Comparative Study of some Numerical Methods for FitzHugh-Nagumo Equation

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Abstract. The FitzHugh-Nagumo equation has various applications in the fields of flame propagation, logistic population growth, neurophysiology, autocatalytic chemical reaction and nuclear theory [1, 6]. In this work, we construct three versions of nonstandard finite difference schemes in order to solve the FitzHugh-Nagumo equation with specified initial and boundary conditions under three different regimes. Properties of the methods such as positivity and boundedness are studied. The performances of the three methods is compared by computing L_1, L_∞ errors and CPU time at given time, $T = 1.0$.

Keywords: FitzHugh-Nagumo equation, Nonstandard finite difference scheme, positivity, boundedness.
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INTRODUCTION

Partial differential equation are widely used to describe or model the complex phenomena in real life and applications are fluid mechanics solid-state, plasma wave and chemical physics . The FitzHugh-Nagumo equation given by

$$u_t = u_{xx} + \beta u(1-u)(u-\gamma), \tag{1}$$

is one form of application in biology where γ controls the global dynamics of the equation and is in the interval $(0,1)$ and $u(x,t)$ is the unknown function which depends on the temporal variable t, and the spatial variable, x. β is parameter. Nonstandard Finite Difference Scheme (NSFD) has been introduced by Mickens [2] to obtain solutions of various partial differential equations. Their derivations are mostly based on the idea of dynamical consistency (positivity, boundedness, monotonicity of the solutions etc) [3]. After generalizing these results, Mickens formulated the following three basic rules in constructing NSFD schemes:
(i) The order of discrete derivatives should be equal to the order of corresponding derivatives appearing in the differential equation.
(ii) Discrete representation for derivatives, in general have non trivial denominator functions, for instance

$$u_t \approx \frac{u_j^{n+1} - u_j^n}{\phi(\Delta t, \lambda)}$$

where $\phi(\Delta t, \lambda) = \Delta t + O[(\Delta t)^2]$.
(iii) Nonlinear terms must be represented by nonlocal discrete representations.

We solve Eq. (1) i.e,

$$u_t = u_{xx} + \beta u(1-u)(u-\gamma),$$

where $u(x,t)$ is the unknown function which depends on spatial variable, $x \in [-10, 10]$ and temporal variable, $t \in [0,1]$. The initial condition is $u(x,0) = \frac{1}{2} - \frac{1}{2}\tanh\left(\frac{\sqrt{\beta}}{2\sqrt{2}}x\right)$ and the boundary conditions are:
$u(-10,t) = \frac{1}{2} - \frac{1}{2}\tanh\left(\frac{\sqrt{\beta}}{2\sqrt{2}}(-10-ct)\right)$ and $u(10,t) = \frac{1}{2} - \frac{1}{2}\tanh\left(\frac{\sqrt{\beta}}{2\sqrt{2}}(10-ct)\right)$.
The exact solution is $u(x,t) = \frac{1}{2} - \frac{1}{2}\tanh\left(\frac{\sqrt{\beta}}{2\sqrt{2}}(x-ct)\right)$ where $\beta > 0$, $\gamma \in \Re$ and $c = -\sqrt{\frac{\beta}{2}}(2\gamma - 1)$. In this work, we consider 3 cases:

Case 1 : $\beta = 1,\ \gamma = 0.2$.

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Case 2 : $0 < \beta < 1\,(\beta = 0.5),\ \gamma = 0.2$.
Case 3 : $\beta > 1\,(\beta = 10),\ \gamma = 0.2$.

We test the performance of the schemes over different values of β over the domain $x \in [-10, 10]$ at time, $T = 1.0$.

NONSTANDARD FINITE DIFFERENCE SCHEME (NSFD)

We present derivation of three versions of NSFD Schemes termed as NSFD1, NSFD2, NSFD3 to discretise

$$u_t = u_{xx} + \beta u(1-u)(u-\gamma). \tag{2}$$

In all the three methods, we use the same discretisation for u_t and u_{xx}. We approximate u_t by $\frac{u_j^{n+1}-u_j^n}{\phi_2(\Delta t)}$ where $\phi_2(\Delta t) = \phi_2(k) = \frac{e^{\beta k}-1}{\beta}$ and u_{xx} by $\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\psi_1(\Delta x)\psi_2(\Delta x)}$ where $\psi_1(\Delta x) = \psi_1(h) = \frac{1-e^{\beta h}}{\beta}$ and $\psi_2(\Delta x) = \psi_2(h) = \frac{e^{\beta h}-1}{\beta}$.

NSFD1 scheme

We extend the scheme constructed in [4] to discretise Eq. (1). The scheme is given by

$$\frac{u_j^{n+1}-u_j^n}{\phi_2(k)} - \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\psi_1(h)\psi_2(h)} = \beta\left(-\frac{3}{2}\left(u_{j-1}^n\right)^2 u_j^{n+1} + \frac{1}{2}\left(u_{j-1}^n\right)^3\right) + \beta\,(1+\gamma)\left(u_{j-1}^n\right)^2 - \beta\,\gamma u_j^{n+1}. \tag{3}$$

A single expression for the scheme is

$$u_j^{n+1} = \frac{(1-2R)u_j^n + R(u_{j+1}^n + u_{j-1}^n) + \beta\phi_2(k)\left((1+\gamma)\left(u_{j-1}^n\right)^2 + \frac{1}{2}\left(u_{j-1}^n\right)^3\right)}{1 + \beta\,\gamma\phi_2(k) + \frac{3}{2}\beta\,\phi_2(k)\left(u_{j-1}^n\right)^2}, \tag{4}$$

where $R = \frac{\phi_2(k)}{\psi_1(h)\,\psi_2(h)}$.

The scheme is positive if $1 - 2R \geq 0$ i.e. ($R \leq \frac{1}{2}$). Positivity is guaranteed if

(a) $k \leq 4.9948 \times 10^{-3}$ for $\beta = 0.5$.
(b) $k \leq 4.9917 \times 10^{-3}$ for $\beta = 1.0$.
(c) $k \leq 5.2880 \times 10^{-3}$ for $\beta = 10$.

Boundedness

$$(u_j^{n+1}-1)\left(1 + \beta\gamma\phi_2(k) + \frac{3}{2}\beta\phi_2(k)\left(u_{j-1}^n\right)^2\right) = (1-2R)u_j^n + R(u_{j+1}^n + u_{j-1}^n) + \beta\phi_2(k)\left[(1+\gamma)(u_{j-1}^n)^2 + \frac{1}{2}(u_{j-1}^n)^3\right] \\ -\ 1 - \beta\gamma\phi_2(k) - \frac{3}{2}\beta\phi_2(k)(u_{j-1}^n)^2. \tag{5}$$

We check if the scheme is bounded. We note that $0 \leq u_j^n \leq 1$. If the scheme is bounded, we need to prove that $0 \leq u_j^{n+1} \leq 1$. We have

$$(u_j^{n+1}-1)\left(1 + \beta\gamma\phi_2(k) + \frac{3}{2}\beta\phi_2(k)\left(u_{j-1}^n\right)^2\right) \leq (1-2R) + 2R + \beta\phi_2(k)(1+\gamma)(u_{j-1}^n)^2 + \frac{1}{2}\beta\phi_2(k)(u_{j-1}^n)^3 - \\ 1 - \beta\gamma\phi_2(k) - \frac{3}{2}\beta\phi_2(k)(u_{j-1}^n)^2, \tag{6}$$

which simplifies as

$$(u_j^{n+1}-1)\left(1 + \beta\gamma\phi_2(k) + \frac{3}{2}\beta\phi_2(k)\left(u_{j-1}^n\right)^2\right) \leq 0.$$

Hence $0 \leq u_j^{n+1} \leq 1$ and therefore NSFD1 is bounded.

NSFD2 scheme

We extend the scheme in [4] to discretise Eq. (1). The scheme is given by

$$\frac{u_j^{n+1}-u_j^n}{\phi_2(k)}-\frac{u_{j+1}^n-2u_j^n+u_{j-1}^n}{\psi_1(h)\,\psi_2(h)}=\beta\left(-u_j^{n+1}\left(u_j^n\right)^2+(1+\gamma)u_j^{n+1}u_j^n-\gamma u_j^{n+1}\right), \tag{7}$$

which can be rewritten as

$$u_j^{n+1}=\frac{(1-2R)u_j^n+R(u_{j+1}^n+u_{j-1}^n)}{1+\beta\;\phi_2(k)\;\gamma-\beta(1+\gamma)\phi_2(k)u_j^n+\beta\;\phi_2(k)\left(u_j^n\right)^2}, \tag{8}$$

where $R=\frac{\phi_2(k)}{\psi_1(h)\,\psi_2(h)}$.

For positivity of NSFD2, we need $u_j^{n+1}\geq 0$ assuming $0\leq u_j^n\leq 1$. This is possible if $1-2R\geq 0$ and $1-\beta\phi_2(k)\gamma^2>0$. This gives

$k\leq\frac{1}{\beta}\ln\left(1+\frac{1}{\gamma^2}\right)$ and $k\leq\frac{1}{\beta}\ln\left(1+\frac{1}{2\beta}\frac{\left(e^{\beta h}-1\right)^2}{e^{\beta h}}\right)$.

(a) $k\leq 6.5162$ and $k\leq 4.9948\times 10^{-3}$ for $\beta=0.5$.
(b) $k\leq 3.2581$ and $k\leq 4.9917\times 10^{-3}$ for $\beta=1.0$.
(c) $k\leq 3.2581\times 10^{-1}$ and $k\leq 5.2880\times 10^{-3}$ for $\beta=10$.

Boundedness

$$(u_j^{n+1}-1)(1+\beta\phi_2(k)-\beta(1+\gamma)\phi_2(k)u_j^n+\beta\phi_2(k)(u_j^n)^2)=(1-2R)u_j^n+R(u_{j+1}^n+u_{j-1}^n)-1-\beta\phi_2(k)\gamma+\beta(1+\gamma)\phi_2(k)u_j^n-\beta\phi_2(k)(u_j^n)^2. \tag{9}$$

Since $0\leq u_j^n\leq 1$ for all values for n and j, we have

$$(u_j^{n+1}-1)\left(1+\beta\phi_2(k)-\beta(1+\gamma)\phi_2(k)u_j^n+\beta\phi_2(k)\left(u_j^n\right)^2\right)\leq 1-2R+2R-1-\beta\gamma\phi_2(k)+\beta(1+\gamma)\phi_2(k)u_j^n-\beta\phi_2(k)\left(u_j^n\right)^2\leq 0. \tag{10}$$

Hence $0\leq u_j^{n+1}\leq 1$ and NSFD2 satisfies the boundedness properties.

NSFD3 scheme

We construct a Nonstandard finite difference scheme using the idea from Roger and Mickens [5]. We propose the following scheme to discretise Eq. (1):

$$\frac{u_j^{n+1}-u_j^n}{\phi_2(k)}-\frac{u_{j+1}^n-2u_j^n+u_{j-1}^n}{\psi_1(h)\,\psi_2(h)}=\beta\left(-\left(2u_j^{n+1}-u_j^n\right)\left(u_j^n\right)^2+(1+\gamma)(u_j^n)^2-\gamma u_j^{n+1}\right). \tag{11}$$

A single expression for NSFD3 scheme is

$$u_j^{n+1}=\frac{(1-2R)u_j^n+R(u_{j+1}^n+u_{j-1}^n)+\beta\phi_2(k)\left(\left(u_j^n\right)^3+(1+\gamma)\left(u_j^n\right)^2\right)}{1+\beta\phi_2(k)\gamma+2\beta\phi_2(k)\left(u_j^n\right)^2}. \tag{12}$$

NSFD3 is positive and definite if $1-2R\geq 0$. We next check if the NSFD3 is bounded.

$$(u_j^{n+1}-1)\left(1+\beta\,\phi_2(k)\gamma+2\beta\phi_2(k)\left(u_j^n\right)^2\right)=(1-2R)u_j^n+R(u_{j+1}^n+u_{j-1}^n)+\beta\phi_2(k)\left(\left(u_j^n\right)^3+(1+\gamma)\left(u_j^n\right)^2\right)-1-\beta\,\gamma\phi_2(k)-2\beta\,\phi_2(k)\left(u_j^n\right)^2. \tag{13}$$

We note that $0 \leq u_j^n \leq 1$ for all values of n and j,

$$\begin{aligned} & (u_j^{n+1}-1)\left(1+\beta\gamma\phi_2(k)+2\beta\phi_2(k)\left(u_j^n\right)^2\right) \\ \leq\, & 1-2R+2R+\beta\phi_2(k)\left((1+\gamma)\left(u_j^n\right)^2+\left(u_j^n\right)^3\right)-1-\beta\gamma\phi_2(k)-2\beta\phi_2(k)\left(u_j^n\right)^2 \leq 0. \end{aligned} \tag{14}$$

Hence the boundedness of NSFD3.
We tabulate L_1, L_∞ and CPU time at $k = 0.0005$ using $\gamma = 0.2$, $h = 0.1$ at time, $T = 1.0$ for 3 cases namely; $\beta = 0.5$, 1.0, 10.0 using NSFD1, NSFD2, NSFD3 schemes in Table (1).

TABLE 1. Computation of L_1, L_∞ errors and CPU time using NSFD1, NSFD2, NSFD3 for three different values of β.

Methods	β	L_1 error	L_∞ error	CPU (s)
NSFD1	0.5	3.4717×10^{-2}	5.2542×10^{-3}	6.276
NSFD2	0.5	8.1163×10^{-5}	1.3847×10^{-5}	6.311
NSFD3	0.5	4.9406×10^{-5}	9.1704×10^{-6}	6.377
NSFD1	1.0	6.9845×10^{-2}	1.4490×10^{-2}	6.214
NSFD2	1.0	3.2693×10^{-4}	6.0565×10^{-5}	7.527
NSFD3	1.0	2.3529×10^{-4}	4.9940×10^{-5}	6.053
NSFD1	10.0	3.7574×10^{-1}	6.4116×10^{-1}	6.461
NSFD2	10.0	4.1558×10^{-2}	2.5652×10^{-2}	6.508
NSFD3	10.0	4.4636×10^{-2}	2.7639×10^{-2}	6.912

CONCLUSION

In this work, we construct three nonstandard finite difference schemes to solve FitzHugh-Nagumo equation under 3 different regimes. We derive conditions such that the scheme are positive definite and bounded. The first order time derivative and second order spatial derivative are approximately in the same way for all the methods and it is only the non-linear polynomial in the partial differential equation which is discretised differently. NSFD2 and NSFD3 are much better than NSFD1 for the 3 cases studied. NSFD3 is better than NSFD2 when $\beta = 0.5$, 1.0 while NSFD2 is slightly better than NSFD3 when $\beta = 10$.

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