A})$ is a nontrivial *variety* of odd Sugihara monoids, by Theorem 2.28. So, $\boldsymbol{S}_3 \in \mathbb{Q}(\boldsymbol{A})$, by Corollary 2.27. Therefore, $\boldsymbol{S}_3 \in \mathbb{Q}(\boldsymbol{A}) \subseteq \mathsf{K}$, since K is a quasivariety. □

Integral and negatively generated [I]RLs

A *bounded lattice* $\langle A; \wedge, \vee, 0, 1\rangle$ is the expansion of a lattice $\langle A; \wedge, \vee\rangle$ by distinguished elements $0, 1 \in A$, where $0 \leqslant a \leqslant 1$ for all $a \in A$. It is said to be *complemented* if, for each $a \in A$, there exists $a' \in A$ such that $a \wedge a' = 0$ and $a \vee a' = 1$. The element a' is uniquely determined (and called the *complement* of a) if $\langle A; \wedge, \vee\rangle$ is distributive.

Definition 2.30. A *Boolean algebra* $\boldsymbol{A} = \langle A; \wedge, \vee, ', 1, 0\rangle$ comprises a complemented distributive lattice $\langle A; \wedge, \vee\rangle$ with extrema $0 < 1$, where a' is the complement of a, for each $a \in A$.

An [I]RL is *integral* if e is its greatest element, in which case it satisfies $e \approx x \to x \approx x \to e$, by (2.17) and (2.15). By (2.21), a square-increasing [I]RL is integral iff its operations $\cdot$ and $\wedge$ coincide. Furthermore, integral S[I]RLs are distributive by (2.11). The following lemma is well known.

Lemma 2.31. *An SIRL is integral iff it is a Boolean algebra (in which the operation $\wedge$ is duplicated by fusion, and the least element is definable as $\neg e$).*[3]

Proof. Sufficiency is clear. Conversely, as we have just seen, the fusionless reduct of an integral SIRL is a distributive lattice. Its greatest element is e, by integrality, and its least element is $\neg e\,(= f)$, since $\neg$ is antitone. Also, it is complemented, by (2.23) and De Morgan's laws. □

Integral S[I]RLs are idempotent, since $\wedge$ is idempotent. In fact, the Sugihara monoid $\boldsymbol{S}_2$ is the two-element Boolean algebra. It is well known that the variety BA of all Boolean algebras coincides with $\mathbb{V}(\boldsymbol{S}_2)\,(= \mathbb{Q}(\boldsymbol{S}_2))$ [24, Cor. 1.14]. In the non-involutive case, integrality is less restrictive.

Definition 2.32. A *Brouwerian algebra* is an integral SRL $\boldsymbol{A}$; it is normally identified with its reduct $\langle A; \wedge, \vee, \to, e\rangle$.

[3] Lemma 2.31 reflects the fact that classical propositional logic is the extension of $\mathbf{R^t}$ by the weakening axiom (see Section 1.2).

A Brouwerian algebra is idempotent and distributive (by the remarks before and after Lemma 2.31) although it need not be bounded. The law of residuation becomes

$$x \wedge y \leqslant z \iff y \leqslant x \to y. \tag{2.29}$$

It follows that every Brouwerian algebra is determined by its lattice reduct, and that every complete (in particular, every finite) distributive lattice is the lattice reduct of a unique Brouwerian algebra. The class of all Brouwerian algebras is a variety denoted by BRA.

Definition 2.33. The variety RSA of *relative Stone algebras* comprises the semilinear Brouwerian algebras; it is generated by the Brouwerian algebra on the chain of non-positive integers.

Definition 2.34. An algebra $\boldsymbol{A} = \langle A; \wedge, \vee, \to, e, \bot \rangle$ is a *Heyting algebra* when its $\bot$-free reduct is a Brouwerian algebra and it satisfies $\bot \leqslant x$.

Thus, $\bot$ belongs to every subalgebra of a Heyting algebra, and homomorphisms between Heyting algebras preserve $\bot$.

We shall undertake a more in-depth study of Brouwerian algebras in Chapter 6, where we explain a categorical duality between the variety of Brouwerian algebras and a certain class of ordered topological spaces.

An element a of an [I]RL $\boldsymbol{A}$ will be called *negative* if $a \leqslant e$, and *positive* if $e \leqslant a$. We denote the set of negative elements of $\boldsymbol{A}$ by

$$A^- := \{a \in A : a \leqslant e\}.$$

The deductive filters of an [I]RL $\boldsymbol{A}$ are determined by their negative elements, in the sense that, for all deductive filters $F, G \in \mathrm{Fil}(\boldsymbol{A})$,

$$\text{if } F \cap A^- = G \cap A^- \text{ then } F = G. \tag{2.30}$$

Indeed, in this case, if $a \in F$ then $a \wedge e \in F \cap A^- = G \cap A^-$, so $a \in G$, because G is upward closed. We shall see in Chapter 6 that the set of negative elements A^- of an S[I]RL $\boldsymbol{A}$ can be given the structure of a Brouwerian algebra, called its *negative cone* (see Definition 6.11).

We say that an [I]RL $\boldsymbol{A}$ is *negatively generated* when it is generated by its negative elements, i.e., $A = \mathrm{Sg}^{\boldsymbol{A}}(A^-)$. As surjective homomorphisms always map generating sets onto generating sets, the following lemma applies.

Lemma 2.35. *If $h \colon \boldsymbol{A} \to \boldsymbol{B}$ is a surjective homomorphism of S[I]RLs and $\boldsymbol{A}$ is negatively generated then so is $\boldsymbol{B}$.*

The Sugihara monoid $\boldsymbol{Z}^*$ satisfies the equation

$$x \approx (x \wedge e) \cdot \neg(\neg x \wedge f), \tag{2.31}$$

because, by De Morgan's laws, $\neg(\neg x \wedge f) \approx x \vee e$, and since every element of $\boldsymbol{Z}^*$ is comparable with e. As $\mathsf{SM} = \mathbb{V}(\boldsymbol{Z}^*)$, *every* Sugihara monoid $\boldsymbol{A}$ satisfies (2.31), and is therefore negatively generated, because $a \wedge e \leqslant e$ and $\neg a \wedge f \leqslant \neg a \wedge e \leqslant e$ for all $a \in A$ (by Theorem 2.11).

Semilinear idempotent RLs

The following abbreviations are useful when working with idempotent RLs:

$$x^* := x \to e \quad \text{and} \quad |x| := x \to x.$$

In the Sugihara monoid $\boldsymbol{Z}^*$ (defined on page 36), the term operation $|x|$ coincides with the natural absolute value operation. By (2.6), (2.7) and (2.17), every [I]RL satisfies

$$x \leqslant x^{**} \quad \text{and} \quad x^{***} \approx x^* \quad \text{and} \quad e \leqslant |x|. \tag{2.32}$$

If an RL is idempotent, then it also satisfies

$$x \leqslant |x|, \tag{2.33}$$

$$x \approx |x| \iff e \leqslant x, \tag{2.34}$$

$$x^* \approx |x| \iff x \leqslant e, \tag{2.35}$$

$$x \approx x^* \iff x \approx e. \tag{2.36}$$

The following theorem shows that the fusion of a totally ordered idempotent RL $\boldsymbol{A}$ resembles that of a Sugihara monoid, and that $\boldsymbol{A}$ is determined by its reduct $\langle A; \wedge, \vee, {}^* \rangle$, and also by its reduct $\langle A; \wedge, \vee, |\text{-}| \rangle$.

Theorem 2.36 ([116, Thms. 12, 14])**.** *Let $\boldsymbol{A}$ be a totally ordered idempotent RL. Then $\boldsymbol{A}$ satisfies*

$$x \cdot y = \begin{cases} x & \text{if } |y| < |x| \\ y & \text{if } |x| < |y| \\ x \wedge y & \text{if } |x| = |y| \end{cases} \quad \text{and} \quad x \to y = \begin{cases} x^* \vee y & \text{if } x \leqslant y \\ x^* \wedge y & \text{if } x > y \end{cases}. \tag{2.37}$$

The structure of totally ordered idempotent RLs is described in [59] (and earlier in [116]) using a representation that will be introduced in Section 6.6 (Theorem 6.28).

Definition 2.37. The variety GSM of *generalized Sugihara monoids* consists of the semilinear idempotent RLs that satisfy

$$(x \vee e)^{**} \approx x \vee e, \tag{2.38}$$

or equivalently, $e \leqslant x \implies x^{**} \approx x$.

The main significance of GSM lies in the next theorem.

Theorem 2.38 ([56, Cor. 3.5])**.** *A semilinear idempotent RL is a generalized Sugihara monoid iff it is negatively generated.*

In the proof of this theorem, one uses the fact that all generalized Sugihara monoids satisfy

$$x \approx (x \wedge e) \cdot (x^* \wedge e)^*. \tag{2.39}$$

A representation of totally ordered generalized Sugihara monoids is given in Corollary 6.29.

Definition 2.39. A *Dunn monoid* is a square-increasing distributive RL.

Dunn monoids originate in [36] and acquired their name in [96].

Theorem 2.40. *Let $\boldsymbol{A}$ be a totally ordered Dunn monoid that is generated by a set X of idempotent elements. Then $\boldsymbol{A}$ is idempotent.*

Proof. Let $a \in A$. Then $a = t^{\boldsymbol{A}}(a_1, \ldots, a_n)$ for some n-ary term $t(x_1, \ldots, x_n)$ and some $a_1, \ldots, a_n \in X$. For brevity, we assume that terms are evaluated in $\boldsymbol{A}$ and let $\vec{a}$ abbreviate $a_1, \ldots, a_n$. We show that $a = a^2$ by induction on the complexity $\#t$ of t.

When $\#t = 0$, clearly $t(\vec{a})^2 = t(\vec{a})$, because $t \in \{e, x_1, \ldots, x_n\}$.

Assume that s and r are terms with $\#s, \#r < \#t$ such that $s(\vec{a})^2 = s(\vec{a})$ and $r(\vec{a})^2 = r(\vec{a})$.

If $t = s \wedge r$ or $t = s \vee r$, then $t(\vec{a}) \in \{s(\vec{a}), r(\vec{a})\}$, since $\boldsymbol{A}$ is totally ordered, and we are done. If $t = s \cdot r$, then $t(\vec{a})^2 = s(\vec{a})^2 \cdot r(\vec{a})^2 = s(\vec{a}) \cdot r(\vec{a}) = t(\vec{a})$, by the induction hypothesis. Lastly, suppose that $t = s \to r$. Note that $t(\vec{a}) \leqslant t(\vec{a})^2$, since $\boldsymbol{A}$ is square-increasing. On the other hand, by (2.6),

$$s(\vec{a}) \cdot (s(\vec{a}) \to r(\vec{a}))^2 = s(\vec{a})^2 \cdot (s(\vec{a}) \to r(\vec{a}))^2 \leqslant r(\vec{a})^2 = r(\vec{a}).$$

So, $t(\vec{a})^2 = (s(\vec{a}) \to r(\vec{a}))^2 \leqslant s(\vec{a}) \to r(\vec{a}) = t(\vec{a})$, by the law of residuation. $\square$

Theorem 2.41. *Let $\boldsymbol{A}$ be an semilinear Dunn monoid. The following are equivalent:*

(i) $\boldsymbol{A}$ *is negatively generated;*

(ii) $\boldsymbol{A}$ *is a generalized Sugihara monoid;*

(iii) $\boldsymbol{A}$ *satisfies equation* (2.39).

Proof. (i) $\Rightarrow$ (ii): By Birkhoff's Subdirect Decomposition Theorem 1.3, $\boldsymbol{A}$ embeds into $\prod_{i \in I} \boldsymbol{A}_i$ for some set $\{\boldsymbol{A}_i \colon i \in I\}$ of totally ordered Dunn monoids, such that each $\boldsymbol{A}_i$ is a homomorphic image of $\boldsymbol{A}$. For each $i \in I$, we have $\boldsymbol{A}_i = \mathbf{Sg}^{\boldsymbol{A}_i} A_i^-$, by Lemma 2.35. By (2.21), every element of A_i^- is idempotent, so $\boldsymbol{A}_i$ is idempotent, by Theorem 2.40. Therefore, each $\boldsymbol{A}_i \in \mathsf{GSM}$, by Theorem 2.38, so $\boldsymbol{A}$ is a generalized Sugihara monoid.

For (ii) $\Rightarrow$ (iii), see the proof of [56, Cor. 3.5]. That (iii) $\Rightarrow$ (i) follows from the form of equation (2.39). In particular, $a \wedge e$ and $a^* \wedge e$ belong to A^-, for every $a \in A$. □

Corollary 2.42. *The negatively generated Dunn monoids form a locally finite variety, namely the variety of generalized Sugihara monoids.*

Indeed, it is shown in [116, Thm. 18] that the variety of semilinear idempotent RLs is locally finite, therefore GSM is as well. For each $n \in \omega$, an n-generated totally ordered idempotent RL has at most $3n+1$ elements. (This is due to the structure theorems that will be exhibited in Section 6.6.) The bound reduces to $n+1$ in the integral case, i.e., in the subvariety of relative Stone algebras.

2.5 De Morgan monoids

In this section we shall show that each FSI De Morgan monoid is either an FSI Sugihara monoid or a 'rigorous extension' of a nontrivial FSI anti-idempotent De Morgan monoid (Theorem 2.57).

The next result is easy and well known, but note that it draws on all the key properties of De Morgan monoids.

Theorem 2.43. *Let* $\boldsymbol{A}$ *be a De Morgan monoid that is FSI, with* $a \in A$. *Then* $e \leqslant a$ *or* $a \leqslant f$. *Thus,* $A = [e) \cup (f\,]$.

Proof. As $\boldsymbol{A}$ is square-increasing, $e \leqslant a \vee \neg a$, by (2.23). So, because $\boldsymbol{A}$ is distributive and FSI, $e \leqslant a$ or $e \leqslant \neg a$, by Lemma 2.16(ii). In the latter case, $a \leqslant f$, because $\neg$ is antitone. □

Corollary 2.44. *Let* $\boldsymbol{A}$ *be a De Morgan monoid that is SI. Let* c *be the largest element of* $\boldsymbol{A}$ *strictly below* e (*which exists, by Lemma* 2.16(iii)). *Then* $c \leqslant f$.

The following result about bounded De Morgan monoids was essentially proved by Meyer [99, Thm. 3], but his argument assumes that the elements $\bot, \top$ are distinguished, or at least definable in terms of generators. To avoid that presupposition, we give a simpler and more direct proof.

Theorem 2.45. *Let $\boldsymbol{A}$ be a bounded FSI De Morgan monoid. Then $\boldsymbol{A}$ is rigorously compact (see Definition 2.5).*

Proof. Let $\bot \neq a \in A$, where $\bot, \top$ are the extrema of $\boldsymbol{A}$. It suffices to show that $\top \cdot a = \top$. As $e \cdot a \nleqslant \bot$, we have $\top \cdot a \nleqslant f$, by (2.1), so

$$e \leqslant \top \cdot a, \tag{2.40}$$

by Theorem 2.43. Recall that $\top^2 = \top$, by Lemma 2.3. Therefore,

$$\top = \top \cdot e \leqslant \top^2 \cdot a \text{ (by (2.40))} \; = \top \cdot a \leqslant \top,$$

whence $\top \cdot a = \top$. □

Corollary 2.46. *If a De Morgan monoid is FSI, then its finitely generated subalgebras are rigorously compact.*

Proof. This follows from Lemma 2.16(i) and Theorems 2.10 and 2.45. □

As the structure of idempotent De Morgan monoids (a.k.a. Sugihara monoids) is very transparent, we concentrate now on De Morgan monoids that are *not* idempotent.

Lemma 2.47. *Let $\boldsymbol{A}$ be a non-idempotent FSI De Morgan monoid, and let a be an idempotent element of $\boldsymbol{A}$. If $a \geqslant f$, then $a > e$. In particular, $f^2 > e$.*

Proof. Suppose $a^2 = a \geqslant f$. As $\boldsymbol{A}$ is not idempotent, $f^2 \neq f$, by Theorem 2.11, so $a \neq f$. Therefore, $a \nleqslant f$, whence $e \leqslant a$, by Theorem 2.43. As $f \leqslant a$, we cannot have $a = e$, by Theorem 2.11, so $e < a$. The last claim follows because f^2 is an idempotent upper bound of f (by Lemma 2.9). □

Lemma 2.48. *Let $\boldsymbol{A}$ be an FSI De Morgan monoid with $f \leqslant a, b \in A$, where a and b are idempotent. Then $a \leqslant b$ or $b \leqslant a$.*

Proof. If $\boldsymbol{A}$ is a Sugihara monoid, the result follows from Lemma 2.23. We may therefore assume that $\boldsymbol{A}$ is not idempotent, so $e < a, b$, by Lemma 2.47. Then $a \cdot \neg a = \neg a$ and $b \cdot \neg b = \neg b$, by Lemma 2.7, so

$$\begin{aligned}
&(a \cdot \neg b) \wedge (b \cdot \neg a) \leqslant (a \cdot \neg b) \cdot (b \cdot \neg a) \text{ (by (2.20))} \\
&= (a \cdot \neg a) \cdot (b \cdot \neg b) = \neg a \cdot \neg b \text{ (by the above)} \\
&= \neg a \wedge \neg b \text{ (by (2.21), as } \neg a, \neg b \leqslant e).
\end{aligned}$$

Therefore, by De Morgan's laws,

$$\neg(\neg a \wedge \neg b) \leqslant \neg((a \cdot \neg b) \wedge (b \cdot \neg a))$$
$$= \neg(a \cdot \neg b) \vee \neg(b \cdot \neg a) = (a \to b) \vee (b \to a)$$

and $e < a \vee b = \neg(\neg a \wedge \neg b)$, so $e < (a \to b) \vee (b \to a)$. Then, since $\boldsymbol{A}$ is FSI, Lemma 2.16(ii) and (2.15) yield $e \leqslant a \to b$ or $e \leqslant b \to a$, i.e., $a \leqslant b$ or $b \leqslant a$. □

Lemma 2.49. *Let $\boldsymbol{A}$ be a De Morgan monoid that is FSI, and let $f \leqslant a \in A$, where $a \not< f^2$. Then a is idempotent.*

Proof. By Lemma 2.9, f^2 is idempotent, so assume that $a \neq f^2$. From $f \leqslant f^2$ and $a \nleqslant f^2$, we infer $a \nleqslant f$. Then $e \leqslant a$, by Theorem 2.43, so $e, f \in [\neg a, a]\,(= \{b \in A : \neg a \leqslant b \leqslant a\})$. Therefore, $\neg(a^2) \leqslant x \leqslant a^2$ for all $x \in \mathrm{Sg}^{\boldsymbol{A}}\{a\}$, by Theorem 2.10. By Corollary 2.46, $\mathbf{Sg}^{\boldsymbol{A}}\{a\}$ is rigorously compact. In particular,

$$a^2 \cdot x = a^2 \text{ whenever } \neg(a^2) < x \in \mathrm{Sg}^{\boldsymbol{A}}\{a\}. \tag{2.41}$$

As $a \leqslant a^2$ and $a \nleqslant f^2$, we have $a^2 \nleqslant f^2$. But a^2 and f^2 are idempotent, by Lemma 2.9, so $f^2 < a^2$, by Lemma 2.48. Thus, $\neg(a^2) < \neg(f^2) \in \mathrm{Sg}^{\boldsymbol{A}}\{a\}$, so

$$a^2 = a^2 \cdot \neg(f^2), \tag{2.42}$$

by (2.41). As $\boldsymbol{A}/[\neg(f^2))$ is idempotent (by Theorem 2.13), $\neg(f^2) \leqslant a^2 \to a$, i.e., $a^2 \cdot \neg(f^2) \leqslant a$, by (2.25) and (2.2). Then (2.42) gives $a^2 \leqslant a$, and so $a^2 = a$. □

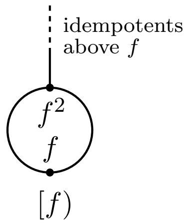
Theorem 2.50. *Let $\boldsymbol{A}$ be a non-idempotent FSI De Morgan monoid, with $f^2 \leqslant a \in A$. Then $\neg a < a$ and the interval $[\neg a, a]$ is a subuniverse of $\boldsymbol{A}$. In particular, $[\neg(f^2), f^2]$ is a subuniverse of $\boldsymbol{A}$.*

Proof. In $\boldsymbol{A}$, we have $\neg(f^2) \leqslant e$, as noted after Lemma 2.12, while $e < f^2$, by Lemma 2.47. Of course, $\neg a \leqslant \neg(f^2)$, so $\neg a < a$. Thus, $[\neg a, a]$ includes e, and it is obviously closed under $\wedge$, $\vee$ and $\neg$. Closure under fusion follows from (2.14) and the square-increasing law, because a is idempotent (by Lemma 2.49). □

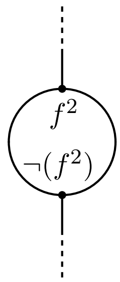
Theorem 2.51. *In any FSI De Morgan monoid, the filter $[f)$ is the union of the interval $[f, f^2]$ and a chain whose least element is f^2. The elements of this chain are just the idempotent upper bounds of f.*

Proof. This follows from Lemma 2.23 when the algebra is idempotent. In the opposite case, the idempotent upper bounds of f are exactly the upper bounds of f^2 (by (2.14) and Lemma 2.49), and they are comparable with all upper bounds of f (by Lemmas 2.49 and 2.48). □

2.51:



2.52:



Theorem 2.52. *Any non-idempotent FSI De Morgan monoid is the union of the interval subuniverse* $[\neg(f^2), f^2]$ *and two chains of idempotents,* $(\neg(f^2)]$ *and* $[f^2)$.

Proof. Let $\boldsymbol{A}$ be a non-idempotent FSI De Morgan monoid. Theorem 2.50 shows that $e, f \in [\neg(f^2), f^2]$ and (with Lemma 2.3) that $\neg(f^2) \cdot f = \neg(f^2)$. Note that $[f^2)$ and $(\neg(f^2)]$ are both chains of idempotents, by Theorem 2.51, involution properties and (2.21).

Suppose, with a view to contradiction, that there exists $a \in A$ such that $a \notin (\neg(f^2)] \cup [\neg(f^2), f^2] \cup [f^2)$. By Theorem 2.43, $e < a$ or $a < f$. By involutional symmetry, we may assume that $e < a$. Then a is incomparable with f^2 (as $a \notin [\neg(f^2), f^2] \cup [f^2)$), so $f^2 \vee a > f^2$. Also, since $f^2, a \geqslant e$, we have $f^2 \cdot a \geqslant f^2 \vee a$, by (2.14), so $f^2 \cdot a > f^2$.

Because $a > e$, we have $f \cdot a \geqslant f$. If $f \cdot a \in [\neg(f^2), f^2]$, then

$$f^2 \cdot a \leqslant (f \cdot a)^2 \leqslant f^4 = f^2 \quad \text{(by Lemma 2.9)},$$

a contradiction. So, by Theorem 2.51, $f \cdot a$ is idempotent and $f \cdot a > f^2$. Then $f \cdot a > e, f$, and by Theorem 2.50, $\neg(f \cdot a) < f \cdot a$. This, with Theorem 2.10, shows that $f \cdot a$ is the greatest element of the algebra $\boldsymbol{C} := \mathbf{Sg}^{\boldsymbol{A}}\{f \cdot a\}$, and $\neg(f \cdot a)$ is the least element of $\boldsymbol{C}$. Note that $\neg(f \cdot a) < \neg(f^2)$, as $f^2 < f \cdot a$. Now $\boldsymbol{C}$ is rigorously compact, by Corollary 2.46, so $\neg(f^2) \cdot (f \cdot a) = f \cdot a > f^2$. Thus, $\neg(f^2) \cdot (f \cdot a) \nleqslant a$, as $f^2 \nleqslant a$.

Nevertheless, as $\neg(f^2) \cdot f = \neg(f^2)$, we have $(\neg(f^2) \cdot f) \cdot a = \neg(f^2) \cdot a \leqslant a$, because $\neg(f^2) \leqslant e$. This contradicts the associativity of fusion in $\boldsymbol{A}$. Therefore, $A = (\neg(f^2)] \cup [\neg(f^2), f^2] \cup [f^2)$. □

Recall from (2.21) that fusion and meet coincide on the lower bounds of e in any De Morgan monoid. For the algebras in Theorem 2.52, the behaviour of fusion is further constrained as follows.

Theorem 2.53. *Let* $\boldsymbol{A}$ *be a non-idempotent FSI De Morgan monoid, and let* $f \leqslant a, b \in A$. *Then*

$$a \cdot b \ = \begin{cases} f^2 & \text{if } a, b \leqslant f^2; \\ \max_{\leqslant}\{a, b\} & \text{otherwise.} \end{cases}$$

If, moreover, $a < b$ *and* $f^2 \leqslant b$, *then* $a \cdot \neg b = \neg b = b \cdot \neg b$ *and* $b \cdot \neg a = b$.

Proof. If $a, b \leqslant f^2$, then $f^2 \leqslant a \cdot b \leqslant f^4 = f^2$, by (2.14) and Lemma 2.9, so $a \cdot b = f^2$. We may therefore assume (in respect of the first claim) that $a \nleqslant f^2$ or $b \nleqslant f^2$. Then a and b are comparable, by Theorem 2.51. By symmetry, we may assume that $a \leqslant b$ and hence that $b \nleqslant f^2$, so $e < f^2 < b = b^2$, by Theorems 2.50 and 2.51.

If $a = b$, then $a \cdot b = b^2 = b = \max_{\leqslant}\{a, b\}$, so we may assume that $a \neq b$. Thus, $b > a \geqslant f$, and so $\neg b < \neg a \leqslant e < b$.

As b is an idempotent upper bound of $e, f, a, \neg a, \neg b$, Theorem 2.10 shows that b is the greatest element of $\mathbf{Sg}^{\boldsymbol{A}}\{a, b\}$, and $\neg b$ is the least element.

By Corollary 2.46, $\mathbf{Sg}^{\boldsymbol{A}}\{a, b\}$ is rigorously compact. We shall therefore have $a \cdot b = b = \max_{\leqslant}\{a, b\}$, provided that $\neg b \neq a$. This is indeed the case, as we have seen that $\neg a < b$.

Finally, suppose $a < b$ and $f^2 \leqslant b$. Again, Theorems 2.50 and 2.51 show that $\neg b, b$ are the (idempotent) extrema of the algebra $\mathbf{Sg}^{\boldsymbol{A}}\{a, b\}$, whose non-extreme elements include $\neg a, a$, so the remaining claims also follow from the rigorous compactness of $\mathbf{Sg}^{\boldsymbol{A}}\{a, b\}$. □

Theorem 2.54. *Let* $\boldsymbol{A}$ *be a non-idempotent FSI De Morgan monoid. Then* $\boldsymbol{A}/[\neg(f^2))$ *is a totally ordered odd Sugihara monoid. Furthermore,* $e/[\neg(f^2))$ *is the interval* $[\neg(f^2), f^2]$, *and* $a/[\neg(f^2)) = \{a\}$ *for any* $a \in A \setminus [\neg(f^2), f^2]$.

Proof. Let $G := [\neg(f^2))$ and $a \in [\neg(f^2), f^2]$. By Theorem 2.50, $[\neg(f^2), f^2]$ is a subuniverse of $\boldsymbol{A}$, so $e \to a$, $a \to e \in [\neg(f^2), f^2] \subseteq G$, whence $a/G = e/G$. Therefore, $[\neg(f^2), f^2] \subseteq e/G$. In particular, since $f \in [\neg(f^2), f^2]$, we have $e/G = f/G$, so $\boldsymbol{A}/G$ is an odd Sugihara monoid, by Theorem 2.25. Furthermore, by Theorem 2.52, $A \setminus [\neg(f^2), f^2]$ is totally ordered, so $\boldsymbol{A}/G$ is as well.

Let $a \in e/G$. Then $\neg(f^2) \leqslant a$ and $\neg(f^2) \leqslant a \to e$. By the law of residuation $a \cdot \neg(f^2) \leqslant e$, so by (2.1), $\neg(f^2) \cdot f \leqslant \neg a$. Since $[\neg(f^2), f^2]$ is a subuniverse of $\boldsymbol{A}$ with least element $\neg(f^2)$, we have $\neg(f^2) = \neg(f^2) \cdot f \leqslant \neg a$, by Lemma 2.3. So, $a \leqslant f^2$. Therefore $e/G = [\neg(f^2), f^2]$.

Lastly, let $a \in A \setminus [\neg(f^2), f^2]$, and suppose that $a/G = b/G$ for some $b \in A$. Notice that $b \notin [\neg(f^2), f^2]$, since $a \notin e/G = [\neg(f^2), f^2]$.

By involutional symmetry, we may assume that $f^2 < a$ (rather than $a < \neg(f^2)$), because otherwise $f^2 < \neg a$, and from $x/G = \{x\}$ and the double negation law, it follows easily that $(\neg x)/G = \{\neg x\}$.

If $b < \neg(f^2)$, then $b < e < a$, but a/G includes a and b, and is an interval of $\boldsymbol{A}$, so it includes e, whence $a/G = e/G$, a contradiction. Therefore, $f^2 < b$. By Theorem 2.53,

$$a \to b = \neg(a \cdot \neg b) \in \{a, b, \neg a, \neg b\} \subseteq A \setminus [\neg(f^2), f^2].$$

As $a/G = b/G$, we have $\neg(f^2) \leqslant a \to b,\ b \to a$, so $e < a \to b$. Similarly, $e < b \to a$, so $a = b$. Therefore, $a/G = \{a\}$. □

To summarise this discussion, we show how any non-idempotent FSI De Morgan monoid can be viewed as a 'rigorous extension' of its anti-idempotent subalgebra on $[\neg(f^2), f^2]$ by the (idempotent) totally ordered odd Sugihara monoid obtained by factoring out $[\neg(f^2))$. This construction is closely related to constructions in [50], [51], [110] and [111].

Let $\boldsymbol{S}$ be a totally ordered odd Sugihara monoid. For any non-constant basic operation φ of $\boldsymbol{S}$ with arity $n > 0$, and for any $a_1, \dots, a_n \in S$,

$$\text{if } \varphi(a_1, \dots, a_n) = e \text{ then } a_i = e \text{ for some } i \leq n. \tag{2.43}$$

When φ is $\neg$, (2.43) follows from the fact that $\boldsymbol{S}$ is odd, and when φ is $\wedge$ or $\vee$, (2.43) holds because $\boldsymbol{S}$ is totally ordered. When φ is $\cdot$, notice that the odd Sugihara monoid $\boldsymbol{Z}$ satisfies the quasi-equation $x \cdot y \approx e \implies x \approx e$, so since $\mathsf{OSM} = \mathbb{Q}(\boldsymbol{Z})$, $\boldsymbol{S}$ satisfies the same quasi-equation, whence (2.43) holds.

Definition 2.55. The *rigorous extension* of a De Morgan monoid $\boldsymbol{A}$ by a totally ordered odd Sugihara monoid $\boldsymbol{S}$ is the algebra

$$\boldsymbol{S}[\boldsymbol{A}] := \langle (S \setminus \{e^{\boldsymbol{S}}\}) \cup A; \wedge', \vee', \cdot', \neg', e^{\boldsymbol{A}} \rangle$$

with the following properties. Let $\star \in \{\wedge, \vee, \cdot\}$. The operations $\neg'$ and $\star'$ extend those of $\boldsymbol{S}$ and $\boldsymbol{A}$, i.e., for every $s, p \in S \setminus \{e^{\boldsymbol{S}}\}$ and $a, b \in A$,

$$\neg' s := \neg^{\boldsymbol{S}} s, \quad \neg' a := \neg^{\boldsymbol{A}} a, \quad s \star' p := s \star^{\boldsymbol{S}} p, \quad \text{and} \quad a \star' b := a \star^{\boldsymbol{A}} b$$

(whence $\{\neg' s,\ s \star' p\} \subseteq S \setminus \{e^{\boldsymbol{S}}\}$, by (2.43)), while

$$a \star' s := s \star' a := \begin{cases} a & \text{if } e^{\boldsymbol{S}} \star^{\boldsymbol{S}} s = e^{\boldsymbol{S}} \\ e^{\boldsymbol{S}} \star^{\boldsymbol{S}} s & \text{otherwise.}^4 \end{cases}$$

Theorem 2.56. *For any De Morgan monoid $\boldsymbol{A}$ and any totally ordered odd Sugihara monoid $\boldsymbol{S}$, the algebra $\boldsymbol{S}[\boldsymbol{A}]$ is a De Morgan monoid having $\boldsymbol{A}$ as a subalgebra.*

Proof. We can describe the order $\leqslant$ on $\boldsymbol{S}[\boldsymbol{A}]$ as the relation that extends $\leqslant^{\boldsymbol{A}}$ and $\leqslant^{\boldsymbol{S}}|_{S\setminus\{e^{\boldsymbol{S}}\}}$, such that for all $s \in S \setminus \{e^{\boldsymbol{S}}\}$ and $a \in A$ we have

$$(a \leqslant s \;\text{ iff }\; e^{\boldsymbol{S}} \leqslant^{\boldsymbol{S}} s) \;\text{ and }\; (s \leqslant a \;\text{ iff }\; s \leqslant^{\boldsymbol{S}} e^{\boldsymbol{S}}).$$

Since $\boldsymbol{S}$ is totally ordered and $\boldsymbol{A}$ distributive, it is easy to see (in light of Theorem 1.22) that $\leqslant$ is a distributive lattice order.

It is straightforward to verify that $\cdot$ is associative and has identity $e^{\boldsymbol{A}}$, and that (2.1) is satisfied. Here, it is helpful to note that there is no element $s \in S \setminus \{e^{\boldsymbol{S}}\}$ such that $e^{\boldsymbol{S}} \cdot^{\boldsymbol{S}} s = e^{\boldsymbol{S}}$. So, $s \cdot a = a \cdot s = s$ for every $s \in S \setminus \{e\}$ and $a \in A$. □

Theorem 2.57. *If $\boldsymbol{A}$ is an FSI De Morgan monoid, then one of the following mutually exclusive conditions holds:*

(i) *$\boldsymbol{A}$ is a Sugihara monoid, or*

(ii) *$\boldsymbol{A} \cong \boldsymbol{S}[\boldsymbol{A}']$, where $\boldsymbol{A}'$ is the nontrivial anti-idempotent subalgebra of $\boldsymbol{A}$ with universe $[\neg(f^2), f^2]$, and $\boldsymbol{S} = \boldsymbol{A}/[\neg(f^2))$ is a totally ordered odd Sugihara monoid.*

Proof. Let $\boldsymbol{A}$ be an FSI De Morgan monoid in which (i) fails. Then $\boldsymbol{A}$ is non-idempotent with $f < f^2$. Let $G = [\neg(f^2))$ and $\boldsymbol{S} = \boldsymbol{A}/G$. Then $\boldsymbol{S}$ is a totally ordered odd Sugihara monoid, by Theorem 2.54. Also, let $\boldsymbol{A}'$ be the nontrivial anti-idempotent subalgebra of $\boldsymbol{A}$ with universe $[\neg(f^2), f^2]$, which exists by Theorem 2.50. We shall show that $\boldsymbol{A} \cong \boldsymbol{S}[\boldsymbol{A}']$, as witnessed by

$$h\colon a \mapsto \begin{cases} a & \text{if } a \in A' \\ a/G & \text{otherwise.} \end{cases}$$

It follows from Theorem 2.54 that h is a bijection. It remains to show that h is a homomorphism. It is clear that h preserves e and $\neg$. Let $\star \in \{\wedge, \vee, \cdot\}$.

[4]Note that in [50, Thm. 6.1] and [51, Sec. 9.6.1], the operations $\wedge$ and $\vee$ are not well defined; there $s \star a := s \star e^{\boldsymbol{S}}$, but in any nontrivial residuated lattice there is an $a \neq e$ for which $e \vee a = e$ or $e \wedge a = e$. In [111, Sec. 3], it is noted that one needs to add the assumption $a \cdot b \neq e$ whenever $a, b \neq e$, for the algebras constructed in [110, Def. 4.7] to be associative.

If $a, b \in A'$ then $h(a) \star^{\boldsymbol{S}[\boldsymbol{A}']} h(b) = a \star^{\boldsymbol{A}'} b = h(a \star^{\boldsymbol{A}} b)$, since $\boldsymbol{A'}$ is a subalgebra of $\boldsymbol{A}$ and of $\boldsymbol{S}[\boldsymbol{A}']$. If $a, b \in A \setminus A'$, then $a \star^{\boldsymbol{A}} b \notin A'$, because otherwise $a/G \star^{\boldsymbol{S}} b/G = e/G$, whence $a/G = e/G$ or $b/G = e/G$, by (2.43), contradicting the fact that $a/G = \{a\}$ and $b/G = \{b\}$ (Theorem 2.54). So,

$$h(a) \star^{\boldsymbol{S}[\boldsymbol{A}']} h(b) = a/G \star^{\boldsymbol{S}[\boldsymbol{A}']} b/G = a/G \star^{\boldsymbol{S}} b/G = (a \star^{\boldsymbol{A}} b)/G = h(a \star^{\boldsymbol{A}} b).$$

Now, let $a \in A'$ and $b \in A \setminus A'$. If $e/G \wedge^{\boldsymbol{S}} b/G = e/G$ then $f^2 < b$, by Theorems 2.52 and 2.54, so $h(a) \wedge^{\boldsymbol{S}[\boldsymbol{A}']} h(b) = a = h(a \wedge^{\boldsymbol{A}} b)$. If $e/G \wedge^{\boldsymbol{S}} b/G \neq e/G$ then $e/G \wedge^{\boldsymbol{S}} b/G = b/G$, since $\boldsymbol{S}$ is totally ordered. Then $b < \neg(f^2)$, so $h(a) \wedge^{\boldsymbol{S}[\boldsymbol{A}']} h(b) = b/G = h(a \wedge^{\boldsymbol{A}} b)$. Similarly, $h(a) \vee^{\boldsymbol{S}[\boldsymbol{A}']} h(b) = h(a \vee^{\boldsymbol{A}} b)$.

It remains to show that $h(a) \cdot^{\boldsymbol{S}[\boldsymbol{A}']} h(b) = h(a \cdot^{\boldsymbol{A}} b)$. Note that $e/G \cdot^{\boldsymbol{S}} b/G = e/G \cdot^{\boldsymbol{S}} b/G = b/G$, so we must show that $a \cdot^{\boldsymbol{A}} b = b$. This follows, as in the proof of Theorem 2.53, from the fact that $\mathbf{Sg}^{\boldsymbol{A}}\{a, b\}$ is rigorously compact with idempotent extrema b and $\neg b$. □

This largely reduces the study of irreducible De Morgan monoids to the anti-idempotent case. We end this section by illuminating some of the properties of rigorous extensions.

Theorem 2.58. *Let $\{\boldsymbol{A}, \boldsymbol{B}\} \cup \{\boldsymbol{A}_i : i \in I\}$ be a family of De Morgan monoids, and $\{\boldsymbol{S}\} \cup \{\boldsymbol{S}_i : i \in I\}$ a family of totally ordered odd Sugihara monoids, for some set I.*

(i) *If $h \colon \boldsymbol{A} \to \boldsymbol{B}$ is a homomorphism, then the map*

$$h' \colon x \mapsto \begin{cases} h(x) & \text{if } x \in A \\ x & \text{otherwise,} \end{cases}$$

is a homomorphism from $\boldsymbol{S}[\boldsymbol{A}]$ to $\boldsymbol{S}[\boldsymbol{B}]$ which extends h.

(ii) *If $\boldsymbol{P}$ is a subalgebra of $\boldsymbol{S}$ and $\boldsymbol{B}$ a subalgebra of $\boldsymbol{A}$, then $\boldsymbol{P}[\boldsymbol{B}]$ is a subalgebra of $\boldsymbol{S}[\boldsymbol{A}]$.*

(iii) *$\prod_{i \in I} (\boldsymbol{S}_i[\boldsymbol{A}_i]) / \mathcal{U} \cong \left(\prod_{i \in I} \boldsymbol{S}_i / \mathcal{U}\right) \left[\prod_{i \in I} \boldsymbol{A}_i / \mathcal{U}\right]$ for every ultrafilter $\mathcal{U}$ over I.*

Proof. For (i), we only show preservation of the binary basic operations with mixed arguments from $S \setminus \{e^{\boldsymbol{S}}\}$ and A, since the other cases are trivial. Let $s \in S \setminus \{e^{\boldsymbol{S}}\}$ and $a \in A$. If $s < a$ then $h'(s \wedge a) = h'(s) = s = h'(s) \wedge h'(a)$ and $h'(s \vee a) = h'(a) = h(a) = s \vee h(a) = h'(s) \vee h'(a)$. When $a < s$ the argument is symmetrical. Also,

$$h'(s \cdot a) = h'(s) = s = s \cdot h(a) = h'(s) \cdot h'(a).$$

Item (ii) follows from the fact that if $p \in P$ and $b \in B$, then for any $\star \in \{\wedge, \vee, \cdot\}$ we get $\{\neg p, \neg b, p \star b, b \star p\} \subseteq \{b, \neg b, p, \neg p\} \subseteq P[B]$.

For (iii), we define a map $h\colon \prod_{i\in I} \boldsymbol{S}_i[\boldsymbol{A}_i] \to \left(\prod_{i\in I} \boldsymbol{S}_i/\mathcal{U}\right)\left[\prod_{i\in I} \boldsymbol{A}_i/\mathcal{U}\right]$ in the following way. Let $\vec{a} \in \prod_{i\in I} S_i[A_i]$. Define $I_{\vec{a}} := \{i \in I \colon a_i \in A_i\}$. When $I_{\vec{a}} \in \mathcal{U}$, we let $h(\vec{a}) = \vec{b}/\mathcal{U} \in \prod_{i\in I} A_i/\mathcal{U}$ where

$$b_i = a_i \text{ if } a_i \in A_i, \text{ and } b_i = e^{\boldsymbol{A}_i} \text{ otherwise.}$$

When $I_{\vec{a}} \notin \mathcal{U}$, then its complement $I^c_{\vec{a}} = \{i \in I \colon a_i \in S_i \setminus \{e^{\boldsymbol{S}_i}\}\} \in \mathcal{U}$, since $\mathcal{U}$ is an ultrafilter. In this case we define $h(\vec{a}) = \vec{s}/\mathcal{U} \in (\prod_{i\in I} S_i/\mathcal{U}) \setminus \{e\}$ where

$$s_i = a_i \text{ if } a_i \in S_i, \text{ and } s_i = e^{\boldsymbol{S}_i} \text{ otherwise.}$$

It can be verified that h is a surjective homomorphism whose kernel is the congruence on $\prod_{i\in I} \boldsymbol{S}_i[\boldsymbol{A}_i]$ associated with $\mathcal{U}$. By the Homomorphism Theorem 1.1, $\prod_{i\in I} (\boldsymbol{S}_i[\boldsymbol{A}_i])/\mathcal{U} \cong \left(\prod_{i\in I} \boldsymbol{S}_i/\mathcal{U}\right)\left[\prod_{i\in I} \boldsymbol{A}_i/\mathcal{U}\right]$. □

Corollary 2.59. *Let $\boldsymbol{A}$ be a De Morgan monoid and $\boldsymbol{S}$ a totally ordered odd Sugihara monoid. If $\boldsymbol{C} \in \mathbb{HSP}_{\mathbb{U}}(\boldsymbol{A})$, then $\boldsymbol{S}[\boldsymbol{C}] \in \mathbb{HSP}_{\mathbb{U}}(\boldsymbol{S}[\boldsymbol{A}])$.*

Proof. Suppose that $h\colon \boldsymbol{B} \to \boldsymbol{C}$ is a surjective homomorphism, with $\boldsymbol{B}$ a subalgebra of $\prod_{i\in I} \boldsymbol{A}/\mathcal{U}$ for some ultrafilter $\mathcal{U}$ over a set I. By Theorem 2.58(i), h can be extended to a surjective homomorphism h' from $\boldsymbol{S}[\boldsymbol{B}]$ to $\boldsymbol{S}[\boldsymbol{C}]$. Recall that any algebra embeds into each of its ultrapowers. In particular, we may identify $\boldsymbol{S}$ with a subalgebra of $\prod_{i\in I} \boldsymbol{S}/\mathcal{U}$. Therefore, by Theorem 2.58(ii), $\boldsymbol{S}[\boldsymbol{B}]$ is a subalgebra of $\left(\prod_{i\in I} \boldsymbol{S}/\mathcal{U}\right)[\prod_{i\in I} \boldsymbol{A}/\mathcal{U}]$. Lastly, by Theorem 2.58(iii), $\left(\prod_{i\in I} \boldsymbol{S}/\mathcal{U}\right)[\prod_{i\in I} \boldsymbol{A}/\mathcal{U}] \cong \prod_{i\in I} \boldsymbol{S}[\boldsymbol{A}]/\mathcal{U}$. So, $\boldsymbol{S}[\boldsymbol{C}] \in \mathbb{HSP}_{\mathbb{U}}(\boldsymbol{S}[\boldsymbol{A}])$. □

Chapter 3

Crystalline and negatively generated algebras

This chapter assembles a number of constructions and representation theorems that will be used throughout the rest of the thesis.

There are, up to isomorphism, only three simple 0-generated De Morgan monoids, namely the two-element Boolean algebra, and two four-element De Morgan monoids called $\boldsymbol{C}_4$ and $\boldsymbol{D}_4$; see Theorem 3.1. Of these, $\boldsymbol{C}_4$ is distinctive, in view of a result of Slaney [130]: $\boldsymbol{C}_4$ is the only 0-generated nontrivial algebra onto which finitely subdirectly irreducible De Morgan monoids may be mapped by non-injective homomorphisms. Algebras that *do* map onto $\boldsymbol{C}_4$ are called *crystalline*. We demonstrate in Section 3.1 that there is a largest variety U of crystalline De Morgan monoids, as well as a largest subvariety M of DMM such that $\boldsymbol{C}_4$ is a *retract* of every nontrivial member of M. Thus, $\mathbb{V}(\boldsymbol{C}_4) \subseteq \mathsf{M} \subseteq \mathsf{U}$. We furnish U and M with finite equational axiomatizations; each has an undecidable equational theory and uncountably many subvarieties.

In Section 3.2, we provide representation theorems for the members of U and M (Corollaries 3.25 and 3.27), involving a 'skew reflection' construction of Slaney [131].

We then present, in Section 3.3, further results of Slaney [129, 130], showing that the free 0-generated De Morgan monoid is finite, and that there are only seven non-isomorphic subdirectly irreducible 0-generated De Morgan monoids. This free algebra and all the non-simple 0-generated SI algebras are skew reflections.

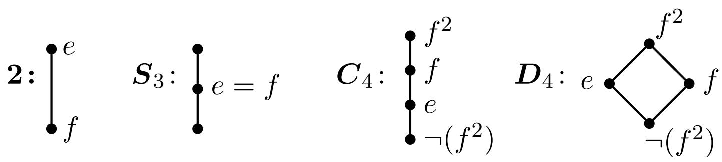
The skew reflection construction generalizes an older (non-skew) *reflection* construction, which is essentially due to Meyer [97]. It was originally used to add an involution to a Dunn monoid, and it may be used to map varieties of Dunn monoids to varieties of De Morgan monoids, in such a way that several important properties are preserved. Section 3.4 therefore focusses on the reflection construction.

In Section 3.5 all the tools that have been developed in this and the previous chapter are applied to provide a representation theorem for negatively generated semilinear De Morgan monoids.

3.1 Crystalline algebras

In this section we introduce two important quasivarieties of De Morgan monoids, which will be called W and N. Their definitions will come after Theorem 3.4, which will make their significance clear. Each of them has a largest subvariety. These varieties are called U and M, respectively, and they contain only anti-idempotent algebras. In the next section (3.2) we shall give representation theorems for the algebras in U and M.

We depict below the two-element Boolean algebra $\mathbf{2}$ ($= \boldsymbol{S}_2$), the three-element Sugihara monoid $\boldsymbol{S}_3$, and two 0-generated four-element De Morgan monoids, $\boldsymbol{C}_4$ and $\boldsymbol{D}_4$. In each case, the labeled Hasse diagram determines the structure, in view of Lemma 2.3, Theorem 2.45 and the definitions. That $\boldsymbol{C}_4$ and $\boldsymbol{D}_4$ are indeed De Morgan monoids was noted long ago in the relevance logic literature, e.g., [98, 99]. All four algebras are simple, by Lemma 2.16(iv).



The next theorem is implicit in findings of Slaney [129, 130], which will be summarized in Section 3.3, but it is convenient here to give a self-contained proof.

Theorem 3.1. *Let* $\boldsymbol{A}$ *be a simple* 0*-generated De Morgan monoid. Then* $\boldsymbol{A} \cong \mathbf{2}$ *or* $\boldsymbol{A} \cong \boldsymbol{C}_4$ *or* $\boldsymbol{A} \cong \boldsymbol{D}_4$.

Proof. Because $\boldsymbol{A}$ is simple (hence nontrivial) and 0-generated, $\{e\}$ is not a subuniverse of $\boldsymbol{A}$, so $e \neq f$ and e has just one strict lower bound in $\boldsymbol{A}$ (Lemma 2.16(iv)). Suppose $\boldsymbol{A} \ncong \mathbf{2}$. As $\mathbf{2}$ is finite, simple and 0-generated, and $\mathsf{BA} = \mathbb{V}(\mathbf{2})$, every FSI (and hence every simple) Boolean algebra is isomorphic to $\mathbf{2}$, by Jónsson's Theorem 1.23. This, together with Lemma 2.31, shows that $\boldsymbol{A}$ is not integral. Equivalently, f is not the least element of $\boldsymbol{A}$, so $f \nleqslant e$. Then by Theorem 2.11, $\boldsymbol{A}$ is not idempotent and $f < f^2$, hence $\neg(f^2) < e$, so $\neg(f^2)$ is the least element of $\boldsymbol{A}$, i.e., f^2 is the greatest element. Consequently, $a \cdot \neg(f^2) = \neg(f^2)$ for all $a \in A$, by Lemma 2.3, and $a \cdot f^2 = f^2$ whenever $\neg(f^2) \neq a \in A$, by Theorem 2.45.

There are two possibilities for the order: $e < f$ or $e \not\leqslant f$. If $e \not\leqslant f$, then $e \wedge f < e$, whence $e \wedge f$ is the extremum $\neg(f^2)$ and, by De Morgan's laws, $e \vee f = f^2$. Otherwise, $\neg(f^2) < e < f < f^2$. Either way, $\{\neg(f^2), e, f, f^2\}$ is the universe of a four-element subalgebra of $\boldsymbol{A}$, having no proper subalgebra of its own, so $A = \{\neg(f^2), e, f, f^2\}$, as $\boldsymbol{A}$ is 0-generated. Thus, $\boldsymbol{A} \cong \boldsymbol{C}_4$ if $e < f$, and $\boldsymbol{A} \cong \boldsymbol{D}_4$ if $e \not\leqslant f$. □

In what follows, some features of $\boldsymbol{C}_4$ will be important.

Lemma 3.2. *Let $\boldsymbol{A}$ be a nontrivial square-increasing IRL, and K a variety of square-increasing IRLs.*

(i) *If $e \leqslant f$ and $a \leqslant f^2$ for all $a \in A$, then $e < f$.*

(ii) *If $e < f$ in $\boldsymbol{A}$, then $\boldsymbol{C}_4$ can be embedded into $\boldsymbol{A}$.*

(iii) *If $\boldsymbol{A}$ is simple and $\boldsymbol{C}_4$ or $\boldsymbol{D}_4$ can be embedded into $\boldsymbol{A}$, then $\boldsymbol{A}$ is anti-idempotent.*

(iv) *If $\boldsymbol{C}_4$ can be embedded into every SI member of K, then K consists of anti-idempotent algebras and satisfies $e \leqslant f$.*

(v) *If $\boldsymbol{D}_4$ can be embedded into every SI member of K, then K consists of anti-idempotent algebras.*

Proof. (i) Suppose $\boldsymbol{A}$ satisfies $e \leqslant f$ and $x \leqslant f^2$. Then $\boldsymbol{A}$ is not idempotent, by Corollary 2.14, so $f \neq e$, by Theorem 2.11, i.e., $e < f$.

(ii) Suppose $e < f$ in $\boldsymbol{A}$. Then $f < f^2$, by Theorem 2.11, i.e., $\neg(f^2) < e$. Thus, $\{\neg(f^2), e, f, f^2\}$ is closed under $\wedge, \vee$ and $\neg$, and $\neg(f^2)$ is idempotent, by (2.21). By Lemma 2.9, f^2 is an idempotent upper bound of e, so $f^2 \cdot \neg(f^2) = \neg(f^2)$, by Lemma 2.7. Closure of $\{\neg(f^2), e, f, f^2\}$ under fusion follows from these observations and (2.14), so $\boldsymbol{C}_4$ embeds into $\boldsymbol{A}$.

(iii) follows from Lemma 2.16(iv), because $\neg(f^2) < e$ in $\boldsymbol{C}_4$ and in $\boldsymbol{D}_4$.

(iv) Suppose $\boldsymbol{C}_4$ embeds into every SI member of K. Then K satisfies $e \leqslant f$, as $\boldsymbol{C}_4$ does. Now let $\boldsymbol{B} \in \mathsf{K}$ be nontrivial. Then $\boldsymbol{B} \in \mathbb{IP}_{\mathbb{S}}\{\boldsymbol{B}_i : i \in I\}$ for suitable SI algebras $\boldsymbol{B}_i \in \mathsf{K}$, by Birkhoff's Subdirect Decomposition Theorem 1.3. As $\boldsymbol{C}_4$ embeds into each $\boldsymbol{B}_i$, it embeds diagonally into $\prod_{i \in I} \boldsymbol{B}_i$ (via the map $a \mapsto \langle a, a, a, \dots \rangle$), and therefore into $\boldsymbol{B}$, because $\boldsymbol{C}_4$ is 0-generated. Thus, no nontrivial $\boldsymbol{B} \in \mathsf{K}$ is idempotent, and so K satisfies $x \leqslant f^2$, by Corollary 2.14.

The proof of (v) is similar. □

The next lemma generalizes [130, Thms. 2, 3] (where it was confined to FSI De Morgan monoids).

Lemma 3.3. *Let* $\boldsymbol{A}$ *be a rigorously compact IRL.*

(i) *There is at most one homomorphism from* $\boldsymbol{A}$ *into* $\boldsymbol{C}_4$.

(ii) *If there is a homomorphism from* $\boldsymbol{A}$ *to* $\boldsymbol{C}_4$*, then* $\neg(f^2) \leqslant a \leqslant f^2$ *for all* $a \in A$.

Proof. Let $\bot, \top$ be the extrema of $\boldsymbol{A}$. Suppose $h_1, h_2 \colon \boldsymbol{A} \to \boldsymbol{C}_4$ are homomorphisms, and note that they are surjective, because $\boldsymbol{C}_4$ is 0-generated. For each $i \in \{1, 2\}$, as h_i is isotone and preserves $\cdot, \neg, e$, we have

$$h_i(f^2) = f^2 = h_i(\top) \text{ and } h_i(\neg(f^2)) = \neg(f^2) = h_i(\bot),$$

so by Lemma 2.6(i), $f^2 = \top$ and $\neg(f^2) = \bot$ (proving (ii)) and

$$h_i^{-1}[\{f^2\}] = \{f^2\} \text{ and } h_i^{-1}[\{\neg(f^2)\}] = \{\neg(f^2)\}. \tag{3.1}$$

Therefore, if $h_1 \neq h_2$, then $h_1(a) = e$ and $h_2(a) = f$ for some $a \in A$. In that case, $h_2(a^2) = f^2$, so $a^2 = f^2$ (by (3.1)), whence $h_1(a^2) = f^2$, contradicting the fact that $h_1(a^2) = (h_1(a))^2 = e^2 = e$. Thus, $h_1 = h_2$, proving (i). □

Theorem 3.4 (Slaney [130, Thm. 1])**.** *Let* $h \colon \boldsymbol{A} \to \boldsymbol{B}$ *be a homomorphism, where* $\boldsymbol{A}$ *is an FSI De Morgan monoid, and* $\boldsymbol{B}$ *is nontrivial and* 0*-generated. Then* h *is an isomorphism or* $\boldsymbol{B} \cong \boldsymbol{C}_4$.

Proof. As $\boldsymbol{B}$ is 0-generated, h is surjective. Suppose h is not an isomorphism. By the remarks preceding Lemma 2.3, $h(a) = e$ for some $a \in A$ with $a < e$. By Theorem 2.43, $a \leqslant f$, so $h(a) \leqslant h(f)$, i.e., $e \leqslant f$ in $\boldsymbol{B}$. As $\boldsymbol{B}$ is 0-generated but not trivial, it cannot satisfy $e = f$, so $e < f$ in $\boldsymbol{B}$. Then $\boldsymbol{C}_4$ embeds into $\boldsymbol{B}$, by Lemma 3.2(ii), so $\boldsymbol{B} \cong \boldsymbol{C}_4$, again because $\boldsymbol{B}$ is 0-generated. □

Generalizing the usage of [130], we say that an IRL $\boldsymbol{A}$ is *crystalline* if there is a homomorphism $h \colon \boldsymbol{A} \to \boldsymbol{C}_4$ (in which case h is surjective).[1]

Recall that an algebra $\boldsymbol{A}$ is said to be a *retract* of an algebra $\boldsymbol{B}$ if there are homomorphisms $g \colon \boldsymbol{A} \to \boldsymbol{B}$ and $h \colon \boldsymbol{B} \to \boldsymbol{A}$ such that $h \circ g$ is the identity function id_A on A. This forces g to be injective and h surjective; we refer to h as a *retraction* (of $\boldsymbol{B}$ onto $\boldsymbol{A}$). The composite of two retractions, when defined, is clearly still a retraction.

Theorem 3.4 motivates the following definitions.

[1] For the sake of Theorem 3.9, we have dropped the requirement in [130] that crystalline algebras be FSI.

Definition 3.5.

(i) $\mathsf{W} := \{\boldsymbol{A} \in \mathsf{DMM} : |A| = 1 \text{ or } \boldsymbol{A} \text{ is crystalline}\}$;

(ii) $\mathsf{N} := \{\boldsymbol{A} \in \mathsf{DMM} : |A| = 1 \text{ or } \boldsymbol{C}_4 \text{ is a retract of } \boldsymbol{A}\} \subseteq \mathsf{W}$.

By Lemma 3.3(ii), the rigorously compact algebras in W are anti-idempotent.

Remark 3.6. Given similar algebras $\boldsymbol{A}$ and $\boldsymbol{B}$, the first canonical projection $\pi_1 \colon \boldsymbol{A} \times \boldsymbol{B} \to \boldsymbol{A}$ is a retraction iff there exists a homomorphism $f \colon \boldsymbol{A} \to \boldsymbol{B}$. (Sufficiency: as id_A and f are homomorphisms, so is the function g from $\boldsymbol{A}$ to $\boldsymbol{A} \times \boldsymbol{B}$ defined by $a \mapsto \langle a, f(a)\rangle$, and clearly $\pi_1 \circ g = \mathrm{id}_A$.) Consequently, if an algebra $\boldsymbol{C}$ is a retract of every member of a class K, then $\boldsymbol{D}$ is a retract of $\boldsymbol{D} \times \boldsymbol{E}$ for all $\boldsymbol{D}, \boldsymbol{E} \in \mathsf{K}$, because there is always a composite homomorphism from $\boldsymbol{D}$ to $\boldsymbol{E}$ (whose image is isomorphic to $\boldsymbol{C}$).

It follows that $\boldsymbol{A}$ is a retract of $\boldsymbol{A} \times \boldsymbol{B}$ for all nontrivial $\boldsymbol{A}, \boldsymbol{B} \in \mathsf{N}$.

Remark 3.7. A 0-generated algebra $\boldsymbol{A}$ is a retract of an algebra $\boldsymbol{B}$ if there exist homomorphisms $g \colon \boldsymbol{A} \to \boldsymbol{B}$ and $h \colon \boldsymbol{B} \to \boldsymbol{A}$. For in this case, every element of A has the form $\alpha^{\boldsymbol{A}}(c_1, \dots, c_n)$ for some term α and some *distinguished* elements $c_i \in A$, whence $h \circ g = \mathrm{id}_A$, because homomorphisms preserve distinguished elements (and respect terms).

Lemma 3.8. *Let K be a variety of finite type, and let $\boldsymbol{A} \in \mathsf{K}$ be finite, simple and 0-generated. Then the following conditions are equivalent.*

(i) *$\boldsymbol{A}$ is a retract of every nontrivial member of K.*

(ii) *Every simple algebra in K is isomorphic to $\boldsymbol{A}$ and embeds into every nontrivial member of K.*

Proof. (i) $\Rightarrow$ (ii): For each simple $\boldsymbol{C} \in \mathsf{K}$, there is a homomorphism h from $\boldsymbol{C}$ onto $\boldsymbol{A}$, by (i), and h must be an isomorphism (as $\boldsymbol{A}$ is nontrivial and $\boldsymbol{C}$ is simple). Thus, the embedding claim also follows from (i).

(ii) $\Rightarrow$ (i): By (ii) and Theorem 1.15, $\boldsymbol{A}$ is a homomorphic image of every finitely generated nontrivial member of K. Consider an arbitrary nontrivial algebra $\boldsymbol{B} \in \mathsf{K}$. By (ii), $\boldsymbol{A} \in \mathbb{IS}(\boldsymbol{B})$. Like any nontrivial algebra, $\boldsymbol{B}$ embeds into an ultraproduct $\boldsymbol{U}$ of finitely generated nontrivial subalgebras $\boldsymbol{B}_i$ of $\boldsymbol{B}$ (Theorem 1.4). As $\boldsymbol{A} \in \mathbb{H}(\boldsymbol{B}_i)$ for all i, and as $\mathbb{P}_{\mathbb{U}}\mathbb{H}(\mathsf{L}) \subseteq \mathbb{HP}_{\mathbb{U}}(\mathsf{L})$ for any class L of similar algebras, there is a homomorphism h from $\boldsymbol{U}$ onto an ultrapower of $\boldsymbol{A}$. But $\boldsymbol{A}$, being finite, is isomorphic to all of its ultrapowers, so h restricts to a homomorphism from $\boldsymbol{B}$ into $\boldsymbol{A}$. Therefore, $\boldsymbol{A}$ is a retract of $\boldsymbol{B}$, by Remark 3.7. $\square$

Theorem 3.9. W *and* N *are quasivarieties.*

Proof. As W and N are isomorphically closed, we must show that they are closed under $\mathbb{S}$, $\mathbb{P}$ and $\mathbb{P}_\mathbb{U}$, bearing Remark 3.7 in mind. If $\boldsymbol{B} \in \mathbb{S}(\boldsymbol{A})$ and $h\colon \boldsymbol{A} \to \boldsymbol{C}_4$ is a homomorphism, then so is $h|_B\colon \boldsymbol{B} \to \boldsymbol{C}_4$, while any embedding $\boldsymbol{C}_4 \to \boldsymbol{A}$ maps into $\boldsymbol{B}$, as $\boldsymbol{C}_4$ is 0-generated. Thus, W and N are closed under $\mathbb{S}$. Let $\{\boldsymbol{A}_i \colon i \in I\}$ be a subfamily of W, where, without loss of generality, $I \neq \emptyset$. For any $j \in I$, the projection $\prod_{i \in I} \boldsymbol{A}_i \to \boldsymbol{A}_j$ can be composed with a homomorphism $\boldsymbol{A}_j \to \boldsymbol{C}_4$, so $\prod_{i \in I} \boldsymbol{A}_i \in \mathsf{W}$. If, moreover, $\boldsymbol{A}_i \in \mathsf{N}$ for all i, then $\boldsymbol{C}_4$ embeds diagonally into $\prod_{i \in I} \boldsymbol{A}_i$, whence $\prod_{i \in I} \boldsymbol{A}_i \in \mathsf{N}$. Every ultraproduct of $\{\boldsymbol{A}_i \colon i \in I\}$ can be mapped into $\boldsymbol{C}_4$, as in the proof of Lemma 3.8 ((ii) $\Rightarrow$ (i)). Also, as $\boldsymbol{C}_4$ is finite and of finite type, the property of having a subalgebra isomorphic to $\boldsymbol{C}_4$ is first order-definable and therefore persists in ultraproducts. Thus, W and N are closed under $\mathbb{P}$ and $\mathbb{P}_\mathbb{U}$. ☐

Nevertheless, W and N are not varieties, i.e., they are not closed under $\mathbb{H}$. To see this, consider any simple De Morgan monoid $\boldsymbol{A}$ of which $\boldsymbol{C}_4$ is a proper subalgebra, and let $\boldsymbol{B} = \boldsymbol{C}_4 \times \boldsymbol{A}$. Then $\boldsymbol{B} \in \mathsf{N}$, by Remark 3.6. Now $\boldsymbol{A} \in \mathbb{H}(\boldsymbol{B})$ but $\boldsymbol{A} \notin \mathsf{W}$, because $\boldsymbol{A}$ is simple and not isomorphic to $\boldsymbol{C}_4$. Concrete examples of finite simple 1-generated De Morgan monoids having $\boldsymbol{C}_4$ as sole proper subalgebra are given in Section 4.2.

The examples in Section 4.2 show that even the semilinear anti-idempotent algebras in W or N do not form a variety. Note that N contains (semilinear) algebras that are not anti-idempotent. For instance, $\boldsymbol{C}_4 \times \boldsymbol{S}_3[\boldsymbol{C}_4] \in \mathsf{N}$ does not satisfy $x \leqslant f^2$, where $\boldsymbol{S}_3[\boldsymbol{C}_4]$ is as in Definition 2.55, i.e., it is the rigorously compact extension of $\boldsymbol{C}_4$ by new extrema.

As W and N are not varieties, it is not obvious that either of them possesses a largest subvariety, but we shall show that both do. Purely equational axioms will be needed in the proof, and the opaque postulate (3.4), which abbreviates an equation, is introduced below for that reason. The following convention helps to eliminate some burdensome notation.

Convention 3.10. In an anti-idempotent IRL, we define

$$\mathit{1} := f^2 \text{ and } \mathit{0} := \neg \mathit{1} = \neg(f^2).$$

(These abbreviations will be used when they enhance readability, rather than always. The typeface distinguishes them from standard uses of 0, 1.)

Definition 3.11. We denote by U the variety of De Morgan monoids satisfying

$$x^2 \vee (\neg x)^2 = \mathit{1} \tag{3.2}$$

$$\mathit{1} \to (x \vee y) \leqslant (\mathit{1} \to x) \vee (\mathit{1} \to y) \tag{3.3}$$

$$\mathit{1} \cdot x \cdot y \cdot q(x) \cdot q(y) \leqslant q(x \cdot y) \wedge q(x \vee y) \wedge q(x \to y) \wedge (\mathit{1} \cdot (x \to y)), \tag{3.4}$$

where $q(x) := \mathit{1} \to (\neg x)^2$. (Note that U consists of anti-idempotent algebras, by (3.2), so our use of the symbol $\mathit{1}$ in this definition is justified.)

Lemma 3.12. *Every rigorously compact member of* W *belongs to* U.

Proof. Let $\boldsymbol{A} \in \mathsf{W}$ be rigorously compact. We may assume that $\boldsymbol{A}$ is nontrivial, so there is a (surjective) homomorphism from $\boldsymbol{A}$ to $\boldsymbol{C}_4$. Because $\boldsymbol{C}_4$ satisfies (3.2),

$$[\mathit{1} \to (x \vee y)] \to [(\mathit{1} \to x) \vee (\mathit{1} \to y)] = \mathit{1} \quad \text{and}$$
$$[\mathit{1} \cdot x \cdot y \cdot q(x) \cdot q(y)] \to [q(x \cdot y) \wedge q(x \vee y) \wedge q(x \to y) \wedge (\mathit{1} \cdot (x \to y))] = \mathit{1},$$

it follows from Lemma 2.6(i) that $\boldsymbol{A}$ satisfies the same laws.[2] Then $\boldsymbol{A}$ satisfies (3.3) and (3.4), by (2.15), because $e \leqslant \mathit{1}$. Thus, $\boldsymbol{A} \in \mathsf{U}$. □

Corollary 3.13. *If* $\boldsymbol{A} \in \mathsf{W}$ *and* $\boldsymbol{A}$ *is FSI, then* $\boldsymbol{A} \in \mathsf{U}$.

Proof. Let $\boldsymbol{A} \in \mathsf{W}_{FSI}$ be nontrivial. Since equations have only a finite number of variables, one can verify that $\boldsymbol{A}$ satisfies every axiom of U, by showing that every finitely generated subalgebra of $\boldsymbol{A}$ belongs to U. Let $\boldsymbol{B}$ be any finitely generated subalgebra of $\boldsymbol{A}$. Recall that $\boldsymbol{B}$ is FSI (by Lemma 2.16(i)). It follows that $\boldsymbol{B}$ is rigorously compact, by Corollary 2.46. Also, $\boldsymbol{B} \in \mathsf{W}$, since W is a quasivariety. Consequently, $\boldsymbol{B} \in \mathsf{U}$, by Lemma 3.12, as required. □

Theorem 3.14. U *is the largest subvariety of* W, *i.e.,* U *is the largest variety of crystalline (or trivial) De Morgan monoids.*

Proof. To see that $\mathsf{U} \subseteq \mathsf{W}$, let $\boldsymbol{A} \in \mathsf{U}$ be SI. It suffices to show that $\boldsymbol{A} \in \mathsf{W}$, because W, like any quasivariety, is closed under $\mathbb{IP}_{\mathbb{S}}$. Now $\boldsymbol{A}$ is nontrivial and bounded by $\mathit{0}, \mathit{1}$ (because of (3.2)), so $\mathit{0} < e \leqslant \mathit{1}$ and $\boldsymbol{A}$ is rigorously compact, by Theorem 2.45. It follows from (3.3), Lemma 2.16(ii) and (2.15) that $\mathit{1}$ is join-irreducible (whence $\mathit{0}$ is meet-irreducible) in $\boldsymbol{A}$. Let

$$B = \{a \in A : \, a \neq \mathit{0} \text{ and } (\neg a)^2 = \mathit{1}\} \text{ and } B' = \{a \in A : \neg a \in B\}.$$

[2] To validate the last equation in $\boldsymbol{C}_4$ quickly, note that the premise of the implication is $\mathit{0}$ unless x and y are both e, in which case both the premise and the conclusion will be $\mathit{1}$.

Then $e \in B$ (by definition of 1) and $1 \notin B$ (as $0^2 = 0 \neq 1$), so $e < 1$.

We claim that B is closed under the operations $\cdot, \rightarrow, \wedge, \vee$ of $\boldsymbol{A}$. Indeed, let $b, c \in B$, so $0 < b, c \in A$ and $(\neg b)^2 = 1 = (\neg c)^2$, i.e., $q(b) = 1 = q(c)$. Then $b \wedge c \neq 0$ and $(\neg(b \wedge c))^2 = 1$ (because $(\neg(b \wedge c))^2 \geqslant (\neg b)^2$), so $b \wedge c \in B$. Clearly, $b \vee c \neq 0$. Also, $b \cdot c \neq 0$, by (2.20), and $1 \cdot b \cdot c \cdot q(b) \cdot q(c) = 1$, by rigorous compactness. Then, by (3.4), each of $q(b \cdot c), q(b \vee c), q(b \rightarrow c)$ and $1 \cdot (b \rightarrow c)$ is 1. Thus, $1 = (\neg(b \cdot c))^2 = (\neg(b \vee c))^2 = (\neg(b \rightarrow c))^2$, again by rigorous compactness, and $b \rightarrow c \neq 0$. This shows that $b \cdot c$, $b \vee c$, $b \rightarrow c \in B$, as claimed.

Let $a \in A \setminus \{0, 1\}$. Since 1 is join-irreducible, (3.2) shows that $a \in B$ or $\neg a \in B$, i.e., $a \in B \cup B'$. Suppose $a \in B \cap B'$, i.e., $a, \neg a \in B$. Then $\neg a \rightarrow a = \neg((\neg a)^2) = \neg 1$ (as $a \in B$) $= 0$, so $(\neg a \rightarrow a)^2 = 0 \neq 1$, so $\neg a \rightarrow a \notin B$, contradicting the fact that B is closed under $\rightarrow$. Therefore, A is the disjoint union of B, B', $\{0\}$ and $\{1\}$.

Suppose $b, c \in B$, with $\neg c \leqslant b$. Then $b \neq 1$, so $\neg b \neq 0$ and $b^2 \geqslant (\neg c)^2 = 1$, so $b^2 = 1$, hence $\neg b \in B$, i.e., $b \in B \cap B' = \emptyset$, a contradiction. Thus, no element of B has a lower bound in B'. This, together with the meet- [resp. join-] irreducibility of 0 [resp. 1], shows that $b \wedge d \in B$ and $b \vee d \in B'$ for all $b \in B$ and $d \in B'$.

Let $h \colon A \rightarrow C_4$ be the function such that $h(0) = 0$ and $h(1) = 1$ and $h(b) = e$ and $h(\neg b) = f$ for all $b \in B$. It follows readily from the above conclusions that h is a homomorphism from $\boldsymbol{A}$ to $\boldsymbol{C}_4$, so $\boldsymbol{A} \in \mathsf{W}$, as required.

Finally, let K be a subvariety of W. The FSI algebras in K belong to U, by Corollary 3.13. Thus, $\mathsf{K} \subseteq \mathsf{U}$. □

Remark 3.15. In $\boldsymbol{C}_4$, we have $f \rightarrow a = 0$ iff $a \in \{0, e\}$, while $a \rightarrow e = 0$ iff $a \in \{f, 1\}$. Therefore, $\boldsymbol{C}_4$ satisfies $(f \rightarrow x) \vee (x \rightarrow e) \neq 0$, and hence also

$$((f \rightarrow x) \vee (x \rightarrow e)) \rightarrow 0 = 0. \tag{3.5}$$

So, because every SI homomorphic image of a member of U is rigorously compact and crystalline, it follows from Lemma 2.6(i) that U satisfies (3.5). Note that N and W do not satisfy (3.5), as (3.5) fails in the algebra $\boldsymbol{C}_4 \times \boldsymbol{S}_3[\boldsymbol{C}_4]$ mentioned before Convention 3.10.

Definition 3.16. We denote by M the variety of anti-idempotent De Morgan monoids satisfying $e \leqslant f$ and (3.5).

Lemma 3.17. *$\boldsymbol{C}_4$ is a retract of every nontrivial member of* M.

Proof. Because M satisfies $e \leqslant f$, it also satisfies

$$x \leqslant f \cdot x, \tag{3.6}$$

and therefore

$$e \leqslant x \implies f \vee x \leqslant f \cdot x. \tag{3.7}$$

As M satisfies (3.5) and $\mathit{0} \to \mathit{0} = \mathit{1}$, its nontrivial members satisfy

$$(f \to x) \vee (x \to e) \neq \mathit{0}, \quad \text{i.e.,} \quad \neg(f \cdot \neg x) \vee \neg(f \cdot x) \neq \mathit{0},$$

or equivalently (by De Morgan's laws),

$$(f \cdot x) \wedge (f \cdot \neg x) \neq \mathit{1}. \tag{3.8}$$

By Lemma 3.2(i),(ii), every nontrivial member of M satisfies $e < f$ and has a subalgebra isomorphic to $\boldsymbol{C}_4$. So, by Lemma 3.8, it suffices to show that every simple member of M is isomorphic to $\boldsymbol{C}_4$. Suppose $\boldsymbol{A} \in \mathsf{M}$ is simple. We may assume that $\boldsymbol{C}_4 \in \mathbb{S}(\boldsymbol{A})$.

We claim that the intervals $[\mathit{0}, e]$, $[e, f]$ and $[f, \mathit{1}]$ of $\boldsymbol{A}$ are doubletons, i.e.,

$$[\mathit{0}, e] = \{\mathit{0}, e\} \text{ and } [e, f] = \{e, f\} \text{ and } [f, \mathit{1}] = \{f, \mathit{1}\}. \tag{3.9}$$

The first and third assertions in (3.9) follow from Lemma 2.16(iv) and involution properties. To prove the middle equality, suppose $a \in A$ with $e < a < f$. As $f = \neg e$, it follows that $e < \neg a < f$ and, by (3.7), $f = f \vee a \leqslant f \cdot a$. As $e \cdot a \nleqslant e$, we have $f \cdot a \nleqslant f$ (by (2.1)), so $f < f \cdot a$. Then $f \cdot a = \mathit{1}$, as $[f, \mathit{1}] = \{f, \mathit{1}\}$. By symmetry, $f \cdot \neg a = \mathit{1}$, so $(f \cdot a) \wedge (f \cdot \neg a) = \mathit{1}$, contradicting (3.8). Therefore, $[e, f] = \{e, f\}$, as claimed.

To complete the proof, it suffices to show that every element of $\boldsymbol{A}$ is comparable with e, as that will imply, by involution properties, that every element is comparable with f, forcing $A = \{\mathit{0}, e, f, \mathit{1}\} = C_4$.

Suppose, on the contrary, that $a \in A$ is incomparable with e, i.e., $\neg a$ is incomparable with f. As $a \nleqslant e$ and $\neg a \nleqslant f$, we have $e < e \vee a$ and $f < f \vee \neg a$, as well as $e \leqslant \neg a$ (by Theorem 2.43), i.e., $a \leqslant f$. So, by (3.7), $f \vee \neg a \leqslant f \cdot \neg a$, hence $f < f \cdot \neg a$, and so $f \cdot \neg a = \mathit{1}$, because $[f, \mathit{1}] = \{f, \mathit{1}\}$.

Again, as $e \cdot a \nleqslant e$, we have $f \cdot a \nleqslant f$, so $e \leqslant f \cdot a$, by Theorem 2.43. This, with $e < f$, gives $e \leqslant f \wedge (f \cdot a)$. Also, $a \leqslant f \cdot a$, by (3.6), so $a \leqslant f \wedge (f \cdot a)$. Therefore, $e \vee a \leqslant f \wedge (f \cdot a)$.

If we can argue that $f \wedge (f \cdot a) < f$, then $e < e \vee a < f$, contradicting the fact that $[e, f] = \{e, f\}$. So, to finish the proof, it suffices to show that f is incomparable with $f \cdot a$, and we have already shown that $f \cdot a \nleqslant f$. If $f < f \cdot a$, then $f \cdot a = \mathit{1}$, as $[f, \mathit{1}] = \{f, \mathit{1}\}$, but since $f \cdot \neg a = \mathit{1}$, this yields $(f \cdot a) \wedge (f \cdot \neg a) = \mathit{1}$, contradicting (3.8). Therefore, f and $f \cdot a$ are indeed incomparable, as required. □

Theorem 3.18. M *is the largest subvariety of* N.

Proof. By Lemma 3.17, M is a subvariety of N. Let K be any subvariety of N. Clearly, K satisfies $e \leqslant f$ and, by Lemma 3.2(iv), its members are anti-idempotent. Now K is a subvariety of U, by Theorem 3.14, because $\mathsf{N} \subseteq \mathsf{W}$. By Remark 3.15, (3.5) is satisfied by U, so it holds in K. Thus, $\mathsf{K} \subseteq \mathsf{M}$. □

Corollary 3.19. M *is the class of all algebras in* U *satisfying* $e \leqslant f$. *In particular,* M *satisfies* (3.2), (3.3) *and* (3.4).

Corollary 3.20. *Every rigorously compact algebra in* N *belongs to* M.

Proof. This follows from Lemma 3.12 and Corollary 3.19. □

At this point in our account, N and M are organizational tools, suggested by Theorem 3.4. They will assume an additional significance when we discuss structural completeness in Chapter 5.

3.2 Skew reflections

In this section we are going to provide a representation theorem for algebras in U or M, using ideas of Slaney [131].[3]

Definition 3.21. Let $\boldsymbol{B} = \langle B; \cdot^{\boldsymbol{B}}, \rightarrow^{\boldsymbol{B}}, \wedge^{\boldsymbol{B}}, \vee^{\boldsymbol{B}}, e \rangle$ be a square-increasing RL, with lattice order $\leqslant^{\boldsymbol{B}}$. Let $B' = \{b' : b \in B\}$ be a disjoint copy of the set B, let $\mathit{0}, \mathit{1}$ be distinct non-elements of $B \cup B'$, and let $S = B \cup B' \cup \{\mathit{0}, \mathit{1}\}$. Let $\leqslant$ be a binary relation on S such that

(i) $\leqslant$ is a lattice order whose restriction to B^2 is $\leqslant^{\boldsymbol{B}}$

(the meet and join operations of $\langle S; \leqslant \rangle$ being denoted by $\wedge$ and $\vee$, respectively), and for all $b, c \in B$,

(ii) $b' \leqslant c'$ iff $c \leqslant b$,

(iii) $b \leqslant c'$ iff $e \leqslant (b \cdot^{\boldsymbol{B}} c)'$,

(iv) $b' \nleqslant c$,

(v) $\mathit{0} \leqslant b \leqslant \mathit{1}$ and $\mathit{0} \leqslant b' \leqslant \mathit{1}$.

The *skew* $\leqslant$*-reflection* $\mathrm{S}^{\leqslant}(\boldsymbol{B})$ *of* $\boldsymbol{B}$ is the algebra $\langle S; \cdot, \wedge, \vee, \neg, e \rangle$ such that

[3] The nomenclature of [131] is untypical. There, 'De Morgan monoids' were not required to be distributive, and likewise the 'Dunn monoids' of Definition 2.39.

(vi) $\cdot$ is a commutative binary operation on S, extending $\cdot^{\boldsymbol{B}}$,

(vii) $a \cdot 0 = 0$ for all $a \in S$, and if $0 \neq a \in S$, then $a \cdot 1 = 1$,

(viii) $b \cdot c' = (b \to^{\boldsymbol{B}} c)'$ and $b' \cdot c' = 1$ for all $b, c \in B$,

(ix) $\neg 0 = 1$ and $\neg 1 = 0$ and $\neg b = b'$ and $\neg(b') = b$ for all $b \in B$.

A *skew reflection of* $\boldsymbol{B}$ is any algebra of the form $\mathrm{S}^{\leqslant}(\boldsymbol{B})$, where $\leqslant$ is a binary relation on S satisfying (i)–(v). (Some examples are pictured in Section 4.2 on page 90.)

Definition 3.21 is essentially due to Slaney [131]. (In [131], (iii) is formulated in an ostensibly more general manner, as

$$\text{for all } a, b, c \in B, \text{ we have } a \cdot^{\boldsymbol{B}} b \leqslant c' \text{ iff } a \leqslant (b \cdot^{\boldsymbol{B}} c)'.$$

This follows from (iii), however. Indeed, for $a, b, c \in B$,

$$a \cdot^{\boldsymbol{B}} b \leqslant c' \text{ iff } e \leqslant ((a \cdot^{\boldsymbol{B}} b) \cdot^{\boldsymbol{B}} c)' = (a \cdot^{\boldsymbol{B}} (b \cdot^{\boldsymbol{B}} c))' \text{ iff } a \leqslant (b \cdot^{\boldsymbol{B}} c)'.)$$

By an *RL-subreduct* of an IRL $\boldsymbol{A} = \langle A; \cdot, \wedge, \vee, \neg, e\rangle$, we mean a subalgebra of the RL-reduct $\langle A; \cdot, \to, \wedge, \vee, e\rangle$ of $\boldsymbol{A}$.

Theorem 3.22 ([131, Fact 1])**.** *A skew reflection* $\mathrm{S}^{\leqslant}(\boldsymbol{B})$ *of a square-increasing RL* $\boldsymbol{B}$ *is a square-increasing IRL, and* $\boldsymbol{B}$ *is an RL-subreduct of* $\mathrm{S}^{\leqslant}(\boldsymbol{B})$.

Remark 3.23. In a skew reflection $\mathrm{S}^{\leqslant}(\boldsymbol{B})$ of a square-increasing RL $\boldsymbol{B}$, we have $f = e'$, hence $f^2 = 1$, so $\mathrm{S}^{\leqslant}(\boldsymbol{B})$ is anti-idempotent and our use of $0, 1$ in Definition 3.21 is consistent with Convention 3.10. By definition, $\mathrm{S}^{\leqslant}(\boldsymbol{B})$ is rigorously compact. Because it has $\boldsymbol{B}$ as an RL-subreduct, $\mathrm{S}^{\leqslant}(\boldsymbol{B})$ satisfies $(f \to x) \vee (x \to e) \neq 0$, and hence also (3.5). It satisfies (3.2) and (3.4) as well.[4] The fact that elements of B lack lower bounds in B' has two easy but important consequences. First,

$$\mathrm{S}^{\leqslant}(\boldsymbol{B}) \text{ is simple iff } \boldsymbol{B} \text{ is trivial (i.e., } e \text{ is the least element of } \boldsymbol{B}),$$

in view of Lemma 2.16(iv). Secondly, by Lemma 2.16(iii),

$$\mathrm{S}^{\leqslant}(\boldsymbol{B}) \text{ is SI iff } \boldsymbol{B} \text{ is SI or trivial.}$$

[4] In verifying (3.4), we may assume that its left-hand side is not 0, so $x, y, q(x), q(y) \neq 0$. This forces $x, y \in B$, whence each conjunct of the right-hand side is 1.

Specifically, when $\boldsymbol{B}$ is not trivial, an element of $\mathsf{S}^{\leqslant}(\boldsymbol{B})$ is the greatest strict lower bound of e in $\mathsf{S}^{\leqslant}(\boldsymbol{B})$ iff it is the greatest strict lower bound of e in $\boldsymbol{B}$.

Elements of B might lack upper bounds in B', e.g., $\boldsymbol{D}_4$ arises in this way from a trivial RL. Such cases are eliminated in the next theorem, however.

Theorem 3.24. *The following two conditions on a square-increasing IRL $\boldsymbol{A}$ are equivalent.*

(i) *There is a homomorphism $h\colon \boldsymbol{A} \to \boldsymbol{C}_4$ and $\boldsymbol{A}$ is rigorously compact.*

(ii) *$\boldsymbol{A}$ is a skew reflection of a square-increasing RL $\boldsymbol{B}$, and 0 is meet-irreducible in $\boldsymbol{A}$.*

In this case, in the notation of Definition 3.21,

(iii) *h is unique and surjective, and 1 is join-irreducible in $\boldsymbol{A}$;*

(iv) *$b \wedge c' \in B$ and $b \vee c' \in B'$ for all $b, c \in B$, so each element of B has an upper bound in B', and elements of B' have lower bounds in B;*

(v) *if $\boldsymbol{B}$ is distributive and $\boldsymbol{A}$ is modular, then $\boldsymbol{A}$ is distributive and therefore a De Morgan monoid, belonging to U.*

Proof. Note first that, in (iii), the uniqueness of h follows from Lemma 3.3(i) (and its surjectivity from the fact that $\boldsymbol{C}_4$ is 0-generated).

(i) $\Rightarrow$ (ii): Being crystalline, $\boldsymbol{A}$ is nontrivial. The set $B := h^{-1}[\{e\}]$ is the universe of an RL-subreduct $\boldsymbol{B}$ of $\boldsymbol{A}$, which inherits the square-increasing law, and $b \mapsto b' := \neg b$ defines an antitone bijection from B onto $B' := h^{-1}[\{f\}]$. Clearly, $B \cap B' = \emptyset$ and no element of B' is a lower bound of an element of B, because h is isotone and $e < f$ in $\boldsymbol{C}_4$. As h fixes 0 and 1, Lemma 2.6 shows that $\boldsymbol{A}$ is anti-idempotent, with $h^{-1}[\{0\}] = \{0\}$ and $h^{-1}[\{1\}] = \{1\}$, and that 0 [resp. 1] is meet- [resp. join-] irreducible in $\boldsymbol{A}$, finishing the proof of (iii). In particular, $A = B \cup B' \cup \{0\} \cup \{1\}$ (disjointly).

We verify that $\boldsymbol{A}$ satisfies conditions (iii) and (viii) of Definition 3.21. Let $b, c \in B$. Because B is closed under the operation $\cdot$ of $\boldsymbol{A}$, we have

$$b \leqslant c' \text{ iff } b \cdot e \leqslant \neg c \text{ iff } b \cdot c \leqslant f \text{ (by (2.1), deployed in } \boldsymbol{A}\text{), iff } e \leqslant (b \cdot c)'.$$

Clearly, $b \cdot c' = (b \to c)'$ and $h(b' \cdot c') = \neg h(b) \cdot \neg h(c) = f^2 = 1$, so $b' \cdot c' = 1$. This completes the proof that $\boldsymbol{A} = \mathsf{S}^{\leqslant}(\boldsymbol{B})$, where $\leqslant$ is the lattice order of $\boldsymbol{A}$.

(ii) $\Rightarrow$ (i): Rigorous compactness was noted in Remark 3.23. Definition 3.21 shows that $\cdot, \neg$ and e are preserved by the function $h\colon \boldsymbol{A} \to \boldsymbol{C}_4$

such that $h(0) = 0$, $h(1) = 1$, $h(b) = e$ and $h(b') = f$ for all $b \in B$. As 0 is meet-irreducible (whence 1 is join-irreducible) in $\boldsymbol{A}$, the map h preserves $\wedge, \vee$ too. Indeed, if $b, c \in B$, then $b \geqslant b \wedge c' \neq 0$ and b has no lower bound in B', so $b \wedge c' \in B$ and, by involution properties, $b \vee c' \in B'$. This proves (i) and (iv).

By (iv), when $\mathrm{S}^{\leqslant}(\boldsymbol{B})$ is modular, it will be distributive iff the five-element lattice $\boldsymbol{M}_3$ doesn't embed into the sublattice $B \cup B'$ of $\mathrm{S}^{\leqslant}(\boldsymbol{B})$; see Theorem 1.22. That is true if $\boldsymbol{B}$ is distributive, as B and B' are then distributive sublattices of $B \cup B'$. This, with Lemma 3.12, proves (v). □

Corollary 3.25. *A De Morgan monoid belongs to* U *iff it is isomorphic to a subdirect product of skew reflections of Dunn monoids, where* 0 *is meet-irreducible in each subdirect factor.*

Proof. The forward implication follows from Theorem 3.24 and Birkhoff's Subdirect Decomposition Theorem 1.3, because the SI homomorphic images of members of U are bounded by $0, 1$, are rigorously compact (Theorem 2.45) and are still crystalline (U being a variety), and because RL-subreducts of De Morgan monoids inherit distributivity. Conversely, by Remark 3.23, skew reflections of Dunn monoids satisfy the defining postulates of U, except possibly for (3.3) and distributivity (which are effectively given here), and U, like any quasivariety, is closed under $\mathbb{IP}_{\mathbb{S}}$. □

Lemma 3.26. *Let* $\boldsymbol{A} = \mathrm{S}^{\leqslant}(\boldsymbol{B})$ *be a skew reflection of a square-increasing RL* $\boldsymbol{B}$*, where* $\boldsymbol{A}$ *satisfies* $e \leqslant f$*. Then, in the notation of Definition* 3.21*,*

(i) $b \leqslant (b \to e)'$ *for all* $b \in B$*, and*

(ii) 0 *is meet-irreducible and* 1 *is join irreducible in* $\boldsymbol{A}$.

Proof. (i) Let $b \in B$. By (2.6), $b \cdot (b \to e) \leqslant e$, so $e \leqslant f \leqslant (b \cdot (b \to e))'$. Then $b \leqslant (b \to e)'$, by Definition 3.21(iii).

(ii) Let $b, c \in B$. By (i), $c \leqslant (c \to e)'$, i.e., $c \to e \leqslant c'$. Because $\boldsymbol{B}$ is an RL-subreduct of $\boldsymbol{A}$ and $0 \notin B$, we have $b \wedge c' \geqslant b \wedge (c \to e) \in B$, so $b \wedge c' \neq 0$. As B and B' are both sublattices of $\boldsymbol{A}$, this shows that 0 is meet-irreducible (whence 1 is join-irreducible) in $\boldsymbol{A}$. □

Corollary 3.27. *A De Morgan monoid belongs to* M *iff it satisfies* $e \leqslant f$ *and is isomorphic to a subdirect product of skew reflections of Dunn monoids.*

Proof. This follows from Lemma 3.26(ii) and Corollaries 3.19 and 3.25. □

3.3 0-generated De Morgan monoids

In this section we recount Slaney's description, in [129], of the free 0-generated De Morgan monoid $\boldsymbol{F}_{\mathsf{DMM}}(0)$.

Infinite 1-generated De Morgan monoids exist. Indeed, there are De Morgan monoids in which chains $x < x^2 < x^3 < \cdots < x^k < x^{k+1} < \ldots$ occur; see Example 4.25. The larger varieties of distributive and of square-increasing IRLs each have infinite 0-generated members as well [131], but Slaney proved that the free 0-generated De Morgan monoid has just 3088 elements [129]. His arguments show that, up to isomorphism, only eight 0-generated De Morgan monoids are FSI; they were exhibited in [130], and are depicted below. As the seven nontrivial 0-generated FSI De Morgan monoids are finite, they are just the 0-generated SI De Morgan monoids.

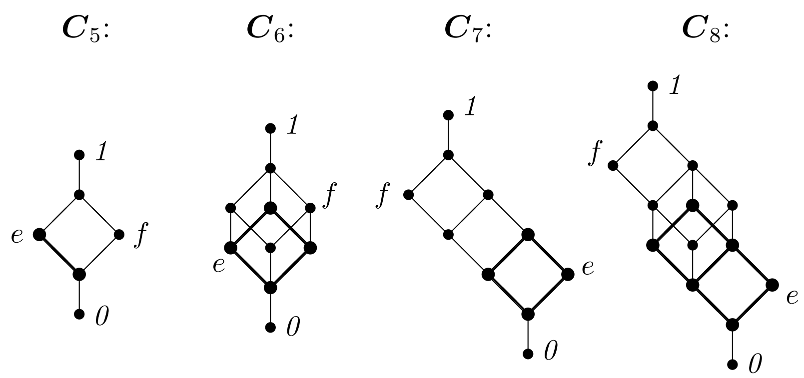
In Theorem 3.1, we already saw that, up to isomorphism, the only simple 0-generated De Morgan monoids are $\mathbf{2}$, $\boldsymbol{C}_4$ and $\boldsymbol{D}_4$. Let $\boldsymbol{A}$ be a nontrivial non-simple 0-generated FSI De Morgan monoid. As $\boldsymbol{A}$ is finitely generated and has finite type, $\boldsymbol{A}$ has a (0-generated) simple homomorphic image, by Theorem 1.15(ii). The simple image must be $\boldsymbol{C}_4$, by Theorem 3.4, so $\boldsymbol{A} \in \mathsf{W}$. In fact, $\boldsymbol{A} \in \mathsf{U}$, by Corollary 3.13, since $\boldsymbol{A}$ is FSI. In particular, $\boldsymbol{A}$ is anti-idempotent.

One can extend this argument to any anti-idempotent FSI De Morgan monoid that is negatively generated:

Theorem 3.28. *Let $\boldsymbol{A}$ be an anti-idempotent negatively generated FSI De Morgan monoid. Then $\boldsymbol{A} \cong \boldsymbol{D}_4$ or $\boldsymbol{A} \in \mathsf{U}$.*

Proof. We may suppose that $\boldsymbol{A}$ is nontrivial. As $\boldsymbol{A}$ is anti-idempotent, $\mathbb{V}(\boldsymbol{A})$ is a Kollár variety, by Corollary 2.15, so $\boldsymbol{A}$ has a simple homomorphic image $\boldsymbol{B}$, by Theorem 1.18. Since $\boldsymbol{A} = \mathbf{Sg}^{\boldsymbol{A}} A^-$, we have $\boldsymbol{B} = \mathbf{Sg}^{\boldsymbol{B}} B^-$ (Lemma 2.35), but B^- is the chain $\neg(f^2) < e$, since $\boldsymbol{B}$ is simple (Lemma 2.16(iv)) and anti-idempotent, so $\boldsymbol{B}$ is 0-generated. Therefore, $\boldsymbol{B}$ is isomorphic to $\boldsymbol{C}_4$ or $\boldsymbol{D}_4$, by Theorem 3.1, as $\mathbf{2}$ is not anti-idempotent. If $\boldsymbol{B} \cong \boldsymbol{D}_4$, then $\boldsymbol{A} \cong \boldsymbol{D}_4$, by Theorem 3.4. Otherwise $\boldsymbol{B} \cong \boldsymbol{C}_4$, in which case $\boldsymbol{A} \in \mathsf{U}$, by Corollary 3.13. □

As 0-generated IRLs are negatively generated, each of the four non-simple 0-generated FSI De Morgan monoids is a skew reflection of some Dunn monoid, by Theorems 3.25 and 3.28. Their lattice reducts are depicted below, with the underlying Dunn monoid highlighted. Slaney [130] calls these algebras *CA6*, *CA10a*, *CA10b* and *CA14*, respectively.



There is already enough information in the diagrams to verify that these algebras are 0-generated. In each case the greatest strict lower bound of e is $c := e \wedge f$, and the greatest element of the highlighted Dunn monoid turns out to be $c^* := (e \wedge f) \to e$. With these elements one can generate the rest of the algebra using only lattice operations and involution.

Let $\mathbf{2}^+$ and $\boldsymbol{D}_4^+$ denote the RL-reducts of $\mathbf{2}$ and $\boldsymbol{D}_4$, respectively. Then $\boldsymbol{C}_5 = \mathrm{S}^{\leqslant}(\mathbf{2}^+)$ and $\boldsymbol{C}_6 = \mathrm{S}^{\leqslant}(\boldsymbol{D}_4^+)$, where the respective orders $\leqslant$ are depicted in the diagrams above. The algebras $\boldsymbol{C}_7$ and $\boldsymbol{C}_8$ are skew reflections of Dunn monoids $\boldsymbol{D}_7$ and $\boldsymbol{D}_8$, respectively, whose fusion tables are given below, where we abbreviate $c^* \wedge \neg(c^*)$ as b.

$\cdot$	c	e	b	c^*
c	c	c	c	c
e	c	e	b	c^*
b	c	b	b	b
c^*	c	c^*	b	c^*

$\cdot$	c	e	b	$b \vee e$	$f \wedge c^*$	c^*
c	c	c	c	c	c	c
e	c	e	b	$b \vee e$	$f \wedge c^*$	c^*
b	c	b	b	b	b	b
$b \vee e$	c	$b \vee e$	b	$b \vee e$	$f \wedge c^*$	c^*
$f \wedge c^*$	c	$f \wedge c^*$	b	$f \wedge c^*$	c^*	c^*
c^*	c	c^*	b	c^*	c^*	c^*

In [129], Slaney showed that $\boldsymbol{F}_{\mathsf{DMM}}(0) \cong \mathbf{2} \times \boldsymbol{D}_4 \times \boldsymbol{A}$, where $\boldsymbol{A}$ is a skew reflection of $\mathbf{2}^+ \times \boldsymbol{D}_4^+ \times \boldsymbol{D}_7 \times \boldsymbol{D}_8$.

Remark 3.29. In each of $\mathbf{2}$, $\boldsymbol{D}_4$, $\mathbf{2}^+$, $\boldsymbol{D}_4^+$, $\boldsymbol{D}_7$ and $\boldsymbol{D}_8$, e has just one strict lower bound. The neutral element in $\mathbf{2}^+ \times \boldsymbol{D}_4^+ \times \boldsymbol{D}_7 \times \boldsymbol{D}_8$ therefore has 2^4 lower bounds, so $e^{\boldsymbol{A}}$ has $2^4 + 1$ lower bounds in $\boldsymbol{A}$, by Definition 3.21(iv),(v). Thus, the number of lower bounds of $e^{\boldsymbol{F}_{\mathsf{DMM}}(0)}$ in $\boldsymbol{F}_{\mathsf{DMM}}(0)$ (including $e^{\boldsymbol{F}_{\mathsf{DMM}}(0)}$ itself) is $2 \times 2 \times (2^4 + 1) = 68$. (This fact will become relevant in Corollary 4.7.)

3.4 Reflections

Definition 3.30. Let $\boldsymbol{B}$ be a square-increasing RL, with lattice order $\leqslant^{\boldsymbol{B}}$, and let $S = B \cup B' \cup \{\mathit{0}, \mathit{1}\}$, where $B' = \{b' : b \in B\}$ is a disjoint copy of B and $\mathit{0}, \mathit{1}$ are distinct non-elements of $B \cup B'$. Let $\leqslant$ be the unique partial order of S whose restriction to B^2 is $\leqslant^{\boldsymbol{B}}$, such that

$$b \leqslant c' \text{ for all } b, c \in B$$

and conditions (ii), (iv) and (v) of Definition 3.21 hold. As (i) and (iii) obviously hold too, we may define the *reflection* $\mathrm{R}(\boldsymbol{B})$ of $\boldsymbol{B}$ to be the resulting skew reflection $\mathrm{S}^{\leqslant}(\boldsymbol{B})$. This definition is essentially due to Meyer; see [97] or [1, pp. 371–373].

In other words, for an SRL $\boldsymbol{B}$, the order of its reflection $\mathrm{R}(\boldsymbol{B})$ extends that of $\boldsymbol{B}$ by placing a reflected copy of $\boldsymbol{B}$ above $\boldsymbol{B}$ and adding fresh extrema $\mathit{0} < \mathit{1}$.

By Theorem 3.22, every Dunn monoid $\boldsymbol{B}$ is an RL-subreduct of its reflection $\mathrm{R}(\boldsymbol{B})$, and $\mathrm{R}(\boldsymbol{B})$ satisfies $e \leqslant f$ (by definition) and is distributive (as $\boldsymbol{B}$ is), so $\mathrm{R}(\boldsymbol{B}) \in \mathsf{M}$, by Corollary 3.27. In particular, $\boldsymbol{C}_4$ is the reflection of a trivial Dunn monoid. Conversely, the RL-reduct of an algebra from M is of course a Dunn monoid, whence so are its subalgebras. This justifies a variant of the 'Crystallization Fact' of [130, p. 124]:

Theorem 3.31. *The variety of Dunn monoids coincides with the class of all RL-subreducts of members of* M.

Corollary 3.32. *The equational theory of* M *is undecidable.*

Proof. This follows from Theorem 3.31, because Urquhart [141, p. 1070] proved that the equational theory of Dunn monoids is undecidable. □

Corollary 3.33. M *is not generated (as a variety) by its finite members.*

Proof. This follows from Corollary 3.32, as M is finitely axiomatized (see the remarks after Theorem 1.39). □

Clearly, in the statements of Theorem 3.31 and Corollary 3.32, we may replace M by any variety K such that $\mathsf{M} \subseteq \mathsf{K} \subseteq \mathsf{DMM}$. The same applies to Corollary 3.33 if K is also finitely axiomatized. In particular, the variety U is not generated by its finite members.

The notational conventions of Definition 3.21 are assumed in the next lemma, which reveals how reflections interact with the class operators $\mathbb{H}$, $\mathbb{S}$ and $\mathbb{P}_{\mathbb{U}}$.

Lemma 3.34. *Let $\boldsymbol{B}$ be a Dunn monoid.*

(i) *If $\boldsymbol{C}$ is a subalgebra of $\boldsymbol{B}$, then $C \cup \{c' \colon c \in C\} \cup \{0, 1\}$ is the universe of a subalgebra of $\mathrm{R}(\boldsymbol{B})$ that is isomorphic to $\mathrm{R}(\boldsymbol{C})$, and every subalgebra of $\mathrm{R}(\boldsymbol{B})$ arises in this way from a subalgebra of $\boldsymbol{B}$.*

(ii) *If θ is a congruence of $\boldsymbol{B}$, then*

$$\mathrm{R}(\theta) := \theta \cup \{\langle a', b'\rangle \colon \langle a, b\rangle \in \theta\} \cup \{\langle 0, 0\rangle, \langle 1, 1\rangle\}$$

is a congruence of $\mathrm{R}(\boldsymbol{B})$, and $\mathrm{R}(\boldsymbol{B})/\,\mathrm{R}(\theta) \cong \mathrm{R}(\boldsymbol{B}/\theta)$. Also, every proper congruence of $\mathrm{R}(\boldsymbol{B})$ has the form $\mathrm{R}(\theta)$ for some $\theta \in \mathrm{Con}(\boldsymbol{B})$.

(iii) *If $\{\boldsymbol{B}_i \colon i \in I\}$ is a family of Dunn monoids and $\mathcal{U}$ is an ultrafilter over I, then $\prod_{i \in I} \mathrm{R}(\boldsymbol{B}_i)/\mathcal{U} \cong \mathrm{R}\big(\prod_{i \in I} \boldsymbol{B}_i/\mathcal{U}\big)$.*

Proof. The first assertions in (i) and (ii) are straightforward. For the final assertions, one shows that if $\boldsymbol{D}$ is a subalgebra and φ a proper congruence of $\mathrm{R}(\boldsymbol{B})$, then $\boldsymbol{D}$ is the reflection of the subalgebra of $\boldsymbol{B}$ on $D \cap B$, while $\varphi = \mathrm{R}(B^2 \cap \varphi)$. To see that $\varphi \subseteq \mathrm{R}(B^2 \cap \varphi)$, observe that if φ identifies a with b' $(a, b \in B)$, and therefore a' with b, it must identify $1 = a' \cdot b'$ with $b \cdot a \in B$. But this contradicts Lemma 2.6(i), because $\mathrm{R}(\boldsymbol{B})$ is rigorously compact.

(iii) For each $i \in I$, let 0_i and 1_i denote the extrema of $\mathrm{R}(\boldsymbol{B}_i)$ and, for convenience, define $\bar{0}_i = \{0_i\}$ and $\bar{1}_i = \{1_i\}$ and $(B')_i = B_i'$. By $0, 1$, we mean (for the moment) the extrema of $\mathrm{R}\big(\prod_{i \in I} \boldsymbol{B}_i/\mathcal{U}\big)$. Consider $\vec{x} \in \prod_{i \in I} \mathrm{R}(B_i)$. As $\mathcal{U}$ is an ultrafilter, there is a unique $F(\vec{x}) \in \{B, B', \bar{0}, \bar{1}\}$ such that

$$\{i \in I : x_i \in F(\vec{x})_i\} \in \mathcal{U},$$

because the four possible sets of indexes above are disjoint from one another, and their union equals the total index set I (see the remarks after Theorem 1.3 on page 4). If $F(\vec{x})$ is $\bar{0}$ [resp. $\bar{1}$], define $h(\vec{x})$ to be 0 [resp. 1]. If $F(\vec{x}) = B$, define $h(\vec{x}) = \vec{z}/\mathcal{U}$, where $\vec{z} \in \prod_{i \in I} B_i$ and, for each $i \in I$,

$$z_i \;=\; \begin{cases} x_i & \text{if } x_i \in B_i; \\ e^{\boldsymbol{B}_i} & \text{otherwise.} \end{cases}$$

If $F(\vec{x}) = B'$, define $h(\vec{x}) = (\vec{z}/\mathcal{U})'$, where $\vec{z} \in \prod_{i \in I} B_i$ and, for each $i \in I$,

$$z_i \;=\; \begin{cases} \text{the unique } b \in B_i \text{ such that } x_i = b', \text{ if this exists;} \\ e^{\boldsymbol{B}_i}, \;\text{ otherwise.} \end{cases}$$

Then h is a homomorphism from $\prod_{i\in I}\mathrm{R}(\boldsymbol{B}_i)$ onto $\mathrm{R}\big(\prod_{i\in I}\boldsymbol{B}_i/\mathcal{U}\big)$, whose kernel is $\Big\{\langle\vec{x},\vec{y}\rangle\in\big(\prod_{i\in I}\mathrm{R}(B_i)\big)^2 : \{i\in I : x_i=y_i\}\in\mathcal{U}\Big\}$, so the result follows from the Homomorphism Theorem 1.1. □

Definition 3.35. Given a variety K of Dunn monoids, the *reflection* $\mathbb{R}(\mathsf{K})$ of K is the subvariety $\mathbb{V}\{\mathrm{R}(\boldsymbol{B}) : \boldsymbol{B}\in\mathsf{K}\}$ of M.

Jónsson's Theorem 1.23, together with Lemma 3.34, yields the next corollary (as every variety is generated by its FSI members).

Corollary 3.36. *Let K be a variety of SRLs, with $\boldsymbol{E}\in\mathbb{R}(\mathsf{K})$. Then $\boldsymbol{E}$ is FSI iff $\boldsymbol{E}\cong\mathrm{R}(\boldsymbol{D})$ for some $\boldsymbol{D}\in\mathsf{K}_{FSI}$. Also, $\boldsymbol{E}$ is SI iff $\boldsymbol{E}\cong\mathrm{R}(\boldsymbol{D})$ for some $\boldsymbol{D}\in\mathsf{K}$ that is either trivial or SI.*

Lemma 3.37. *A variety K of SRLs is locally finite iff $\mathbb{R}(\mathsf{K})$ is locally finite.*

Proof. ($\Rightarrow$): As K is locally finite, there is a function $p\colon\omega\to\omega$ such that, for each $n\in\omega$, every n-generated member of K_{SI} has at most $p(n)$ elements (Fact 1.21). It suffices to show that, for each $n\in\omega$, every n-generated $\boldsymbol{E}\in\mathbb{R}(\mathsf{K})_{SI}$ has at most $2+2p(n)$ elements (see Fact 1.21). By Corollary 3.36, any such $\boldsymbol{E}$ may be assumed to be $\mathrm{R}(\boldsymbol{D})$ for some $\boldsymbol{D}\in\mathsf{K}_{SI}$. Let G be an irredundant generating set for $\boldsymbol{E}$, with $|G|\leq n$. Then $\mathit{0},\mathit{1}\notin G$. Let $H=(G\cap D)\cup\{\neg g : g\in G\cap D'\}$, so $|H|\leq n$ and $\boldsymbol{C}:=\mathrm{Sg}^{\boldsymbol{D}}H$ has at most $p(n)$ elements. By Lemma 3.34(i), $\mathrm{R}(\boldsymbol{C})$ may be identified with a subalgebra of $\boldsymbol{E}$, but then $G\subseteq\mathrm{R}(C)$, so $\mathrm{R}(\boldsymbol{C})=\boldsymbol{E}$, whence $|E|\leq 2+2p(n)$.

($\Leftarrow$): Use the fact that an SIRL of the form $\mathrm{R}(\boldsymbol{A})$ is generated by A. □

As a function from the lattice of varieties of Dunn monoids to the subvariety lattice of M, the operator $\mathbb{R}$ is obviously isotone.

Lemma 3.38. $\mathbb{R}$ *is order-reflecting and therefore injective.*

Proof. Let $\mathbb{R}(\mathsf{K})\subseteq\mathbb{R}(\mathsf{L})$, where K and L are varieties of Dunn monoids. We must show that $\mathsf{K}\subseteq\mathsf{L}$. Let $\boldsymbol{A}\in\mathsf{K}$ be SI. It suffices to show that $\boldsymbol{A}\in\mathsf{L}$. By assumption, $\mathrm{R}(\boldsymbol{A})\in\mathbb{R}(\mathsf{L})$. Also, $\mathrm{R}(\boldsymbol{A})$ is SI (because $\boldsymbol{A}$ is), so by Jónsson's Theorem 1.23, $\mathrm{R}(\boldsymbol{A})\in\mathbb{HSP}_{\mathbb{U}}\{\mathrm{R}(\boldsymbol{B}) : \boldsymbol{B}\in\mathsf{L}\}$. Because L is closed under $\mathbb{H}$, $\mathbb{S}$ and $\mathbb{P}_{\mathbb{U}}$, it follows from Lemma 3.34 that $\mathrm{R}(\boldsymbol{A})\cong\mathrm{R}(\boldsymbol{B})$ for some $\boldsymbol{B}\in\mathsf{L}$, whence $\boldsymbol{A}\cong\boldsymbol{B}$, and so $\boldsymbol{A}\in\mathsf{L}$. □

Note that Brouwerian algebras are just the integral Dunn monoids. Recall that every variety of countable type has at most $2^{\aleph_0}$ subvarieties (see the remarks after Jónsson's Theorem 1.23 on page 14). It is known that there are $2^{\aleph_0}$ distinct varieties of Brouwerian algebras [150]. So, the injectivity of $\mathbb{R}$ in Lemma 3.38 yields the following conclusion.

Theorem 3.39. *The variety* M *has* $2^{\aleph_0}$ *distinct subvarieties.*

This is the first of a number of theorems where we use reflections of Brouwerian algebras to obtain results about varieties of De Morgan monoids. That is in fact the theme of Chapter 7.

3.5 Negatively generated semilinear De Morgan monoids

In this section we apply all the tools developed in this and the previous chapter to obtain a representation theorem for all negatively generated semilinear De Morgan monoids (using rigorous extensions of skew reflections of generalized Sugihara monoids). A representation of generalized Sugihara monoids to be presented in Section 6.6 (Corollary 6.29) will therefore further illuminate the structure of negatively generated semilinear De Morgan monoids, but here we do not rely on that representation.

Recall, from Theorem 3.28, that every FSI negatively generated anti-idempotent De Morgan monoid $\boldsymbol{A}$ belongs to $\mathbb{I}(\{\boldsymbol{D}_4\}) \cup \mathsf{U}$. Therefore, by Theorem 3.25, $\boldsymbol{A}$ is skew reflection of some Dunn monoid $\boldsymbol{D}$.

Now suppose that $\boldsymbol{A}$ is semilinear (so it is totally ordered by Lemma 2.22). It follows that $\boldsymbol{A} \in \mathsf{M}$, because $\boldsymbol{A}$ satisfies $e \leqslant f$, but we can be more specific. Recall that, since $\boldsymbol{A}$ is a skew reflection, it satisfies $b' \nleqslant c$ for every $b, c \in D$ (see Definition 3.21(iv)). But then $c < b'$ for all $b, c \in D$, i.e., $\boldsymbol{A}$ is the *reflection* of $\boldsymbol{D}$ (see Definition 3.30). Furthermore, $\boldsymbol{D}$ is itself totally ordered, because $\boldsymbol{D}$ is a subalgebra of the RL-reduct of $\boldsymbol{A}$, by Theorem 3.22. Thus, we obtain:

Lemma 3.40. *Every totally ordered negatively generated anti-idempotent De Morgan monoid is a reflection of a totally ordered Dunn monoid* (*and it belongs to* M).

The underlying Dunn monoid in the statement above is itself negatively generated, because of the next lemma. We shall see in the proof of Theorem 3.42 that the converse of Lemma 3.40 holds for such (negatively generated) Dunn monoids.

Lemma 3.41. *Let* $\boldsymbol{A} = \mathrm{R}(\boldsymbol{D})$ *for some Dunn monoid* $\boldsymbol{D}$. *If* $\boldsymbol{A} = \mathbf{Sg}^{\boldsymbol{A}} X$ *for some* $X \subseteq D$, *then* $\boldsymbol{D} = \mathbf{Sg}^{\boldsymbol{D}} X$.

Proof. Let $\boldsymbol{B}$ be the subalgebra of $\boldsymbol{D}$ generated by X. We argue that $\boldsymbol{B} = \boldsymbol{D}$. By Lemma 3.34(i), $\mathrm{R}(\boldsymbol{B})$ can be identified with a subalgebra of $\boldsymbol{A}$. (Note that Lemma 3.34(i) does not hold for skew reflections. For example, the De Morgan monoids $\boldsymbol{D}_7$ and $\boldsymbol{D}_8$ on page 65 each have a three-element subalgebra, but their respective skew reflections $\boldsymbol{C}_7$ and $\boldsymbol{C}_8$ have no subalgebra.) But then $\mathrm{R}(\boldsymbol{B}) = \boldsymbol{A} = \mathrm{R}(\boldsymbol{D})$, since $X \subseteq B \subseteq \mathrm{R}(B)$ and $\boldsymbol{A} = \mathbf{Sg}^{\boldsymbol{A}}X$. It follows that $B = D$, because $\boldsymbol{A}$ is a reflection of a Dunn monoid whose universe must be $\{a \in A \colon a \neq \mathit{0} \text{ and } a^2 \neq \mathit{1}\}$. □

Lemma 3.41 becomes false if we replace 'reflection' with 'skew reflection'; this can be shown using the algebras depicted before Lemma 4.19. We can now give a representation of semilinear negatively generated anti-idempotent De Morgan monoids. Define the following unary terms:

$$\begin{aligned} d'(x) &:= (f^2 \to (x \cdot f)) \wedge (f^2 \cdot \neg x), \\ \sigma(x) &:= (x \wedge e) \cdot (x^* \wedge e)^*, \\ d(x) &:= d'(\neg x) \text{ and } \sigma'(x) := \neg\sigma(\neg x). \end{aligned}$$

Recall that $\sigma(x) \approx x$ is equation (2.39) on page 42, which is satisfied by every generalized Sugihara monoid. Consider the equation

$$x \approx \big(d(\sigma(x)) \wedge \sigma(x)\big) \vee \big(d'(\sigma'(x)) \wedge \sigma'(x)\big) \vee \big((f^2 \vee \neg(f^2)) \to \sigma'(x)\big)\,. \quad (3.10)$$

Note that we do not abbreviate f^2 and $\neg(f^2)$ above as $\mathit{1}$ and $\mathit{0}$, respectively. The reason for this (and for not rewriting $f^2 \vee \neg(f^2)$ as f^2) is that the next theorem will be generalized in Theorem 3.45 to accommodate De Morgan monoids that need not be anti-idempotent. In that situation the intuition behind the abbreviations collapses. We retain the use of $\mathit{1}$ and $\mathit{0}$, however, to indicate the extrema of a reflection.

Theorem 3.42. *Let* $\boldsymbol{A}$ *be an anti-idempotent semilinear De Morgan monoid. The following are equivalent:*

(i) $\boldsymbol{A}$ *is negatively generated;*

(ii) $\boldsymbol{A}$ *is a subdirect product of reflections of totally ordered generalized Sugihara monoids;*

(iii) $\boldsymbol{A}$ *satisfies equation* (3.10).

Proof. (i) ⇒ (ii): As in the proof Theorem 2.41, by Birkhoff's Subdirect Decomposition Theorem 1.3, we need only show that every *totally ordered* anti-idempotent De Morgan monoid $\boldsymbol{B}$ that is negatively generated is a reflection of a totally ordered generalized Sugihara monoid. By Lemma 3.40, $\boldsymbol{B} \cong \mathrm{R}(\boldsymbol{D})$ for some totally ordered Dunn monoid $\boldsymbol{D}$. Note that $\mathrm{R}(\boldsymbol{D})$ is generated by D^-, because $\mathrm{R}(D)^- = D^- \cup \{0\}$ and $0 = \neg(f^2) \in \mathrm{Sg}^{\boldsymbol{B}}\{e\}$. But then $\boldsymbol{D} = \mathbf{Sg}^{\boldsymbol{D}} D^-$, by Lemma 3.41. It follows, by Theorem 2.41, that $\boldsymbol{D} \in \mathsf{GSM}$.

(ii) ⇒ (iii): We claim that every reflection of a totally ordered generalized Sugihara monoid satisfies (3.10), and so $\boldsymbol{A}$ does as well. Let $\boldsymbol{B} = \mathrm{R}(\boldsymbol{D})$ for some totally ordered $\boldsymbol{D} \in \mathsf{GSM}$. For any $a \in B$, it follows from the definition of reflection that

$$d(a) = \begin{cases} 1 & \text{if } a \in D; \\ 0 & \text{otherwise,} \end{cases} \qquad d'(a) = \begin{cases} 1 & \text{if } a \in D'; \\ 0 & \text{otherwise,} \end{cases}$$

$$(f^2 \vee \neg(f^2)) \to a = f^2 \to a = \begin{cases} 1 & \text{if } a = 1; \\ 0 & \text{otherwise,} \end{cases}$$

$$\sigma(a) = \begin{cases} 1 & \text{if } a \in D'; \\ a & \text{otherwise,} \end{cases} \quad \text{and} \quad \sigma'(a) = \begin{cases} 0 & \text{if } a \in D; \\ a & \text{otherwise.} \end{cases}$$

It is then easy to verify that $\boldsymbol{B}$ satisfies (3.10), by checking the cases where $a \in D$, $a \in D'$, $a = 1$ and $a = 0$.

(iii) ⇒ (i): This follows directly from the shape of equation (3.10), because σ is built up from the terms $x \wedge e$ and $x^* \wedge e$, and σ' is built up from $\neg x \wedge e$ and $(\neg x)^* \wedge e$. For any assignment to x of an element of $\boldsymbol{A}$, these terms clearly evaluate into A^-. □

Corollary 3.43. *Let* K *be the class of negatively generated semilinear anti-idempotent De Morgan monoids. Then*

(i) K *is a variety that is axiomatized relative to semilinear De Morgan monoids by* $x \leqslant f^2$ *and (3.10);*

(ii) $\mathsf{K} = \mathbb{R}(\mathsf{GSM})$*;*

(iii) K *is locally finite.*

(iv) *If* $\boldsymbol{A} \in \mathsf{K}$ *is totally ordered and* n*-generated, then* $|A| \leq 6n + 4$.

Proof. (i) follows immediately from Theorem 3.42.

For (ii), it follow straightforwardly from Theorem 3.42 that $\mathsf{K} \subseteq \mathbb{R}(\mathsf{GSM})$. To establish the converse, it is enough to show that $\mathbb{R}(\mathsf{GSM})_{\mathrm{SI}} \subseteq \mathsf{K}$, because K is closed under $\mathbb{IP}_{\mathbb{S}}$ (by (i)). By Corollary 3.36, this reduces to showing that $\mathbb{R}(\mathsf{GSM}_{\mathrm{SI}}) \subseteq \mathsf{K}$, which follows from Theorem 3.42.

(iii) follows from item (ii), because GSM is locally finite (Corollary 2.42), and the reflection operator preserves local finiteness (Lemma 3.37).

Recall from the remarks after Corollary 2.42 that if $\boldsymbol{B} \in \mathsf{GSM}$ is totally ordered and n-generated then $|B| \leq 3n + 1$. Let $\boldsymbol{A}$ be a totally ordered n-generated member of K. Since $\boldsymbol{A}$ is FSI (by Lemma 2.22) and finite (by (iii)), $\boldsymbol{A}$ is SI. Interrogating the proof of Lemma 3.37, we find that $|A| \leq 2(3n + 1) + 2 = 6n + 4$, proving (iv). □

Corollary 3.44. *Let* K *be any nontrivial variety of negatively generated semilinear anti-idempotent De Morgan monoids. Then* $\mathsf{K} = \mathbb{R}(\mathsf{L})$ *for some variety* L *of generalized Sugihara monoids.*

Proof. Consider the class $\mathsf{D} := \{\boldsymbol{D} \in \mathsf{GSM} \colon \mathrm{R}(\boldsymbol{D}) \in \mathsf{K}_{\mathrm{SI}}\}$. Let $\mathsf{L} = \mathbb{V}(\mathsf{D})$. By Birkhoff's Subdirect Decomposition Theorem 1.3, it suffices to show that $\mathsf{K}_{\mathrm{SI}} = \mathbb{R}(\mathsf{L})_{\mathrm{SI}}$.

Let $\boldsymbol{A} \in \mathsf{K}_{\mathrm{SI}}$. By (i) $\Rightarrow$ (ii) of Theorem 3.42, $\boldsymbol{A} \cong \mathrm{R}(\boldsymbol{D})$ for some $\boldsymbol{D} \in \mathsf{GSM}$. But then $\boldsymbol{D} \in \mathsf{D}$, so $\boldsymbol{A} \cong \mathrm{R}(\boldsymbol{D}) \in \mathbb{R}(\mathsf{L})$.

Conversely, let $\boldsymbol{A} \in \mathbb{R}(\mathsf{L})_{\mathrm{SI}}$. By Corollary 3.36, $\boldsymbol{A} \cong \mathrm{R}(\boldsymbol{D})$ for some $\boldsymbol{D} \in \mathsf{L}$ that is either trivial or SI. In the first case $\boldsymbol{A} \cong \boldsymbol{C}_4$, and $\boldsymbol{C}_4 \in \mathsf{K}$, because K is a nontrivial subvariety of M (Lemma 3.40). In the second case, $\boldsymbol{D} \in \mathbb{V}(\mathsf{D})_{\mathrm{SI}} \subseteq \mathbb{HSP}_{\mathbb{U}}(\mathsf{D})$, by Jónsson's Theorem 1.23. So, by Lemma 3.34,

$$\boldsymbol{A} \cong \mathrm{R}(\boldsymbol{D}) \in \mathbb{HSP}_{\mathbb{U}}(\{\mathrm{R}(\boldsymbol{B}) : \boldsymbol{B} \in \mathsf{D}\}) \subseteq \mathsf{K}. \qquad \square$$

We can now describe all semilinear De Morgan monoids that are negatively generated, using the characterization of FSI De Morgan monoids (in Theorem 2.57) by means of rigorous extensions.

Theorem 3.45. *Let* $\boldsymbol{A}$ *be a semilinear De Morgan monoid. The following are equivalent:*

(i) $\boldsymbol{A}$ *is negatively generated;*

(ii) $\boldsymbol{A}$ *is a subdirect product of totally ordered Sugihara monoids and De Morgan monoids of the form* $\boldsymbol{S}[\mathrm{R}(\boldsymbol{D})]$, *where* $\boldsymbol{S} \in \mathsf{OSM}_{FSI}$ *and* $\boldsymbol{D} \in \mathsf{GSM}_{FSI}$;

(iii) $\boldsymbol{A}$ *satisfies equation* (3.10).

Proof. (i) $\Rightarrow$ (ii): Let $\boldsymbol{B}$ be a totally ordered negatively generated De Morgan monoid. If $\boldsymbol{B}$ is a Sugihara monoid we are done, so suppose $\boldsymbol{B}$ is not a Sugihara monoid. Then, by Theorem 2.57, $\boldsymbol{B} \cong \boldsymbol{S}[\boldsymbol{B}']$ for a nontrivial anti-idempotent subalgebra $\boldsymbol{B}'$ of $\boldsymbol{B}$ and an odd Sugihara monoid $\boldsymbol{S}$ (both totally ordered). Suppose, with a view to contradiction, that $\boldsymbol{B}'$ is not negatively generated. Then $\boldsymbol{B}'' := \mathbf{Sg}^{\boldsymbol{B}'} B^-$ is a proper subalgebra of $\boldsymbol{B}'$. But then, by Theorem 2.58(ii), $\boldsymbol{S}[\boldsymbol{B}'']$ is a proper subalgebra of $\boldsymbol{S}[\boldsymbol{B}']$ containing

$S[B']^-$, contradicting the fact that $\boldsymbol{S}[\boldsymbol{B}']$ is negatively generated. So, $\boldsymbol{B}'$ is negatively generated, totally ordered, and anti-idempotent, which implies that $\boldsymbol{B}' \cong \mathrm{R}(\boldsymbol{D})$ for some totally ordered $\boldsymbol{D} \in \mathsf{GSM}$, by Theorem 3.42.

(ii) $\Rightarrow$ (iii): First we show that (3.10) holds for every Sugihara monoid, using the fact that $\mathsf{SM} = \mathbb{V}(\boldsymbol{Z}^*)$. For $a \in Z^*$, we have $d(a) = a \wedge \neg a = d'(a)$, $\sigma(a) = a = \sigma'(a)$, and $(f^2 \vee \neg(f^2)) \to a = e \to a = a$. Therefore, the right-hand side of (3.10) simplifies to $(a \wedge \neg a) \vee a$, which clearly equals a.

Lastly, let $\boldsymbol{B} = \boldsymbol{S}[\mathrm{R}(\boldsymbol{D})]$ for some totally ordered $\boldsymbol{S} \in \mathsf{OSM}$ and some totally ordered $\boldsymbol{D} \in \mathsf{GSM}$. We have just seen that $\boldsymbol{S}$ satisfies (3.10). From Theorem 3.42, the subalgebra $\mathrm{R}(\boldsymbol{D})$ of $\boldsymbol{B}$ also satisfies (3.10).

Let $a \in B \setminus \mathrm{R}(D)$, and let b be the right-hand side of (3.10) when x is assigned the value of a. Recall from Theorems 2.54 and 2.57 that there is a homomorphism from $\boldsymbol{B}$ onto $\boldsymbol{S}$, whose kernel identifies two distinct elements iff they belong to $\mathrm{R}(D)$. So, if $a \neq b$, then since $a \notin \mathrm{R}(D)$, it follows that $h(a)$ is not $h(b)$, contradicting the fact that $\boldsymbol{S}$ satisfies (3.10).

The proof of (iii) $\Rightarrow$ (i) is similiar to that of Theorem 3.42. □

Corollary 3.46. *Let* K *by the class of all negatively generated semilinear De Morgan monoids.*

(i) K *is a variety and it is axiomatized relative to semilinear De Morgan monoids by (3.10).*

(ii) *If* $\boldsymbol{A} \in \mathsf{K}$ *is totally ordered and* n*-generated, then* $|A| \leq 6n + 4$.

(iii) K *is locally finite.*

Proof. (i) follows directly from Theorem 3.45.

Let $\boldsymbol{A} \in \mathsf{K}$ be totally ordered an n-generated, where $n \in \omega$. If $\boldsymbol{A}$ is a Sugihara monoid, then $|A| \leq 2n+2 \leq 6n+4$ (see Theorem 2.26). If $\boldsymbol{A}$ is not a Sugihara monoid, then $\boldsymbol{A} \cong \boldsymbol{S}[\boldsymbol{A}']$ for an anti-idempotent subalgebra $\boldsymbol{A}'$ of $\boldsymbol{A}$, and a totally ordered odd Sugihara monoid $\boldsymbol{S}$, by Theorem 2.57. Let us divide the n generators of $\boldsymbol{S}[\boldsymbol{A}']$ into $X \subseteq A'$ and $Y \subseteq S \setminus \{e^{\boldsymbol{S}}\}$, so that when $|X| = p$ and $|Y| = q$, we have $p + q \leq n$. Since $\boldsymbol{A}'$ is totally ordered, anti-idempotent and negatively generated, $|A'| \leq 6p+4$, by Corollary 3.43. Note that $\boldsymbol{S}$ is generated by Y, because if there was a proper subalgebra $\boldsymbol{P}$ of $\boldsymbol{S}$ still containing Y, then, by Theorem 2.58(ii), $\boldsymbol{P}[\boldsymbol{A}']$ would be a proper subalgebra of $\boldsymbol{S}[\boldsymbol{A}']$ containing $X \cup Y$, a contradiction. So, by Theorem 2.26, $\left|S \setminus \{e^{\boldsymbol{S}}\}\right| \leq 2q$. But then $|A| \leq 2q+6p+4 \leq 6(p+q)+4 \leq 6n+4$, proving (ii).

Therefore, K is locally finite, by Fact 1.21 (since the SI algebras in K are totally ordered). □

Chapter 4

The subvariety lattice of DMM

As was mentioned in the introduction, much of the interest in De Morgan monoids stems from the fact that DMM algebraizes the relevance logic $\mathbf{R^t}$, whence the axiomatic extensions of $\mathbf{R^t}$ and the subvarieties of DMM form anti-isomorphic lattices (see Theorem 1.38). This motivates analysis of the lattice of varieties of De Morgan monoids. The present chapter is primarily a study of the lower part of that lattice.

We prove that a variety of De Morgan monoids consists of Sugihara monoids iff it omits $\boldsymbol{C}_4$ and $\boldsymbol{D}_4$ (Theorem 4.1). This implies that DMM has just four minimal subvarieties, all of which are finitely generated (Theorem 4.2). Sugihara monoids encompass two of the minimal varieties, viz. the variety BA of Boolean algebras and $\mathbb{V}(\boldsymbol{S}_3)$. The remaining two are generated, respectively, by $\boldsymbol{C}_4$ and $\boldsymbol{D}_4$.

In the literature of substructural logics, subvariety lattices are more prominent than subquasivariety lattices, because they mirror the extensions of a logic by new axioms, as opposed to new inference rules. Nevertheless, some natural logical problems call for a consideration of quasivarieties if they are to be approached algebraically, e.g., the identification of the structurally complete strengthenings of a logic (see Chapter 5). Each of the four minimal varieties of De Morgan monoids is also minimal as a quasivariety, but they are not alone in this. Indeed, we prove that DMM has just 68 minimal subquasivarieties (Corollary 4.7). The proof exploits Slaney's description of the free 0-generated De Morgan monoid (see Section 3.3).

Sections 4.2–4.5 investigate the covers of the four atoms within the subvariety lattice of DMM. It suffices to consider the join-irreducible covers, as the subvariety lattice of DMM is distributive. We show, in Section 4.2, that BA has no join-irreducible cover within DMM, and that $\mathbb{V}(\boldsymbol{S}_3)$ has just one; the situation for $\mathbb{V}(\boldsymbol{C}_4)$ and $\mathbb{V}(\boldsymbol{D}_4)$ is much more complex (see Theorem 4.9).

The covers of $\mathbb{V}(\boldsymbol{C}_4)$ are distinctive, because $\boldsymbol{C}_4$ has more interesting homomorphic pre-images than $\mathbf{2}$ or $\boldsymbol{D}_4$, by Theorem 3.4. With the help of the skew reflection representations in Section 3.2, we identify all of the covers of $\mathbb{V}(\boldsymbol{C}_4)$ within U (they are varieties whose nontrivial members are homomorphic pre-images of $\boldsymbol{C}_4$; see Theorem 3.14). There are just ten such covers, of which exactly six fall within M (i.e., their nontrivial members have $\boldsymbol{C}_4$ as a retract; see Theorem 3.18). All ten of these varieties are finitely generated (see Theorem 4.23 and Corollary 4.24 in Section 4.3).

Within DMM, every cover of $\mathbb{V}(\boldsymbol{D}_4)$ is semisimple. The same applies to the covers of $\mathbb{V}(\boldsymbol{C}_4)$ that are not contained in U. In both cases, we identify infinitely many such covers that are finitely generated, and some that are not even generated by their finite members (see Sections 4.4 and 4.5).

4.1 Atoms

The fact that the subvariety lattice of DMM has just four atoms was first proved in the author's MSc thesis [146]. A simpler proof was subsequently published in Moraschini, Raftery and Wannenburg [103]. We reproduce the second proof below.

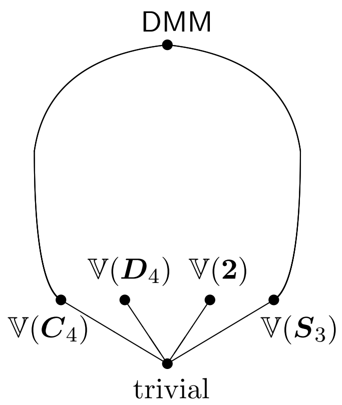
A quasivariety is said to be *minimal* if it is nontrivial and has no nontrivial proper subquasivariety. If we say that a variety is *minimal* (without further qualification), we mean that it is nontrivial and has no nontrivial proper subvariety. When we mean instead that it is *minimal as a quasivariety*, we shall say so explicitly, thereby avoiding ambiguity. (Recall from the remarks after Theorem 1.23 on page 14 that every nontrivial [quasi]variety has a minimal sub[quasi]variety.)

Theorem 4.1. *A variety* K *of De Morgan monoids consists of Sugihara monoids iff it excludes* $\boldsymbol{C}_4$ *and* $\boldsymbol{D}_4$.

Proof. Necessity is clear, as $\boldsymbol{C}_4$ and $\boldsymbol{D}_4$ are not idempotent. Conversely, suppose $\boldsymbol{C}_4, \boldsymbol{D}_4 \notin \mathsf{K}$ and let $\boldsymbol{A} \in \mathsf{K}_{\mathrm{SI}}$. It suffices to show that $\boldsymbol{A}$ is a Sugihara monoid. Suppose not. Then, by Theorem 2.50, $\neg(f^2) < f^2$ and the subalgebra $\boldsymbol{B}$ of $\boldsymbol{A}$ on $[\neg(f^2), f^2]$ is nontrivial, whence the 0-generated subalgebra $\boldsymbol{E}$ of $\boldsymbol{A}$ is nontrivial. Recall that every nontrivial finitely generated algebra of finite type has a simple homomorphic image (Theorem 1.15). Let $\boldsymbol{G}$ be a simple homomorphic image of $\boldsymbol{E}$, so $\boldsymbol{G} \in \mathsf{K}$. By assumption, neither $\boldsymbol{C}_4$ nor $\boldsymbol{D}_4$ is isomorphic to $\boldsymbol{G}$, but $\boldsymbol{G}$ is 0-generated, so $\mathbf{2} \cong \boldsymbol{G}$, by Theorem 3.1. Thus, $\mathbf{2} \in \mathbb{HS}(\boldsymbol{B})$. Then $\mathbf{2}$ must inherit from $\boldsymbol{B}$ the anti-idempotent identity $x \leqslant f^2$. This is false, however, so $\boldsymbol{A}$ is a Sugihara monoid. ∎

Theorem 4.2. *The distinct classes* $\mathbb{V}(\mathbf{2})$, $\mathbb{V}(\boldsymbol{S}_3)$, $\mathbb{V}(\boldsymbol{C}_4)$ *and* $\mathbb{V}(\boldsymbol{D}_4)$ *are precisely the minimal varieties of De Morgan monoids.*

Proof. Each $\boldsymbol{X} \in \{\mathbf{2}, \boldsymbol{S}_3, \boldsymbol{C}_4, \boldsymbol{D}_4\}$ is finite and simple, with no proper nontrivial subalgebra, so the nontrivial members of $\mathbb{HS}(\boldsymbol{X})$ are isomorphic to $\boldsymbol{X}$. Thus, the SI members of $\mathbb{V}(\boldsymbol{X})$ belong to $\mathbb{I}(\boldsymbol{X})$, by Jónsson's Theorem 1.23, because DMM is a congruence distributive variety. As varieties are determined by their SI members, this shows that $\mathbb{V}(\boldsymbol{X})$ has no proper nontrivial subvariety, and that $\mathbb{V}(\boldsymbol{X}) \neq \mathbb{V}(\boldsymbol{Y})$ for distinct $\boldsymbol{X}, \boldsymbol{Y} \in \{\mathbf{2}, \boldsymbol{S}_3, \boldsymbol{C}_4, \boldsymbol{D}_4\}$. As $\mathbb{V}(\mathbf{2})$ and $\mathbb{V}(\boldsymbol{S}_3)$ are the only minimal varieties of Sugihara monoids, Theorem 4.1 shows that they, together with $\mathbb{V}(\boldsymbol{C}_4)$ and $\mathbb{V}(\boldsymbol{D}_4)$, are the only minimal subvarieties of DMM. □



With a view to axiomatizing the varieties in Theorem 4.2, consider the following (abbreviated) equations.

$$e \leqslant (x \to (y \vee \neg y)) \vee (y \wedge \neg y) \tag{4.1}$$
$$e \leqslant (f^2 \to x) \vee (x \to e) \vee \neg x \tag{4.2}$$
$$x \wedge (x \to f) \leqslant (f \to x) \vee (x \to e) \tag{4.3}$$
$$x \to e \leqslant x \vee (f^2 \to \neg x) \tag{4.4}$$

Theorem 4.3 ([146, 103])**.**

(i) $\mathbb{V}(\mathbf{2})$ *is axiomatized by adding* $x \leqslant e$ *to the axioms of* DMM;

(ii) $\mathbb{V}(\boldsymbol{S}_3)$ *by adding* $e \approx f$, (2.28) *and* (4.1);

(iii) $\mathbb{V}(\boldsymbol{D}_4)$ *by adding* $x \leqslant f^2$, $x \wedge \neg x \leqslant y$ *and* (4.2);

(iv) $\mathbb{V}(\boldsymbol{C}_4)$ *by adding* $x \leqslant f^2$, $e \leqslant f$, (2.28), (4.3) *and* (4.4).[1]

As the proof of Theorem 4.3 in [103] is not substantially different from the corresponding proof in [146], it will not be reproduced here.

Theorem 4.2 says, in effect, that for each axiomatic consistent extension $\mathbf{L}$ of $\mathbf{R^t}$, there exists $\boldsymbol{B} \in \{\mathbf{2}, \boldsymbol{S}_3, \boldsymbol{C}_4, \boldsymbol{D}_4\}$ such that the theorems of $\mathbf{L}$ all take values $\geqslant e$ on any evaluation of their variables in $\boldsymbol{B}$. Postulates for the four maximal consistent axiomatic extensions of $\mathbf{R^t}$ follow systematically from Theorem 4.3. For example, (2.28) becomes the axiom $(p \to q) \vee (q \to p)$, while (4.4) becomes $(p \to \mathbf{t}) \to (p \vee (\mathbf{f}^2 \to \neg p))$.

We shall now describe the minimal *subquasivarieties* of DMM.

Bergman and McKenzie [8] showed that every locally finite congruence modular minimal variety is also minimal as a quasivariety. Thus, by Theorem 4.2, $\mathbb{V}(\mathbf{2})$, $\mathbb{V}(\boldsymbol{S}_3)$, $\mathbb{V}(\boldsymbol{C}_4)$ and $\mathbb{V}(\boldsymbol{D}_4)$ are minimal as quasivarieties. We proceed to show that the total number of minimal subquasivarieties of DMM is still finite, but much greater than four.

Lemma 4.4. *Let* $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *be nontrivial algebras, where* $\boldsymbol{A}$ *is* 0*-generated.*

(i) *If* $\boldsymbol{B} \in \mathbb{Q}(\boldsymbol{A})$, *then* $\boldsymbol{A}$ *can be embedded into* $\boldsymbol{B}$, *whence* $\mathbb{Q}(\boldsymbol{A}) = \mathbb{Q}(\boldsymbol{B})$.

(ii) $\mathbb{Q}(\boldsymbol{A})$ *is a minimal quasivariety.*

(iii) *If* $\boldsymbol{B} \in \mathbb{Q}(\boldsymbol{A})$ *and* $\boldsymbol{B}$ *is* 0*-generated, then* $\boldsymbol{A} \cong \boldsymbol{B}$.

(iv) *If* $\boldsymbol{A}$ *has finite type and* $\mathbb{Q}(\boldsymbol{A})$ *is a variety, then* $\boldsymbol{A}$ *is simple.*

Proof. (i) Recall that $\mathbb{Q}(\boldsymbol{A}) = \mathbb{ISPP}_{\mathbb{U}}(\boldsymbol{A})$. Let $\boldsymbol{B} \in \mathbb{Q}(\boldsymbol{A})$. So, $\boldsymbol{B}$ embeds into a direct product $\boldsymbol{D}$ of ultrapowers of $\boldsymbol{A}$, where the index set of the direct product is not empty (because $\boldsymbol{B}$ is nontrivial). Clearly, if a variable-free equation ε is true in $\boldsymbol{A}$, then it is true in $\boldsymbol{B}$. Conversely, if ε is true in $\boldsymbol{B}$, then it is true in $\boldsymbol{D}$, as variable-free equations persist in extensions. In that case, since ε persists in homomorphic images, it is true in an ultrapower $\boldsymbol{U}$ of $\boldsymbol{A}$, whence it is true in $\boldsymbol{A}$, because all first order sentences persist in ultraroots, by Theorem 1.9. There is therefore a well defined injection $k\colon A \to B$, given by

$$\alpha^{\boldsymbol{A}}(c_1^{\boldsymbol{A}}, c_2^{\boldsymbol{A}}, \dots) \;\mapsto\; \alpha^{\boldsymbol{B}}(c_1^{\boldsymbol{B}}, c_2^{\boldsymbol{B}}, \dots),$$

where $c_1, c_2, \dots$ are the nullary operation symbols of the signature and α is any term. Clearly, k is a homomorphism from $\boldsymbol{A}$ into $\boldsymbol{B}$, so $\boldsymbol{A} \in \mathbb{IS}(\boldsymbol{B})$.

[1] Of course, (i) is well known. We have not encountered (ii)–(iv) in earlier literature, but a variant of (ii) could be derived from [37, Cor. 2].

(ii) follows immediately from (i).

(iii) In the proof of (i), the image of the embedding k is a subalgebra of $\boldsymbol{B}$. So, if $\boldsymbol{B}$ is 0-generated, then k is surjective, i.e., $k\colon \boldsymbol{A} \cong \boldsymbol{B}$.

(iv) Suppose $\boldsymbol{A}$ has finite type and is not simple. As $\boldsymbol{A}$ is 0-generated and nontrivial, it has a simple homomorphic image $\boldsymbol{C}$, by Theorem 1.15, and $\boldsymbol{C}$ is still 0-generated. If $\boldsymbol{C} \in \mathbb{Q}(\boldsymbol{A})$, then $\boldsymbol{A} \cong \boldsymbol{C}$, by (iii), contradicting the non-simplicity of $\boldsymbol{A}$. So, $\boldsymbol{C} \notin \mathbb{Q}(\boldsymbol{A})$, whence $\mathbb{Q}(\boldsymbol{A})$ is not a variety (by Birkhoff's Theorem 1.7). □

Theorem 4.5. *A quasivariety of De Morgan monoids is minimal iff it is* $\mathbb{V}(\boldsymbol{S}_3)$ *or* $\mathbb{Q}(\boldsymbol{A})$ *for some nontrivial* 0*-generated De Morgan monoid* $\boldsymbol{A}$.

Proof. Sufficiency follows from Lemma 4.4(ii) and previous remarks about $\mathbb{V}(\boldsymbol{S}_3)$. Conversely, let K be a minimal subquasivariety of DMM. Being minimal, K is $\mathbb{Q}(\boldsymbol{A})$ for some nontrivial De Morgan monoid $\boldsymbol{A}$. Let $\boldsymbol{B}$ be the smallest subalgebra of $\boldsymbol{A}$. If $\boldsymbol{B}$ is trivial, then $\boldsymbol{A}$ satisfies $e \approx f$, so K is a variety, by Theorems 2.11 and 2.28. In this case, as K is a minimal variety of odd Sugihara monoids, it is $\mathbb{V}(\boldsymbol{S}_3)$, by Corollary 2.27. On the other hand, if $\boldsymbol{B}$ is nontrivial, then $\mathsf{K} = \mathbb{Q}(\boldsymbol{B})$ (again by the minimality of K), and this completes the proof, because $\boldsymbol{B}$ is 0-generated. □

Theorem 4.6. *The minimal subquasivarieties of* DMM *form a finite set, whose cardinality is the number of lower bounds of* e *in the free* 0*-generated De Morgan monoid* $\boldsymbol{F}_{\mathsf{DMM}}(0)$.

Proof. Let $\boldsymbol{F} = \boldsymbol{F}_{\mathsf{DMM}}(0)$. As we noted in Section 3.3, Slaney [129] proved that $\boldsymbol{F}$ has just 3088 elements; its bottom element is $e^{\boldsymbol{F}} \leftrightarrow f^{\boldsymbol{F}}$, by Theorem 2.10. By the Homomorphism Theorem 1.1, every 0-generated De Morgan monoid is isomorphic to a factor algebra of $\boldsymbol{F}$, so DMM has only finitely many minimal subquasivarieties, by Theorem 4.5.

Now consider a factor algebra $\boldsymbol{F}/G$, where G is a deductive filter of $\boldsymbol{F}$. As $\boldsymbol{F}$ is finite, $G = [\alpha^{\boldsymbol{F}})$ for some nullary term α in the language of IRLs, where $\alpha^{\boldsymbol{F}} \leqslant e^{\boldsymbol{F}}$. If $\boldsymbol{F}/G$ is nontrivial, i.e., $\alpha^{\boldsymbol{F}} \nleqslant e^{\boldsymbol{F}} \leftrightarrow f^{\boldsymbol{F}}$, then $\boldsymbol{F}/G$ is not odd (by (2.16)), whence $\mathbb{Q}(\boldsymbol{F}/G) \neq \mathbb{V}(\boldsymbol{S}_3)$. The function $\alpha^{\boldsymbol{F}} \mapsto \mathbb{Q}(\boldsymbol{F}/[\alpha^{\boldsymbol{F}}))$ is therefore a well defined surjection from the lower bounds of $e^{\boldsymbol{F}}$ in $\boldsymbol{F}$ to the set consisting of the trivial subvariety (corresponding to the bottom element of $\boldsymbol{F}$) and the minimal subquasivarieties of DMM, *other* than $\mathbb{V}(\boldsymbol{S}_3)$. It remains only to show that this map is injective. To that end, suppose $\boldsymbol{F}/[\alpha^{\boldsymbol{F}})$ and $\boldsymbol{F}/[\beta^{\boldsymbol{F}})$ generate the same quasivariety, where $\alpha^{\boldsymbol{F}}, \beta^{\boldsymbol{F}} \leqslant e^{\boldsymbol{F}}$. Then there is an isomorphism $g\colon \boldsymbol{F}/[\alpha^{\boldsymbol{F}}) \cong \boldsymbol{F}/[\beta^{\boldsymbol{F}})$, by Lemma 4.4(iii). As $\beta^{\boldsymbol{F}} \leqslant e^{\boldsymbol{F}}$, we have $\beta^{\boldsymbol{F}} \leftrightarrow e^{\boldsymbol{F}} = \beta^{\boldsymbol{F}} \in [\beta^{\boldsymbol{F}})$, by (2.15) and (2.17), so

$\beta^{\boldsymbol{F}/[\beta^{\boldsymbol{F}})} = e^{\boldsymbol{F}/[\beta^{\boldsymbol{F}})}$. Now

$$g(\beta^{\boldsymbol{F}}/[\alpha^{\boldsymbol{F}})) = g(\beta^{\boldsymbol{F}/[\alpha^{\boldsymbol{F}})}) = \beta^{\boldsymbol{F}/[\beta^{\boldsymbol{F}})} = e^{\boldsymbol{F}/[\beta^{\boldsymbol{F}})} = g(e^{\boldsymbol{F}}/[\alpha^{\boldsymbol{F}})),$$

but g is injective, so $\beta^{\boldsymbol{F}}/[\alpha^{\boldsymbol{F}}) = e^{\boldsymbol{F}}/[\alpha^{\boldsymbol{F}})$, i.e., $\beta^{\boldsymbol{F}} = \beta^{\boldsymbol{F}} \leftrightarrow e^{\boldsymbol{F}} \in [\alpha^{\boldsymbol{F}})$. This means that $\alpha^{\boldsymbol{F}} \leqslant \beta^{\boldsymbol{F}}$ and, by symmetry, $\alpha^{\boldsymbol{F}} = \beta^{\boldsymbol{F}}$, completing the proof. □

Corollary 4.7. *There are exactly* 68 *minimal quasivarieties of De Morgan monoids.*

Proof. This follows immediately from Theorem 4.6 and Remark 3.29. □

4.2 Covers of atoms

Recall that when a lattice $\boldsymbol{L}$ has a least element $\bot$, its *atoms* are the covers of $\bot$. Provided that $\boldsymbol{L}$ is modular, the join of any two distinct atoms covers each join-and, so a cover c of an atom is interesting when it is *not* the join of two atoms. If $\boldsymbol{L}$ is distributive, that is equivalent to the ostensibly stronger demand that c be join-irreducible.

Recall that the lattice of subvarieties of a congruence distributive variety E is itself distributive. Therefore, once the atoms of this lattice have been determined, the immediate concern is to identify the join-irreducible covers of each atom E'; we refer to these as covers of E' *within* E. In particular, it behoves us to investigate the join-irreducible covers, within DMM, of the four varieties in Theorem 4.2.

Let S be a nontrivial variety of De Morgan monoids. By Theorem 4.2, S includes an algebra $\boldsymbol{X} \in \{\mathbf{2}, \boldsymbol{S}_3, \boldsymbol{C}_4, \boldsymbol{D}_4\}$. So, a *cover of* $\mathbb{V}(\boldsymbol{X})$ *within* S is a variety K, with $\mathbb{V}(\boldsymbol{X}) \subsetneq \mathsf{K} \subseteq \mathsf{S}$, such that no proper subvariety of K properly contains $\mathbb{V}(\boldsymbol{X})$.[2]

By Corollary 2.27, $\mathbb{V}(\boldsymbol{S}_5)$ is a join-irreducible cover of $\mathbb{V}(\boldsymbol{S}_3)$ within DMM.

For each $\boldsymbol{X} \in \{\mathbf{2}, \boldsymbol{S}_3, \boldsymbol{C}_4, \boldsymbol{D}_4\}$ and each variety K of De Morgan monoids, if $\boldsymbol{A} \in (\mathsf{K} \setminus \mathbb{I}(\boldsymbol{X}))_{\mathrm{FSI}}$ is nontrivial, then $\boldsymbol{A} \notin \mathbb{V}(\boldsymbol{X})$, by Jónsson's Theorem 1.23, because the nontrivial members of $\mathbb{HS}(\boldsymbol{X})$ belong to $\mathbb{I}(\boldsymbol{X})$. In this case, if K covers $\mathbb{V}(\boldsymbol{X})$, then $\mathsf{K} = \mathbb{V}(\boldsymbol{A}, \boldsymbol{X})$, so if K is also join-irreducible, it coincides with $\mathbb{V}(\boldsymbol{A})$. In other words:

[2] For $\mathsf{S} = \mathsf{DMM}$, the logic $\vdash_{\mathsf{K}}$ (see p. 23) corresponding to K is then *pre-maximal* in the lattice of axiomatic extensions of $\mathbf{R^t}$, i.e., it is not a co-atom of this lattice, but each of its consistent axiomatic proper extensions is a co-atom.

Fact 4.8. *If* $\boldsymbol{X} \in \{\mathbf{2}, \boldsymbol{S}_3, \boldsymbol{C}_4, \boldsymbol{D}_4\}$, *then every join-irreducible cover of* $\mathbb{V}(\boldsymbol{X})$ *within* DMM *is generated by each of its nontrivial FSI members, other than the isomorphic copies of* $\boldsymbol{X}$.

In the subvariety lattice of U, the only atom is $\mathbb{V}(\boldsymbol{C}_4)$ (as $\mathsf{U} \subseteq \mathsf{W}$ by Theorem 3.14 and every algebra in W maps onto $\boldsymbol{C}_4$; see Definition 3.5). So, every cover of $\mathbb{V}(\boldsymbol{C}_4)$ within U is join-irreducible.

Recall from Section 3.3 that the four nontrivial non-simple 0-generated De Morgan monoids, $\boldsymbol{C}_5, \ldots, \boldsymbol{C}_8$, belong to U. For $n \in \{5, 6, 7, 8\}$, $\boldsymbol{C}_n$ violates $e \leqslant f$, so $\boldsymbol{C}_n \in \mathsf{U} \setminus \mathsf{M}$. Moreover, $\boldsymbol{C}_n$ has just three deductive filters, and hence just three factor algebras, since $|(e]| = 3$ in $\boldsymbol{C}_n$. The class of nontrivial members of $\mathbb{HS}(\boldsymbol{C}_n)$ is therefore $\mathbb{I}(\boldsymbol{C}_n, \boldsymbol{C}_4)$, because $\boldsymbol{C}_n$ is 0-generated. Thus, $\mathbb{V}(\boldsymbol{C}_n)$ is a (join-irreducible) cover of $\mathbb{V}(\boldsymbol{C}_4)$ within U, by Jónsson's Theorem 1.23.

Theorem 4.9.

(i) $\mathbb{V}(\mathbf{2})$ *has no join-irreducible cover within* DMM.

(ii) $\mathbb{V}(\boldsymbol{S}_5)$ *is the only join-irreducible cover of* $\mathbb{V}(\boldsymbol{S}_3)$ *within* DMM.

(iii) *If* K *is a join-irreducible cover of* $\mathbb{V}(\boldsymbol{C}_4)$ *within* DMM, *then* K *consists of anti-idempotent algebras and exactly one of the following holds.*

   (1) $\mathsf{K} \subseteq \mathsf{M}$.

   (2) $\mathsf{K} = \mathbb{V}(\boldsymbol{C}_n)$ *for some* $n \in \{5, 6, 7, 8\}$.

   (3) $\mathsf{K} = \mathbb{V}(\boldsymbol{A})$ *for some simple* 1*-generated De Morgan monoid* $\boldsymbol{A}$, *where* $\boldsymbol{C}_4$ *is a proper subalgebra of* $\boldsymbol{A}$.

(iv) *If* K *is a join-irreducible cover of* $\mathbb{V}(\boldsymbol{D}_4)$ *within* DMM, *then* $\mathsf{K} = \mathbb{V}(\boldsymbol{A})$ *for some simple* 1*-generated De Morgan monoid* $\boldsymbol{A}$, *where* $\boldsymbol{D}_4$ *is a proper subalgebra of* $\boldsymbol{A}$. *In this case,* K *consists of anti-idempotent algebras.*

Proof. Let $\boldsymbol{X} \in \{\mathbf{2}, \boldsymbol{S}_3, \boldsymbol{C}_4, \boldsymbol{D}_4\}$, and let K be a join-irreducible cover of $\mathbb{V}(\boldsymbol{X})$ within DMM. As $\mathbb{V}(\boldsymbol{X}) \subsetneq \mathsf{K}$, there exists a finitely generated SI algebra $\boldsymbol{A} \in \mathsf{K} \setminus \mathbb{V}(\boldsymbol{X})$. Then $\mathsf{K} = \mathbb{V}(\boldsymbol{A})$, by Fact 4.8. Note that $\boldsymbol{A}$ is rigorously compact, by Corollary 2.46. Let $\boldsymbol{B}$ be the 0-generated subalgebra of $\boldsymbol{A}$, so $\boldsymbol{B}$ is FSI, by Lemma 2.16(i). Now $\boldsymbol{B}$ is finite, since $\boldsymbol{F}_{\mathsf{DMM}}(0)$ is finite, so $\boldsymbol{B}$ is SI or trivial.

If $\boldsymbol{B}$ is trivial, then $\boldsymbol{A}$ is an odd Sugihara monoid (by Theorem 2.11), whence K consists of odd Sugihara monoids, forcing $\boldsymbol{X} = \boldsymbol{S}_3$ and $\mathsf{K} = \mathbb{V}(\boldsymbol{S}_5)$ (by Corollary 2.27), as K covers $\mathbb{V}(\boldsymbol{X})$.

We may therefore assume that $\boldsymbol{B}$ is nontrivial, in view of the present theorem's statement. By Theorems 1.15 and 3.4, $\boldsymbol{B}$ is simple or crystalline, so by Theorem 3.1, we may assume that $\boldsymbol{B} \in \{\mathbf{2}, \boldsymbol{D}_4\}$ or $\boldsymbol{C}_4 \in \mathbb{H}(\boldsymbol{B})$.

If $\boldsymbol{B} = \mathbf{2}$, then $\boldsymbol{A}$ is idempotent (by Theorem 2.11). In this case, if $\boldsymbol{X} \neq \mathbf{2}$, then $\mathsf{K} = \mathbb{V}(\boldsymbol{X}, \mathbf{2})$, while if $\boldsymbol{X} = \mathbf{2}$, then $\boldsymbol{A} \ncong \mathbf{2}$ (as $\boldsymbol{A} \notin \mathbb{V}(\boldsymbol{X})$), so $\boldsymbol{S}_3 \in \mathbb{H}(\boldsymbol{A})$ (by the remark preceding Corollary 2.27), whereupon $\mathsf{K} = \mathbb{V}(\mathbf{2}, \boldsymbol{S}_3)$. Either way, this contradicts the join-irreducibility of K, so $\boldsymbol{B} \neq \mathbf{2}$, whence $\boldsymbol{D}_4 = \boldsymbol{B}$ or $\boldsymbol{C}_4 \in \mathbb{H}(\boldsymbol{B})$.

For the same reason, the cases $\boldsymbol{X} \neq \boldsymbol{D}_4 = \boldsymbol{B}$ and $\boldsymbol{X} \neq \boldsymbol{C}_4 \in \mathbb{H}(\boldsymbol{B})$ are ruled out, as K would be $\mathbb{V}(\boldsymbol{X}, \boldsymbol{D}_4)$ in the first of these, and $\mathbb{V}(\boldsymbol{X}, \boldsymbol{C}_4)$ in the second. If $\boldsymbol{X} = \boldsymbol{C}_4 \in \mathbb{H}(\boldsymbol{B}) \setminus \mathbb{I}(\boldsymbol{B})$, then $\mathsf{K} = \mathbb{V}(\boldsymbol{B})$, instantiating (iii)(2), as $\boldsymbol{C}_5, \ldots, \boldsymbol{C}_8$ are, up to isomorphism, the only SI 0-generated De Morgan monoids that map homomorphically onto $\boldsymbol{C}_4$ but are not isomorphic to it (see Section 3.3). The assertion '$\boldsymbol{X} = \boldsymbol{C}_4 \in \mathbb{H}(\boldsymbol{B}) \setminus \mathbb{I}(\boldsymbol{B})$' may therefore be assumed false. (The exclusivity claim in (iii) will be proved separately below.)

It follows that $\boldsymbol{B} \cong \boldsymbol{X} \in \{\boldsymbol{C}_4, \boldsymbol{D}_4\}$. We identify $\boldsymbol{B}$ with $\boldsymbol{X}$ and refer henceforth only to the latter. Thus, $\boldsymbol{X}$ is a subalgebra of $\boldsymbol{A}$, and $\boldsymbol{X} \neq \boldsymbol{A}$ (as $\boldsymbol{A} \notin \mathbb{V}(\boldsymbol{X})$), so $\boldsymbol{A}$ is not 0-generated. Also, K has no nontrivial idempotent member (otherwise K would be $\mathbb{V}(\boldsymbol{X}, \mathbf{2})$ or $\mathbb{V}(\boldsymbol{X}, \boldsymbol{S}_3)$), so K consists of anti-idempotent algebras, by Corollary 2.14.

By Theorem 1.15, there is a surjective homomorphism $h\colon \boldsymbol{A} \to \boldsymbol{E}$ for some simple $\boldsymbol{E} \in \mathsf{K}$. Now $\boldsymbol{E} \ncong \boldsymbol{D}_4$, by Theorem 3.4, because $\boldsymbol{A}$ is not 0-generated.

If $\boldsymbol{X} = \boldsymbol{C}_4$, then $\boldsymbol{C}_4 \in \mathbb{S}(\boldsymbol{A})$. If, moreover, $\boldsymbol{C}_4 \in \mathbb{H}(\boldsymbol{A})$, then $\boldsymbol{A} \in \mathsf{N}$, by Remark 3.7, so $\boldsymbol{A} \in \mathsf{M}$, by Corollary 3.20, because $\boldsymbol{A}$ is rigorously compact. In this case, $\mathsf{K} \subseteq \mathsf{M}$, because $\mathsf{K} = \mathbb{V}(\boldsymbol{A})$.

We may therefore assume that $\boldsymbol{X} = \boldsymbol{D}_4$ or $\boldsymbol{X} = \boldsymbol{C}_4 \notin \mathbb{H}(\boldsymbol{A})$. In both cases, $\boldsymbol{E} \ncong \boldsymbol{X}$. As $\boldsymbol{E}$ is a nontrivial member of K, it is not idempotent, so the subalgebra $h[\boldsymbol{X}]$ of $\boldsymbol{E}$ cannot be trivial (by Theorem 2.11). Therefore, $h|_X$ embeds $\boldsymbol{X}$ into $\boldsymbol{E}$, because $\boldsymbol{X}$ is simple. Since $\boldsymbol{X}$ is 0-generated and finite, it is isomorphic to a proper subalgebra of a 1-generated subalgebra $\boldsymbol{E}'$ of $\boldsymbol{E}$. As $\boldsymbol{E}$ is simple, so is $\boldsymbol{E}'$, by Theorem 1.24, since IRLs have the CEP. Thus, because $\boldsymbol{X} \ncong \boldsymbol{E}' \in \mathsf{K}$, Fact 4.8 gives $\mathsf{K} = \mathbb{V}(\boldsymbol{E}')$, witnessing (iii)(3) or (iv).

For the mutual exclusivity claim in (iii), note that (1) precludes (2) (as $\boldsymbol{C}_n \notin \mathsf{M}$) and (3) (as $\boldsymbol{C}_4 \notin \mathbb{H}(\boldsymbol{A})$ for the simple generator $\boldsymbol{A}$ of K in (3)). Also, (2) precludes (3), by Corollary 2.17, because $\boldsymbol{C}_n$ is SI but not simple. □

If K and $\boldsymbol{A}$ are as in Theorem 4.9(iii)(3) [resp. 4.9(iv)], then K is semisimple, by Corollary 2.17. If, moreover, $\boldsymbol{A}$ is finite, then the class of simple members of K is $\mathbb{I}(\boldsymbol{C}_4, \boldsymbol{A})$ [resp. $\mathbb{I}(\boldsymbol{D}_4, \boldsymbol{A})$], by Jónsson's Theorem 1.23 and the CEP. The options for $\boldsymbol{A}$ are discussed in Sections 4.4 and 4.5.

An immediate consequence of Theorem 4.9(iii) is the following.

Corollary 4.10. *The varieties $\mathbb{V}(\boldsymbol{C}_5)$, $\mathbb{V}(\boldsymbol{C}_6)$, $\mathbb{V}(\boldsymbol{C}_7)$ and $\mathbb{V}(\boldsymbol{C}_8)$ are exactly the covers of $\mathbb{V}(\boldsymbol{C}_4)$ within* U *that are not within* M.

In the next section, we shall show that $\mathbb{V}(\boldsymbol{C}_4)$ has just six covers within M. Some preparatory results will be required.

Lemma 4.11. *Let K be a cover of $\mathbb{V}(\boldsymbol{C}_4)$ within U. Then $\mathsf{K} = \mathbb{V}(\boldsymbol{A})$ for some skew reflection $\boldsymbol{A}$ of an SI Dunn monoid $\boldsymbol{B}$, where 0 is meet-irreducible in $\boldsymbol{A}$, and $\boldsymbol{A}$ is generated by the greatest strict lower bound of e in $\boldsymbol{B}$.*

Proof. By assumption, there is an SI algebra $\boldsymbol{G} \in \mathsf{K} \setminus \mathbb{V}(\boldsymbol{C}_4)$, and K is join-irreducible in the subvariety lattice of DMM. As $\boldsymbol{G} \in \mathsf{U}$, Corollary 3.25 shows that $\boldsymbol{G}$ is a skew reflection of a Dunn monoid $\boldsymbol{H}$, and *0* is meet-irreducible in $\boldsymbol{G}$. Now $\boldsymbol{H}$ is nontrivial, because $\boldsymbol{G} \ncong \boldsymbol{C}_4$, so Remark 3.23 shows that $\boldsymbol{H}$ is SI, and that $\boldsymbol{H}$ includes the greatest strict lower bound of e in $\boldsymbol{G}$, which we denote by c. Then $\boldsymbol{A} := \mathbf{Sg}^{\boldsymbol{G}}\{c\} \in \mathsf{K}$ is SI, by Lemma 2.16(iii), and $\boldsymbol{A} \ncong \boldsymbol{C}_4$, as $0 < c < e$. Consequently, $\mathsf{K} = \mathbb{V}(\boldsymbol{A})$, by Fact 4.8. Clearly, $\boldsymbol{A}$ is the skew reflection of the SI Dunn monoid $\boldsymbol{B} := \mathbf{Sg}^{\boldsymbol{A}}(H \cap A)$, with respect to the restricted order of $\boldsymbol{A}$, and *0* is meet-irreducible in $\boldsymbol{A}$. □

A partial converse of Lemma 4.11 is supplied below. It extends the claim about $\boldsymbol{C}_5, \ldots, \boldsymbol{C}_8$ preceding Theorem 4.9.

Lemma 4.12. *If a skew reflection $\boldsymbol{A} \in \mathsf{U}$ of a finite simple Dunn monoid $\boldsymbol{B}$ is generated by the least element of $\boldsymbol{B}$, then $\mathbb{V}(\boldsymbol{A})$ is a (join-irreducible) cover of $\mathbb{V}(\boldsymbol{C}_4)$ within U.*

Proof. Let $\bot$ be the least element of $\boldsymbol{B}$. By Lemma 2.16(iv), $\bot$ is the only strict lower bound of e in $\boldsymbol{B}$. The lower bounds of e in $\boldsymbol{A}$ therefore form the chain $0 < \bot < e$, so $\boldsymbol{A} \ncong \boldsymbol{C}_4$, but $\boldsymbol{A}$ is SI, by Remark 3.23. Therefore, $\boldsymbol{A} \notin \mathbb{V}(\boldsymbol{C}_4)$, by Jónsson's Theorem 1.23, and so $\mathbb{V}(\boldsymbol{C}_4) \subsetneq \mathbb{V}(\boldsymbol{A}) \subseteq \mathsf{U}$. To see that $\mathbb{V}(\boldsymbol{A})$ covers $\mathbb{V}(\boldsymbol{C}_4)$, let $\boldsymbol{E} \in \mathbb{V}(\boldsymbol{A}) \setminus \mathbb{V}(\boldsymbol{C}_4)$ be SI. We must show that $\boldsymbol{A} \in \mathbb{V}(\boldsymbol{E})$. Since $\boldsymbol{A}$ is finite, Jónsson's Theorem 1.23 gives $\boldsymbol{E} \in \mathbb{HS}(\boldsymbol{A})$. Any subalgebra $\boldsymbol{D}$ of $\boldsymbol{A}$ is nontrivial, so e has a strict lower bound in $\boldsymbol{D}$, by (2.16). If *0* is the only strict lower bound of e in $\boldsymbol{D}$, then $\boldsymbol{D}$ is a simple member of U (by Lemma 2.16(iv)), whence $\boldsymbol{D} \cong \boldsymbol{C}_4$ (as $\mathsf{U} \subseteq \mathsf{W}$).

Otherwise, $\bot \in D$, in which case $\boldsymbol{D} = \boldsymbol{A}$, as $\boldsymbol{A}$ is generated by $\bot$. Thus, $\mathbb{S}(\boldsymbol{A}) \subseteq \{\boldsymbol{C}_4, \boldsymbol{A}\}$, and so $\boldsymbol{E}$ is a homomorphic image of $\boldsymbol{A}$ (as $\boldsymbol{C}_4$ is simple). Now $\boldsymbol{A}$ has only three deductive filters (because $|(e]| = 3$ in $\boldsymbol{A}$), so $\boldsymbol{A}$ has just three factor algebras, of which $\boldsymbol{A}$ and a trivial algebra are two. The other is isomorphic to $\boldsymbol{C}_4$, as $\boldsymbol{A} \in \mathsf{U} \subseteq \mathsf{W}$. Therefore, $\boldsymbol{E} \cong \boldsymbol{A}$, whence $\boldsymbol{A} \in \mathbb{V}(\boldsymbol{E})$, as required. □

The RL-reducts of $\mathbf{2}$, $\boldsymbol{S}_3$ and $\boldsymbol{C}_4$ shall be denoted by $\mathbf{2}^+$, $\boldsymbol{S}_3^+$ and $\boldsymbol{C}_4^+$, respectively. (In fact, $\boldsymbol{S}_3$ and $\boldsymbol{S}_3^+$ are termwise equivalent, because $\neg x$ is definable as $x \to e$ in $\boldsymbol{S}_3^+$.) The following result will be needed later.

Theorem 4.13. *Let $\boldsymbol{B}$ be a square-increasing RL that is SI. Let c be the greatest strict lower bound of e in $\boldsymbol{B}$ (which exists, by Lemma 2.16(iii)).*

If $c \to e = e$, then $\mathrm{Sg}^{\boldsymbol{B}}\{c\} = \{c, e\}$ and $\mathbf{Sg}^{\boldsymbol{B}}\{c\} \cong \mathbf{2}^+$.

If $c \to e \neq e$, then $\mathbf{Sg}^{\boldsymbol{B}}\{c\} \cong \boldsymbol{S}_3^+$, its lattice reduct being $c < e < c \to e$.

Proof. As $c < e$, we have $c^2 = c$, by (2.21), and $e \leqslant c \to e$, by (2.15). Then $c \to c = c \to e$, because (2.14), (2.17) and (2.22) yield

$$c \to c \;\leqslant\; c \to e \;\leqslant\; c \to (c \to c) \;\leqslant\; c \to c.$$

Therefore, in view of (2.17), if $c \to e = e$, then $\{c, e\}$ is the universe of a subalgebra of $\boldsymbol{B}$, isomorphic to $\mathbf{2}^+$.

We may now assume that $e < c \to e$. Then $(c \to e) \to e \leqslant e \to e = e$, by (2.14), whereas $e \nleqslant (c \to e) \to e$, by (2.15), so $(c \to e) \to e < e$. Then $(c \to e) \to e \leqslant c$, by definition of c, so $(c \to e) \to e = c$, by (2.6). It suffices, therefore, to show that the chain $(c \to e) \to e < e < c \to e$ constitutes a subalgebra of $\boldsymbol{B}$, isomorphic to $\boldsymbol{S}_3^+$, but this was already proved by Galatos [50, Thm. 5.7]. Although its statement in [50] assumes idempotence (and a weak form of commutativity) for fusion, all appeals to idempotence in the proof require only the square-increasing law. □

4.3 Covers of $\mathbb{V}(\mathbf{C}_4)$ within M

If K is a cover of $\mathbb{V}(\boldsymbol{C}_4)$ within M, then by Lemma 4.11 and Remark 3.23, there exist $\boldsymbol{A}$, $\boldsymbol{B}$ and $\bot$ such that $\mathsf{K} = \mathbb{V}(\boldsymbol{A})$,

- $\boldsymbol{B}$ is an SI Dunn monoid, $\boldsymbol{A}$ is a skew reflection of $\boldsymbol{B}$ in which $e < f$, and $\boldsymbol{A} = \mathbf{Sg}^{\boldsymbol{A}}\{\bot\}$, where $\bot \in B$ is the greatest strict lower bound of e in $\boldsymbol{A}$.

The displayed properties of $\boldsymbol{A}$, $\boldsymbol{B}$ and $\bot$ will now be *assumed*, until the 'conclusions' after Lemma 4.22. By Lemma 3.26(ii), they imply that $\mathit{0}$ is meet-irreducible (and $\mathit{1}$ join-irreducible) in $\boldsymbol{A}$. We shall prove that they also force $\boldsymbol{A}$ to be finite and $\boldsymbol{B}$ simple, with $|A| \leq 14$ (i.e., $|B| \leq 6$).

We define $\top = \bot \to e$, so $\top \in B$. By Theorem 4.13, $\mathbf{Sg}^{\boldsymbol{B}}\{\bot\}$ consists of $\bot, e, \top$ and is isomorphic to $\mathbf{2}^+$ (with $e = \top$) or to $\boldsymbol{S}_3^+$ (with $e < \top$). The respective tables for $\cdot$ and $\to$ in $\mathbf{Sg}^{\boldsymbol{B}}\{\bot\}$ are recalled below. (There is no guarantee that $\mathbf{Sg}^{\boldsymbol{B}}\{\bot\}$ exhausts $\boldsymbol{B}$.)

$\cdot$	$\bot$	$\top$
$\bot$	$\bot$	$\bot$
$\top$	$\bot$	$\top$

$\to$	$\bot$	$\top$
$\bot$	$\top$	$\top$
$\top$	$\bot$	$\top$

or

$\cdot$	$\bot$	e	$\top$
$\bot$	$\bot$	$\bot$	$\bot$
e	$\bot$	e	$\top$
$\top$	$\bot$	$\top$	$\top$

$\to$	$\bot$	e	$\top$
$\bot$	$\top$	$\top$	$\top$
e	$\bot$	e	$\top$
$\top$	$\bot$	$\bot$	$\top$

Warning. Although $\bot, \top$ will turn out to be extrema for $\boldsymbol{B}$, that fact will emerge only after Lemma 4.22. Until then, our use of these symbols should not be taken to justify claims like '$b \cdot \bot = \bot$ for all $b \in B$' on the basis of Lemma 2.3 alone. Such claims will be justified directly when needed.

All this notation will remain fixed until the 'conclusions' after Lemma 4.22. We use freely the notation from Definition 3.21 as well, e.g., $\bot' = \neg^{\boldsymbol{A}}\bot \in B'$ and $\top' = \neg^{\boldsymbol{A}}\top \in B'$. The superscript $\boldsymbol{A}$ will normally be omitted.

Theorem 4.14. *The algebra* $\boldsymbol{A}$ *is the reflection of* $\boldsymbol{B}$ *iff* e *and* $\top'$ *are comparable. In this case,* $e < \top'$ *and* $\boldsymbol{B} = \mathbf{Sg}^{\boldsymbol{B}}\{\bot\}$, *so* $\boldsymbol{B}$ *consists of* $\bot, e, \top$ *only, and*

(i) $e = \top$ *iff* $\boldsymbol{B} \cong \mathbf{2}^+$, *iff* $\boldsymbol{A} \cong \mathrm{R}(\mathbf{2}^+)$;

(ii) $e \neq \top$ *iff* $\boldsymbol{B} \cong \boldsymbol{S}_3^+$, *iff* $\boldsymbol{A} \cong \mathrm{R}(\boldsymbol{S}_3^+)$.

Proof. In the first assertion, necessity follows from the definition of reflection. Conversely, suppose e and $\top'$ are comparable, and let h be the unique homomorphism from $\boldsymbol{A}$ to $\boldsymbol{C}_4$. As h is isotone and $h(e) = e < f = h(\top')$, we can't have $\top' \leqslant e$, so $e < \top' = (\top \cdot \top)'$. Then, by Definition 3.21(iii), $\top \leqslant \top'$, so

$$\mathit{0} < \bot < e \leqslant \top < \top' \leqslant f < \bot' < \mathit{1}, \tag{4.5}$$

where $e = \top$ iff $\top' = f$. The elements $\bot, e, \top \in B$ are closed under $\cdot, \to$, so items (vi)–(ix) of Definition 3.21 ensure that the elements listed in (4.5) are closed under $\cdot, \wedge, \vee, \neg$. They include e, so they constitute a subalgebra of $\boldsymbol{A}$. As they also include $\bot$, which generates $\boldsymbol{A}$, they exhaust $\boldsymbol{A}$. Consequently, $\boldsymbol{A} = \mathrm{R}(\boldsymbol{B})$, where $\boldsymbol{B}$ consists of $\bot, e, \top$ only, and is therefore generated by $\bot$. Then (i) and (ii) follow from Theorem 4.13. $\square$

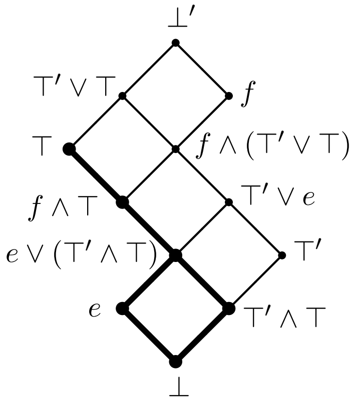
Corollary 4.15. *If* $\mathbf{Sg}^{\boldsymbol{B}}\{\bot\} \ncong \boldsymbol{S}_3^+$, *then* $\boldsymbol{A} \cong \mathrm{R}(\mathbf{2}^+)$.

Proof. In this case, $\mathbf{Sg}^{\boldsymbol{B}}\{\bot\} \cong \mathbf{2}^+$, by Theorem 4.13, so $e = \top$. As $\boldsymbol{A} \in \mathsf{M}$, we have $e < f = e' = \top'$, so the result follows from Theorem 4.14. □

By Lemma 4.12, $\mathrm{R}(\mathbf{2}^+)$ and $\mathrm{R}(\boldsymbol{S}_3^+)$ generate covers of $\mathbb{V}(\boldsymbol{C}_4)$ within M, as each is generated by its own unique atom. Because our aim is now to isolate the *other* covers of $\mathbb{V}(\boldsymbol{C}_4)$ within M, the previous two results allow us to assume, until further notice, that

- $\mathbf{Sg}^{\boldsymbol{B}}\{\bot\}$ is isomorphic to $\boldsymbol{S}_3^+$ and has universe $\bot < e < \top$, and
- $\boldsymbol{A}$ is *not* the reflection of $\boldsymbol{B}$, i.e., e and $\top'$ are incomparable.

Consider the formal diagram below.



By Theorem 3.24(iv), the labels on the thicker points all identify elements of B, but we do *not* claim that the twelve depicted elements are distinct in $\boldsymbol{A}$. (It will turn out that they exhaust $\boldsymbol{A}$, but that is not yet obvious.)

Lemma 4.16. *The subset of A comprising the elements depicted above is closed under the operations* $\wedge, \vee$ *and* $\neg$ *of* $\boldsymbol{A}$.

Moreover, the label on the diagrammatic join of any two elements is the actual join in $\boldsymbol{A}$ *of the labels on those elements, and similarly for meets.*

Proof. Note first that the diagram order is sound, in the sense that wherever x is depicted as a lower bound of y, then $x \leqslant y$ in $\boldsymbol{A}$. This is easy to see, except perhaps for the $\boldsymbol{A}$-inequalities $\bot \leqslant \top'$ and $\top' \wedge \top \leqslant f$ $(= e')$. The first of these follows from Lemma 3.26(i), as $\top = \bot \to e$, and the second from Definition 3.21(iii), because $e \leqslant \top \leqslant \top \vee \top' = (\top' \wedge \top)' = ((\top' \wedge \top) \cdot e)'$ in $\boldsymbol{A}$.

Closure under $\neg$ follows from the identity $\neg\neg x \approx x$ and De Morgan's laws.

Let x, y be expressions from the diagram. We shall show that $x \vee y$ (computed in $\boldsymbol{A}$) is equal (in $\boldsymbol{A}$) to the label on the diagrammatic join of x, y. In view of the chosen labels and De Morgan's laws, the same will then follow for meets. We go through the possible values of x. Because the diagram order is sound, we can eliminate cases where y is comparable (according to the diagram) with x. This eliminates $\bot$ and $\bot'$ as values for x and for y.

If x is e, then the uneliminated values of y are $\top'$ and $\top' \wedge \top$. These cases are disposed of by noting that $e \vee \top'$ and $e \vee (\top' \wedge \top)$ appear (up to commutativity of $\vee$) in the diagram, and that the diagram order makes them the least upper bounds, respectively, of $e, \top'$ and of $e, \top' \wedge \top$.

When x is $\top' \wedge \top$, the only uneliminated value of y is e, which we have just considered.

When x is $e \vee (\top' \wedge \top)$, the uneliminated possibilities for y are $e, \top' \wedge \top, \top'$, of which all but $\top'$ have been considered. As $e \vee (\top' \wedge \top) \vee \top' = \top' \vee e$, which appears (in the correct place) in the diagram, we are done with this case.

When x is $\top'$, the only uneliminated choices for y, not already considered, are $\top$ and $f \wedge \top$. Now $\top' \vee \top$ is well-placed in the diagram, and in $\boldsymbol{A}$, we have $\top' \vee (f \wedge \top) = (\top' \vee f) \wedge (\top' \vee \top) = f \wedge (\top' \vee \top)$, which is also well-placed.

When x is $\top' \vee e$, the only interesting possibilities for y are $\top$ and $f \wedge \top$. And in $\boldsymbol{A}$, we have well-placed values $(\top' \vee e) \vee \top = \top' \vee \top$ and

$$(\top' \vee e) \vee (f \wedge \top) = (\top' \vee e \vee f) \wedge (\top' \vee e \vee \top) = f \wedge (\top' \vee \top).$$

From cases already considered, it follows that $(f \wedge \top) \vee y$ is well-placed in the diagram, for every y.

When x is $f \wedge (\top' \vee \top)$, the only interesting y is $\top$. In $\boldsymbol{A}$, we have

$$(f \wedge (\top' \vee \top)) \vee \top = (f \vee \top) \wedge ((\top' \vee \top) \vee \top) = (f \vee \top) \wedge (\top' \vee \top).$$

This expression will simplify to the well-placed $\top' \vee \top$, provided that $f \vee \top = \bot'$, or equivalently, $e \wedge \top' = \bot$ (in $\boldsymbol{A}$), which we now show. We have already verified that $\bot \leqslant \top'$, so $\bot \leqslant e \wedge \top'$. As $e \nleqslant \top'$, we have $e \wedge \top' < e$, whence $e \wedge \top' \leqslant \bot$ (by definition of $\bot$), and so $e \wedge \top' = \bot$, as required.

When x is $\top$, the only new y to consider is f, but we have just shown that $\top \vee f = \bot'$, which is well-placed.

When x is f, the only new y is $\top' \vee \top$, and $\top \leqslant \top' \vee \top \leqslant \bot'$. In $\boldsymbol{A}$, we have seen that $f \vee \top = \bot' = f \vee \bot'$, so $f \vee (\top' \vee \top) = \bot'$, which is well-placed.

When x is $\top' \vee \top$, there is no longer any unconsidered option for y. $\square$

Lemma 4.17. *Fusion in **A** behaves as in the following table.*

$\cdot$	$\bot$	e	$\top' \wedge \top$	$e \vee (\top' \wedge \top)$	$f \wedge \top$	$\top$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
e		e	$\top' \wedge \top$	$e \vee (\top' \wedge \top)$	$f \wedge \top$	$\top$
$\top' \wedge \top$			$\top' \wedge \top$	$\top' \wedge \top$	$\top' \wedge \top$	$\top' \wedge \top$
$e \vee (\top' \wedge \top)$				$e \vee (\top' \wedge \top)$	$f \wedge \top$	$\top$
$f \wedge \top$						$\top$
$\top$						$\top$

Proof. The submatrix involving only $\bot, e, \top$ is justified, because $\mathbf{Sg}^{\boldsymbol{B}}\{\bot\}$ is an RL-subreduct of $\boldsymbol{A}$. If $x \in \{\top' \wedge \top,\ e \vee (\top' \wedge \top),\ f \wedge \top\}$, then $\bot \leqslant x \leqslant \top$, so $\bot \cdot x = \bot$, by (2.14), since $\bot^2 = \bot = \bot \cdot \top$. This justifies the first row; the second records the neutrality of e in $\boldsymbol{A}$.

To see that $(\top' \wedge \top) \cdot \top = \top' \wedge \top$, note the following consequences of (2.14), Definition 3.21(viii) and the tables for $\mathbf{Sg}^{\boldsymbol{B}}\{\bot\}$:

$$(\top' \wedge \top) \cdot \top \leqslant (\top' \cdot \top) \wedge \top^2 = (\top \to \top)' \wedge \top$$
$$= \top' \wedge \top = (\top' \wedge \top) \cdot e \leqslant (\top' \wedge \top) \cdot \top.$$

For any $x \in \{\top' \wedge \top,\ e \vee (\top' \wedge \top),\ f \wedge \top\}$, we now have

$$\top' \wedge \top \leqslant (\top' \wedge \top)^2 \leqslant (\top' \wedge \top) \cdot x \leqslant (\top' \wedge \top) \cdot \top = \top' \wedge \top,$$

so $(\top' \wedge \top) \cdot x = \top' \wedge \top$. If $y \in \{e \vee (\top' \wedge \top),\ f \wedge \top,\ \top\}$, then

$$\top = e \cdot \top \leqslant y \cdot \top \leqslant \top^2 = \top,$$

whence $y \cdot \top = \top$. By (2.11) and the idempotence of $\top' \wedge \top$,

$$(e \vee (\top' \wedge \top))^2 = e \vee (\top' \wedge \top) \vee (\top' \wedge \top)^2 = e \vee (\top' \wedge \top).$$

Finally, by (2.11) and since $\top' \leqslant f$, we have

$$(e \vee (\top' \wedge \top)) \cdot (f \wedge \top) = (e \cdot (f \wedge \top)) \vee ((\top' \wedge \top) \cdot (f \wedge \top))$$
$$= (f \wedge \top) \vee (\top' \wedge \top) = f \wedge \top.$$ $\square$

Lemma 4.18. *Residuation in **A** behaves as in the following table.*

$\rightarrow$	$\bot$	e	$\top' \wedge \top$	$e \vee (\top' \wedge \top)$	$f \wedge \top$	$\top$
$\bot$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$
e	$\bot$	e	$\top' \wedge \top$	$e \vee (\top' \wedge \top)$	$f \wedge \top$	$\top$
$\top' \wedge \top$			$\top$	$\top$	$\top$	$\top$
$e \vee (\top' \wedge \top)$	$\bot$		$\top' \wedge \top$	$e \vee (\top' \wedge \top)$	$f \wedge \top$	$\top$
$f \wedge \top$	$\bot$		$\top' \wedge \top$		$e \vee (\top' \wedge \top)$	$\top$
$\top$	$\bot$	$\bot$	$\top' \wedge \top$	$\top' \wedge \top$	$\top' \wedge \top$	$\top$

Proof. All elements in the table lie between $\bot$ and $\top$. The submatrix involving only $\bot, e, \top$ documents residuation in $\mathbf{Sg}^{\boldsymbol{B}}\{\bot\}$, so the first row and last column follow from (2.14), because $\top = \bot \rightarrow \bot = \bot \rightarrow \top = \top \rightarrow \top$. The second row is justified by (2.17). Then $(\top' \wedge \top) \rightarrow \top' = \neg((\top' \wedge \top) \cdot \top) = (\top' \wedge \top)'$ (Lemma 4.17) $= \top' \vee \top$, so by (2.12),

$$(\top' \wedge \top) \rightarrow (\top' \wedge \top) = ((\top' \wedge \top) \rightarrow \top') \wedge ((\top' \wedge \top) \rightarrow \top) = (\top' \vee \top) \wedge \top = \top.$$

Now, for each $x \in \{e \vee (\top' \wedge \top),\ f \wedge \top\}$,

$$\top = (\top' \wedge \top) \rightarrow (\top' \wedge \top) \leqslant (\top' \wedge \top) \rightarrow x \leqslant (\top' \wedge \top) \rightarrow \top = \top,$$

by (2.14), whence $(\top' \wedge \top) \rightarrow x = \top$.

Clearly, $e \vee (\top' \wedge \top) \leqslant \top = \bot \rightarrow \bot$, so (2.8) and (2.14) yield

$$\bot \leqslant (e \vee (\top' \wedge \top)) \rightarrow \bot \leqslant e \rightarrow \bot = \bot,$$

whence $(e \vee (\top' \wedge \top)) \rightarrow \bot = \bot$. By Lemma 4.17, $e \vee (\top' \wedge \top)$ is an idempotent upper bound of e, so $(e \vee (\top' \wedge \top)) \rightarrow (e \vee (\top' \wedge \top)) = e \vee (\top' \wedge \top)$, by Lemma 2.7. Also, by (2.13),

$$\begin{aligned}(e \vee (\top' \wedge \top)) \rightarrow (\top' \wedge \top) &= (e \rightarrow (\top' \wedge \top)) \wedge ((\top' \wedge \top) \rightarrow (\top' \wedge \top)) \\ &= (\top' \wedge \top) \wedge \top = \top' \wedge \top;\end{aligned}$$

$$\begin{aligned}(e \vee (\top' \wedge \top)) \rightarrow (f \wedge \top) &= (e \rightarrow (f \wedge \top)) \wedge ((\top' \wedge \top) \rightarrow (f \wedge \top)) \\ &= (f \wedge \top) \wedge \top = f \wedge \top.\end{aligned}$$

Similarly, from $f \wedge \top \leqslant \top = \bot \rightarrow \bot$ and $e \leqslant f \wedge \top$ and (2.8), (2.14), we obtain $\bot \leqslant (f \wedge \top) \rightarrow \bot \leqslant e \rightarrow \bot = \bot$, hence $(f \wedge \top) \rightarrow \bot = \bot$. Also, by (2.12),

$$\begin{aligned}(f \wedge \top) \rightarrow (\top' \wedge \top) &= ((f \wedge \top) \rightarrow \top') \wedge ((f \wedge \top) \rightarrow \top) \\ &= \neg((f \wedge \top) \cdot \top) \wedge \top = \top' \wedge \top \ \text{(by Lemma 4.17)};\end{aligned}$$

$$(f \wedge \top) \to (f \wedge \top) = ((f \wedge \top) \to f) \wedge ((f \wedge \top) \to \top) = \neg(f \wedge \top) \wedge \top$$
$$= (e \vee \top') \wedge \top = e \vee (\top' \wedge \top) \ \text{(by distributivity, since } e \leqslant \top);$$

$$\top \to (\top' \wedge \top) = (\top \to \top') \wedge (\top \to \top) = \neg(\top \cdot \top) \wedge \top = \top' \wedge \top;$$
$$\top \to (f \wedge \top) = (\top \to f) \wedge (\top \to \top) = \neg\top \wedge \top = \top' \wedge \top,$$

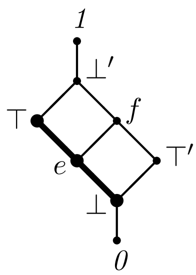
so, because $\top' \wedge \top \leqslant e \vee (\top' \wedge \top) \leqslant f \wedge \top$, (2.14) yields

$$\top \to (e \vee (\top' \wedge \top)) = \top' \wedge \top.$$ □

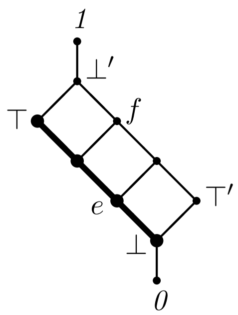
Recall that $\boldsymbol{A}$ has the following properties (under present assumptions):

- $[\top') \cap (\top] = \emptyset$, by the definition of a skew reflection,
- $\bot < e < \top$ and $\top' < f < \bot'$, as $\mathbf{Sg}^{\boldsymbol{B}}\{\bot\} \cong \boldsymbol{S}_3^+$, and
- e and $\top'$ are incomparable, by Theorem 4.14, as $\boldsymbol{A} \neq \mathrm{R}(\boldsymbol{B})$.

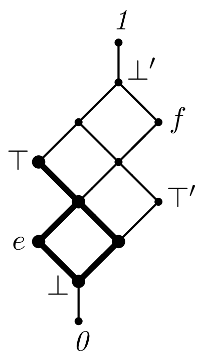
There are only four ways to identify elements from the diagram preceding Lemma 4.16 while respecting these rules. Thus, $\boldsymbol{A}$ must have one of the four Hasse diagrams below, where the thicker points denote the elements of $\boldsymbol{B}$.



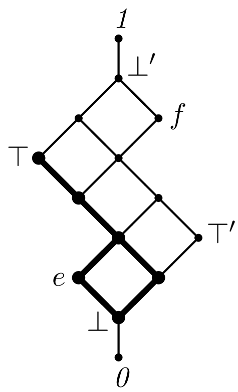
Case I



Case II



Case III



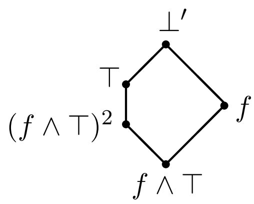
Case IV

Lemma 4.19. *In Cases II, III and IV, we have* $(f \wedge \top) \to e = \bot$.

Proof. From $f \wedge \top \leqslant \top = \bot \to e$ and (2.8) we infer $\bot \leqslant (f \wedge \top) \to e$. As $e \leqslant f \wedge \top$, (2.14) shows that $(f \wedge \top) \to e \leqslant e \to e = e$. In Cases II, III and IV, $f \wedge \top \nleqslant e$, so (2.15) shows that $(f \wedge \top) \to e < e$. Therefore, $(f \wedge \top) \to e \leqslant \bot$ (by definition of $\bot$), hence the result. □

Lemma 4.20. *In Cases II and IV, we have* $(f \wedge \top)^2 = \top$.

Proof. By (2.14), $(f \wedge \top)^2 \leqslant \top^2 = \top$. By Lemma 4.18, $(f \wedge \top) \to (f \wedge \top) = e \vee (\top' \wedge \top)$. In Cases II and IV, therefore, $(f \wedge \top) \to (f \wedge \top) \neq f \wedge \top \geqslant e$, so $f \wedge \top$ is not idempotent (by Lemma 2.7), i.e., $f \wedge \top < (f \wedge \top)^2$. Suppose, with a view to contradiction, that $(f \wedge \top)^2 < \top$. Then the diagram below depicts a five-element subposet of $\langle A; \leqslant \rangle$, which we claim is a sublattice of $\langle A; \wedge, \vee \rangle$. By Theorem 1.22, that will contradict the distributivity of $\boldsymbol{A}$, finishing the proof.



The claim amounts to the assertion that $f \vee (f \wedge \top)^2 = \bot'$. As f and $(f \wedge \top)^2$ are incomparable, we have $f < f \vee (f \wedge \top)^2 \leqslant \bot'$. In $\boldsymbol{A}$, however, $\bot'$ is the smallest strict upper bound of f (because $\bot$ is the greatest strict lower bound of e). Therefore, $f \vee (f \wedge \top)^2 = \bot'$. □

Lemma 4.21. *In Cases III and IV,* $\bot$ *is the value of all three of*

$$(\top' \wedge \top) \to e, \ \ (\top' \wedge \top) \to \bot \ \ \textit{and} \ \ (e \vee (\top' \wedge \top)) \to e.$$

Proof. As $\top' \wedge \top \leqslant \top = \bot \to \bot$ and $\bot \leqslant e$, we have

$$\bot \leqslant (\top' \wedge \top) \to \bot \leqslant (\top' \wedge \top) \to e,$$

by (2.8) and (2.14). Thus, the first claim subsumes the second. Suppose, with a view to contradiction, that

$$\bot < (\top' \wedge \top) \to e. \tag{4.6}$$

Since $e \leqslant \top$, it follows from (2.14) that

$$(\top' \wedge \top) \to e \leqslant \top \cdot ((\top' \wedge \top) \to e). \tag{4.7}$$

On the other hand, $(\top' \wedge \top) \cdot \top = \top' \wedge \top$, by Lemma 4.17, so

$$(\top' \wedge \top) \cdot \top \cdot ((\top' \wedge \top) \to e) = (\top' \wedge \top) \cdot ((\top' \wedge \top) \to e) \leqslant e,$$

by (2.6), whence $\top \cdot ((\top' \wedge \top) \to e) \leqslant (\top' \wedge \top) \to e$, by (2.2). Then, by (4.7),

$$\top \cdot ((\top' \wedge \top) \to e) = (\top' \wedge \top) \to e = d, \text{ say.} \tag{4.8}$$

In Cases III and IV, $\top' \wedge \top \nleqslant e$, so $e \nleqslant d$, by (2.15). Consequently, $d \leqslant f$, by Theorem 2.43, because $\boldsymbol{A}$ is SI. Therefore, $e \leqslant \neg d = \neg(d \cdot \top)$ (by (4.8)) $= d \to \top'$, so $d \leqslant \top'$, i.e., $(\top' \wedge \top) \to e \leqslant \top'$. Also, $\bot \leqslant \top' \wedge \top$, so by (2.14), $(\top' \wedge \top) \to e \leqslant \bot \to e = \top$. Therefore,

$$(\top' \wedge \top) \to e \leqslant \top' \wedge \top. \tag{4.9}$$

Now, by (2.2),

$$\begin{aligned} e \geqslant (\top' \wedge \top) \cdot ((\top' \wedge \top) \to e) \geqslant ((\top' \wedge \top) \to e)^2 \ \text{(by (4.9))} \\ \geqslant (\top' \wedge \top) \to e > \bot \ \text{(by (4.6))}. \end{aligned}$$

This forces

$$(\top' \wedge \top) \cdot ((\top' \wedge \top) \to e) = e, \tag{4.10}$$

by definition of $\bot$. Then, by Lemma 4.17,

$$\top' \wedge \top = (\top' \wedge \top)^2 \geqslant (\top' \wedge \top) \cdot ((\top' \wedge \top) \to e) \ \text{(by (4.9))} = e \ \text{(by (4.10))},$$

contradicting the diagrams for Cases III and IV. Thus, $(\top' \wedge \top) \to e = \bot$.

Finally, by (2.13) and the claim just proved,

$$(e \vee (\top' \wedge \top)) \to e = (e \to e) \wedge ((\top' \wedge \top) \to e) = e \wedge \bot = \bot. \qquad \square$$

The next lemma applies to Case IV. Its statement remains true in Case II, but is redundant there, as $\top' \wedge \top = \bot$ and $e \vee (\top' \wedge \top) = e$ in Case II (cf. Lemma 4.19).

Lemma 4.22. *In Case IV, we have* $(f \wedge \top) \to (e \vee (\top' \wedge \top)) = \top' \wedge \top$.

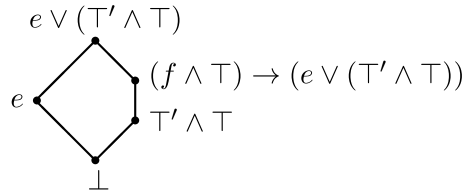
Proof. Observe that

$$\begin{aligned} \top' \wedge \top &= (f \wedge \top) \to (\top' \wedge \top) \ \text{(by Lemma 4.18)} \\ &\leqslant (f \wedge \top) \to (e \vee (\top' \wedge \top)) \ \text{(by (2.14))} \\ &\leqslant e \to (e \vee (\top' \wedge \top)) \ \text{(by (2.14), as } e \leqslant f \wedge \top) \\ &= e \vee (\top' \wedge \top) \ \text{(by (2.17))}. \end{aligned}$$

In Case IV, $f \wedge \top \nleqslant e \vee (\top' \wedge \top)$, so $e \nleqslant (f \wedge \top) \to (e \vee (\top' \wedge \top))$, by (2.15). Thus, $(f \wedge \top) \to (e \vee (\top' \wedge \top)) \neq e \vee (\top' \wedge \top)$. Suppose, with a view to contradiction, that $(f \wedge \top) \to (e \vee (\top' \wedge \top)) \neq \top' \wedge \top$. Then

$$\top' \wedge \top < (f \wedge \top) \to (e \vee (\top' \wedge \top)) < e \vee (\top' \wedge \top),$$

so the Hasse diagram below depicts a five-element subposet of $\langle A; \leqslant \rangle$.



Using the fact that $\perp$ is the greatest strict lower bound of e in $\boldsymbol{A}$, we obtain

$$e \wedge ((f \wedge \top) \rightarrow (e \vee (\top' \wedge \top))) \leqslant \perp$$

(cf. the proof of Lemma 4.20). On the other hand, by Lemma 4.17,

$$(f \wedge \top) \cdot \perp = \perp \leqslant e \vee (\top' \wedge \top),$$

so by (2.2), $\perp \leqslant (f \wedge \top) \rightarrow (e \vee (\top' \wedge \top))$. Also, $\perp \leqslant e$, so

$$\perp \leqslant e \wedge ((f \wedge \top) \rightarrow (e \vee (\top' \wedge \top))).$$

Therefore, $e \wedge ((f \wedge \top) \rightarrow (e \vee (\top' \wedge \top))) = \perp$, whence the elements depicted above form a sublattice of $\langle A; \wedge, \vee \rangle$, contradicting the distributivity of $\boldsymbol{A}$. $\square$

This completes the tables from Lemmas 4.17 and 4.18 in all cases.

Conclusions.

The above arguments put constraints on $\boldsymbol{B}$ and on the order $\leqslant$ if $\boldsymbol{A} = \mathrm{S}^{\leqslant}(\boldsymbol{B})$ is to generate a cover of $\mathbb{V}(\boldsymbol{C}_4)$ within M. In particular, $\boldsymbol{B}$ must be finite and simple, with $|B| \leq 6$ (i.e., $|A| \leq 14$), and in each of Cases I–IV, there is at most one way to choose $\leqslant$ and the operations $\cdot, \rightarrow$ on $\boldsymbol{B}$ if this is to happen, in view of Lemmas 4.16–4.22. It remains, however, to check that in each case, $\boldsymbol{B}$ really is a Dunn monoid for which $\mathbf{Sg}^{\boldsymbol{A}}\{\perp\} = \boldsymbol{A}$. If so, then since $\boldsymbol{B}$ is finite and simple, $\mathbb{V}(\boldsymbol{A})$ will indeed be a cover of $\mathbb{V}(\boldsymbol{C}_4)$ within M, by Lemma 4.12, and the resulting varieties will be the only covers of $\mathbb{V}(\boldsymbol{C}_4)$ within M, apart from $\mathrm{R}(\mathbf{2}^+)$ and $\mathrm{R}(\boldsymbol{S}_3^+)$.

In Case I, the intended $\boldsymbol{B}$ is clearly the Dunn monoid $\boldsymbol{S}_3^+$, which is generated by $\perp$, so $\mathbf{Sg}^{\boldsymbol{A}}\{\perp\} = \boldsymbol{A}$.

In Case II, $\boldsymbol{B} \cong \boldsymbol{C}_4^+$, so $\boldsymbol{B}$ is a Dunn monoid. Its elements form the chain

$$\perp < e < f \wedge \top < \top.$$

As the co-atom of $\boldsymbol{B}$ is $f \wedge \top$, it is clear that $\perp$ generates the skew reflection $\boldsymbol{A}$ of $\boldsymbol{B}$ shown in the diagram for Case II.

In Case III, the intended elements of $\boldsymbol{B}$ are

$$\bot,\ e,\ \top' \wedge \top,\ \top \ \text{ and } \ f \wedge \top = e \vee (\top' \wedge \top).$$

That the operations in the lemmas turn this into a Dunn monoid (actually, an idempotent one) with neutral element e can be verified mechanically, the only real issues being the associativity of fusion and the law of residuation; we omit the details.

We shall call this Dunn monoid $\boldsymbol{T}_5$. It is clear from the above description of its elements that its skew reflection $\boldsymbol{A}$, in the diagram for Case III, is generated by $\bot$.

Finally, in Case IV, the intended elements of $\boldsymbol{B}$ are

$$\bot,\ e,\ \top' \wedge \top,\ e \vee (\top' \wedge \top),\ f \wedge \top \ \text{ and } \ \top.$$

We suppress the mechanical verification that this becomes a Dunn monoid, with neutral element e, when equipped with the operations in the lemmas.

We denote this Dunn monoid by $\boldsymbol{T}_6$. Again, the above description of its elements shows that its skew reflection $\boldsymbol{A}$, in the diagram for Case IV, is generated by $\bot$.

We have now proved the following.

Theorem 4.23. *The covers of* $\mathbb{V}(\boldsymbol{C}_4)$ *within* M *are just*

$$\mathbb{V}(\mathrm{R}(\boldsymbol{2}^+)),\ \mathbb{V}(\mathrm{R}(\boldsymbol{S}_3^+)),\ \mathbb{V}(\mathrm{S}^{\leqslant}(\boldsymbol{S}_3^+)),\ \mathbb{V}(\mathrm{S}^{\leqslant}(\boldsymbol{C}_4^+)),\ \mathbb{V}(\mathrm{S}^{\leqslant}(\boldsymbol{T}_5)) \text{ and } \mathbb{V}(\mathrm{S}^{\leqslant}(\boldsymbol{T}_6)),$$

for the last four of which $\leqslant$ *is as in the respective diagrams of Cases I–IV.*

We shall see in Theorem 5.43 that these varieties in fact cover $\mathbb{V}(\boldsymbol{C}_4)$ as quasivarieties.

Combining Theorem 4.23 and Corollary 4.10, we obtain the following. (Recall that $\boldsymbol{C}_5, \ldots, \boldsymbol{C}_8$ are, up to isomorphism, the non-simple SI 0-generated De Morgan monoids from Section 3.3.)

Corollary 4.24. *There are just ten covers of* $\mathbb{V}(\boldsymbol{C}_4)$ *within* U, *viz. the six listed in Theorem* 4.23 *and* $\mathbb{V}(\boldsymbol{C}_5), \ldots, \mathbb{V}(\boldsymbol{C}_8)$.

In contrast with Theorem 5.43, for each $n \in \{5, 6, 7, 8\}$, the quasivariety $\mathbb{Q}(\boldsymbol{C}_n)$ omits $\boldsymbol{C}_4$, and is therefore strictly smaller than $\mathbb{V}(\boldsymbol{C}_n)$. Indeed, the quasi-equation $e \approx e \wedge f \implies x \approx y$ holds in $\boldsymbol{C}_n$ but not in $\boldsymbol{C}_4$.

By Theorem 4.9 and Corollaries 2.17 and 4.24, the non-semisimple covers of atoms in the subvariety lattice of DMM (regardless of join-irreducibility) are just $\mathbb{V}(\boldsymbol{S}_5)$ and the ones contained in U. All of these are finitely generated varieties. Example 4.29 will show, however, that $\mathbb{V}(\boldsymbol{C}_4)$ has at least one join-irreducible cover within DMM (but not within U) that is not finitely generated.

4.4 Other covers of $\mathbb{V}(\mathbf{C}_4)$

We have seen that each cover of $\mathbb{V}(\boldsymbol{C}_4)$ within U is generated by a finite non-simple algebra. By Lemma 3.2(iii), a *simple* De Morgan monoid $\boldsymbol{A}$ is anti-idempotent if it has $\boldsymbol{C}_4$ as a subalgebra (cf. Theorem 4.9(iii)(3)). If $\boldsymbol{A}$ is finite as well, then it generates a cover of $\mathbb{V}(\boldsymbol{C}_4)$ exactly when $\boldsymbol{C}_4$ is its *only* proper subalgebra, by Jónsson's Theorem 1.23, and Theorem 1.24. In that case, by the same arguments, $\mathbb{V}(\boldsymbol{A})$ is join-irreducible in the subvariety lattice of DMM.

In fact, $\mathbb{V}(\boldsymbol{C}_4)$ has infinitely many finitely generated covers within DMM witnessing Theorem 4.9(iii)(3), as the next example shows.

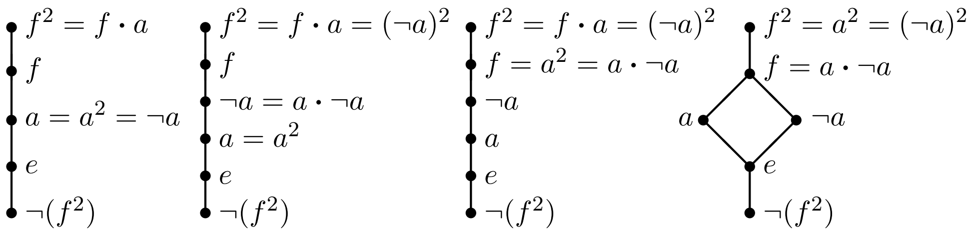
Example 4.25. For each positive integer p, let $\boldsymbol{A}_p$ be the rigorously compact De Morgan monoid on the chain $0 < 1 < 2 < 4 < 8 < \ldots < 2^p < 2^{p+1}$, where fusion is multiplication, truncated at 2^{p+1}. Thus, $|A_p| = p + 3$ and e is the integer 1, while $f = 2^p$ and $\neg(2^k) = 2^{p-k}$ for all $k \in \{0, 1, \ldots, p\}$. Clearly, $\boldsymbol{A}_p$ is simple and generated by 2, and we may identify $\boldsymbol{C}_4$ with the subalgebra of $\boldsymbol{A}_p$ on $\{0, 1, 2^p, 2^{p+1}\}$.

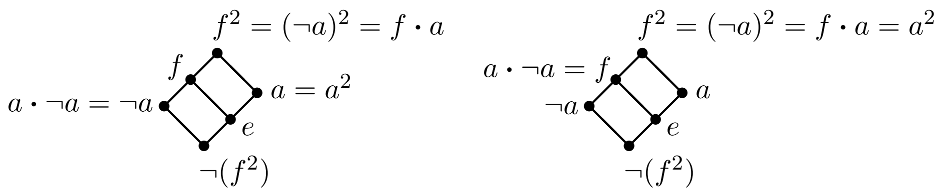
Now suppose p is prime. We claim that $\boldsymbol{C}_4$ is the only proper subalgebra of $\boldsymbol{A}_p$.

As $\boldsymbol{A}_p = \mathbf{Sg}^{\boldsymbol{A}_p}\{2\}$, it suffices to show that, whenever $k \in \{1, 2, \ldots, p-1\}$, then $2 \in \mathrm{Sg}^{\boldsymbol{A}_p}\{2^k\}$. The proof is by induction on k and the base case is trivial, so let $k > 1$. As p is prime, it is not divisible by k, whence there is a positive integer n such that $kn \in \{p-1, p-2, \ldots, p-(k-1)\}$, so $\neg(2^{kn}) \in \{2, 4, \ldots, 2^{k-1}\} \cap \mathrm{Sg}^{\boldsymbol{A}_p}\{2^k\}$. Because $\neg(2^{kn}) = 2^r$, where $1 \leq r < k$, the induction hypothesis implies that $2 \in \mathrm{Sg}^{\boldsymbol{A}_p}\{\neg(2^{kn})\} \subseteq \mathrm{Sg}^{\boldsymbol{A}_p}\{2^k\}$, as required.

Thus, $\mathbb{V}(\boldsymbol{A}_p)$ is a (join-irreducible) cover of $\mathbb{V}(\boldsymbol{C}_4)$ within DMM. And by Jónsson's Theorem 1.23, $\mathbb{V}(\boldsymbol{A}_p) \neq \mathbb{V}(\boldsymbol{A}_q)$ for distinct primes p, q, vindicating the claim preceding this example.

The $\wedge, \vee$ reduct of a simple De Morgan monoid is a self-dual distributive lattice in which e is an atom and f a co-atom. It is therefore not difficult to verify that, up to isomorphism, there are just eight simple De Morgan monoids $\boldsymbol{A}$ on at most 6 elements (and none on 7 elements) such that $\boldsymbol{C}_4$ is the only proper subalgebra of $\boldsymbol{A}$. Six such algebras are depicted below; the other two are $\boldsymbol{A}_2$ and $\boldsymbol{A}_3$ from Example 4.25. Each of these eight De Morgan monoids is 1-generated and generates a (join-irreducible) cover of $\mathbb{V}(\boldsymbol{C}_4)$ exemplifying Theorem 4.9(iii)(3).





The exhaustiveness of this eight-item list will not be proved here, as we shall not rely on it below.[3] Some features of the covers of $\mathbb{V}(\boldsymbol{C}_4)$ consisting of semilinear algebras deserve to be established, however. We consider first the case where $\boldsymbol{A}$ has an idempotent element outside $\boldsymbol{C}_4$.

Theorem 4.26. *Let $\boldsymbol{A}$ be a simple totally ordered De Morgan monoid, having $\boldsymbol{C}_4$ as a proper subalgebra, and suppose $a^2 = a \in A \setminus C_4$. Then a generates a subalgebra of $\boldsymbol{A}$ isomorphic to one of the first two algebras pictured above.*

Proof. By Lemmas 2.16(iv) and 3.2(iii), $\boldsymbol{A}$ is anti-idempotent, with $e < a < f$ and $e < \neg a < f$. Now $a \leqslant \neg a$, by (2.1), as $a^2 \leqslant f$. Also, $a \cdot \neg a = \neg a$, by Lemma 2.7, and $f \cdot a = f^2 = f \cdot \neg a$, by Corollary 2.19. If $\neg a = a$, then $\mathbf{Sg}^{\boldsymbol{A}}\{a\}$ matches the first of the two pictured algebras. If $\neg a \nleqslant a$ then $(\neg a)^2 \nleqslant f$, by (2.1), whence $(\neg a)^2 = f^2$ and $\mathbf{Sg}^{\boldsymbol{A}}\{a\}$ matches the second pictured algebra. □

Now we consider the case where $\boldsymbol{A}$ has no idempotent element outside $\boldsymbol{C}_4$, assuming that $\boldsymbol{A}$ is finite.

[3] Readers wanting to confirm it should note that all self-dual distributive lattices on 5, 6 or 7 elements are pictured above, except for the seven-element chain (ruled out by Theorem 4.28) and the seven-element lattice that stacks one four-element diamond on another, gluing them at the juncture. The latter supports several simple De Morgan monoids $\boldsymbol{A}$ that extend $\boldsymbol{C}_4$, but in each case, the vertical 'midpoint' a of $\boldsymbol{A}$ is a fixed point of $\neg$, and $a \cdot f = f^2$, by Corollary 2.19 and Lemma 3.2(iii), while $a \leqslant a^2 = a \cdot \neg a \leqslant f$, so $a^2 \in [a, f] = \{a, f\}$. Thus, $\mathbf{Sg}^{\boldsymbol{A}}\{a\}$ is a proper subalgebra of $\boldsymbol{A}$, strictly containing $\boldsymbol{C}_4$, so $\mathbb{V}(\boldsymbol{A})$ does not cover $\mathbb{V}(\boldsymbol{C}_4)$. The arguments for the lattices depicted above are no more difficult.

Theorem 4.27. *Let $\boldsymbol{A}$ be a finite simple totally ordered De Morgan monoid, having $\boldsymbol{C}_4$ as a proper subalgebra, where no element of $A \backslash C_4$ is idempotent.*

Let c be the cover of e in $\boldsymbol{A}$, and n the smallest positive integer such that $c^{n+1} = c^{n+2}$. Then

(i) $c^n = f$ *and* $c^{n+1} = f^2$;

(ii) $\neg(c^{m+1}) < c^{n-m} \leqslant \neg(c^m)$ *for each positive integer* $m < n$;

(iii) $b \cdot \neg b = f$ *for all* $b \in A \setminus \{f^2, \neg(f^2)\}$.

If, moreover, $|A|$ is odd, then $\mathbf{Sg}^{\boldsymbol{A}}\{a\} \cong \boldsymbol{A}_2$ (as defined in Example 4.25), where a is the fixed point of $\neg$ in $\boldsymbol{A}$.

Proof. Again, recall that $\boldsymbol{A}$ is anti-idempotent, with $A = \{\neg(f^2), f^2\} \cup [e, f]$, and note that f covers $\neg c$, by definition of c.

(i) As c^{n+1} is idempotent, we have $e < c^{n+1} \in C_4$ (by assumption), whence $c^{n+1} = f^2$. As c^n is not idempotent, $c^n < f^2$, i.e., $c^n \leqslant f$ (since $\boldsymbol{A}$ is totally ordered and simple). But $c^n \nleqslant \neg c$ (by (2.1), since $c^{n+1} \nleqslant f$), so $c^n = f$, because f covers $\neg c$.

(ii) Consider a positive integer $m < n$. We cannot have $c^{n-m} \leqslant \neg(c^{m+1})$, otherwise $c^{n+1} = c^{m+1} \cdot c^{n-m} \leqslant c^{m+1} \cdot \neg(c^{m+1}) \leqslant f$ (by (2.3)), a contradiction. Thus, $\neg(c^{m+1}) < c^{n-m}$. By (2.2), $c^{n-m} \leqslant c^m \to c^n = c^m \to f = \neg(c^m)$.

(iii) Let $b \in A \setminus \{f^2, \neg(f^2)\}$. Then $b \cdot \neg b \leqslant f$, by (2.3). Since $b \cdot \neg b = f$ for $b \in \{e, f\}$, we may assume that $e < b < f$, i.e., $c \leqslant b \leqslant \neg c$. Suppose $b \cdot \neg b < f$, i.e., $b \cdot \neg b \leqslant \neg c$. Then $b \cdot c \leqslant b$, by (2.1). As $c \leqslant b < f = c^n$, we have $c^p \leqslant b < c^{p+1}$ for some positive integer $p < n$. Then $c^{p+1} \leqslant b \cdot c \leqslant b < c^{p+1}$, a contradiction. Thus, $b \cdot \neg b = f$.

Finally, let $|A|$ be odd, so $\neg a = a$ for some (unique) $a \in A$, as $\neg$ is a bijection. Then $a \notin C_4$, so $a^2 = a \cdot \neg a = f$, by (iii), whence $Sg^{\boldsymbol{A}}\{a\} = C_4 \cup \{a\}$ and $\mathbf{Sg}^{\boldsymbol{A}}\{a\} \cong \boldsymbol{A}_2$. □

The third example pictured above shows that, in Theorem 4.27, when $\boldsymbol{A}$ has even cardinality, it need not have a subalgebra of the form $\boldsymbol{A}_p$ for $p > 1$.

Theorem 4.28. *If $\mathbb{V}(\boldsymbol{A})$ is a cover of $\mathbb{V}(\boldsymbol{C}_4)$ within* DMM, *where $\boldsymbol{A}$ is finite, simple and totally ordered, then $|A|$ is 5 or an even number.*

Proof. The hypothesis implies that $\boldsymbol{C}_4$ is the only proper subalgebra of $\boldsymbol{A}$, as noted earlier. If $|A|$ is odd, then $|A| \neq 6$, so by Theorems 4.26 and 4.27, $\boldsymbol{A}$ has a five-element subalgebra, which cannot be proper, so $|A| = 5$. □

In the statement of Theorem 4.9(iii)(3), the algebra $\boldsymbol{A}$ cannot always be chosen finite, in view of the following example.

Example 4.29. The set $B = \{0\} \cup \{2^n : n \in \omega\} \cup \{\infty\}$ is the universe of a Dunn monoid $\boldsymbol{B}$ whose lattice order is the conventional total order, and whose fusion is ordinary multiplication on the finite elements of B, while $0 \cdot \infty = 0$ and $b \cdot \infty = \infty$ whenever $0 \neq b \in B$ (hence $e = 1$). For finite nonzero $b, c \in B$, the value of $b \to c$ is c/b if b divides c; otherwise it is 0. It is well known that there is a unique totally ordered De Morgan monoid $\boldsymbol{A}_\infty$, having $\boldsymbol{B}$ as an RL-subreduct and having exactly the additional elements indicated and ordered below:

$$0 < 1 < 2 < 4 < 8 < 16 < \ldots < \neg 16 < \neg 8 < \neg 4 < \neg 2 < \neg 1 < \infty.$$

Here, $b \cdot \neg c = \neg(b \to c)$ and $\neg b \cdot \neg c = \infty$ for all finite nonzero $b, c \in B$. Note that $\boldsymbol{A}_\infty$ is generated by 2. The subalgebra of $\boldsymbol{A}_\infty$ on $\{0, 1, \neg 1, \infty\}$ is isomorphic to $\boldsymbol{C}_4$. Clearly, $\boldsymbol{A}_\infty$ is simple, so $\boldsymbol{A}_\infty \notin \mathsf{W}$, whence $\boldsymbol{A}_\infty$ is not the reflection of a Dunn monoid.

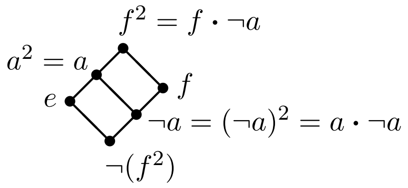
By Corollary 2.17, every SI algebra $\boldsymbol{C} \in \mathbb{V}(\boldsymbol{A}_\infty)$ embeds into an ultrapower of $\boldsymbol{A}_\infty$, and it is easily deduced that $\boldsymbol{C}$ contains an isomorphic copy of $\boldsymbol{A}_\infty$, unless $\boldsymbol{C} \cong \boldsymbol{C}_4$ (take the subalgebra generated by an element $a \in C \setminus C_4$ for which $a^2 \neq f^2$). In particular, $\mathbf{2}, \boldsymbol{S}_3, \boldsymbol{D}_4 \notin \mathbb{V}(\boldsymbol{A}_\infty)$, and $\mathbb{V}(\boldsymbol{A}_\infty)$ is not generated by its finite members. This establishes that $\mathbb{V}(\boldsymbol{A}_\infty)$ is a join-irreducible cover of $\mathbb{V}(\boldsymbol{C}_4)$ within DMM, exemplifying Theorem 4.9(iii)(3), and that $\mathbb{V}(\boldsymbol{A}_\infty)$ is not finitely generated.

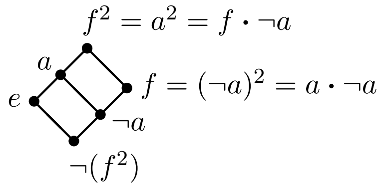
Actually, $\boldsymbol{A}_\infty$ embeds naturally into an ultraproduct of the algebras $\boldsymbol{A}_p$ (p a positive prime) from Example 4.25, so $\mathbb{V}(\boldsymbol{A}_\infty)$ is contained in the varietal join of the $\mathbb{V}(\boldsymbol{A}_p)$.

4.5 Covers of $\mathbb{V}(\mathbf{D}_4)$

Suppose $\boldsymbol{D}_4$ is a subalgebra of an FSI De Morgan monoid $\boldsymbol{A}$. Then A is the *disjoint* union of the anti-isomorphic sublattices $[e)$ and $(f]$ of $\langle A; \wedge, \vee\rangle$, by Theorem 2.43. Consequently, if $\boldsymbol{A}$ is finite, then $|A|$ is even. Also, if $\boldsymbol{A}$ is simple (cf. Theorem 4.9(iv)), then it is anti-idempotent, by Lemma 3.2(iii). When $\boldsymbol{A}$ is both finite and simple, then $\boldsymbol{D}_4$ is the sole proper subalgebra of $\boldsymbol{A}$ iff $\mathbb{V}(\boldsymbol{A})$ is a cover of $\mathbb{V}(\boldsymbol{D}_4)$ within DMM, in which case $\mathbb{V}(\boldsymbol{A})$ is join-irreducible (the arguments being just as for $\boldsymbol{C}_4$).

Example 4.30. Each of the simple De Morgan monoids depicted below generates a cover of $\mathbb{V}(\boldsymbol{D}_4)$ within DMM, witnessing Theorem 4.9(iv). Up to isomorphism, they are the only two such algebras on 6 elements. (The one on the left is a subalgebra of the eight-element 'Belnap lattice'; it is obtained by removing from that structure the elements labeled 2 and -2 in [1, p. 252]. The one on the right is isomorphic to the algebra $\boldsymbol{B}_2$ in Example 4.31 below.)

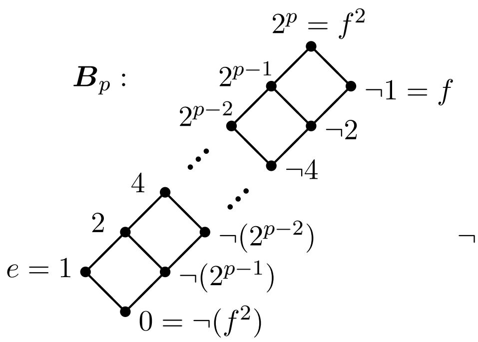




In analogy with the case of $\boldsymbol{C}_4$, there are infinitely many finitely generated covers of $\mathbb{V}(\boldsymbol{D}_4)$ within DMM, as well as a cover that is not finitely generated (nor even generated by its finite members). This is shown by the next two examples.

Example 4.31. For each positive integer p, it can be checked that there is a unique rigorously compact (simple) De Morgan monoid $\boldsymbol{B}_p$ having the labeled Hasse diagram and fusion indicated below, where it is understood that $m, n, k, \ell \in \omega$ with $m, n \leq p$ and $k, \ell < p$.

$\boldsymbol{B}_p$:



$$2^m \cdot 2^n = 2^{\min\{m+n,p\}}$$

$$2^m \cdot \neg(2^\ell) = \begin{cases} \neg(2^{\ell-m}) & \text{if } \ell \geq m; \\ 2^p & \text{if } \ell < m. \end{cases}$$

$$\neg(2^k) \cdot \neg(2^\ell) = \begin{cases} \neg(2^{k+\ell-p}) & \text{if } k+\ell \geq p; \\ 2^p & \text{if } k+\ell < p. \end{cases}.$$

The subalgebra of $\boldsymbol{B}_p$ on $\{0, 1, \neg 1, 2^p\}$ may be identified with $\boldsymbol{D}_4$.

Now suppose p is prime. We claim that $\boldsymbol{B}_p$ has no proper subalgebra other than $\boldsymbol{D}_4$. It suffices (by involution properties) to show, by induction on k, that $2 \in Y := \mathrm{Sg}^{\boldsymbol{B}_p}\{2^k\}$ for each positive integer $k < p$. The base case is trivial, so let $k > 1$. As k does not divide p, we have $r := p - kn \in \{1, 2, \dots, k-1\}$ for some positive integer n, so $\neg(2^{p-r}) = \neg(2^{kn}) \in Y$, whence $2^r = e \vee \neg(2^{p-r}) \in Y$. By the induction hypothesis, $2 \in \mathrm{Sg}^{\boldsymbol{B}_p}\{2^r\}$, so $2 \in Y$, completing the proof. Thus, $\mathbb{V}(\boldsymbol{B}_p)$ is a cover of $\mathbb{V}(\boldsymbol{D}_4)$ within DMM and, by Jónsson's Theorem 1.23, $\mathbb{V}(\boldsymbol{B}_p) \neq \mathbb{V}(\boldsymbol{B}_q)$ for distinct primes p, q.

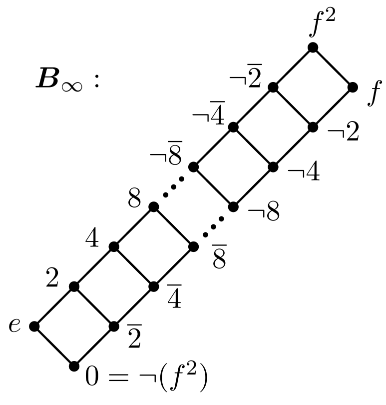
Example 4.32. In each $\boldsymbol{B}_p$ above, the element e has a unique cover. That is a first order property, so it persists in the rigorously compact simple ultraproduct $\prod_p \boldsymbol{B}_p/\mathcal{F}$, for each nonprincipal ultrafilter $\mathcal{F}$ over the set of positive primes. By similar applications of Łos' Theorem 1.8, in any such ultraproduct, the rigorously compact simple subalgebra $\boldsymbol{B}_\infty$ generated by the cover of e (still denoted by 2) has the infinite lattice reduct shown in the next diagram, and its fusion is determined by the following additional information, where m, n are positive integers:

$$f \cdot x = f^2 \text{ whenever } x \in B_\infty \setminus \{0, e\}$$
$$2^m \cdot 2^n = 2^{m+n}$$
$$2^m \cdot \overline{2^n} = \overline{2^{m+n}} = \overline{2^m} \cdot \overline{2^n}$$
$$\neg(2^m) \cdot \neg\overline{2^n} = f^2 = \neg\overline{2^m} \cdot \neg\overline{2^n} = \neg(2^m) \cdot \neg(2^n)$$
$$2^m \cdot \neg\overline{2^n} = \left\{ \begin{array}{ll} \neg\overline{2^{n-m}} & \text{if } m \leq n \\ f^2 & \text{if } m > n \end{array} \right\} = 2^m \cdot \neg(2^n) = \overline{2^m} \cdot \neg(2^n) = \overline{2^m} \cdot \neg\overline{2^n}.$$

We claim that $\mathbb{V}(\boldsymbol{B}_\infty)$ is a join-irreducible cover of $\mathbb{V}(\boldsymbol{D}_4)$ within DMM, not generated by its finite members. For this, it suffices, as in Example 4.29, to establish the following.

Fact 4.33. *Let $\boldsymbol{D}$ be a subalgebra of an ultrapower of $\boldsymbol{B}_\infty$, where $\boldsymbol{D} \ncong \boldsymbol{D}_4$. Then $\boldsymbol{B}_\infty$ can be embedded into $\boldsymbol{D}$.*

$\boldsymbol{B}_\infty$:



Proof. (Sketch) Identifying $\boldsymbol{D}_4$ with the 0-generated subalgebra of the ultrapower $\boldsymbol{U}$ (and hence of $\boldsymbol{D}$), we see that $\boldsymbol{D}$ is anti-idempotent, rigorously compact and simple, and that D is the disjoint union of its subsets $[e)$ and $(f]$. We may choose $a \in D \setminus D_4$, because $\boldsymbol{D} \ncong \boldsymbol{D}_4$ and $\boldsymbol{D}_4$ is finite. Membership and non-membership of $\boldsymbol{D}_4$ are first order properties, because

e is distinguished. We can arrange that

$$e < a < a^2 < f^2,$$

because the following properties of $\boldsymbol{B}_\infty$ are expressible as universal first order sentences (which therefore persist in both $\boldsymbol{U}$ and $\boldsymbol{D}$):

$$x \approx x^2 \implies x \in D_4;$$
$$(x \notin D_4 \;\;\&\;\; x^2 \approx f^2) \implies e < (e \vee \neg x) < (e \vee \neg x)^2 < f^2.$$

(So, we may replace a by $e \vee \neg a \in D$ if $a^2 = f^2$, and by $e \vee a \in D$ if $e \not< a$ with $a^2 < f^2$.) As 2 generates $\boldsymbol{B}_\infty$, there is at most one homomorphism $h\colon \boldsymbol{B}_\infty \to \boldsymbol{D}$ sending 2 to a. To see that h is well defined and injective, it suffices (by (2.16)) to show that for any unary term α in $\cdot, \wedge, \neg, e$, we have

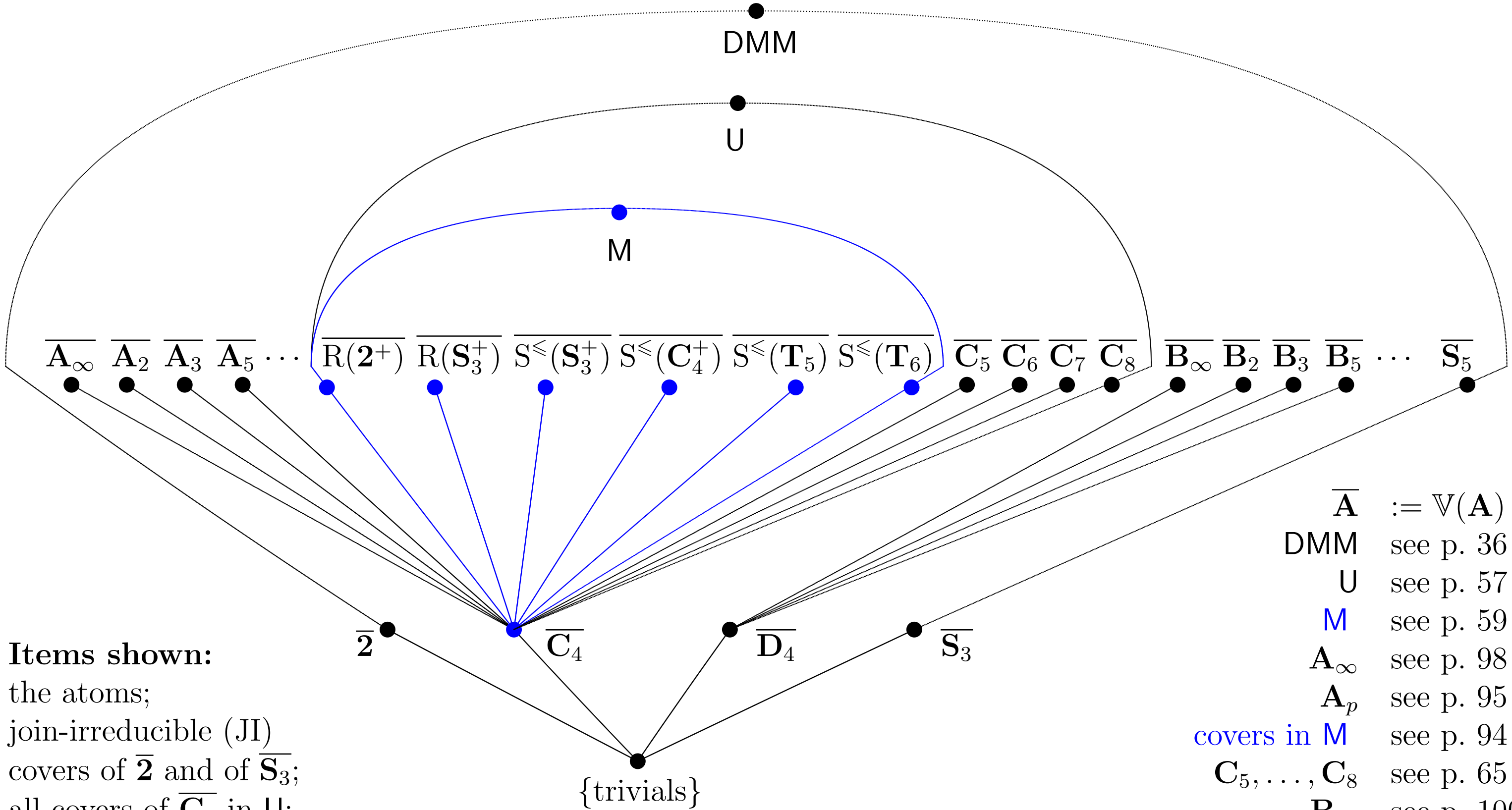
$$e \leqslant \alpha^{\boldsymbol{B}_\infty}(2) \text{ iff } e \leqslant \alpha^{\boldsymbol{D}}(a).$$

This can be shown by induction on the complexity of α. At the inductive step, the case of $\wedge$ is trivial, while $\neg$ is straightforward, because D is the disjoint union of $[e)$ and $(f]$. Fusion requires an examination of subcases, which is aided by noting that $\boldsymbol{B}_\infty$ (and hence $\boldsymbol{D}$) has properties of the following kind, where n, m, p are any *fixed* positive integers with $p \geq m$:

$$e < x < x^2 < f^2 \implies (x^m \cdot x^n \approx x^{m+n} \;\;\&\;\; x^m \cdot (e \vee \neg(x^p)) \approx e \vee \neg(x^{p-m})).$$

□

The subvariety lattice of De Morgan monoids



Items shown:
the atoms;
join-irreducible (JI)
covers of $\overline{\mathbf{2}}$ and of $\overline{\mathbf{S}_3}$;
all covers of $\overline{\mathbf{C}_4}$ in U;
some JI covers of $\overline{\mathbf{C}_4}$ outside U
(infinitely many, not all finitely generated);
similarly for covers of $\overline{\mathbf{D}_4}$.

$\overline{\mathbf{A}}$	$:= \mathbb{V}(\mathbf{A})$
DMM	see p. 36
U	see p. 57
M	see p. 59
$\mathbf{A}_\infty$	see p. 98
$\mathbf{A}_p$	see p. 95
covers in M	see p. 94
$\mathbf{C}_5, \ldots, \mathbf{C}_8$	see p. 65
$\mathbf{B}_\infty$	see p. 100
$\mathbf{B}_p$	see p. 99
$\mathbf{S}_n$	see p. 38
atoms	see p. 77

Chapter 5

Singly generated quasivarieties

In this chapter, we further our understanding of the lattice of varieties of De Morgan monoids by investigating some of its connections with the sub*quasivariety* lattice of DMM. When analysing a variety K of De Morgan monoids, it may help to know, for instance, that every subquasivariety of K is a variety or, more generally, that each proper subquasivariety of K generates a proper subvariety of K or—still more generally—that K is a *singly generated quasivariety*, i.e., that $\mathsf{K} = \mathbb{Q}(\boldsymbol{A})$ for some algebra $\boldsymbol{A}$. Each of these properties has a logical significance, as will be explained below.

When a logic is algebraized by a quasivariety K, the derivable rules of the logic may or may not be determined by a single set of 'truth tables', i.e., by the operation tables of a single algebra $\boldsymbol{A} \in \mathsf{K}$. If some member of K determines the *finite* rules of the logic, then another member determines *all* of the rules (see Remark 5.17), so what is needed is only that K be generated by a single algebra. Even when K is a variety, it must be generated *as a quasivariety* by one of its members, if the generator is to determine rules (as opposed to theorems only), i.e., K must be singly generated.

Obviously, classical propositional logic (**CPL**) has this property: its algebraic counterpart BA is generated as a quasivariety by its unique two-element member **2**. More surprisingly, the same holds for the intuitionistic propositional logic **IPL** (though not with a finite algebra), and for the relevance logic **R** [137], but not for its conservative expansion $\mathbf{R}^{\mathbf{t}}$. In the intuitionistic case, the algebra determining the (possibly infinite) rules cannot be countable [151].

Maltsev [89] proved that a quasivariety K is singly generated iff it has the *joint embedding property* (JEP), i.e., any two nontrivial members of K can both be embedded into some third member. By [42, Thm. 3], the JEP amounts to a syntactic 'relevance principle' (Definition 5.12 below), which stems from the so-called Łoś-Suszko Theorem 5.13.

Various strengthenings of the JEP have received attention in the literature and they are the focus of this chapter. Their names reflect logical origins, but we choose maximally transparent characterizations here as definitions. One such strengthening, called *structural completeness*, asks (in effect) that a quasivariety be generated by its free $\aleph_0$-generated member. A quasivariety is *hereditarily structurally complete* if each of its subquasivarieties is structurally complete. When a variety K has one of these properties, the structure of its subvariety lattice is illuminated by the following characterizations: K is structurally complete [resp. hereditarily structurally complete] iff each proper subquasivariety of K generates a proper subvariety of K [resp. each subquasivariety of K is a variety]; see Theorem 5.20.

A weaker variant of structural completeness, now called *passive structural completeness* (PSC), amounts to the demand that any two nontrivial members of K have the same existential positive theory. This hereditary property still implies the JEP (Theorem 5.19).

When we started to investigate these properties for classes of De Morgan monoids, it became clear that, in many of our results, large parts of the proofs had a general universal algebraic (or even model-theoretic) character. The first two sections of this chapter largely concern such generalities. Recall that a quasivariety K is called a *Kollár quasivariety* (Definition 1.16) if its nontrivial members lack trivial subalgebras. We prove that if such a quasivariety has the JEP, then its relatively simple members all belong to the universal class generated by one of them (Theorem 5.7). If, in addition, K is relatively semisimple, then it is generated (as a quasivariety) by one K-*simple* algebra. We prove that a quasivariety of finite type with a finite nontrivial member is PSC iff its nontrivial members have a common retract (Theorem 5.28).

Before characterizing the varieties of De Morgan monoids with the JEP (Theorem 5.37), we describe completely those that are PSC (Theorem 5.34). The structurally complete varieties of De Morgan monoids fall into two classes—a denumerable family that is fully transparent and a more opaque collection of subvarieties of M (see Definition 3.16 and Theorem 3.18). Within M, however, there are also $2^{\aleph_0}$ structurally incomplete varieties; this will be proved in Chapter 7.

We show, in Section 5.3, that in the varietal join J of the six covers of $\mathbb{V}(\boldsymbol{C}_4)$ within M, every finite subdirectly irreducible algebra is projective. It follows that J is (hereditarily) structurally complete.

Some of the results in this chapter can be generalized to a model-theoretic setting (i.e., they may hold for structures with relations as well as operations). For consistency, however, we restrict the scope of the current discussion to the algebraic setting. (An interested reader may consult

Moraschini, Raftery and Wannenburg [106] for the more general case.)

5.1 The joint embedding property

Definition 5.1. A class K of similar algebras is said to have the *joint embedding property* (JEP) if, for any two nontrivial algebras $\boldsymbol{A}, \boldsymbol{B} \in \mathsf{K}$, there exists $\boldsymbol{C} \in \mathsf{K}$ such that $\boldsymbol{A}$ and $\boldsymbol{B}$ can both be embedded into $\boldsymbol{C}$.

For quasivarieties, the characterization of the JEP given below was proved in [89, Thm. 4] (also see [91, p. 288] or [63, Prop. 2.1.19]).

Theorem 5.2 (Maltsev)**.** *A quasivariety* K *has the JEP iff it is generated by a single algebra (i.e., there exists* $\boldsymbol{A} \in \mathsf{K}$ *such that* $\mathsf{K} = \mathbb{Q}(\boldsymbol{A})$*).*

Additional characterizations of the JEP for a quasivariety K can be found in [42, Thm. 3] and implicitly in [70, Thm. 1.2]. They include the following.[1]

(i) For each *set* S of nontrivial members of K, there exists a member of K into which every member of S embeds.

(ii) Whenever Φ and Ψ are universal sentences whose disjunction $\Phi \sqcup \Psi$ is true in all nontrivial members of K, then there exists $\Xi \in \{\Phi, \Psi\}$ such that Ξ is true in every nontrivial member of K.

(iii) Whenever Φ and Ψ are existential sentences, each of which is true in some nontrivial member of K, then their conjunction $\Phi \;\&\; \Psi$ is true in some nontrivial member of K.

(iv) Whenever Σ is a set of existential sentences, each of which is true in at least one nontrivial member of K, then there is a nontrivial member of K in which all sentences from Σ are true.

Easily, (ii) and (iii) follow from the JEP, and (iv) from (i). To prove (i), we apply the Compactness Theorem 1.10 to $\Sigma \cup \{\overline{\mathrm{Diag}}(\boldsymbol{A}) : \boldsymbol{A} \in \mathsf{S}\}$, where Σ is a set of sentences axiomatizing K and each $\overline{\mathrm{Diag}}(\boldsymbol{A})$ is the set of atomic

[1] In [70], the JEP is formulated for arbitrary first order theories, without the restriction to nontrivial models, and its equivalence with each of (i)–(iv) (likewise unrestricted) is inferred from a result proved in [153].

or negated atomic sentences that are true in $\boldsymbol{A}_A$.[2] (We arrange first that the members of S are disjoint.)[3]

The next result allows us to restrict attention to relatively subdirectly irreducible algebras when testing a quasivariety for the JEP.

Proposition 5.3. *Let K be a quasivariety, and suppose that, whenever $\boldsymbol{A}, \boldsymbol{B} \in \mathsf{K}_{\mathrm{RSI}}$, then there exists $\boldsymbol{C} \in \mathsf{K}$ such that $\boldsymbol{A}$ and $\boldsymbol{B}$ can both be embedded into $\boldsymbol{C}$. Then K has the JEP.*

Proof. Let $\boldsymbol{A}, \boldsymbol{B} \in \mathsf{K}$ be nontrivial. Then

$$\boldsymbol{A} \in \mathbb{IP}_{\mathbb{S}}\{\boldsymbol{A}_i : i \in I\} \text{ and } \boldsymbol{B} \in \mathbb{IP}_{\mathbb{S}}\{\boldsymbol{B}_j : j \in J\}$$

for suitable $\boldsymbol{A}_i, \boldsymbol{B}_j \in \mathsf{K}_{\mathrm{RSI}}$, where I and J are non-empty sets. We may assume that $I \subseteq J$. Fixing $\ell \in I$ and defining $\boldsymbol{A}_j = \boldsymbol{A}_\ell$ for all $j \in J \setminus I$, we find that $\boldsymbol{A} \in \mathbb{IP}_{\mathbb{S}}\{\boldsymbol{A}_j : j \in J\}$. By assumption, for each $j \in J$, there exists $\boldsymbol{C}_j \in \mathsf{K}$ such that $\boldsymbol{A}_j, \boldsymbol{B}_j \in \mathbb{IS}(\boldsymbol{C}_j)$. Then $\prod_J \boldsymbol{A}_j$ and $\prod_J \boldsymbol{B}_j$ both embed into $\boldsymbol{C} := \prod_J \boldsymbol{C}_j \in \mathsf{K}$, so $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{IS}(\boldsymbol{C})$. □

Corollary 5.4. *The JEP is decidable for finitely generated quasivarieties of finite type.*

Proof. Let $\mathsf{K} = \mathbb{Q}(\boldsymbol{A}_1, \dots, \boldsymbol{A}_m)$, where $\boldsymbol{A}_1, \dots, \boldsymbol{A}_m$ are finitely many finite algebras of finite type. Let $\boldsymbol{A} = \boldsymbol{A}_1 \times \dots \times \boldsymbol{A}_m$, and let $k = |A|$. Then $\mathbb{V}(\mathsf{K}) = \mathbb{V}(\boldsymbol{A})$. For any cardinal n, the free n-generated algebra in $\mathbb{V}(\boldsymbol{A})$ belongs to K and embeds into the direct power $\boldsymbol{A}^{(k^n)}$, so it has at most $f(n) := k^{(k^n)}$ elements. So, every n-generated member of $\mathbb{V}(\mathsf{K})$ (in particular, of K) has at most $f(n)$ elements.

Recall that $\mathbb{Q} = \mathbb{IP}_{\mathbb{S}}\mathbb{SP}_{\mathbb{U}}$, so if $\boldsymbol{A}, \boldsymbol{B} \in \mathsf{K}_{\mathrm{RSI}}$, then $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{IS}(\boldsymbol{A}_1, \dots, \boldsymbol{A}_m)$ (because the $\boldsymbol{A}_i$ are finite). Therefore, by Proposition 5.3, K has the JEP iff some $\boldsymbol{C} \in \mathsf{K}$ contains isomorphic copies of $\boldsymbol{A}_1, \dots, \boldsymbol{A}_m$ as subalgebras. In this case, because K is closed under $\mathbb{S}$, the algebra $\boldsymbol{C}$ can be chosen n-generated, where n is the (finite) sum of the cardinalities of $A_1, \dots, A_m$, so $\boldsymbol{C}$ can be assumed to have at most $f(n)$ elements. So, the JEP for K can be settled by taking an arbitrary $f(n)$-element set X and examining the members of K whose universes are subsets of X. There are only finitely

[2] Recall that atomic formulas in an algebraic language are just equations, so atomic sentences are equations that contain no variable. Also recall that for $\boldsymbol{A} = \langle A; F \rangle$, the algebra $\boldsymbol{A}_A$ is defined on page 16 to be $\langle A; F \cup A_0 \rangle$, where A_0 consists of the elements of A, treated as nullary operations on A.

[3] In fact, the JEP implies that the (categorical) K-free product of the members of any set $\mathsf{S} \subseteq \mathsf{K}$ exists in K, so it can serve as the common extension in (i); see [42, Thm. 3] and [89, Cor. 3].

many of these, as K has finite type. Moreover, each candidate for $\boldsymbol{C}$, being finite, can be checked mechanically for membership of K, because C is the domain of only finitely many functions into each of $A_1, \ldots, A_m$. □

Remark 5.5. Let m be the maximum of $\aleph_0$ and the cardinalities of the respective sets of operation and relation symbols of a quasivariety K with the JEP. Then $\mathsf{K} = \mathbb{Q}(\boldsymbol{A})$ for some structure $\boldsymbol{A}$ for which $|A| \leq m$. To see this, let Σ be the set of all quasi-equations over *Var* (our denumerable set of variables) that are not satisfied by K, so $|\Sigma| \leq m$. For each $\Phi \in \Sigma$, we can choose $\boldsymbol{A}_\Phi \in \mathsf{K}$ such that $\boldsymbol{A}_\Phi$ is finitely generated (whence $|A_\Phi| \leq m$) and $\boldsymbol{A}_\Phi \not\models \Phi$. As K has the JEP, $\{\boldsymbol{A}_\Phi : \Phi \in \Sigma\} \subseteq \mathbb{IS}(\boldsymbol{A})$ for some $\boldsymbol{A} \in \mathsf{K}$, by item (i) after Theorem 5.2. Clearly, we may choose $\boldsymbol{A}$ to be generated by the union of the images of the structures $\boldsymbol{A}_\Phi$, whence $|A| \leq m$. Now $\boldsymbol{A}$ refutes every formula from Σ, whence $\mathsf{K} = \mathbb{Q}(\boldsymbol{A})$.

Proposition 5.6. *Let K be a quasivariety with the JEP.*

(i) ([81]) *Any two nontrivial* 0*-generated members of K are isomorphic.*

(ii) *If K has a constant symbol, then K is a Kollár quasivariety or* every *member of K has a trivial subalgebra.*

(iii) *Every nontrivial* 0*-generated member of K is relatively simple.*

Proof. (i) Let $\boldsymbol{A}, \boldsymbol{B} \in \mathsf{K}$ be nontrivial and 0-generated. By the JEP, there exist $\boldsymbol{C} \in \mathsf{K}$ and embeddings $g \colon \boldsymbol{A} \to \boldsymbol{C}$ and $h \colon \boldsymbol{B} \to \boldsymbol{C}$. As $g[\boldsymbol{A}]$ and $h[\boldsymbol{B}]$ are 0-generated substructures of $\boldsymbol{C}$, they coincide, so $h^{-1}|_{h[B]} \circ g \colon \boldsymbol{A} \cong \boldsymbol{B}$.

(ii) Let c be a constant symbol of K. If $\boldsymbol{A} \in \mathsf{K}$ has no trivial subalgebra, then for some basic operation symbol f of $\boldsymbol{A}$, the equation $f(c, c, \ldots, c) \approx c$ (briefly, Φ) is false in $\boldsymbol{A}$. In that case, if $\boldsymbol{B} \in \mathsf{K}$ has a proper trivial subalgebra, then Φ is true in $\boldsymbol{B}$, so $\boldsymbol{A}$ and $\boldsymbol{B}$ have no common extension, contradicting the JEP.

(iii) Let $\boldsymbol{A} \in \mathsf{K}$ be nontrivial and 0-generated. Then K has a constant symbol and $\boldsymbol{A}$ has no trivial subalgebra, so K is a Kollár quasivariety, by (ii). Therefore, $\boldsymbol{A}$ has a homomorphic image $\boldsymbol{B} \in \mathsf{K}_{\mathrm{RS}}$, by Corollary 1.18. Since $\boldsymbol{B}$ is also 0-generated and nontrivial, it is isomorphic to $\boldsymbol{A}$, by (i), so $\boldsymbol{A}$ is relatively simple. □

The assumption that K has a constant symbol cannot be dropped from (ii), even when K is a variety (see Example 5.16).

Theorem 5.7. *Let K be a nontrivial Kollár quasivariety with the JEP. Then there is a relatively simple algebra $\boldsymbol{A} \in \mathsf{K}$ such that $\mathbb{ISP}_{\mathbb{U}}(\boldsymbol{A})$ includes every relatively simple member of K.*

Consequently, $\mathbb{Q}(\mathsf{K}_{\mathrm{RS}}) = \mathbb{Q}(\boldsymbol{A})$, *so* $\mathbb{Q}(\mathsf{K}_{\mathrm{RS}})$ *also has the JEP.*

Proof. For any algebra $\boldsymbol{B}$, let $\mathrm{EPS}(\boldsymbol{B})$ denote the set of existential positive sentences that are true in $\boldsymbol{B}$. As K has the JEP, Theorem 5.2 shows that $\mathsf{K} = \mathbb{Q}(\boldsymbol{C})$ for some $\boldsymbol{C} \in \mathsf{K}$. Since K is nontrivial, so is $\boldsymbol{C}$. By Corollary 1.18, $\boldsymbol{C}$ has a homomorphic image $\boldsymbol{A} \in \mathsf{K}_{\mathrm{RS}}$. Observe that

$$\boldsymbol{A} \models \mathrm{EPS}(\boldsymbol{C}), \tag{5.1}$$

as $\boldsymbol{A} \in \mathbb{H}(\boldsymbol{C})$. We claim, moreover, that

$$\boldsymbol{C} \models \mathrm{EPS}(\boldsymbol{B}), \text{ for every } \boldsymbol{B} \in \mathsf{K}_{\mathrm{RS}}. \tag{5.2}$$

Indeed, because $\mathsf{K} = \mathbb{Q}(\boldsymbol{C}) = \mathbb{IP}_{\mathbb{S}}\mathbb{SP}_{\mathbb{U}}(\boldsymbol{C})$, we have $\mathsf{K}_{\mathrm{RS}} \subseteq \mathsf{K}_{\mathrm{RSI}} \subseteq \mathbb{ISP}_{\mathbb{U}}(\boldsymbol{C})$, so $\boldsymbol{C} \in \mathbb{R}_{\mathbb{U}}\mathbb{EH}(\boldsymbol{B})$ for all $\boldsymbol{B} \in \mathsf{K}_{\mathrm{RS}}$. Thus, (5.2) follows from Corollary 1.30.

Now let $\boldsymbol{B} \in \mathsf{K}_{\mathrm{RS}}$. Then $\boldsymbol{A} \models \mathrm{EPS}(\boldsymbol{B})$, by (5.1) and (5.2), so there is a homomorphism $h \colon \boldsymbol{B} \to \boldsymbol{U}$ for some ultrapower $\boldsymbol{U}$ of $\boldsymbol{A}$, by Theorem 1.29. Since $\boldsymbol{A}$ is nontrivial, so is $\boldsymbol{U}$. Then h is an embedding, by Fact 1.19, as K is a Kollár quasivariety. Thus, $\boldsymbol{B} \in \mathbb{ISP}_{\mathbb{U}}(\boldsymbol{A})$, as claimed.

This shows that $\mathbb{Q}(\mathsf{K}_{\mathrm{RS}}) = \mathbb{Q}(\boldsymbol{A})$, which has the JEP, by Theorem 5.2. □

Corollary 5.8. *Let* K *be a nontrivial relatively semisimple Kollár quasivariety with the JEP. Then* $\mathsf{K} = \mathbb{Q}(\boldsymbol{A})$ *for some relatively simple* $\boldsymbol{A} \in \mathsf{K}$.

Proof. This follows from Theorem 5.7, as $\mathsf{K} = \mathbb{Q}(\mathsf{K}_{\mathrm{RSI}})$ and $\mathsf{K}_{\mathrm{RSI}} = \mathsf{K}_{\mathrm{RS}}$. □

Corollary 5.9. *Let* K *be a nontrivial Kollár quasivariety with the JEP. If the class of all relatively simple members of* K *is elementary, then it too has the JEP.*

Proof. Let $\boldsymbol{A}, \boldsymbol{B} \in \mathsf{K}_{\mathrm{RS}}$. By Theorem 5.7, there exist $\boldsymbol{C} \in \mathsf{K}_{\mathrm{RS}}$ and embeddings $\boldsymbol{A} \to \boldsymbol{U}$ and $\boldsymbol{B} \to \boldsymbol{V}$, where $\boldsymbol{U}, \boldsymbol{V} \in \mathbb{P}_{\mathbb{U}}(\boldsymbol{C})$. Because $\boldsymbol{U}$ and $\boldsymbol{V}$ are elementarily equivalent, some ultrapower $\boldsymbol{W}$ of $\boldsymbol{U}$ is isomorphic to an ultrapower of $\boldsymbol{V}$, by the Keisler-Shelah Theorem 1.11, and of course $\boldsymbol{W} \in \mathsf{K}$. Then $\boldsymbol{A}$ and $\boldsymbol{B}$ both embed into $\boldsymbol{W}$. Moreover, as $\boldsymbol{W}$ is elementarily equivalent to $\boldsymbol{C}$, and as K_{RS} is elementary, $\boldsymbol{W} \in \mathsf{K}_{\mathrm{RS}}$. □

In view of Theorem 5.2 and Corollary 5.8, it is natural to ask whether a quasivariety with the JEP must be generated by a relatively finitely subdirectly irreducible algebra. This is not the case, as the next example shows (also see Example 5.40).

Example 5.10. The Dunn monoid reduct of a De Morgan monoid $\boldsymbol{A}$ shall be denoted by $\boldsymbol{A}^+$. We then denote by $X(\boldsymbol{A})$ the De Morgan monoid that extends the reflection $\mathrm{R}(\boldsymbol{A}^+)$ by just one element x, where $a < x < b'$ for all $a, b \in A$, and $x \cdot \neg(f^2) = \neg(f^2)$ and $x = \neg x = x \cdot c$ and $x \cdot d = f^2$ whenever $\neg(f^2) < c \leqslant x < d \leqslant f^2$. (It is easily checked that this $X(\boldsymbol{A})$ is indeed a De Morgan monoid, with $\mathrm{R}(\boldsymbol{A}^+) \in \mathbb{S}(X(\boldsymbol{A}))$.)

Let $\mathsf{K} = \mathbb{V}(X(\mathbf{2} \times \boldsymbol{S}_3))$. As K is generated by one finite algebra, its finitely subdirectly irreducible members are finite and can be computed mechanically, by Jónsson's Theorem 1.23. None of them has the property that its $\mathbb{HS}$-closure contains all the others, but all of them embed into $X(\mathbf{2} \times \boldsymbol{S}_3)$ (excepting the trivial algebra). Therefore, K is not generated as a variety by a single (finitely) subdirectly irreducible algebra, but $\mathsf{K} = \mathbb{Q}(X(\mathbf{2} \times \boldsymbol{S}_3))$, so K has the JEP, by Theorem 5.2. □

Theorem 5.11. *Let K be a variety with EDPC (defined on page 15), and $\boldsymbol{A} \in \mathsf{K}$ a simple algebra. Then $\mathbb{V}(\boldsymbol{A}) = \mathbb{Q}(\boldsymbol{A})$, so the variety $\mathbb{V}(\boldsymbol{A})$ has the JEP.*

Proof. As K has EDPC, its class of simple members is closed both under $\mathbb{P}_{\mathbb{U}}$ and (by the CEP) under nontrivial subalgebras (see Theorem 1.25). So, when $\boldsymbol{A} \in \mathsf{K}$ is simple, the nontrivial members of $\mathbb{HSP}_{\mathbb{U}}(\boldsymbol{A})$ belong to $\mathbb{ISP}_{\mathbb{U}}(\boldsymbol{A})$. In this case, by Jónsson's Theorem 1.23, $\mathbb{V}(\boldsymbol{A}) = \mathbb{Q}(\boldsymbol{A})$, which has the JEP, by Theorem 5.2. □

Recall from Section 2.3 (page 35) that every variety of De Morgan monoids has EDPC. By Theorem 4.9, every join-irreducible cover of an atom in the lattice of subvarieties of DMM that is not contained in U or OSM is generated as a variety by a simple algebra, and therefore has the JEP. We shall give a full characterization of the join-irreducible covers with the JEP in Corollary 5.39.

The JEP has a syntactic meaning in algebraic logic. For a set Γ of formulas, we denote by $\mathit{Var}(\Gamma)$ the set of all variables x such that x occurs in at least one member of Γ.

Definition 5.12. A finitary deductive system $\vdash$ is said to respect the *abstract relevance principle* if the following is true whenever $\Gamma \cup \Delta \cup \{\alpha\}$ is a finite set of formulas, with $\mathit{Var}(\Delta) \cap \mathit{Var}(\Gamma \cup \{\alpha\}) = \emptyset$, and Δ is consistent over $\vdash$ (i.e., there exists a formula β such that $\Delta \nvdash \beta$):

$$\text{if } \Gamma \cup \Delta \vdash \alpha, \text{ then } \Gamma \vdash \alpha.$$

The Łoś-Suszko Theorem 5.13 ([82, p. 182], corrected in [149])**.** *Let $\vdash$ be a finitary deductive system that is algebraized by a quasivariety K. Then $\vdash$ respects the abstract relevance principle iff K is singly generated.*

(More exactly, this 'bridge theorem' is the specialization of the Łoś-Suszko Theorem to elementarily algebraizable finitary logics. Variants of the Łoś-Suszko Theorem for special families of deductive systems are discussed in [3, 51, 75, 84, 86, 137].)

We say that a quasivariety K respects the *abstract relevance principle* when its equational consequence relation $\models_{\mathsf{K}}$ satisfies the conditions of Definition 5.12, where 'formulas' are replaced there by 'equations'. The following result is therefore (via Theorem 5.2) an analogue of the Łoś-Suszko Theorem for the equational consequence relations of quasivarieties.

Theorem 5.14 ([42])**.** *A quasivariety has the JEP iff it respects the abstract relevance principle.*

Proof. ($\Rightarrow$) This follows from item (ii) after Theorem 5.2, because quasi-equations are essentially disjunctions, and because the sentences $\forall \bar{x}\,(\Phi \sqcup \Psi)$ and $(\forall \bar{x}\,\Phi) \sqcup (\forall \bar{x}\,\Psi)$ are logically equivalent when Φ and Ψ involve different variables and are quantifier-free.

($\Leftarrow$) If $\boldsymbol{A}, \boldsymbol{B}$ are disjoint nontrivial members of a quasivariety K, then the respective identity functions on A and B extend to surjective homomorphisms $\pi_A \colon \boldsymbol{F}_{\mathsf{K}}(A) \to \boldsymbol{A}$ and $\pi_B \colon \boldsymbol{F}_{\mathsf{K}}(B) \to \boldsymbol{B}$ (recalling that $\boldsymbol{F}_{\mathsf{K}}(X)$ denotes a member of K that is K-free over X). In $\boldsymbol{F} := \boldsymbol{F}_{\mathsf{K}}(A \cup B)$, let θ be the K-congruence generated by the union of the kernels of π_A and π_B, and let $\boldsymbol{C} = \boldsymbol{F}/\theta$, so $\boldsymbol{C} \in \mathsf{K}$. The map $h_A \colon a \mapsto a/\theta$ [resp. $h_B \colon b \mapsto b/\theta$] is a homomorphism from $\boldsymbol{A}$ [resp. $\boldsymbol{B}$] into $\boldsymbol{C}$. To prove the injectivity of h_A suppose that $a/\theta = a'/\theta$. Then $\Theta^{\boldsymbol{F}}_{\mathsf{K}}(a, a') \subseteq \theta = \Theta^{\boldsymbol{F}}_{\mathsf{K}}(\ker \pi_A \cup \ker \pi_B)$. By the algebraicity of the lattice $\mathbf{Con}_{\mathsf{K}}(\boldsymbol{F})$ (Lemma 1.13), there is a finite set $Y \subseteq \ker \pi_A \cup \ker \pi_B$, such that $\Theta^{\boldsymbol{F}}_{\mathsf{K}}(a, a') \subseteq \Theta^{\boldsymbol{F}}_{\mathsf{K}}\, Y$. We then apply Lemma 1.14 and the abstract relevance principle to show that $\langle a, a' \rangle \in \ker \pi_A$, which implies that $a = a'$. The argument for h_B is the same. $\square$

Definition 5.15. A *relevant algebra* is an e-free subreduct of a De Morgan monoid (i.e., a subalgebra of the reduct $\langle A; \cdot, \wedge, \vee, \neg \rangle$ of some $\boldsymbol{A} \in \mathsf{DMM}$).

These algebras form a variety RA, algebraizing the $\mathbf{t}$-free fragment $\mathbf{R}$ of $\mathbf{R}^{\mathbf{t}}$ (see Section 1.2). A finite equational basis for RA is given in [47] (also see [41], [69, Cor. 4.11] and [103, Sec. 7]). Note that Boolean algebras may be regarded as relevant algebras, since they satisfy $e \approx x \vee \neg x$.

As we mentioned in the introduction, relevance logic was originally designed to avoid the paradoxes of material implication, as exemplified by the weakening axiom $p \to (q \to p)$. The relevance logicians wanted a logic where α would imply β only if α is *relevant* to β. This demand found its expression in the form of a variable sharing principle, called the (concrete) *relevance principle*, which holds for 'relevant' implication:

$$\text{if } \vdash_{\mathbf{R}} \alpha \to \beta, \text{ then } \alpha \text{ and } \beta \text{ have a common variable [5].} \tag{5.3}$$

The corresponding claim for $\mathbf{R^t}$ is false, for example, $\mathbf{R^t}$ satisfies axiom **A11** ($\mathbf{t} \to (p \to p)$) of Definition 1.42. Although a study of RA accommodates the relevance principle, it has some forbidding features for the algebraist. It lacks the congruence extension property (CEP) [35], whereas DMM has EDPC, and therefore the CEP. Also, De Morgan monoids have much in common with abelian groups (the residual being a partial surrogate for multiplicative inverses), but relevant algebras are less intuitive, being semigroup-based, rather than monoid-based. Finally, the study of De Morgan monoids can simplify the analysis of relevant algebras; see for example [103, Sec. 7].

Since RA lacks the CEP, it does not have EDPC, by Theorem 1.25. Therefore, $\mathbf{R}$ does not have a deduction-detachment theorem, by Theorem 1.41. The relevance principle (5.3) is therefore tied to the connective $\to$, and it does not lead straightforwarly to a proof of the abstract relevance principle for $\mathbf{R}$ (and RA). Nevertheless, we have:

Example 5.16. The variety RA respects the abstract relevance principle of Definition 5.12, by [84, Thm. 6].[4] Therefore, RA has the JEP, by Theorem 5.14. In other words (by Theorem 5.2), $\mathsf{RA} = \mathbb{Q}(\boldsymbol{A})$ for some $\boldsymbol{A}$ (cf. [137, Thm. 5]). By the Łoś-Suszko Theorem 5.14, therefore, $\mathbf{R}$ respects not only the relevance principle, but the abstract relevance principle as well.

In contrast, the abstract relevance principle fails for $\mathbf{R^t}$, because DMM lacks the JEP (by Proposition 5.6(i), as it has non-isomorphic 0-generated nontrivial members).

Because $\mathbf{2}$ has no trivial subalgebra, while the e-free reduct of $\boldsymbol{S}_3$ has a trivial subalgebra (and so belongs to no Kollár quasivariety), RA would violate Proposition 5.6(ii) if we dropped the demand there for a constant symbol in the signature.

In the variety K generated by the (simple) relevant algebra reducts of $\boldsymbol{C}_4$ and $\boldsymbol{D}_4$, these reducts and $\mathbf{2}$ are the only subdirectly irreducible algebras, by Jónsson's Theorem 1.23. Thus, K is a Kollár variety that lacks the JEP, by Theorem 5.7. This shows that the JEP is not a hereditary property. □

[4]It should be noted here that every finite set of equations in the signature of RA is consistent over RA, as follows from a consideration of the locally finite e-free reduct of the odd Sugihara monoid $\boldsymbol{Z}$.

Remark 5.17. For a class K of similar algebras, let $\mathbb{U}(\mathsf{K})$ be the class of all algebras $\boldsymbol{B}$ such that every $|Var|$-generated subalgebra of $\boldsymbol{B}$ belongs to K. In general, $\mathbb{UISP}(\mathsf{K}) \subseteq \mathbb{Q}(\mathsf{K})$, and the two need not be equal. Now suppose K is a quasivariety with the JEP. Then $\mathsf{K} = \mathbb{UISP}(\boldsymbol{A})$ for some $\boldsymbol{A} \in \mathsf{K}$, by item (i) after Theorem 5.2, because the $|Var|$-generated members of K form a set, up to isomorphism. Thus, if we allowed quasi-equations (over Var) to have infinitely many premises, their validity in $\boldsymbol{A}$ would still entail their validity throughout K. In fact, if we generalized Definition 5.12 to arbitrary sets $\Gamma \cup \Delta$ of formulas, then Theorem 5.14 would remain true. This point is made in [42, Thm. 3(vi)].

We shall see in Examples 5.31 that the variety HA of Heyting algebras has the JEP. When HA is represented as $\mathbb{Q}(\boldsymbol{C}) = \mathbb{UISP}(\boldsymbol{D})$, the algebra $\boldsymbol{C}$ can be chosen countable (by Remark 5.5), but $\boldsymbol{D}$ cannot (see [151]).

5.2 Passive structural completeness

Recall Corollary 1.31, which states that the following (hereditary) demands on a quasivariety K are equivalent.

(i) The nontrivial members of K all satisfy the same existential positive sentences.

(ii) For any two nontrivial members of K, each can be mapped homomorphically into an ultrapower of the other.

Definition 5.18. A quasivariety is said to be *passively structurally complete* (PSC) if it satisfies the equivalent conditions of Corollary 1.31. (The reasons for this name will emerge from remarks made after Definitions 5.21 and 5.23.)

Because the JEP need not persist in subvarieties (see Example 5.16), the following result is of interest.

Theorem 5.19. *If a quasivariety is PSC, then it has the JEP, and so do all of its subquasivarieties.*

Proof. Let $\boldsymbol{A}, \boldsymbol{B}$ be nontrivial members of a PSC quasivariety K. Then there are homomorphisms $f\colon \boldsymbol{A} \to \boldsymbol{B}_u$ and $g\colon \boldsymbol{B} \to \boldsymbol{A}_u$, for suitable ultrapowers $\boldsymbol{A}_u$ and $\boldsymbol{B}_u$ of $\boldsymbol{A}$ and $\boldsymbol{B}$, respectively. Recall that there are (elementary) embeddings $e_A\colon \boldsymbol{A} \to \boldsymbol{A}_u$ and $e_B\colon \boldsymbol{B} \to \boldsymbol{B}_u$. Consider the maps

$$\langle e_A, f\rangle\colon \boldsymbol{A} \to \boldsymbol{A}_u \times \boldsymbol{B}_u \text{ and } \langle g, e_B\rangle\colon \boldsymbol{B} \to \boldsymbol{A}_u \times \boldsymbol{B}_u$$

defined by the following rules: for every $a \in A$ and $b \in B$,

$$\langle e_A, f \rangle(a) = \langle e_A(a), f(a) \rangle \text{ and } \langle g, e_B \rangle(b) = \langle g(b), e_B(b) \rangle.$$

Clearly, $\langle e_A, f \rangle$ and $\langle g, e_B \rangle$ are embeddings, so $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{IS}(\boldsymbol{A}_u \times \boldsymbol{B}_u)$, and $\boldsymbol{A}_u \times \boldsymbol{B}_u \in \mathbb{Q}(\boldsymbol{A}, \boldsymbol{B}) \subseteq \mathsf{K}$. Thus, K has the JEP, as do its subquasivarieties, in view of the argument—or by heredity of the PSC condition. □

Recall that every variety K is generated by its free $\aleph_0$-generated algebra, i.e., $\mathsf{K} = \mathbb{V}(\boldsymbol{F}_{\mathsf{K}}(\aleph_0))$, but K need not coincide with the quasivariety $\mathbb{Q}(\boldsymbol{F}_{\mathsf{K}}(\aleph_0))$ (which has the JEP, by Theorem 5.2).

Theorem 5.20 ([6, Prop. 2.3])**.** *The following conditions on a quasivariety* K *are equivalent.*

(i) $\mathsf{K} = \mathbb{Q}(\boldsymbol{F}_{\mathsf{K}}(\aleph_0))$.

(ii) *Whenever* K' *is a proper subquasivariety of* K, *then* K' *and* K *generate distinct varieties, i.e.,* $\mathbb{H}(\mathsf{K}') \subsetneq \mathbb{H}(\mathsf{K})$.

(iii) *For each quasi-equation* $(\varphi_1 \approx \psi_1 \;\&\; \cdots \;\&\; \varphi_n \approx \psi_n) \Longrightarrow \varphi \approx \psi$ *that is invalid in (some member of)* K, *there exists a substitution* h *(i.e., an endomorphism of the absolutely free algebra over Var) such that* $\mathsf{K} \models h(\varphi_i) \approx h(\psi_i)$ *for* $i = 1, \ldots, n$, *but* $\mathsf{K} \not\models h(\varphi) \approx h(\psi)$.

Definition 5.21. A quasivariety K is said to be *structurally complete* (SC) if it satisfies the equivalent conditions of Theorem 5.20. It is *hereditarily structurally complete* (HSC) if, in addition, its subquasivarieties are all SC.

In particular, a variety K is SC iff each of its proper subquasivarieties generates a proper subvariety of K; it is HSC iff its subquasivarieties are all varieties [6, Prop. 2.4]. Note that Theorem 2.28 therefore states that the variety OSM of odd Sugihara monoids is HSC.

Theorem 5.22 (Gorbunov [61]; also see [112, Sec. 9])**.** *A locally finite variety* K *is HSC iff every finite SI member of* K *embeds into each of its homomorphic pre-images in* K.

The logical significance of these notions is as follows: when a quasivariety K algebraizes a finitary logic $\vdash$, then K is SC iff every proper extension of $\vdash$ has some new *theorem* (as opposed to having nothing but new derivable rules); K is HSC iff every extension of $\vdash$ is an *axiomatic* extension (see page 21, and, for instance, [113]).

Every SC quasivariety K is PSC in the sense of Definition 5.18. Indeed, if $\boldsymbol{A}, \boldsymbol{B} \in \mathsf{K}$ are nontrivial, then $\boldsymbol{A}$ has a homomorphic image $\boldsymbol{C} \in \mathsf{K}_{\mathrm{RSI}}$, while $\boldsymbol{B}$ is an extension of a 1-generated homomorphic image of $\boldsymbol{F}_{\mathsf{K}}(\aleph_0)$, so it suffices to show that $\boldsymbol{F}_{\mathsf{K}}(\aleph_0)$ satisfies the existential positive sentences that are true in $\boldsymbol{C}$. This is indeed the case, by Theorem 1.29, as $\boldsymbol{C} \in \mathbb{ISP}_{\mathbb{U}}(\boldsymbol{F}_{\mathsf{K}}(\aleph_0))$ (because $\mathsf{K} = \mathbb{IP}_{\mathbb{S}}\mathbb{SP}_{\mathbb{U}}(\boldsymbol{F}_{\mathsf{K}}(\aleph_0))$, by Theorem 5.20(i)). Alternatively, condition (iii) of Theorem 5.20 clearly entails the characterization of passive structural completeness in Theorem 5.24 below.

The above argument and Theorem 5.19 establish the implications

$$\text{HSC} \implies \text{SC} \implies \text{PSC} \implies \text{JEP},$$

none of which is reversible. A variety of lattices that is SC but not HSC is exhibited in [6, Ex. 2.14.4]. It is well known (and follows, for instance, from [102]) that the variety of Heyting algebras is not SC, but it is PSC (see Examples 5.31). As we noted in Example 5.16, RA has the JEP, but it is not PSC. In fact, RA has no nontrivial PSC subvariety, other than $\mathbb{V}(\mathbf{2})$ [121, Thm. 6].

Definition 5.23. A set Γ of equations in the signature of a quasivariety K is said to be *unifiable* over K if there is a substitution h such that $\mathsf{K} \models h(\varphi) \approx h(\psi)$ for every equation $\varphi \approx \psi$ from Γ. A quasi-equation

$$(\varphi_1 \approx \psi_1 \ \& \ \cdots \ \& \ \varphi_n \approx \psi_n) \Longrightarrow \varphi \approx \psi$$

in the same signature is said to be *active* [resp. *passive*] *over* K if its set of premises $\{\varphi_i \approx \psi_i : i = 1, \ldots, n\}$ is [resp. is not] unifiable over K.

The next result amplifies the logical meaning of passive structural completeness. (It strengthens an earlier finding of Bergman [6, Thm. 2.7].)

Theorem 5.24 (Wroński [152, Fact 2, p. 68])**.** *A quasivariety K is PSC iff every quasi-equation that is passive over K is valid in (all members of) K.*

Theorem 5.24 motivates the 'passive' terminology used above, which is adapted from [30]. A complementary demand, now called 'active structural completeness' (ASC) and analysed in [26, 40], asks that condition (iii) of Theorem 5.20 should hold for all active quasi-equations; also see [128]. Computational aspects of these notions are explored in [39, 95, 135]. [5]

[5] Obviously, an ASC quasivariety is SC iff it is PSC. As Corollary 1.31 and Theorems 5.25 and 5.35 (below) do not assume active structural completeness, they cast a more general light on items 3.2–3.4 and 4.1–4.3 of [26].

Evidently, a quasi-equation is passive over a quasivariety K iff it is passive over the variety $\mathbb{V}(\mathsf{K})$. It may happen that K is PSC for the vacuous reason that no quasi-equation is passive over K (as applies, for instance, to every quasivariety of lattices). The next theorem and its corollary decode this case in model-theoretic terms. The conditions mentioned in these results persist, of course, under varietal generation, unlike passive structural completeness itself.

Theorem 5.25. *Let K be a quasivariety. Then the following conditions are equivalent.*

(i) *No quasi-equation is passive over K (i.e., every finite set of equations in the signature of K is unifiable over K).*

(ii) *K is PSC and is either trivial or not a Kollár quasivariety.*

(iii) *Every member of K has an ultrapower with a trivial subalgebra.*

(iv) *$\boldsymbol{F}_{\mathsf{K}}(1)$ has an ultrapower with a trivial subalgebra.*

Proof. (i) $\Rightarrow$ (ii): Certainly, K is PSC, by (i) and Theorem 5.24. If $\boldsymbol{F}_{\mathsf{K}}(1)$ is trivial, then every member of K has a trivial subalgebra (by Theorem 1.6), so we may assume that $\boldsymbol{F}_{\mathsf{K}}(1)$ is nontrivial.

Let Σ be the set of all existential positive sentences in the first order signature of K, and let $\{f_1,\dots,f_n\}$ be any finite set of basic operation symbols of K. By (i), the equations $f_i(x,\dots,x)\approx x$ $(i=1,\dots,n)$ are unifiable, i.e., there is a term φ such that $\mathsf{K}\models f_i(\varphi,\dots,\varphi)\approx\varphi$ for $i=1,\dots,n$. Identifying variables, we see that φ may be chosen unary, whence

$$\boldsymbol{F}_{\mathsf{K}}(1)\models\exists x\,(x\approx f_1(x,\dots,x)\approx\dots\approx f_n(x,\dots,x)).$$

As $\{f_1,\dots,f_n\}$ was arbitrary, this implies that $\boldsymbol{F}_{\mathsf{K}}(1)\models\Sigma$. Let $\boldsymbol{C}\in\mathsf{K}$ be trivial. Of course, Σ is the set of all existential positive sentences that hold in $\boldsymbol{C}$, so by Theorem 1.29, $\boldsymbol{C}$ can be mapped homomorphically into an ultrapower $\boldsymbol{U}$ of $\boldsymbol{F}_{\mathsf{K}}(1)$, i.e., $\boldsymbol{U}$ has a trivial subalgebra. Now $\boldsymbol{U}$ is nontrivial (because $\boldsymbol{F}_{\mathsf{K}}(1)\in\mathbb{IS}(\boldsymbol{U})$), so K is not a Kollár quasivariety.

(ii) $\Rightarrow$ (iii): Let $\boldsymbol{A}\in\mathsf{K}$. We may assume that K is nontrivial (otherwise, (iii) is immediate). Then, by (ii), some nontrivial $\boldsymbol{B}\in\mathsf{K}$ has a trivial subalgebra $\boldsymbol{C}$, and K is PSC, so there is a homomorphism $h\colon\boldsymbol{B}\to\boldsymbol{U}$ for some $\boldsymbol{U}\in\mathbb{P}_{\mathbb{U}}(\boldsymbol{A})$ (see Corollary 1.31). Now $h[\boldsymbol{C}]$ is a trivial subalgebra of $\boldsymbol{U}$.

(iii) $\Rightarrow$ (iv) is immediate, since $\boldsymbol{F}_{\mathsf{K}}(1)\in\mathsf{K}$.

(iv) $\Rightarrow$ (i): Let $\boldsymbol{U}\in\mathbb{P}_{\mathbb{U}}(\boldsymbol{F}_{\mathsf{K}}(1))$, where $\boldsymbol{U}$ has a trivial subalgebra. Then, for any finite set Γ of equations in the signature of K, the sentence $\exists\,\vec{x}\,(\&\,\Gamma)$ is true in $\boldsymbol{U}$, so it is true in $\boldsymbol{F}_{\mathsf{K}}(1)$. Therefore, Γ is unifiable over K. □

Corollary 5.26. *Let* K *be a quasivariety, either of finite type or whose free* 1*-generated algebra is finite. Then no quasi-equation is passive over* K *iff every member of* K *has a trivial subalgebra.*

Proof. Sufficiency follows from Theorem 5.25. Conversely, suppose that no quasi-equation is passive over K. Then some ultrapower $\boldsymbol{A}$ of $\boldsymbol{F}_{\mathsf{K}}(1)$ has a trivial subalgebra, again by Theorem 5.25. It clearly suffices to show that $\boldsymbol{F}_{\mathsf{K}}(1)$ has a trivial subalgebra. If $\boldsymbol{F}_{\mathsf{K}}(1)$ is finite, then it is isomorphic to $\boldsymbol{A}$, and we are done. If the signature of K is finite then, for its models, the property of having a trivial subalgebra is expressed by an existential positive sentence (which persists in ultraroots by Theorem 1.8). In that case, $\boldsymbol{F}_{\mathsf{K}}(1)$ has a trivial subalgebra, because $\boldsymbol{A}$ does. □

In general, however, the ultrapowers in Theorem 5.25 cannot be eliminated, because of the next example.

Example 5.27. For $n \in \mathbb{N} = \{1, 2, 3, \dots\}$, let $f_n \colon \mathbb{N} \to \mathbb{N}$ be the function such that $f_n(m) = m + n$ for $m = 1, \dots, n$ and $f_n(m) = m$ whenever $n < m \in \mathbb{N}$. Let $\boldsymbol{A}$ be the algebra with universe $\mathbb{N}$, whose set of basic operations is $\{f_n : n \in \mathbb{N}\}$, and let $\mathsf{K} = \mathbb{V}(\boldsymbol{A})$. In this signature, every term that is not a variable has the form $f_{i_1} \dots f_{i_k}(x)$ for some $i_1, \dots, i_k \in \mathbb{N}$. Therefore, since $\boldsymbol{A}$ generates K, every finite set of equations can be unified over K by substituting $f_r(x)$ for every variable, where r is sufficiently large. Thus, no quasi-equation is passive over K, but $\boldsymbol{A}$ is a nontrivial member of K that has no trivial subalgebra. (For each non-principal ultrafilter $\mathcal{U}$ over $\mathbb{N}$, the ultrapower $\boldsymbol{A}^{\mathbb{N}}/\mathcal{U}$ has a trivial subuniverse, viz. $\{\langle 1, 2, 3, \dots \rangle/\mathcal{U}\}$.) □

In the context of De Morgan monoids, by Corollary 5.26, the subquasivarieties of DMM over which no quasi-equation is passive are exactly the sub(quasi)varieties of OSM, because every De Morgan monoid with a trivial subalgebra is an odd Sugihara monoid, by Theorem 2.25.

The next result identifies the PSC quasivarieties of finite type containing at least one finite nontrivial algebra. (It strengthens Lemma 3.8.)

Theorem 5.28. *Let* K *be a quasivariety of finite type, with a finite nontrivial member. Then the following conditions are equivalent.*

(i) K *is PSC.*

(ii) *The nontrivial members of* K *have a common retract.*

(iii) *Each nontrivial member of* K *can be mapped homomorphically into every member of* K.

In this case, the nontrivial members of K *have a finite common retract that has no nontrivial proper subalgebra and is either trivial or relatively simple.*

Moreover, when K *is PSC, its nontrivial members have at most one nontrivial common retract, and they have at most one* 0*-generated common retract (up to isomorphism).*

Proof. By assumption, K has a finite nontrivial member, and that algebra has a relatively simple (finite nontrivial) homomorphic image $\boldsymbol{A} \in \mathsf{K}$, by Theorem 1.15(ii).

(i) $\Rightarrow$ (ii): Possession of a trivial subalgebra is expressible, over K, by an existential positive sentence, because K has finite type. Therefore, since K is PSC, if some nontrivial member of K has a trivial subalgebra, then so does every member of K. In that case, every member of K has a trivial retract.

We may therefore assume that K is a Kollár quasivariety. In particular, $\boldsymbol{A}$ has no trivial subalgebra. To complete the proof of (ii), we shall show that $\boldsymbol{A}$ is a retract of every nontrivial member of K.

Accordingly, let $\boldsymbol{B} \in \mathsf{K}$ be nontrivial, so $\boldsymbol{B}$ has no trivial subalgebra. Since $\boldsymbol{A}$ is finite and of finite type, there is an existential positive sentence Φ such that an algebra in the signature of K satisfies Φ iff it has a subalgebra that is a homomorphic image of $\boldsymbol{A}$. As Φ is true in $\boldsymbol{A}$, it is true in $\boldsymbol{B}$, because K is PSC (and since $\boldsymbol{A}$ and $\boldsymbol{B}$ are nontrivial). Therefore, there is a homomorphism $g\colon \boldsymbol{A} \to \boldsymbol{B}$. As $\boldsymbol{A}$ is relatively simple and $\boldsymbol{B}$ has no trivial subalgebra, g is an embedding, by Fact 1.19. Moreover, since K is PSC, there is a homomorphism h from $\boldsymbol{B}$ into an ultrapower of $\boldsymbol{A}$, but $\boldsymbol{A}$ is finite, so $h\colon \boldsymbol{B} \to \boldsymbol{A}$. Thus, $h \circ g$ is an endomorphism of $\boldsymbol{A}$.

Because $\boldsymbol{A}$ has no trivial subalgebra, the argument for the injectivity of g applies equally to $h \circ g$. Then, since $h \circ g$ is an injection from the finite set A to itself, it is surjective, i.e., $h \circ g$ is an automorphism of $\boldsymbol{A}$.

As the automorphism group of $\boldsymbol{A}$ is finite, $(h \circ g)^{n+1} = \mathrm{id}_A$ for some $n \in \omega$. Then, for the homomorphism $k := g \circ (h \circ g)^n \colon \boldsymbol{A} \to \boldsymbol{B}$, we have $h \circ k = (h \circ g)^{n+1} = \mathrm{id}_A$. Thus, $\boldsymbol{A}$ is a retract of $\boldsymbol{B}$, as claimed.

We have shown that a finite common retract $\boldsymbol{A}'$ of the nontrivial members of K exists and can be chosen relatively simple or trivial. Being finite, $\boldsymbol{A}'$ cannot be a retract of a proper subalgebra of itself, so it has no such nontrivial subalgebra. In particular, if $\boldsymbol{A}'$ is nontrivial, then it is isomorphic to any other nontrivial common retract of the nontrivial members of K. Consequently, if $\boldsymbol{A}'$ is 0-generated, then it is isomorphic to *any* other common retract of the nontrivial members of K, because it is either trivial or has no trivial subalgebra.

(ii) ⇒ (iii): Let $\boldsymbol{C}, \boldsymbol{D} \in \mathsf{K}$, where $\boldsymbol{C}$ is nontrivial. We may assume that $\boldsymbol{D}$ is nontrivial, so there is a common retract $\boldsymbol{A}$ of $\boldsymbol{C}, \boldsymbol{D}$, by (ii). Then there exist a surjective homomorphism $\boldsymbol{C} \to \boldsymbol{A}$ and an embedding $\boldsymbol{A} \to \boldsymbol{D}$, whose composition is a homomorphism $\boldsymbol{C} \to \boldsymbol{D}$.

(iii) ⇒ (i): Let $\boldsymbol{C}, \boldsymbol{D} \in \mathsf{K}$ be nontrivial. By (iii), $\boldsymbol{C}$ can be mapped homomorphically into (an ultrapower of) $\boldsymbol{D}$, so K is PSC. □

Note 5.29. In Theorem 5.28, the finiteness of the signature and the presence of a finite nontrivial algebra in K are needed only for the implication (i) ⇒ (ii).

It follows easily from Theorem 5.28(iii) that passive structural completeness is a decidable property for finitely generated quasivarieties of finite type. Also, Theorem 5.28(iii) amounts to the demand that each nontrivial member of K is a retract of its direct product with any member of K.

Recall that the quasivariety N, from Definition 3.5, comprises all De Morgan monoids that are either trivial or have $\boldsymbol{C}_4$ as a retract. So, it follows from Theorem 5.28 that N is PSC, as are all of its subquasivarieties (including its largest subvariety M; see Theorem 3.18).

Corollary 5.30. *Let K be a PSC Kollár quasivariety of finite type, with a finite nontrivial member. Then K has a unique relatively simple member (up to isomorphism), and that algebra is a finite common retract of the nontrivial members of K.*

Proof. This follows from Theorem 5.28, because a relatively simple member of a Kollár quasivariety is isomorphic to each of its retracts (by Fact 1.19). □

Examples 5.31. It follows from Theorem 5.28 (and Note 5.29) that every variety consisting of groups or of Heyting algebras is PSC (and therefore has the JEP, by Theorem 5.19). Indeed, every group has a trivial retract, while the two-element Boolean algebra is a retract of every nontrivial Heyting algebra. The class of all distributive lattices is a PSC variety whose nontrivial members have both a trivial and a nontrivial common retract, the latter being the two-element lattice (see [7, Cor. 2.45]). In Corollary 5.30, we cannot drop the demand that K be a Kollár quasivariety, as the variety of abelian groups satisfies the other hypotheses, but includes all the simple groups $\mathbb{Z}_p$ (p a positive prime). □

Notation. For a quasivariety K, with $\boldsymbol{A} \in \mathsf{K}$, we define

$$\text{Ret}(\mathsf{K}, \boldsymbol{A}) = \{\boldsymbol{B} \in \mathsf{K} : \boldsymbol{B} \text{ is trivial or } \boldsymbol{A} \text{ is a retract of } \boldsymbol{B}\}.$$

(In this notation, $\mathsf{N} = \mathrm{Ret}(\mathsf{DMM}, \boldsymbol{C}_4)$.)

Theorem 5.32. *Let* K *be a quasivariety of finite type, and* $\boldsymbol{A} \in \mathsf{K}$ *a finite* 0*-generated algebra.*

(i) $\mathrm{Ret}(\mathsf{K}, \boldsymbol{A})$ *is a PSC quasivariety.*

(ii) *If* $\boldsymbol{A}$ *is nontrivial or* K *is not a Kollár quasivariety, then* $\mathrm{Ret}(\mathsf{K}, \boldsymbol{A})$ *is a maximal PSC subquasivariety of* K.

(iii) *If* K' *is a maximal PSC subquasivariety of* K, *and if* $\boldsymbol{B}' \in \mathsf{K}'$ *is finite and nontrivial, then* $\mathsf{K}' = \mathrm{Ret}(\mathsf{K}, \boldsymbol{A}')$, *where* $\boldsymbol{A}'$ *is the* 0*-generated subalgebra of* $\boldsymbol{B}'$.

(iv) *Every PSC subquasivariety of* K *that has a finite nontrivial member is contained in just one maximal PSC subquasivariety of* K.

Proof. Let $\mathsf{L} = \mathrm{Ret}(\mathsf{K}, \boldsymbol{A})$.

(i) It suffices, by Note 5.29, to show that L is a quasivariety. We can use the proof of Theorem 3.9 (for N), with $\boldsymbol{A}$ in the role of $\boldsymbol{C}_4$, because in that proof, we used only the fact that $\boldsymbol{C}_4$ is finite, 0-generated and of finite type.

(ii) Suppose $\mathsf{L} \subseteq \mathsf{K}' \subseteq \mathsf{K}$, where K' is a PSC quasivariety. Then $\boldsymbol{A} \in \mathsf{K}'$. If $\boldsymbol{A}$ is nontrivial, then Theorem 5.28 applies to K' (because $\boldsymbol{A}$ is finite) and it shows that, for every nontrivial $\boldsymbol{C} \in \mathsf{K}'$, there are homomorphisms $\boldsymbol{A} \to \boldsymbol{C}$ and $\boldsymbol{C} \to \boldsymbol{A}$ (as K' is PSC). In this case $\mathsf{K}' \subseteq \mathsf{L}$, by Remark 3.7 (as $\boldsymbol{A}$ is 0-generated). We may therefore assume that $\boldsymbol{A}$ is trivial. Now suppose K is not a Kollár quasivariety. Then $\boldsymbol{A}$ embeds into some nontrivial $\boldsymbol{B} \in \mathsf{K}$, whence $\boldsymbol{B} \in \mathsf{L}$, and so $\boldsymbol{B} \in \mathsf{K}'$. Thus, K' is not a Kollár quasivariety. Then $\mathsf{K}' \subseteq \mathsf{L}$, by Proposition 5.6(ii) and Theorem 5.19.

(iii) Let $\mathsf{K}', \boldsymbol{B}', \boldsymbol{A}'$ be as described. By (i), it is enough to show that $\mathsf{K}' \subseteq \mathrm{Ret}(\mathsf{K}, \boldsymbol{A}')$. This will be true if every member of K' has a trivial subalgebra (in which case $\boldsymbol{A}'$ is trivial). We may therefore assume, by Proposition 5.6(ii) and Theorem 5.19, that K' is a Kollár quasivariety (as K' is PSC). Then $\boldsymbol{A}'$ is nontrivial, so it is K'-simple, by Proposition 5.6(iii). Thus, $\mathsf{K}' \subseteq \mathrm{Ret}(\mathsf{K}, \boldsymbol{A}')$, by Corollary 5.30.

(iv) follows from Theorem 5.28, together with (i)–(iii). □

Notice that N is a maximal PSC subquasivariety of DMM, by Theorem 5.32(ii), since $\boldsymbol{C}_4$ is finite and 0-generated. We shall now show that every maximal PSC subquasivariety of DMM is $\mathrm{Ret}(\mathsf{DMM}, \boldsymbol{A})$ for some 0-generated De Morgan monoid $\boldsymbol{A}$.

Recall from Remark 3.29 that the free 0-generated De Morgan monoid has 68 factor algebras, no two of which are isomorphic (see the proof of Theorem 4.6). Let $\boldsymbol{A}_1, \ldots, \boldsymbol{A}_{68}$ denote the factor algebras, where $\boldsymbol{A}_1$ is trivial. By the Homomorphism Theorem 1.1, these are all of the 0-generated De Morgan monoids, up to isomorphism. As passive structural completeness persists in subquasivarieties, the next result is a characterization of the PSC quasivarieties of De Morgan monoids.

Theorem 5.33. *The maximal PSC subquasivarieties of* DMM *are just the distinct classes* $\mathrm{Ret}(\mathsf{DMM}, \boldsymbol{A}_i)$, $i = 1, \ldots, 68$, *and every nontrivial PSC quasivariety of De Morgan monoids is contained in just one of these.*

Moreover, $\mathrm{Ret}(\mathsf{DMM}, \boldsymbol{A}_1)$ *is the variety of odd Sugihara monoids. For* $i > 1$, *each relatively simple member of* $\mathrm{Ret}(\mathsf{DMM}, \boldsymbol{A}_i)$ *is isomorphic to* $\boldsymbol{A}_i$.

Proof. A De Morgan monoid has a trivial subalgebra iff it is an odd Sugihara monoid, by Theorem 2.11, so $\mathrm{Ret}(\mathsf{DMM}, \boldsymbol{A}_1) = \mathsf{OSM}$, and DMM is not a Kollár variety. Therefore, $\mathrm{Ret}(\mathsf{DMM}, \boldsymbol{A}_i)$ is a maximal PSC subquasivariety of DMM, for $i = 1, \ldots, 68$, by Theorem 5.32(i), (ii). Every maximal PSC subquasivariety K' of DMM, other than OSM, has a finite nontrivial member (viz. the 0-generated subalgebra of any member of $\mathsf{K}' \setminus \mathsf{OSM}$), so $\mathsf{K}' = \mathrm{Ret}(\mathsf{DMM}, \boldsymbol{A}_i)$ for some $i \in \{2, \ldots, 68\}$, by Theorem 5.32(iii), and every nontrivial PSC subquasivariety of DMM is contained in $\mathrm{Ret}(\mathsf{DMM}, \boldsymbol{A}_i)$ for exactly one $i \in \{1, \ldots, 68\}$, by Theorem 5.32(iv). For $i > 1$, the common retract $\boldsymbol{A}_i$ of $\mathrm{Ret}(\mathsf{DMM}, \boldsymbol{A}_i)$ is unique (up to isomorphism) and relatively simple, by Theorem 5.28, since it is 0-generated and nontrivial. □

The PSC sub*varieties* of DMM are more limited.

Theorem 5.34. *Let* K *be a variety of De Morgan monoids. Then* K *is PSC iff one of the following four (mutually exclusive) conditions holds:*

(i) K *is the variety* $\mathbb{V}(\mathbf{2})$ *of all Boolean algebras;*

(ii) $\mathsf{K} = \mathbb{V}(\boldsymbol{D}_4)$;

(iii) K *consists of odd Sugihara monoids;*

(iv) K *is a nontrivial subvariety of* M.

Proof. By Theorem 5.33, a nontrivial variety of De Morgan monoids is PSC iff it lies within $\mathrm{Ret}(\mathsf{DMM}, \boldsymbol{A}_i)$ for some $i \in \{1, \ldots, 68\}$ (in which case i is unique). This includes all the varieties mentioned in the present theorem, because $\mathbf{2}$, $\boldsymbol{C}_4$, $\boldsymbol{D}_4$ and the trivial De Morgan monoid are 0-generated and finite. Conversely, consider a nontrivial PSC variety $\mathsf{K} \subseteq \text{Ret}(\mathsf{DMM}, \boldsymbol{A}_i)$. As $\text{Ret}(\mathsf{DMM}, \boldsymbol{A}_1) = \mathsf{OSM}$, we may assume that $i > 1$. Theorem 5.33 also asserts that $\boldsymbol{A}_i$ is relatively simple in the quasivariety $\text{Ret}(\mathsf{DMM}, \boldsymbol{A}_i)$, so it is a simple member of K. Therefore, $\boldsymbol{A}_i \in \mathbb{I}(\mathbf{2}, \boldsymbol{C}_4, \boldsymbol{D}_4)$, by Theorem 3.1. If $\boldsymbol{A}_i \cong \boldsymbol{C}_4$ then $\mathsf{K} \subseteq \mathsf{M}$, by Theorem 3.18, so suppose $\boldsymbol{A}_i \cong \mathbf{2}$ [resp. $\boldsymbol{A}_i \cong \boldsymbol{D}_4$]. Let $\boldsymbol{B} \in \mathsf{K}$ be subdirectly irreducible. As $\boldsymbol{A}_i \in \mathbb{H}(\boldsymbol{B})$, Theorem 3.4 shows that $\boldsymbol{B} \cong \boldsymbol{A}_i$. Consequently, K is $\mathbb{V}(\mathbf{2})$ [resp. $\mathbb{V}(\boldsymbol{D}_4)$]. □

Notice that any minimal quasivariety is HSC, and hence (P)SC. Of the 68 minimal subquasivarieties of De Morgan monoids, the four that are *varieties* are generated by simple algebras. This fact instantiates the following general theorem, in view of Corollary 2.17.

Theorem 5.35. *A relatively semisimple quasivariety* K *is PSC iff it is a minimal quasivariety or has no passive quasi-equation.*

Proof. Sufficiency is obvious. Conversely, let K be PSC and suppose that some quasi-equation is passive over K. Then K is a nontrivial Kollár quasivariety, by Theorem 5.25. Let $\boldsymbol{A} \in \mathsf{K}$ be nontrivial. As K is relatively semisimple, its minimality will follow if we can show that $\mathsf{K}_{\text{RS}} \subseteq \mathbb{Q}(\boldsymbol{A})$, so let $\boldsymbol{B} \in \mathsf{K}_{\text{RS}}$. Since $\boldsymbol{B}$ is nontrivial and K is PSC, there is a homomorphism h from $\boldsymbol{B}$ into an ultrapower $\boldsymbol{C}$ of $\boldsymbol{A}$. Of course, $\boldsymbol{C}$ is also nontrivial, so h is an embedding, by Fact 1.19, because K is a Kollár quasivariety. Thus, $\boldsymbol{B} \in \mathbb{IS}(\boldsymbol{C}) \subseteq \mathbb{ISP}_{\mathbb{U}}(\boldsymbol{A}) \subseteq \mathbb{Q}(\boldsymbol{A})$, as required. □

The proof of Theorem 5.35 yields the following.

Corollary 5.36. *If a relatively semisimple quasivariety with a passive quasi-equation is PSC, then it is both a Kollár quasivariety and a minimal quasivariety (and is therefore HSC).*

We can now describe the varieties of De Morgan monoids with the JEP.

Theorem 5.37. *Let* K *be a variety of De Morgan monoids. Then* K *has the JEP iff one of the following (mutually exclusive) conditions is met.*

(i) K *is PSC (see Theorem 5.34).*

(ii) $\mathsf{K} = \mathbb{V}(\boldsymbol{A})$ *for some simple De Morgan monoid* $\boldsymbol{A}$ *such that* $\boldsymbol{D}_4$ *is a proper subalgebra of* $\boldsymbol{A}$*.*

(iii) *There exist* $\boldsymbol{A}, \boldsymbol{B}$ *such that* $\mathsf{K} = \mathbb{Q}(\boldsymbol{B})$*,* $\boldsymbol{A}$ *is a simple subalgebra of* $\boldsymbol{B}$*, and* $\boldsymbol{C}_4$ *is a proper subalgebra of* $\boldsymbol{A}$*.*

In (iii), '$\mathsf{K} = \mathbb{Q}(\boldsymbol{B})$' *can be paraphrased as* '$\mathsf{K} = \mathbb{V}(\boldsymbol{B})$ *and every finitely generated subdirectly irreducible member of* $\mathbb{HP}_{\mathbb{U}}(\boldsymbol{B})$ *belongs to* $\mathbb{ISP}_{\mathbb{U}}(\boldsymbol{B})$'.

Proof. Sufficiency follows from Theorems 5.2, 5.19 and 5.11, since DMM has EDPC.

Conversely, suppose that K has the JEP but is not PSC. Then K is nontrivial and, by Theorem 5.34, K does not consist solely of Boolean algebras, nor solely of odd Sugihara monoids. In particular, not every member of K has a trivial subalgebra. Therefore, K is a Kollár variety, by Proposition 5.6(ii), so $\boldsymbol{S}_3 \notin \mathsf{K}$. As we observed before Corollary 2.27, every finitely generated subdirectly irreducible Sugihara monoid that is not a Boolean algebra maps homomorphically onto $\boldsymbol{S}_3$, so no such algebra belongs to K, whence every idempotent member of K is Boolean. Consequently, if K has an idempotent nontrivial member, then the 0-generated subalgebras of its nontrivial members are all isomorphic to $\mathbf{2}$, by Proposition 5.6(i). In that case, K consists of idempotent algebras, by Theorem 2.11, and so coincides with $\mathbb{V}(\mathbf{2})$, a contradiction. This shows that K has no nontrivial idempotent member.

Being nontrivial, K therefore includes $\boldsymbol{C}_4$ or $\boldsymbol{D}_4$, so $\mathbb{I}(\boldsymbol{C}_4)$ or $\mathbb{I}(\boldsymbol{D}_4)$ is the class of all 0-generated nontrivial members of K, by Proposition 5.6(i). Also, by Corollary 2.14, K satisfies $x \leqslant f^2$ (and hence $\neg(f^2) \leqslant x$ as well). On the other hand, $\mathsf{K} \nsubseteq \mathsf{M}$ and $\mathsf{K} \neq \mathbb{V}(\boldsymbol{D}_4)$, by Theorem 5.34, as K is not PSC.

By Theorem 5.7, there is a simple De Morgan monoid $\boldsymbol{A} \in \mathsf{K}$ such that

$$\text{every simple member of } \mathsf{K} \text{ belongs to } \mathbb{ISP}_{\mathbb{U}}(\boldsymbol{A}). \tag{5.4}$$

By Theorem 5.2, there exists $\boldsymbol{E} \in \mathsf{K}$ such that $\mathsf{K} = \mathbb{Q}(\boldsymbol{E}) = \mathbb{IP}_{\mathbb{S}}\mathbb{SP}_{\mathbb{U}}(\boldsymbol{E})$, whence $\boldsymbol{A} \in \mathbb{ISP}_{\mathbb{U}}(\boldsymbol{E})$ (as $\boldsymbol{A}$ is simple). Choose $\boldsymbol{B} \in \mathbb{IP}_{\mathbb{U}}(\boldsymbol{E})$ with $\boldsymbol{A} \in \mathbb{S}(\boldsymbol{B})$. As $\boldsymbol{B}$ is an ultrapower of $\boldsymbol{E}$, we have $\boldsymbol{E} \in \mathbb{IS}(\boldsymbol{B})$, whence $\mathsf{K} = \mathbb{Q}(\boldsymbol{B})$.

Suppose first that $\mathbb{I}(\boldsymbol{C}_4)$ is the class of 0-generated nontrivial members of K. As K is a Kollár variety, the 0-generated subalgebra of $\boldsymbol{A}$ is nontrivial, so it can be identified with $\boldsymbol{C}_4$. If $\boldsymbol{A} = \boldsymbol{C}_4$, then every simple member of K is isomorphic to $\boldsymbol{C}_4$, by (5.4), so $\boldsymbol{C}_4$ is a retract of every nontrivial member of K (by Corollary 1.18 and Remark 3.7), i.e., $\mathsf{K} \subseteq \mathsf{M}$, a contradiction. This shows that $\boldsymbol{C}_4$ is a proper subalgebra of $\boldsymbol{A}$, so (iii) holds.

We may now assume that $\mathbb{I}(\boldsymbol{D}_4)$ is the class of 0-generated nontrivial members of K. Let $\boldsymbol{G}$ be any subdirectly irreducible member of K. Again, since K is a Kollár variety, the 0-generated subalgebras of $\boldsymbol{A}, \boldsymbol{G}$ are nontrivial, so we may assume that $\boldsymbol{D}_4 \in \mathbb{S}(\boldsymbol{A}) \cap \mathbb{S}(\boldsymbol{G})$. Therefore, as $\boldsymbol{D}_4$ satisfies $e \wedge f \approx \neg(f^2)$, so does $\boldsymbol{G}$. Consequently, as $\neg(f^2)$ is the least element of $\boldsymbol{G}$, it follows from Theorem 2.43 that $\neg(f^2)$ is the sole strict lower bound of e in $\boldsymbol{G}$, whence $\boldsymbol{G}$ is simple, by Lemma 2.16(iv). This shows that K is a semisimple variety, so $\mathsf{K} = \mathbb{Q}(\boldsymbol{A})$, by (5.4). Since $\mathsf{K} \neq \mathbb{V}(\boldsymbol{D}_4) = \mathbb{Q}(\boldsymbol{D}_4)$, we must have $\boldsymbol{A} \neq \boldsymbol{D}_4$, and so (ii) holds.

Note that (i) precludes both (ii) and (iii), by Theorem 5.28, because each of $\boldsymbol{C}_4, \boldsymbol{D}_4$ has no retract other than its isomorphic images, and cannot be a retract of a strictly larger simple algebra. Also, (ii) precludes (iii), by Proposition 5.6(i), as $\boldsymbol{C}_4$ and $\boldsymbol{D}_4$ are both 0-generated and nontrivial.

Since every variety is generated as such by its finitely generated subdirectly irreducible members, the paraphrase in the last claim is justified by Jónsson's Theorem 1.23, and the fact that $\mathbb{HS}(\boldsymbol{P}) \subseteq \mathbb{SH}(\boldsymbol{P})$ for all $\boldsymbol{P} \in \mathsf{DMM}$ (by Theorem 1.24, because De Morgan monoids have the CEP). □

Corollary 5.38. *A variety of Sugihara monoids has the JEP iff it is PSC.*

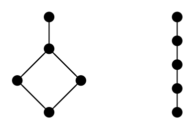
Corollary 5.39. *In the lattice of varieties of De Morgan monoids, all but four of the join-irreducible covers of atoms have the JEP.*

Proof. From Theorem 4.9, with four exceptions, each join-irreducible cover of an atom has the JEP, as it is either a subvariety of M or of OSM (and is thus PSC) or has the form $\mathbb{V}(\boldsymbol{A})$ for a simple algebra $\boldsymbol{A}$. The exceptions are covers of $\mathbb{V}(\boldsymbol{C}_4)$ that lack the JEP, by Proposition 5.6(i), because they are the varietal closures of the 0-generated algebras $\boldsymbol{C}_5, \ldots, \boldsymbol{C}_8$ (respectively), each of which has more elements than $\boldsymbol{C}_4$. □

We showed in Example 5.10 that $\mathbb{V}(X(\mathbf{2} \times \boldsymbol{S}_3)) = \mathbb{Q}(X(\mathbf{2} \times \boldsymbol{S}_3))$. This instantiates Theorem 5.37(iii), because for the trivial De Morgan monoid $\boldsymbol{E}$, the five-element simple algebra $X(\boldsymbol{E})$ belongs to $\mathbb{V}(X(\mathbf{2} \times \boldsymbol{S}_3))$ and has $\boldsymbol{C}_4$ as its smallest subalgebra. So, in Theorem 5.37(iii), it can happen that K is not generated, even as a variety, by one finitely subdirectly irreducible algebra, and that remains the case when $\mathsf{K} = \mathbb{Q}(\boldsymbol{B})$ for some *finite* $\boldsymbol{B}$.

As cases (i) and (iii) of Theorem 5.37 are mutually exclusive, the variety $\mathbb{V}(X(\mathbf{2} \times \boldsymbol{S}_3))$ is not PSC. But it is still the case that a PSC variety (which has the JEP, by Theorem 5.19) need not be generated as a quasivariety by a finitely subdirectly irreducible algebra, as the next example shows.

Example 5.40. Let $\mathsf{K} = \mathbb{V}(\boldsymbol{A}, \boldsymbol{B})$, where $\boldsymbol{A}$ and $\boldsymbol{B}$ are the only two non-isomorphic subdirectly irreducible five-element Heyting algebras (depicted below).



Like every variety of Heyting algebras, K is PSC and therefore has the JEP (see Examples 5.31). Suppose $\mathsf{K} = \mathbb{Q}(\boldsymbol{C})$, where $\boldsymbol{C}$ is finitely subdirectly irreducible. By Jónsson's Theorem 1.23, $\boldsymbol{C} \in \mathbb{HSP}_{\mathbb{U}}(\boldsymbol{A}, \boldsymbol{B}) = \mathbb{HS}(\boldsymbol{A}, \boldsymbol{B})$ (as $\boldsymbol{A}$ and $\boldsymbol{B}$ are finite), whence $|C| \leq 5$. Now $\boldsymbol{A}$ and $\boldsymbol{B}$ are subdirectly irreducible members of $\mathbb{Q}(\boldsymbol{C}) = \mathbb{IP}_{\mathbb{S}}\mathbb{SP}_{\mathbb{U}}(\boldsymbol{C})$, so $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{ISP}_{\mathbb{U}}(\boldsymbol{C}) = \mathbb{IS}(\boldsymbol{C})$ (as $\boldsymbol{C}$ is finite). Since $|C| \leq |A|, |B|$, this forces $\boldsymbol{A} \cong \boldsymbol{C} \cong \boldsymbol{B}$, a contradiction. Thus, no finitely subdirectly irreducible algebra generates K as a quasivariety. □

5.3 Structural completeness

Recall that the four minimal varieties of De Morgan monoids are minimal as quasivarieties (see Theorem 4.5). In particular, $\mathbb{V}(\mathbf{2})$, $\mathbb{V}(\boldsymbol{C}_4)$ and $\mathbb{V}(\boldsymbol{D}_4)$ are HSC, and so is OSM (because of Theorem 2.28).

By Theorem 5.34, every remaining SC variety of De Morgan monoids must be a subvariety of M. We shall see, in Section 7.1, that there is also a continuum of structurally *in*complete subvarieties of M. This shows that M and the quasivariety N are not HSC. (We conjecture that M and N are not SC.)

Recall from Theorem 4.23 that, in the lattice of subvarieties of M, the unique atom $\mathbb{V}(\boldsymbol{C}_4)$ has just six covers. We shall now show that the varietal join of those six covers is HSC. To facilitate the proof, let $\boldsymbol{G}_1, \ldots, \boldsymbol{G}_6$ abbreviate the six algebras mentioned in Theorem 4.23, so that $\mathbb{V}(\boldsymbol{G}_i)$, $i = 1, \ldots, 6$, are the covers of $\mathbb{V}(\boldsymbol{C}_4)$ within M. Their varietal join $\mathbb{V}(\boldsymbol{G}_1, \ldots, \boldsymbol{G}_6)$ is locally finite, like any finitely generated variety (Theorem 1.20). For each $i \in \{1, \ldots, 6\}$, recall that $\boldsymbol{G}_i$ and $\boldsymbol{C}_4$ are the only subalgebras of $\boldsymbol{G}_i$ and are also, up to isomorphism, the only nontrivial homomorphic images of $\boldsymbol{G}_i$ (because $|(e]| = 3$ in $\boldsymbol{G}_i$). By Jónsson's Theorem 1.23,

$$\text{if } \emptyset \neq \mathsf{X} \subseteq \{\boldsymbol{G}_1, \ldots, \boldsymbol{G}_6, \boldsymbol{C}_4\}, \text{ then } \mathbb{V}(\mathsf{X})_{\mathrm{SI}} = \mathbb{I}(\mathsf{X} \cup \{\boldsymbol{C}_4\}). \tag{5.5}$$

Lemma 5.41. *Let $\boldsymbol{Z}_1, \ldots, \boldsymbol{Z}_n \in \{\boldsymbol{G}_1, \ldots, \boldsymbol{G}_6, \boldsymbol{C}_4\}$, where $0 < n \in \omega$. Then $\boldsymbol{Z}_1, \ldots, \boldsymbol{Z}_n$ are retracts of each algebra that embeds subdirectly into $\prod_{i=1}^n \boldsymbol{Z}_i$.*

Proof. The proof is by induction on n. The case $n = 1$ is trivial, so let $n > 1$. For $\boldsymbol{Z} := \prod_{i=1}^n \boldsymbol{Z}_i$, suppose that $\boldsymbol{A} \in \mathbb{S}(\boldsymbol{Z})$ and that $\pi_i[A] = Z_i$ for each canonical projection $\pi_i \colon \boldsymbol{Z} \to \boldsymbol{Z}_i$.

Let $\boldsymbol{B} = \pi[\boldsymbol{A}]$, where $\pi \colon \boldsymbol{Z} \to \prod_{i=1}^{n-1} \boldsymbol{Z}_i$ is the homomorphism

$$\langle z_1, \ldots, z_{n-1}, z_n \rangle \mapsto \langle z_1, \ldots, z_{n-1} \rangle.$$

Then $\boldsymbol{B}$ embeds subdirectly into $\prod_{i=1}^{n-1} \boldsymbol{Z}_i$, so by the induction hypothesis,

$$\boldsymbol{Z}_1, \dots, \boldsymbol{Z}_{n-1} \text{ are retracts of } \boldsymbol{B}. \tag{5.6}$$

Also, $\boldsymbol{A}$ embeds subdirectly into $\boldsymbol{B} \times \boldsymbol{Z}_n$ and shall be identified here with the image of the obvious embedding.

By Fleischer's Lemma 1.28 (and since IRLs are congruence permutable), there exist an algebra $\boldsymbol{C}$ and surjective homomorphisms $g\colon \boldsymbol{B} \to \boldsymbol{C}$ and $h\colon \boldsymbol{Z}_n \to \boldsymbol{C}$ such that

$$A = \{\langle x, y\rangle \in B \times Z_n : g(x) = h(y)\}. \tag{5.7}$$

As $\boldsymbol{C} \in \mathbb{H}(\boldsymbol{Z}_n)$, we may assume that $\boldsymbol{C}$ is $\boldsymbol{Z}_n$ or $\boldsymbol{C}_4$ or a trivial algebra.

If $\boldsymbol{C}$ is trivial, then $\boldsymbol{A} = \boldsymbol{B} \times \boldsymbol{Z}_n$, by (5.7), so the retracts of $\boldsymbol{A}$ include $\boldsymbol{B}$ and $\boldsymbol{Z}_n$ (by Remark 3.6) and hence all of $\boldsymbol{Z}_1, \dots, \boldsymbol{Z}_n$, by (5.6).

If $\boldsymbol{C} = \boldsymbol{Z}_n \ncong \boldsymbol{C}_4$, then h is an isomorphism and

$$\boldsymbol{Z}_n \in \mathbb{H}(\boldsymbol{B}) \subseteq \mathbb{HP}_{\mathbb{S}}(\boldsymbol{Z}_1, \dots, \boldsymbol{Z}_{n-1}) \subseteq \mathbb{V}(\boldsymbol{Z}_1, \dots, \boldsymbol{Z}_{n-1}),$$

but $\boldsymbol{Z}_n$ is SI, so $\boldsymbol{Z}_n \in \mathbb{I}(\boldsymbol{Z}_1, \dots, \boldsymbol{Z}_{n-1})$, by (5.5), whence $\boldsymbol{Z}_1, \dots, \boldsymbol{Z}_n$ are retracts of $\boldsymbol{B}$, by (5.6). In this case, therefore, it suffices to show that $\boldsymbol{B}$ is a retract of $\boldsymbol{A}$. As $\mathrm{id}_B\colon \boldsymbol{B} \to \boldsymbol{B}$ and $h^{-1} \circ g\colon \boldsymbol{B} \to \boldsymbol{Z}_n$ are homomorphisms, so is the function $k\colon \boldsymbol{B} \to \boldsymbol{B} \times \boldsymbol{Z}_n$ defined by $x \mapsto \langle x, h^{-1}g(x)\rangle$, and $k[B] \subseteq A$, by (5.7). Obviously, $\pi \circ k = \mathrm{id}_B$, so $\pi|_A$ is the desired retraction.

We may therefore assume, for the remainder of the proof, that $\boldsymbol{C} = \boldsymbol{C}_4$.

First, let $i \in \{1, \dots, n-1\}$. By (5.6), there are homomorphisms $r\colon \boldsymbol{Z}_i \to \boldsymbol{B}$ and $s\colon \boldsymbol{B} \to \boldsymbol{Z}_i$ (and hence $s \circ \pi|_A\colon \boldsymbol{A} \to \boldsymbol{Z}_i$) with $s \circ r = \mathrm{id}_{Z_i}$. Because $g \circ r\colon \boldsymbol{Z}_i \to \boldsymbol{C}_4 \in \mathbb{S}(\boldsymbol{Z}_n)$ is a homomorphism, so is the map $p\colon \boldsymbol{Z}_i \to \boldsymbol{B} \times \boldsymbol{Z}_n$ defined by $x \mapsto \langle r(x), gr(x)\rangle$. Now $h|_{C_4}$ is an endomorphism of $\boldsymbol{C}_4$, which can only be id_{C_4}, so $gr(x) = hgr(x)$ for all $x \in \boldsymbol{Z}_i$, whence $p[Z_i] \subseteq A$, by (5.7). Clearly, $s \circ \pi|_A \circ p = \mathrm{id}_{Z_i}$, so $\boldsymbol{Z}_i$ is a retract of $\boldsymbol{A}$.

It remains to show that $\boldsymbol{Z}_n$ is a retract of $\boldsymbol{A}$. As $h\colon \boldsymbol{Z}_n \to \boldsymbol{C}_4 \in \mathbb{S}(\boldsymbol{B})$ is a homomorphism, so is the function $t\colon \boldsymbol{Z}_n \to \boldsymbol{B} \times \boldsymbol{Z}_n$ given by $x \mapsto \langle h(x), x\rangle$. Since the endomorphism $g|_{C_4}$ of $\boldsymbol{C}_4$ is the identity map, we have $gh(x) = h(x)$ for all $x \in Z_n$. Therefore, $t[Z_n] \subseteq A$, by (5.7), while $\pi_n|_A \circ t = \mathrm{id}_{Z_n}$. □

A member of a variety K is said to be *projective* in K if it is a retract of each of its homomorphic pre-images in K. It follows that a finitely generated member of K is projective in K iff it is a retract of its finitely generated homomorphic pre-images in K (see for instance [112, Lem. 8.2]).

Theorem 5.42. *In the variety $\mathbb{V}(\boldsymbol{G}_1, \dots, \boldsymbol{G}_6)$, every finite subdirectly irreducible algebra $\boldsymbol{E}$ is projective.*

Proof. Let $\boldsymbol{A} \in \mathsf{J} := \mathbb{V}(\boldsymbol{G}_1, \dots, \boldsymbol{G}_6)$ be a finitely generated homomorphic pre-image of $\boldsymbol{E}$. Then $\boldsymbol{A}$ is finite (as J is locally finite) and nontrivial. Also, $\mathsf{J}_{\text{SI}} = \mathbb{I}(\boldsymbol{G}_1, \dots, \boldsymbol{G}_6, \boldsymbol{C}_4)$, by (5.5), and there are only finitely many maps from $\boldsymbol{A}$ to members of $\{\boldsymbol{G}_1, \dots, \boldsymbol{G}_6, \boldsymbol{C}_4\}$. Therefore, by Birkhoff's Subdirect Decomposition Theorem 1.3, there exist an integer $n > 0$ and (not necessarily distinct) algebras $\boldsymbol{Z}_1, \dots, \boldsymbol{Z}_n \in \{\boldsymbol{G}_1, \dots, \boldsymbol{G}_6, \boldsymbol{C}_4\}$ such that $\boldsymbol{A}$ embeds subdirectly into $\prod_{i=1}^{n} \boldsymbol{Z}_i$. By Lemma 5.41, $\boldsymbol{Z}_1, \dots, \boldsymbol{Z}_n$ are retracts of $\boldsymbol{A}$. Now $\boldsymbol{E}$ is an SI member of $\mathbb{H}\mathbb{P}_{\mathbb{S}}(\boldsymbol{Z}_1, \dots, \boldsymbol{Z}_n) \subseteq \mathbb{V}(\boldsymbol{Z}_1, \dots, \boldsymbol{Z}_n)$, so $\boldsymbol{E} \in \mathbb{I}(\boldsymbol{Z}_1, \dots, \boldsymbol{Z}_n, \boldsymbol{C}_4)$, by (5.5). As $\boldsymbol{C}_4$ is a retract of each nontrivial member of M, this show that $\boldsymbol{E}$ is a retract of $\boldsymbol{A}$, and hence that $\boldsymbol{E}$ is projective in J. □

Theorem 5.43. *Every subquasivariety of* $\mathbb{V}(\boldsymbol{G}_1, \dots, \boldsymbol{G}_6)$ *is a variety, i.e.,* $\mathbb{V}(\boldsymbol{G}_1, \dots, \boldsymbol{G}_6)$ *is HSC.*

Proof. Since $\mathbb{V}(\boldsymbol{G}_1, \dots, \boldsymbol{G}_6)$ is locally finite (by Theorem 1.20), it is enough to show, by Theorem 5.22, that every finite SI member of $\mathbb{V}(\boldsymbol{G}_1, \dots, \boldsymbol{G}_6)$ embeds into each of its homomorphic pre-images. But this follows immediately from Theorem 5.42. □

Theorem 5.43 shows that not only is each $\mathbb{V}(\boldsymbol{G}_i)$ $(i = 1, \dots, 6)$ a cover of $\mathbb{V}(\boldsymbol{C}_4)$ in the sub*variety* lattice of DMM (Theorem 4.23), but each is a cover of $\mathbb{V}(\boldsymbol{C}_4)$ in the sub*quasivariety* lattice of DMM.

Chapter 6

Surjectivity of epimorphisms

The 'bridge theorems' of abstract algebraic logic include connections between the so-called 'definability properties' of a logic and the surjectivity of suitable epimorphisms between its models. In this chapter, after recalling the precise definitions and connections, we explore the prevalence of these conditions in varieties of residuated structures—and particularly in varieties of De Morgan monoids.

Given a class K of similar algebras, a K-*morphism* is a homomorphism $f\colon \boldsymbol{A} \to \boldsymbol{B}$, where $\boldsymbol{A}, \boldsymbol{B} \in \mathsf{K}$. It is called a K-*epimorphism* provided that, whenever $g, h\colon \boldsymbol{B} \to \boldsymbol{C}$ are K-morphisms with $g \circ f = h \circ f$, then $g = h$.

$$\boldsymbol{A} \xrightarrow{f} \boldsymbol{B} \underset{h}{\overset{g}{\rightrightarrows}} \boldsymbol{C}$$

Clearly, surjective K-morphisms are K-epimorphisms.

The converse need not hold. Indeed, rings and distributive lattices each form varieties in which non-surjective epimorphisms arise. As it happens, this reflects the absence of unary terms defining multiplicative inverses in rings, and complements in distributive lattices, despite the uniqueness of those entities when they exist. Such constructs are said to be *implicitly* (and not *explicitly*) *definable*.

Definition 6.1. A class K of algebras has the *epimorphism-surjectivity (ES) property* if all K-epimorphisms are surjective. A K-morphism $f\colon \boldsymbol{A} \to \boldsymbol{B}$ is said to be *almost onto* if there is a *finite* subset $C \subseteq B$ such that $f[A] \cup C$ generates $\boldsymbol{B}$. The class K is said to have the *weak ES property* if all almost onto K-epimorphisms are surjective.

The notion of a set $\boldsymbol{\rho}(x, y)$ of *equivalence formulas* for a logic was defined on page 20. Recall that, in the substructural logics under discussion here, we can always take $\{x \to y, y \to x\}$ (or $\{x \leftrightarrow y\}$) for $\boldsymbol{\rho}$. Also recall that, in this thesis, all deductive systems are formulated over a fixed denumerable set *Var* of variables.

Definition 6.2. Let $\vdash$ be a deductive system with a set $\boldsymbol{\rho}$ of equivalence formulas. We shall say that $\vdash$ has the *infinite (deductive) Beth (definability) property (w.r.t. Var)*[1] provided that the following is true for all disjoint sets X and Z of variables, and all sets Γ of formulas over $X \cup Z$, such that $T(X) \neq \emptyset$ [2] and $Var \setminus (X \cup Z)$ is infinite: if

> for each $z \in Z$ and each substitution h such that $h(x) = x$ for all $x \in X$, we have $\Gamma \cup h[\Gamma] \vdash \boldsymbol{\rho}(z, h(z))$,

then, for each $z \in Z$, there exists a formula φ_z over X such that $\Gamma \vdash \boldsymbol{\rho}(z, \varphi_z)$.

The displayed assumption is pronounced as 'Γ *defines* Z *implicitly in terms of* X in $\vdash$'. The term φ_z in the conclusion is called an *explicit definition* of z in terms of X, with respect to Γ, in $\vdash$.

Definition 6.3. If, in Definition 6.2, we ask only that the implicit definability of Z (by Γ in terms of X) entails the explicit definability of Z (by Γ in terms of X) for *finite* sets Z of variables (still assuming the other conditions on Γ, X and Z in Definition 6.2), then $\vdash$ is said to have the *finite Beth property*.

The assumption '$Var \setminus (X \cup Z)$ is infinite' is not normally included in definitions of the Beth properties. It is required for the bridge theorem below, which depends crucially on the denumerability of *Var* and the countability of $\vdash$'s signature [105]. In lieu of these assumptions, one may formulate more general versions of the Beth properties with a similar bridge theorem, but only under the awkward assumption that $\vdash$ be formulated with a *proper class* of variables [13]. The definitions agree in the restricted setting of Theorem 6.4.

Theorem 6.4 ([105, Cor. 7.8, Thm. 7.9])**.** *Let* $\vdash$ *be a finitary logic with a countable signature and* K *a quasivariety that algebraizes* $\vdash$*. Then*

(i) $\vdash$ *has the infinite Beth property iff* K *has the ES property, and*

(ii) $\vdash$ *has the finite Beth property iff* K *has the weak ES property.*

[1] The allusion is to E.W. Beth's theorem concerning classical predicate logic in [9].
[2] This means that $X \neq \emptyset$ or the signature includes a constant symbol.

In this situation, if K is a variety, then its subvarieties algebraize the axiomatic extensions of $\vdash$, but the ES property need not persist in subvarieties. For example, the variety of all lattices has the ES property, while its subvariety comprising the distributive lattices does not. It is therefore a well-motivated (but often nontrivial) task to determine which subvarieties of K have surjective epimorphisms. The present chapter addresses this question in the context of (possibly involutive) square-increasing residuated lattices (S[I]RLs), i.e., the algebraic models of the substructural logics $\mathbf{FL}_{ec}$ and $\mathbf{FL}_{ec}^{+}$. (The arguments in Section 6.5 do not require distributivity and apply regardless of the presence or absence of an involution, so De Morgan monoids are special cases.)

Recall that Brouwerian algebras are integral SRLs (Definition 2.32). The next theorem states two classical ES results for varieties of Brouwerian algebras; the first was proved by Maksimova [88] (also see [45, 49, 87]), and the second (in effect) by Kreisel [78].

Theorem 6.5.

(i) *The variety* BRA *of Brouwerian algebras has the ES property.*

(ii) *Every subvariety of* BRA *has the weak ES property.*

(In Chapter 7, we shall investigate varieties of Brouwerian algebras in which the ES property fails.)

Recall from Section 2.4 that, given an S[I]RL $\boldsymbol{A}$, its set of negative elements is denoted by $A^- = \{a \in A : a \leqslant e\}$. The set A^- can be given the structure of a Brouwerian algebra $\boldsymbol{A}^-$ (see Section 6.2), called the *negative cone* of $\boldsymbol{A}$. This allows us to ascribe to $\boldsymbol{A}$ a 'depth', which is either a non-negative integer or ∞ (see Definition 6.20).

One can construct a functor from a variety of S[I]RLs to a variety of Brouwerian algebras that sends S[I]RLs to their negative cones (and restricts morphisms accordingly). Such a functor is not a category equivalence, except in quite special cases; see [48, 53, 55, 56]. Partly for these reasons, we cannot systematically reduce ES problems for arbitrary varieties of S[I]RLs to an examination of negative cones.

Nevertheless, $\boldsymbol{A}^-$ contains enough information about $\boldsymbol{A}$ to facilitate the main results of this chapter, which provide sufficient conditions for the surjectivity of epimorphisms in varieties of S[I]RLs whose FSI members are negatively generated.

The main result of Section 6.5, namely Theorem 6.22, states that, in a variety K of S[I]RLs (e.g. De Morgan monoids or Dunn monoids), epimorphisms will be surjective if each finitely subdirectly irreducible member of K is negatively generated and has finite depth. Theorem 6.22 is a generalization of a recent result, in [11], that epimorphisms are surjective in varieties of Brouwerian algebras with finite depth. The assumptions of Theorem 6.22 persist in subvarieties and under varietal joins, so the result is labour-saving.

Section 6.6 culminates in the result that every variety of negatively generated *semilinear* De Morgan or Dunn monoids has surjective epimorphisms (even those that contain algebras with infinite depth); see Corollary 6.46. This result strengthens earlier findings that established the ES property for all varieties of relative Stone algebras and Sugihara monoids [11, Thms. 5.7, 8.5], and the weak ES property for every variety K of generalized Sugihara monoids [56, Thm. 13.1].

Before proving the main results of this chapter (in Sections 6.5 and 6.6), it is convenient to collect some tools (in Sections 6.1–6.4) that will be used throughout this and the next chapter. These include a categorical duality (due to Esakia) between the variety of Brouwerian algebras and a certain class of ordered topological spaces, which is recounted briefly in Section 6.4.

6.1 Epic subalgebras

Consider a class K of similar algebras. A subalgebra $\boldsymbol{A}$ of an algebra $\boldsymbol{B} \in \mathsf{K}$ is said to be K-*epic* (in $\boldsymbol{B}$) when, for every pair of morphisms $g, h \colon \boldsymbol{B} \to \boldsymbol{C}$ in K,

$$\text{if } g|_A = h|_A, \text{ then } g = h.$$

Lemma 6.6. *Let* K *be a class of algebras such that* $\mathbb{S}(\mathsf{K}) \subseteq \mathsf{K}$*. Then* K *has the ES property if and only if all its members lack proper* K*-epic subalgebras.*

Proof. Observe that if there is a non-surjective epimorphism $f \colon \boldsymbol{A} \to \boldsymbol{B}$ in K, then $f[\boldsymbol{A}]$ is a proper K-epic subalgebra of $\boldsymbol{B}$. Conversely, if $\boldsymbol{A}$ is a proper K-epic subalgebra of $\boldsymbol{B}$, then the inclusion map $\boldsymbol{A} \hookrightarrow \boldsymbol{B}$ is a non-surjective epimorphism in K. □

Theorem 6.7 ([105, Thm. 5.1])**.** *A quasivariety* K *has the weak ES property iff no finitely generated member of* K *has a proper* K*-epic subalgebra.*

Recall that if $\boldsymbol{A}$ is a subalgebra of an algebra $\boldsymbol{B}$, and $\mu \in \mathrm{Con}(\boldsymbol{B})$, then the relation $\mu|_A := A^2 \cap \mu$ is a congruence of $\boldsymbol{A}$.

Lemma 6.8. *Let* K *be a variety of algebras, and* $\boldsymbol{A}$ *a* K*-epic subalgebra of* $\boldsymbol{B} \in \mathsf{K}$*. Then, for any* $\mu \in \mathrm{Con}(\boldsymbol{B})$*, the map* $f \colon \boldsymbol{A}/(\mu|_A) \to \boldsymbol{B}/\mu$ *defined by* $a/(\mu|_A) \mapsto a/\mu$ *is an injective* K*-epimorphism.*

Proof. As K is a variety, $\boldsymbol{B}/\mu$, $\boldsymbol{A}/(\mu|_A) \in \mathsf{K}$. Let $i\colon \boldsymbol{A} \to \boldsymbol{B}$ be the inclusion homomorphism and $q\colon \boldsymbol{B} \to \boldsymbol{B}/\mu$ the surjective homomorphism $b \mapsto b/\mu$. Clearly, $\mu|_A$ is the kernel of the homomorphism $q \circ i\colon \boldsymbol{A} \to \boldsymbol{B}/\mu$, so f is an injective K-morphism.

Suppose $g, h\colon \boldsymbol{B}/\mu \to \boldsymbol{C} \in \mathsf{K}$ are homomorphisms with $g \circ f = h \circ f$. Then $g \circ q$ and $h \circ q$ are homomorphisms from $\boldsymbol{B}$ to $\boldsymbol{C}$. For each $a \in A$,

$$g(q(a)) = g(a/\mu) = g(f(a/(\mu|_A))) = h(f(a/(\mu|_A))) = h(q(a)),$$

i.e., $g \circ q \circ i = h \circ q \circ i$. Therefore, $g \circ q = h \circ q$, as i is a K-epimorphism. Since q is surjective, it follows that $g = h$, as required. ☐

The following theorem will allow us to focus on FSI algebras when investigating the ES property.

Theorem 6.9 (Campercholi [25, Thm. 6.8])**.** *If a congruence permutable variety K with EDPM lacks the ES property, then some FSI member of K has a K-epic proper subalgebra.*

Recall from Section 2.3 that every variety of [I]RLs is congruence permutable and has EDPM. Therefore, Theorem 6.9 applies to all such varieties, and we shall make extensive use of it in this chapter.

Lemma 6.10. *Let K be a variety of algebras and let $\boldsymbol{B}$ be a subalgebra of $\boldsymbol{A} \in \mathsf{K}$. Then $\boldsymbol{B}$ is K-epic in $\boldsymbol{A}$ iff, whenever $\boldsymbol{C} \in \mathsf{K}_{\mathrm{SI}}$ and $g, h\colon \boldsymbol{A} \to \boldsymbol{C}$ are homomorphisms that agree on B, then $g = h$.*

Proof. The forward implication is clear. Conversely, suppose that $\boldsymbol{C}$ is an arbitrary member of K and that $g, h\colon \boldsymbol{A} \to \boldsymbol{C}$ are homomorphisms that agree on B. By Birkhoff's Subdirect Decomposition Theorem 1.3, there is a subdirect embedding $i\colon \boldsymbol{C} \to \prod_{j \in J} \boldsymbol{C}_j$, where $\boldsymbol{C}_j \in \mathsf{K}_{\mathrm{SI}}$ for every $j \in J$. Let $\pi_j\colon \prod_{k \in J} \boldsymbol{C}_k \to \boldsymbol{C}_j$ denote the j-th projection map, for every $j \in J$. If $g(a) \neq h(a)$ for some $a \in \boldsymbol{A}$, then $i(g(a)) \neq i(h(a))$, because i is an embedding. So, there exists $j \in J$ such that $\pi_j(i(g(a))) \neq \pi_j(i(h(a)))$. Then $\pi_j \circ i \circ g$ and $\pi_j \circ i \circ h$ are two different homomorphisms into $\boldsymbol{C}_j$ that agree on B, a contradiction. So, $g = h$. ☐

6.2 Negative cones

Definition 6.11. The *negative cone* of an S[I]RL $\boldsymbol{A} = \langle A; \wedge, \vee, \cdot, \to, e\, [, \neg] \rangle$ is the Brouwerian algebra

$$\boldsymbol{A}^- = \big\langle A^-;\ \wedge|_{(A^-)^2},\ \vee|_{(A^-)^2},\ \to^-,\ e \big\rangle,$$

where $a \to^- b = (a \to b) \wedge e$ for all $a, b \in A^-$.

When $\boldsymbol{A}$ is a Brouwerian algebra, we refer to its deductive filters just as 'filters', because they coincide with the filters of $\langle A; \wedge, \vee\rangle$.

Lemma 6.12. *Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be S[I]RLs, and F a deductive filter of $\boldsymbol{A}$.*

(i) *If $h\colon \boldsymbol{A} \to \boldsymbol{B}$ is a homomorphism, then $h|_{A^-}$ is a homomorphism from $\boldsymbol{A}^-$ to $\boldsymbol{B}^-$. If, moreover, h is surjective, then so is $h|_{A^-}$.*

(ii) *$A^- \cap F$ is a filter of $\boldsymbol{A}^-$, and $(\boldsymbol{A}/F)^- \cong \boldsymbol{A}^-/(A^- \cap F)$, the isomorphism and its inverse being*

$$a/F \mapsto (a \wedge e)/(A^- \cap F) \text{ and } a/(A^- \cap F) \mapsto a/F.$$

Proof. (i) Since homomorphisms between S[I]RLs are isotone maps that preserve e, we have $h[A^-] \subseteq B^-$. Also, as h is a homomorphism, it is clear from the definitions of the operations on the negative cone that $h|_{A^-}$ is a homomorphism from $\boldsymbol{A}^-$ to $\boldsymbol{B}^-$. Now suppose h is onto. For each $b \in B^-$, there exists $a \in A$ with $h(a) = b$, and since $b \leqslant e$, we have $b = h(a) \wedge e = h(a \wedge e)$. As $a \wedge e \in A^-$, this shows that $B^- = h[A^-]$.

(ii) Clearly, $A^- \cap F$ is a filter of $\boldsymbol{A}^-$. Let $q\colon \boldsymbol{A} \to \boldsymbol{A}/F$ be the canonical surjection. By (i), $q|_{A^-}\colon \boldsymbol{A}^- \to (\boldsymbol{A}/F)^-$ is a surjective homomorphism. For all $a, b \in A^-$, we have $a \leftrightarrow b \in F$ iff $(a \leftrightarrow b) \wedge e \in A^- \cap F$ (since F is upward closed and contains e), i.e., the kernel of $q|_{A^-}$ is $\boldsymbol{\Omega}^{\boldsymbol{A}^-}(A^- \cap F)$; see page 33. Thus, by the Homomorphism Theorem 1.1, $a/(A^- \cap F) \mapsto a/F$ defines an isomorphism from $\boldsymbol{A}^-/(A^- \cap F)$ onto $(\boldsymbol{A}/F)^-$. For any $a \in A$, if $a/F \in (A/F)^-$, then $a \wedge e \in A^-$ and $(a \wedge e)/F = (a/F) \wedge (e/F) = a/F$, so $a/F \mapsto (a \wedge e)/(A^- \cap F)$ defines the inverse of the above isomorphism. $\square$

Given an S[I]RL $\boldsymbol{A}$, if F is a filter of $\boldsymbol{A}^-$, then (by (2.27) on page 33)

$$\mathrm{Fg}^{\boldsymbol{A}} F = \{a \in A\colon a \geqslant b \text{ for some } b \in F\}, \text{ so } A^- \cap \mathrm{Fg}^{\boldsymbol{A}} F = F. \quad (6.1)$$

On the other hand, for any deductive filter G of an S[I]RL $\boldsymbol{A}$, we have $G = \mathrm{Fg}^{\boldsymbol{A}}(G \cap A^-)$, by (2.30) on page 40. So, the deductive filters of an S[I]RL $\boldsymbol{A}$ are completely determined by the filters of $\boldsymbol{A}^-$. In fact, $\mathbf{Fil}(\boldsymbol{A}) \cong \mathbf{Fil}(\boldsymbol{A}^-)$ via the map $G \mapsto G \cap A^-$, whose inverse is $F \mapsto \mathrm{Fg}^{\boldsymbol{A}}(F)$.

As we mentioned in the introduction to this chapter, the 'negative cone functor' that sends an algebra $\boldsymbol{A}$ from a variety K of S[I]RLs to $\boldsymbol{A}^-$ and a homomorphism $h\colon \boldsymbol{A} \to \boldsymbol{B}$ between algebras in K to $h|_{A^-}$, is not normally a category equivalence. One exception (proved in [55, Thm. 5.8]) is that the variety OSM of odd Sugihara monoids is categorically equivalent, via the negative cone functor, to the variety RSA of relative Stone algebras (i.e., the variety of semilinear Brouwerian algebras). An analogous but more complex result for arbitrary Sugihara monoids is proved in [56, Thm. 10.5] and refined in [48, Thm. 2.24].

6.3 Prerequisites for Esakia duality

Although there is no advance guarantee that ES problems involving (possibly non-integral) S[I]RLs can be reduced to an examination of (Brouwerian) negative cones, the availability of Esakia duality for Brouwerian algebras is an incentive, because it allows one to work with simpler structures. We therefore give a brief account of Esakia duality over the next two sections. (Relational duals for non-integral S[I]RLs also exist [142], but they are rather complicated.)

The following notation departs from the conventions of Chapter 2, but is standard in duality theory. In an indicated poset $\langle X; \leqslant \rangle$, we define

$$\uparrow x := \{y \in X : x \leqslant y\} \ (= [x)) \quad \text{and} \quad \uparrow U := \bigcup\nolimits_{u \in U} \uparrow u,$$

for $U \cup \{x\} \subseteq X$, and if $U = \uparrow U$, we call U an *upset* of $\langle X; \leqslant \rangle$. We define $\downarrow x$, $\downarrow U$ and *downset* dually. For $x, y \in X$, the notation $x \prec y$ signifies that y covers x.

If $Y \subseteq X$ and $x \in X$, we sometimes need to refer to $Y \cap \uparrow x$, which we then denote as $\uparrow^Y x$, even if $x \notin Y$.

A filter F of a lattice $\langle L; \wedge, \vee \rangle$ is said to be *prime* if its complement $L \setminus F$ is closed under the binary operation $\vee$. It follows that, for any filters F, G, H of $\langle L; \wedge, \vee \rangle$,

$$\text{if } F \cap G \subseteq H \text{ and } H \text{ is prime, then } F \subseteq H \text{ or } G \subseteq H. \qquad (6.2)$$

Prime Filter Extension Theorem 6.13 ([4, Thm. III.6.5]). *Let $\langle K; \wedge, \vee \rangle$ be a sublattice of a distributive lattice $\langle L; \wedge, \vee \rangle$. Then the prime filters of $\langle K; \wedge, \vee \rangle$ are exactly the non-empty sets $K \cap F$ such that F is a prime filter of $\langle L; \wedge, \vee \rangle$.*

Prime Filter Lemma 6.14 ([7, Thm. 4.1]). *Let $\langle A; \wedge, \vee \rangle$ be a distributive lattice and let $a, b \in A$. If $a \nleqslant b$, then there is a prime filter F of $\langle A; \wedge, \vee \rangle$ such that $a \in F$ and $b \notin F$.*

We use $\mathrm{Pr}(\boldsymbol{A})$ to denote the set of all prime deductive filters of an S[I]RL $\boldsymbol{A}$, *including* A itself. We always consider $\mathrm{Pr}(\boldsymbol{A})$ to be partially ordered by set inclusion. For a deductive filter F of $\boldsymbol{A}$, we write

$$\uparrow^{\boldsymbol{A}} F = \{H \in \mathrm{Pr}(\boldsymbol{A}) : F \subseteq H\},$$

i.e., $\uparrow^{\boldsymbol{A}} F$ abbreviates $\uparrow^{\mathrm{Pr}(\boldsymbol{A})} F$, within $\langle \mathrm{Fil}(\boldsymbol{A}); \subseteq \rangle$.

Remark 6.15. Suppose $h\colon \boldsymbol{A} \to \boldsymbol{B}$ is a surjective homomorphism of S[I]RLs. Recall (from page 33) that the kernel of h is $\boldsymbol{\Omega}^{\boldsymbol{A}}K$ for some deductive filter K of $\boldsymbol{A}$. If G is a deductive filter of $\boldsymbol{A}$, with $K \subseteq G$, then $h[G] := \{h(g) : g \in G\}$ is a deductive filter of $\boldsymbol{B}$, and by the Correspondence Theorem 1.12,

$$H \mapsto h^{-1}[H] := \{a \in A : h(a) \in H\}$$

is a lattice isomorphism from the deductive filter lattice of $\boldsymbol{B}$ onto the lattice of deductive filters of $\boldsymbol{A}$ that contain K; the inverse isomorphism is given by $G \mapsto h[G]$. In particular,

$$h^{-1}[h[G]] = G \text{ for all deductive filters } G \text{ of } \boldsymbol{A} \text{ such that } K \subseteq G. \tag{6.3}$$

Clearly, a deductive filter H of $\boldsymbol{B}$ is prime iff the filter $h^{-1}[H]$ of $\boldsymbol{A}$ is prime, so $H \mapsto h^{-1}[H]$ also defines an isomorphism of partially ordered sets from $\mathrm{Pr}(\boldsymbol{B})$ onto $\uparrow^{\boldsymbol{A}} K$ (both ordered by inclusion).

6.4 Esakia duality

A well known duality between Heyting algebras and Esakia spaces was established by Esakia in [43, 44]. It entails an analogous duality between the variety BRA of Brouwerian algebras (considered as a concrete category, equipped with all algebraic homomorphisms) and the category PESP of 'pointed Esakia spaces' defined below, i.e., there is a category equivalence between BRA and the opposite category of PESP. This is explained, for instance, in [11, Sec. 3], but we recall the key definitions here.

A structure $\boldsymbol{X} = \langle X; \tau, \leqslant, m\rangle$ is a *pointed Esakia space* if $\langle X; \leqslant\rangle$ is a partially ordered set with a greatest element m, and $\langle X; \tau\rangle$ is a compact Hausdorff space in which

(i) every open set is a union of clopen (i.e., closed and open) sets;

(ii) $\uparrow x$ is closed, for all $x \in X$;

(iii) $\downarrow V$ is clopen, for all clopen $V \subseteq X$.

In this case, the *Priestley separation axiom* of [115] holds: for all $x, y \in X$,

(iv) if $x \nleqslant y$, then $x \in U$ and $y \notin U$, for some clopen upset $U \subseteq X$.

The morphisms of PESP are the so-called *Esakia morphisms* between these objects. They are the isotone continuous functions $g\colon \boldsymbol{X} \to \boldsymbol{Y}$ such that,

$$\text{whenever } x \in X \text{ and } g(x) \leqslant y \in Y, \text{ then } y = g(z) \text{ for some } z \in {\uparrow} x. \quad (6.4)$$

It follows that $g(m) = m$, and if g is bijective then $g^{-1}\colon \boldsymbol{Y} \to \boldsymbol{X}$ is also an Esakia morphism, so g is a (categorical) isomorphism.[3]

Given $\boldsymbol{A} \in \mathsf{BRA}$ and $a \in A$, let $\gamma^{\boldsymbol{A}}(a)$ denote $\{F \in \mathrm{Pr}(\boldsymbol{A}) : a \in F\}$ and $\gamma^{\boldsymbol{A}}(a)^c$ its complement $\{F \in \mathrm{Pr}(\boldsymbol{A}) : a \notin F\}$. The *dual* (in PESP) of $\boldsymbol{A}$ is $\boldsymbol{A}_* = \langle \mathrm{Pr}(\boldsymbol{A}); \tau, \subseteq, A\rangle$, where τ is the topology on $\mathrm{Pr}(\boldsymbol{A})$ with sub-basis

$$\{\gamma^{\boldsymbol{A}}(a) : a \in A\} \cup \{\gamma^{\boldsymbol{A}}(a)^c : a \in A\}.$$

The *dual* of a morphism $h\colon \boldsymbol{A} \to \boldsymbol{B}$ in BRA is the Esakia morphism $h_*\colon \boldsymbol{B}_* \to \boldsymbol{A}_*$, defined by $F \mapsto h^{-1}[F]$. Thus, the contravariant functor $(-)_*\colon \mathsf{BRA} \to \mathsf{PESP}$ in the duality is given by $\boldsymbol{A} \mapsto \boldsymbol{A}_*$; $h \mapsto h_*$.

The contravariant reverse functor $(-)^*\colon \mathsf{PESP} \to \mathsf{BRA}$ works as follows. Let $g\colon \boldsymbol{X} \to \boldsymbol{Y}$ be an Esakia morphism, where $\boldsymbol{X}, \boldsymbol{Y} \in \mathsf{PESP}$. Then

$$\boldsymbol{X}^* = \langle \mathrm{Cpu}(\boldsymbol{X}); \cap, \cup, \to, X\rangle \in \mathsf{BRA},$$

where $\mathrm{Cpu}(\boldsymbol{X})$ is the set of all *non-empty* clopen upsets of $\boldsymbol{X}$, and

$$U \to V \;:=\; X \setminus {\downarrow}(U \setminus V)$$

for all $U, V \in \mathrm{Cpu}(\boldsymbol{X})$, while the homomorphism $g^*\colon \boldsymbol{Y}^* \to \boldsymbol{X}^*$ is given by $U \mapsto g^{-1}[U]$. We refer to $\boldsymbol{X}^*$ [resp. g^*] as the *dual* of $\boldsymbol{X}$ [resp. g] in BRA.

For $\boldsymbol{A} \in \mathsf{BRA}$ and $\boldsymbol{X} \in \mathsf{PESP}$, the respective canonical isomorphisms from $\boldsymbol{A}$ to $\boldsymbol{A}_*{}^*$ and from $\boldsymbol{X}$ to $\boldsymbol{X}^*{}_*$ are given by $a \mapsto \gamma^{\boldsymbol{A}}(a)$ and

$$x \mapsto \{U \in \mathrm{Cpu}(\boldsymbol{X}) : x \in U\}.$$

Given a variety K of Brouwerian algebras, we denote by K_* the full subcategory of PESP whose class of objects is the isomorphic closure of $\{\boldsymbol{A}_*\colon \boldsymbol{A} \in \mathsf{K}\}$. It is clear that the functors $(-)_*$ and $(-)^*$ restrict to a dual equivalence between K and K_*.

In PESP, there is a notion of substructure: an *E-subspace* of $\boldsymbol{X} \in \mathsf{PESP}$ is a non-empty closed upset U of $\boldsymbol{X}$. It is the universe of a pointed Esakia space $\boldsymbol{U}$, with the restricted order and the subspace topology, so the inclusion $\boldsymbol{U} \to \boldsymbol{X}$ is an Esakia morphism.

A *correct partition* on $\boldsymbol{X}$ (sometimes called an *Esakia relation* or *bisimulation equivalence* in the literature) is an equivalence relation R on X such that for every $x, y, z \in X$,

[3]We use here the fact that a continuous bijection from a compact space to a Hausdorff space always has a continuous inverse function.

(i) if $\langle x, y\rangle \in R$ and $x \leqslant z$, then $\langle z, w\rangle \in R$ for some $w \geqslant y$, and

(ii) if $\langle x, y\rangle \notin R$, then there is a clopen U, such that $x \in U$ and $y \notin U$, which moreover is a union of equivalence classes of R.

In this case, we denote by $\boldsymbol{X}/R$ the pointed Esakia space consisting of the quotient space of $\boldsymbol{X}$ with respect to R,[4] equipped with the partial order $\leqslant^{\boldsymbol{X}/R}$ defined as follows: for every $x, y \in X$,

$$x/R \leqslant^{\boldsymbol{X}/R} y/R \iff \text{there are } x', y' \in X \text{ such that } \langle x, x'\rangle, \langle y, y'\rangle \in R \text{ and } x' \leqslant^{\boldsymbol{X}} y'.$$

Notice that m/R is the greatest element of $\boldsymbol{X}/R$. The map $q\colon x \mapsto x/R$ is an Esakia morphism from $\boldsymbol{X}$ to $\boldsymbol{X}/R$, and for every Esakia morphism $f\colon \boldsymbol{X} \to \boldsymbol{Y}$, the kernel of f is a correct partition on $\boldsymbol{X}$. If, moreover, f is surjective, then there is a PESP-isomorphism $i\colon \boldsymbol{X}/\ker f \cong \boldsymbol{Y}$, such that $i \circ q = f$.

Lemma 6.16. *Let $\boldsymbol{A} \in$ BRA, and $\boldsymbol{X} \in$ PESP.*

(i) *$\boldsymbol{A}$ is FSI if and only if the poset underlying $\boldsymbol{A}_*$ is* rooted *(i.e., has a least element).*

(ii) *A homomorphism h between Brouwerian algebras is injective iff h_* is surjective. Also, h is surjective iff h_* is injective.*

(iii) *The image of an Esakia morphism is an E-subspace of the co-domain, and the restriction of an Esakia morphism to an E-subspace is still an Esakia morphism.*

(iv) *Let R be a correct partition on $\boldsymbol{X}$. If $\boldsymbol{Y}$ is an E-subspace of $\boldsymbol{X}$, then $R \cap Y^2$ is a correct partition on $\boldsymbol{Y}$.*

(v) *Every non-empty chain in $\boldsymbol{X}$ has an infimum and a supremum. Moreover, infima and suprema of chains are preserved by Esakia morphisms.*

Proof. The statements of (i)–(v) are essentially contained in [44]. See [11, Lem. 3.4] for brief English explanations of (i), (ii) and the first part of (iii). The right-to-left implication of the first assertion of (ii) relies on the fact that BRA has the ES property (Theorem 6.5(i)). The forward implication

[4] This means that $A \subseteq X/R$ is open in $\boldsymbol{X}/R$ iff $q^{-1}[A]$ is open in $\boldsymbol{X}$, where $q : X \to X/R$ is the canonical surjection.

in the second assertion of (ii) employs the Prime Filter Extension Theorem 6.13. The second part of (iii) follows from the fact that the inclusion map from an E-subspace to its parent space is an Esakia morphism.

To prove (iv), notice that $R \cap Y^2$ is the kernel of the Esakia morphism obtained by composing the canonical surjection from $\boldsymbol{X}$ to $\boldsymbol{X}/R$ with the inclusion from $\boldsymbol{Y}$ to $\boldsymbol{X}$.

Finally, (v) is a consequence of the duality, together with the observation that unions and intersections of chains of prime filters are still prime filters (and that these unions and intersections are preserved by inverse images of homomorphisms). □

Note that, owing to (ii), if K is a variety of Brouwerian algebras then K_* is closed under taking E-subspaces and quotients by correct partitions.

In the absence of a convenient reference, a proof of the next lemma is supplied below; the result is presumably well known.

Lemma 6.17. *Let F be a filter of a Brouwerian algebra $\boldsymbol{A}$, and $q\colon \boldsymbol{A} \to \boldsymbol{A}/F$ the canonical surjection. Then q_* is an isomorphism from $(\boldsymbol{A}/F)_*$ onto the E-subspace $q_*[(\boldsymbol{A}/F)_*]$ of $\boldsymbol{A}_*$ whose universe is $\uparrow^{\boldsymbol{A}} F$. Also, the map*

$$\gamma_F^{\boldsymbol{A}}\colon a/F \mapsto \{H \in \mathrm{Pr}(\boldsymbol{A}) : F \cup \{a\} \subseteq H\}$$

is an isomorphism from $\boldsymbol{A}/F$ onto $(q_*[(\boldsymbol{A}/F)_*])^*$ and the following diagram commutes, where $i_1\colon q_*[(\boldsymbol{A}/F)_*] \to \boldsymbol{A}_*$ is the inclusion map.

$$\begin{array}{ccc} \boldsymbol{A} & \xrightarrow{\quad q \quad} & \boldsymbol{A}/F \\ \wr\big\downarrow \gamma^{\boldsymbol{A}} & & \wr\big\downarrow \gamma_F^{\boldsymbol{A}} \\ {\boldsymbol{A}_*}^* & \xrightarrow{\quad i_1^* \quad} & (q_*[(\boldsymbol{A}/F)_*])^* = (\uparrow^{\boldsymbol{A}} F)^* \end{array}$$

Furthermore, if G is a filter of $\boldsymbol{A}$, with $F \subseteq G$, then the following diagram commutes, where $q'\colon a/F \mapsto a/G$, and i_2 is the inclusion map.

$$\begin{array}{ccc} \boldsymbol{A}/F & \xrightarrow{\quad q' \quad} & \boldsymbol{A}/G \\ \wr\big\downarrow \gamma_F^{\boldsymbol{A}} & & \wr\big\downarrow \gamma_G^{\boldsymbol{A}} \\ (\uparrow^{\boldsymbol{A}} F)^* = (q_*[(\boldsymbol{A}/F)_*])^* & \xrightarrow{\quad i_2^* \quad} & (q_*[(\boldsymbol{A}/G)_*])^* = (\uparrow^{\boldsymbol{A}} G)^* \end{array}$$

Proof. As $q\colon \boldsymbol{A} \to \boldsymbol{A}/F$ is a surjective BRA-morphism, Lemma 6.16(ii) shows that $q_*\colon (\boldsymbol{A}/F)_* \to \boldsymbol{A}_*$ is an injective Esakia morphism; its image is the E-subspace $\uparrow^{\boldsymbol{A}} F$ of $\boldsymbol{A}_*$, by Lemma 6.16(iii) and Remark 6.15. Let $k = {q_*}^{-1}\big|_{\uparrow^{\boldsymbol{A}} F}$, so $k\colon \uparrow^{\boldsymbol{A}} F \cong (\boldsymbol{A}/F)_*$ is defined by

$$k(H) = q[H] = H/F \ := \{a/F : a \in H\} \ \ (H \in \uparrow^{\boldsymbol{A}} F),$$

and $k^*\colon {(\boldsymbol{A}/F)_*}^* \cong (\uparrow^{\boldsymbol{A}} F)^*$. Since $\gamma^{\boldsymbol{A}/F}\colon \boldsymbol{A}/F \cong {(\boldsymbol{A}/F)_*}^*$, we have

$$k^* \circ \gamma^{\boldsymbol{A}/F}\colon \boldsymbol{A}/F \cong (\uparrow^{\boldsymbol{A}} F)^*.$$

For each $a \in A$,

$$\begin{aligned}(k^* \circ \gamma^{\boldsymbol{A}/F})(a/F) &= k^*(\{H \in \mathrm{Pr}(\boldsymbol{A}/F) : a/F \in H\}) \\ &= \{G \in \uparrow^{\boldsymbol{A}} F : a/F \in G/F\} \\ &= \{G \in \uparrow^{\boldsymbol{A}} F : a \in G\} \ \text{(by (6.3))} = \gamma^{\boldsymbol{A}}_F(a/F),\end{aligned}$$

so $k^* \circ \gamma^{\boldsymbol{A}/F} = \gamma^{\boldsymbol{A}}_F$, whence $\gamma^{\boldsymbol{A}}_F\colon \boldsymbol{A}/F \cong (\uparrow^{\boldsymbol{A}} F)^* = (q_*[(\boldsymbol{A}/F)_*])^*$.

Commutativity of the second diagram subsumes that of the first (after identification of $\boldsymbol{A}/\{e\}$ with $\boldsymbol{A}$, and $\gamma^{\boldsymbol{A}}_{\{e\}}$ with $\gamma^{\boldsymbol{A}}$).

Accordingly, let G be a filter of $\boldsymbol{A}$, with $F \subseteq G$, so $q'\colon a/F \mapsto a/G$ is a homomorphism from $\boldsymbol{A}/F$ onto $\boldsymbol{A}/G$. For each $a \in A$, the respective left and right hand sides of the equation $\gamma^{\boldsymbol{A}}_G(q'(a/F)) = i_2^*(\gamma^{\boldsymbol{A}}_F(a/F))$ are, by definition,

$$\{H \in \mathrm{Pr}(\boldsymbol{A}) : G \cup \{a\} \subseteq H\} \ \text{ and } \ (\uparrow^{\boldsymbol{A}} G) \cap \{H \in \mathrm{Pr}(\boldsymbol{A}) : F \cup \{a\} \subseteq H\},$$

which are clearly equal, so $i_2^* \circ \gamma^{\boldsymbol{A}}_F = \gamma^{\boldsymbol{A}}_G \circ q'$. □

The next result is a topological reformulation of Lemma 6.6 for varieties of Brouwerian algebras. Notice that, because of the dual equivalence, the dual of every epimorphism between Brouwerian algebras is a *monomorphism* between pointed Esakia spaces (i.e., a morphism f such that, for any Esakia morphisms g and h, if $f \circ g = f \circ h$ then $g = h$).

Lemma 6.18. *A variety* K *of Brouwerian algebras lacks the ES property if and only if there is an Esakia space* $\boldsymbol{X} \in \mathsf{K}_*$ *with a correct partition* R*, different from the identity relation, such that for every* $\boldsymbol{Y} \in \mathsf{K}_*$ *and every pair of Esakia morphisms* $g, h\colon \boldsymbol{Y} \to \boldsymbol{X}$*, if* $\langle g(y), h(y)\rangle \in R$ *for every* $y \in Y$*, then* $g = h$*.*

Proof. First suppose that K lacks the ES property. By Lemma 6.6, there exists $\boldsymbol{A} \in \mathsf{K}$ with a proper K-epic subalgebra $\boldsymbol{B}$. Let i denote the inclusion map from $\boldsymbol{B}$ to $\boldsymbol{A}$. Note that i is an injective, non-surjective epimorphism. From the duality and Lemma 6.16(ii), it follows that $i_*\colon \boldsymbol{A}_* \to \boldsymbol{B}_*$, is a surjective, non-injective monomorphism. Let $R = \ker i_*$, $q\colon \boldsymbol{A}_* \to \boldsymbol{A}_*/R$ the canonical surjection, and $\sigma\colon \boldsymbol{A}_*/R \cong \boldsymbol{B}_*$ the isomorphism such that $\sigma \circ q = i_*$. So, R is a non-identity correct partition on $\boldsymbol{A}_*$.

Let $\boldsymbol{Y} \in \mathsf{K}_*$ and let $g, h\colon \boldsymbol{Y} \to \boldsymbol{A}_*$ be Esakia morphisms such that $\langle g(y), h(y)\rangle \in R$ for every $y \in Y$. Then $q \circ g = q \circ h$. It follows that $\sigma \circ q \circ g = \sigma \circ q \circ h$, i.e., $i_* \circ g = i_* \circ h$. Therefore $g = h$, since i_* is a monomorphism.

Conversely, suppose that $\boldsymbol{X} \in \mathsf{K}_*$ and R is a non-identity correct partition on $\boldsymbol{X}$ as in the statement of the Lemma. Let $q\colon \boldsymbol{X} \to \boldsymbol{X}/R$ be the canonical surjection. It follows that q is a surjective, non-injective monomorphism. But then, by the duality and Lemma 6.16(ii), $q^*\colon (\boldsymbol{X}/R)^* \to \boldsymbol{X}^*$ is a non-surjective epimorphism. □

6.5 Finite depth

For $\boldsymbol{X} = \langle X; \tau, \leqslant, m\rangle \in \mathsf{PESP}$ and $x \in X$, we define $\mathrm{depth}(x)$ (the *depth* of x in $\boldsymbol{X}$) to be the greatest $n \in \omega$ (if it exists) such that there is a chain

$$x = x_0 < x_1 < \cdots < x_n = m$$

in $\boldsymbol{X}$. (Thus, m has depth 0 in $\boldsymbol{X}$.) If no greatest such n exists, we set $\mathrm{depth}(x) = \infty$. We define $\mathrm{depth}(\boldsymbol{X}) = \sup\{\mathrm{depth}(x) : x \in X\}$.

For $\boldsymbol{A} \in \mathsf{BRA}$ and any subvariety K of BRA, we define

$$\mathrm{depth}(\boldsymbol{A}) = \mathrm{depth}(\boldsymbol{A}_*) \text{ and } \mathrm{depth}(\mathsf{K}) = \sup\{\mathrm{depth}(\boldsymbol{B}) : \boldsymbol{B} \in \mathsf{K}\}.$$

The following claims are explained in [11, Sec. 4], where their antecedents are also discussed.

If a subvariety of BRA is finitely generated (i.e., of the form $\mathbb{V}(\boldsymbol{A})$ for some finite $\boldsymbol{A} \in \mathsf{BRA}$) then it has finite depth, and if it has finite depth then it is locally finite (i.e., its finitely generated members are finite). Both converses are false. For each $n \in \omega$, let

$$\mathsf{D}_n := \{\boldsymbol{A} \in \mathsf{BRA} : \mathrm{depth}(\boldsymbol{A}) \leq n\}.$$

Notice that D_0 is the class of trivial Brouwerian algebras.

Theorem 6.19. *Let $n \in \omega$. A Brouwerian algebra $\boldsymbol{A}$ has depth at most n if and only if it satisfies the equation $d_n \approx e$, where*

$$d_0 := y$$
$$d_{n+1} := x_{n+1} \vee (x_{n+1} \to d_n), \text{ for all } n \in \omega.$$

As a consequence, D_n is a finitely axiomatizable variety.

Definition 6.20. For any S[I]RL $\boldsymbol{A}$ and any variety K of S[I]RLs, we define the *depth* of $\boldsymbol{A}$ to be the depth of its negative cone $\boldsymbol{A}^-$, and the *depth* of K to be $\sup\{\text{depth}(\boldsymbol{B}) : \boldsymbol{B} \in \mathsf{K}\}$.

Consequently, every finitely generated variety of S[I]RLs has finite depth.

For each $n \in \omega$, an S[I]RL has depth at most n iff it satisfies the equations that result from the axioms for D_n when we replace $\to$ by $\to^-$ and x by $x \wedge e$, for every apparent variable x. Thus, the S[I]RLs of depth at most n also form a finitely axiomatizable variety.

Theorem 6.21. *Let K be a variety of S[I]RLs such that each FSI member of K has finite depth. Then K has finite depth.*

Proof. Suppose, with a view to contradiction, that K doesn't have finite depth. Then, for each finite n, there exists $\boldsymbol{A}_n \in \mathsf{K} \setminus \mathsf{D}_n$. Moreover, $\boldsymbol{A}_n$ can be chosen SI, because D_n is a variety (and by Birkhoff's Subdirect Decomposition Theorem 1.3). Let $\mathcal{U}$ be a non-principal ultrafilter over ω, and let $\boldsymbol{A}$ be the corresponding ultraproduct of the algebras $\boldsymbol{A}_n$ $(n \in \omega)$. Then $\boldsymbol{A}$ is FSI (by Lemma 2.16(i)) and belongs to K, so by assumption, $\boldsymbol{A}$ has finite depth, say depth $m \in \omega$. But $\{i \in \omega \colon \text{depth}(\boldsymbol{A}_i) > m\} \in \mathcal{U}$, since the former is a co-finite set and the latter contains the Fréchet filter. So, since the property 'depth$(\boldsymbol{A}) > m$' is expressible by a first order sentence (by Theorem 6.19), $\boldsymbol{A}$ has depth greater than m (by Łoś' Theorem 1.8), a contradiction. □

We can now formulate the main result of this section.

Theorem 6.22. *Let K be a variety of S[I]RLs, such that each FSI member of K is negatively generated and has finite depth. Then every K-epimorphism is surjective.*

The proof of Theorem 6.22 is by contradiction, and it proceeds via a sequence of claims. Let K be as postulated, and suppose that K lacks the ES property. By Theorem 6.9, some $\boldsymbol{A} \in \mathsf{K}_{\text{FSI}}$ has a proper K-epic subalgebra. Now $\boldsymbol{A}/\theta \in \mathsf{K}$ for all $\theta \in \text{Con}(\boldsymbol{A})$, as K is a variety. We shall define a congruence θ of $\boldsymbol{A}$ such that the following is true.

Claim 1: *There exist $a \in A$ and a K-epic proper subalgebra $\boldsymbol{C}$ of $\boldsymbol{A}/\theta$ such that $\boldsymbol{C}$ is generated by its negative cone C^-, and $\boldsymbol{A}/\theta$ is generated by $C^- \cup \{a/\theta\}$, and $a/\theta \prec e/\theta$ in $\boldsymbol{A}/\theta$.*

Once θ has been identified and Claim 1 proved, we shall contradict the fact that $\boldsymbol{C}$ is K-epic in $\boldsymbol{A}/\theta$, by constructing a non-identity homomorphism $\ell\colon \boldsymbol{A}/\theta \to \boldsymbol{A}/\theta$, such that $\ell|_C = \text{id}_C$, as follows.

Let $b = a/\theta$. Then $\mathrm{Fg}^{\boldsymbol{A}/\theta}\{b\} = \{d \in A/\theta : b \leqslant d\}$, because $b < e/\theta$. Let $\alpha = \boldsymbol{\Omega}^{\boldsymbol{A}/\theta}\,\mathrm{Fg}^{\boldsymbol{A}/\theta}\{b\}$ (see page 33). For any $u, v \in A/\theta$, we have

$$u \equiv_\alpha v \text{ iff } b \leqslant u \leftrightarrow v. \tag{6.5}$$

In particular, $e/\theta \equiv_\alpha b$, by (2.15) and (2.17).

Let $\{a_i : i \in I\}$ be an indexing of A/θ, and $\vec{c} = c_0, c_1, \ldots$ a well-ordering of the elements of C^-. Since $C \neq A/\theta$ and $\boldsymbol{A}/\theta$ is generated by $C^- \cup \{b\}$, it follows that $b \notin C$ and, for each $i \in I$, we have $a_i = t_i^{\boldsymbol{A}/\theta}(b, \vec{c})$ for a suitable S[I]RL-term $t_i(x, \vec{y})$, where $\vec{y} = y_0, y_1, \ldots$. When a_i is some $c_j \in C^-$, we can (and do) choose t_i to be the variable y_j.

We define $\ell \colon A/\theta \to A/\theta$ by

$$\ell(a_i) = t_i^{\boldsymbol{A}/\theta}(e/\theta, \vec{c}) \text{ for all } i \in I.$$

By the above choice, $\ell(c_j) = c_j$ for $j = 0, 1, \ldots$, while $\ell[A/\theta] \subseteq C$, because $\boldsymbol{C}$ is a subalgebra of $\boldsymbol{A}/\theta$. We claim that ℓ is a homomorphism.

To see this, let σ be a basic S[I]RL-operation symbol, and let $a_{i_1}, \ldots, a_{i_n} \in A/\theta$, where n is the arity of σ. Then $\sigma^{\boldsymbol{A}/\theta}(a_{i_1}, \ldots, a_{i_n}) = a_j$ for some $j \in I$. For this j, we perform the following calculation, where every term is evaluated in $\boldsymbol{A}/\theta$:

$$\begin{aligned}
\sigma(\ell(a_{i_1}), \ldots, \ell(a_{i_n})) &= \sigma(t_{i_1}(e/\theta, \vec{c}), \ldots, t_{i_n}(e/\theta, \vec{c})) \\
&\equiv_\alpha \sigma(t_{i_1}(b, \vec{c}), \ldots, t_{i_n}(b, \vec{c})) \\
&= \sigma(a_{i_1}, \ldots, a_{i_n}) \\
&= a_j = t_j(b, \vec{c}) \\
&\equiv_\alpha t_j(e/\theta, \vec{c}) = \ell(a_j) = \ell(\sigma(a_{i_1}, \ldots, a_{i_n})).
\end{aligned}$$

By (6.5), therefore,

$$b \leqslant (\sigma(\ell(a_{i_1}), \ldots, \ell(a_{i_n})) \leftrightarrow \ell(\sigma(a_{i_1}, \ldots, a_{i_n}))) \wedge (e/\theta).$$

Note that

$$(\sigma(\ell(a_{i_1}), \ldots, \ell(a_{i_n})) \leftrightarrow \ell(\sigma(a_{i_1}, \ldots, a_{i_n}))) \wedge (e/\theta) \in C$$

(because $\ell[A/\theta] \subseteq C$), but $b \notin C$, so

$$b < (\sigma(\ell(a_{i_1}), \ldots, \ell(a_{i_n})) \leftrightarrow \ell(\sigma(a_{i_1}, \ldots, a_{i_n}))) \wedge (e/\theta) \leqslant e/\theta.$$

Since $b \prec e/\theta$ in $\boldsymbol{A}/\theta$, this forces

$$e/\theta = (\sigma(\ell(a_{i_1}), \ldots, \ell(a_{i_n})) \leftrightarrow \ell(\sigma(a_{i_1}, \ldots, a_{i_n}))) \wedge (e/\theta),$$

i.e., $e/\theta \leqslant \sigma(\ell(a_{i_1}), \dots, \ell(a_{i_n})) \leftrightarrow \ell(\sigma(a_{i_1}, \dots, a_{i_n}))$. Then, by (2.16),

$$\sigma(\ell(a_{i_1}), \dots, \ell(a_{i_n})) = \ell(\sigma(a_{i_1}, \dots, a_{i_n})),$$

confirming that ℓ is a homomorphism.

For each $c \in C$, we have $c = t^{\boldsymbol{A}/\theta}(\vec{c})$ for some S[I]RL-term t (as $\boldsymbol{C}$ is generated by C^-), so

$$\ell(c) = \ell(t(\vec{c})) = t(\ell(c_0), \ell(c_1), \dots) = t(c_0, c_1, \dots) = c.$$

This shows that $\ell|_C = \mathrm{id}_C$, but $\ell(b) \neq b$, since $\ell(b) \in C$. As intended, this contradicts the fact that $\boldsymbol{C}$ is K-epic in $\boldsymbol{A}/\theta$.

It remains to construct θ and to prove Claim 1. The construction of θ exploits the assumption that members of K_{FSI} have finite depth.

Recall that $\boldsymbol{A}$ has a proper K-epic subalgebra, $\boldsymbol{B}$ say. As $\boldsymbol{A}$ is FSI, so is $\boldsymbol{B}$ (by Lemma 2.16(i)). By assumption, therefore, $\boldsymbol{A}$ and $\boldsymbol{B}$ are both negatively generated, so $\boldsymbol{B}^- \neq \boldsymbol{A}^-$, because $\boldsymbol{B} \neq \boldsymbol{A}$.

For each $F \in \mathrm{Pr}(\boldsymbol{A}^-)$, we clearly have $B \cap F = B^- \cap F = i_*(F)$, where i is the inclusion map $i\colon \boldsymbol{B}^- \to \boldsymbol{A}^-$, considered as a BRA-morphism.

As i is not surjective, its dual $i_*\colon {\boldsymbol{A}^-}_* \to {\boldsymbol{B}^-}_*$ is not injective, by Lemma 6.16(ii), i.e., the following set is not empty:

$$W \;:=\; \{\langle F_1, F_2\rangle \in (\mathrm{Pr}(\boldsymbol{A}^-))^2 : F_1 \neq F_2 \text{ and } F_1 \cap B = F_2 \cap B\}.$$

By assumption, $\boldsymbol{A}^-$ has finite depth, so

$$\{\min\{\mathrm{depth}(F_1), \mathrm{depth}(F_2)\} : \langle F_1, F_2\rangle \in W\}$$

is a non-empty subset of ω, and therefore has a least element, n say. Pick $F_1 \in \mathrm{Pr}(\boldsymbol{A}^-)$ such that $\mathrm{depth}(F_1) = n$ and $\langle F_1, G\rangle \in W$ for some G. Now,

$$\text{whenever } \langle F_1', F_2'\rangle \in W, \text{ then } \mathrm{depth}(F_1) \leq \mathrm{depth}(F_1'), \mathrm{depth}(F_2'). \tag{6.6}$$

Having fixed F_1 in this way, we similarly choose $F_2 \in \mathrm{Pr}(\boldsymbol{A}^-) \setminus \{F_1\}$ such that $F_1 \cap B = F_2 \cap B$ and

$$\text{whenever } \langle F_1, F_2'\rangle \in W, \text{ then } \mathrm{depth}(F_2) \leq \mathrm{depth}(F_2'). \tag{6.7}$$

As $\langle F_1, F_2\rangle \in W$, we have $\mathrm{depth}(F_1) \leq \mathrm{depth}(F_2)$ (by (6.6)), so F_1 is not a proper subset of F_2.

Lemma 6.23. *If* $F_1 \subsetneq G \in \mathrm{Pr}(\boldsymbol{A}^-)$, *then* $F_2 \subsetneq G$.

Proof. Let $F_1 \subsetneq G \in \mathrm{Pr}(\boldsymbol{A}^-)$, so $\mathrm{depth}(G) < \mathrm{depth}(F_1)$. As i_* is an Esakia morphism and $i_*(F_2) = i_*(F_1) \subseteq i_*(G)$, there exists $H \in \mathrm{Pr}(\boldsymbol{A}^-)$ such that $F_2 \subseteq H$ and $i_*(G) = i_*(H)$, by (6.4), i.e., $G \cap B = H \cap B$. Therefore, if $G \neq H$, then $\mathrm{depth}(F_1) \leq \mathrm{depth}(G)$, by (6.6). This is a contradiction, so $G = H$, whence $F_2 \subseteq G$. If $F_2 = G$, then

$$\mathrm{depth}(F_2) = \mathrm{depth}(G) < \mathrm{depth}(F_1),$$

contradicting the fact that $\mathrm{depth}(F_1) \leq \mathrm{depth}(F_2)$. Therefore, $F_2 \subsetneq G$. □

Lemma 6.24. *If* $F_2 \subsetneq G \in \mathrm{Pr}(\boldsymbol{A}^-)$, *then* $F_1 \subseteq G$.

Proof. Let $F_2 \subsetneq G \in \mathrm{Pr}(\boldsymbol{A}^-)$. Again, $i_*(F_1) = i_*(F_2) \subseteq i_*(G)$, so there exists $H \in \mathrm{Pr}(\boldsymbol{A}^-)$ such that $F_1 \subseteq H$ and $i_*(G) = i_*(H)$. Suppose $G \neq H$. If $F_1 = H$, then $F_1 \cap B = G \cap B$, so, by (6.7), $\mathrm{depth}(F_2) \leq \mathrm{depth}(G)$, contradicting the fact that $F_2 \subsetneq G$. Therefore, $F_1 \subsetneq H$, so $\mathrm{depth}(H) < \mathrm{depth}(F_1)$. Then, by (6.6), $\mathrm{depth}(F_1) \leq \mathrm{depth}(H)$, since $G \cap B = H \cap B$. This is a contradiction, so $G = H$, whence $F_1 \subseteq G$. □

Recalling that F_1, F_2 are distinct and that F_1 is not properly contained in F_2, we make the following claim:

Claim 2: *There are just two possibilities:*

(A) $F_2 \subsetneq F_1$, *in which case* $F_2 \prec F_1$ *(in fact,* F_1 *is the least strict upper bound of* F_2 *in* $\mathrm{Pr}(\boldsymbol{A}^-)$*);*

(B) F_1 *and* F_2 *are incomparable, in which case they have the same depth, the same strict upper bounds and, therefore, the same covers in* $\mathrm{Pr}(\boldsymbol{A}^-)$.

Proof. If $F_2 \subsetneq F_1$, then F_1 is the least strict upper bound of F_2, by Lemma 6.24, i.e., F_1 is the unique cover of F_2. We may therefore assume that F_2 is not a proper subset of F_1, i.e., that F_1 and F_2 are incomparable. Then, by Lemmas 6.23 and 6.24, F_1 and F_2 have the same strict upper bounds, and hence the same covers in $\mathrm{Pr}(\boldsymbol{A}^-)$, from which it clearly follows that they also have the same depth. □

By (6.1), $\mathrm{Fg}^{\boldsymbol{A}}(F_1 \cap F_2) = \{d \in A : d \geqslant c \text{ for some } c \in F_1 \cap F_2\}$. We define

$$\theta = \boldsymbol{\Omega}^{\boldsymbol{A}} \, \mathrm{Fg}^{\boldsymbol{A}}(F_1 \cap F_2).$$

Claim 3: *The following diagram commutes, where the maps will be defined in the proof.*

$$\begin{array}{ccccc}
(\boldsymbol{B}/(\theta|_B))^- & & \xhookrightarrow{\quad j \quad} & & (\boldsymbol{A}/\theta)^- \\
\wr\downarrow i_2 & & & & \wr\downarrow i_1 \\
\boldsymbol{B}^-/(F_1 \cap B) & & \boldsymbol{X}^* & \xleftarrow[\sim]{\gamma^{\boldsymbol{A}^-}_{F_1 \cap F_2}} & \boldsymbol{A}^-/(F_1 \cap F_2) \\
\wr\downarrow \gamma^{\boldsymbol{B}^-}_{F_1 \cap B} & & \downarrow (i_Y)^* & & \downarrow\!\!\downarrow q \\
\boldsymbol{Z}^* & \xrightarrow[\sim]{(i_*|_Y)^*} & \boldsymbol{Y}^* & \xleftarrow[\sim]{\gamma^{\boldsymbol{A}^-}_{F_1}} & \boldsymbol{A}^-/F_1
\end{array}$$

Proof. The map $j\colon \boldsymbol{B}/(\theta|_B) \to \boldsymbol{A}/\theta$, defined by $b/(\theta|_B) \mapsto b/\theta$, is an injective K-epimorphism, by Lemma 6.8. By Lemma 6.12(i), the restriction of j to $(B/(\theta|_B))^-$ is a BRA-morphism from $(\boldsymbol{B}/(\theta|_B))^-$ into $(\boldsymbol{A}/\theta)^-$. We shall not distinguish notationally between j and this restriction. Whenever $b \in B$ and $b/(\theta|_B) \leqslant e/(\theta|_B)$, then $b/(\theta|_B) = (b \wedge e)/(\theta|_B)$ and $b/\theta = (b \wedge e)/\theta$, so

$$(B/(\theta|_B))^- = \{b/(\theta|_B) : b \in B^-\} \tag{6.8}$$

and $j[(B/(\theta|_B))^-] = \{b/\theta : b \in B^-\}$.

Let $K = F_1 \cap F_2$, so $\theta = \boldsymbol{\Omega}^{\boldsymbol{A}} \operatorname{Fg}^{\boldsymbol{A}} K$. By (6.1), $A^- \cap \operatorname{Fg}^{\boldsymbol{A}} K = K$, so $B^- \cap \operatorname{Fg}^{\boldsymbol{A}} K = B^- \cap K = B \cap K = B \cap F_1$ (since $B \cap F_1 = B \cap F_2$). By (2.26), $\theta|_B = \boldsymbol{\Omega}^{\boldsymbol{B}}(B \cap \operatorname{Fg}^{\boldsymbol{A}} K)$, so Lemma 6.12(ii) supplies isomorphisms

$$i_1\colon (\boldsymbol{A}/\theta)^- \cong \boldsymbol{A}^-/(F_1 \cap F_2) \quad\text{and}\quad i_2\colon (\boldsymbol{B}/(\theta|_B))^- \cong \boldsymbol{B}^-/(F_1 \cap B),$$

defined by $a/\theta \mapsto (a \wedge e)/(F_1 \cap F_2)$ and $b/(\theta|_B) \mapsto (b \wedge e)/(F_1 \cap B)$.

By Lemma 6.17, ${\uparrow}^{\boldsymbol{B}^-}(F_1 \cap B)$ is the universe of an E-subspace, $\boldsymbol{Z}$ say, of $\boldsymbol{B}^-{}_*$, and $\gamma^{\boldsymbol{B}^-}_{F_1 \cap B}\colon \boldsymbol{B}^-/(F_1 \cap B) \cong \boldsymbol{Z}^*$. Also, $q\colon a/(F_1 \cap F_2) \mapsto a/F_1$ defines a homomorphism from $\boldsymbol{A}^-/(F_1 \cap F_2)$ onto $\boldsymbol{A}^-/F_1$. Let $\boldsymbol{X}$ [resp. $\boldsymbol{Y}$] be the E-subspace of $\boldsymbol{A}^-{}_*$ with universe ${\uparrow}^{\boldsymbol{A}^-}(F_1 \cap F_2)$ [resp. ${\uparrow}^{\boldsymbol{A}^-} F_1$]. Let $i_Y\colon \boldsymbol{Y} \to \boldsymbol{X}$ be the inclusion morphism in PESP. By Lemma 6.17, the following diagram commutes.

$$\begin{array}{ccc}
\boldsymbol{A}^-/(F_1 \cap F_2) & \xrightarrow[\sim]{\gamma^{\boldsymbol{A}^-}_{F_1 \cap F_2}} & \boldsymbol{X}^* \\
\downarrow\!\!\downarrow q & & \downarrow\!\!\downarrow (i_Y)^* \\
\boldsymbol{A}^-/F_1 & \xrightarrow[\sim]{\gamma^{\boldsymbol{A}^-}_{F_1}} & \boldsymbol{Y}^*
\end{array}$$

Recall that the BRA-morphism $i\colon \boldsymbol{B}^- \to \boldsymbol{A}^-$ is the inclusion map. As i is injective, its dual $i_*\colon \boldsymbol{A}^-{}_* \to \boldsymbol{B}^-{}_*$ is surjective, by Lemma 6.16(ii).

The above definitions clearly imply that $i_*[Y] \subseteq Z$. To establish the reverse inclusion, let $G \in Z$. By the Prime Filter Extension Theorem 6.13, $G = H \cap B$ for some $H \in \operatorname{Pr}(\boldsymbol{A}^-)$. Now, $i_*[F_1] = B \cap F_1 \subseteq G = i_*[H]$, so by

(6.4), $i_*[H] = i_*[H']$ for some $H' \in {\uparrow}^{\boldsymbol{A}^-} F_1 = Y$, whence $G = i_*[H'] \in i_*[Y]$. Therefore, $Z = i_*[Y]$.

We claim that $i_*|_Y$ is injective. Suppose, on the contrary, that $H_1, H_2 \in Y$, with $H_1 \neq H_2$ and $i_*[H_1] = i_*[H_2]$. For each $k \in \{1,2\}$, (6.6) shows that $\text{depth}(F_1) \leq \text{depth}(H_k)$, but $F_1 \subseteq H_k$, so $H_k = F_1$, whence $H_1 = H_2$. This contradiction confirms that $i_*|_Y$ is injective, whence $i_*|_Y \colon \boldsymbol{Y} \cong \boldsymbol{Z}$ in PESP. In BRA, therefore, $(i_*|_Y)^* \colon \boldsymbol{Z}^* \cong \boldsymbol{Y}^*$.

A composition of isomorphisms in BRA is an isomorphism, so

$$g \;:=\; (i_*|_Y)^* \circ \gamma^{\boldsymbol{B}^-}_{F_1 \cap B} : \boldsymbol{B}^-/(F_1 \cap B) \;\cong\; \boldsymbol{Y}^*. \tag{6.9}$$

To show that the diagram in Claim 3 commutes, it remains to prove that $g \circ i_2 = \gamma^{\boldsymbol{A}^-}_{F_1} \circ q \circ i_1 \circ j$. And indeed, if $b \in B$ and $b/(\theta|_B) \in (B/(\theta|_B))^-$, then

$$\begin{aligned}
(g \circ i_2)(b/(\theta|_B)) &= g((b \wedge e)/(F_1 \cap B)) \\
&= (i_*|_Y)^*(\{H \in \text{Pr}(\boldsymbol{B}^-) : (F_1 \cap B) \cup \{b \wedge e\} \subseteq H\}) \\
&= \{H \in \text{Pr}(\boldsymbol{A}^-) : F_1 \subseteq H \text{ and } (F_1 \cap B) \cup \{b \wedge e\} \subseteq H \cap B\} \\
&= \{H \in \text{Pr}(\boldsymbol{A}^-) : F_1 \cup \{b \wedge e\} \subseteq H\} \\
&= \gamma^{\boldsymbol{A}^-}_{F_1}((b \wedge e)/F_1) \;=\; (\gamma^{\boldsymbol{A}^-}_{F_1} \circ q \circ i_1 \circ j)(b/(\theta|_B)).
\end{aligned}$$
□

Claim 4: *Suppose $k \in \{1,2\}$ and $a \in A^-$ and $b \in B^-$, where $a \equiv_\theta b$. Then $a \in F_k$ iff $b \in F_k$. Consequently, $a \notin (F_1 \setminus F_2) \cup (F_2 \setminus F_1)$.*

Proof. As $a \equiv_\theta b$, we have $a \leftrightarrow b \in \text{Fg}^{\boldsymbol{A}}(F_1 \cap F_2)$, so

$$a \leftrightarrow^- b \;:=\; (a \leftrightarrow b) \wedge e \in A^- \cap \text{Fg}^{\boldsymbol{A}}(F_1 \cap F_2) = F_1 \cap F_2,$$

by (6.1). As $a \to^- b,\, b \to^- a \geqslant a \leftrightarrow^- b$, it follows that $a \to^- b,\, b \to^- a \in F_k$. Thus, $a \in F_k$ iff $b \in F_k$. In particular, if $a \in (F_1 \setminus F_2) \cup (F_2 \setminus F_1)$ then $b \in B \cap ((F_1 \setminus F_2) \cup (F_2 \setminus F_1))$, contradicting the fact that $B \cap F_1 = B \cap F_2$. □

By Claim 3, $h \;:=\; \gamma^{\boldsymbol{A}^-}_{F_1 \cap F_2} \circ i_1 : (\boldsymbol{A}/\theta)^- \cong \boldsymbol{X}^*$ and, for each $a \in A$ such that $a/\theta \in (A/\theta)^-$, we have

$$h(a/\theta) = \{H \in \text{Pr}(\boldsymbol{A}^-) : (F_1 \cap F_2) \cup \{a \wedge e\} \subseteq H\}.$$

By Claim 2, $F_1 \setminus F_2 \neq \emptyset$. In fact,

$$h(d/\theta) = {\uparrow}^{\boldsymbol{A}^-} F_1 \text{ for all } d \in F_1 \setminus F_2.$$

To confirm this, let $d \in F_1 \setminus F_2$. Clearly, ${\uparrow}^{\boldsymbol{A}^-} F_1 \subseteq h(d/\theta)$. Conversely, let $H \in h(d/\theta)$. If $F_1 \nsubseteq H$, then since $F_1 \cap F_2 \subseteq H \in \text{Pr}(\boldsymbol{A}^-)$, we have $F_2 \subseteq H$, by (6.2). In that case, $F_2 \subsetneq H$, because $d \in H \setminus F_2$, but this contradicts Lemma 6.24. Thus, $F_1 \subseteq H$, and so $h(d/\theta) = {\uparrow}^{\boldsymbol{A}^-} F_1$, as claimed.

In Case (A) of Claim 2, we have ${\uparrow}^{\boldsymbol{A}^-} F_2 = {\uparrow}^{\boldsymbol{A}^-}(F_1 \cap F_2) = X = h(e/\theta)$.

In Case (B), we have $F_2 \setminus F_1 \neq \emptyset$, and we claim that

$$h(d/\theta) = {\uparrow}^{\boldsymbol{A}^-} F_2 \text{ for every } d \in F_2 \setminus F_1.$$

To see this, let $d \in F_2 \setminus F_1$. It is clear that ${\uparrow}^{\boldsymbol{A}^-} F_2 \subseteq h(d/\theta)$, so consider $H \in h(d/\theta)$. If $F_2 \nsubseteq H$ then, since $F_1 \cap F_2 \subseteq H$, we have $F_1 \subseteq H$, by (6.2). In that case, $F_1 \subsetneq H$, as $d \in H \setminus F_1$, but this contradicts Lemma 6.23. Thus, $F_2 \subseteq H$, and so $h(d/\theta) \subseteq {\uparrow}^{\boldsymbol{A}^-} F_2$.

We define $M = (A/\theta)^- \setminus j[(B/(\theta|_B))^-]$.

Claim 5: *Fix any $a_1 \in F_1 \setminus F_2$. Choose a_2 to be e in Case* (A) *of Claim* 2*, and an arbitrary element of $F_2 \setminus F_1$ in Case* (B)*. Then*

$$M = \{a_1/\theta\} \textit{ in Case } (\mathrm{A}), \textit{ and } M = \{a_1/\theta, a_2/\theta\} \textit{ in Case } (\mathrm{B}).$$

Moreover, $a/\theta \prec e/\theta$ (in $\boldsymbol{A}/\theta$) for all $a \in A$ such that $a/\theta \in M$.

Proof. Observe that $a_1, a_2 \leqslant e$ and, as we showed above,

$$h(a_1/\theta) = {\uparrow}^{\boldsymbol{A}^-} F_1 \text{ and } h(a_2/\theta) = {\uparrow}^{\boldsymbol{A}^-} F_2.$$

By Claim 4 and (6.8), we have $a_1/\theta \in M$ and, in Case (B), $a_2/\theta \in M$. In Case (A), $a_2/\theta \notin M$, since $e \in B$.

Because h is an isomorphism, $h[M] = X^* \setminus h[j[(B/(\theta|_B))^-]]$ and, for the first assertion of Claim 5, it suffices to prove that $h[M] \subseteq \{{\uparrow}^{\boldsymbol{A}^-} F_1, {\uparrow}^{\boldsymbol{A}^-} F_2\}$.

Suppose, with a view to contradiction, that there exists $\mathcal{U} \in h[M]$ with $\mathcal{U} \notin \{{\uparrow}^{\boldsymbol{A}^-} F_1, {\uparrow}^{\boldsymbol{A}^-} F_2\}$. Then $\mathcal{U} \subseteq X$, but $\mathcal{U} \neq X$, because

$$X = h(j(e/(\theta|_B))) \in h[j[(B/(\theta|_B))^-]].$$

We show first that $\mathcal{U} \subsetneq {\uparrow}^{\boldsymbol{A}^-} F_1$.

Suppose $F_1, F_2 \in \mathcal{U}$. For each $H \in X$, we have $F_1 \subseteq H$ or $F_2 \subseteq H$, by (6.2), so $H \in \mathcal{U}$ (since $\mathcal{U}$ is upward closed). This shows that $X \subseteq \mathcal{U}$, a contradiction. Therefore, F_1 and F_2 don't both belong to $\mathcal{U}$.

Suppose $F_2 \in \mathcal{U}$. Then $F_1 \notin \mathcal{U}$ and ${\uparrow}^{\boldsymbol{A}^-} F_2 \subseteq \mathcal{U}$, as $\mathcal{U}$ is upward closed. If $H \in \mathcal{U}$, then $H \neq F_1$ and $F_1 \cap F_2 \subseteq H$ (as $\mathcal{U} \subseteq X$). In that case, $F_2 \subseteq H$ (otherwise, $F_1 \subseteq H$, by (6.2), whence $F_1 \subsetneq H$, but then $F_2 \subsetneq H$, by Lemma 6.23). This shows that $\mathcal{U} \subseteq {\uparrow}^{\boldsymbol{A}^-} F_2$, so $\mathcal{U} = {\uparrow}^{\boldsymbol{A}^-} F_2$, contrary to our initial assumptions about $\mathcal{U}$. Therefore, $F_2 \notin \mathcal{U}$.

We claim that $\mathcal{U} \subseteq {\uparrow}^{\boldsymbol{A}^-} F_1$. For otherwise, $F_1 \nsubseteq H$ for some $H \in \mathcal{U}$, whence $H \neq F_2$ and, by (6.2), $F_2 \subseteq H$, i.e., $F_2 \subsetneq H$, whereupon Lemma 6.24 delivers the contradiction $F_1 \subseteq H$. Thus, $\mathcal{U} \subsetneq {\uparrow}^{\boldsymbol{A}^-} F_1$ (since $\mathcal{U} \neq {\uparrow}^{\boldsymbol{A}^-} F_1$, by assumption).

Now we shall argue that $\mathcal{U} \in h[j[(B/\theta|_B)^-]]$ (contradicting the fact that $\mathcal{U} \in h[M]$, and thereby confirming the relation $h[M] \subseteq \{{\uparrow}^{\boldsymbol{A}^-} F_1, {\uparrow}^{\boldsymbol{A}^-} F_2\}$).

As $\mathcal{U} \in X^*$ and $\mathcal{U} \subsetneq {\uparrow}^{\boldsymbol{A}^-} F_1 = Y$, we have

$$\mathcal{U} = \mathcal{U} \cap Y = (i_Y)^*(\mathcal{U}) \in Y^*,$$

so by Claim 3, there exists $b \in B$ with $b/(\theta|_B) \in (B/(\theta|_B))^-$ such that

$$\mathcal{U} = g(i_2(b/(\theta|_B))) = (i_Y)^*(h(j(b/(\theta|_B)))) = Y \cap h(b/\theta) = Y \cap \mathcal{V},$$

where g is as in (6.9), and $\mathcal{V} := h(b/\theta)$. By (6.8), we may assume that $b \leqslant e$.

Now $\mathcal{V} \in X^*$, so $\mathcal{V}$ is an upward-closed subset of X. Note that $F_1 \notin \mathcal{V}$ (otherwise $Y = {\uparrow}^{\boldsymbol{A}^-} F_1 \subseteq \mathcal{V}$, yielding the contradiction $\mathcal{U} \subsetneq Y = Y \cap \mathcal{V} = \mathcal{U}$). It follows that $Y \nsubseteq \mathcal{V}$ (as $F_1 \in Y$).

For any $H \in \mathcal{V}$, if $F_1 \subseteq H$, then $F_1 \subsetneq H$ (as $F_1 \notin \mathcal{V}$), whence $F_2 \subsetneq H$ (by Lemma 6.23), whereas if $F_1 \nsubseteq H$, then $F_2 \subseteq H$ (by (6.2)). This shows that $\mathcal{V} \subseteq {\uparrow}^{\boldsymbol{A}^-} F_2$.

We now argue that $\mathcal{V} \subseteq Y$.

Suppose, on the contrary, that there exists $H \in \mathcal{V} \setminus Y$. Then $F_1 \nsubseteq H$ (by definition of Y), so $F_2 \subseteq H$, by (6.2). Now Lemma 6.24 prevents F_2 from being a proper subset of H, so $F_2 = H$. In particular, $F_2 \in \mathcal{V}$, so ${\uparrow}^{\boldsymbol{A}^-} F_2 \subseteq \mathcal{V}$, whence $\mathcal{V} = {\uparrow}^{\boldsymbol{A}^-} F_2$.

In Case (A) of Claim 2, it would follow that $Y = {\uparrow}^{\boldsymbol{A}^-} F_1 \subseteq {\uparrow}^{\boldsymbol{A}^-} F_2 = \mathcal{V}$, a contradiction.

In Case (B), we have $e \geqslant a_2 \in F_2 \setminus F_1$ and $h(a_2/\theta) = {\uparrow}^{\boldsymbol{A}^-} F_2 = \mathcal{V} = h(b/\theta)$. Then, since h is injective, $a_2/\theta = b/\theta$, contradicting Claim 4.

This confirms that $\mathcal{V} \subseteq Y$, and so $\mathcal{V} = Y \cap \mathcal{V} = \mathcal{U}$. Therefore,

$$\mathcal{U} = h(b/\theta) = h(j(b/(\theta|_B))) \in h[j[(B/\theta|_B)^-]],$$

completing the proof that M is $\{a_1/\theta\}$ in Case (A), and is $\{a_1/\theta, a_2/\theta\}$ in Case (B).

It remains to show that $a/\theta \prec e/\theta$ in $\boldsymbol{A}/\theta$, whenever $a/\theta \in M$.

To establish that $a_1/\theta \prec e/\theta$ in $\boldsymbol{A}/\theta$ (i.e., in $(\boldsymbol{A}/\theta)^-$), it suffices to show that ${\uparrow}^{\boldsymbol{A}^-} F_1 \prec X$ in $\boldsymbol{X}^*$, because h is an isomorphism.

Suppose ${\uparrow}^{\boldsymbol{A}^-} F_1 \subsetneq \mathcal{W} \subsetneq X$, where $\mathcal{W} \in X^*$. Then $F_1 \nsubseteq H$ for some $H \in \mathcal{W}$, whence $F_2 \subseteq H$, by (6.2). We cannot have $F_2 = H$, otherwise

$\uparrow^{\boldsymbol{A}^-} F_2 \subseteq \mathcal{W}$, in which case every element G of X belongs to $\mathcal{W}$ (as G contains F_1 or F_2, again by (6.2)). Therefore, $F_2 \subsetneq H$, and so $F_1 \subseteq H$, by Lemma 6.24. This contradiction confirms that $a_1/\theta \prec e/\theta$ in $\boldsymbol{A}/\theta$.

We may now assume that Case (B) applies. The desired conclusion $a_2/\theta \prec e/\theta$ amounts similarly to the claim that $\uparrow^{\boldsymbol{A}^-} F_2 \prec X$ in $\boldsymbol{X}^*$. Suppose, on the contrary, that $\uparrow^{\boldsymbol{A}^-} F_2 \subsetneq \mathcal{W} \subsetneq X$, for some $\mathcal{W} \in X^*$. Then $F_2 \not\subseteq H$ for some $H \in \mathcal{W}$. Now $H \in X$, so $F_1 \subseteq H$, by (6.2). Then $F_1 = H$, by Lemma 6.23, so $F_1 \in \mathcal{W}$, whence $\uparrow^{\boldsymbol{A}^-} F_1 \subseteq \mathcal{W}$. By (6.2) again, $X \subseteq (\uparrow^{\boldsymbol{A}^-} F_1) \cup (\uparrow^{\boldsymbol{A}^-} F_2) \subseteq \mathcal{W}$, contradicting the fact that $\mathcal{W} \subsetneq X$. Therefore, $a_2/\theta \prec e/\theta$ in $\boldsymbol{A}/\theta$. □

We are now in a position to prove Claim 1 (and hence Theorem 6.22).

Proof of Claim 1. Since $\boldsymbol{A}$ and $\boldsymbol{B}$ are negatively generated, so are $\boldsymbol{A}/\theta$ and $\boldsymbol{B}/(\theta|_B)$, by Lemma 2.35. The subalgebra

$$\boldsymbol{J} := j[\boldsymbol{B}/(\theta|_B)]$$

of $\boldsymbol{A}/\theta$ is isomorphic to $\boldsymbol{B}/(\theta|_B)$, so $\boldsymbol{J}$ is also generated by its negative cone J^-. By Lemma 6.12(i), $J^- = j[(B/(\theta|_B))^-]$, whence $M = (A/\theta)^- \setminus J^-$. As $\boldsymbol{B}$ is K-epic in $\boldsymbol{A}$, Lemma 6.8 shows that $\boldsymbol{J}$ is K-epic in $\boldsymbol{A}/\theta$. Moreover,

$$\boldsymbol{S} := \mathbf{Sg}^{\boldsymbol{A}/\theta}(J^- \cup \{a_1/\theta\})$$

is negatively generated (since $J^- \cup \{a_1/\theta\} \subseteq S^-$), and $\boldsymbol{J}$ is a subalgebra of $\boldsymbol{S}$ (as $J = \mathrm{Sg}^{\boldsymbol{A}/\theta}(J^-)$), so $\boldsymbol{S}$ is K-epic in $\boldsymbol{A}/\theta$ (because $\boldsymbol{J}$ is).

Observe that $J \neq A/\theta$, because $a_1/\theta \notin J$ (by Claim 4 and (6.8), since $a_1 \in F_1 \setminus F_2$), and that $A/\theta = \mathrm{Sg}^{\boldsymbol{A}/\theta}((A/\theta)^-) = \mathrm{Sg}^{\boldsymbol{A}/\theta}(J^- \cup M)$.

We choose $\boldsymbol{C} = \boldsymbol{J}$ and $a = a_1$ in Case (A). We make the same choices in Case (B) *if* $a_2/\theta \in S$. Under these conditions, $J^- \cup M = J^- \cup \{a_1/\theta\}$ (by Claim 5) and $A/\theta = \mathrm{Sg}^{\boldsymbol{A}/\theta}(J^- \cup M) \subseteq S = \mathrm{Sg}^{\boldsymbol{A}/\theta}(C^- \cup \{a/\theta\})$, so $\boldsymbol{A}/\theta$ is generated by $C^- \cup \{a/\theta\}$, as required.

In Case (B), if $a_2/\theta \notin S$ (whence $S \neq A/\theta$), we choose $\boldsymbol{C} = \boldsymbol{S}$ and $a = a_2$, whereupon $J^- \cup M = J^- \cup \{a_1/\theta, a_2/\theta\}$ (by Claim 5) and

$$A/\theta = \mathrm{Sg}^{\boldsymbol{A}/\theta}(J^- \cup M) \subseteq \mathrm{Sg}^{\boldsymbol{A}/\theta}(S^- \cup \{a_2/\theta\}) = \mathrm{Sg}^{\boldsymbol{A}/\theta}(C^- \cup \{a/\theta\}),$$

so again, $\boldsymbol{A}/\theta$ is generated by $C^- \cup \{a/\theta\}$. □

Applications

Recall from Theorem 4.2 that the four minimal varieties of De Morgan monoids satisfy the conditions of Theorem 6.22 (their FSI members have finite depth and are negatively generated), and thus have the ES property. Furthermore, by Corollary 4.24, each of the ten covers of $\mathbb{V}(\boldsymbol{C}_4)$ within U is generated by a finite De Morgan monoid that is itself generated by one of its negative elements. Therefore, the conditions of Theorem 6.22 obtain in all ten covers, and hence in their varietal join, so we deduce:

Theorem 6.25. *The join of the ten covers of* $\mathbb{V}(\boldsymbol{C}_4)$ *within* U *has the ES property, and so do all of its subvarieties.*

As was mentioned in the introduction to this chapter, Theorem 6.22 can be viewed as a generalization (to certain non-integral varieties of SRLs) of the following recent result:

Corollary 6.26 ([11, Thm. 5.4])**.** *Every variety of Brouwerian algebras with finite depth has the ES property.*

(This follows directly from Theorem 6.22, because every Brouwerian algebra is negatively generated.)

Theorem 6.22 remains true for varieties of S[I]RLs with distinguished extrema; its proof requires no essential alteration.[5] In this form, it generalizes the finding that every variety of Heyting algebras of finite depth has surjective epimorphisms [11, Thm. 5.3].

In the same paper, Corollary 6.26 is used to prove that epimorphisms are surjective in every variety of semilinear Brouwerian algebras (i.e., every variety of relative Stone algebras) [11, Cor. 5.7]. It therefore follows immediately from the category equivalence mentioned at the end of Section 6.2 that every variety of odd Sugihara monoids has the ES property. More generally, the next theorem is proved in [11] using similar categorical methods.

Theorem 6.27 ([11, Thm. 8.5])**.** *Every variety of Sugihara monoids has surjective epimorphisms.*

The same is true of all varieties of *positive* Sugihara monoids (i.e., RL-subreducts of Sugihara monoids) [11, Thm. 8.6]. For the finitely generated varieties of these kinds, the surjectivity of epimorphisms could alternatively be deduced (immediately) from Theorem 6.22. (Note that [positive] Sugihara monoids are always negatively generated, as they satisfy equation (2.39).)

[5] More precisely, the (slight) alterations needed are to definitions in the underlying duality theory (see page 186), not to the proof strategy of Theorem 6.22 itself.

6.6 Semilinearity and epimorphisms

We proved in the previous section that the ES property holds in all varieties of S[I]RLs whose FSI members are negatively generated and have finite depth (Theorem 6.22). A number of varieties of S[I]RLs that have infinite depth are known to have the ES property. We have already mentioned, for example, BRA, RSA, SM, OSM and the variety of positive Sugihara monoids (see Theorem 6.5 and the end of the last section). All these varieties consist of negatively generated idempotent algebras, and (with the exception of BRA) they all consist of semilinear algebras. In this section, we shall establish more general sufficient conditions for the surjectivity of epimorphisms, encompassing the semilinear cases just mentioned.

We begin this section with a representation theorem (from [59]) for totally ordered idempotent RLs (Theorem 6.28 below), and a characterization of homomorphisms between algebras of this kind (Theorem 6.32). These results will be used to prove that epimorphisms are surjective in a number of interesting varieties of semilinear idempotent RLs. (The idempotence assumption will be relaxed in the following section, where involutive algebras, such as semilinear De Morgan monoids, are also considered.)

Let $\boldsymbol{A}$ be a totally ordered idempotent RL. (Recall that x^* was defined to be $x \to e$ on page 41.) Then

$$A^{**} := \{a^{**} \colon a \in A\}$$

is the universe of a subalgebra $\boldsymbol{A}^{**}$ of $\boldsymbol{A}$ which, moreover, is termwise equivalent to a (totally ordered) odd Sugihara monoid, where $\neg x := x^*$ [59, Lem. 3.3, Prop. 3.4]. For every $c \in A^{**}$, the set

$$A_c := \{a \in A \colon a^{**} = c\}$$

is an interval of $\boldsymbol{A}$ with greatest element c [59, Prop. 3.4]. For any $\boldsymbol{A}$ as above, we define

$$\mathcal{A} := \{\langle A_c; \leqslant|_{A_c}\rangle \colon c \in A^{**}\}.$$

Let $\boldsymbol{S}$ be a totally ordered odd Sugihara monoid and let

$$\mathcal{X} = \{\langle X_c; \leqslant_c\rangle \colon c \in S\}$$

be an S-indexed family of disjoint chains such that each $c \in S$ is the greatest element of X_c. For all $a, b \in S$ with $x \in X_a$ and $y \in X_b$, we define

$$x \preccurlyeq y \text{ iff } a < b \text{ or } (a = b \text{ and } x \leqslant_a y).$$

Thus, $\preccurlyeq$ is the lexicographic total order on $S \otimes \mathcal{X} := \bigcup\{X_c \colon c \in S\}$.

We let $\wedge$ and $\vee$ denote the meet and join operations for $\preccurlyeq$ and define

$$\boldsymbol{S} \otimes \mathcal{X} := \langle S \otimes \mathcal{X}; \wedge, \vee, \cdot, \to, e \rangle,$$

where for $a, b \in S$ and $x \in X_a$, $y \in X_b$,

$$x \cdot y = \begin{cases} x \wedge y & \text{if } a = b \leqslant e \\ x \vee y & \text{if } e < a = b \\ x & \text{if } a \neq b \text{ and } a \cdot^{\boldsymbol{S}} b = a \\ y & \text{if } a \neq b \text{ and } a \cdot^{\boldsymbol{S}} b = b \end{cases} \quad \text{and} \quad x \to y = \begin{cases} a^* \vee y & \text{if } x \leqslant y \\ a^* \wedge y & \text{if } y < x \end{cases}.$$

Recall that $a \cdot b \in \{a, b\}$ for all elements a, b of the Sugihara monoid $\boldsymbol{Z}^*$ (see page 36). As this property is expressible as a positive universal sentence, it holds for every totally ordered Sugihara monoid, by Jónsson's Theorem 1.23. This shows why the definition of $\cdot$ above is exhaustive. The following representation theorem for totally ordered idempotent RLs from [59] has an antecedent in [116].

Theorem 6.28 ([59, Thm. 3.5])**.** *For $\boldsymbol{S}$ and $\mathcal{X}$ as above, the algebra $\boldsymbol{S} \otimes \mathcal{X}$ is a totally ordered idempotent RL satisfying $\boldsymbol{S} = (\boldsymbol{S} \otimes \mathcal{X})^{**}$ and $(S \otimes \mathcal{X})_c = X_c$ for every $c \in S$. Moreover, every totally ordered idempotent RL has this form. In particular, $\boldsymbol{A} = \boldsymbol{A}^{**} \otimes \mathcal{A}$, for any totally ordered idempotent RL $\boldsymbol{A}$ (where $\mathcal{A}$ is as on page* 150).

Recall that the variety GSM of generalized Sugihara monoids was introduced in Definition 2.37 and characterized in Theorem 2.38.

Corollary 6.29 ([56])**.** *A totally ordered idempotent RL $\boldsymbol{A}$ is a generalized Sugihara monoid iff $A_c = \{c\}$ for every $c > e$.*

Proof. ($\Rightarrow$): Let $e < c \in A$. As $\boldsymbol{A} \in \mathsf{GSM}$, we have $c^{**} = c$, and therefore $c \in A_c$. Now let $d \in A_c$, i.e., $d^{**} = c$, so $d^* = d^{***} = c^* \leqslant e$. We must show that $d = c$. If $d \leqslant e$, then $e \leqslant d^*$, so $d^* = e$, whence $c = d^{**} = e$, a contradiction. Consequently, $e < d$. Then, since $\boldsymbol{A} \in \mathsf{GSM}$, we have $d = d^{**} = c$, as required.

($\Leftarrow$): Suppose $A_c = \{c\}$ whenever $e < c \in A$. To see that $\boldsymbol{A} \in \mathsf{GSM}$, let $e \leqslant a \in A$. If $a = e$, then $a^{**} = e = a$, so assume that $e < a$. Then $e < a^{**}$ (because $a \leqslant a^{**}$), so $A_{a^{**}} = \{a^{**}\}$, by assumption. But $a \in A_{a^{**}}$, so $a = a^{**}$, as required. □

The next two lemmas will assist in proving a characterization of homomorphisms between totally ordered idempotent RLs (Theorem 6.32).

Lemma 6.30 ([111, Prop. 2.5])**.** *Let $\boldsymbol{A}$ be a totally ordered idempotent RL and let $F \in \mathrm{Fil}(\boldsymbol{A})$. If a and b are* distinct *elements of A such that $a/F = b/F$, then $b/F = e/F$.*

Proof. By definition, $a/F = b/F$ means that $a \to b,\ b \to a \in F$. By semilinearity and symmetry, we may assume that $a < b$. Then $e \nleqslant b \to a$, by (2.15), so $b \to a < e$, because $\boldsymbol{A}$ is totally ordered. It follows from (2.14) that $b \to (b \to a) \leqslant b \to e$. Now, $b \to (b \to a) = (b \cdot b) \to a = b \to a$, by (2.9) and the fact that b is idempotent. Therefore $b \to a \leqslant b \to e$, whence $b \to e \in F$, because $b \to a \in F$. On the other hand, $b \nleqslant b \to a$, because otherwise $b = b \cdot b \leqslant a$. So, $b \to a < b$, since $\boldsymbol{A}$ is totally ordered. As $b \to a \in F$, we have $e \to b = b \in F$, so $b/F = e/F$. □

Recall that any congruence class of an algebra with a lattice reduct is an interval; specifically, if F is a deductive filter of an [I]RL $\boldsymbol{A}$, then the set $e/F = \{a \in A \colon e \to a,\ a \to e \in F\} = \{a \in A \colon a, a^* \in F\}$ is an interval subuniverse of $\boldsymbol{A}$ (see [66] or [51, Thm. 4.47]). When $\boldsymbol{A}$ is totally ordered, then e/F is the convex closure of $\{a \colon e \geqslant a \in F\} \cup \{a^* \colon e \geqslant a \in F\}$, because if $e < a \in A$, then $a \to e < e$ and $a \leqslant (a \to e)^*$.

Lemma 6.31. *Let $\boldsymbol{A}$ be a totally ordered idempotent RL and let I be an interval of $\boldsymbol{A}$, containing e, that is closed under *. Define*

$$I_* := \{a \in A \colon a \notin I \text{ and } a^* \in I\}.$$

(i) *I is a subuniverse of $\boldsymbol{A}$;*

(ii) *$I_* \cap A^{**} = \emptyset$;*

(iii) *Every element of I_* is strictly below every element of I;*

(iv) *If $b \in I_*$ then b^* is the greatest element of I;*

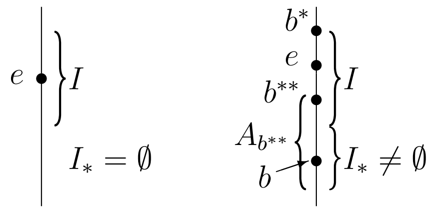
(v) *$I \cup I_*$ is an interval of $\boldsymbol{A}$ that is closed under *.*

Proof. Item (i) follows from the fact that $\{a \cdot b, a \to b\} \subseteq \{a, b, a^*, b^*\}$ for any $a, b \in I$, by Theorem 2.36 (and the fact the $e \in I$ by assumption). Item (ii) holds, because otherwise $a^{**} \in I_*$ for some $a \in A$, but then $a^{**} = a^{****} \in I$, a contradiction.

Let $b \in I_*$. Then $b^* \in I$ and thus $b^{**} \in I$. Suppose, with a view to contradiction, that $a \leqslant b$ for some $a \in I$. Then $a \leqslant b \leqslant b^{**}$, by (2.32). Since I is an interval, $b \in I$, a contradiction. Therefore, (iii) holds.

For (iv), suppose $a > b^*$ for some $a \in I$. If $b \leqslant a^*$, then $a \leqslant a^{**} \leqslant b^*$, contrary to the supposition, so $a^* < b \leqslant b^{**}$. Then $b \in I$, a contradiction.

To show (v), notice that $I \cup I_*$ is clearly closed under *, so it remains to show that $I \cup I_*$ is an interval. If $I_* = \emptyset$ we are done, so let b be an arbitrary element of I_*. For any $a \in I_*$, we have $a^* = b^*$, by (iv), so $a \in A_{b^{**}}$. It follows that $I \cup I_*$ is the union of the overlapping intervals I and $A_{b^{**}}$. □



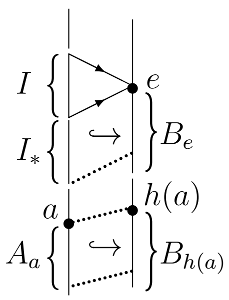
Theorem 6.32. *Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be totally ordered idempotent RLs. A map $h\colon A \to B$ is a homomorphism from $\boldsymbol{A}$ to $\boldsymbol{B}$ iff the following holds:*

(i) *The set $I = h^{-1}[\{e\}]$ is an interval of $\boldsymbol{A}$, which contains e and is closed under *.*

(ii) *h is an order embedding from I_* into $B_e \setminus \{e\}$.*

(iii) *h is an order embedding from $A^{**} \setminus I$ into $B^{**} \setminus \{e\}$, preserving *.*

(iv) *For every $a \in A^{**} \setminus I$, h is an order embedding from A_a into $B_{h(a)}$.*



Proof. Suppose $h\colon \boldsymbol{A} \to \boldsymbol{B}$ is a homomomorphism. The congruence $\theta := \ker h$ corresponds to a deductive filter F of $\boldsymbol{A}$, so that $\boldsymbol{A}/\theta = \boldsymbol{A}/F$. Let $I = h^{-1}[\{e\}]$. Then $I = e/F$, which we have already noted is an interval and a subuniverse of $\boldsymbol{A}$. In particular, I is closed under * and contains e.

Let $a, b \in A \setminus I$ such that $a \neq b$. Then $h(a) \neq h(b)$, because otherwise $a/F = b/F$, which would imply that $b/F = e/F$, by Lemma 6.30, i.e., that $h(b) = h(e) = e$, contradicting $b \notin I$. Therefore, h is injective outside of I.

By [59, Lem. 3.3], h restricts to a homomorphism $h|_{A^{**}}\colon \boldsymbol{A}^{**} \to \boldsymbol{B}^{**}$, which in particular preserves *. Then (iii) holds, because $h(a) \neq e$ for any $a \in A^{**} \setminus I$.

For any $a \in I_*$, we have $a^* \in I$, so $e = e^* = h(a^*)^* = h(a)^{**}$ and $h(a) \neq e$. Therefore, $h[I_*] \subseteq B_e \setminus \{e\}$, so (ii) holds. For any $a \in A^{**} \setminus I$ and $x \in A_a$, we have $x^{**} = a = a^{**}$, so $h(x)^{**} = h(a)^{**}$. Therefore, $h[A_a] \subseteq B_{h(a)}$, so (iv) holds.

Conversely, let h be as in the theorem and let $I = h^{-1}[\{e\}]$. By Theorem 6.28, $\boldsymbol{A} = \boldsymbol{A}^{**} \otimes \mathcal{A}$ and $\boldsymbol{B} = \boldsymbol{B}^{**} \otimes \mathcal{B}$, so the families $\{A_a \colon a \in A^{**}\}$ and $\{B_b \colon b \in B^{**}\}$ are partitions of A and B, respectively. Note that $I \cup I_* = \bigcup \{A_a \colon a \in I \cap A^{**}\}$, because for each $c \in A$, we have $c \in I \cup I_*$ iff $c^{**} \in I$ (using (2.32)). So, the sets I, I_*, and A_a ($a \in A^{**} \setminus I$) form a partition of A. It follows from properties (i)–(iv) that h is injective outside of I, and that h preserves order (and hence the lattice operations), in view of the definitions of $\boldsymbol{A}^{**} \otimes \mathcal{A}$ and $\boldsymbol{B}^{**} \otimes \mathcal{B}$.

Let $a \in A$. If $a \in I \cup I_*$ then $a^* \in I$, so $h(a^*) = e = h(a)^*$, by (i) and (ii). If $a \in A \setminus (I \cup I_*)$, then $a^{**} \in A^{**} \setminus I$. From (iii) and (iv), it follows that $h(a^{***}) = h((a^{**})^*) = h(a^{**})^*$ and $h(a) \in h[A_{a^{**}}] \subseteq B_{h(a^{**})}$, i.e., $h(a)^{**} = h(a^{**})$. So, $h(a^*) = h(a^{***}) = h(a^{**})^* = h(a)^{***} = h(a)^*$. Therefore, h preserves *.

For $a, b \in A$, if we consider the characterization of $a \to b$ in Theorem 2.36, the preservation of $\to$ follows from the preservation of $\wedge$, $\vee$ and *, except when $a > b$ but $h(a) = h(b)$. In this situation $a, b \in I$, because h is injective outside of I, so $a \to b \in I$, by Lemma 6.31(i), whence

$$h(a) \to h(b) = e \to e = e = h(a \to b).$$

Therefore, h also preserves $|\text{-}|$.

Again, by Theorem 2.36, since h preserves $\vee$, $\wedge$ and $|\text{-}|$, to show preservation of $\cdot$, we need only consider cases where $|a| > |b|$ but $|h(a)| = |h(b)|$ (i.e., $h(|a|) = h(|b|)$) for some $a, b \in A$. Then $|a|, |b| \in I$, so $a, b \in I \cup I_*$, because for any $c \in A$, $|c| \in \{c, c^*\}$, by Theorem 2.36. We have $h(a \cdot b) = h(a)$, since $a \cdot b = a$, and $h(a) \cdot h(b) = h(a) \wedge h(b)$, so it remains to show that $h(a) \leqslant h(b)$.

Note that we cannot have $b \in I_*$, because in this case $b \leqslant e$, by Lemma 6.31(iii), so that $|b| = b^* \geqslant |a|$, by Lemma 6.31(iv), since $|a| \in I$. If $a, b \in I$, then $h(a) = e = h(b)$. Lastly, if $a \in I_*$ and $b \in I$, then $a \leqslant e$, so $h(a) \leqslant h(e) = e = h(b)$. □

Theorem 6.33. *Epimorphisms are surjective in the variety of all idempotent semilinear RLs.*

Proof. Let $\boldsymbol{B}$ be a proper subalgebra of a totally ordered idempotent RL $\boldsymbol{A}$. Let $a \in A \setminus B$. We shall show that $\boldsymbol{B}$ is not epic in $\boldsymbol{A}$ by constructing a totally ordered idempotent RL $\boldsymbol{C}$ and two homomorphisms from $\boldsymbol{A}$ into

$\boldsymbol{C}$ which agree on $\boldsymbol{B}$ but differ at a. It then follows from Theorem 6.9 that the variety of all idempotent semilinear RLs has the ES property. We split into two cases: $a \in A^{**}$ and $a \notin A^{**}$.

First suppose that $a \in A^{**}$. Without loss of generality, $a < e$. Indeed, if $e \leqslant a$, then $e \geqslant a^* = a^{***} \in A^{**}$; moreover, $a^* \notin B$, because otherwise, $a = a^{**} \in B$. Then $F := \{b \in A \colon b > a\}$ is a deductive filter of $\boldsymbol{A}$. Let q be the canonical surjection from $\boldsymbol{A}$ to the totally ordered algebra $\boldsymbol{C} := \boldsymbol{A}/F$. Consider the set

$$I := [a, a^*] \cup [a, a^*]_* = [a, a^*] \cup \{x \in A \colon x \notin [a, a^*] \text{ and } x^* \in [a, a^*]\}.$$

Define the map $h \colon A \to A/F$ by

$$h(x) = \begin{cases} e^{\boldsymbol{C}} & \text{if } x \in I \\ q(x) & \text{otherwise.} \end{cases}$$

Note that $h(a) = e^{\boldsymbol{C}}$, since $a \in I$, so $h(a) \neq q(a)$ (because, if $a \in e/F$ then $a = e \to a \in F$, which is not the case). We now show that h is a homomorphism. As $[a, a^*]$ is an interval containing e that is closed under * (because $a \in A^{**}$), the same of true of I, by Lemma 6.31(v). Furthermore, $h^{-1}[\{e^{\boldsymbol{C}}\}] = I$, because $q^{-1}[\{e^{\boldsymbol{C}}\}] = e/F \subseteq \{b \in A \colon a < b \leqslant a^*\} \subseteq I$. So, condition (i) of Theorem 6.32 holds. Note that $a^* = \max I$, by Lemma 6.31(iii).

If $b \in I_*$ then $b \notin I$ and b^* is the greatest element of I, by Lemma 6.31(iv), so $b^* = a^*$. But, since $b \notin [a, a^*]$ and $b^* \in [a, a^*]$, we have $b \in [a, a^*]_*$, a contradiction. So, $I_* = \emptyset$, and condition (ii) of Theorem 6.32 is vacuously satisfied.

As q is a homomorphism between totally ordered idempotent RLs, Theorem 6.32 applies to q. In particular, the following conditions hold:

(iii) q is an order embedding from $A^{**} \setminus (e/F)$ into $C^{**} \setminus \{e\}$, preserving *;

(iv) for every $a \in A^{**} \setminus (e/F)$, q is an order embedding from A_a into $C_{q(a)}$.

As $e/F \subseteq I$, conditions (iii) and (iv) also hold for h. So, h is a homomorphism, by Theorem 6.32.

To show that $h|_B = q|_B$, we let $b \in B \cap I$ and prove that $q(b) = e^{\boldsymbol{C}}$, i.e., that $b, b^* \in F$. Note that $b^* \in [a, a^*]$. If $b^* \notin F$ then $b^* \leqslant a$ by the definition of F, so $a = b^* \in B$, a contradiction. Therefore $b^* \in F$, as required. Suppose that $b \in [a, a^*]_*$. By Lemma 6.31(iv), $b^* = a^*$, so $a = a^{**} = b^{**} \in B$, a contradiction. So, $b \in [a, a^*]$. Since $a \neq b$, we get $a < b$, i.e., $b \in F$.

Now suppose that $a \notin A^{**}$. Let $\boldsymbol{A}'_s = \langle A_s; \leqslant|_{A_s} \rangle$ whenever $a^{**} \neq s \in A^{**}$. Define $\boldsymbol{A}'_{a^{**}} = \langle A'_{a^{**}}; \leqslant' \rangle$ where $A'_{a^{**}} = A_{a^{**}} \cup \{c\}$ for some fresh element $c \notin A$ and $\leqslant'$ is the total order on $A'_{a^{**}}$ that extends $\leqslant|_{A_{a^{**}}}$ with $c <' a$ and $b <' c$ whenever $a > b \in A_{a^{**}}$. Let $\mathcal{A}' = \{\boldsymbol{A}'_s : s \in A^{**}\}$ and $\boldsymbol{C} = \boldsymbol{A}^{**} \otimes \mathcal{A}'$. By Theorem 6.28, $\boldsymbol{C}$ is a totally ordered idempotent RL. By Theorem 6.32, the inclusion map $i\colon \boldsymbol{A} \to \boldsymbol{C}$ is a homomorphism, and so is the map

$$h\colon x \mapsto \begin{cases} c & \text{if } x = a \\ x & \text{otherwise.} \end{cases}.$$

Note that h and i differ only at a, so $h|_B = i|_B$. □

Definition 6.34.

(i) The *strong ES property* for a class K of algebras asks that, whenever $\boldsymbol{A}$ is a subalgebra of $\boldsymbol{B} \in \mathsf{K}$ and $b \in B \setminus A$, then there exist $\boldsymbol{C} \in \mathsf{K}$ and homomorphisms $g, h\colon \boldsymbol{B} \to \boldsymbol{C}$ such that $g|_A = h|_A$ and $g(b) \neq h(b)$.

(ii) The *amalgamation property* for a variety $\boldsymbol{K}$ is the demand that, for any two embeddings $g_B\colon \boldsymbol{A} \to \boldsymbol{B}$ and $g_C\colon \boldsymbol{A} \to \boldsymbol{C}$ between algebras in K, there exist embeddings $h_B\colon \boldsymbol{B} \to \boldsymbol{D}$ and $h_C\colon \boldsymbol{C} \to \boldsymbol{D}$, with $\boldsymbol{D} \in \mathsf{K}$, such that $h_B \circ g_B = h_C \circ g_C$.

(iii) The *strong amalgamation property* for K asks, in addition to the demands of (ii), that $\boldsymbol{D}$, h_B and h_C can be chosen so that $(h_B \circ g_B)[A] = h_B[B] \cap h_C[C]$.

These conditions are linked as follows (see [71, 125, 76] and [68, Sec. 2.5.3]).

Theorem 6.35. *A variety has the strong amalgamation property iff it has the amalgamation property and the weak ES property. In that case, it also has the strong ES property.*

It was recently shown in [59, Thm. 6.6] that the variety of semilinear idempotent RLs has the amalgamation property. Combining this observation with Theorems 6.33 and 6.35, we obtain:

Corollary 6.36. *The variety of semilinear idempotent RLs has the strong amalgamation property and hence the strong ES property.*

Note that the proof of Theorem 6.33 essentially showed that the class of *totally ordered* idempotent RLs has the strong ES property. Nevertheless, we cannot deduce from this fact alone that the whole variety of *semilinear* idempotent RLs has the strong ES property, because there is no analogue of Theorem 6.9 for the strong ES property. For instance, the strong ES property holds for $\mathsf{L}^4{}_{\mathrm{FSI}}$ but fails for L^4, where L^4 is the variety generated by the four-element totally ordered Brouwerian algebra. In fact, Maksimova showed in [88, Thm. 4.3] that there are just six nontrivial varieties of Brouwerian algebras with the strong ES property, only three of which are semilinear, namely the variety of all relative Stone algebras and the varieties generated, respectively, by the two-element and three-element relative Stone algebras.

The fact that $\mathsf{L}^4{}_{\mathrm{FSI}}$ has the strong ES property can be deduced from the proof of the next theorem.

Not all varieties of semilinear idempotent RLs have the ES property, as we shall see in Example 6.40. But the following theorem shows that epimorphisms are surjective in all varieties of *negatively generated* semilinear idempotent RLs, i.e., all varieties of generalized Sugihara monoids (even those with infinite depth). This theorem strengthens [56, Thm. 13.1], which states that every variety of generalized Sugihara monoids has the *weak* ES property, and it also strengthens [11, Thm. 8.9], which shows that the ES property holds in all varieties of positive Sugihara monoids.

Theorem 6.37. *All varieties of generalized Sugihara monoids have surjective epimorphisms.*

Proof. Assume, with a view to contradiction, that K is a subvariety of GSM without the ES property. Then, by Theorem 6.9, there exists $\boldsymbol{A} \in \mathsf{K}_{\mathrm{FSI}}$ (i.e., a totally ordered $\boldsymbol{A} \in \mathsf{K}$) with a proper K-epic subalgebra $\boldsymbol{B}$.

Since $\boldsymbol{A}$ is negatively generated, there exists $a \in A^- \setminus B$, so $a < e$. So, as in Theorem 6.33, $F := \{b \in A \colon a < b\}$ is a deductive filter of $\boldsymbol{A}$. Let $\boldsymbol{C} := \boldsymbol{A}/F$, and let $q \colon \boldsymbol{A} \to \boldsymbol{C}$ be the canonical surjection. Note that $\boldsymbol{C}$ is totally ordered and $\boldsymbol{C} \in \mathsf{K}$, because K is a variety.

Recall that $a \leqslant a^{**}$. If $a = a^{**}$, then $a \in A^{**}$, so we can use the first homomorphism in the proof of Theorem 6.33, to show that $\boldsymbol{B}$ is not K-epic in $\boldsymbol{A}$, a contradiction.

So, we may suppose that $a < a^{**}$. In this case, define $h \colon A \to C$ by

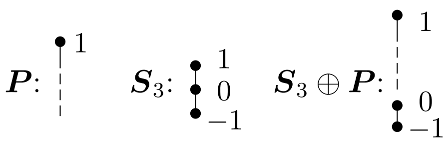
$$h(x) = \begin{cases} e^{\boldsymbol{C}} & \text{if } x = a \\ q(x) & \text{otherwise.} \end{cases}$$

Then $h^{-1}[\{e^{\boldsymbol{C}}\}] = (e/F) \cup \{a\}$. We claim that $(e/F) \cup \{a\} = [a, a^*]$, which is clearly an interval of $\boldsymbol{A}$ containing e that is closed under *. If $b \in [a, a^*]$ and $b \neq a$, we must show that $b/F = e/F$, i.e., that $a < b, b^*$. Clearly $a < b$ and $a \leqslant b^*$. If $a = b^*$, then $a^{**} = b^{***} = b^* = a$, contradicting the assumption that $a < a^{**}$. So, $a < b^*$, as required.

Because q satisfies conditions (ii)–(iv) of Theorem 6.32, and $q^{-1}[\{e^{\boldsymbol{C}}\}] = e/F \subseteq h^{-1}[\{e^{\boldsymbol{C}}\}]$, it is easy to see that h satisfies the conditions of Theorem 6.32. So, h is a homomorphism. Clearly, $h|_B = q|_B$, but $h(a) \neq q(a)$. Therefore, $\boldsymbol{B}$ is not K-epic in $\boldsymbol{A}$, a contradiction. □

Every variety K of RLs with the ES property exhibited thus far has at least one of the following two properties: (i) K is generated by algebras that are negatively generated (as in Theorems 6.22 and 6.37), or (ii) K has infinite depth (as in Theorem 6.33). In the next theorem we supply an example of a variety with surjective epimorphisms which satisfies neither (i) nor (ii).

Let $\mathbf{2}^+$ denote the two-element Brouwerian algebra. Recall that the three-element Sugihara monoid $\boldsymbol{S}_3$ has universe $\{-1, 0, 1\}$. For any chain $\boldsymbol{P}$ with greatest element 1, we abbreviate $\boldsymbol{S}_3 \otimes \{\{-1\}, \{0\}, \boldsymbol{P}\}$ as $\boldsymbol{S}_3 \oplus \boldsymbol{P}$.



Lemma 6.38 ([110, Thm. 3.7])**.** *A semilinear idempotent RL is simple iff it is isomorphic to* $\mathbf{2}^+$ *or* $\boldsymbol{S}_3 \oplus \boldsymbol{P}$ *for some chain* $\boldsymbol{P}$ *with top element* 1.

Let S be the class of all simple totally ordered idempotent RLs.

Theorem 6.39. *Epimorphisms are surjective in* $\mathbb{V}(\mathsf{S})$.

Proof. We use the same strategy as in Theorems 6.33 and 6.37. Let $\boldsymbol{A} \in \mathbb{V}(\mathsf{S})_{\mathrm{FSI}}$ and $\boldsymbol{B}$ a proper subalgebra of $\boldsymbol{A}$. It follows that $\boldsymbol{A}$ is nontrivial. By Corollary 2.17, $\boldsymbol{A}$ is simple. So, by Lemma 6.38, $\boldsymbol{A}$ is isomorphic to $\mathbf{2}^+$ or $\boldsymbol{S}_3 \oplus \boldsymbol{P}$ for some chain $\boldsymbol{P}$ with greatest element 1.

In the first case, $B = \{e\}$. The identity map from $\boldsymbol{A}$ to itself, and the map sending $\boldsymbol{A}$ onto the trivial subalgebra of $\boldsymbol{A}$, are two different homomorphisms that agree on B. So, $\boldsymbol{B}$ is not $\mathbb{V}(\mathsf{S})$-epic in $\boldsymbol{A}$.

We may therefore suppose that $\boldsymbol{A} = \boldsymbol{S}_3 \oplus \boldsymbol{P}$. If $\boldsymbol{B}$ is trivial, we are done, as in the previous paragraph. So, we may assume that $\boldsymbol{B}$ is nontrivial, in which case $S_3 \subseteq B$. Let $c \in A \setminus B$. Then $c \in P \setminus \{1\}$.

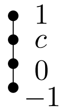
As in Theorem 6.33, let $P' = P \cup \{d\}$ for some fresh element $d \notin A$ and extend the total order of $\boldsymbol{P}$ to $\boldsymbol{P}'$ by defining d to be the immediate predecessor of c. By Theorem 6.32, the inclusion map i from $\boldsymbol{S}_3 \oplus \boldsymbol{P}$ to $\boldsymbol{S}_3 \oplus \boldsymbol{P}' \in \mathsf{S}$ is a homomorphism, and the map

$$h \colon x \mapsto \begin{cases} d & \text{if } x = c \\ x & \text{otherwise} \end{cases}$$

is also homomorphism, differing from i only at c, so that $h|_B = i|_B$. Thus, $\boldsymbol{B}$ is not $\mathbb{V}(\mathsf{S})$-epic in $\boldsymbol{A}$. □

We now exhibit a subvariety of $\mathbb{V}(\mathsf{S})$ which does not have the ES property. Let $\mathit{2}$ be the two-element chain with elements $c < 1$.

$\boldsymbol{S}_3 \oplus \mathit{2}$:



Example 6.40. $\mathbb{V}(\boldsymbol{S}_3 \oplus \mathit{2})$ does not have the ES property.

Proof. We show that $\boldsymbol{S}_3$ is an epic subalgebra of $\boldsymbol{S}_3 \oplus \mathit{2}$.

Let $g, h \colon \boldsymbol{S}_3 \oplus \mathit{2} \to \boldsymbol{C}$ be two homomorphisms into $\boldsymbol{C} \in \mathbb{V}(\boldsymbol{S}_3 \oplus \mathit{2})_{SI}$ such that $g|_{S_3} = h|_{S_3}$. By Lemma 6.10, it suffices to show that $g = h$.

Since $\boldsymbol{S}_3 \oplus \mathit{2}$ is simple, and g and h agree on a non-neutral element, g and h are either both embeddings or they both have range $\{e\}$. In the second case, clearly $g = h$. So, we assume that g and h are embeddings.

By Jónsson's theorem 1.23, $\boldsymbol{C} \in \mathbb{HSP}_{\mathbb{U}}(\boldsymbol{S}_3 \oplus \mathit{2})$. Since $\boldsymbol{S}_3 \oplus \mathit{2}$ is finite and simple, $\boldsymbol{C}$ is isomorphic to $\boldsymbol{S}_3$ or to $\boldsymbol{S}_3 \oplus \mathit{2}$. Since g and h are embeddings, the first case is ruled out on cardinality grounds, so $\boldsymbol{C} \cong \boldsymbol{S}_3 \oplus \mathit{2}$. But then $g = h$ because $\boldsymbol{S}_3 \oplus \mathit{2}$ has no nontrivial automorphism. □

Note that $\boldsymbol{S}_3 \oplus \mathit{2}$ is not negatively generated (as the subuniverse generated by $\{-1, 0\}$ excludes c). Also, since $\boldsymbol{S}_3 \oplus \mathit{2}$ is finite and has a proper $\mathbb{V}(\boldsymbol{S}_3 \oplus \mathit{2})$-epic subalgebra, $\mathbb{V}(\boldsymbol{S}_3 \oplus \mathit{2})$ fails to have even the *weak* ES property, by Theorem 6.7.

We now relax the condition of idempotence and consider varieties of semilinear RLs that are merely square-increasing.

Recall from Theorem 2.41 that the class of negatively generated semilinear Dunn monoids coincides with the variety of generalized Sugihara monoids. The following is therefore a paraphrase of Theorem 6.37.

Corollary 6.41. *Let* D *be a variety of negatively generated semilinear Dunn monoids. Then* D *has surjective epimorphisms.*

The variety of *all* semilinear Dunn monoids does not have the ES property, however. This is illustrated by the following examples.

Theorem 6.42. *Let* K *be a variety of semilinear Dunn or De Morgan monoids containing a totally ordered algebra* $\boldsymbol{A}$ *which is generated by some* $a \in A$ *that satisfies* $a = a^n \to a^{n+1}$ *for some positive integer* n*, such that* a^{n+1} *generates a proper subalgebra of* $\boldsymbol{A}$*. Then* K *does not have the ES property.*

Proof. We show that $\boldsymbol{B} = \mathbf{Sg}^{\boldsymbol{A}}\{a^{n+1}\}$ is a proper K-epic subalgebra of $\boldsymbol{A}$. Let $h, g \colon \boldsymbol{A} \to \boldsymbol{C}$ be two different homomorphisms that agree at a^{n+1}. Because of Lemma 6.10, we may suppose that $\boldsymbol{C} \in \mathsf{K}_{\mathrm{SI}}$, so $\boldsymbol{C}$ is a totally ordered algebra. Note that $h(a) \neq g(a)$, because $\boldsymbol{A}$ is generated by a. Since $\boldsymbol{C}$ is totally ordered, we may suppose by symmetry that $h(a) < g(a)$. Note that $h(a^n) = h(a)^n \leqslant g(a)^n = g(a^n)$. Then

$$g(a) \cdot h(a^n) \leqslant g(a) \cdot g(a^n) = g(a^{n+1}) = h(a^{n+1}).$$

But then $g(a) \leqslant h(a^n) \to h(a^{n+1}) = h(a^n \to a^{n+1}) = h(a)$, by the law of residuation (2.2), a contradiction. So, $h = g$. □

By Theorem 6.7, none of these varieties has even the weak ES property, because the algebra in Theorem 6.42 is finitely generated.

Recall from Example 4.25 that for every positive integer p, we defined a totally ordered De Morgan monoid called $\boldsymbol{A}_p$ on the chain $0 < 1 < 2 < \cdots < 2^{p+1}$, where fusion is multiplication, truncated at 2^{p+1}. For each $p > 2$, $\boldsymbol{A}_p$ is generated by 2, and satisfies $2 = 2^{p-1} \to 2^p$. Furthermore, $f = 2^p$, and the subalgebra $\mathbf{Sg}^{\boldsymbol{A}_p}\{f\}$ has universe $\{0, 1, 2^p, 2^{p+1}\}$ and is isomorphic to $\boldsymbol{C}_4$. So, $\boldsymbol{A}_p$ satisfies the conditions of Theorem 6.42 with $n = p - 1$.

Recall that for distinct primes $p, q > 2$, the algebras $\boldsymbol{A}_p, \boldsymbol{A}_q$ generate distinct covers of $\mathbb{V}(\boldsymbol{C}_4)$ in the subvariety lattice of De Morgan monoids. There are therefore infinitely many covers of $\mathbb{V}(\boldsymbol{C}_4)$ that lack the (weak) ES property and consist of semilinear algebras.

An analogous situation holds for the involution-less reducts of these algebras (except that they do not generate covers of atoms in the subvariety lattice of Dunn monoids). For every positive integer p, let $\boldsymbol{A}_p^+$ denote the Dunn monoid reduct of $\boldsymbol{A}_p$. Note that 2 still generates $\boldsymbol{A}_p^+$, and $2 = 2^p \to 2^{p+1}$. Moreover, 2^{p+1} is idempotent in $\boldsymbol{A}_p^+$, so it generates an idempotent subalgebra of $\boldsymbol{A}_p^+$, by Theorem 2.40, which must therefore be a proper subalgebra. In fact, $\mathrm{Sg}^{\boldsymbol{A}_p^+}\{2^{p+1}\} = \{0, 1, 2^{p+1}\}$. Therefore, $\boldsymbol{A}_p^+$ satisfies the conditions of Theorem 6.42, with $a = 2$ and $n = p$.

For distinct primes p and q, it follows from Jónsson's Theorem 1.23 that $\mathbb{V}(\boldsymbol{A}_{p-1}^+) \neq \mathbb{V}(\boldsymbol{A}_{q-1}^+)$, since the only proper nontrivial subalgebra of $\boldsymbol{A}_{p-1}^+$ has universe $\{0, 1, 2^p\}$, and each $\boldsymbol{A}_{p-1}^+$ is simple and finite. We therefore obtain infinitely many varieties of semilinear Dunn monoids in which the (weak) ES property fails.

6.7 Epimorphisms in semilinear varieties with involution

We shall now use the representation theorems of Section 3.5 to prove some positive ES results for semilinear varieties with involution.

The effect on epimorphisms of the reflection construction ($\boldsymbol{A} \mapsto \mathrm{R}(\boldsymbol{A})$ and $\mathsf{K} \mapsto \mathbb{R}(\mathsf{K})$; see Section 3.4) is described in the next theorem.

Theorem 6.43. *Let* K *be a variety of SRLs, let* $\boldsymbol{B}$ *be a subalgebra of* $\boldsymbol{A} \in \mathsf{K}$, *and identify* $\mathrm{R}(\boldsymbol{B})$ *with the subalgebra of* $\mathrm{R}(\boldsymbol{A})$ *given in Lemma* 3.34(i). *Then*

(i) $\boldsymbol{B}$ *is* K*-epic in* $\boldsymbol{A}$ *iff* $\mathrm{R}(\boldsymbol{B})$ *is* $\mathbb{R}(\mathsf{K})$*-epic in* $\mathrm{R}(\boldsymbol{A})$;

(ii) K *has the ES property iff* $\mathbb{R}(\mathsf{K})$ *has the ES property;*

(iii) K *has the weak ES property iff* $\mathbb{R}(\mathsf{K})$ *has the weak ES property;*

Proof. (i) ($\Rightarrow$): Let $g, h\colon \mathrm{R}(\boldsymbol{A}) \to \boldsymbol{E} \in \mathbb{R}(\mathsf{K})$ be homomorphisms that agree on $\mathrm{R}(\boldsymbol{B})$. In showing that $g = h$, we may assume that $\boldsymbol{E}$ is subdirectly irreducible (by Lemma 6.10), whence $\boldsymbol{E} = \mathrm{R}(\boldsymbol{D})$ for some $\boldsymbol{D} \in \mathsf{K}_{FSI}$, by Corollary 3.36. Since g, h preserve $e, \cdot, \neg$, they preserve $\mathit{1}$ $(= f^2)$ and $\mathit{0}$ $(= \neg(f^2))$. If $a, b \in A$, then $g(a), h(a) \neq \mathit{0}$ (otherwise, the kernel of g or h would identify $\mathit{1} = a \cdot \mathit{1}$ with $\mathit{0} \cdot \mathit{1} = \mathit{0}$), and $g(a), h(a) \neq \mathit{1}$ (because the kernels don't identify $\mathit{1} = \mathit{1} \to \mathit{1}$ with $\mathit{1} \to a = \mathit{0}$), while $g(a), h(a) \neq b'$ (because the kernels don't identify $a^2 \in A$ with $\mathit{1} = (b')^2$). Thus, $g[A], h[A] \subseteq D$, and so $g|_A, h|_A$ are homomorphisms from $\boldsymbol{A}$ to $\boldsymbol{D}$, which agree on $\boldsymbol{B}$. As $\boldsymbol{D} \in \mathsf{K}$ and $\boldsymbol{B}$ is K-epic in $\boldsymbol{A}$, we conclude that $g|_A = h|_A$. Then $g|_{A'} = h|_{A'}$, since g, h preserve $\neg$. Consequently, $g = h$.

($\Leftarrow$): Let $g, h\colon \boldsymbol{A} \to \boldsymbol{D} \in \mathsf{K}$ be homomorphisms that agree on $\boldsymbol{B}$. Then $\mathrm{R}(\boldsymbol{D}) \in \mathbb{R}(\mathsf{K})$. Let $\overline{g}, \overline{h}\colon \mathrm{R}(\boldsymbol{A}) \to \mathrm{R}(\boldsymbol{D})$ be the respective extensions of g, h, preserving $\mathit{0}, \mathit{1}$, such that $\overline{g}(a') = g(a)'$ and $\overline{h}(a') = h(a')$ for all $a \in A$. Then $\overline{g}, \overline{h}$ are homomorphisms that agree on $\mathrm{R}(\boldsymbol{B})$, so by assumption, $\overline{g} = \overline{h}$, whence $g = h$.

(ii) Obviously, $\boldsymbol{B} = \boldsymbol{A}$ iff $\mathrm{R}(\boldsymbol{B}) = \mathrm{R}(\boldsymbol{A})$. Therefore, the implication from right to left follows from (i). For the converse, use Theorem 6.9, Corollary 3.36, Lemma 3.34(i) and item (i) of the present theorem.

(iii) This is similar to the previous item, except that it also uses Theorem 6.7 and the fact that $\boldsymbol{A}$ is finitely generated iff $\mathrm{R}(\boldsymbol{A})$ is as well (see the proof of Lemma 3.37). ☐

Theorem 6.43 will be used throughout the next chapter to infer ES results for varieties of De Morgan monoids from corresponding results about varieties of Brouwerian algebras.

Recall that S is the class of all simple totally ordered idempotent RLs. It follows immediately from Theorem 6.43(ii) that $\mathbb{R}(\mathbb{V}(\mathsf{S}))$ has the ES property. It also has finite depth and its members are not all negatively generated.

Recall from Corollary 3.44 that every nontrivial variety of negatively generated semilinear *anti-idempotent* De Morgan monoids is $\mathbb{R}(\mathsf{L})$ for some variety L of generalized Sugihara monoids.

Theorem 6.44. *Let* K *be any variety of negatively generated semilinear anti-idempotent De Morgan monoids. Then* K *has surjective epimorphisms.*

Proof. We may suppose without loss of generality that K is nontrivial, so $\mathsf{K} = \mathbb{R}(\mathsf{L})$ for some variety L of generalized Sugihara monoids, by Corollary 3.44. By Theorem 6.37, L has surjective epimorphisms. But then, by Lemma 6.43(ii), $\mathbb{R}(\mathsf{L}) = \mathsf{K}$ has as well. □

Now we can strengthen Theorem 6.44 as follows:

Theorem 6.45. *Let* K *be any variety of negatively generated semilinear De Morgan monoids. Then* K *has surjective epimorphisms.*

Proof. Suppose not. By Theorem 6.9, there exists $\boldsymbol{A} \in \mathsf{K}_{\mathrm{FSI}}$ with a proper K-epic subalgebra $\boldsymbol{B}$.

Let K^{SM} be the class of all idempotent members of K. Note that K^{SM} is a variety of Sugihara monoids, and so has surjective epimorphisms, by Theorem 6.27. Therefore, $\boldsymbol{A}$ is not a Sugihara monoid. But then, by Theorem 2.57, we may suppose that $\boldsymbol{A} = \boldsymbol{S}[\boldsymbol{A}']$ for some nontrivial anti-idempotent $\boldsymbol{A}' \in \mathsf{K}$ and some odd Sugihara monoid $\boldsymbol{S}$, both totally ordered.

Let $\boldsymbol{B}'$ be the subalgebra of $\boldsymbol{A}'$ with universe $A' \cap B$. We conclude the proof by showing that $\boldsymbol{B}'$ is a proper $\mathbb{V}(\boldsymbol{A}')$-epic subalgebra of $\boldsymbol{A}'$, which will contradict the fact that, by Theorem 6.44, $\mathbb{V}(\boldsymbol{A}')$ has surjective epimorphisms.

First, we claim that $\boldsymbol{B} = \boldsymbol{S}[\boldsymbol{B}']$. Evidently $B \subseteq S[B']$. Suppose, with a view to contradiction, that $a \in S \setminus B$. Note that $a \in A$. Let $h\colon \boldsymbol{A} \to \boldsymbol{S}$ be the extension, from Theorem 2.58(i), of the homomorphism which maps $\boldsymbol{A}'$ onto the trivial algebra. Then $h(a) \notin h[B]$, by definition of h, since $a \in S$. It therefore follows from the surjectivity of h that $h[B]$ is a proper K^{SM}-epic subalgebra of $\boldsymbol{S}$ (since $\boldsymbol{B}$ is K-epic in $\boldsymbol{A}$ and compositions of epimorphisms are epimorphisms). But then K^{SM} does not have the ES property, a contradiction. This confirms that $\boldsymbol{B} = \boldsymbol{S}[\boldsymbol{B}']$.

Since $B \subsetneq A = S[A']$, it follows from the claim just proved that $B' \subsetneq A'$, so it remains only to show that $\boldsymbol{B'}$ is $\mathbb{V}(\boldsymbol{A'})$-epic in $\boldsymbol{A'}$. Let $h, g \colon \boldsymbol{A'} \to \boldsymbol{C}$ be homomorphisms into some $\boldsymbol{C} \in \mathbb{V}(\boldsymbol{A'})_{\mathrm{SI}}$ such that $h|_{B'} = g|_{B'}$. By Jónsson's Theorem 1.23, $\boldsymbol{C} \in \mathbb{HSP}_{\mathbb{U}}(\boldsymbol{A'})$. By Corollary 2.59, $\boldsymbol{S}[\boldsymbol{C}] \in \mathbb{HSP}_{\mathbb{U}}(\boldsymbol{S}[\boldsymbol{A'}]) \subseteq \mathsf{K}$. We extend h and g to homomorphisms h' and g' from $\boldsymbol{S}[\boldsymbol{A'}]$ to $\boldsymbol{S}[\boldsymbol{C}]$, as in Theorem 2.58(i). Note that $h'|_B = g'|_B$, because $B = S[B']$, and $h'|_S = g'|_S$, by construction. But then $h' = g'$, since $\boldsymbol{B}$ is K-epic in $\boldsymbol{A}$. Therefore, $h = g$, so $\boldsymbol{B'}$ is $\mathbb{V}(\boldsymbol{A'})$-epic in $\boldsymbol{A'}$, by Lemma 6.10. □

The conjunction of Corollary 6.41 and Theorem 6.45 can be expressed as follows:

Corollary 6.46. *Every variety of negatively generated semilinear S[I]RLs has surjective epimorphisms.*

Chapter 7

Some uncountability results

The variety M (which consists of De Morgan monoids) was introduced in Definition 3.16 and characterized in Corollary 3.27. In Lemma 3.38 we proved the injectivity of the reflection operator $\mathbb{R}$ which maps varieties of Dunn monoids to subvarieties of M. Since Brouwerian algebras are Dunn monoids, we concluded in Theorem 3.39 that M has $2^{\aleph_0}$ distinct subvarieties, using the fact that there are $2^{\aleph_0}$ distinct subvarieties of BRA [150] (and that every variety of countable type has at most $2^{\aleph_0}$ subvarieties).

In this chapter we shall prove a number of results of a similar kind, where we use $\mathbb{R}$ to transfer findings about varieties of Brouwerian algebras to corresponding findings about De Morgan monoids.

The first main result (Theorem 7.6) establishes the extent to which structural completeness can fail in subvarieties of M (and hence of DMM). The second (Theorem 7.23) draws analogous conclusions about the surjectivity of epimorphisms. In both cases, we prove new results about Brouwerian algebras first, using the Esakia duality summarized in Section 6.4.

7.1 Structural completeness

Recall from Section 1.2 that Heyting and Brouwerian algebras model intuitionistic propositional logic and its positive fragment, respectively. We noted in Example 5.31 that all varieties of Heyting algebras are *passively* structurally complete; the same applies to Brouwerian algebras, by Theorem 5.28, as they have trivial retracts. Citkin has determined the *hereditarily* structurally complete varieties of Heyting algebras [31] and of Brouwerian algebras [32]. There are denumerably many of each. In Section 5.2 we determined which extensions of relevance logic in its full signature ($\mathbf{R}^{\mathbf{t}}$) are modelled by PSC varieties (Theorem 5.34).

Currently, no transparent characterization of the structurally complete subvarieties of BRA (or of HA) is known. The same is true for varieties of De Morgan monoids, although here, because of Theorem 5.34, the undiagnosed SC varieties all belong to M. Moreover, we saw in Section 5.3 that the join of the six covers of the atom in the subvariety lattice of M is HSC.

In this section we contribute to these open problems by showing that there are $2^{\aleph_0}$ structurally *in*complete varieties of Brouwerian algebras. We then show that the operator $\mathbb{R}$ preserves structural incompleteness, and infer that there is a continuum of structurally incomplete varieties of De Morgan monoids, even within M. The cardinality of the set of structurally complete varieties is not known in either case, so far as we are aware.

For each positive integer n, let $\boldsymbol{K}_n$ be the pointed Esakia space whose underlying poset is the subposet of the n-th direct power of the two-element chain, consisting of the least element, the greatest element, the n atoms and the n co-atoms. Note that $\boldsymbol{K}_n$ is rooted and has depth 3. Each atom of $\boldsymbol{K}_n$ is covered by just $n-1$ co-atoms, and each co-atom covers just $n-1$ atoms. Let

$$\mathcal{K} := \text{ the power set of } \{\boldsymbol{K}_n : 3 \leq n \in \mathbb{N}\},$$

so $|\mathcal{K}| = 2^{\aleph_0}$. Kuznetsov [79] proved that there are $2^{\aleph_0}$ distinct varieties of Brouwerian algebras of depth 3, by establishing the following:

$$\text{for any } \mathsf{C}, \mathsf{D} \in \mathcal{K}, \text{ if } \mathsf{C} \neq \mathsf{D}, \text{ then } \mathbb{V}(\mathsf{C}^*) \neq \mathbb{V}(\mathsf{D}^*), \tag{7.1}$$

where C^* abbreviates $\{\boldsymbol{X}^* : \boldsymbol{X} \in \mathsf{C}\}$, i.e., the set comprising the algebraic duals of the spaces in C. (Ostensibly, [79] deals with Heyting algebras, but its argument applies equally to Brouwerian algebras.)

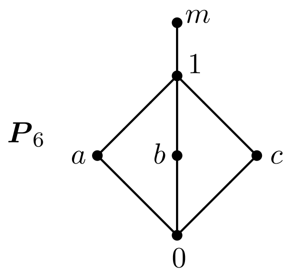
By (6.4), for any Esakia morphism $g\colon \boldsymbol{X} \to \boldsymbol{Y}$ between pointed Esakia spaces,

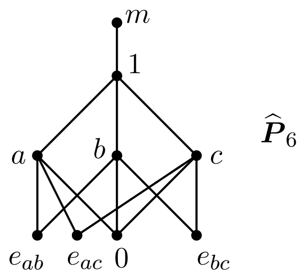
$$\text{if } x \in X \text{ has depth } n \in \omega, \text{ then } g(x) \text{ has depth at most n.} \tag{7.2}$$

Each finite $\boldsymbol{X} \in \mathsf{PESP}$ is an E-subspace of a pointed Esakia space $\widehat{\boldsymbol{X}}$, which differs from $\boldsymbol{X}$ only as follows: whenever a, b are distinct elements of depth 2 in $\boldsymbol{X}$, then $\widehat{\boldsymbol{X}}$ has a (new) element e_{ab} that has no strict lower bound; the strict upper bounds of e_{ab} are just the elements of $\uparrow^X\{a, b\}$, i.e., elements of X that are upper bounds of both a and b. (Note: e_{ab} and e_{ba} are the same element.)

Observe that if $\boldsymbol{X}$ has depth n, then so does $\widehat{\boldsymbol{X}}$, unless $n = 2$ (in which case $\widehat{\boldsymbol{X}}$ has depth 3). Also, since $X \in \text{Cpu}(\widehat{\boldsymbol{X}})$, we always have $\boldsymbol{X}^* \in \mathbb{H}(\widehat{\boldsymbol{X}}^*)$, by Lemma 6.16(ii).

The hat construction is illustrated below for a pointed Esakia space $\boldsymbol{P}_6$ that will play a role in subsequent arguments.





Lemma 7.1. *Let* $\boldsymbol{Y}, \boldsymbol{Z}, \boldsymbol{W} \in \{\boldsymbol{P}_6\} \cup \{\boldsymbol{K}_n : n \geq 3\}$, *where* $\boldsymbol{Y} \notin \{\boldsymbol{P}_6, \boldsymbol{Z}\}$. *Then* $\boldsymbol{Y}^* \notin \mathbb{SH}(\widehat{\boldsymbol{Z}}^*)$ *and* $\boldsymbol{W}^* \notin \mathbb{IS}(\widehat{\boldsymbol{Z}}^*)$.

Proof. Suppose $\boldsymbol{Y}^* \in \mathbb{SH}(\widehat{\boldsymbol{Z}}^*)$. Dualizing the injective/surjective homomorphisms, we infer from Lemma 6.16(ii) that there exist $U \in \mathrm{Cpu}(\widehat{\boldsymbol{Z}})$ and a surjective Esakia morphism $g \colon \boldsymbol{U} \to \boldsymbol{Y}$. Now $\boldsymbol{Y}$ and $\widehat{\boldsymbol{Z}}$ have depth 3, and $\boldsymbol{Y}$ has a unique element of depth 3, viz. its least element, w say. As g is surjective, $w = g(u)$ for some $u \in U$. Then u has depth 3, by (7.2). As g is an Esakia morphism (see (6.4) on page 135) and w has at least three distinct covers, each of depth 2, the same is true of u, by (7.2). This prevents u from having the form e_{xy}, so u belongs to $\boldsymbol{Z}$. As the least element of $\boldsymbol{Z}$ is its sole element of depth 3, it is u. Therefore, $Z \subseteq U$, as $U \in \mathrm{Cpu}(\widehat{\boldsymbol{Z}})$. Moreover, ${\uparrow} w = {\uparrow} g(u) = g[{\uparrow}^{\widehat{Z}} u]$, i.e., $\boldsymbol{Y} = g[\boldsymbol{Z}]$.

Then $\boldsymbol{Z} \neq \boldsymbol{P}_6$, because $\boldsymbol{Y}$ has at least eight elements, while $\boldsymbol{P}_6$ has only six. Thus, $\boldsymbol{Y}, \boldsymbol{Z}$ are distinct elements of $\{\boldsymbol{K}_n : n \geq 3\}$. As $g|_Z$ is a surjective Esakia morphism $\boldsymbol{Z} \to \boldsymbol{Y}$, the homomorphism $(g|_Z)^* \colon \boldsymbol{Y}^* \to \boldsymbol{Z}^*$ is injective, by Lemma 6.16(ii), so $\boldsymbol{Y}^* \in \mathbb{IS}(\boldsymbol{Z}^*)$, whence $\mathbb{V}(\boldsymbol{Y}^*, \boldsymbol{Z}^*) = \mathbb{V}(\boldsymbol{Z}^*)$. This contradicts (7.1), because $\boldsymbol{Y} \neq \boldsymbol{Z}$, so $\boldsymbol{Y}^* \notin \mathbb{SH}(\widehat{\boldsymbol{Z}}^*)$.

Now suppose $\boldsymbol{W}^* \in \mathbb{IS}(\widehat{\boldsymbol{Z}}^*)$. Then the situation in the first paragraph of the present proof obtains, but with $\boldsymbol{U} = \widehat{\boldsymbol{Z}}$ and $\boldsymbol{Y} = \boldsymbol{W}$. Let p, q, r be distinct covers of w in $\boldsymbol{W}$. As we saw above, u has distinct covers p', q', r' (of depth 2) that are mapped by g to p, q, r, respectively. As g is isotone, $g(e_{p'q'})$ is a common lower bound of the set $\{g(p'), g(q')\} = \{p, q\}$, so $g(e_{p'q'}) = w$. Then, because w has three distinct covers (of depth 2) in $\boldsymbol{W}$, it follows from (6.4) and (7.2) that $e_{p'q'}$ has three distinct covers (of depth 2) in $\widehat{\boldsymbol{Z}}$, but this contradicts the definitions of $e_{p'q'}$ and $\widehat{\boldsymbol{Z}}$. Thus, $\boldsymbol{W}^* \notin \mathbb{IS}(\widehat{\boldsymbol{Z}}^*)$. □

Lemma 7.2. *Let* $\mathsf{C} = \{\widehat{\boldsymbol{P}_6}\} \cup \{\widehat{\boldsymbol{Z}} \colon \boldsymbol{Z} \in \mathsf{E}\}$, *where* $\mathsf{E} \in \mathcal{K}$. *Then the variety* $\mathbb{V}(\mathsf{C}^*)$ *is structurally incomplete.*

Proof. Let $\boldsymbol{D}$ be the direct product of the members of C^*. Then $\mathbb{V}(\mathsf{C}^*) = \mathbb{V}(\boldsymbol{D})$, so it suffices to show that $\mathbb{V}(\mathsf{C}^*) \neq \mathbb{Q}(\boldsymbol{D})$. As ${\boldsymbol{P}_6}^* \in \mathbb{H}(\widehat{\boldsymbol{P}}_6^*) \subseteq \mathbb{V}(\mathsf{C}^*)$, it is enough to prove that ${\boldsymbol{P}_6}^* \notin \mathbb{Q}(\boldsymbol{D})$. Observe that

$$ {\boldsymbol{P}_6}^* \in \mathbb{Q}(\boldsymbol{D}) \text{ iff } {\boldsymbol{P}_6}^* \in \mathbb{I}\mathbb{S}\mathbb{P}_{\mathbb{U}}(\boldsymbol{D}) \text{ iff } {\boldsymbol{P}_6}^* \in \mathbb{I}\mathbb{S}(\boldsymbol{D}). $$

The first equivalence obtains because $\mathbb{Q} = \mathbb{I}\mathbb{P}_{\mathbb{S}}\mathbb{S}\mathbb{P}_{\mathbb{U}}$ and ${\boldsymbol{P}_6}^*$ is subdirectly irreducible; the second because ${\boldsymbol{P}_6}^*$ is finite and of finite type (so that having a copy of ${\boldsymbol{P}_6}^*$ as a subalgebra is a first order property, persisting under $\mathbb{P}_{\mathbb{U}}$ and $\mathbb{R}_{\mathbb{U}}$, by Łoś' Theorem 1.8). We must therefore show that ${\boldsymbol{P}_6}^* \notin \mathbb{I}\mathbb{S}(\boldsymbol{D})$.

Suppose ${\boldsymbol{P}_6}^* \in \mathbb{I}\mathbb{S}(\boldsymbol{D})$. Then ${\boldsymbol{P}_6}^* \in \mathbb{I}\mathbb{S}\mathbb{P}(\mathsf{C}^*)$. As $\mathbb{S}\mathbb{P} = \mathbb{P}_{\mathbb{S}}\mathbb{S}$, it follows (again from the subdirect irreducibility of ${\boldsymbol{P}_6}^*$) that ${\boldsymbol{P}_6}^*$ embeds into $\widehat{\boldsymbol{Z}}^*$ for some $\boldsymbol{Z} \in \{\boldsymbol{P}_6\} \cup \mathsf{E}$. This contradicts Lemma 7.1, so ${\boldsymbol{P}_6}^* \notin \mathbb{I}\mathbb{S}(\boldsymbol{D})$. □

Lemma 7.3. *The set* $\{\mathbb{V}(\mathsf{C}^*) : \mathsf{C} = \{\widehat{\boldsymbol{P}}_6\} \cup \{\widehat{\boldsymbol{Z}} : \boldsymbol{Z} \in \mathsf{E}\}$ *for some* $\mathsf{E} \in \mathcal{K}\}$ *is a continuum of varieties of Brouwerian algebras of depth* 3.

Proof. Suppose $\mathsf{E} \cup \{\boldsymbol{Y}\} \in \mathcal{K}$, where $\boldsymbol{Y} \notin \{\boldsymbol{P}_6\} \cup \mathsf{E}$. It suffices to show that $\widehat{\boldsymbol{Y}}^* \notin \mathbb{V}(\{\widehat{\boldsymbol{P}}_6^*\} \cup \{\widehat{\boldsymbol{Z}}^* : \boldsymbol{Z} \in \mathsf{E}\})$. As $\boldsymbol{Y}^* \in \mathbb{H}(\widehat{\boldsymbol{Y}}^*)$, it is enough to prove that $\boldsymbol{Y}^* \notin \mathbb{V}(\{\widehat{\boldsymbol{P}}_6^*\} \cup \{\widehat{\boldsymbol{Z}}^* : \boldsymbol{Z} \in \mathsf{E}\})$.

Let G be the class of all Brouwerian algebras $\boldsymbol{B}$ such that $\boldsymbol{Y}^* \notin \mathbb{S}\mathbb{H}(\boldsymbol{B})$. We noted before Corollary 2.17 that the variety of SRLs has EDPC. As Brouwerian algebras are SRLs, all varieties of Brouwerian algebras have EDPC. So, since $\boldsymbol{Y}^*$ is finite and subdirectly irreducible, Theorem 1.26 shows that, for any variety K of Brouwerian algebras, we have $\boldsymbol{Y}^* \notin \mathsf{K}$ iff $\mathsf{K} \subseteq \mathsf{G}$. Therefore, it remains only to confirm that $\{\widehat{\boldsymbol{P}}_6^*\} \cup \{\widehat{\boldsymbol{Z}}^* : \boldsymbol{Z} \in \mathsf{E}\} \subseteq \mathsf{G}$, i.e., that $\boldsymbol{Y}^* \notin \mathbb{S}\mathbb{H}(\widehat{\boldsymbol{P}}_6^*)$ and $\boldsymbol{Y}^* \notin \mathbb{S}\mathbb{H}(\widehat{\boldsymbol{Z}}^*)$ for all $\boldsymbol{Z} \in \mathsf{E}$. This is indeed the case, by Lemma 7.1, because $\boldsymbol{Y} \notin \{\boldsymbol{P}_6\} \cup \mathsf{E}$. □

Theorem 7.4. *The variety of Brouwerian algebras has* $2^{\aleph_0}$ *structurally incomplete subvarieties (of depth* 3*).*

Proof. Use Lemmas 7.2 and 7.3. □

Theorem 7.5. *Let* K *be a variety of Dunn monoids. If* $\mathbb{R}(\mathsf{K})$ *is structurally complete, then so is* K *(i.e.,* $\mathbb{R}$ *preserves structural* in*completeness).*

Proof. Suppose K is not SC, so $\mathsf{K} = \mathbb{H}(\mathsf{L})$ for some quasivariety $\mathsf{L} \subsetneq \mathsf{K}$. Now $\mathsf{L}^\dagger := \mathbb{I}\{\mathrm{R}(\boldsymbol{B}) : \boldsymbol{B} \in \mathsf{L}\}$ is closed under $\mathbb{S}$ and $\mathbb{P}_{\mathbb{U}}$, by Lemma 3.34(i),(iii), so $\mathbb{Q}(\mathsf{L}^\dagger) = \mathbb{I}\mathbb{P}_{\mathbb{S}}(\mathsf{L}^\dagger) \subseteq \mathbb{R}(\mathsf{K})$. As $\mathsf{L} \subsetneq \mathsf{K}$, and because all quasivarieties are closed under subdirect products, there is an algebra $\boldsymbol{A} \in \mathsf{K}_{\mathrm{SI}} \setminus \mathsf{L}$. Then $\mathrm{R}(\boldsymbol{A})$ belongs to $\mathbb{R}(\mathsf{K})$ and is subdirectly irreducible. So, if $\mathrm{R}(\boldsymbol{A}) \in \mathbb{Q}(\mathsf{L}^\dagger)$, then $\mathrm{R}(\boldsymbol{A}) \cong \mathrm{R}(\boldsymbol{B})$ for some $\boldsymbol{B} \in \mathsf{L}$, whence $\boldsymbol{A} \cong \boldsymbol{B}$, contradicting the fact that $\boldsymbol{A} \notin \mathsf{L}$. Therefore, $\mathrm{R}(\boldsymbol{A}) \notin \mathbb{Q}(\mathsf{L}^\dagger)$, and so $\mathbb{Q}(\mathsf{L}^\dagger) \neq \mathbb{R}(\mathsf{K})$.

We claim that $\mathbb{R}(\mathsf{K}) = \mathbb{V}(\mathsf{L}^\dagger)$. To see this, let $\boldsymbol{C} \in \mathbb{R}(\mathsf{K})_{\mathrm{SI}}$. By Jónsson's Theorem and Lemma 3.34, $\boldsymbol{C} \cong \mathrm{R}(\boldsymbol{D})$ for some $\boldsymbol{D} \in \mathsf{K}$. As $\mathsf{K} = \mathbb{H}(\mathsf{L})$, we may assume that $\boldsymbol{D} = \boldsymbol{E}/\theta$ for some $\boldsymbol{E} \in \mathsf{L}$ and some $\theta \in \mathrm{Con}(\boldsymbol{E})$. Then $\boldsymbol{C} \cong \mathrm{R}(\boldsymbol{E}/\theta) \cong \mathrm{R}(\boldsymbol{E})/\,\mathrm{R}(\theta)$, by Lemma 3.34(ii), whence $\boldsymbol{C} \in \mathbb{H}(\mathsf{L}^\dagger) \subseteq \mathbb{V}(\mathsf{L}^\dagger)$. This vindicates the claim.

In summary, $\mathbb{Q}(\mathsf{L}^\dagger)$ is a proper subquasivariety of $\mathbb{R}(\mathsf{K})$ that fails to generate a *proper* subvariety of $\mathbb{R}(\mathsf{K})$, so $\mathbb{R}(\mathsf{K})$ is not SC. □

The variety of semilinear Dunn monoids is structurally incomplete [113, Thm. 9.4]. Its reflection is just the variety SLM of semilinear members of M, by Lemma 3.34 and Corollary 3.27, so SLM is not structurally complete either, by Theorem 7.5. This confirms that M is not HSC (and likewise N), but we can say more:

Theorem 7.6. *The variety* M *has* $2^{\aleph_0}$ *structurally incomplete subvarieties.*

Proof. This follows from Theorems 7.4 and 7.5, because the operator $\mathbb{R}$ is injective (by Lemma 3.38). □

7.2 Epimorphism surjectivity

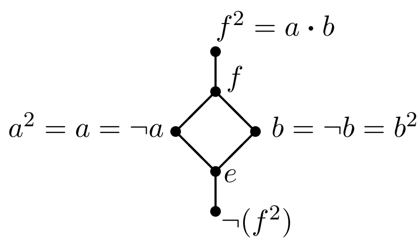
Recall that in the previous section (on page 165) we exhibited $2^{\aleph_0}$ distinct subvarieties of BRA that have depth 3. Since each of these varieties has finite depth, they all have the ES property (by Corollary 6.46). We saw in Theorem 6.43 that $\mathbb{R}$ preserves the ES property, and in Lemma 3.37 that it preserves local finiteness. The following can then be inferred from the injectivity of $\mathbb{R}$ (Lemma 3.38).

Theorem 7.7. *There are* $2^{\aleph_0}$ *distinct locally finite varieties of De Morgan monoids with the ES property.*

We saw after Theorem 6.42 that there are infinitely many varieties of De Morgan monoids that lack even the weak ES property, and that these varieties can be chosen to generate covers of $\mathbb{V}(\boldsymbol{C}_4)$ in the subvariety lattice of DMM.

Beyond the covers of $\mathbb{V}(\boldsymbol{C}_4)$, there are in fact uncountably many further examples. Indeed, by an argument of Urquhart [143] (also see [13, Cor. 4.15]), a variety of De Morgan monoids lacks the weak ES property if it contains a certain six-element algebra $\boldsymbol{C}$, called the *crystal lattice* (which is not generated by its negative cone). That algebra is depicted below.

(Deletion of b leaves an epic subalgebra behind, owing to the uniqueness of existent relative complements in distributive lattices.) The argument adapts to Dunn monoids, using the RL-reduct of $\boldsymbol{C}$.



In particular, $\boldsymbol{C}$ is absent from each of the $2^{\aleph_0}$ varieties K of De Morgan monoids in Theorem 7.7, while the corresponding varieties $\mathbb{V}(\mathsf{K} \cup \{\boldsymbol{C}\})$ lack the weak ES property, and by Jónsson's Theorem 1.23, they are distinct. Each such variety $\mathbb{V}(\mathsf{K} \cup \{\boldsymbol{C}\})$ is also locally finite, because by (1.4) and Jónsson's Theorem 1.23, every finitely generated $\boldsymbol{A} \in \mathbb{V}(\mathsf{K} \cup \{\boldsymbol{C}\})_{\mathrm{SI}}$ belongs either to K or $\mathbb{HSP}_{\mathbb{U}}(\boldsymbol{C})$. In the first case $\boldsymbol{A}$ is finite by the local finiteness of K, and in the second case finiteness of $\boldsymbol{A}$ follows from the fact that $\boldsymbol{C}$ is finite. Thus, we obtain:

Theorem 7.8. *There is a continuum of distinct locally finite varieties of De Morgan monoids* without *the weak ES property.*

The situation is different for varieties of Heyting and Brouwerian algebras. Recall from Theorem 6.5(ii), that *every* variety of Brouwerian algebras has the weak ES property (the same is true for every variety of Heyting algebras). It follows from a result of Campercholi [25, Cor. 6.5] that, in any finitely generated variety with a majority term (e.g., one generated by a finite lattice-based algebra), the weak ES property entails the ES property. This provides a different explanation of the slightly earlier finding, in [11], that all finitely generated varieties of Brouwerian (or Heyting) algebras have the ES property (see the end of Section 6.5).

Maksimova [49, 87, 88] established that only finitely many varieties of Brouwerian (or Heyting) algebras enjoy the *strong* ES property (see Definition 6.34(i) and the remarks after Corollary 6.36 on page 156).

It was shown in [11, Cor. 6.2] that even a locally finite variety of Brouwerian algebras need not have the ES property. (The counter-example confirmed Blok and Hoogland's conjecture [13] that the weak ES property really is strictly weaker than the ES property, and therefore that the finite Beth property does not entail the infinite Beth property; see Theorem 6.4.) In the last two sections of this chapter we show that the ES property fails in uncountably many further locally finite varieties of Brouwerian algebras (Theorem 7.22). We then apply the fact that $\mathbb{R}$ preserves the weak ES property and also *failure* of the ES property (Theorem 6.43) to conclude that there are $2^{\aleph_0}$ locally finite varieties of De Morgan monoids that lack the ES property, and yet have the weak ES property.

The results in these last two sections are adapted from Moraschini and Wannenburg [108], which proves analogous results for varieties of Heyting algebras, but there it is also shown that, for every finite $n > 2$, the ES property fails in the variety of all Heyting algebras with *width* at most n. That manuscript also establishes a test for the ES property in subvarieties of the so-called *Kuznetsov-Gerčiu* variety KG. (KG is the variety generated by finite linear sums of 1-generated Heyting algebras; see (7.6) on page 186.)

7.3 Width and incomparability

Definition 7.9. Let n be a positive integer. A pointed Esakia space $\boldsymbol{X} = \langle X; \tau, \leqslant, m\rangle$ has *width at most* n if for every $x \in X$, the poset $\uparrow x$ does not contain an antichain of $n+1$ elements.

A Brouwerian algebra $\boldsymbol{A}$ has *width at most* n when its dual space $\boldsymbol{A}_*$ has.

For $0 < n \in \omega$, let W_n denote the class of Brouwerian algebras with width at most n.

Since Brouwerian algebras are integral, (2.15) simplifies to the law

$$x \to y \approx e \iff x \leqslant y$$

in this context.

Theorem 7.10. *Let $0 < n \in \omega$. A Brouwerian algebra $\boldsymbol{A}$ has width at most n if and only if it satisfies the equation $w_n \approx e$, where*

$$w_n := \bigvee_{i=0}^{n} \Big(x_i \to \bigvee_{i \neq j \in \{0,\dots,n\}} x_j\Big).$$

As a consequence, W_n is a variety.

Proof. The axiomatization for W_n can be found in [28, p. 43] in the context of Heyting algebras. In lieu of a convenient reference to a proof which survives the transition from Heyting to Brouwerian algebras, we provide such a proof here.

First assume that $\boldsymbol{A}$ has width at most n. Suppose, with a view to contradiction, that $\boldsymbol{A}$ does not satisfy $w_n \approx e$. By Birkhoff's Subdirect Decomposition Theorem 1.3, some SI homomorphic image $\boldsymbol{B}$ of $\boldsymbol{A}$ fails to satisfy $w_n \approx e$. Let $h\colon \boldsymbol{A} \to \boldsymbol{B}$ be a surjective homomorphism. Then, by Lemma 6.16(iii), $h_*[\boldsymbol{B}_*]$ is an E-subspace of $\boldsymbol{A}_*$, and, since h_* is injective by Lemma 6.16(ii), $h_*[\boldsymbol{B}_*]$ is isomorphic to $\boldsymbol{B}_*$. It follows plainly from the definition that $\boldsymbol{B}$ also has width at most n.

Since $\boldsymbol{B}$ falsifies $w_n \approx e$, there exist $a_0, a_1, \dots, a_n \in B$ such that $a_i \nleqslant \bigvee_{j \neq i} a_j$ for every $i \in \{0, 1, \dots, n\}$. By the Prime Filter Lemma 6.14, for every $i = 0, 1, \dots, n$, there exists $F_i \in \mathrm{Pr}(\boldsymbol{B})$ such that $a_i \in F_i$ and $\bigvee_{j \neq i} a_j \notin F_i$. Since F_i is prime, $a_j \notin F_i$ for every $j \in \{0, 1, \dots, n\} \setminus \{i\}$. It follows that $F_0, F_1, \dots, F_n$ form an antichain of $n+1$ elements in $\boldsymbol{B}_*$, which contradicts the fact that $\boldsymbol{B}_*$ has width at most n, because $\boldsymbol{B}_*$ is rooted by Lemma 6.16(i), since $\boldsymbol{B}$ is FSI.

Conversely, assume that $\boldsymbol{A}$ satisfies $w_n \approx e$. Suppose with a view to contradiction that $\boldsymbol{A}$ does not have width at most n. There exist $G, G_0, G_1, \dots, G_n \in \mathrm{Pr}(\boldsymbol{A})$, such that $G_0, G_1, \dots, G_n$ form an antichain whose members all contain G. Because $\boldsymbol{A}_*$ is a pointed Esakia space, $\uparrow G$ is closed in $\boldsymbol{A}_*$, so it forms a rooted E-subspace of $\boldsymbol{A}_*$. By Lemma 6.16(ii), $(\uparrow G)^*$ is an FSI homomorphic image of $\boldsymbol{A}$, which therefore also satisfies $w_n \approx e$. Thus, we may assume without loss of generality that $\boldsymbol{A}$ is FSI.

Let $i \in \{0, 1, \dots, n\}$. For every $j \in \{0, 1, \dots, n\} \setminus \{i\}$, let $g_j \in G_i \setminus G_j$. Then set $a_i = \bigwedge_{j \neq i} g_j$. It follows that $a_i \in G_i$ and $a_i \notin G_j$ for every $j \in \{0, 1, \dots, n\} \setminus \{i\}$. Since $\boldsymbol{A}$ is FSI, e is join-irreducible in $\boldsymbol{A}$, by Lemma 2.16(i), so as $\boldsymbol{A}$ satisfies $w_n \approx e$, there exists $i \in \{0, 1, \dots, n\}$ such that $a_i \leqslant \bigvee_{j \neq i} a_j$. But then $\bigvee_{j \neq i} a_j \in G_i$, so, since G_i is prime, $a_j \in G_i$ for some $j \neq i$, a contradiction. $\square$

Note that $w_1 = (x_0 \to x_1) \vee (x_1 \to x_0)$, so $w_1 \approx e$ amounts to equation (2.28) on page 37. Therefore, W_1 is the variety of semilinear Brouwerian algebras, i.e., the variety of relative Stone algebras, by Theorem 7.10 (as one would expect).

Definition 7.11. Let $n \in \omega$. A pointed Esakia space $\boldsymbol{X}$ is said to have *incomparability degree* at most n if, for every $x \in X$, the set $\uparrow x$ does not contain any point which is incomparable with $n+1$ elements of $\uparrow x$.

Clearly, pointed Esakia spaces of incomparability degree at most n also have width at most $n+1$, but the converse is not true in general (since elements incomparable with a given element may be comparable with each other).

Definition 7.12. A Brouwerian algebra $\boldsymbol{A}$ has *incomparability degree* at most n when its dual space $\boldsymbol{A}_*$ has. We denote by ID_n the class of all Brouwerian algebras of incomparability degree at most n.

Notice that $\mathsf{ID}_0 = \mathsf{W}_1$, the variety of relative Stone algebras. We shall see that ID_n is a (finitely axiomatizable) variety.

Let $n \in \omega$. Consider a set of variables $Z_n = \{y_1, \dots, y_{n+1}\}$. We let $\mathbb{Z}_{n,1}, \dots, \mathbb{Z}_{n,k_n}$ be a fixed enumeration of all possible posets with universe Z_n. For each such $\mathbb{Z}_{n,k} = \langle Z_n, \leqslant_k \rangle$, with $k \leqslant k_n$, define the terms

$$\psi_{n,k} \;:=\; \bigvee_{i=1}^{n+1} (y_i \to (x \vee \bigvee_{j\colon y_i \nleqslant_k y_j} y_j)).$$

When the set $\{j\colon y_i \nleqslant_k y_j\}$ is empty for some $i \leq n+1$ in the display above, we follow the convention that the disjunction with $\bigvee_{j\colon y_i \nleqslant_k y_j} y_j$ is ignored. Moreover, we set

$$\delta_{n,k} \;:=\; \psi_{n,k} \vee (x \to \bigvee_{i=1}^{n+1} y_i)$$

and

$$\Sigma_n \;:=\; \{\delta_{n,k} \approx e \colon k = 1, \dots, k_n\}.$$

Theorem 7.13. *For every $n \in \omega$, the class ID_n of Brouwerian algebras with incomparability degree at most n is axiomatized by the set of equations Σ_n. As a consequence, ID_n is a variety.*

Proof. First we show that for every Brouwerian algebra $\boldsymbol{A} \notin \mathsf{ID}_n$, we have $\boldsymbol{A} \not\models \Sigma_n$. Note that we need only exhibit the failure of some equation of Σ_n in some homomorphic image of $\boldsymbol{A}$. Since $\boldsymbol{A} \notin \mathsf{ID}_n$, there is an $x \in \boldsymbol{A}_*$ such that $\uparrow x$ contains a point which is incomparable with $n+1$ points. Thus, as in the proof of Theorem 7.10, we may, without loss of generality, suppose that $\boldsymbol{A}$ is FSI, otherwise we replace $\boldsymbol{A}$ with its FSI homomorphic image whose dual is isomorphic to the subspace $\uparrow x$.

Since $\boldsymbol{A} \notin \mathsf{ID}_n$, there are distinct $F, G_1, \dots, G_{n+1} \in \mathrm{Pr}(\boldsymbol{A})$ such that F is incomparable with each of $G_1, \dots, G_{n+1}$. Then for every $i \leq n+1$ we can choose an element $a_i \in F \setminus G_i$. We set

$$\hat{a} \;:=\; a_1 \wedge \dots \wedge a_{n+1}.$$

Observe that

$$\hat{a} \in F \setminus (G_1 \cup \dots \cup G_{n+1}). \tag{7.3}$$

Given $i \leq n+1$, we can choose $b_i \in G_i \setminus F$. Moreover, for every $j \leq n+1$ such that $G_i \nsubseteq G_j$, we choose $b_{i_j} \in G_i \setminus G_j$ and set

$$b'_i := b_i \wedge \bigwedge_{j\colon G_i \nsubseteq G_j} b_{i_j}.$$

Finally for every $j \leq n+1$, we define

$$\hat{b}_j := \bigwedge_{i\colon G_i \subseteq G_j} b'_i.$$

Observe that for every $i \leq n+1$,

$$\hat{b}_i \in G_i \setminus (F \cup \bigcup_{j\colon G_i \nsubseteq G_j} G_j). \tag{7.4}$$

From (7.3), (7.4) and the fact that $G_1, \ldots, G_{n+1}$ are different, we deduce that the elements $\hat{a}, \hat{b}_1, \ldots, \hat{b}_{n+1}$ are different one from the other. Then consider the subposet $\mathbb{X}$ of $\boldsymbol{A}$ with universe $\{\hat{b}_1, \ldots, \hat{b}_{n+1}\}$. Clearly there is a $k \leq k_n$ such that $\mathbb{Z}_{n,k}$ is isomorphic to $\mathbb{X}$ under the map $y_i \mapsto \hat{b}_i$ $(i \leq n+1)$.

For every $i, j \leq n+1$,

$$\hat{b}_i \leqslant_k \hat{b}_j \text{ iff } \hat{b}_j \in G_i. \tag{7.5}$$

The forward implication follows from the fact that $\hat{b}_i \in G_i$. Conversely, if $\hat{b}_i \in G_j$, then by (7.4), $G_j \subseteq G_i$. In this case, whenever $G_k \subseteq G_j$, then $G_k \subseteq G_i$, so $\hat{b}_i \leqslant_k \hat{b}_j$ (by definition).

We show that $\boldsymbol{A}$ fails to satisfy $\delta_{n,k}(x, y_1, \ldots .y_{n+1}) \approx e$, by showing that $\delta^{\boldsymbol{A}}_{n,k}(\hat{a}, \hat{b}_1, \ldots, \hat{b}_{n+1}) \neq e$. Suppose the contrary. Then

$$\psi^{\boldsymbol{A}}_{n,k}(\hat{a}, \hat{b}_1, \ldots, \hat{b}_{n+1}) \vee (\hat{a} \to \bigvee_{i=1}^{n+1} \hat{b}_i) = \delta^{\boldsymbol{A}}_{n,k}(\hat{a}, \hat{b}_1, \ldots, \hat{b}_{n+1}) = e.$$

As $\boldsymbol{A}$ is FSI, e is join-irreducible in $\boldsymbol{A}$, by Lemma 2.16(i). The display above thus decomposes into two cases:

$$\text{either } \psi^{\boldsymbol{A}}_{n,k}(\hat{a}, \hat{b}_1, \ldots, \hat{b}_{n+1}) = e \text{ or } \hat{a} \leqslant \bigvee_{i=1}^{n+1} \hat{b}_i.$$

First consider the case where $\hat{a} \leqslant \bigvee_{i=1}^{n+1} \hat{b}_i$. Since $\hat{a} \in F$ by (7.3), this implies $\bigvee_{i=1}^{n+1} \hat{b}_i \in F$. As F is prime, we conclude that $\hat{b}_i \in F$ for some $i \in \{1, \ldots, n+1\}$. But this contradicts (7.4).

Then we conclude that

$$e = \psi^{\boldsymbol{A}}_{n,k}(\hat{a}, \hat{b}_1, \ldots, \hat{b}_{n+1}) = \bigvee_{i=1}^{n+1}(\hat{b}_i \to (\hat{a} \vee \bigvee_{j:\hat{b}_i \not\leqslant_k \hat{b}_j} \hat{b}_j)).$$

By the join-irreducibility of e, there exists $i \in \{1, \ldots, n+1\}$ such that

$$\hat{b}_i \leqslant \hat{a} \vee \bigvee_{j:\hat{b}_i \not\leqslant_k \hat{b}_j} \hat{b}_j.$$

As $\hat{b}_i \in G_i$, the right-hand side of the inequality above also belongs to G_i. So, as G_i is a prime filter and $\hat{a} \notin G_i$ by (7.3), there exists $j \in \{1, \ldots, n+1\}$ such that $\hat{b}_j \in G_i$ and $\hat{b}_i \not\leqslant_k \hat{b}_j$. But this contradicts (7.5). We have therefore established that $\boldsymbol{A} \not\models \Sigma_n$.

Conversely, consider a Brouwerian algebra $\boldsymbol{A}$ such that $\boldsymbol{A} \not\models \Sigma_n$. We show that $\boldsymbol{A} \notin \mathsf{ID}_n$. There exists $k \leq k_n$ such that the equation $\delta_{n,k} \approx e$ in Σ_n is not valid in $\boldsymbol{A}$, so there are $a, b_1, \ldots, b_{n+1}$ such that

$$\bigvee_{i=1}^{n+1}(b_i \to (a \vee \bigvee_{j:\, y_i \not\leqslant_k y_j} b_j)) \vee (a \to \bigvee_{i=1}^{n+1} b_i) \neq e.$$

Therefore, for every $i \leq n+1$,

$$b_i \not\leqslant a \vee \bigvee_{j:\, y_i \not\leqslant_k y_j} b_j \quad \text{and} \quad a \not\leqslant \bigvee_{j=1}^{n+1} b_j.$$

By the Prime Filter Lemma 6.14, there are prime filters $F, G_1, \ldots, G_{n+1}$ of $\boldsymbol{A}$ such that

$$a \in F \text{ and } b_1, \ldots, b_{n+1} \notin F$$

and for every $i \leq n+1$,

$$b_i \in G_i \text{ and } \{a\} \cup \{b_j \colon y_i \not\leqslant_k y_j\} \subseteq A \setminus G_i.$$

It follows easily from these properties and the pairwise distinctness of $y_1, \ldots, y_{n+1}$ that $G_1, \ldots, G_{n+1}$ are pairwise different, and that F is incomparable with G_i for every $i \in \{1, \ldots, n+1\}$. Therefore $\boldsymbol{A} \notin \mathsf{ID}_n$ as required. □

Sums of Brouwerian algebras

Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be Brouwerian algebras. The *(linear) sum* $\boldsymbol{A}+\boldsymbol{B}$ is the Brouwerian algebra obtained by pasting $\boldsymbol{B}$ *below* $\boldsymbol{A}$, gluing the top element of $\boldsymbol{B}$ to the bottom element of $\boldsymbol{A}$ when it exists. To give a more formal definition, it is convenient to assume that the universes of $\boldsymbol{A}$ and $\boldsymbol{B}$ are disjoint. Moreover, let us denote by $\leqslant^{\boldsymbol{A}}$ and $\leqslant^{\boldsymbol{B}}$ the lattice orders of $\boldsymbol{A}$ and $\boldsymbol{B}$, respectively. Let $A' = A \setminus \{0\}$ if $\boldsymbol{A}$ has a bottom element 0, and $A' = A$, otherwise. Then $\boldsymbol{A}+\boldsymbol{B}$ is the unique Brouwerian algebra with universe $A' \cup B$ whose lattice order $\leqslant$ is defined as follows: for every $a, b \in A' \cup B$,

$$\begin{aligned} b \leqslant a \ \text{ iff } \ &(a, b \in A \text{ and } b \leqslant^{\boldsymbol{A}} a) \text{ or } (a, b \in B \text{ and } b \leqslant^{\boldsymbol{B}} a) \\ &\text{or } (b \in B \text{ and } a \in A). \end{aligned}$$

As $+$ is clearly associative, there is no ambiguity in writing $\boldsymbol{A}_1 + \cdots + \boldsymbol{A}_n$ for the descending chain of finitely many Brouwerian algebras $\boldsymbol{A}_1, \ldots, \boldsymbol{A}_n$, each glued to the previous one where possible.

To obtain interesting results about epimorphisms in Brouwerian varieties that are not consequences of Theorem 6.22, we need to consider algebras with unbounded depth. It is therefore useful to introduce an infinite generalization of the above construction. Let $\{\boldsymbol{A}_n : n \in \omega\}$ be a family of Brouwerian algebras with disjoint universes. The *sum* $\sum \boldsymbol{A}_n$ is the unique Brouwerian algebra with universe $\bigcup_{n\in\omega} A'_n$ and whose lattice order is defined as follows: for every $a, b \in \bigcup_{n\in\omega} A'_n$,

$$\begin{aligned} a \leqslant b \ \text{ iff } \ &(a, b \in A_n \text{ for some } n \in \omega \text{ and } a \leqslant^{\boldsymbol{A}_n} b) \\ &\text{or } (a \in A_n \text{ and } b \in A_m \text{ for some } n, m \in \omega \text{ such that } n > m). \end{aligned}$$

In words, $\sum \boldsymbol{A}_n$ is a tower of algebras, each pasted below the previous, gluing the top element to the bottom previous algebra where possible. When $\{\boldsymbol{A}_n : n \in \omega\}$ is a family consisting of copies of the same algebra $\boldsymbol{A}$, we write $\boldsymbol{A}^\infty$ instead of $\sum \boldsymbol{A}_n$.

These constructions are analogous to the more familiar notion of sums of Heyting algebras (see page 186), which has found various applications in the study of intermediate logics. See for instance [10, 58, 80, 85], but note that in the usual definition, subsequent algebras are added on top, instead of below. For finitely many summands this amounts to only a notational difference.

For present purposes, it is convenient to describe the dual spaces of sums of Brouwerian algebras as well.

Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be two pointed Esakia spaces with disjoint universes. (Recall that m denotes the maximum element of a pointed Esakia space.)

Let $X' = X \setminus \{m\}$ when $\{m\}$ is open in $\boldsymbol{X}$, and $X' = X$ otherwise. Note that $\{m\}$ is always closed, because $\boldsymbol{X}$ is Hausdorff, and is obviously an upset of $\boldsymbol{X}$. So, $\{m\}$ is open iff it is the smallest non-empty clopen upset of $\boldsymbol{X}$, i.e., the least element of the algebra $\boldsymbol{X}^*$. The *sum* $\boldsymbol{X} + \boldsymbol{Y}$ has universe $X' \cup Y$, and order relation $\leqslant$ defined as follows: for every $x, y \in X' \cup Y$,

$$\begin{aligned} x \leqslant y \ \text{ iff } \ &(x, y \in X \text{ and } x \leqslant^{\boldsymbol{X}} y) \text{ or } (x, y \in Y \text{ and } x \leqslant^{\boldsymbol{Y}} y) \\ &\text{or } (x \in X \text{ and } y \in Y). \end{aligned}$$

In words, the poset of $\boldsymbol{X} + \boldsymbol{Y}$ is obtained by placing $\boldsymbol{Y}$ *above* the restriction of $\boldsymbol{X}$ to X', so its top element is $m^{\boldsymbol{Y}}$. The topology of $\boldsymbol{X} + \boldsymbol{Y}$ consists of the sets $U \subseteq X' \cup Y$ such that $U \cap X'$ and $U \cap Y$ are open, respectively, in $\boldsymbol{X}$ and $\boldsymbol{Y}$. Then $\boldsymbol{X} + \boldsymbol{Y}$ is a pointed Esakia space.

Then let $\{\boldsymbol{X}_n : n \in \omega\}$ be a family of pointed Esakia spaces with disjoint universes, and let m be a fresh element. The *sum* $\sum \boldsymbol{X}_n$ has universe

$$\{m\} \cup \bigcup_{n \in \omega} X'_n$$

and order relation $\leqslant$ defined as follows: for every $x, y \in \sum X_n$,

$$\begin{aligned} x \leqslant y \ \text{ iff } \ &y = m \text{ or } (x, y \in X_n \text{ for some } n \in \omega \text{ and } x \leqslant^{\boldsymbol{X}_n} y) \\ &\text{or } (x \in X_n \text{ and } y \in X_p \text{ for some } n, p \in \omega \text{ such that } n < p). \end{aligned}$$

Hence, the poset of $\sum \boldsymbol{X}_n$ is obtained by placing the primed restriction of each successive poset above the previous and adding a new top element.

The topology of $\sum \boldsymbol{X}_n$ is

$$\begin{aligned} \tau = \{U : \, &U \cap X'_n \text{ is open in } \boldsymbol{X}_n \text{ for all } n \in \omega, \text{ and} \\ &\text{if } m \in U, \text{ then there exists } n \in \omega \text{ with } \bigcup_{n \leq p} X'_p \subseteq U\}. \end{aligned}$$

Then $\sum \boldsymbol{X}_n$ is a pointed Esakia space. When $\{\boldsymbol{X}_n : n \in \omega\}$ consists of copies of the same pointed Esakia space $\boldsymbol{X}$, we write $\boldsymbol{X}^\infty$ instead of $\sum \boldsymbol{X}_n$.

Lemma 7.14. *Let $\{\boldsymbol{A}, \boldsymbol{B}\} \cup \{\boldsymbol{A}_n : n \in \omega\}$ be a family of Brouwerian algebras. The pointed Esakia spaces $(\boldsymbol{A} + \boldsymbol{B})_*$ and $(\sum \boldsymbol{A}_n)_*$ are isomorphic, respectively, to $\boldsymbol{A}_* + \boldsymbol{B}_*$ and $\sum \boldsymbol{A}_{n*}$.*

Proof. We sketch the proof only for the case of $\sum \boldsymbol{A}_n$. We define a map

$$f : \sum \boldsymbol{A}_{n*} \to (\sum \boldsymbol{A}_n)_*,$$

setting $f(m) := \sum A_n \,(= m^{(\sum \boldsymbol{A}_n)_*})$ and for every $n \in \omega$ and $F \in (\boldsymbol{A}_{n*})'$,

$$f(F) := \{a \in \sum \boldsymbol{A}_n \colon a \geqslant b \text{ for some } b \in F\}.$$

It is not difficult to see that f is order-preserving. That f is bijective follows from the fact that a Brouwerian algebra $\boldsymbol{A}$ has a bottom element iff $\{m^{\boldsymbol{A}_*}\}$ is open (and thus clopen) in $\boldsymbol{A}_*$. Therefore, it remains only to prove that f is continuous and satisfies (6.4).

To show the latter, suppose that $m \neq F \in \sum \boldsymbol{A}_{n*}$ and $\sum A_n \neq G \in (\sum \boldsymbol{A}_n)_*$ such that $f(F) \leqslant G$. Then $F \in (\boldsymbol{A}_{j*})'$ for some $j \in \omega$. We may therefore let $k \in \omega$ be the least $k \geq j$ such that $G \cap A_p \neq \emptyset$ for every $p \geq k$. If we let $G' = G \cap A_k$, it follows that $G' \in (\boldsymbol{A}_{k*})' \subseteq \sum \boldsymbol{A}_{n*}$, with $F \leqslant G'$ and $f(G') = G$.

To prove that f is continuous, first consider some subbasic clopen set of the form $\gamma^{\sum \boldsymbol{A}_n}(a)\,(= \{F \in \mathrm{Pr}(\sum \boldsymbol{A}_n) : a \in F\})$ with $a \in \sum A_n$. We have $a \in A_k'$ for some $k \in \omega$ and, therefore,

$$f^{-1}[\gamma^{\sum \boldsymbol{A}_n}(a)] = \{m\} \cup \gamma^{\boldsymbol{A}_k}(a) \cup \bigcup_{p>k} (A_{p*})'.$$

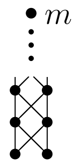
Clearly the sets $\gamma^{\boldsymbol{A}_k}(a)$ and $\{m\} \cup \bigcup_{p>k} A_{p*}$ are open in $\sum \boldsymbol{A}_{n*}$. Now, similarly, consider a subbasic clopen set of the form $\gamma^{\sum \boldsymbol{A}_n}(a)^c$. Then

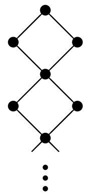
$$f^{-1}[\gamma^{\sum \boldsymbol{A}_n}(a)^c] = f^{-1}[\gamma^{\sum \boldsymbol{A}_n}(a)]^c = \gamma^{\boldsymbol{A}_k}(a)^c \cup \bigcup_{p=0}^{k-1} (A_{p*})',$$

which is also clearly open in $\sum \boldsymbol{A}_{n*}$. This shows that f is continuous. Therefore, f is a bijective Esakia morphism, and hence an isomorphism. $\square$

As an example, we can now construct an algebra $(\boldsymbol{D}_2^\infty)^*$, which witnesses the failure of the ES property in the variety generated by it. We let $\boldsymbol{D}_2$ be the pointed Esakia space with two incomparable elements below the maximum. It follows that $\boldsymbol{D}_2{}^* \cong \mathbf{2}^+ \times \mathbf{2}^+$, where $\mathbf{2}^+$ is the two-element Brouwerian algebra. From Lemma 7.14, it follows that $(\boldsymbol{D}_2^\infty)^* \cong (\mathbf{2}^+ \times \mathbf{2}^+)^\infty$, i.e., it comprises denumerably many copies of the four-element diamond, each pasted below the previous one.[1]

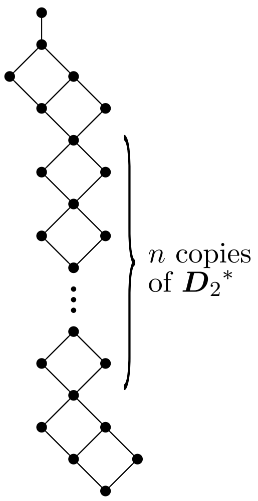
[1]The algebra used in [11, Cor. 6.2] to generate a variety of Brouwerian algebras without the ES property is in fact $\mathbf{2}^+ + (\boldsymbol{D}_2^\infty)^*$, which has the virtue of being SI. For us, however, this difference is immaterial, since $\mathbf{2}^+ + (\boldsymbol{D}_2^\infty)^*$ and $(\boldsymbol{D}_2^\infty)^*$ generate the same variety, and both have epic subalgebras in this variety.





$\boldsymbol{D}_2^{\infty}$ $(\boldsymbol{D}_2^{\infty})^*$

We can also use this construction to show that $\mathsf{W}_2 \cap \mathsf{ID}_2$ has a continuum of subvarieties. Let $\boldsymbol{A}$ be the direct product of the two-element and three-element Brouwerian algebras. For $n \in \omega$, let $\boldsymbol{B}_n$ be the algebra $\mathbf{2}^+ + \boldsymbol{A} + \boldsymbol{C}_1 + \cdots + \boldsymbol{C}_n + \boldsymbol{A}$, where $\boldsymbol{C}_1, \ldots, \boldsymbol{C}_n$ are copies of the four-element diamond $\boldsymbol{D}_2{}^*$. The algebra $\boldsymbol{B}_n$ (depicted below) is SI, and belongs to $\mathsf{W}_2 \cap \mathsf{ID}_2$.[2]



Lemma 7.15. *Let $n, m \in \omega$ such that $n \neq m$. Then $\boldsymbol{B}_n \notin \mathbb{HS}(\boldsymbol{B}_m)$.*

Proof. First, notice that if $n > m$, we are done on cardinality grounds. So, suppose with a view to contradiction that $n < m$, and there exist $\boldsymbol{S} \in \mathbb{S}(\boldsymbol{B}_m)$ and a surjective homomorphism $h \colon \boldsymbol{S} \to \boldsymbol{B}_n$.

There exist $a, b, c \in B_n$ such that a is incomparable with both b and c, and $a \vee b \vee c$ is the co-atom of $\boldsymbol{B}_n$. Since h is surjective, there exist distinct $a', b', c' \in S \subseteq B_m$, such that $h(a') = a$, $h(b') = b$ and $h(c') = c$. Then a' is incomparable with both b' and c', because h is order-preserving. Therefore, a', b' and c' must be in either the top or the bottom copy of $\boldsymbol{A}$ in $\boldsymbol{B}_m$. If

[2] These algebras are adapted from similar ones used in [10, Lem. 5.38(5), Thm. 5.39(1)] in the context of Heyting algebras.

they are in the bottom copy then $a' \wedge b' \wedge c'$ is the bottom element of $\boldsymbol{B}_m$. Consider the bottom element $\bot$ of $\boldsymbol{B}_n$. Since h is surjective there exists $\bot' \in S$ such that $h(\bot') = \bot$, but then $\bot = h(\bot') \geq h(a' \wedge b' \wedge c') = a \wedge b \wedge c$. This contradicts the fact that $a \vee b \vee c$ is the co-atom of $\boldsymbol{B}_n$. So, a', b' and c' must be in the top copy of $\boldsymbol{A}$ in $\boldsymbol{B}_m$. Therefore, $a' \vee b' \vee c'$ is the co-atom d of $\boldsymbol{B}_m$.

Suppose that h is not injective. Then $h(d) = h(e^{\boldsymbol{S}})$, so $a \vee b \vee c = h(a' \vee b' \vee c') = h(d) = e^{\boldsymbol{B}_n}$, a contradiction. So, h is injective, and hence an isomorphism.

Therefore, $\boldsymbol{B}_n$ embeds into $\boldsymbol{B}_m$. Since incomparable elements remain incomparable under a lattice embedding, similar arguments to the above show that the two copies of $\boldsymbol{A}$ in $\boldsymbol{B}_n$ must be mapped to the respective copies in $\boldsymbol{B}_m$. Every proper subalgebra of $\boldsymbol{B}_m$ that keeps the top structure of $\mathbf{2}^+ + \boldsymbol{A}$ and the bottom structure of $\boldsymbol{A}$ intact introduces a two-element chain in the middle, making the embedding impossible. □

Corollary 7.16. *Let $F := \{\boldsymbol{B}_n \colon n \in \omega\}$. For every pair of distinct subsets $S, T \subseteq F$, we have $\mathbb{V}(S) \neq \mathbb{V}(T)$. Consequently, $\mathsf{W}_2 \cap \mathsf{ID}_n$ has $2^{\aleph_0}$ subvarieties.*

Proof. For $\boldsymbol{B}_n \in S \setminus T$, Lemma 7.15 shows that T is contained in $\mathsf{K} := \{\boldsymbol{A} \in \mathsf{BRA} \colon \boldsymbol{B}_n \notin \mathbb{HS}(\boldsymbol{A})\}$, and Theorem 1.26 shows that this K is a variety, so $\mathbb{V}(T)$ is contained in K, whence $\boldsymbol{B}_n \notin \mathbb{V}(T)$ (as required). □

7.4 Separating the ES and weak ES properties

In this section we show that there is a continuum of locally finite subvarieties of $\mathsf{W}_2 \cap \mathsf{ID}_2$ lacking the ES property. Each of these varieties distinguishes the ES property from the weak ES property, because every variety of Brouwerian algebras has the weak ES property (Theorem 6.5(ii)). We then draw some conclusions for varieties of De Morgan monoids.

We shall need the following technical lemma.

Lemma 7.17. *Let $0 < n \in \omega$ and let $f \colon \boldsymbol{Y} \to \boldsymbol{X}$ be an Esakia morphism between pointed Esakia spaces of width at most n such that*

(i) *$\boldsymbol{Y}$ has a minimum $\bot$, and*

(ii) *for every $z \in X \setminus \{m\}$, if $f(\bot) < z$, then there is an antichain of n elements in ${\uparrow} f(\bot)$, which contains z.*

Then there is a subposet $\langle Z; \leqslant^{\boldsymbol{Y}} \rangle$ of $\boldsymbol{Y}$ such that the restriction

$$f\colon \langle Z; \leqslant^{\boldsymbol{Y}} \rangle \to \langle \uparrow f(\bot); \leqslant^{\boldsymbol{X}} \rangle$$

is a poset isomorphism.

Proof. Observe that, since f is an Esakia morphism, $\uparrow f(\bot)$ coincides with $f[Y]$. Suppose $z \in \uparrow f(\bot) \setminus \{m, f(\bot)\}$ and define

$$T_z := \{a \in Y\colon f(a) = z\}.$$

Observe that $T_z \neq \emptyset$.

We claim that T_z is a chain in $\boldsymbol{X}$. Indeed, since $f(\bot) < z$, assumption (ii) shows that z belongs to an antichain $\{x_1, \ldots, x_{n-1}, z\}$ of n elements in $\uparrow f(\bot)$. Since f is an Esakia morphism, there are $y_1, \ldots, y_{n-1} \in Y$ such that $f(y_i) = x_i$ for $i = 1, \ldots, n-1$. Together with the fact that f is order-preserving, this implies that the set $\{y_1, \ldots, y_{n-1}, a\}$ is an antichain of n elements in $\boldsymbol{Y}$, for every $a \in T_z$.

Now, suppose with a view to contradiction that T_z is not a chain. Then there are two incomparable elements $a, c \in T_z$. Hence $\{y_1, \ldots, y_{n-1}, a, c\}$ is an antichain of $n+1$ elements in $\boldsymbol{Y}$. Together with the fact that $\boldsymbol{Y}$ has a minimum element by assumption (i), we conclude that $\boldsymbol{Y}$ does not have width at most n. But this contradicts the assumptions, thus establishing the claim.

By Lemma 6.16(v), the chain T_z has a maximum element, which we denote by $\max(T_z)$. Consider the set

$$Z := \{\max(T_z)\colon z \in \uparrow f(\bot) \setminus \{m, f(\bot)\}\} \cup \{\bot, m\}.$$

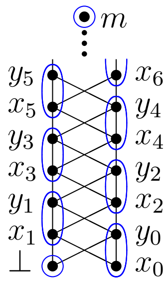
Clearly $Z \subseteq Y$, and it is easy to verify that the restriction

$$f\colon \langle Z; \leqslant^{\boldsymbol{Y}} \rangle \to \langle \uparrow f(\bot); \leqslant^{\boldsymbol{X}} \rangle$$

is a surjective order-preserving map. In order to prove that f is a poset isomorphism, it remains only to show that f is order-reflecting. To this end, consider $z_1, z_2 \in Z$ such that $f(z_1) \leqslant^{\boldsymbol{X}} f(z_2)$. If $z_1 = \bot$, then clearly $z_1 = \bot \leqslant^{\boldsymbol{Y}} z_2$, and we are done. Also, if $z_2 = m$, then $z_1 \leq z_2 = m$. So, consider the case where $z_1 \neq \bot$ and $z_2 \neq m$. In particular, this implies that $f(z_1), f(z_2) \in \uparrow f(\bot) \setminus \{m, f(\bot)\}$, and therefore that $z_2 = \max(T_{f(z_2)})$. Now, since $f(z_1) \leqslant^{\boldsymbol{X}} f(z_2)$ and f is an Esakia morphism, there exists $z_3 \in Y$ such that $z_1 \leqslant^{\boldsymbol{Y}} z_3$ and $f(z_3) = f(z_2)$. But then $z_3 \in T_{f(z_2)}$, which implies that $z_3 \leqslant^{\boldsymbol{Y}} z_2$ and, therefore, that $z_1 \leqslant^{\boldsymbol{Y}} z_3 \leqslant^{\boldsymbol{Y}} z_2$. Thus, we conclude that f is order-reflecting, as desired. □

Lemma 7.18. *Let* K *be a subvariety of* $\mathsf{ID}_2 \cap \mathsf{W}_2$*. If* $\boldsymbol{D}_2^\infty \in \mathsf{K}_*$*, then* K *lacks the ES property.*

Proof. Suppose with a view to contradiction that there is a variety $\mathsf{K} \subseteq \mathsf{ID}_2 \cap \mathsf{W}_2$ with the ES property and such that $\boldsymbol{D}_2^\infty \in \mathsf{K}_*$. Let R be the equivalence relation on $\boldsymbol{D}_2^\infty$ whose corresponding partition is depicted in the diagram below:



We start by showing that R is a correct partition on $\boldsymbol{D}_2^\infty$. It is easily verified that R satisfies condition (i) in the definition of correct partitions on page 136. Notice that if U is any finite subset of $D_2^\infty \setminus \{m\}$, then U is clopen in $\boldsymbol{D}_2^\infty$. Indeed, it is open by the definition of the topology on $\boldsymbol{D}_2^\infty$, since the topology on $\boldsymbol{D}_2$ is discrete. That U is closed follows from the fact that singletons are closed in Hausdorff spaces. To prove condition (ii), we consider two distinct points $x, y \in D_2^\infty$ such that $\langle x, y\rangle \notin R$. If $x \neq m$, then the equivalence class x/R is a finite subset of $\boldsymbol{D}_2^\infty \setminus \{m\}$, and thus clopen. Also, x/R is clearly a union of equivalence classes of R with $x \in x/R$ and $y \notin x/R$, as required. Now suppose that $x = m$. Then there is some $k \in \omega$ such that $y \notin {\uparrow} x_k$. Note that $x \in {\uparrow} x_k$ and ${\uparrow} x_k$ is a union of R-equivalence classes. Also, ${\uparrow} x_k$ is clopen, since $({\uparrow} x_k)^c$ is a finite subset of $D_2^\infty \setminus \{m\}$.

Since K has the ES property, we can apply Lemma 6.18, so there exist $\boldsymbol{Y} \in \mathsf{K}_*$ and a pair of different Esakia morphisms $f, g \colon \boldsymbol{Y} \to \boldsymbol{D}_2^\infty$ such that $\langle f(y), g(y)\rangle \in R$ for every $y \in Y$.

Since $f \neq g$, there exists $\bot \in Y$ such that $f(\bot) \neq g(\bot)$. Together with the fact that $\langle f(\bot), g(\bot)\rangle \in R$, this implies that $\{f(\bot), g(\bot)\} = \{x_n, y_n\}$ for some $n \in \omega$. We can assume without loss of generality that $f(\bot) = x_n$ and $g(\bot) = y_n$. Note that ${\uparrow}\bot$ is an E-subspace of $\boldsymbol{Y}$ which belongs to K_*, since K is closed under homomorphic images. Also, by Lemma 6.16(iii), the restrictions of f and g to ${\uparrow}\bot$ are Esakia morphisms. We may therefore assume without loss of generality that $\boldsymbol{Y} = {\uparrow}\bot$ (otherwise we replace $\boldsymbol{Y}$ with ${\uparrow}\bot$).

Observe $\boldsymbol{D}_2^\infty$ and $\boldsymbol{Y}$ have width at most 2. Moreover, $\boldsymbol{Y}$ and $\boldsymbol{D}_2^\infty$, and the Esakia morphism $f \colon \boldsymbol{Y} \to \boldsymbol{D}_2^\infty$ satisfy the assumptions of Lemma 7.17.

Therefore, $\boldsymbol{Y}$ has a subposet $\langle Z; \leqslant^{\boldsymbol{Y}} \rangle$ such that the restriction

$$f\colon \langle Z; \leqslant^{\boldsymbol{Y}} \rangle \to \langle \uparrow f(\bot); \leqslant^{\boldsymbol{D}_2^{\infty}} \rangle$$

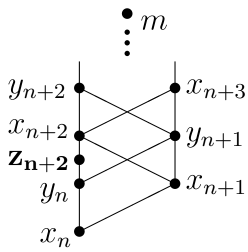
is a poset isomorphism. For the sake of simplicity, we denote the elements of Z exactly as their alter egos in $\uparrow f(\bot)$. Under this convention,

$$Z = \{x_{n+p} \colon p \in \omega\} \cup \{y_{n+p} \colon p \in \omega\} \cup \{m\},$$

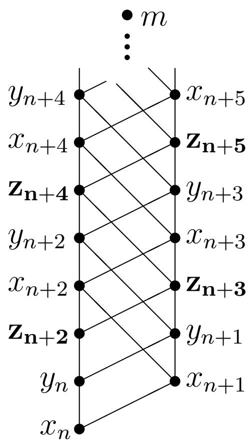
and $f(x_i) = x_i$ and $f(y_i) = y_i$, for every $x_i, y_i \in Z \setminus \{m\}$. On the other hand, since g is order-preserving and $g(\bot) = y_n$, we have $g(x_i) = g(y_i) = y_i$, for all $x_i, y_i \in Z \setminus \{m\}$. In summary, for all $x_i, y_i \in Z \setminus \{m\}$, we have

$$f(x_i) = x_i \text{ and } f(y_i) = g(x_i) = g(y_i) = y_i.$$

We shall now interrogate the structure of $\boldsymbol{Y}$ to produce a sequence of elements $z_{n+2}, z_{n+3}, \dots \in Y \setminus Z$ and describe how they are ordered with respect to the elements of Z. First, observe that $g(y_n) = y_n \leqslant^{\boldsymbol{D}_2^{\infty}} x_{n+2}$. Since g is an Esakia morphism, there is an element $z_{n+2} \in Y$ such that $y_n \leqslant^{\boldsymbol{Y}} z_{n+2}$ and $g(z_{n+2}) = x_{n+2}$. Let us describe the structure of the poset $\langle Z \cup \{z_{n+2}\}; \leqslant^{\boldsymbol{Y}} \rangle$. First observe that z_{n+2} is incomparable with x_{n+1} and y_{n+1} with respect to $\leqslant^{\boldsymbol{Y}}$, since g is order-preserving and $g(z_{n+2}) = x_{n+2}$ is incomparable with $g(x_{n+1}) = g(y_{n+1}) = y_{n+1}$ in $\boldsymbol{D}_2^{\infty}$. Moreover, $z_{n+2} \leqslant^{\boldsymbol{Y}} x_{n+2}$. To prove this, observe that x_{n+2} and y_{n+1} are incomparable in $\boldsymbol{Y}$. Since $\boldsymbol{Y}$ has width at most 2, this implies that z_{n+2} must be comparable with one of them. Since z_{n+2} is incomparable with y_{n+1}, if follows that z_{n+2} is comparable with x_{n+2}. Keeping in mind that $g(x_{n+2}) = y_{n+2} \nleqslant^{\boldsymbol{D}_2^{\infty}} x_{n+2} = g(z_{n+2})$ and that g is order-preserving, we obtain $x_{n+2} \nleqslant^{\boldsymbol{Y}} z_{n+2}$. As a consequence, we conclude that $z_{n+2} <^{\boldsymbol{Y}} x_{n+2}$ as desired. Summing up, the structure of $\langle Z \cup \{z_{n+2}\}; \leqslant^{\boldsymbol{Y}} \rangle$ is described exactly by the following picture:



Now, observe that $g(z_{n+2}) = x_{n+2} \leqslant^{\boldsymbol{D}_2^{\infty}} x_{n+3}$. Since g is an Esakia morphism, there is an element $z_{n+3} \in Y$ with $z_{n+2} \leqslant^{\boldsymbol{Y}} z_{n+3}$ such that $g(z_{n+3}) = x_{n+3}$. We can replicate the previous argument, used to describe the structure of the poset $\langle Z \cup \{z_{n+2}\}; \leqslant^{\boldsymbol{Y}}\rangle$, to show that z_{n+3} is incomparable with x_{n+2} and y_{n+2}, and that $z_{n+3} <^{\boldsymbol{Y}} x_{x+3}$. Then, as in the previous argument, $y_{n+1} <^{\boldsymbol{Y}} z_{n+3}$. Iterating this process we construct a series of elements $\{z_{n+p} \colon 2 \leq p \in \omega\} \subseteq Y$ such that $g(z_i) = x_i$, for all $i \geq 2$. The structure of the poset $\mathbb{Z}' := \langle Z \cup \{z_{n+p} \colon 2 \leq p \in \omega\}; \leqslant^{\boldsymbol{Y}}\rangle$ is as depicted below:



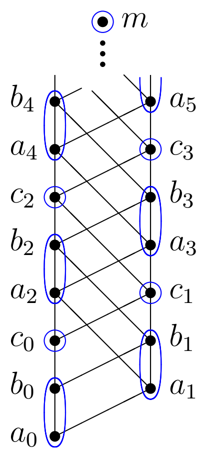
We claim that for every $a \in Y$ such that $x_{n+2} \leqslant^{\boldsymbol{Y}} a$, either $a \in Z'$ or $b \leqslant^{\boldsymbol{Y}} a$ for every $b \in Z' \setminus \{m\}$. To prove this, consider $a \in Y$ such that $x_{n+2} \leqslant^{\boldsymbol{Y}} a$ and $a \notin Z'$. It will be enough to show that $x_{n+p} \leqslant^{\boldsymbol{Y}} a$ for $2 < p \in \omega$. Suppose, with a view to contradiction, that there is a smallest integer $p > 2$ such that $x_{n+p} \nleqslant^{\boldsymbol{Y}} a$. Looking at the figure above, it is easy to see that every point in $Z' \setminus \{x_n, y_n, m\}$ is incomparable with two elements in ${\uparrow} x_n$. Since $\boldsymbol{Y}$ has incomparability degree at most 2, it follows that every element in ${\uparrow} x_n \setminus Z'$ is comparable with all the elements of $Z' \setminus \{x_n, y_n\}$. We shall make extensive use of this observation. First recall that $x_{n+p} \nleqslant^{\boldsymbol{Y}} a$. As a is comparable with x_{n+p}, this implies that $a <^{\boldsymbol{Y}} x_{n+p}$. Moreover, a is comparable with y_{n+p-1}. Since $y_{n+p-1} \nleqslant^{\boldsymbol{Y}} x_{n+p}$ and $a <^{\boldsymbol{Y}} x_{n+p}$, it follows that $a <^{\boldsymbol{Y}} y_{n+p-1}$. Now, a is comparable with z_{n+p}. If $z_{n+p} \leqslant^{\boldsymbol{Y}} a$, then $z_{n+p} \leqslant^{\boldsymbol{Y}} y_{n+p-1}$, which is false. Thus, $a <^{\boldsymbol{Y}} z_{n+p}$. By minimality of p we have $x_{n+p-1} \leqslant^{\boldsymbol{Y}} a$. This yields that $x_{n+p-1} \leqslant^{\boldsymbol{Y}} z_{n+p}$. This contradiction establishes the claim.

From the definition of a pointed Esakia space we know that the upset ${\uparrow} x_{n+2}$ in $\boldsymbol{Y}$ is closed and, therefore, an E-subspace of $\boldsymbol{Y}$. Moreover, ${\uparrow} x_{n+2} \in \mathsf{K}_*$, since K is closed under homomorphic images. Consider the equivalence relation S on ${\uparrow} x_{n+2}$ defined as follows: for every $a, b \in {\uparrow} x_{n+2}$,

$$\langle a, b\rangle \in S \text{ iff } \text{ either } a = b \text{ or } a, b \notin Z' \setminus \{m\}.$$

We shall prove that S is a correct partition on $\uparrow x_{n+2}$. To this end, observe that from the claim it follows that S satisfies condition (i) in the definition of a correct partition (page 136). In order to prove condition (ii), consider $a, b \in \uparrow x_{n+2}$ such that $\langle a, b\rangle \notin S$. We can assume without loss of generality that $b \in Z' \setminus \{m\}$. If $b \in \{x_{n+2}, y_{n+2}\}$, let $b' = z_{n+4}$; otherwise, let $b' = b$. Let c be the minimum element of $\uparrow x_{n+2}$ that is incomparable with b'. By the Priestley separation axiom (item (iv) on page 134), since $c \nleqslant^{\boldsymbol{Y}} b'$, there is a clopen upset U such that $c \in U$ and $b' \notin U$. Looking at the above picture, it is easy to see that $U = \uparrow c$ and $U^c = \downarrow b'$. In particular, $b \in \downarrow b' = U^c$. By the claim above, $a \in \uparrow c = U$. The fact that U and U^c are unions of equivalence classes of S follows from the definition of S. This establishes condition (ii) and, therefore, that S is a correct partition on $\uparrow x_{n+2}$.

Then let $\boldsymbol{W}$ be the pointed Esakia space $(\uparrow x_{n+2})/S$. Observe that $\boldsymbol{W} \in \mathsf{K}_*$, since K is closed under homomorphic images. Moreover, the poset underlying $\boldsymbol{W}$ is isomorphic to $Z' \cap \uparrow x_{n+2}$, because by the claim and the definition of S all elements of $\uparrow x_{n+2} \setminus Z'$ are identified with m. Now, consider the equivalence relation T on W whose corresponding partition is depicted below:



An argument, similar to the one detailed in the case of S, shows that the relation T is a correct partition on $\boldsymbol{W}$, except that in this case we let b be such that $a \nleqslant b$, and let b' be c_0 if $b \in \{a_0, b_0\}$ and the maximum of the equivalence class b/T otherwise.

Since K has the ES property, we can apply Lemma 6.18, so there exist $\boldsymbol{V} \in \mathsf{K}_*$ and a pair of different Esakia morphisms $f, g \colon \boldsymbol{V} \to \boldsymbol{W}$ such that $\langle f(v), g(v)\rangle \in T$ for every $v \in V$. As above, since $f \neq g$, there are $\bot \in V$ and $n \in \omega$ such that $\{f(\bot), g(\bot)\} = \{a_n, b_n\}$. We can assume without loss of generality that $f(\bot) = a_n$ and $g(\bot) = b_n$, and that $V = \uparrow \bot$. Moreover, we can find a subposet $\langle Q; \leqslant^{\boldsymbol{V}} \rangle$ of $\boldsymbol{V}$ such that the restriction

$$f \colon \langle Q; \leqslant^{\boldsymbol{V}} \rangle \to \langle \uparrow f(\bot); \leqslant^{\boldsymbol{W}} \rangle$$

is a poset isomorphism. We denote the elements of Q exactly as their alter egos in $\uparrow f(\bot)$. Under this convention,

$$Q = \{a_{n+p} \colon p \in \omega\} \cup \{b_{n+p} \colon p \in \omega\} \cup \{c_{n+p} \colon p \in \omega\} \cup \{m\},$$

and for every $a_i, b_i, c_i \in Q \setminus \{m\}$,

$$f(a_i) = a_i \text{ and } f(b_i) = g(a_i) = g(b_i) = b_i \text{ and } f(c_i) = g(c_i) = c_i.$$

Observe that $g(b_n) = b_n \leqslant^{\boldsymbol{W}} a_{n+2}$. Since g is an Esakia morphism, there exists $v \in V$ such that $b_n \leqslant^{\boldsymbol{V}} v$ and $g(v) = a_{n+2}$. So,

$$\{g(v), g(a_{n+2}), g(b_{n+2})\} = \{a_{n+2}, b_{n+2}\} \text{ and } g(c_{n+1}) = c_{n+1}.$$

Therefore, the elements $g(v), g(a_{n+2}), g(b_{n+2})$ are incomparable with $g(c_{n+1})$ in $\boldsymbol{W}$. Since g is order-preserving, v, a_{n+2}, b_{n+2} are incomparable with c_{n+1} in $\boldsymbol{V}$. Because $a_{n+2} \neq b_{n+2}$ and $\boldsymbol{V}$ has incomparability degree ≤ 2, we conclude that either $v = a_{n+2}$ or $v = b_{n+2}$. Observe that if $v = a_{n+2}$, then $a_{n+2} = g(v) = g(a_{n+2}) = b_{n+2}$, which is false. A similar argument rules out the case where $v = b_{n+2}$. We have therefore reached a contradiction, as desired. □

As was already mentioned, until now the only published example of a variety of Brouwerian algebras without the ES property was precisely $\mathbb{V}((\boldsymbol{D}_2^{\infty})^*)$. This example is now subsumed by the above lemma.

Before showing that there is a continuum of locally finite varieties in the interval $[\mathbb{V}((\boldsymbol{D}_2^{\infty})^*), \mathsf{W}_2 \cap \mathsf{ID}_2]$, we need to recall some facts about the connection between Brouwerian algebras and Heyting algebras, and to introduce the 'Kuznetsov-Gerčiu variety'.

If $\boldsymbol{A}$ is a Brouwerian algebra, we let $\boldsymbol{A}_{\bot}$ denote the unique Heyting algebra whose lattice reduct is got by adding a new least element $\bot$ to $\langle A; \wedge, \vee \rangle$. For a variety K of Brouwerian algebras let $\mathsf{K}_{\bot} := \mathbb{V}(\{\boldsymbol{A}_{\bot} \colon \boldsymbol{A} \in \mathsf{K}\})$. The proof of the following lemma can be found in [88, Lems. 3.2, 3.5].

Lemma 7.19. *Let K be a variety of Brouwerian algebras and $\boldsymbol{A}$ a Brouwerian algebra.*

(i) $\boldsymbol{A} \in \mathsf{K}$ *iff* $\boldsymbol{A}_{\bot} \in \mathsf{K}_{\bot}$.

(ii) *If $\mathsf{K} = \mathbb{V}(\mathsf{L})$ for some class L of Brouwerian algberas, then*

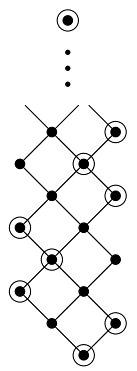
$$\mathsf{K}_{\bot} = \mathbb{V}(\{\boldsymbol{A}_{\bot} : \boldsymbol{A} \in \mathsf{L}\}).$$

The original Esakia duality of [44] supplies an equivalence between the category HA of Heyting algebras (and their homomorphisms—which must preserve $\bot$) and the opposite of the category ESP of *Esakia spaces.*

The objects of ESP are like those of PESP, except that they need not have maximum elements; the definition of morphisms is unaffected. For $\boldsymbol{A} \in \mathsf{HA}$ and $\boldsymbol{X} \in \mathsf{ESP}$, we *re-define* $\mathrm{Pr}(\boldsymbol{A})$ as the set of prime *proper* filters of $\boldsymbol{A}$, and $\mathrm{Cpu}(\boldsymbol{X})$ as the set of *all* clopen upsets of $\boldsymbol{X}$, including $\emptyset$. After these changes, the definitions of $\boldsymbol{A}_*$, $\boldsymbol{X}^*$, the duals of morphisms, and the canonical isomorphisms remain the same (but note that $\boldsymbol{A}_*$ is empty when $|A| = 1$). The definition of depth is adjusted so that a Heyting algebra and its Brouwerian reduct have the same depth. (In particular, the depth of the Esakia space reduct of a pointed Esakia space $\boldsymbol{X}$ exceeds that of $\boldsymbol{X}$ by 1.)

If $\boldsymbol{A}$ is a Heyting algebra, we let $\boldsymbol{A}^+$ denote its Brouwerian reduct. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be Heyting algebras. Notice that $\boldsymbol{A}^+ + \boldsymbol{B}^+$ has a least element, namely the least element of $\boldsymbol{B}^+$. Also, the least element of $\boldsymbol{A}^+$ is identified with the greatest element of $\boldsymbol{B}^+$ in $\boldsymbol{A}^+ + \boldsymbol{B}^+$. We define $\boldsymbol{A} + \boldsymbol{B}$ be the unique Heyting algebra whose Brouwerian reduct is $\boldsymbol{A}^+ + \boldsymbol{B}^+$. Notice that if $\boldsymbol{A}$ is a Heyting algebra then $(\boldsymbol{A}^+)_\bot = \boldsymbol{A} + \mathbf{2}$, where $\mathbf{2}$ is the two-element Boolean algebra, considered as a Heyting algebra.

It is well known that the free 1-generated Heyting algebra is the *Rieger-Nishimura* lattice $\boldsymbol{RN}$, depicted below [109, 124]. As a consequence, $\mathbb{H}(\boldsymbol{RN})$ is the class of 1-generated Heyting algebras.



The *Kuznetsov-Gerčiu* variety is defined as follows:

$$\mathsf{KG} := \mathbb{V}(\{\boldsymbol{A}_1 + \dots + \boldsymbol{A}_n \colon 0 < n \in \omega \text{ and } \boldsymbol{A}_1, \dots, \boldsymbol{A}_n \in \mathbb{H}(\boldsymbol{RN})\}). \quad (7.6)$$

The variety KG was introduced by Kuznetsov and Gerčiu [58, 80] in a study of varieties of Heyting algebras that are finitely axiomatized and/or generated by their finite members (also see [10, 12]). Remarkably, a continuum of subvarieties of KG are generated by their finite members, and also a continuum fail to have this property [10, Thm. 5.39(1), Cor. 5.41]. We characterized the subvarieties of KG that have the ES property in Moraschini and Wannenburg [108].

Theorem 7.20 (Bezhanishvili et al. [10, Thms. 8.49, 8.54])**.** *Let* K *be a subvariety of* KG*. The following conditions are equivalent:*

(i) K *is locally finite.*

(ii) K *excludes an algebra of the form* $\boldsymbol{A}+\boldsymbol{2}$ *where* $\boldsymbol{A}$ *is a finite SI member of* $\mathbb{H}(\boldsymbol{RN})$.

Lemma 7.21. $(\boldsymbol{D}_2^\infty)^* \in \mathbb{V}(\boldsymbol{RN}^+)$.

Proof. Recall that every algebra embeds into a ultraproduct of its finitely generated subalgebras (Theorem 1.4). Now, observe that the finitely generated subalgebras of $(\boldsymbol{D}_2^\infty)^*$ coincide with finite sums, where each summand is the RL-reduct of either the two-element or the four-element Boolean algebra. It is therefore not hard to see, when considering the subalgebra of encircled elements in the figure of $\boldsymbol{RN}$ above, that every finitely generated subalgebra of $(\boldsymbol{D}_2^\infty)^*$ belongs to $\mathbb{HS}(\boldsymbol{RN}^+)$. As a consequence, $(\boldsymbol{D}_2^\infty)^* \in \mathbb{SP}_{\mathbb{U}}\mathbb{HS}(\boldsymbol{RN}^+) \subseteq \mathbb{V}(\boldsymbol{RN}^+)$. □

Theorem 7.22. *There are* $2^{\aleph_0}$ *locally finite subvarieties of* $\mathsf{W}_2 \cap \mathsf{ID}_2$ *without the ES property.*

Proof. Define $F := \{\boldsymbol{B}_n \colon n \in \omega\}$, where each algebra $\boldsymbol{B}_n$ is as defined before Lemma 7.15. From Corollary 7.16, $\mathbb{V}(S) \neq \mathbb{V}(T)$, for every pair of different subsets $S, T \subseteq F$.

We claim that $\mathbb{V}(S, (\boldsymbol{D}_2^\infty)^*) \neq \mathbb{V}(T, (\boldsymbol{D}_2^\infty)^*)$, for every pair of different subsets $S, T \subseteq F$. To prove this, consider two different $S, T \subseteq F$. Since $\mathbb{V}(S) \neq \mathbb{V}(T)$, we can assume without loss of generality that $\boldsymbol{B}_n \in \mathbb{V}(S) \setminus \mathbb{V}(T)$ for some $n \in \omega$. Suppose with a view to contradiction that $\mathbb{V}(S, (\boldsymbol{D}_2^\infty)^*) = \mathbb{V}(T, (\boldsymbol{D}_2^\infty)^*)$. In particular, $\boldsymbol{B}_n \in \mathbb{V}(T, (\boldsymbol{D}_2^\infty)^*)_{\mathrm{FSI}}$. Now, from (1.4) on page 15, it follows that $\mathbb{V}(T, (\boldsymbol{D}_2^\infty)^*)_{\mathrm{FSI}} = \mathbb{V}(T)_{\mathrm{FSI}} \cup \mathbb{V}((\boldsymbol{D}_2^\infty)^*)_{\mathrm{FSI}}$. Since $\boldsymbol{B}_n \notin \mathbb{V}(T)_{\mathrm{FSI}}$, we have $\boldsymbol{B}_n \in \mathbb{V}((\boldsymbol{D}_2^\infty)^*)$. Now, observe that $(\boldsymbol{D}_2^\infty)^* \in \mathsf{ID}_1$. As a consequence, $\boldsymbol{B}_n \in \mathsf{ID}_1$. But this is easily seen to be false. So, we have reached a contradiction, thus establishing the claim.

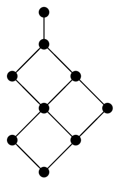
From the claim it follows that the set

$$G := \{\mathbb{V}(T, \boldsymbol{D}_2^\infty) \colon T \subseteq F\}$$

has the cardinality of the continuum. Consider $T \subseteq F$. By Lemma 7.18, $\mathbb{V}(T, (\boldsymbol{D}_2^\infty)^*) \subseteq \mathsf{W}_2 \cap \mathsf{ID}_2$ lacks the ES property. It remains to show that $\mathbb{V}(T, (\boldsymbol{D}_2^\infty)^*)$ is locally finite.

We first claim that $\mathbb{V}(T, (\boldsymbol{D}_2^\infty)^*)_\perp \subseteq \mathsf{KG}$. From Lemmas 7.19(i) and 7.21, it follows that $((\boldsymbol{D}_2^\infty)^*)_\perp \in \mathbb{V}((\boldsymbol{RN}^+)_\perp)$. But $(\boldsymbol{RN}^+)_\perp = \boldsymbol{RN} + \mathbf{2} \in \mathsf{KG}$. So, $((\boldsymbol{D}_2^\infty)^*)_\perp \in \mathsf{KG}$. Consider $\boldsymbol{B}_n \in T$, for some $n \in \omega$. Then $(\boldsymbol{B}_n)_\perp \cong \mathbf{2} + \boldsymbol{A}' + \boldsymbol{C}'_1 + \cdots + \boldsymbol{C}'_n + \boldsymbol{A}' + \mathbf{2}$, where $\boldsymbol{A}'$ the direct product of the two-element and three-element Heyting algebras and $\boldsymbol{C}'_i \cong \mathbf{2} \times \mathbf{2}$ for every $i \leq n$. It is easy to see that $\mathbf{2}, \mathbf{2} \times \mathbf{2}$, and $\boldsymbol{A}'$ are 1-generated, whence they belong to $\mathbb{H}(\boldsymbol{RN})$, from which it follows that $\boldsymbol{B}_n \in \mathsf{KG}$. So, the claim follows from Lemma 7.19(ii).

Then we turn to prove that $\mathbb{V}(T, (\boldsymbol{D}_2^\infty)^*)_\perp$ is locally finite. Let $\boldsymbol{D}$ be the Brouwerian algebra depicted below:



Observe that the equation

$$\bigvee_{i=1}^{3} (x \to y_i) \vee (y_i \to x) \approx e$$

holds in $\mathbb{V}(T, (\boldsymbol{D}_2^\infty)^*)$ but fails in $\boldsymbol{D}$. As a consequence, $\boldsymbol{D} \notin \mathbb{V}(T, (\boldsymbol{D}_2^\infty)^*)$. Now, from Lemma 7.19(i), it follows that $\boldsymbol{D}_\perp \notin \mathbb{V}(T, (\boldsymbol{D}_2^\infty)^*)_\perp$. But $\boldsymbol{D}_\perp \cong \boldsymbol{D}' + \mathbf{2}$, where $\boldsymbol{D}'$ is $\boldsymbol{D}$ considered as a Heyting algebra. Since $\boldsymbol{D}_\perp$ has the form of one of the algebras in condition (ii) of the statement of Theorem 7.20, we conclude that $\mathbb{V}(T, (\boldsymbol{D}_2^\infty)^*)_\perp$ is locally finite.

Finally, let $\boldsymbol{E} \in \mathbb{V}(T, (\boldsymbol{D}_2^\infty)^*)$ be n-generated, for some $n \in \omega$. Then $\boldsymbol{E}_\perp \in \mathbb{V}(T, (\boldsymbol{D}_2^\infty)^*)_\perp$, by Lemma 7.19(i). Note that $\boldsymbol{E}_\perp$ is also n-generated. Since $\mathbb{V}(T, (\boldsymbol{D}_2^\infty)^*)_\perp$ is locally finite, $\boldsymbol{E}_\perp$ is finite. But then $\boldsymbol{E}$ is also finite. Therefore, $\mathbb{V}(T, (\boldsymbol{D}_2^\infty)^*)$ is locally finite. □

Recall that every variety of Brouwerian algebras has the weak ES property (Theorem 6.5(ii)). The next theorem therefore follows from the fact that $\mathbb{R}$ is injective (Lemma 3.38), and that $\mathbb{R}$ preserves local finiteness, the weak ES property and failure of the ES property (see Lemma 3.37 and Theorem 6.43).

Theorem 7.23. *There are $2^{\aleph_0}$ locally finite varieties of De Morgan monoids that lack the ES property, but that do have the weak ES property.*

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List of Symbols

$\mathbf{2}$	The two-element Boolean algebra	53
$\boldsymbol{A}_*$	Dual space of $\boldsymbol{A}$	135
$\boldsymbol{A}^{**}$	Odd Sugihara monoid subalgebra of a totally ordered idempotent RL $\boldsymbol{A}$	150
$\prod_{i\in I} \boldsymbol{A}_i$	Direct product of a family $\{\boldsymbol{A}_i : i \in I\}$	2
$\prod_{i\in I} \boldsymbol{A}_i/\mathcal{U}$	Ultraproduct over I	5
A^-	Negative elements of [I]RL $\boldsymbol{A}$	40
$\boldsymbol{A}^-$	Negative cone of [I]RL $\boldsymbol{A}$	131
A_c	Interval $\{a \in A\colon a^{**} = c\}$ of a totally ordered idempotent RL $\boldsymbol{A}$	150
$\boldsymbol{A}/F$	Abbreviates $\boldsymbol{A}/\boldsymbol{\Omega}^{\boldsymbol{A}}F$	33
$\boldsymbol{A}^I$	Direct power	3
$\boldsymbol{A}_S$	The algebra where the elements of S are added to $\boldsymbol{A}$ as new constant symbols	16
$\boldsymbol{A}/\theta$	Factor algebra	3
$(a]$	Set of lower bounds of an element a (including a itself) in a partially ordered set	9
$[a)$	Set of upper bounds of an element a (including a itself) in a partially ordered set	9
$[a, b]$	The interval $\{c \in P\colon a \leqslant c \leqslant b\}$ where a and b are elements of a partially ordered set $\langle P; \leqslant\rangle$	9
a/F	The $\boldsymbol{\Omega}^{\boldsymbol{A}}F$ equivalence class of a	33
a/θ	The θ equivalence class of a	3
$a/\mathcal{U}$	Element of an ultraproduct	5
BA	Class of Boolean algebras	39
BRA	Class of Brouwerian algebras	40
$\boldsymbol{C}_4$	The four-element simple 0-generated De Morgan monoid chain	53
$\boldsymbol{C}_5, \dots, \boldsymbol{C}_8$	The non-simple SI 0-generated De Morgan monoids	65
CEP	Congruence extension property	15
$\mathrm{Con}(\boldsymbol{A})$	Set of congruences of algebra $\boldsymbol{A}$	4
$\mathrm{Con}_{\mathsf{K}}(\boldsymbol{A})$	Set of K-congruences of algebra $\boldsymbol{A}$	4

$\mathbf{Con}(\boldsymbol{A})$	Congruence lattice of algebra $\boldsymbol{A}$	11
$\mathbf{Con}_{\mathsf{K}}(\boldsymbol{A})$	K-congruence lattice of algebra $\boldsymbol{A}$	11
CPL	Classical propositional logic	25
$\mathrm{Cpu}(\boldsymbol{X})$	Set of (non-empty) clopen upsets of $\boldsymbol{X}$	135
$\boldsymbol{D}_4$	The four-element simple 0-generated De Morgan monoid diamond	53
$\mathrm{depth}(\boldsymbol{A})$	Depth of a Brouwerian algebra $\boldsymbol{A}$	139
$\mathrm{Diag}(\boldsymbol{A})$	Set of all atomic sentences that are true in $\boldsymbol{A}_A$ (cf. $\boldsymbol{A}_S$)	16
DMM	Class of De Morgan monoids	36
D_n	Class of Brouwerian algebras with depth at most n	139
$\mathbb{E}(\mathsf{K})$	Class of extensions of members of K	5
EDPC	Equationally definable principal congruences	15
EDPM	Equationally definable principal meets	15
ES	Epimorphism surjectivity	127
$\mathrm{Fg}^{\boldsymbol{A}}\,X$	Smallest deductive filter of [I]RL $\boldsymbol{A}$ containing $X \subseteq A$	33
$\mathbf{Fil}(\boldsymbol{A})$	Lattice of deductive filters of [I]RL $\boldsymbol{A}$	33
$\boldsymbol{F}_{\mathsf{K}}(X)$	K-free algebra over X, belonging to $\mathbb{ISP}(\mathsf{K})$	7
$\mathbf{FL}_e$	Full Lambek calculus with exchange	24
$\mathbf{FL}_{ec}$	Full Lambek calculus with exchange and contraction	25
FSI	Finitely subdirectly irreducible	4
GSM	Class of generalized Sugihara monoids	42
$\mathbb{H}(\mathsf{K})$	Class of homomorphic images of members of K	5
HSC	Hereditary structural completeness	113
$\mathbb{I}(\mathsf{K})$	Class of structures isomorphic to ones in K	5
id_X	Identity map on set X	11
ID_n	Class of Brouwerian algebras with incomparability degree at most n	172
$\inf X$	Infimum of X	10

IPL	Intuitionistic propositional logic	25
$\mathbf{IPL}^+$	Positive intuitionistic propositional logic	25
IRL	Involutive (commutative) residuated lattice	28
JEP	Joint embedding property	105
K_*	Isomorphic closure of $\{\boldsymbol{A}_* \colon \boldsymbol{A} \in \mathsf{K}\}$	135
$\ker h$	Kernel of h	3
$\mathsf{K}_{\mathrm{FSI}}$	Finitely subdirectly irreducible members of class K	13
$\mathsf{K}_{\mathrm{RFSI}}$	Relatively finitely subdirectly irreducible members of class K	13
K_ρ	Class of algebras with similarity type ρ	7
K_{RS}	Relatively simple members of class K	13
$\mathsf{K}_{\mathrm{RSI}}$	Relatively subdirectly irreducible members of class K	13
K_{S}	Simple members of class K	13
K_{SI}	Subdirectly irreducible members of class K	13
$\mathbf{L}^+$	Positive fragment of logic $\mathbf{L}$	25
$\mathbf{LR}^{\mathbf{t}}$	Non-distributive relevance logic with Ackermann constants	25
M	Largest subvariety of N	59
$\boldsymbol{M}_3$	The five-element non-distributive modular lattice	14
N	Class of De Morgan monoids that are either trivial or have $\boldsymbol{C}_4$ as a retract	55
$\boldsymbol{N}_5$	The five-element non-modular lattice	14
OSM	Class of odd Sugihara monoids	38
$\mathbb{P}(\mathsf{K})$	Class of direct products of members of K	5
PESP	Class of pointed Esakia spaces	134
$\mathrm{Pr}(\boldsymbol{A})$	Set of (non-empty possibly total) prime deductive filters of S[I]RL $\boldsymbol{A}$	133

$\mathbb{P}_{\mathbb{S}}(\mathsf{K})$	Class of subdirect products of members of K	5
PSC	Passive structural completeness	112
$\mathbb{P}_{\mathbb{U}}(\mathsf{K})$	Class of ultraproducts of members of K	5
$\mathbb{Q}(\mathsf{K})$	Quasivariety generated by class K	8
$\mathbb{R}(\mathsf{K})$	Variety generated by the reflections of members of K	69
$\mathbf{R}$	Relevance logic (without Ackermann constants)	25
$\mathrm{R}(\boldsymbol{A})$	Reflection of algebra $\boldsymbol{A}$	67
RA	Class of relevant algebras	110
$\mathrm{Ret}(\mathsf{K}, \boldsymbol{A})$	$\{\boldsymbol{B} \in \mathsf{K} : \boldsymbol{B}$ is trivial or $\boldsymbol{A}$ is a retract of $\boldsymbol{B}\}$	118
RL	(Commutative) residuated lattice	28
$\mathbf{RM^t}$	Relevance logic with Ackermann constants and mingle	25
$\boldsymbol{RN}$	Rieger-Nishimura lattice	186
RSA	Class of relative Stone algebras	40
$\mathbf{R^t}$	Relevance logic with Ackermann constants	25
$\mathbb{R}_{\mathbb{U}}(\mathsf{K})$	Class of ultraroots of members of K	5
$\mathbb{S}(\mathsf{K})$	Class of subalgebras of members of K	5
$\mathrm{S}^{\leqslant}(\boldsymbol{A})$	Skew $\leqslant$-reflection of algebra $\boldsymbol{A}$	61
$\boldsymbol{S}[\boldsymbol{A}]$	Rigorous extension of De Morgan monoid $\boldsymbol{A}$ by totally ordered odd Sugihara monoid $\boldsymbol{S}$	48
SC	Structural completeness	113
$\mathrm{Sg}^{\boldsymbol{A}}\, X$	Universe of the subalgebra of $\boldsymbol{A}$ generated by $X \subseteq A$	2
$\mathbf{Sg}^{\boldsymbol{A}} X$	Subalgebra of $\boldsymbol{A}$ generated by $X \subseteq A$	2
SI	Subdirectly irreducible	4
SIRL	Square-increasing involutive (commutative) residuated lattice	31
SM	Class of Sugihara monoids	36
SRL	Square-increasing (commutative) residuated lattice	31
$\sup X$	Supremum of X	10
$\boldsymbol{S}_3$	The three-element Sugihara monoid	53
$\boldsymbol{S}_n$	The n-element Sugihara monoid chain	38

$\mathrm{Th}(\boldsymbol{A})$	Set of all first order sentences that are true in $\boldsymbol{A}_A$ (cf. $\boldsymbol{A}_S$)	16
$\boldsymbol{T}_\rho(X)$	Term algebra of type ρ over X	6
$T_\rho(X)$	Set of terms of type ρ over X	6
$\#t$	Complexity of term t	6
$T(X)$	Abbreviates $T_\rho(X)$ when ρ is understood	6
U	Largest subvariety of W	58
$\mathbb{V}(\mathsf{K})$	Variety generated by class K	8
Var	A fixed denumerably infinite set of variables	5
$\mathit{Var}(\Gamma)$	The set of all variables $x \in \mathit{Var}$ such that x occurs in at least one member of Γ	109
W	Class of De Morgan monoids that are either trivial or crystalline	55
W_n	Class of Brouwerian algebras with width at most n	170
$\uparrow X$	Upward closure of X	133
$\downarrow X$	Downward closure of X	133
$\boldsymbol{X}^*$	Dual algebra of $\boldsymbol{X}$	135
X^c	Complement of X	135
$\boldsymbol{X}/R$	Quotient space of $\boldsymbol{X}$	136
x^*	Abbreviates $x \to e$	41
$\lvert x \rvert$	Abbreviates $x \to x$	41
$\bigwedge X$	Infimum of X	10
$\bigvee X$	Supremum of X	10
$\boldsymbol{Z}$	The Sugihara monoid on $\mathbb{Z}$ that generates OSM	38
$\mathbb{Z}$	The set of integers	36
$\boldsymbol{Z}^*$	The Sugihara monoid on $\mathbb{Z} \setminus \{0\}$ that generates the variety SM	36

Index