Radial symmetry and mass-independent boundedness of stationary states of aggregation-diffusion models

by

Chelsea Bright

Submitted in fulfilment of the requirements for the degree

Magister Scientiae

In the Faculty of Natural & Agricultural Sciences

University of Pretoria

Pretoria

April 2020

UNIVERSITEIT VAN PRETORIA
UNIVERSITY OF PRETORIA
YUNIBESITHI YA PRETORIA

Declaration

I, Chelsea Bright declare that the thesis/dissertation, which I hereby submit for the degree Magister Scientiae at the University of Pretoria, is my own work and has not previously been submitted by me for a degree at this or any other tertiary institution.

SIGNATURE:

DATE: April 2020

Acknowledgements

I would first like to thank my supervisor Prof. Roumen Anguelov for his continual support and guidance during my MSc research. I am highly grateful for his patience and commitment in helping me understand the topic. His enthusiasm and expertise drove my curiosity and motivation to learn.

I would also like to acknowledge my lecturers Dr. Labuschagne and Prof. Banasiak for teaching me necessary mathematical fundamentals during my MSc study. I would not have arrived where I am now without the help of the devoted academic and support staff in the Department of Mathematics and Applied Mathematics. I am also immensely grateful to my family - my mom, dad, and brother - for supporting me emotionally throughout my journey, helping me to believe in myself and my abilities.

Lastly, I acknowledge the financial support that has aided me in the completion of my MSc study, namely, the the postgraduate research bursary provided by the University of Pretoria.

Contents

Abstract

General aggregation diffusion equations have been used in a variety of different settings, including the modelling of chemotaxis and the biological aggregation of insects and herding of animals. We consider a non-local aggregation diffusion equation, where the repulsion is modelled by nonlinear diffusion (Laplace operator applied to mth power of the spatial density) and attraction modelled by non-local interaction. The competition between these forces gives rise to characteristic time-independent morphologies. When the attractive interaction kernel is radially symmetric and strictly increasing with respect to the norm in the n-dimensional linear space of the space variable, it is previously known that all stationary solutions are radially symmetric and decreasing up to a translation. We extend this result to attractive kernels with compact support, where a wider variety of time-independent patterns occur. We prove that for compactly supported attractive kernels and for power in the diffusion term $m > 1$, all stationary states are radially symmetric and decreasing up to a translation on each connected component of their support. Furthermore, for $m > 2$, we prove analytically that stationary states have an upper-bound independent of the initial data, confirming previous numerical results given in the literature. This result is valid for both attractive kernels with compact support and unbounded support. Finally, we investigate a model that incorporates both non-local attraction and non-local repulsion. We show that this model may be considered as a generalization of the aggregation diffusion equation and we present numerical results showing that $m = 2$ is a threshold value such that, for $m > 2$, stationary states of the fully non-local model possess a mass-independent upper-bound.

1

Introduction

1.1 General overview

The formation of aggregations in nature, such as insect swarms, fish schools, animal herds, and bacteria colonies often arise as a result of the self-organization of the individual members of the population. This self-organization is driven by social or sensory mechanisms of the individuals, including sight or smell. The individuals of the group self-organize through forces such as attraction and repulsion. Attractive forces allow for individuals to aggregate together, which pose certain benefits. Firstly, large aggregations allow for protection against predators. In particular, more individuals will be positioned in the interior of the group, which decreases their likelihood of being targeted by a predator, see [38] and references therein. Secondly, larger aggregations increase an individual's probability of finding a mate. On the other hand, repulsive forces play an important role in the well-being of each individual, as it inhibits crowding and thus decreases the potential to spread a disease. In addition, repulsive effects reduce predation, as densely populated aggregations are known to attract predators [45].

Additionally, formation of aggregations can occur as a result of exogenous forces such as wind, gravity, food or light sources; however, in this dissertation we will focus mainly on endogenous forces given by attraction and repulsion. We note that at low densities, attraction should dominate over repulsion to allow for a group to form, while at high densities, repulsion should dominate to prevent over-crowding. Hence, both attraction and repulsion should be density dependent with repulsion possessing a more non-linear density dependence compared to attraction.

It has been observed that physical and biological aggregations possess certain key characteristics, such as sharp edges and a constant internal density that does not change as the population size increases, see [37, 38, 39] and references therein. Hence, a requirement for a realistic model of an aggregation is that the resulting population density should have an upper-bound that is independent of the size of the swarm, which represents the preferred maximum density. In addition, when the swarm is large enough, the internal density should be constant with the height equivalent to the preferred maximum density. Furthermore, as the population grows, the height should remain constant and the spatial extent of the swarm should increase.

1.2 Modelling of collective behaviour

General aggregation diffusion equations have been used in a variety of different settings, including the modelling of chemotaxis and the biological aggregation of insects and herding animals, see [7, 36, 37, 42, 1]. Additionally, use of these equations can be seen in the modelling of opinion dynamics, see [15] and references therein. The behaviour in these settings are typically driven by long-range attraction and short-range repulsion. The competition between these forces gives rise to characteristic time-independent morphologies. These resulting equilibria are the focus of this dissertation. In particular, we consider a continuum description of collective behaviour where we analyze the evolution of the population density $\rho(x,t)$ at some location $x \in \mathbb{R}^d$ and at time $t \geq 0$.

We consider the initial value problem given by the following non-local integro-differential equation

$$\partial_t \rho = \varepsilon \Delta \rho^m + \nabla \cdot (\rho \nabla (W * \rho)) \qquad x \in \mathbb{R}^d, t > 0 \tag{1.1}$$

with initial condition $\rho_0 \in L^1_+(\mathbb{R}^d) \cap L^m(\mathbb{R}^d)$. The local repulsion is modelled using nonlinear diffusion with $m > 1$ and where ε is the diffusion coefficient. The non-local attraction arises from the second term on the right, where $W * \rho = \int_{\mathbb{R}^d} W(x-y)\rho(y)\, dy$. In essence, the presence of individuals at position $y \in \mathbb{R}^d$ creates a force, proportional to $-\nabla W(x-y)$, that acts on the individuals positioned at $x \in \mathbb{R}^d$. The interaction kernel $W : \mathbb{R}^d \to \mathbb{R}$ is given and is assumed to be radially symmetric and non-decreasing from its centre. That is, there is a function $\omega : [0, \infty) \to \mathbb{R}$ such that $W(x) = \omega(||x||)$ for all $x \in \mathbb{R}^d$ and $\omega'(x) \geq 0$, $x \in \mathbb{R}_+$. Thus, the interaction kernel only takes into account attractive effects.

1.3 Research questions

In this dissertation, we address the following questions:

1. For which combination of attractive and repulsive terms do we obtain stationary solutions of Equation (1.1) with characteristics that are physically and biologically realistic?
2. What effect does the range of interaction of the attractive force have on the properties of the stationary solutions?
3. What mechanisms allow for the formation of patterns over time?
4. Is there a benefit in replacing non-linear diffusion with a non-local term to model repulsion?

1.4 Aims and objectives

In this dissertation, we aim to present a mathematical model of collective behaviour based on interactions between group members, which results in pattern formation that is realistic to aggregations in nature. Furthermore, we extend this model to incorporate terms modelling the group's interaction with its environment as well as to incorporate interaction kernels that are dependent on both time and space.

Our first main result in this work is that, for $m > 1$ and the case where ω' has bounded support, a stationary solution of (1.1) is radially symmetric and decreasing up to a translation on each connected component of its support.

Our second main result is that for $m > 2$ the stationary states have an upper-bound independent of the initial data for both cases of ω' with bounded and unbounded support. This is a notable characteristic of the model as it is a natural property of physical and biological aggregations, where individuals will aggregate together up to a maximum density and no further. The value of the diffusion coefficient m plays a key role in the emergence of this boundedness, where $m = 2$ is a threshold. More precisely, we prove analytically that for $m > 2$ we obtain boundedness of stationary states independent of the initial data, while for $m \leq 2$ the maximal density grows with the total mass $\int_{\mathbb{R}^d} \rho_0(x)\, dx$. This boundedness property for the special case of $m = 3$ is shown numerically in [42].

1.5 Organization of the dissertation

This dissertation is made up of seven chapters. In Chapter 2, we provide some mathematical preliminaries from measure and integration theory, functional analysis and the theory of decreasing rearrangements, which will be used for analysis and proving the results in later chapters. In Chapters 3, 4, and 5, we provide a theoretical analysis of Equation (1.1), which incorporates non-local aggregation and diffusion. We first provide a summary of the results obtained in the existing literature in Chapter 3, which includes existence and basic properties of solutions and stationary states of (1.1).

In Chapter 4, we present our first main result of this dissertation, namely, the radial symmetry property of stationary solutions to (1.1) for a compactly supported attractive kernel. In Chapter 5, we prove the existence of a mass-independent upper-bound for $m > 2$, where there is no restriction on the support of the attractive kernel. Furthermore, we present numerical simulations which support these theoretical results and illustrate how patterns can be formed by considering a compactly supported interaction kernel as well as an interaction kernel that is dependent on both space and time.

In Chapter 6, we present a model incorporating both non-local attraction and non-local repulsion, where the range of interaction of the repulsive kernel is less than that of the attractive kernel. We first provide a summary of results of the model obtained in the existing literature. We then present numerical simulations of stationary solutions and compare our numerical results with the results obtained for stationary states of Equation (1.1). In addition, we show that for $m > 2$ we are able to derive a mass-independent upper-bound for stationary solutions of the fully non-local model using its corresponding energy functional. We conclude the dissertation in Chapter 7 and outline possibilities for future work that build upon the results presented here.

2

Mathematical Preliminaries

In this chapter we provide the mathematical background needed to prove the results given in the chapters that follow. In Section 2.1, we provide some mathematical preliminaries on measure and integration theory, which follows mostly from the book [2]. In Section 2.2, we introduce some concepts from functional analysis, based mainly on the book [11], and in Section 2.3, we provide some basic properties of the convolution function from [32].

Furthermore, in Section 2.4, we give an introduction on decreasing rearrangements and their basic properties. We then use the theory introduced in Section 2.4 to define the Steiner symmetrization of a set or function in Section 2.5. Finally, in Section 2.6, we use the concept of Steiner Symmetrization to define the continous Steiner symmetrization of a set or function, which is a necessary tool in proving the results given in the subsequent chapters. The last three sections of the chapter follow mostly from the books [30] and [35].

2.1 Measure and Integration

We now provide some relevant definitions and theorems from measure and integration theory. We begin by introducing the basic notion of a measure space and then introduce some useful results on integration with respect to a measure. This background allows us to give an appropriate introduction to integration with respect to the Dirac measure and the Lebesgue measure on $\mathbb{R}^d$, which plays a role in the development of the theory in Sections 2.4, 2.5, and 2.6. This theory is necessary to obtain the properties of solutions to Equation (1.1), including the radial symmetry property and the compactness of the support for stationary solutions.

2.1.1 Abstract measure spaces

Definition 2.1 (σ-Algebra)**.** *A class Σ of subsets of a set X is called a σ-algebra if $X \in \Sigma$ and the class is closed under the formation of countable unions and of complements.*

Remark 2.1. If we consider the class to be closed under the formation of finite unions, the class is known as an algebra.

Theorem 2.2. *[2] Let $\mathcal{A}$ be a class of subsets of a set X. Then there exists a smallest σ-algebra $\mathcal{S}$, containing $\mathcal{A}$, such that $\mathcal{S}$ is called the σ-algebra generated by $\mathcal{A}$.*

Definition 2.2 (Borel σ-Algebra). *The σ-algebra, denoted $\mathcal{B}$, generated by the class of intervals of the form $[a, b)$ is known as the Borel σ-algebra. Its members are called the Borel sets of $\mathbb{R}$.*

Definition 2.3 (Measurable space). *Let X be a set and let Σ be a σ-algebra over X. Then the pair (X, Σ) is a measurable space. If $E \in \Sigma$, then E is called a measurable set.*

Definition 2.4 (Measure). *A measure $\mu : \Sigma \to [0, \infty]$ is a function defined on a σ-algebra Σ with the following properties:*

1. $\mu(\emptyset) = 0$.

2. *If $\{E_n\}_{n=1}^{\infty}$ is any sequence of disjoint measurable sets, that is $E_i \cap E_j = \emptyset$ for any $E_i \neq E_j$, we have that*

$$\mu(\bigcup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} \mu(E_n).$$

Example 2.1. Some examples of measures defined on $\mathbb{R}^d$ include:

1. The Lebesgue measure, (see Subsection 2.1.3).

2. The Dirac delta-measure, (see Subsection 2.1.4).

Definition 2.5 (Measure space). *If X is a set, Σ is a σ-algebra over X, and $\mu : \Sigma \to [0, \infty]$ is a measure on Σ, then the triple (X, Σ, μ) is called a measure space.*

Definition 2.6 (Almost everywhere). *If a property holds except on a set of measure zero, then it holds almost everywhere, abbreviated a.e.*

2.1.2 Integration with respect to a measure

Let $(X, \mathcal{S}, \mu)$ be a measure space. For the development of the theory, we first define the integral of a non-negative simple function, where a simple function is defined as follows:

Definition 2.7 (Simple function). *A function $g : X \to [0, \infty)$ is a simple function if it takes only finitely many distinct values $a_1, ..., a_n$ and is defined by*

$$g(x) = \sum_{i=1}^{n} a_i \chi_{A_i}(x),$$

where the sets $A_i = \{x \in X : g(x) = a_i\}$ are measurable.

Definition 2.8 (Integral of a simple function)**.** *Let $g : X \to [0, \infty)$ be a non-negative simple function. Then the integral of g with respect to μ is defined by*

$$\int g \; d\mu = \sum_{i=1}^{n} a_i \mu(A_i).$$

We now define what it means for a non-negative function to be measurable, in order to obtain the definition of its integral.

Definition 2.9 (Measurable function)**.** *Let f be a real-valued function defined on a measurable set E. Then f is a measurable function if, for each $\alpha \in \mathbb{R}$, the set $\{x \in E : f(x) > \alpha\}$ is measurable.*

We note that measurable functions are closed under addition and multiplication, as proved in [2].

Definition 2.10 (Integral of a non-negative function)**.** *Let $f : X \to [0, \infty]$ be a non-negative measurable function. Then the integral of f with respect to μ is defined by*

$$\int f \; d\mu = \sup\{\int g \; d\mu : g \text{ is a non-negative simple function, } g \leq f\}.$$

Now, that we have defined the integral of a non-negative function with respect to a measure μ, we can define what it means for a non-negative function to be integrable with respect to μ.

Definition 2.11. *A function $f : X \to [0, \infty]$ is integrable if it is measurable and $\int f \; d\mu < \infty$.*

In the theorems that follow, we provide some of the basic properties of the integral of a non-negative function, relevant for the results given in later chapters.

Theorem 2.3. *[2] Let f and g be non-negative measurable functions.*

(a) *If $f \leq g$, then $\int f \; d\mu \leq \int g \; d\mu$.*

(b) *If A is a measurable set and $f \leq g$ on A, then $\int_A f \; d\mu \leq \int_A g \; d\mu$.*

(c) *If $a \geq 0$, then $\int af \; d\mu = a \int f d\mu$.*

(d) *If A and B are measurable sets and $A \subseteq B$, then $\int_A f \; d\mu \leq \int_B f d\mu$.*

Theorem 2.4. *If $f : X \to [0, \infty]$ is measurable, then*

$$\mu(\{x \in X : f(x) \geq a\}) \leq \frac{1}{a} \int f \; d\mu, \text{ for all } a > 0.$$

Proof. Fix $a > 0$. Then, since $a\chi_{\{f \geq a\}}(x) \leq f(x)$ for all $x \in X$,

$$a\mu(\{x \in X : f(x) \geq a\}) = a \int \chi_{\{f \geq a\}} \; d\mu \leq \int f \; d\mu,$$

by Theorem 2.3. □

Corollary 2.5. Suppose $f : X \to [0, \infty]$ is measurable. If $\int f \, d\mu = 0$, then $f = 0$ a.e.

Proof. Assume $\int f \, d\mu = 0$. By Theorem 2.4, we have that

$$\mu(\{x \in X : f(x) \geq a\}) \leq \frac{1}{a} \int f \, d\mu = 0, \text{ for all } a > 0.$$

Hence, since $\mu(E) \geq 0$ for any $E \in \mathcal{S}$, by definition, it follows that

$$\mu(\{x \in X : f(x) \geq a\}) = 0, \text{ for all } a > 0.$$

Therefore, $f = 0$ a.e. □

Theorem 2.6. *[2] Let $f_n : X \to [0, \infty]$ be a sequence of non-negative measurable functions. Then,*

$$\int \sum_{n=1}^{\infty} f_n \, d\mu = \sum_{n=1}^{\infty} \int f_n \, d\mu.$$

Theorem 2.7. *[2] Let $f : X \to [0, \infty]$ be a non-negative measurable function. Then, there exists an increasing sequence of simple functions g_n such that $g_n \uparrow f$ pointwise as $n \to \infty$ and*

$$\int f \, d\mu = \lim_{n \to \infty} \int g_n \, d\mu.$$

We have now given the definition of the integral of a non-negative measurable function and have outlined some of its relevant properties. However, it is still necessary to extend the definition of integrability to any measurable function.

In order to do this, we consider a real-valued function f defined on X and define $f^+(x) = \max(f(x), 0)$ and $f^-(x) = \max(-f(x), 0)$. It is easy to see that $f = f^+ - f^-$ and $|f| = f^+ + f^-$. Furthermore, by the definition of a measurable function, we have that f is measurable if and only if f^+ and f^- are measurable. Hence, we are able to define an integrable function on X as follows:

Definition 2.12. *Let $f : X \to \mathbb{R}$ be any measurable function. If $\int f^+ \, d\mu < \infty$ and $\int f^- \, d\mu < \infty$, then f is integrable and*

$$\int f \, d\mu = \int f^+ \, d\mu - \int f^- \, d\mu.$$

Definition 2.13. *If E is a measurable set, f is a measurable function, and $\chi_E f$ is integrable, then f is integrable over E and $\int_E f \, d\mu = \int f \chi_E \, d\mu$.*

Remark 2.8. We note that, since $|f| = f^+ + f^-$, we have that f is integrable if and only if $|f|$ is. We denote the class of functions integrable with respect to μ by $L^1(X, \mu)$. If $X = \mathbb{R}$ and μ is the Lebesgue measure, then we write $L^1(\mathbb{R})$. Furthermore, we denote the set of non-negative integrable functions with respect to the Lebesgue measure on $\mathbb{R}$ by $L^1_+(\mathbb{R})$.

Definition 2.14. *For $p > 0$, we define $L^p(X, \mu)$ to be the class of measurable functions $\{f : \int |f|^p \, d\mu < \infty\}$.*

2.1.3 Lebesgue measure on $\mathbb{R}^d$

In this subsection, we provide an introduction to the Lebesgue measure on $\mathbb{R}^d$. We first give a definition of the Lebesgue measure on the real line and then extend this definition to define the Lebesgue measure on $\mathbb{R}^d$.

The Lebesgue measure on $\mathbb{R}$

Definition 2.15 (Lebesgue outer measure)**.** *Let $\ell(I)$ denote the length of the interval I. Then, the Lebesgue outer measure of a set is given by*

$$m^*(E) = \inf \Big\{ \sum_{n=1}^{\infty} \ell(I_n) : E \subseteq \bigcup_{n=1}^{\infty} I_n \text{ and } I_n = (a_n, b_n), n \geq 1 \Big\}.$$

Definition 2.16 (Lebesgue measurable set)**.** *The set E is Lebesgue measurable if, for each set A, we have*

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c).$$

We denote by $\mathcal{M}$ the class of Lebesgue measurable sets, which is a σ-algebra, as proved in [2]. We recall the notation of the Borel σ-algebra given by $\mathcal{B}$. We see that, by properties of a σ-algebra, $\mathcal{B}$ is generated by the class of open intervals and open sets. Additionally, we have the following useful theorem:

Theorem 2.9. *[2] Every Borel set is measurable. That is, $\mathcal{B} \subset \mathcal{M}$.*

We now outline some properties of the Lebesgue outer measure on the real line.

Theorem 2.10. *[2] Given sets $A, B \subseteq \mathbb{R}$, the Lebesgue outer measure has the following properties:*

1. *$m^*(A) \geq 0$ and $m^*(A) \leq m^*(B)$ if $A \subseteq B$,*

2. *$m^*(\emptyset) = 0$ and $m^*(\{x\}) = 0$ for any $x \in \mathbb{R}$,*

3. *For any sequence of sets $\{E_n\}_{n=1}^{\infty}$,*

$$m^*\Big(\bigcup_{n=1}^{\infty} E_n\Big) \leq \sum_{n=1}^{\infty} m^*(E_n).$$

Theorem 2.11. *[2] If $\{E_n\}_{n=1}^{\infty}$ is any sequence of disjoint measurable sets, that is $E_i \cap E_j = \emptyset$ for any $E_i \neq E_j$, we have that*

$$m^*\Big(\bigcup_{n=1}^{\infty} E_n\Big) = \sum_{n=1}^{\infty} m^*(E_n).$$

Remark 2.12. If E is a Lebesgue measurable set we write $m(E) := m^*(E)$, where m denotes the Lebesgue measure and is defined on the σ-algebra $\mathcal{M}$ of Lebesgue measurable sets.

Example 2.2. $(\mathbb{R}, \mathcal{M}, m)$ is a measure space, where $\mathcal{M}$ denotes the class of all Lebesgue measurable sets and m denotes the Lebesgue measure, defined for all sets $E \in \mathcal{M}$.

Definition 2.17. *For any sequence of sets* $\{E_n\}_{n=1}^{\infty}$*,*

$$\lim_{n\to\infty} \inf E_n \;=\; \bigcup_{n=1}^{\infty}\bigcap_{m=n}^{\infty} E_m$$

and

$$\lim_{n\to\infty} \sup E_n \;=\; \bigcap_{n=1}^{\infty}\bigcup_{m=n}^{\infty} E_m.$$

By Definition 2.17, we have that $\lim_{n\to\infty} \inf E_n \subseteq \lim_{n\to\infty} \sup E_n$. If equality holds, then the set is denoted by $\lim_{n\to\infty} E_n$. Furthermore, it is easy to see that if $E_1 \subseteq E_2 \subseteq ...$, then $\lim_{n\to\infty} E_n = \cup_{n=1}^{\infty} E_n$ and if $E_1 \supseteq E_2 \supseteq ...$, then $\lim_{n\to\infty} E_n = \cap_{n=1}^{\infty} E_n$.

Theorem 2.13. *[2] Let* $\{E_n\}_{n=1}^{\infty}$ *be a sequence of measurable sets. Then,*

a) *If* $E_1 \subseteq E_2 \subseteq ...$*, we have that* $m(\lim_{n\to\infty} E_n) = \lim_{n\to\infty} m(E_n)$*.*

b) *If* $E_1 \supseteq E_2 \supseteq ...$ *and* $m(E_n) < \infty$ *for each* $n \in \mathbb{N}$*, we have that* $m(\lim_{n\to\infty} E_n) = \lim_{n\to\infty} m(E_n)$*.*

Lastly, we state the following theorem regarding a Lebesgue measurable function defined on $\mathbb{R}$, which will be used in the development of the theory in Section 2.4.

Theorem 2.14. *[41] A monotone increasing function is Lebesgue measurable.*

The Lebesgue measure on $\mathbb{R}^d$

We now use the definition of the Lebesgue measure on the real line to construct the Lebesgue measure on $\mathbb{R}^d$. In order to do this, we first need to define what it means for a measure to be σ - finite and complete.

Definition 2.18 (σ - finite measure)**.** *Let* $(X, \mathcal{S}, \mu)$ *be a measure space. If there exists countably many sets* $E_1, E_2, ... \in \mathcal{S}$ *such that* $\mu(E_k) < \infty$ *for each* k *and* $X = \cup_{k=1}^{\infty} E_k$*, then* μ *is a* σ*-finite measure and* $(X, \mathcal{S}, \mu)$ *is a* σ*-finite measure space.*

Definition 2.19 (Complete measure)**.** *If, for* $E \in \mathcal{S}$*,* $F \subseteq E$ *and* $\mu(E) = 0$*, we have that* $F \in \mathcal{S}$*, then* μ *is a complete measure and* $(X, \mathcal{S}, \mu)$ *is a complete measure space.*

Example 2.3. [2] The measure space $(\mathbb{R}, \mathcal{M}, m)$ is σ-finite and complete.

Suppose μ is a measure, defined on a σ-algebra $\mathcal{S}$, that is not complete. This measure may be extended to a complete measure by adjoining to $\mathcal{S}$ the subsets of the sets $E \in \mathcal{S}$ where $\mu(E) = 0$. This is outlined in Theorem 2.15, which is used to construct the Lebesgue measure on $\mathbb{R}^d$. We denote the symmetric difference of two sets A, B by $A\Delta B$, where $A\Delta B = (A - B) \cup (B - A)$.

Theorem 2.15. *[2] Let μ be a measure on a σ-algebra $\mathcal{S}$. Let E and N be any sets such that $E \in \mathcal{S}$ and N is contained in some set in $\mathcal{S}$ of zero measure. Then the class $\bar{\mathcal{S}}$ of sets of the form $E\Delta N$ is a σ-algebra and $\bar{\mu}$, defined by $\bar{\mu}(E\Delta N) = \mu(E)$, is a complete measure on $\bar{\mathcal{S}}$.*

In order to construct the Lebesgue measure on $\mathbb{R}^d$, we must give the definition of a product measure. We first introduce some necessary preliminaries.

Definition 2.20. *If X and Y are spaces, then the product space $X \times Y$ is the space of ordered pairs $\{(x, y) : x \in X, y \in Y\}$.*

To define measures on $X \times Y$ we assume that $(X, \mathcal{S})$ and $(Y, \mathcal{A})$ are measurable spaces.

Definition 2.21. *A space $\Omega \subset X \times Y$ is called a rectangle if there exists $A \subset X$ and $B \subset Y$ such that $\Omega = A \times B$. Furthermore, we say that a space $A \times B$ in $X \times Y$ is a measurable rectangle if $A \in \mathcal{S}$ and $B \in \mathcal{A}$.*

We denote $\mathcal{S} \times \mathcal{A}$ to be the σ-algebra generated by the class of measurable rectangles. Furthermore, we denote $(X \times Y, \mathcal{S} \times \mathcal{A})$ to be the product of the measurable spaces $(X, \mathcal{S})$ and $(Y, \mathcal{A})$.

Now, if $E \subseteq X \times Y$, we define the sets $E_x := \{y : (x, y) \in E\}$ and $E^y := \{x : (x, y) \in E\}$, known as the x-section and y-section of the set E, respectively. The following theorem states that if the set E is measurable then the x-section and y-section of E are also measurable.

Theorem 2.16. *[2] If $E \in \mathcal{S} \times \mathcal{A}$, then for each $x \in X$ and $y \in Y$, $E_x \in \mathcal{A}$ and $E^y \in \mathcal{S}$.*

Similarly, for a function f defined on $X \times Y$, we define the x-section of f by $f_x(y) = f(x, y)$, for a fixed $x \in X$ and the y-section of f by $f_y(x) = f(x, y)$, for a fixed $y \in Y$. We say that a function $f : X \to Y$ is $\mathcal{S}$-measurable if the set $\{x \in X : f(x) \in E\} \in \mathcal{S}$ for all $E \in \mathcal{A}$. As a result of Theorem 2.16, we have the following theorem, given a measurable function f on $X \times Y$:

Theorem 2.17. *[2] Let f be an $\mathcal{S} \times \mathcal{A}$-measurable function on $X \times Y$. Then, for each $x \in X$ and $y \in Y$, we have that f_x is $\mathcal{A}$-measurable and f_y is $\mathcal{S}$-measurable.*

We now introduce the main theorem that is used to define the product measure on the σ-algebra $\mathcal{S} \times \mathcal{A}$.

Theorem 2.18. *[2] Let $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{A}, \nu)$ be σ-finite measure spaces. For $V \in \mathcal{S} \times \mathcal{A}$, let $\phi(x) = \nu(V_x)$, and $\psi(y) = \mu(V^y)$, for each $x \in X$, $y \in Y$. Then ϕ is $\mathcal{S}$-measurable, ψ is $\mathcal{A}$-measurable, and*

$$\int_X \phi \; d\mu = \int_Y \psi \; d\nu.$$

As a result of Theorem 2.18, we can define the product measure $\mu \times \nu$ on $\mathcal{S} \times \mathcal{A}$ as follows:

Definition 2.22. *Let $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{A}, \nu)$ be σ-finite measure spaces. Then, the product measure $\mu \times \nu$ on $\mathcal{S} \times \mathcal{A}$ is given by*

$$(\mu \times \nu)(V) = \int_X \nu(V_x)\, d\mu = \int_Y \mu(V^y)\, d\nu,$$

for each $V \in \mathcal{S} \times \mathcal{A}$.

We now have all the tools necessary to construct the Lebesgue measure on $\mathbb{R}^d$. Indeed, since $(\mathbb{R}, \mathcal{M}, m)$ is a σ-finite measure space, we can apply Theorem 2.18 to define the product measure $m \times m$ on $\mathcal{M} \times \mathcal{M}$ in terms of Definition 2.22. Furthermore, using Theorem 2.15, we can form the completion of the product measure $m \times m$, which we define to be the Lebesgue measure on $\mathbb{R}^2$.

Definition 2.23 (Lebesgue measure on $\mathbb{R}^2$)**.** *The completion of the product measure $m \times m$ on $\mathcal{M} \times \mathcal{M}$ is the Lebesgue measure m_2 on $\mathbb{R}^2$. Furthermore, $\mathcal{M}_2$ denotes the class of Lebesgue measurable sets that are measurable with respect to m_2.*

Now, using induction, we are able to define the Lebesgue measure on $\mathbb{R}^d$ as follows:

Definition 2.24 (Lebesgue measure on $\mathbb{R}^d$)**.** *For $d > 1$, define $m^{(d)} = m^{(d-1)} \times m$ and $\mathcal{M}^{(d)} = \mathcal{M}^{(d-1)} \times \mathcal{M}$. We define the Lebesgue measure m_d on $\mathcal{M}_d$ to be the completion of $m^{(d)}$ on $\mathcal{M}^{(d)}$.*

In order to be consistent with the notation used in the relevant literature on aggregation diffusion equations, for the remainder of the dissertation we denote the Lebesgue measure on $\mathbb{R}^d$ by $|E|_d$, where E is any Lebesgue measurable set in $\mathbb{R}^d$. Furthermore, we denote $|E|_1 = |E|$.

2.1.4 The Dirac measure

In this subsection we give the definition of the Dirac measure, an example of a measure defined on $\mathbb{R}^d$. The Dirac measure is used in the development of the theory on decreasing rearrangements, provided in Section 2.4.

Definition 2.25 (The Dirac measure)**.** *Fix $a \in \mathbb{R}^d$. The Dirac measure on $\mathbb{R}^d$, located at a, is defined for any set $A \in \mathcal{M}_d$ by*

$$\delta_a(A) := \begin{cases} 1, & \text{if } a \in A \\ 0 & \text{if } a \notin A. \end{cases}$$

We see that the Dirac measure is a probability measure, that is, $\delta_a(\mathbb{R}^d) = 1$, and thus is finite. A notable property of the Dirac measure is the following:

Let f be any measurable function. Then,

$$\int f\, d\delta_a = f(a).$$

We can prove this property as follows:

From the definition of the Dirac measure we have that

$$\int \chi_A \, d\delta_a = \begin{cases} 1, & \text{if } a \in A \\ 0 & \text{if } a \notin A. \end{cases}$$

Hence, considering a simple function

$$g(x) = \sum_{i=1}^{n} a_i \chi_{A_i}(x),$$

we have that, for $a \in A_k$,

$$\int g(x) \, d\delta_a(x) = \sum_{i=1}^{n} a_i \int \chi_{A_i}(x) \, d\delta_a(x) = a_k.$$

Here, we have used Theorem 2.6 and the fact that each $A_i \cap A_j = \emptyset$ for any $i \neq j$. Now, since $a \in A_k$, we have that

$$g(a) = \sum_{i=1}^{n} a_i \chi_{A_i}(a) = a_k,$$

and so

$$\int g(x) \, d\delta_a(x) = g(a).$$

Hence, the property holds for any measurable simple function. Now, by Theorem 2.7, for any non-negative measurable function f, there exists an increasing sequence of simple functions g_n such that $g_n \uparrow f$ pointwise as $n \to \infty$ and

$$\int f \, d\delta_a = \lim_{n \to \infty} \int g_n \, d\delta_a.$$

Hence, since $g_n(a) \uparrow f(a)$ and $g_n(a) = \int g_n(x) \, d\delta_a(x)$ for each n, we have that

$$\int f(x) \, d\delta_a(x) = \lim_{n \to \infty} \int g_n(x) \, d\delta_a(x) = \lim_{n \to \infty} g_n(a) = f(a).$$

Finally, since $f = f^+ - f^-$, where $f^+(x) = \max(f(x), 0) \geq 0$ and $f^-(x) = \max(-f(x), 0) \geq 0$, it follows that

$$\begin{aligned} \int f(x) \, d\delta_a(x) &= \int f^+(x) \, d\delta_a(x) - \int f^-(x) \, d\delta_a(x) \\ &= f^+(a) - f^-(a) = f(a). \end{aligned}$$

Hence, the property holds for any measurable function f.

2.2 Some results from functional analysis

In this section, we provide some definitions and mathematical theories from functional analysis used in the results that follow. A major part of this section is to define the notion of a Sobolev space, which plays an important role in the theory of partial differential equations, as they usually contain the weak solutions of PDEs, that is, solutions that may not necessarily have classical derivatives.

The function space, given in Definition 2.27, is important in deriving the weak formulation of a partial differential equation, which is solved to obtain the weak solutions of the PDE.

Definition 2.26. $C^\infty(\mathbb{R}^d)$
Let $C^m(\mathbb{R}^d)$ be the class of functions with continuous derivatives up to order m in $\mathbb{R}^d$. Then $C^\infty(\mathbb{R}^d)$ is the class of functions in $C^m(\mathbb{R}^d)$ for each m.

Definition 2.27. $C_0^\infty(\mathbb{R}^d)$
A function f is in $C_0^\infty(\mathbb{R}^d)$ if it is in $C^\infty(\mathbb{R}^d)$ and it has compact support.

In order to give the definition of a Sobolev space, the notion of a weak derivative must first be defined.

Definition 2.28. *A function $f \in L^1(\mathbb{R}^d)$ has a weak derivative $\partial^\alpha f \in L^1(\mathbb{R}^d)$ if*

$$\int_{\mathbb{R}^d} \partial^\alpha f \phi \, dx = (-1)^{|\alpha|} \int_{\mathbb{R}^d} f \partial^\alpha \phi \, dx, \quad \textit{for all} \quad \phi \in C_0^\infty(\mathbb{R}^d). \tag{2.1}$$

Here, we have set $\partial^\alpha = (\partial_{x_1})^{\alpha_1} ... (\partial_{x_d})^{\alpha_d}$ where $|\alpha| = \sum_{i=1}^d \alpha_i$. Note that we denote the weak gradient of f by ∇f where $\nabla f = (\partial_{x_1} f, ..., \partial_{x_d} f)$.

Now that we have the definition of a weak derivative, we can introduce the notion of a Sobolev space.

Definition 2.29. *Suppose $k \in \mathbb{N}$ and $1 \leq p \leq \infty$. Then, we define the Sobolev space $\mathcal{W}^{k,p}(\mathbb{R}^d)$ to be the space of functions $f \in L^p(\mathbb{R}^d)$ such that*

$$\partial^\alpha f \in L^p(\mathbb{R}^d), \textit{ for } |\alpha| \leq k.$$

Furthermore, we write $\mathcal{W}^{k,2}(\mathbb{R}^d) = H^k(\mathbb{R}^d)$.

We note that the space $\mathcal{W}^{k,p}(\mathbb{R}^d)$ is a Banach space equipped with the norm

$$||f||_{\mathcal{W}^{k,p}(\mathbb{R}^d)} = \sum_{|\alpha| \leq k} \Big(\int_{\mathbb{R}^d} |\partial^\alpha f|^p \, dx \Big)^{1/p}$$

for $1 \leq p < \infty$ and

$$||f||_{\mathcal{W}^{k,\infty}(\mathbb{R}^d)} = \max_{|\alpha| \leq k} \big(\operatorname*{ess\,sup}_{\mathbb{R}^d} |\partial^\alpha f| \big),$$

where

$$\operatorname*{ess\,sup}_X f = \inf\{a \in \mathbb{R} : |\{x \in X : f(x) > a\}|_d = 0\} \tag{2.2}$$

is the essential supremum of a Lebesgue measurable function $f : X \to \mathbb{R}$.

In particular, $H^k(\mathbb{R}^d)$ is a Hilbert space with inner product defined as

$$(f, g) = \sum_{|\alpha| \leq k} \int_{\mathbb{R}^d} (\partial^\alpha f)(\partial^\alpha g)\ dx.$$

Theorem 2.19 (Hölder's inequality). *[35] Suppose E is a Lebesgue measurable set. Let $\frac{1}{p} + \frac{1}{q} = 1$ with $1 \leq p \leq \infty$ and let $f \in L^p(E)$ and $g \in L^q(E)$. Then $fg \in L^1(E)$ and*

$$\int_E |fg|\ dx \leq ||f||_{L^p}||g||_{L^q}.$$

Theorem 2.20. *Let E be a Lebesgue measurable set with finite measure. If $0 < p < q \leq \infty$, then $L^q(E) \subset L^p(E)$.*

Proof. Let $f \in L^q(E)$. Then $f^p \in L^{q/p}(E)$ and so by Hölder's inequality,

$$\begin{aligned}
\int_E |f|^p\ dx &= \int_E 1 \cdot |f|^p\ dx \\
&\leq \Big(\int_E |f|^{pq/p}\ dx\Big)^{p/q} \Big(\int_E\ dx\Big)^{1-p/q} \\
&= |E|_d^{1-p/q} ||f||_{L^q}^p < \infty.
\end{aligned}$$

□

2.3 Definition and basic properties of convolution

In this section, we will give a formal definition of the convolution function defined on $\mathbb{R}^d$ and state some basic properties that will be used in the analysis of Equation (1.1).

Definition 2.30 (Convolution). *Let $f \in L^1(\mathbb{R}^d)$ and let $g \in L^p(\mathbb{R}^d)$, where $1 \leq p \leq \infty$. Then the convolution function is defined by*

$$(f * g)(x) = \int_{\mathbb{R}^d} f(x - y)g(y)\ dy, \quad x \in \mathbb{R}^d.$$

Theorem 2.21 (Young's convolution inequality). *[35] Let $f, g \in L^1(\mathbb{R}^d)$ so that the convolution of f and g exists. Furthermore, let $f \in L^p(\mathbb{R}^d)$, $g \in L^q(\mathbb{R}^d)$, and $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$, where $1 \leq p, q \leq r \leq \infty$. Then,*

$$||f * g||_r \leq ||f||_p||g||_q.$$

We now introduce an important theorem regarding the gradient of the convolution function.

Theorem 2.22. *Suppose $f \in \mathcal{W}^{1,1}(\mathbb{R}^d)$ and $g \in L^1(\mathbb{R}^d)$. Then,*

$$\nabla(f * g) = (\nabla f) * g.$$

In order to prove Theorem 2.22, we need to introduce some preliminary results. It is first necessary to define the Fourier transform of a function $f \in L^1(\mathbb{R}^d)$.

Definition 2.31. *If $f \in L^1(\mathbb{R}^d)$, then the Fourier transform of f is the bounded continuous function in $\mathbb{R}^d$ defined by*

$$\mathcal{F}\{f\}(\xi) = \int_{\mathbb{R}^d} e^{-i\langle x,\xi\rangle} f(x)\ dx, \quad \xi \in \mathbb{R}^d.$$

If the Fourier transform of f is integrable, we can express f in terms of its Fourier transform, as follows:

$$f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i\langle x,\xi\rangle} \mathcal{F}\{f\}(\xi)\ d\xi. \tag{2.3}$$

We now provide the following theorem which states that the Fourier transform of the convolution of two integrable functions is equal to the product of their Fourier transforms.

Theorem 2.23 (Convolution theorem)**.** *Let f and g be two integrable functions with convolution $f * g$. Then*

$$\mathcal{F}\{f * g\} = \mathcal{F}\{f\} \cdot \mathcal{F}\{g\}.$$

Proof.

$$\begin{aligned} \mathcal{F}\{f * g\} &= \int_{\mathbb{R}^d} e^{-i\langle x,\xi\rangle} \int_{\mathbb{R}^d} f(x-y)g(y)\ dy\ dx \\ &= \int_{\mathbb{R}^d} g(y) \int_{\mathbb{R}^d} e^{-i\langle x,\xi\rangle} f(x-y)\ dx\ dy \end{aligned}$$

Setting $\tau = x - y$, we have that

$$\begin{aligned} \mathcal{F}\{f * g\}(\xi) &= \int_{\mathbb{R}^d} g(y) \int_{\mathbb{R}^d} e^{-i\langle \tau+y,\xi\rangle} f(\tau)\ d\tau\ dy \\ &= \mathcal{F}\{f\}(\xi) \int_{\mathbb{R}^d} g(y) e^{-i\langle y,\xi\rangle}\ dy \\ &= \mathcal{F}\{f\}(\xi) \cdot \mathcal{F}\{g\}(\xi). \end{aligned}$$

□

The last result needed for the proof of Theorem 2.22 is as follows:

Theorem 2.24. *[32] Suppose $f \in \mathcal{W}^{1,1}(\mathbb{R}^d)$. Then,*

$$\mathcal{F}\{\nabla f\}(\xi) = 2i\pi\xi \mathcal{F}\{f\}(\xi).$$

Proof of Theorem 2.22.
The proof follows directly by applying Theorems 2.23 and 2.24. Indeed, we have that

$$\mathcal{F}\{\nabla(f * g)\} = 2i\pi\xi\mathcal{F}\{f * g\} = 2i\pi\xi\mathcal{F}\{f\} \cdot \mathcal{F}\{g\} = \mathcal{F}\{\nabla f\} \cdot \mathcal{F}\{g\} = \mathcal{F}\{\nabla f * g\}.$$

Hence, applying the inverse Fourier transform, we obtain

$$\nabla(f * g) = \nabla f * g.$$

□

2.4 Decreasing rearrangements

In this section, the definition and basic properties of decreasing rearrangements for non-negative functions are introduced, which are used to define the continuous Steiner symmetrization of a non-negative function.

Definition 2.32 (Vanishing at infinity)**.** *A non-negative measurable function f on $\mathbb{R}^d$ is said to vanish at infinity if $|\{x \in \mathbb{R}^d : f(x) > t\}|_d$ is finite for all $t > 0$.*

The functions that are appropriate for the definition of rearrangements are Lebesgue measurable functions that vanish at infinity. By Theorem 2.4, we have that functions in $L^1_+(\mathbb{R}^d)$ vanish at infinity. Hence, we restrict our attention to the definition of decreasing rearrangements for any $f \in L^1_+(\mathbb{R}^d)$.

We are now able to introduce the following definitions:

Definition 2.33. *A non-negative function f on $\mathbb{R}^d$ is radially symmetric if there is a function $\tilde{f}$ defined on $[0, \infty)$ such that $f(x) = \tilde{f}(||x||)$ for all $x \in \mathbb{R}^d$.*

Definition 2.34. *A non-negative function f on $\mathbb{R}^d$ is radially decreasing up to a translation if there exists some $x_0 \in \mathbb{R}^d$ such that $f(\cdot - x_0)$ is radially symmetric and $\tilde{f}(||x - x_0||)$ is non-increasing in $||x - x_0||$.*

Here, $|| \cdot ||$ denotes the Euclidean norm.

Definition 2.35 (Rearranged function)**.** *A non-negative function f is rearranged if it is radially symmetric and $\tilde{f}$ is a non-negative, right-continuous, non-increasing function on $(0, \infty)$.*

Definition 2.36 (Connected component)**.** *The connected components of a topological space X are closed, disjoint, non-empty subsets of X such that their union is the whole space X.*

Definition 2.37. *Denote the level set of a function f by $\{f > \tau\} := \{x \in \mathbb{R}^d : f(x) > \tau\}$. Then the distribution function of $f \in L^1_+(\mathbb{R}^d)$ is given by*

$$\zeta_f(\tau) := |\{f > \tau\}|_d \quad \textit{for all} \quad \tau > 0.$$

Note that the distribution function $\zeta_f(\tau)$ of f is a monotonically decreasing function of τ. We use this fact to define the Hardy-Littlewood one-dimensional decreasing rearrangement of f, as follows:

Definition 2.38. *The Hardy-Littlewood one-dimensional decreasing rearrangement of f is the function $f^* : [0, \infty) \to [0, \infty]$ such that*

$$f^*(s) = \sup\{\tau > 0 : \zeta_f(\tau) > s\} \textit{ for all } s \in [0, \infty).$$

Proposition 2.25. [30] The functions f and f^* are equimeasurable. That is, they have the same distribution function.

Corollary 2.26. [30] If $f \in L^p(\mathbb{R}^d)$ for $1 \leq p \leq \infty$, then $f^* \in L^p(0,\infty)$ and

$$||f||_{L^p(\mathbb{R}^d)} = ||f^*||_{L^p(0,\infty)}.$$

Making use of the definition of the Hardy-Littlewood one-dimensional decreasing rearrangement, we can define the following symmetric decreasing rearrangement on $\mathbb{R}^d$.

Definition 2.39. *The Schwarz spherical decreasing rearrangement of f is the function*

$$f^{\#}(x) = f^*(S_{d-1}||x||^d), \;\; x \in \mathbb{R}^d,$$

where S_{d-1} is the surface area of the unit sphere in $\mathbb{R}^d$.

By definition of $f^{\#}$ and by the properties of f^* outlined above, we can easily see that $f^{\#}$ is radially symmetric and decreasing, f, f^*, and $f^{\#}$ are equimeasurable, and if $f \in L^p(\mathbb{R}^d)$, we obtain the invariance property of the L^p norms for f, f^*, and $f^{\#}$.

Furthermore, let $E^{\#}$ be the ball centred at the origin with $|E^{\#}|_d = |E|_d$, for any measurable set E. Since $f^{\#}$ is radially symmetric and decreasing with the same distribution function as f, we have that, for $\Omega_f = \{x \in \mathbb{R}^d : f(x) > 0\}$, $f^{\#}$ is supported in the ball $\Omega_f^{\#}$.

Theorem 2.27 (Layer cake representation)**.** *[35] Let ν be a measure on the Borel sets of $[0,\infty)$ such that $\phi(t) := \nu([0,t))$ is finite for every $t > 0$. Let (Ω, Σ, μ) be a measure space and f any non-negative measurable function on Ω. Then*

$$\int_\Omega \phi(f(x))d\mu(x) = \int_0^\infty \mu(\{x : f(x) > t\})d\nu(t).$$

In particular, if ν is chosen to be the Lebesgue measure, then, by Theorem 2.9 and the definition of ϕ, $\phi(0) = 0$ and ϕ is monotone increasing. Hence, ϕ is a non-negative measurable function by Theorem 2.14 and

$$\int_\Omega \phi(f(x))d\mu(x) = \int_\Omega \nu([0, f(x)))d\mu(x) = \int_\Omega f(x)d\mu(x).$$

Now, choosing μ to be the Dirac measure at some point $y \in \Omega$, we have that

$$\int_\Omega f(x)d\mu(x) = \int_\Omega f(x)d\delta_y(x) = f(y)$$

and

$$\int_0^\infty \mu(\{x : f(x) > t\})d\nu(t) = \int_0^\infty \delta_y(\{x : f(x) > t\})dt = \int_0^\infty \chi_{\{f>t\}}(y)dt.$$

Hence, from the layer-cake representation theorem, we obtain

$$f(y) = \int_0^\infty \chi_{\{f>t\}}(y)dt. \tag{2.4}$$

Equation (2.4) is known as the layer-cake representation formula of f. Notably, using (2.4), we are able to obtain the following representation of the Schwarz decreasing rearrangement of f:

$$f^{\#}(x) = \int_0^\infty \chi_{\{f>\tau\}^{\#}} \, d\tau,$$

where we have used the fact that $\{f^{\#} > \tau\} = \{f > \tau\}^{\#}$, proved in [30].

We conclude this section by giving an example of the Schwarz decreasing rearrangement of a non-negative integrable function on $\mathbb{R}$.

Example 2.4. Consider

$$f(x) = \begin{cases} -|x-1|+1, & 0 \leq x \leq 2 \\ 0, & \text{otherwise} \end{cases}$$

From the definition of f^*, we see that

$$f^*(s) = \begin{cases} -\frac{s}{2}+1, & 0 \leq s \leq 2 \\ 0, & \text{otherwise} \end{cases}$$

and therefore,

$$f^{\#}(x) = f^*(2|x|) = \begin{cases} -|x|+1, & |x| \leq 1 \\ 0, & \text{otherwise.} \end{cases}$$

Indeed, we see that $f^{\#}$ is symmetric about 0 and decreasing for $x > 0$. Furthermore, we have that $|\Omega_f| = |\Omega_f^{\#}| = 2$ and it can be easily seen that the invariance property of the L^p norms is satisfied.

2.5 Steiner Symmetrization

We now use the theory on decreasing rearrangements given in the previous section to introduce the Steiner symmetrization of both a measurable set $E \subset \mathbb{R}^d$ and of a non-negative function f on $\mathbb{R}^d$. The concept of Steiner symmetrization can then be used to define the continuous Steiner symmetrization, which is necessary for the results given in Section 4.2. We first introduce some notation.

Denote $x \in \mathbb{R}^d$ by $x = (x_1, x')$ where $x' = (x_2, ..., x_d) \in \mathbb{R}^{d-1}$. Additionally, for any measurable set $E \subset \mathbb{R}^d$, denote

$$E_{x'} = \{x_1 \in \mathbb{R} : (x_1, x') \in E\}.$$

We first define the Steiner symmetrization of a set $E \subset \mathbb{R}$.

Definition 2.40. *The Steiner symmetrization of a measurable set $E \subset \mathbb{R}$ is the symmetric interval*

$$S(E) = \left\{ x \in \mathbb{R} : |x| < \frac{|E|}{2} \right\}.$$

We can extend this definition such that

$$S_{\tilde{x}}(E) = \left\{ x \in \mathbb{R} : |x - \tilde{x}| < \frac{|E|}{2} \right\}$$

is an open interval, symmetric about some $\tilde{x} \in \mathbb{R}$.

Now, consider $d \geq 2$. We define the Steiner symmetrization of a measurable set $E \subset \mathbb{R}^d$ in the direction corresponding to the unit vector $e_1 = (1, 0, ..., 0)$. We note that the definition can be modified to consider any other direction in $\mathbb{R}^d$.

Definition 2.41. *The Steiner symmetrization of a measurable set $E \subset \mathbb{R}^d$ in the direction x_1 is the set $S(E)$ which is symmetric about the hyperplane $\{x_1 = 0\}$ and defined by*

$$S(E) = \{(x_1, x') \in \mathbb{R}^d : x_1 \in S(E_{x'})\}.$$

We can extend this definition by defining $S_{\tilde{x}}(E)$ to be the set which is symmetric about the hyperplane $\{x_1 = \tilde{x}\}$ and defined by

$$S_{\tilde{x}}(E) = \{(x_1, x') \in \mathbb{R}^d : x_1 \in S_{\tilde{x}}(E_{x'})\}.$$

In particular, we see that $|E|_d = |S(E)|_d$ and thus, $|E|_d = |S_{\tilde{x}}(E)|_d$.

Now, consider a non-negative function $f \in L^1_+(\mathbb{R}^d)$. For the special case of $d = 1$, we define the distribution function of f by

$$\zeta_f(h) = |U^h|, \quad h > 0,$$

where

$$U^h = \{\alpha \in \mathbb{R} : f(\alpha) > h\}.$$

Similarly, for f defined on $\mathbb{R}^d$, $d \geq 2$, we have that the distribution function of $f(\cdot, x')$ is

$$\zeta_f(h, x') = |U^h_{x'}|, \quad h > 0, \; x' \in \mathbb{R}^{d-1},$$

where

$$U^h_{x'} = \{x_1 \in \mathbb{R} : f(x_1, x') > h\}.$$

We first give the definition of the Steiner symmetrization of a non-negative function on $\mathbb{R}$.

Definition 2.42. *The Steiner symmetrization of a non-negative function f on $\mathbb{R}$ is given by*

$$Sf(\alpha) = \sup\{h > 0 : \zeta_f(h) > 2|\alpha|\}, \; \alpha \in \mathbb{R}.$$

Now, we consider the Steiner symmetrization of a non-negative function on $\mathbb{R}^d$ in a particular direction.

Definition 2.43. *The Steiner symmetrization of a non-negative function f on $\mathbb{R}^d$ in the direction x_1 is given by*

$$Sf(x_1, x') = \sup\{h > 0 : \zeta_f(h, x') > 2|x_1|\}.$$

From the above definition, it can be seen that the Steiner symmetrization of $f(\cdot, x')$ is equal to the Schwarz rearrangement of $f(\cdot, x')$. Since this is the case, the Steiner symmetrization of $f(\cdot, x')$ has the same properties as the Schwarz rearrangement of $f(\cdot, x')$. In particular, we have that Sf and f have the same distribution function, yielding the invariance of the L^p norms. That is,

$$||Sf||_{L^p(\mathbb{R}^d)} = ||f||_{L^p(\mathbb{R}^d)} \text{ for all } 1 \leq p \leq \infty.$$

Furthermore, by the layer-cake representation formula, we have that, for $d = 1$,

$$Sf(\alpha) = \int_0^\infty \chi_{S(U^h)}(\alpha)\ dh$$

and for $d \geq 2$,

$$Sf(x_1, x') = \int_0^\infty \chi_{S(U^h_{x'})}(x_1)\ dh.$$

Now, consider the following definition of a function that is symmetric decreasing about a hyperplane:

Definition 2.44. *A function $f \in L^1_+(\mathbb{R}^d)$ is symmetric decreasing about a hyperplane $H \subset \mathbb{R}^d$, with normal vector e, if for any $x \in H$, the function $g(t) = f(x + te)$ is rearranged. That is, $g = g^\#$.*

From Definitions 2.43 and 2.44, we see that the Steiner symmetrization $Sf(x_1, x')$ is a function that is symmetric decreasing about the hyperplane $\{x_1 = 0\}$.

Similar to the case of a measurable set E in $\mathbb{R}^d$, we can extend the definition of the Steiner symmetrization of a non-negative function f on $\mathbb{R}^d$ in such a way that we obtain a function that is symmetric about the hyperplane $\{x_1 = \tilde{x}\}$, for $\tilde{x} \in \mathbb{R}$ arbitrary. To do this, we fix $\tilde{x} \in \mathbb{R}$ and define $S_{\tilde{x}}f$ in the direction x_1 as

$$S_{\tilde{x}}f(x_1, x') = Sf(\tilde{x} - x_1, x') = \sup\{h > 0 : \zeta_f(h, x') > 2|x_1|\}.$$

Hence, $|\{S_{\tilde{x}}f > h\}|_d = |\{Sf > h\}|_d = |\{f > h\}|_d$ and so

$$||S_{\tilde{x}}f||_{L^p(\mathbb{R}^d)} = ||f||_{L^p(\mathbb{R}^d)} \text{ for all } 1 \leq p \leq \infty.$$

Furthermore, by the layer-cake representation formula,

$$\begin{aligned} S_{\tilde{x}}f(x_1, x') &= f^\#(\tilde{x} - x_1, x') \\ &= \int_0^\infty \chi_{\{f>\tau\}^\#}(\tilde{x} - x_1)\ dh \\ &= \int_0^\infty \chi_{S_{\tilde{x}}(U^h_{x'})}(x_1)\ dh. \end{aligned}$$

The reasoning behind extending the definition of the Steiner symmetrization to that of the function $S_{\tilde{x}}$ is so that it can be used in the arguments given in Section 4.2. Indeed, in Section 4.2, we investigate the radial symmetry of stationary solutions of Equation (1.1) for compactly supported attractive kernels. In this case, the support of the of stationary solutions may consist of multiple connected components. In particular, we investigate whether the stationary states are radially symmetric and decreasing on each connected component of their support. Hence, it is necessary to extend the definition of Steiner symmetrization to that of $S_{\tilde{x}}$ in order to consider a function $S_{\tilde{x}}f(x_1, x')$ that is symmetric about the hyperplane $\{x_1 = \tilde{x}\}$, where $\tilde{x}$ may not necessarily be zero.

Lastly, we introduce Riesz's Rearrangement Inequality, which is used in Section 4.2

Theorem 2.28 (Riesz's Rearrangement Inequality). *[35] Let $f, g,$ and h be non-negative Lebesgue measurable functions defined on $\mathbb{R}^d$ that vanish at infinity. Then,*

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x)g(x-y)h(y)\ dxdy \leq \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f^{\#}(x)g^{\#}(x-y)h^{\#}(y)\ dxdy.$$

2.6 Continuous Steiner symmetrization

In this section, we introduce the concept of continuous Steiner symmetrization. Recall that, for W with compact support, the stationary states of (1.1) may have a support with multiple connected components. In Section 4.2, we are interested in investigating the radial symmetry of stationary states restricted to a single connected component of the support. This made it necessary in the previous section to extend the definition of the Steiner symmetrization of a set to be symmetric about an arbitrary point $\tilde{x} \in \mathbb{R}$. Similarly, we extended the definition of the Steiner symmetrization of a function on $\mathbb{R}^d$ in the direction x_1 to be symmetric about the hyperplane $\{x_1 = \tilde{x}\}$, for some arbitrary $\tilde{x} \in \mathbb{R}$.

In this section, we define the continuous Steiner symmetrization of a set $U \subset \mathbb{R}$ such that it represents an interpolation between the set U and $S_{\tilde{x}}(U)$. Similarly, for a function f on $\mathbb{R}^d$, we define its continuous Steiner symmetrization in a particular direction, say x_1, as an interpolation between $f(\cdot, x')$ and $S_{\tilde{x}}f(\cdot, x')$. We use this concept in our proof of the radial symmetry of stationary states of Equation (1.1), given in Section 4.2, for the special case where the attractive kernel W has compact support.

Note that, for simplicity, we henceforth denote $S_{\tilde{x}}$ by S.

Definition 2.45. *The continuous Steiner symmetrization of any open set $U \subset \mathbb{R}$, denoted $M^\tau(U)$, $\tau \geq 0$, is defined as below. We denote $I(\tilde{x} + c, r) := (\tilde{x} + c - r, \tilde{x} + c + r)$, where $\tilde{x}, c, r \in \mathbb{R}$.*

1. If $U = I(\tilde{x} + c, r)$, then

$$M^\tau(I(\tilde{x}+c,r)) := \begin{cases} I(\tilde{x}+c-\tau\,\mathrm{sgn}\,c,r), & \text{if } 0\le\tau<|c| \\ I(\tilde{x},r), & \text{if } \tau\ge|c|. \end{cases}$$

2. If $U=\cup_{i=1}^N I(\tilde{x}+c_i,r_i)$, where all $I(\tilde{x}+c_i,r_i)$ are disjoint from each other, then $M^\tau(U):=\cup_{i=1}^N M^\tau(I(\tilde{x}+c_i,r_i))$ for $0\le\tau<\tau_1$, where τ_1 is the first time two intervals $M^\tau(I(\tilde{x}+c_i,r_i))$ share a common endpoint. Once this occurs, we merge the two intervals sharing a common endpoint into one open interval and then define $M^\tau(U)$ in the same way, starting from $\tau=\tau_1$.

3. If $U=\cup_{i=1}^\infty I(\tilde{x}+c_i,r_i)$, let $U_N=\cup_{i=1}^N I(\tilde{x}+c_i,r_i)$, for each $N\ge1$, and define $M^\tau(U):=\cup_{N=1}^\infty M^\tau(U_N)$.

We see that, for all three cases of U outlined above, as τ tends to infinity, $M^\tau(U)$ ultimately becomes a single open interval that is symmetric about $\tilde{x}$. Furthermore, case 3 can be considered as a limit of case 2. Indeed, suppose that $U=\cup_{i=1}^\infty I(\tilde{x}+c_i,r_i)$. For $N_1<N_2$, we have that $U_{N_1}\subset U_{N_2}$ and so

$$\begin{aligned} M^\tau(U_{N_1}) &= M^\tau\big(\cup_{i=1}^{N_1} I(\tilde{x}+c_i,r_i)\big) \\ &\subset M^\tau\big((\cup_{i=1}^{N_1} I(\tilde{x}+c_i,r_i)) \,\cup\, (\cup_{i=N_1}^{N_2} I(\tilde{x}+c_i,r_i))\big) = M^\tau(U_{N_2}), \end{aligned}$$

for all $\tau\ge0$. It then follows directly from Definition 2.17 that

$$M^\tau(U)=\cup_{N=1}^\infty M^\tau(U_N)=\lim_{n\to\infty} M^\tau(U_N).$$

We now state some properties of the continuous Steiner symmetrization of an open set in $\mathbb{R}$.

Lemma 2.29. *[21] Let $U\subset\mathbb{R}$ be any open set with $M^\tau(U)$ as defined in Definition 2.45. Then,*

1. *$M^0(U)=U$ and $M^\infty(U)=S(U)$.*
2. *$|M^\tau(U)|=|U|$ for all $\tau\ge0$.*
3. *If $U_1\subset U_2$ then $M^\tau(U_1)\subset M^\tau(U_2)$ for all $\tau\ge0$.*
4. *M^τ has the semigroup property. That is, $M^{\tau+s}(U)=M^\tau(M^s(U))$ for any $\tau,s\ge0$.*

Using the definition of the continuous Steiner symmetrization of an open set in $\mathbb{R}$ given in Definition 2.45, we are now able to define the continuous Steiner symmetrization of a non-negative function f defined on $\mathbb{R}$.

Definition 2.46. *Let $f\in L^1_+(\mathbb{R})$. The continuous Steiner symmetrization for f is given by*

$$S^\tau f(\alpha):=\int_0^\infty \chi_{M^\tau(U^h)}(\alpha)\;dh.$$

Similarly, we may define the continuous Steiner symmetrization of a non-negative function f defined on $\mathbb{R}^d$, with respect to a specific direction, as follows:

Definition 2.47. *Fix $x_1 \in \mathbb{R}$, $x' \in \mathbb{R}^{d-1}$, and $h > 0$. Let $f \in L^1_+(\mathbb{R}^d)$. The continuous Steiner symmetrization for f in the direction x_1 is given by*

$$S^\tau f(x_1, x') := \int_0^\infty \chi_{M^\tau(U^h_{x'})}(x_1)\, dh.$$

We can now introduce some properties of the continuous Steiner symmetrization of a function, as defined in Definition 2.47.

Lemma 2.30. *[21] Let $f \in L^1_+(\mathbb{R}^d)$. Then the continuous Steiner symmetrization of f, with respect to a specific direction, has the following properties:*

1. *$S^0 f = f$ and $S^\infty f = Sf$.*
2. *For any $h > 0$, $|\{S^\tau f > h\}| = |\{f > h\}|$, and $||S^\tau f||_{L^p(\mathbb{R}^d)} = ||f||_{L^p(\mathbb{R}^d)}$ for all $p \in [1, \infty]$.*
3. *S^τ has the semigroup property. That is, $S^{\tau+s} f = S^\tau(S^s f)$ for any $\tau, s \geq 0$.*

Lastly, we give a theorem that is used in Section 4.2 for the proof regarding the radial symmetry of stationary states of (1.1).

Theorem 2.31. *[27] Let $f \in C(\mathbb{R}^d)$. Suppose that for every unit vector e, there exists a hyperplane $H \subset \mathbb{R}^d$ with normal vector e such that f is symmetric decreasing about H. Then, f is radially decreasing up to a translation.*

3

Model of non-local aggregation and local repulsion

As previously stated, our main interest is the long-time behaviour of solutions to the aggregation-diffusion equation (1.1). In particular, our aim is to investigate what conditions should be placed on the aggregation and diffusion terms that are biologically reasonable and produce spatial patterns over time. The model for physical and biological aggregations, given by (1.1), incorporates long-range attraction and short-range repulsion. It is interesting to note that in absence of any attractive effects, Equation (1.1) reduces to the well-known porous medium equation, given by

$$\partial_t \rho = \varepsilon \Delta(\rho^m), \qquad x \in \mathbb{R}^d,\ t > 0,\ m > 1.$$

A notable feature of the porous medium equation is that for a compactly supported initial condition, the solutions remain compactly supported for positive time. This is a contrast to the case when $m = 1$, the heat equation, where for any $t > 0$ the support of the solution is $\mathbb{R}^d$.
Furthermore, it is proved in [44] that weak solutions of the porous medium equation, with initial condition $\rho_0 \in L^1_+(\mathbb{R}^d)$, converge uniformly in time to the self-similar Barenblatt solution, given by

$$\mathcal{U}(x, t; C) = t^{-\alpha}(C - \kappa ||x||^2 t^{-2\beta})_+^{\frac{1}{m-1}},$$

where $(s)_+ = \max(s, 0)$ and

$$\alpha = \frac{d}{d(m-1)+2}, \quad \beta = \frac{\alpha}{d}, \quad \kappa = \frac{\beta(m-1)}{2m}.$$

The constant C can be changed to adjust the mass M of the solution, which is independent of time. More precisely, we have

$$M = \int_{\mathbb{R}^d} \mathcal{U}(x, t; C)\ dx = \int_{\mathbb{R}^d} \rho(x, t)\ dx = \int_{\mathbb{R}^d} \rho_0(x)\ dx > 0.$$

From the definition of the Barenblatt solution, we see that the solutions of the porous medium equation converge uniformly to zero as $t \to \infty$, resulting in the absence of patterns for large enough time.

In contrast, setting $\varepsilon = 0$, Equation (1.1) takes into account only attractive effects. It is proved that, depending on the initial condition and the strength of the attractive kernel, solutions may blow up in finite or infinite time. Considering weak measure solutions, the solution will aggregate to Dirac measures in either finite or infinite time, resulting in a scenario that is biologically unrealistic [23].

Hence, a natural next step would be to investigate the existence and long-term behaviour of solutions to Equation (1.1), which combines both attractive and repulsive effects. We note that, in the existing literature, significant work has been done on the aggregation diffusion equation where the interaction kernel is the attractive power-law kernel of the form

$$W_k(x) = \begin{cases} \frac{|x|^k}{k}, & k \neq 0 \\ \ln|x|, & k = 0, \end{cases}$$

for $2 - d \leq k \leq 2$ [21, 6, 5, 3, 10, 17] . A special case of a power-law kernel where $k = 2 - d$ is the Newtonian kernel. In fact, if the convolution operator $W*$ acting on ρ in Equation (1.1) is the Newtonian potential then we obtain the nonlinear parabolic elliptic Keller-Segel model of Chemotaxis, see [9, 13]. This follows from the fact that the Newtonian potential is the inverse of the negative Laplacian.

3.1 Basic properties of the model

To begin our analysis of Equation (1.1), in this section we outline results on the existence of solutions to Equation (1.1) as well as some basic properties of the model.

3.1.1 Existence

Mathematically, the first question regarding Equation (1.1) is the existence of solutions. In [6], the local and global existence of solutions to (1.1) are investigated for the case of bounded domains for dimension $d \geq 2$ and for the whole space $\mathbb{R}^d$ for $d \geq 3$. In particular, the authors show global existence of weak solutions to Equation (1.1) with initial data $\rho_0 \in L^\infty(\mathbb{R}^d) \cap L^1_+(\mathbb{R}^d)$ for $m > 2 - \frac{2}{d}$ and for W no more singular at the origin than the Newtonian kernel.

In what follows we derive the weak formulation of Equation (1.1), which allows us to give a formal definition of the weak solutions of Equation (1.1).

Let $H^1_0(\mathbb{R}^d)$ denote the completion of $C^\infty_0(\mathbb{R}^d)$ with respect to the inner product

$$(f, g)_H := \int_{\mathbb{R}^d} \nabla f(x) \cdot \nabla g(x) \; dx.$$

Furthermore, define $\Omega_T = \mathbb{R}^d \times [0,T]$ and fix $\phi \in H_0^1(\Omega_T)$ such that $\phi(\cdot,0) = \phi(\cdot,T) = 0$. Then, multiply Equation (1.1) by ϕ and integrate to obtain

$$\int_{\mathbb{R}^d} \rho_t \phi \; dx = \int_{\mathbb{R}^d} \big[\varepsilon \Delta \rho^m + \nabla \cdot (\rho \nabla W * \rho)\big] \phi \; dx.$$

Assume ρ^m, $\nabla \rho^m \in L^2((0,T); L^2(\mathbb{R}^d))$, $\rho \in L^\infty(\Omega_T) \cap L^\infty((0,T); L^1(\mathbb{R}^d))$, and $\rho \nabla W * \rho \in L^2((0,T); L^2(\mathbb{R}^d))$. Under these assumptions, we have that

$$\begin{aligned} \int_{\mathbb{R}^d} \rho_t \phi \; dx &= \int_{\mathbb{R}^d} \nabla \cdot \big[\varepsilon \nabla \rho^m + \rho \nabla W * \rho\big] \phi \; dx \\ &= -\int_{\mathbb{R}^d} \big[\varepsilon \nabla \rho^m + \rho \nabla W * \rho\big] \nabla \phi \; dx. \end{aligned}$$

Now, integrating with respect to t, we obtain

$$\int_0^T \int_{\mathbb{R}^d} \rho_t \phi \; dxdt = -\int_0^T \int_{\mathbb{R}^d} \big[\varepsilon \nabla \rho^m + \rho \nabla W * \rho\big] \nabla \phi \; dxdt.$$

In order to swap the integrals so that

$$\int_0^T \int_{\mathbb{R}^d} \rho_t \phi \; dxdt = \int_{\mathbb{R}^d} \int_0^T \rho_t \phi \; dtdx,$$

we must have that

$$\int_0^T |\rho_t \phi| \; dt < \infty.$$

Consequently, applying integration by parts,

$$\int_{\mathbb{R}^d} \int_0^T \rho_t \phi \; dtdx = -\int_{\mathbb{R}^d} \int_0^T \rho \phi_t \; dtdx,$$

yielding

$$\int_0^T \int_{\mathbb{R}^d} \rho \phi_t \; dxdt = \int_0^T \int_{\mathbb{R}^d} \varepsilon \nabla \rho^m \nabla \phi + \rho \nabla W * \rho \nabla \phi \; dxdt.$$

We can now define weak solutions to Equation (1.1) as follows:

Definition 3.1. *Let $\rho_0 \in L^\infty(\mathbb{R}^d) \cap L^1_+(\mathbb{R}^d)$ and suppose W is no more singular at the origin than the Newtonian kernel. Then, $\rho : \mathbb{R}^d \times [0,T] \to [0,\infty)$ is a weak solution of (1.1) if $\rho \in L^\infty(\Omega_T) \cap L^\infty((0,T); L^1(\mathbb{R}^d))$, $\nabla \rho^m \in L^2((0,T); L^2(\mathbb{R}^d))$, $\rho_t \in L^2((0,T); H_0^{-1}(\mathbb{R}^d))$, and*

$$\int_0^T \int_{\mathbb{R}^d} \rho \phi_t \; dxdt = \int_0^T \int_{\mathbb{R}^d} \varepsilon \nabla \rho^m \nabla \phi + \rho \nabla W * \rho \nabla \phi \; dxdt,$$

for all $\phi \in H_0^1(\Omega_T)$. Furthermore, it is required that

$$\rho(\cdot, t) \to \rho_0 \ in \ H_0^1(\mathbb{R}^d) \ as \ t \to 0.$$

Note that we have denoted the dual space of $H_0^1(\mathbb{R}^d)$ by $H_0^{-1}(\mathbb{R}^d)$.

The existence result given in [6] is extended in [5] to obtain global existence of weak solutions in $\mathbb{R}^d$ for $d \geq 2$, where a slightly stronger restriction is placed on the initial data, namely that $\rho_0 \in L^\infty(\mathbb{R}^d) \cap L_+^1(\mathbb{R}^d)$ has finite second moment, that is,

$$\int_{\mathbb{R}^d} ||x||^2 \rho_0 \, dx < \infty.$$

In [25], a further extension of the existence is given for solutions to Equation (1.1) with initial data $\rho_0 \in \mathcal{P}_2(\mathbb{R}^d) \cap L^m(\mathbb{R}^d)$, where $\mathcal{P}_2(\mathbb{R}^d)$ is the space of probability densities with finite second moment. We note that the assumption that ρ_0 is a probability density is not restrictive.

Indeed, for ρ a probability density that satisfies equation (1.1), we consider $\tilde{\rho} = M\rho$ such that $\int_{\mathbb{R}^d} \tilde{\rho} \, dx = M$. Then,

$$\tilde{\rho}_t = M\rho_t, \quad \Delta(\tilde{\rho}^m) = M^m \Delta(\rho^m), \quad \text{and} \quad \nabla \cdot (\tilde{\rho} \nabla W * \tilde{\rho}) = M^2 \nabla \cdot (\rho \nabla W * \rho).$$

Hence,

$$\begin{aligned} M^{-1}\tilde{\rho}_t &= \rho_t \\ &= \varepsilon \Delta(\rho^m) + \nabla \cdot (\rho \nabla W * \rho) \\ &= \varepsilon M^{-m} \Delta(\tilde{\rho}^m) + M^{-2} \nabla \cdot (\tilde{\rho} \nabla W * \tilde{\rho}). \end{aligned}$$

This then yields

$$\tilde{\rho}_t = \tilde{\varepsilon} \Delta(\tilde{\rho}^m) + \nabla \cdot (\tilde{\rho} \nabla \tilde{W} * \tilde{\rho}),$$

where $\tilde{\varepsilon} = \varepsilon M^{1-m}$ and $\tilde{W} = M^{-1}W$. Since $\tilde{\varepsilon} > 0$ and $\tilde{W}$ satisfies the same assumptions placed on W, we can conclude that the existence result given in [25] holds for solutions with any positive mass.

In the sequel, we assume the conditions that allow for the existence of solutions to Equation (1.1) hold. The assumptions on W made so far can be given as follows:

***W*1** $W(x) \in C^1(\mathbb{R}^d \setminus \{0\})$ is radially symmetric and non-decreasing. That is, there exists a function $\omega : (0, \infty) \to \mathbb{R}$ such that $W(x) = \omega(||x||) = \omega(r)$ and $\omega'(r) \geq 0$ for all $r > 0$.

***W*2** There exists some $C_\omega > 0$ such that $\omega'(r) \leq C_\omega r^{1-d}$ for $r \leq 1$.

It is easy to see that Assumption $(W2)$ restricts the kernel to be no more singular at the origin than the Newtonian kernel. We can now formulate the existence of solutions to Equation (1.1) as follows:

Theorem 3.1. *Let $m > 2 - \frac{2}{d}$, and let $\rho_0 \in L_+^1(\mathbb{R}^d) \cap L^m(\mathbb{R}^d)$ have finite second moment. Furthermore, suppose W satisfies assumptions ($W1$) and ($W2$). Then, Equation (1.1) has a weak solution $\rho \in L^\infty((0, \infty); L_+^1(\mathbb{R}^d) \cap L^m(\mathbb{R}^d))$.*

3.1.2 Conservation properties

Now that we have confirmed the existence of solutions to Equation (1.1), we can state some of the equation's conservation properties, where the solution is defined on the whole space $\mathbb{R}^d$. The proofs of these properties may be found in [14].

Theorem 3.2 (Preservation of positivity). *If $\rho_0 \in L^\infty(\mathbb{R}^d)$ is non-negative, then $\rho(x,t) \geq 0$ for all $t > 0$.*

Theorem 3.3 (Conservation of mass). *Let ρ be a weak solution of (1.1). If $\int_{\mathbb{R}^d} \rho(x,0)\ dx = M$ then $\int_{\mathbb{R}^d} \rho(x,t) = M$ for all $t \in [0,T]$.*

Theorem 3.4 (Conservation of the centre of mass). *Let ρ be a weak solution of (1.1). Then,*

$$\int_{\mathbb{R}^d} x\rho(x,0)\ dx = \int_{\mathbb{R}^d} x\rho(x,t)\ dx \ \ \textit{for all} \ \ t \in [0,T].$$

The preservation of positivity is expected as ρ represents the population density. The conservation of mass is also physically relevant, as the equation does not incorporate terms modelling growth or decay of the total population. In addition, since the centre of mass is conserved, Equation (1.1) cannot be used to model travelling swarms or a change in swarm behaviour due to exogenous forces. Some results regarding such behaviour may be found in [8] and [36]. Further, in the numerical simulations presented in Section 5.3, we investigate the interplay between a time dependent attractive kernel and exogenous forces.

3.1.3 Energy

A further property, which is particularly useful in determining stable stationary solutions of (1.1), is that there is a Lyapunov functional for the evolution of Equation (1.1) given by the energy functional,

$$\mathcal{E}[\rho] = \frac{\varepsilon}{m-1}\int_{\mathbb{R}^d} \rho^m\ dx + \frac{1}{2}\int_{\mathbb{R}^d} \rho(W * \rho)\ dx =: \mathcal{S}[\rho] + \mathcal{I}[\rho]. \tag{3.1}$$

The first term of (3.1) arises from repulsion and the second from aggregation. We see that the energy dissipates under the dynamics of (1.1) by considering the time derivative of $\mathcal{E}$. Indeed, if we assume that ρ is a classical solution of (1.1), we have that

$$\begin{aligned}
\partial_t \mathcal{E}[\rho] &= \int_{\mathbb{R}^d} \frac{\varepsilon}{m-1}\partial_t(\rho^m) + \frac{1}{2}(W * \rho)\partial_t\rho + \frac{1}{2}\rho W * (\partial_t \rho)\ dx \\
&= \int_{\mathbb{R}^d} \frac{m\varepsilon}{m-1}\rho^{m-1}\partial_t\rho + \frac{1}{2}(W*\rho)\partial_t\rho\ dx + \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \rho(x,t)W(x-y)\partial_t\rho(y,t)\ dydx
\end{aligned}$$

Now, since W is symmetric it follows that

$$\begin{aligned}\int_{\mathbb{R}^d} \rho(x,t) W * (\partial_t \rho)(x,t)\, dx &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \rho(x,t) W(x-y) \partial_t \rho(y,t)\, dydx \\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \partial_t \rho(y,t) W(y-x) \rho(x,t)\, dxdy \\ &= \int_{\mathbb{R}^d} \partial_t \rho(y,t) W * \rho(y,t)\, dy.\end{aligned}$$

Hence,

$$\partial_t \mathcal{E}[\rho] = \int_{\mathbb{R}^d} \frac{m\varepsilon}{m-1} \rho^{m-1} \partial_t \rho + (W * \rho) \partial_t \rho\, dx.$$

Since ρ is a solution of Equation (1.1), we have that

$$\begin{aligned}\partial_t \mathcal{E}[\rho] &= \int_{\mathbb{R}^d} \nabla \cdot \left(\varepsilon \nabla \rho^m + \rho \nabla (W * \rho)\right) \left(\frac{m\varepsilon}{m-1} \rho^{m-1} + W * \rho\right) dx \\ &= \int_{\mathbb{R}^d} \nabla \cdot \left(\frac{m\varepsilon}{m-1} \rho \nabla \rho^{m-1} + \rho \nabla (W * \rho)\right) \left(\frac{m\varepsilon}{m-1} \rho^{m-1} + W * \rho\right) dx \\ &= -\int_{\mathbb{R}^d} \rho \left(\frac{m\varepsilon}{m-1} \nabla \rho^{m-1} + \nabla (W * \rho)\right)^2 dx \leq 0,\end{aligned}$$

where we have applied the divergence theorem and used the fact that, by Theorem 2.4, $\rho \in L^1_+(\mathbb{R}^d)$ vanishes at infinity, as defined in Definition 2.32.

This is extended in [6], where it is proved that weak solutions of (1.1), as defined in Definition 3.1, satisfy the energy dissipation inequality, for almost all $t \in (0, T)$, given by

$$\mathcal{E}[\rho(t)] + \int_0^t \int_{\mathbb{R}^d} \rho \left(\frac{m\varepsilon}{m-1} \nabla \rho^{m-1} + \nabla (W * \rho)\right)^2 dxdt \leq \mathcal{E}[\rho_0].$$

3.2 Stationary states of the aggregation diffusion equation

Once existence of solutions is known, an important next step is to consider the long-time asymptotics of the model. We recall that one of our objectives is to investigate what choice of diffusion coefficient m and attractive kernel W allows for the emergence of non-trivial patterns which exhibit characteristics of physical and biological aggregations. In order to do this, one must first consider the existence of stationary states to Equation (1.1).

3.2.1 Stationary state equation

Assuming that the density ρ does not depend on time, we obtain the stationary problem of (1.1), given by

$$\varepsilon\Delta\rho^m + \nabla\cdot(\rho\nabla(W*\rho)) = 0, \qquad x\in\mathbb{R}^d. \tag{3.2}$$

We first derive the variational form of Equation (1.1). We consider test functions $\phi\in C_0^\infty(\mathbb{R}^d)$. As before, we multiply (3.2) by ϕ and integrate to obtain

$$\int_{\mathbb{R}^d} \varepsilon\Delta\rho^m\phi + \nabla\cdot(\rho\nabla W*\rho)\phi\; dx = 0.$$

Now, by the divergence theorem and using the fact that ϕ vanishes at infinity, we have that

$$\int_{\mathbb{R}^d} \varepsilon\nabla\rho^m\nabla\phi + (\rho\nabla W*\rho)\nabla\phi\; dx = 0.$$

Hence, considering weak solutions of Problem (1.1), we have that $\rho_s\in L^1_+(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$ is a stationary state of Equation (1.1) if $\rho_s^m\in H^1(\mathbb{R}^d)$, $\nabla W*\rho_s\in L^1(\mathbb{R}^d)$, and it satisfies

$$\varepsilon\nabla\rho_s^m = -\rho_s(\nabla W*\rho_s) \text{ in } \operatorname{supp}\rho_s, \tag{3.3}$$

in the sense of distributions in $\mathbb{R}^d$.

3.2.2 Stationary states for kernels with infinite support

A general method to prove existence of stationary states of (1.1) is by showing that the global minimizer of (3.1) corresponds to a stationary state of (1.1). It is proved in [21] that a global minimizer of (3.1) exists for W with infinite support.

Indeed, under assumptions $(W1)$ and $(W2)$, as well as the following additional assumption on the interaction kernel W, given by $(W3)$, it is proved in [21] that in the diffusion dominated regime, that is $m>\max\{2-\frac{2}{d},1\}$, a global minimizer exists for any given mass, and this global minimizer is uniformly bounded and corresponds to a stationary solution of (1.1) in the weak sense. We state the additional assumption as follows:

***W*3** $\omega'(r)>0$ for all $r>0$ and there exists some $C_\omega>0$ such that $\omega'(r)\le C_\omega$ for $r>1$. Moreover, $\lim_{r\to\infty}\omega(r)=0$ and there exists an $\alpha\in(0,d)$ for which $m>1+\frac{\alpha}{d}$ and $\omega(\tau r)\le\tau^{-\alpha}\omega(r)$ for all $\tau\ge 1$ and $r>0$.

Assumption $(W3)$ combines assumptions $(K3)$, $(K4)$, and $(K6)$ given in [21], allowing us to refer to the results obtained therein. Note that in [21] it is assumed that $\lim_{r\to\infty}\omega(r)=\ell\in(0,\infty)$. However, it is not restrictive to assume that $\ell=0$ since adding a constant to the potential W does not change Equation (1.1).

Note that the threshold value $r = 1$ in $(W2)$ and $(W3)$ can be replaced by any positive value since, in essence, $(W2)$ places restriction on ω' when r is small, while $(W3)$ places a restriction on ω' when r is large.

Furthermore, under assumptions $(W1)$, $(W2)$ and $(W3)$, it is proved in [21] that stationary states of (1.1) are radially symmetric and decreasing up to a translation, as defined in Definition 2.34. We note that this result cannot be applied to kernels with compact support, as Assumption $(W3)$ requires that $\omega(r)$ is strictly increasing in r for all $r > 0$. In terms of the notations adopted here, the results in [21, Theorem 2.2, Theorem 3.1, Theorem 3.7, Lemma 3.8, Lemma 3.9] can be formulated as follows:

Theorem 3.5. *[21] For $m > \max\{2 - \frac{2}{d}, 1\}$, assume that conditions $(W1)$, $(W2)$, and $(W3)$ hold. Then, for any positive mass M, there exists a stationary state $\rho_s \in L^1_+(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ of (1.1). Furthermore, any stationary state in $L^1_+(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ is radially symmetric and decreasing up to a translation and compactly supported.*

As a result of the radial symmetry of stationary solutions, in [26] it is proved that stationary states are unique up to a translation, for a fixed mass. The proof relies on the fact that all stationary solutions of (1.1), where $(W1)$, $(W2)$, and $(W3)$ are satisfied, are radially symmetric and decreasing. Thus, we are not able to extend this uniqueness result given in [26] to the case where W has compact support.

It is therefore an important question to ask whether stationary solutions of Equation (1.1), for W compactly supported, are also radially symmetric and decreasing, as it may allow for the extension of the uniqueness result to a wider class of interaction kernels. In fact, to our knowledge, radial symmetry of stationary solutions for the case when W has compact support is an open question.

4

Stationary states for compactly supported kernels

In this chapter, we consider stationary solutions of Equation (1.1) where the attractive kernel W has compact support. We prove, for this case of attractive kernel, that continuous, compactly supported stationary solutions of (1.1) exist. Our main result of this chapter is that, for $m > 1$, stationary states are radially decreasing up to a translation on each connected component of their support, where the definition of a connected component is given by Definition 2.36. Furthermore, we prove that if the support of a stationary state has more than one connected component then the distance between any two components is at least the radius of $\operatorname{supp} W$.

4.1 General setting and basic properties

Throughout this chapter, we assume that W satisfies assumptions $(W1)$ and $(W2)$ as well as the following assumption given below.

$\hat{W}$4 There exists $q > 0$ such that $\omega'(r) > 0$ for all $0 < r < q$ and $\omega'(r) = 0$ for all $r \geq q$.

The threshold value $r = q$ in $(\hat{W}4)$ can be assumed to be 1, as proven in the following theorem.

Theorem 4.1. *Consider Equation (1.1) with initial condition ρ_0 and suppose the assumptions $(W1)$, $(W2)$, and $(\hat{W}4)$ hold. Then we may assume, without loss of generality, that $q = 1$.*

Proof. Let $z = \frac{x}{q}$ and $\theta(z,t) = \rho(x,t)$. Then we have, by the chain rule, that

$$\partial_t \theta(z,t) = \partial_t \rho(x,t), \quad \nabla_z^2 \theta^m(z,t) = q^2 \nabla_x^2 \rho^m(x,t),$$
$$\nabla_z \cdot (\theta(z,t) \nabla_z (W * \theta(z,t))) = q^2 \nabla_x \cdot (\rho(x,t) \nabla_x (W * \rho(x,t))).$$

Hence,

$$\begin{aligned}\partial_t\theta &= \varepsilon\Delta\rho^m + \nabla\cdot(\rho\nabla(W*\rho)) \\ &= \frac{\varepsilon}{q^2}\Delta\theta^m + \frac{1}{q^2}\nabla\cdot(\theta\nabla(W*\theta)) \\ &= \tilde{\varepsilon}\Delta\theta^m + \nabla\cdot(\theta\nabla(\tilde{W}*\theta))\end{aligned}$$

where $\tilde{W}(z) = \frac{1}{q^2}W(x)$ and $\tilde{\varepsilon} = \dfrac{\varepsilon}{q^2}$. Now, letting $\tilde{r} = ||z||$ and $\tilde{W}(z) = \tilde{\omega}(\tilde{r})$ we have that

$$\nabla_z\tilde{W}(z) = \frac{1}{q}\nabla_x W(x) = \frac{x}{q||x||}\omega'(||x||)$$

and

$$\nabla_z\tilde{W}(z) = \frac{z}{||z||}\tilde{\omega}'(||z||).$$

Therefore, using the fact that $r = q||z|| = q\tilde{r}$, we obtain

$$\tilde{\omega}'(\tilde{r}) = \frac{1}{q}\omega'(r).$$

Hence, it follows that

$$\omega'(r) > 0 \text{ for all } 0 < r < q \text{ implies } \tilde{\omega}'(\tilde{r}) > 0 \text{ for all } 0 < \tilde{r} < 1 \text{ and}$$
$$\omega'(r) = 0 \text{ for all } r > q \text{ implies } \tilde{\omega}'(\tilde{r}) = 0 \text{ for all } \tilde{r} > 1.$$

Since $\tilde{\varepsilon} \in \mathbb{R}$, $\tilde{\varepsilon} > 0$, and $\tilde{W}$ satisfies $(W1)$, $(W2)$, and $(\hat{W}4)$ with $q = 1$, we may assume that $q = 1$ without loss of generality. □

Therefore, for simplicity, we reformulate $(\hat{W}4)$ as follows:

***W*4** $\omega'(r) > 0$ for all $0 < r < 1$ and $\omega'(r) = \omega(r) = 0$ for all $r \geq 1$.

Assumption $(W4)$ implies that W is integrable. This property will be used in the sequel. Hence, we prove it in the theorem below.

Theorem 4.2. *If W satisfies assumptions $(W1)$, $(W2)$ and $(W4)$, then $W \in L^1(\mathbb{R}^d)$.*

Proof. Firstly, using assumption $(W2)$ we see that, for $0 < r < 1$,

$$\left|\int_r^1 \omega'(s)\ ds\right| \leq C_\omega\left|\int_r^1 s^{1-d}\ ds\right|,$$

which implies that

$$|\omega(r)| \leq C_\omega\left|\int_r^1 s^{1-d}\ ds\right|,$$

by the fundamental theorem of calculus. Computing the integral on the right we see that there exists some $C_1 > 0$ such that $|\omega(r)| \leq C_1\phi(r)$ for all $0 < r < 1$ where

$$\phi(r) := \begin{cases} r^{2-d} - 1 & \text{if } d \geq 3 \\ -\log(r) & \text{if } d = 2 \\ 1 - r & \text{if } d = 1. \end{cases}$$

Now, consider the case $d = 1$. Following from $(W1)$, $(W2)$ and $(W4)$, we have that

$$\begin{aligned} \int_{\mathbb{R}} |W(x)|\, dx &= \int_0^\infty |\omega(|x|)|\, dx \\ &= \int_0^1 |\omega(r)|\, dr \\ &\leq C_1 \int_0^1 (1-r)\, dr < \infty. \end{aligned}$$

Similarly, for $d \geq 2$ we obtain

$$\begin{aligned} \int_{\mathbb{R}^d} |W(x)|\, dx &= \int_{B_1(0)} |\omega(||x||)|\, dx \\ &= S_{d-1} \int_0^1 r^{d-1} |\omega(r)|\, dr \\ &\leq C_1 S_{d-1} \int_0^1 r^{d-1} \phi(r)\, dr. \end{aligned} \tag{4.1}$$

Applying integration by parts, we see that for any $d \geq 2$, the integral (4.1) is finite. Note that for $d = 2$ we applied L'Hospital's rule to obtain the limit $\lim_{r \to 0^+} \log(r) r^2 = 0$. Hence, for any $d \geq 1$, it follows that

$$\int_{\mathbb{R}^d} |W(x)|\, dx < \infty,$$

as required. □

We summarize our main results of this chapter in the following theorem.

Theorem 4.3. *Let $m > 2$ and let W satisfy assumptions $(W1)$, $(W2)$, and $(W4)$. Then there exists a stationary solution $\rho_s \in L^1_+(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ of (1.1). Furthermore, $\rho_s \in C(\mathbb{R}^d)$, and is radially symmetric, decreasing, and compactly supported on each connected component of $\operatorname{supp} \rho_s$. Additionally, if $\operatorname{supp} \rho_s$ has more than one connected component then the distance between any two components is at least the radius of $\operatorname{supp} W$.*

As previously mentioned, global minimizers of the energy functional can be used to determine existence of stationary states of (1.1). Hence, we start by stating a previous result on the existence of radially decreasing global minimizers of (3.1).

Theorem 4.4. *[4] For $m > 2$, $W \in L^1(\mathbb{R}^d)$ radially symmetric and non-decreasing, and for any $M > 0$, there exists a radially symmetric and decreasing global minimizer of the energy functional (3.1) defined in*

$$\mathcal{Y}_M := \{\rho \in L^1_+(\mathbb{R}^d) \cap L^m(\mathbb{R}^d) : ||\rho||_{L^1(\mathbb{R}^d)} = M\}.$$

Since $W \in L^1(\mathbb{R}^d)$ under assumptions $(W1)$, $(W2)$, and $(W4)$ by Theorem 4.2, we have existence of a radially symmetric and decreasing global minimizer of (3.1). We further show, under these assumptions on the interaction kernel, that all global minimizers of (3.1) defined in $\mathcal{Y}_M$, whose support consists of a single connected component, are radially symmetric and decreasing, compactly supported, uniformly bounded, and correspond to stationary states of (1.1) in the weak sense.

Theorem 4.5. *Let $m > 2$ and let W satisfy assumptions $(W1)$, $(W2)$, and $(W4)$. If $\bar{\rho}$ is a global minimizer of (3.1) in $\mathcal{Y}_M$ whose support consists of a single connected component, then $\bar{\rho}$ is radially symmetric and decreasing up to a translation.*

Proof. Suppose for a contradiction that $\bar{\rho}$ is not radially symmetric and decreasing under any translation. We show that the energy decreases strictly when $\bar{\rho}$ is replaced with its Schwarz decreasing rearrangement

$$\bar{\rho}^{\#}(x) = \sup\{\tau > 0 : |\{\bar{\rho} > \tau\}|_d > S_{d-1}||x||^d\}, \ x \in \mathbb{R}^d,$$

as defined in Definition 2.39. By Corollary 2.26, we have the invariance property of the L^p norms for $\bar{\rho}$ and $\bar{\rho}^{\#}$, that is, for every $m > 1$,

$$\int_{\mathbb{R}^d} (\bar{\rho}^{\#})^m \, dx = \int_{\mathbb{R}^d} (\bar{\rho})^m \, dx. \tag{4.2}$$

Furthermore, by Riesz's rearrangement inequality, it follows that

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x)g(x-y)h(y) \, dydx \le \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f^{\#}(x)g^{\#}(x-y)h^{\#}(y) \, dydx,$$

for all $f, g, h \in L^1_+(\mathbb{R}^d)$. Now, suppose $g(x) = \tilde{g}(||x||)$ is radially symmetric and non-increasing on $||x|| > 0$. Also, suppose f and h are radially decreasing up to the same translation, as defined in Definition 2.34, such that there exists an $x_0 \in \mathbb{R}^d$ where $f^{\#}(x) = f(x + x_0)$ and $h^{\#}(x) = h(x + x_0)$. Hence, we obtain

$$\begin{aligned}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x)g(x-y)h(y) \, dydx &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x+x_0)g(x-y)h(y+x_0) \, dydx \\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f^{\#}(x)g^{\#}(x-y)h^{\#}(y) \, dydx.\end{aligned}$$

Now, since $-W(x) = -\omega(||x||)$ is non-negative, radially symmetric, and non-increasing on $||x|| > 0$, then

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \bar{\rho}(x)W(x-y)\bar{\rho}(y) \, dydx \ge \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \bar{\rho}^{\#}(x)W(x-y)\bar{\rho}^{\#}(y) \, dydx. \tag{4.3}$$

If $\bar{\rho}$ were radially decreasing up to a translation we would have equality in (4.3), but by assumption $\bar{\rho}$ is not radially decreasing under any translation. Hence, it must be that

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \bar{\rho}(x)W(x-y)\bar{\rho}(y)\ dydx > \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \bar{\rho}^{\#}(x)W(x-y)\bar{\rho}^{\#}(y)\ dydx. \tag{4.4}$$

Combining (4.2) and (4.4) yields $E[\bar{\rho}^{\#}] < E[\bar{\rho}]$, contradicting the assumption that $\bar{\rho}$ is a global minimizer. Hence, $\bar{\rho}$ is radially symmetric and decreasing up to a translation. □

The proof of Theorem 4.6 below follows similarly to the proof of [22, Theorem 3.1], which considers global minimizers of the energy functional corresponding to the $2D$ Keller-Segel equation, that is, where W is the Newtonian kernel. Theorem 4.6 is used to show that global minimizers of the energy functional are in fact stationary states of (1.1).

Theorem 4.6. *Let $m > 2$ and let W satisfy assumptions (W1), (W2), and (W4). If $\bar{\rho}$ is a global minimizer of (3.1) in $\mathcal{Y}_M$ whose support consists of a single connected component, then there exists a constant $D[\bar{\rho}]$ such that*

$$\frac{m\varepsilon}{m-1}\bar{\rho}^{m-1}(x) + (W * \bar{\rho})(x) = D[\bar{\rho}],\ \ a.e\ in\ \operatorname{supp}\bar{\rho} \tag{4.5}$$

and

$$\frac{m\varepsilon}{m-1}\bar{\rho}^{m-1}(x) + (W * \bar{\rho})(x) \geq D[\bar{\rho}],\ \ a.e\ outside\ \operatorname{supp}\bar{\rho} \tag{4.6}$$

where

$$D[\bar{\rho}] = \frac{2}{M}\mathcal{E}[\bar{\rho}] + \frac{m-2}{M(m-1)}||\bar{\rho}||_m^m.$$

That is,

$$\frac{m\varepsilon}{m-1}\bar{\rho}^{m-1}(x) = \big((-W * \bar{\rho})(x) + D[\bar{\rho}]\big)_+$$

for all $x \in \mathbb{R}^d$ where $(a)_+ = \max\{a, 0\}$.

Proof. To prove (4.5), consider $\delta > 0$ and a test function $\psi \in C_0^\infty(\mathbb{R}^d)$ such that $\psi(x) = \psi(-x)$ and define

$$\varphi(x) = \left(\psi(x) - \frac{1}{M}\int_{\mathbb{R}^d}\psi(z)\bar{\rho}(z)\ dz\right)\bar{\rho}(x).$$

It is easy to see that $\varphi \in L^1(\mathbb{R}^d)$ since

$$\begin{aligned}\int_{\mathbb{R}^d}|\varphi(x)|\ dx &\leq \int_{\mathbb{R}^d}|\psi(x)|\bar{\rho}(x)\ dx + \frac{1}{M}\int_{\mathbb{R}^d}\bar{\rho}(x)\int_{\mathbb{R}^d}|\psi(z)|\bar{\rho}(z)\ dzdx \\ &\leq ||\psi||_{L^\infty}||\bar{\rho}||_{L^1} + \frac{1}{M}||\psi||_{L^\infty}||\bar{\rho}||_{L^1}^2 < \infty.\end{aligned}$$

Similarly, it can be shown that $\varphi \in L^m(\mathbb{R}^d)$. Furthermore,

$$\int_{\mathbb{R}^d}\varphi(x)\ dx = 0$$

and $\operatorname{supp}\varphi \subseteq \operatorname{supp}\bar{\rho}$. Moreover, for $\delta < \delta_0 := \dfrac{1}{2m||\psi||_{L^\infty}}$,

$$\begin{aligned}\bar{\rho} + m\delta\varphi &\geq \bar{\rho}\big[1 + m\delta(\psi - ||\psi||_{L^\infty})\big]\\ &\geq \bar{\rho}\big[1 - 2m\delta||\psi||_{L^\infty})\big] \geq 0.\end{aligned}$$

For the same choice of δ, it also follows that $\bar{\rho} + \delta\varphi \geq 0$, since $m > 1$. Hence, $\bar{\rho} + \delta\varphi \in \mathcal{Y}_M$, allowing us to calculate the first variation of $\mathcal{E}$ defined by

$$\frac{\delta\mathcal{E}}{\delta\varphi}[\bar{\rho}] = \lim_{\delta\to 0}\frac{\mathcal{E}[\bar{\rho}+\delta\varphi] - \mathcal{E}[\bar{\rho}]}{\delta}.$$

We find that

$$\frac{\mathcal{E}[\bar{\rho}+\delta\varphi] - \mathcal{E}[\bar{\rho}]}{\delta} = \int_{\operatorname{supp}\bar{\rho}} \frac{\varepsilon}{\delta(m-1)}\big[(\bar{\rho}+\delta\varphi)^m - \bar{\rho}^m\big]\,dx + \int_{\mathbb{R}^d} \varphi W * \bar{\rho}\,dx + \delta\mathcal{W}[\varphi],$$

where

$$\mathcal{W}[\varphi] = \frac{1}{2}\int_{\mathbb{R}^d}\varphi(W*\varphi)\,dx.$$

Now, it must be shown that $\lim_{\delta\to 0}\dfrac{\mathcal{E}[\bar{\rho}+\delta\varphi] - \mathcal{E}[\bar{\rho}]}{\delta}$ exists. We see that

$$\frac{1}{\delta}\int_{\operatorname{supp}\bar{\rho}}\big[(\bar{\rho}+\delta\varphi)^m - \bar{\rho}^m\big]\,dx \;=\; m\int_0^1\int_{\operatorname{supp}\bar{\rho}}(\bar{\rho}+\delta t\varphi)^{m-1}\varphi\,dxdt.$$

By Hölder's inequality, we find that, for all $t \in [0,1]$ and $\delta < \delta_0$,

$$\begin{aligned}\int_{\operatorname{supp}\bar{\rho}}(\bar{\rho}+\delta t\varphi)^{m-1}\varphi\,dx &\leq \left(\int_{\operatorname{supp}\bar{\rho}}\left((\bar{\rho}+\delta t\varphi)^{m-1}\right)^{m/(m-1)}dx\right)^{(m-1)/m}\left(\int_{\operatorname{supp}\bar{\rho}}\varphi^m\,dx\right)^{1/m}\\ &\leq \left(||\bar{\rho}||_{L^m(\mathbb{R}^d)} + \delta_0||\varphi||_{L^m(\mathbb{R}^d)}\right)^{m-1}||\varphi||_{L^m(\mathbb{R}^d)}.\end{aligned}$$

Using the fact that $\left(||\bar{\rho}||_{L^m(\mathbb{R}^d)} + \delta_0||\varphi||_{L^m(\mathbb{R}^d)}\right)^{m-1}||\varphi||_{L^m(\mathbb{R}^d)}$ is Lebesgue integrable with respect to t on $[0,1]$ and that the first order Taylor expansion of $(\bar{\rho}+\delta\varphi)^m$ at $\delta = 0$ is given by $\bar{\rho}^m + m\delta\bar{\rho}^{m-1}\varphi$, it follows by Lebesgue's dominated convergence theorem that

$$\lim_{\delta\to 0}\frac{1}{\delta}\int_{\mathbb{R}^d}\left((\bar{\rho}+\delta\varphi)^m - \bar{\rho}^m\right)dx = \int_{\mathbb{R}^d} m\bar{\rho}^{m-1}\varphi\,dx.$$

Now, if $W \in L^{\frac{m}{m-1}}(\mathbb{R}^d)$ then, by Young's convolution inequality,

$$|(W*\varphi)(x)| \leq ||W||_{L^{\frac{m}{m-1}}(\mathbb{R}^d)}||\varphi||_{L^m(\mathbb{R}^d)},$$

for all $x \in \mathbb{R}^d$. Hence, we need to show that $W \in L^{\frac{m}{m-1}}(\mathbb{R}^d)$. But, this follows from Hölder's inequality, since

$$\begin{aligned}\int_{\mathbb{R}^d}|W(x)|^{\frac{m}{m-1}}\,dx &\leq \left(\int_{B_1(0)}\left(|W(x)|^{\frac{m}{m-1}}\right)^{\frac{m-1}{m}}dx\right)^{\frac{m}{m-1}}\left(\int_{B_1(0)}1\,dx\right)^{\frac{1}{1-m}}\\ &= \left(\frac{S_{d-1}}{d}\right)^{\frac{1}{1-m}}||W||_{L^1(\mathbb{R}^d)}^{\frac{m}{m-1}} < \infty.\end{aligned}$$

Consequently, we have that

$$\int_{\mathbb{R}^d} \varphi(x)(W*\varphi)(x)\ dx \leq ||W*\varphi||_{L^\infty(\mathbb{R}^d)}||\varphi||_{L^1(\mathbb{R}^d)},$$

and so $\lim_{\delta\to 0}\delta\mathcal{W}[\varphi]=0$. As a result,

$$\lim_{\delta\to 0}\frac{\mathcal{E}[\bar\rho+\delta\varphi]-\mathcal{E}[\bar\rho]}{\delta}=\int_{\mathbb{R}^d}\left(\frac{m\varepsilon}{m-1}\bar\rho^{m-1}+W*\bar\rho\right)\varphi\ dx\geq 0,$$

where the non-negativity follows from the fact that $\bar\rho$ is a global minimizer of $\mathcal{E}$ and so $\mathcal{E}[\bar\rho+\delta\varphi]\geq\mathcal{E}[\bar\rho]$.

Now, by performing the same argument using $-\psi$ instead of ψ, we obtain

$$\int_{\mathbb{R}^d}\left(\frac{m\varepsilon}{m-1}\bar\rho^{m-1}+W*\bar\rho\right)\varphi\ dx=0,$$

which gives us

$$\int_{\mathbb{R}^d}\left(\frac{m\varepsilon}{m-1}\bar\rho^{m-1}+W*\bar\rho-D[\bar\rho]\right)\bar\rho\psi\ dx=0,$$

for all even functions $\psi\in C_0^\infty(\mathbb{R}^d)$. Hence, using Theorem 4.5, we have that

$$\frac{m\varepsilon}{m-1}\bar\rho^{m-1}(x)+(W*\bar\rho)(x)=D[\bar\rho],\ \text{a.e in}\ \operatorname{supp}\bar\rho.$$

To show (4.6), we consider an even function $\psi\in C_0^\infty(\mathbb{R}^d)$ with $\psi\geq 0$ such that $\psi(x)\in[0,1]$. Define

$$\varphi=\psi-\frac{\bar\rho}{M}\int_{\mathbb{R}^d}\psi(x)\ dx.$$

Then $\varphi\in L^1(\mathbb{R}^d)\cap L^m(\mathbb{R}^d)$ and $\int_{\mathbb{R}^d}\varphi\ dx=0$. Furthermore,

$$\begin{aligned}\bar\rho+m\delta\varphi &\geq \bar\rho\left(1-\frac{m\delta}{M}\int_{\operatorname{supp}\psi}\psi(x)\ dx\right)\\ &\geq \bar\rho\left(1-\frac{m\delta}{M}|\operatorname{supp}\psi|_d\right).\end{aligned}$$

Choose $\delta<\delta_0:=\frac{M}{m|\operatorname{supp}\psi|_d}$. Then $\bar\rho+m\delta\varphi\geq 0$. Similarly, for the same choice of δ, $\bar\rho+\delta\varphi\geq 0$. Hence $\bar\rho+\delta\varphi\in\mathcal{Y}_M$. Using the same argument as for the proof of (4.5), we find that

$$\int_{\mathbb{R}^d}\left[\frac{m\varepsilon}{m-1}\bar\rho^{m-1}+(W*\bar\rho)-D[\bar\rho]\right]\psi\ dx\geq 0,$$

for all ψ as defined above. Again, by Theorem 4.5, we have that

$$\frac{m\varepsilon}{m-1}\bar\rho^{m-1}(x)+(W*\bar\rho)(x)-D[\bar\rho]\geq 0,$$

for a.e. $x\in\mathbb{R}^d$, yielding (4.6). □

Using Theorem 4.5 and 4.6, we are able to obtain the following result:

Theorem 4.7. *Let $m > 2$ and let W satisfy assumptions (W1), (W2), and (W4). If $\bar{\rho}$ is a global minimizer of (3.1) in $\mathcal{Y}_M$ whose support consists of a single connected component, then $\bar{\rho}$ is compactly supported.*

Proof. Suppose for a contradiction that $\operatorname{supp}\bar{\rho}$ is not compact. Then $\operatorname{supp}\bar{\rho} = \mathbb{R}^d$ since, by Theorem 4.5, $\bar{\rho}$ is radially decreasing from some $x_0 \in \mathbb{R}^d$ acting as a centre. Hence, from (4.5), we have that there exists a constant C such that

$$\frac{m\varepsilon}{m-1}\bar{\rho}^{m-1}(x) + (W * \bar{\rho})(x) = C, \tag{4.7}$$

for a.e. $x \in \mathbb{R}^d$. Since $\bar{\rho}$ is radially decreasing and in $L^1(\mathbb{R}^d)$, there is a function $\bar{\rho}_*$ where $\bar{\rho}(x) = \bar{\rho}_*(||x||)$ and where $\lim_{||x||\to\infty} \bar{\rho}_*(||x||) = 0$.

In addition, we claim that $\lim_{||x||\to\infty}(W * \bar{\rho})(x) = 0$. To show this, let $\mathcal{A} = \{y \in \mathbb{R}^d : ||x - y|| < 1\}$ and fix $||x|| > ||x_0|| + 1$. Then, since $||x - y|| < 1$ implies $||x_0|| < ||x|| - 1 < ||y||$, we have that $\bar{\rho}_*(||x_0||) \geq \bar{\rho}_*(||x|| - 1) \geq \bar{\rho}_*(||y||)$. Therefore, for $||x|| > ||x_0|| + 1$,

$$\begin{aligned}
\int_{\mathbb{R}^d} W(x-y)\bar{\rho}(y)\,dy &= \int_A W(x-y)\bar{\rho}_*(||y||)\,dy \\
&\leq \bar{\rho}_*(||x|| - 1)\int_A W(x-y)\,dy \\
&\leq \bar{\rho}_*(||x|| - 1)||W||_{L^1(\mathbb{R}^d)}.
\end{aligned}$$

Taking the limit as $||x|| \to \infty$ yields our claim.

Now, taking the limit as $||x|| \to \infty$ in (4.7), we obtain

$$\frac{m\varepsilon}{m-1}\bar{\rho}^{m-1}(x) = (-W * \bar{\rho})(x),$$

for a.e. $x \in \mathbb{R}^d$. Fix $x \in \mathbb{R}^d$ and define $\mathcal{B} = \{y \in \mathcal{A} : ||x|| > ||y||\}$. Since $W(x) = \omega(||x||) \leq 0$ for all $||x|| > 0$, we have that $|W(x)| = -W(x)$ for all $x \in \mathbb{R}^d$ and so there exists $C > 0$ such that

$$\begin{aligned}
\int_{\mathbb{R}^d} -W(x-y)\bar{\rho}(y)\,dy &\geq \int_{\mathcal{B}} |W(x-y)|\bar{\rho}_*(||y||)\,dy \\
&\geq \bar{\rho}_*(||x||)\int_{\mathcal{B}} |W(x-y)|\,dy \\
&= C\bar{\rho}_*(||x||) > 0.
\end{aligned}$$

Thus, it follows that

$$\frac{m\varepsilon}{m-1}\bar{\rho}^{m-1}(x) \geq C\bar{\rho}(x) > 0,$$

and so

$$\bar{\rho}(x) \geq \left(\frac{C(m-1)}{m\varepsilon}\right)^{\frac{1}{m-2}} > 0,$$

for a.e $x \in \mathbb{R}^d$. Hence,

$$\int_{\mathbb{R}^d} \bar{\rho}(x)\, dx \geq \int_{\mathbb{R}^d} \left(\frac{C(m-1)}{m\varepsilon}\right)^{\frac{1}{m-2}} dx = \infty,$$

This contradicts the fact that $\bar{\rho} \in L^1(\mathbb{R}^d)$. Hence, $\operatorname{supp} \bar{\rho}$ is compact.

□

Theorem 4.8. *Let $m > 2$ and let W satisfy assumptions ($W1$), ($W2$), and ($W4$) and let $\bar{\rho} \in \mathcal{Y}_M$ be a global minimizer of (3.1) whose support consists of a single connected component. Then $\bar{\rho} \in L^\infty(\mathbb{R}^d)$.*

Theorems 4.5, 4.6, and 4.7 are applied to obtain the above result. The method of proof for Theorem 4.8 follows similarly to that of [21, Lemma 3.9], where it is proved that the boundedness property of $\bar{\rho}$ holds under the assumptions ($W1$)-($W3$). Indeed, if ($W4$) holds, then $\lim_{r\to\infty} \omega(r) = 0$ and there exists $C_\omega > 0$ such that $\omega'(r) \leq C_\omega$ for all $r > 1$, as given in ($W3$). Furthermore, the proof of [21, Lemma 3.9] does not rely on the assumptions made in ($W3$) that $\omega'(r) > 0$ for all $r > 0$ and that there exists an $\alpha \in (0, d)$ for which $m > 1 + \frac{\alpha}{d}$ and $\omega(\tau r) \leq \tau^{-\alpha}\omega(r)$ for all $\tau \geq 1$ and $r > 0$.

Theorem 4.9. *Let $m > 2$ and let W satisfy assumptions ($W1$), ($W2$), and ($W4$) and let $\bar{\rho} \in \mathcal{Y}_M$ be a global minimizer of (3.1) whose support consists of a single connected component. Then $W * \bar{\rho} \in \mathcal{W}^{1,\infty}(\mathbb{R}^d)$.*

Proof. We show first that $\nabla W * \bar{\rho}$ is globally bounded. Fix $x \in \mathbb{R}^d$ and let $\mathcal{A} = \{y \in \mathbb{R}^d : ||x-y|| < 1\}$. Then,

$$\begin{aligned}
|(\nabla W * \bar{\rho})(x)| &\leq \int_{\mathbb{R}^d} \omega'(||x-y||)\bar{\rho}(y)\, dy \\
&\leq C_\omega \int_{\mathcal{A}} \frac{1}{||x-y||^{d-1}} \bar{\rho}(y)\, dy \\
&\leq C_\omega ||\bar{\rho}||_{L^\infty(\mathbb{R}^d)} \int_{\mathcal{A}} \frac{1}{||x-y||^{d-1}}\, dy \\
&= C := \text{Const.}
\end{aligned}$$

Furthermore, we see that $W * \bar{\rho}$ is globally bounded since

$$\begin{aligned}
|(W * \bar{\rho})(x)| &\leq \int_{\mathbb{R}^d} |W(x-y)|\bar{\rho}(y)\, dy \\
&\leq ||\bar{\rho}||_{L^\infty(\mathbb{R}^d)} ||W||_{L^1(\mathbb{R}^d)}, \text{ for all } x \in \mathbb{R}^d.
\end{aligned}$$

□

The following theorem relates global minimizers of the energy functional to stationary solutions of Equation (1.1).

Theorem 4.10. *Let $m > 2$ and let W satisfy assumptions $(W1)$, $(W2)$, and $(W4)$ and let $\bar{\rho} \in \mathcal{Y}_M$ be a global minimizer of (3.1) whose support consists of a single connected component. Then $\bar{\rho}$ is a stationary solution of (1.1) in the weak sense.*

Proof. From Theorem 4.6 we have that

$$\frac{m\varepsilon}{m-1}\bar{\rho}^{m-1} + (W * \bar{\rho}) = D[\bar{\rho}], \text{ a.e in } \operatorname{supp}\bar{\rho}. \tag{4.8}$$

Furthermore, from Theorem 4.9 it follows that $W * \bar{\rho} \in \mathcal{W}^{1,\infty}(\mathbb{R}^d)$. Hence, we can take gradients on both sides of (4.8) and multiply by $\bar{\rho}$ to obtain

$$\frac{m\varepsilon}{m-1}\bar{\rho}\nabla\bar{\rho}^{m-1} = -\bar{\rho}\nabla(W * \bar{\rho}), \text{ a.e in } \operatorname{supp}\bar{\rho}.$$

Using the fact that $\bar{\rho}\nabla\bar{\rho}^{m-1} = \frac{m-1}{m}\nabla\bar{\rho}^m$, we have that

$$\varepsilon\nabla\bar{\rho}^m = -\bar{\rho}\nabla(W * \bar{\rho}), \text{ a.e in } \operatorname{supp}\bar{\rho},$$

yielding (3.3), as required. □

From Theorem 4.10 we have the existence of a stationary solution ρ_s of (1.1) in $L^1_+(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. We now show that, for any stationary state of (1.1) in $L^1_+(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ with support possibly made up of more than one connected component (as defined in Definition 2.36), we have that on each connected component of $\operatorname{supp}\rho_s$, ρ_s is radially symmetric and decreasing up to a translation and compactly supported. Additionally, the distance between any two components is at least the radius of $\operatorname{supp} W$.

From this point onwards we restrict our attention to the case where $m > 2$, where we have existence of stationary states of (1.1) for any W satisfying assumptions $(W1)$, $(W2)$ and $(W4)$.

In order to prove our result on radial symmetry of stationary solutions for compactly supported W, given in the next section, we need Lemma 4.11 given below. A similar result, for the case when $(W1) - (W3)$ are satisfied, is given in [21, Lemma 2.3]. Note that in [21, Lemma 2.3] an extra assumption is made, that is, $\omega(1 + ||x||)\rho_s \in L^1(\mathbb{R}^d)$.

This assumption need not be made for the case of ω satisfying $(W1)$, $(W2)$, and $(W4)$. This is because $\omega(r) = 0$ for all $r \geq 1$, by assumption $(W4)$, which yields $\omega(1 + ||x||)\rho_s \in L^1(\mathbb{R}^d)$, as required.

Lemma 4.11. *Let $\rho_s \in L^1_+(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ be a non-negative stationary state of (1.1) where $m > 2$ and where W satisfies assumptions $(W1)$, $(W2)$, and $(W4)$. Then $\rho_s \in C(\mathbb{R}^d)$ and*

$$\frac{m\varepsilon}{m-1}\rho_s^{m-1}(x) + (W * \rho_s)(x) = C_j, \ \textit{for } x \in D_j, \tag{4.9}$$

where C_j may be different on each connected component D_j of $\operatorname{supp}\rho_s$*.*

Furthermore, there exists some $C = C(||\rho_s||_{L^1}, ||\rho_s||_{L^\infty}, C_\omega, d) > 0$ such that

$$\frac{m\varepsilon}{m-1}|\nabla(\rho_s^{m-1})| \leq C \text{ in } \operatorname{supp}\rho_s.$$

Proof. The proof follows the same approach as that of [21, Lemma 2.3]. It is only necessary to prove, under the assumptions $(W1)$, $(W2)$, and $(W4)$, that $\nabla W * \rho_s$ and $W * \rho_s$ are globally bounded. This is proved by using the same argument as in the proof of Theorem 4.9.

□

From Theorem 4.10, we have that global minimizers of the energy, whose supports are connected, are stationary states of Equation (1.1). The converse is not necessarily true; however, it is the case that a stationary state of (1.1) is a stationary point of the energy, as stated in the following theorem.

Theorem 4.12. *If ρ_s is a stationary state of Equation (1.1), then ρ_s is a stationary point of the energy functional (3.1).*

Proof. Consider $\delta > 0$ and a test function $\psi \in C_0^\infty(\mathbb{R}^d)$. Similarly to the proof of Theorem 4.6, we define

$$\varphi(x) = \left(\psi(x) - \frac{1}{M}\int_{\mathbb{R}^d}\psi(z)\rho_s(z)\ dz\right)\rho_s(x).$$

In order to prove the result we must show that the first variation of $\mathcal{E}$ vanishes. That is,

$$\lim_{\delta\to 0}\frac{1}{\delta}(\mathcal{E}[\rho_s + \delta\varphi] - \mathcal{E}[\rho_s]) = 0.$$

From the proof of Theorem 4.6, we know that $\varphi \in L^1(\mathbb{R}^d) \cap L^m(\mathbb{R}^d)$ and

$$\int_{\mathbb{R}^d}\varphi(x)\ dx = 0.$$

Furthermore, for $\delta < \delta_0 := \dfrac{1}{2m||\psi||_{L^\infty}}$, we have that $\rho_s + m\delta\varphi \geq 0$ and $\rho_s + \delta\varphi \geq 0$, since $m > 1$. Calculating the first variation, we obtain

$$\frac{\mathcal{E}[\rho_s + \delta\varphi] - \mathcal{E}[\rho_s]}{\delta} = \int_{\operatorname{supp}\rho_s}\frac{\varepsilon}{\delta(m-1)}\left[(\rho_s + \delta\varphi)^m - \rho_s^m\right]\ dx + \int_{\mathbb{R}^d}\varphi W * \rho_s\ dx + \delta\mathcal{W}[\varphi],$$

where

$$\mathcal{W}[\varphi] := \frac{1}{2}\int_{\mathbb{R}^d}\varphi(W * \varphi)\ dx.$$

Using the same arguments as in the proof of Theorem 4.6, we find that

$$\lim_{\delta\to 0}\frac{\mathcal{E}[\rho_s+\delta\varphi]-\mathcal{E}[\rho_s]}{\delta}=\int_{\mathbb{R}^d}\left(\frac{m\varepsilon}{m-1}\rho_s^{m-1}+W*\rho_s\right)\varphi\,dx.$$

Now, since $\int_{\mathbb{R}^d}\varphi(x)\,dx=0$, for the first variation to vanish it must be that

$$\frac{m\varepsilon}{m-1}\rho_s^{m-1}(x)+W*\rho_s(x)=C_j \text{ for } x\in\operatorname{supp}\rho_s,$$

where C_j is a constant that may be different on each connected component of $\operatorname{supp}\rho_s$. Hence, since ρ_s satisfies (4.9) by Lemma 4.11, we have that ρ_s is a stationary point of $\mathcal{E}$. □

4.2 Radial symmetry property of stationary states

In this section, we prove our main result of this chapter. Namely that, for $m>2$ and W satisfying assumptions $(W1)$, $(W2)$, and $(W4)$, all stationary solutions of (1.1) are radially symmetric and decreasing when restricted to a single connected component of their support. Furthermore, we show that, for any stationary state whose support has more than one connected component, the distance between any two components is at least the radius of the support of W. These results are summarized in the theorem below.

Theorem 4.13. *Let $m>2$ and let W satisfy assumptions $(W1)$, $(W2)$, and $(W4)$. Let $\rho_s\in L^1_+(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$ be a stationary state of (1.1) and let $D\subset\mathbb{R}^d$ be a connected component of $\operatorname{supp}\rho_s$, as given in Definition 2.36. That is, $D\cap\text{closure}\,(\operatorname{supp}\rho_s\setminus D)=\emptyset$. Then the following holds:*

1. *There exists an $x_0\in D$ such that $\rho_s|_D$ is radially symmetric and decreasing from x_0 as a centre.*
2. *For all $x\in\text{interior}\,(D)$ and for any $y\in\text{interior}(\operatorname{supp}\rho_s\setminus D)$, we have $||x-y||\geq 1$.*

Proof outline for Theorem 4.13.1

We prove Theorem 4.13.1 by contradiction, assuming ρ_s is not radially symmetric and decreasing under any translation.

Similarly to the proof of [21, Theorem 2.2], we use continuous Steiner symmetrization to construct a family of densities $\mu(\tau,\cdot)$ with $\rho_s|_D=\mu(0,\cdot)=:\mu_0$ such that $\mathcal{E}[\mu(\tau)]-\mathcal{E}[\mu_0]<-C_1\tau$, for some $C_1>0$ and any sufficiently small $\tau>0$.

However; since μ_0 is a stationary state of $\mathcal{E}$, it can be shown that $|\mathcal{E}[\mu(\tau)]-\mathcal{E}[\mu_0]|\leq C_2\tau^2$ for some $C_2>0$ and for all τ small enough. Combining these two inequalities results in a contradiction for τ sufficiently small.

In order to prove Theorem 4.13.1, we must prove the preliminary results, Lemma 4.14 and 4.15, given below, where we investigate how the interaction energy, denoted by $\mathcal{I}$ in (3.1), between two densities $\mu_1, \mu_2 \in C(\mathbb{R})$ changes under their continuous Steiner symmetrizations. That is, we consider how

$$I_K[\mu_1, \mu_2](\tau) := \int_{\mathbb{R}\times\mathbb{R}} S^\tau \mu_1(\alpha) S^\tau \mu_2(\beta) K(\alpha - \beta)\, d\alpha d\beta, \tag{4.10}$$

changes with respect to τ, given some $K \in C^1(\mathbb{R})$ to be defined later.

Lemma 4.14. *Assume $K \in C^1(\mathbb{R})$ is an even function with $K'(z) < 0$ for all $0 < z < R$ and $K'(z) = 0$ for all $|z| \geq R$. Let $\mu_i := \chi_{I(\tilde{x}+c_i, r_i)}$ for $i = 1, 2$. Then, for $I(\tau) := I_K[\mu_1, \mu_2](\tau)$,*

1. $\frac{d^+}{d\tau} I(0) \geq 0$.
2. *In addition, if* $\operatorname{sgn} c_1 \neq \operatorname{sgn} c_2$,

$$|c_2 - c_1| < r_2 + r_1 + R, \quad \text{and} \quad |r_2 - r_1| < |c_2 - c_1| + R, \tag{4.11}$$

then

$$\tfrac{d^+}{d\tau} I(0) \geq \tfrac{1}{6}\varphi(c_1, r_1, c_2, r_2, R) \min_{r \in [\frac{R}{3\sqrt{2}}, \frac{R}{\sqrt{2}}]} K'(r) =: c > 0,$$

where

$$\varphi(c_1, r_1, c_2, r_2, R) = \min\{-|r_1 - r_2| + |c_2 - c_1| + R, -|c_2 - c_1| + r_1 + r_2 + R, R\} \tag{4.12}$$

Proof. Without loss of generality, assume $c_2 \geq c_1$. For the case where $c_2 < c_1$ the roles of x and y in the proof are reversed.

For $x \in \mathbb{R}$, we have that $S^\tau \mu_i = \int_0^\infty \chi_{M^\tau(U^h(\mu_i))}(x)\, dh = \chi_{M^\tau(I(\tilde{x}+c_i, r_i))}$ for $i = 1, 2$. Then for any $\tau \geq 0$ it follows that

$$\begin{aligned} I(\tau) &= I_K[\chi_{I(\tilde{x}+c_1, r_1)}, \chi_{I(\tilde{x}+c_2, r_2)}](\tau) \\ &= \int_{-r_1+\tilde{x}+c_1-\tau \operatorname{sgn} c_1}^{r_1+\tilde{x}+c_1-\tau \operatorname{sgn} c_1} \int_{-r_2+\tilde{x}+c_2-\tau \operatorname{sgn} c_2}^{r_2+\tilde{x}+c_2-\tau \operatorname{sgn} c_2} K(x-y)\, dydx \\ &= \int_{-r_1}^{r_1} \int_{-r_2}^{r_2} K(x - y + (c_1 - c_2) + \tau(\operatorname{sgn} c_2 - \operatorname{sgn} c_1))\, dydx \end{aligned}$$

If $\operatorname{sgn} c_1 = \operatorname{sgn} c_2$, then

$$\begin{aligned} \frac{d^+}{d\tau} I(0) &= (\operatorname{sgn} c_2 - \operatorname{sgn} c_1) \int_{-r_1}^{r_1} \int_{-r_2}^{r_2} K'(x - y + (c_1 - c_2))\, dydx \\ &= 0. \end{aligned}$$

If $\operatorname{sgn} c_1 \neq \operatorname{sgn} c_2$ we have that $\operatorname{sgn} c_2 - \operatorname{sgn} c_1$ is either 2 or 1. Hence,

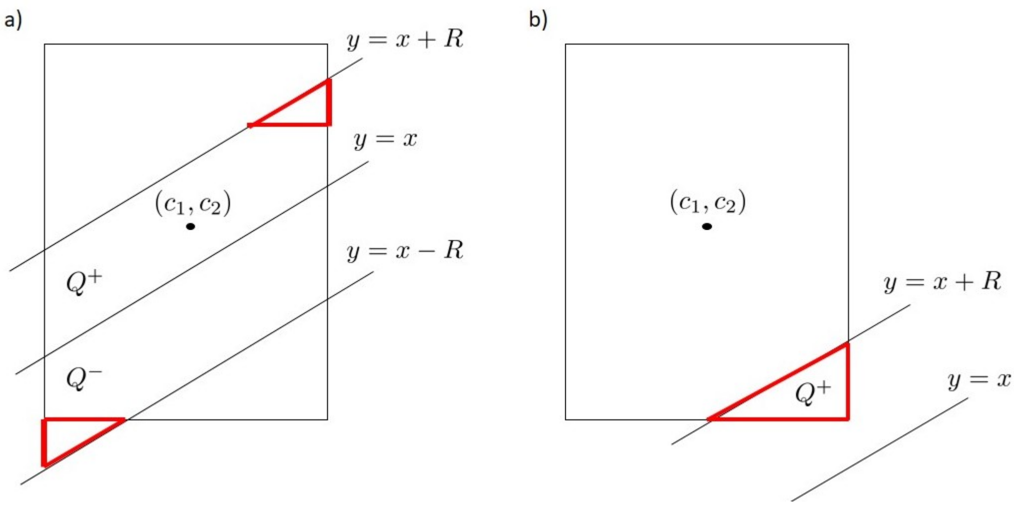

Figure 4.1: Illustration of the rectangle Q given in the proof of Lemma 4.14.

$$\begin{aligned} \frac{d^+}{d\tau} I(0) &= (\operatorname{sgn} c_2 - \operatorname{sgn} c_1) \int_{-r_1}^{r_1} \int_{-r_2}^{r_2} K'(x - y + (c_1 - c_2))\, dydx \\ &= (\operatorname{sgn} c_2 - \operatorname{sgn} c_1) \int_Q K'(x-y)\, dydx \end{aligned} \tag{4.13}$$

where Q is the rectangle $[c_1 - r_1, c_1 + r_1] \times [c_2 - r_2, c_2 + r_2]$. Define $Q^- := Q \cap \{0 < x - y < R\}$ and $Q^+ := Q \cap \{-R < x - y < 0\}$. Note that $K'(x-y) < 0$ in Q^- and $K'(x-y) > 0$ in Q^+. Since $K'(x-y) = 0$ for $|x - y| \geq R$, we have that

$$\frac{d^+}{d\tau} I(0) \geq \int_{Q^+} K'(x-y)\, dydx + \int_{Q^-} K'(x-y)\, dydx.$$

Regardless of the choice of r_1 and r_2, since $\operatorname{sgn} c_1 \neq \operatorname{sgn} c_2$ and $c_2 > c_1$, Q forms a rectangle with its centre, given by (c_1, c_2), lying above the line $y = x$. Hence, for any $h > 0$, the length of the line segment $Q^+ \cap \{x - y = -h\}$ will be greater or equal to the length of $Q^- \cap \{x - y = h\}$. This implies that $|Q^+| \geq |Q^-|$ and so

$$\frac{d^+}{d\tau} I(0) \geq 0,$$

which proves 1.

Now, assume that $\operatorname{sgn} c_1 \neq \operatorname{sgn} c_2$ and (4.11) holds. Furthermore, assume that $r_2 \geq r_1$. Under these assumptions we obtain three possibilities, given in Figure 4.1 and 4.2 (a).

We see in Figure 4.1 (a) that, since $r_2 - r_1 < |c_2 - c_1| + R$, the bottom left-hand corner of the rectangle must be above the line $y = x - R$. Similarly, in Figure 4.1 (b), since $|c_2 - c_1| < r_2 + r_1 + R$, we see that the bottom right-hand corner of the rectangle must be below the line $y = x + R$.

Under our assumptions we will always have that the area of Q^+ will be strictly greater than that of Q^-, regardless of the choice of r_1 and r_2, where $r_2 \geq r_1$. Furthermore, in all cases the difference in area will be at least the size of the triangle denoted D in Figure 4.2 (b) (outlined red in Figure 4.1 (a), (b), and Figure 4.2 (a)).

The vertices of D are given by $(c_1 + r_1, c_1 + r_1 + R)$, $(c_1 + r_1, z)$ and $(z - R, z)$, where $z := \max\{2c_1 - c_2 + r_2, c_1 + r_1, c_2 - r_2\}$. Now, consider the trapezium $\Omega \subset D$ where the bases of Ω lie parallel to the hypotenuse of D and the longer base intersects the medicentre of D, as illustrated in Figure 4.2 (b). Since $K'(x - y) > 0$ for all $x, y \in \Omega$, we have that

$$\begin{aligned} \frac{d^+}{d\tau} I(0) &\geq \int_D K'(x - y)\, dydx \\ &\geq \int_\Omega K'(x - y)\, dydx \\ &\geq |\Omega| \min_{(x,y)\in\Omega} K'(x - y). \end{aligned}$$

Now, substituting $(c_1 + r_1, z)$ and the medicentre of D, given by $\frac{1}{3}(2c_1 + 2r_1 + z - R, c_1 + r_1 + R + 2z)$, into the equation of the unit normal to $y = x$, given by $\frac{1}{\sqrt{2}}(y - x) = 0$, we find that

$$\min_{(x,y)\in\Omega} K'(x - y) = \min_{r\in[z_1,z_2]} K'(r),$$

where

$$z_1 = \tfrac{1}{\sqrt{2}}(z - c_1 - r_1) \geq \tfrac{R}{3\sqrt{2}}$$

and

$$z_2 = \tfrac{1}{3\sqrt{2}}(z - c_1 - r_1 + 2R) \leq \tfrac{1}{3\sqrt{2}}(c_1 + r_1 + R - c_1 - r_1 + 2R) = \tfrac{R}{\sqrt{2}}.$$

Hence,

$$\min_{r\in[z_1,z_2]} K'(r) \geq \min_{r\in[\frac{R}{3\sqrt{2}},\frac{R}{\sqrt{2}}]} K'(r) > 0.$$

Furthermore, by the properties of the medicentre, the longer base of Ω divides both the base and the height of D in a ratio $1 : 2$ as illustrated in Figure 4.2 (b). Therefore, by the definition of Ω, we find that $|\Omega| = \frac{1}{6}(c_1 + r_1 - z + R)^2$. Now, denoting

$$\begin{aligned} \varphi^*(c_1, r_1, c_2, r_2, R) &:= c_1 + r_1 - z + R \\ &= \min\{r_1 - r_2 + c_2 - c_1 + R, c_1 - c_2 + r_1 + r_2 + R, R\}, \end{aligned}$$

by our assumptions we have that $\varphi^*(c_1, r_1, c_2, r_2, R) > 0$. Hence, we may conclude that

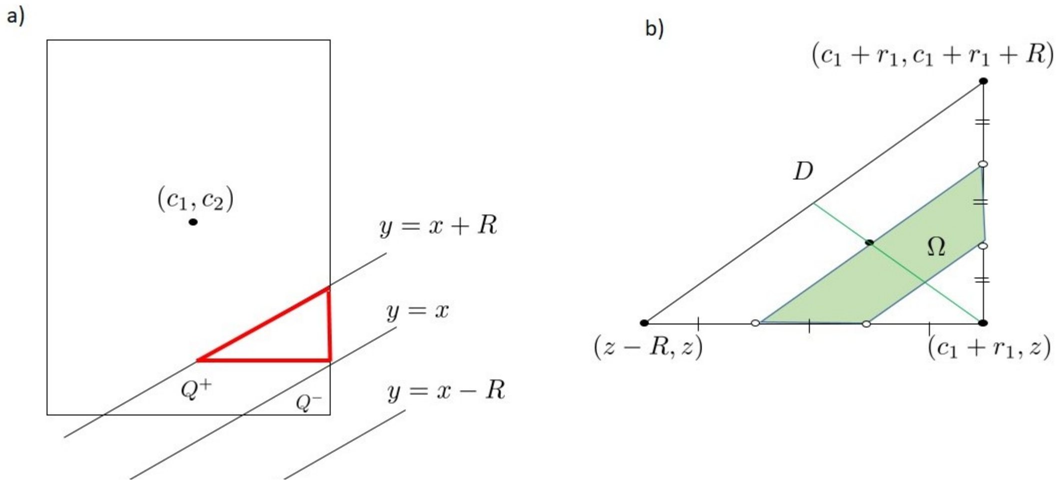

Figure 4.2: a) Illustration of the rectangle Q. b) Illustration of the trapezium Ω contained in D.

$$\frac{d^+}{d\tau} I(0) > \frac{1}{6} \varphi^*(c_1, r_1, c_2, r_2, R) \min_{r \in [\frac{R}{3\sqrt{2}}, \frac{R}{\sqrt{2}}]} K'(r) > 0. \qquad (4.14)$$

For $r_1 > r_2$, one can obtain in a similar way the inequality (4.14), where r_1 and r_2 are swapped. Accommodating also the case of $c_1 > c_2$ the inequality (4.14) holds with

$$\varphi^*(c_1, r_1, c_2, r_2, R) = \varphi(c_1, r_1, c_2, r_2, R),$$

as defined in (4.12) □

Lemma 4.15. *Assume $K \in C^1(\mathbb{R})$ is as defined in Lemma 1. For any open sets U_1, $U_2 \subset \mathbb{R}$, let $\mu_i := \chi_{U_i}$ for $i = 1, 2$ and $I(\tau) = I_K[\mu_1, \mu_2](\tau)$. Then*

$$\frac{d}{d\tau} I(\tau) \geq 0 \text{ for all } \tau \geq 0.$$

Proof. Suppose first that U_1 and U_2 consist of a finite union of disjoint open intervals. Fix $\tau_0 \geq 0$. Define $M^{\tau_0}(U_1) =: \bigcup_{k=1}^{N_1} I(\tilde{x} + c_k^1, r_k^1)$ and $M^{\tau_0}(U_2) =: \bigcup_{k=1}^{N_2} I(\tilde{x} + c_k^2, r_k^2)$ where $I(\tilde{x} + c_k^1, r_k^1)$ and $I(\tilde{x} + c_k^2, r_k^2)$ are disjoint for all $k \in \{1, ..., N_1\}$ and $k \in \{1, ..., N_2\}$, respectively.

Now, S^τ has the semigroup property. That is, $S^{\tau+s}\mu_i = S^\tau(S^s \mu_i)$ for $i = 1, 2$ and any $\tau, s \geq 0$. Hence, for $s \geq 0$ small enough, where we set $\tau = \tau_0 + s$, we have that $M^\tau(U_1) = \bigcup_{k=1}^{N_1} M^s(I(\tilde{x} + c_k^1, r_k^1))$ and $M^\tau(U_2) = \bigcup_{k=1}^{N_2} M^s(I(\tilde{x} + c_k^2, r_k^2))$. Assume without loss of generality that $c_k^1 \leq c_\ell^2$ for all $k \in \{1, ..., N_1\}$ and $\ell \in \{1, ..., N_2\}$. Denote $I^{(k,\ell)}(s) = I_K[\chi_{(\tilde{x}+c_k^1, r_k^1)}, \chi_{I(\tilde{x}+c_\ell^2, r_\ell^2)}](s)$. Then,

$$\begin{aligned}
I(\tau) &= I_K[\chi_{M^{\tau_0}(U_1)}, \chi_{M^{\tau_0}(U_2)}](s) \\
&= \int_{\mathbb{R}\times\mathbb{R}} \chi_{M^{\tau}(U_1)}\chi_{M^{\tau}(U_2)} K(x-y)\, dydx \\
&= \sum_{k=1}^{N_1}\sum_{\ell=1}^{N_2} \int_{M^s(I(\tilde{x}+c_k^1, r_k^1))} \int_{M^s(I(\tilde{x}+c_\ell^2, r_\ell^2))} K(x-y)\, dydx \\
&= \sum_{k=1}^{N_1}\sum_{\ell=1}^{N_2} I_K[\chi_{(\tilde{x}+c_k^1, r_k^1)}, \chi_{I(\tilde{x}+c_\ell^2, r_\ell^2)}](s) \\
&= \sum_{k=1}^{N_1}\sum_{\ell=1}^{N_2} I^{(k,\ell)}(s)
\end{aligned}$$

Hence,

$$\frac{d}{d\tau} I(\tau) = \sum_{k=1}^{N_1}\sum_{\ell=1}^{N_2} \frac{d}{ds} I^{(k,\ell)}(s)$$

Taking $s = 0$, we obtain

$$\frac{d}{d\tau} I(\tau_0) = \sum_{k=1}^{N_1}\sum_{\ell=1}^{N_2} \frac{d}{ds} I^{(k,\ell)}(0) \geq 0, \tag{4.15}$$

by Lemma 4.14. Therefore, since $\tau_0 \geq 0$ is arbitrary, the result holds.

Now, if U_1, U_2 each consist of an infinite union of disjoint open intervals then, for $\tau = \tau_0 + s$, we have that

$$\begin{aligned}
I(\tau) &= \int_{M^{\tau}(U_1)} \int_{M^{\tau}(U_2)} K(x-y)\, dxdy \\
&= I_K[\chi_{M^{\tau_0}(U_1)}, \chi_{M^{\tau_0}(U_2)}](s),
\end{aligned} \tag{4.16}$$

where

$$M^{\tau_0}(U_1) = \cup_{N=1}^{\infty} M^{\tau_0}(U_N) = \cup_{N=1}^{\infty} \bigcup_{k=1}^{N} I(\tilde{x}+c_k^1, r_k^1)$$

and

$$M^{\tau_0}(U_2) = \cup_{M=1}^{\infty} M^{\tau_0}(U_M) = \cup_{M=1}^{\infty} \bigcup_{k=1}^{M} I(\tilde{x}+c_k^2, r_k^2).$$

Now, if $N_1 < N_2$ then $M^{\tau_0}(U_{N_1}) \subset M^{\tau_0}(U_{N_2})$ for all $\tau_0 \geq 0$ and so $\lim_{N\to\infty} M^{\tau_0}(U_N) = \cup_{N=1}^{\infty} M^{\tau_0}(U_N)$ and $\lim_{M\to\infty} M^{\tau_0}(U_M) = \cup_{M=1}^{\infty} M^{\tau_0}(U_M)$, for any $\tau_0 \geq 0$. Hence, for s sufficiently small, we have that

$$\begin{aligned}
I(\tau) &= \lim_{N\to\infty}\lim_{M\to\infty} \int_{M^s(M^{\tau_0}(U_N))} \int_{M^s(M^{\tau_0}(U_M))} K(x-y)\, dxdy \\
&= \lim_{N\to\infty}\lim_{M\to\infty} I_K[\chi_{M^{\tau_0}(U_N)}, \chi_{M^{\tau_0}(U_M)}](s).
\end{aligned} \tag{4.17}$$

Therefore, combining (4.16) and (4.17), we see that

$$\lim_{N\to\infty}\lim_{M\to\infty} I_K[\chi_{M^{\tau_0}(U_N)}, \chi_{M^{\tau_0}(U_M)}](s)|_{s=0} = I_K[\chi_{M^{\tau_0}(U_1)}, \chi_{M^{\tau_0}(U_2)}](s)|_{s=0}.$$

Furthermore, $I_K[\chi_{M^{\tau_0}(U_N)}, \chi_{M^{\tau_0}(U_M)}](s)$ is differentiable and

$$\begin{aligned}
&\frac{d}{ds} I_K[\chi_{M^{\tau_0}(U_N)}, \chi_{M^{\tau_0}(U_M)}](s)|_{s=0} \\
=\ & \sum_{k=1}^{N}\sum_{\ell=1}^{M} (\operatorname{sgn} c_\ell^2 - \operatorname{sgn} c_k^1) \int_{c_k^1 - r_k^1}^{c_k^1 + r_k^1} \int_{c_\ell^2 - r_\ell^2}^{c_\ell^2 + r_\ell^2} K'(x-y)\, dydx \\
\leq\ & 2\sum_{k=1}^{N}\sum_{\ell=1}^{M} \int_{c_k^1 - r_k^1}^{c_k^1 + r_k^1} \int_{c_\ell^2 - r_\ell^2}^{c_\ell^2 + r_\ell^2} K'(x-y)\, dydx \\
=\ & 2\int_{M^{\tau_0}(U_N)} \int_{M^{\tau_0}(U_M)} K'(x-y)\, dydx.
\end{aligned}$$

That is, $\frac{d}{ds} I_K[\chi_{M^{\tau_0}(U_N)}, \chi_{M^{\tau_0}(U_M)}](s)|_{s=0}$ is bounded above and is increasing as $N, M \to \infty$, implying that $\lim_{N\to\infty}\lim_{M\to\infty} \frac{d}{ds} I_K[\chi_{M^{\tau_0}(U_N)}, \chi_{M^{\tau_0}(U_M)}](s)|_{s=0}$ exists. Hence,

$$\frac{d}{ds} I_K[\chi_{M^{\tau_0}(U_1)}, \chi_{M^{\tau_0}(U_2)}](s)|_{s=0} = \lim_{N\to\infty}\lim_{M\to\infty} \frac{d}{ds} I_K[\chi_{M^{\tau_0}(U_N)}, \chi_{M^{\tau_0}(U_M)}](s)|_{s=0} \geq 0,$$

by (4.15), yielding

$$\frac{d}{d\tau} I(\tau_0) = \frac{d}{ds} I_K[\chi_{M^{\tau_0}(U_1)}, \chi_{M^{\tau_0}(U_2)}](s)|_{s=0} \geq 0,$$

as required. □

Proof of Theorem 4.13.1.
Assume, for a contradiction, that $\mu_0|_D$ is not radially decreasing with respect to any $x_0 \in D$ considered as a centre. Then by [21, Lemma 2.18] there exists a unit vector e such that μ_0 is not symmetric decreasing about any hyperplane with normal vector e. We set $e = (1, 0, ..., 0)$ without loss of generality.

Recall Definition 2.47, where we define the continuous Steiner symmetrization of a non-negative function on $\mathbb{R}^d$ with respect to a specific direction. In order to prove Theorem 4.13.1, we modify $S^\tau \mu_0$ in such a way that $U_{x'}^h$ travels at the speed $v(h)$ instead of a constant speed 1, where

$$v(h) := \begin{cases} 1, & \text{if } h \geq h_0 \\ \left(\frac{h}{h_0}\right)^{m-1}, & \text{if } 0 < h < h_0. \end{cases}$$

for some h_0 sufficiently small, defined later. We let $\mu(\tau, \cdot) = \tilde{S}^\tau \mu_0$ where

$$\tilde{S}^\tau \mu_0 := \int_0^\infty \chi_{M^{v(h)\tau}(U^h_{x'})}(x_1)\ dh.$$

Now, in [21, Proposition 2.8] it is shown that there exists some $\delta_1 > 0$ and $C > 0$, depending on m, μ_0, and W, such that, for any $\tau \in [0, \delta_1]$,

$$|\mu(\tau, x) - \mu_0(x)| \leq C\mu_0(x)\tau \text{ for all } x \in \mathbb{R}^d, \tag{4.18}$$
$$\int_{D_i} (\mu(\tau, x) - \mu_0(x))\ dx = 0, \tag{4.19}$$

for any connected component D_i of $\operatorname{supp} \mu_0$.

The proof of [21, Proposition 2.8] does not rely on the infinite support of ω' assumed in [21]. As a result, we see that (4.18) and (4.19) hold for W satisfying assumptions $(W1)$, $(W2)$, and $(W4)$.

Since (4.18) and (4.19) hold, we can use the same argument as in the proof of [21, Theorem 2.2] to obtain that there exists some $C_2 > 0$ and $\delta_0 > 0$ with $\delta_0 \leq \delta_1$ such that

$$|\mathcal{E}[\mu(\tau, \cdot)] - \mathcal{E}[\mu_0]| \leq C_2\tau^2 \text{ for all } \tau \in [0, \delta_0]. \tag{4.20}$$

It remains to be shown that there exists a $C_1 > 0$ and some τ_1, τ_2 with $0 \leq \tau_1 < \tau_2 \leq \delta_0$ such that

$$\mathcal{E}[\mu(\tau, \cdot)] - \mathcal{E}[\mu_0] \leq -C_1\tau \text{ for all } \tau \in [\tau_1, \tau_2]. \tag{4.21}$$

This will allow us to conclude that (4.20) and (4.21) hold for all $\tau \in [\tau_1, \tau_2]$. Combining (4.20) and (4.21) will then lead to a contradiction of τ_1.

In the proof of [21, Proposition 2.8], it is shown that $\mathcal{S}[\mu(\tau, \cdot)] \leq \mathcal{S}[\mu_0]$ for all $\tau > 0$. Hence, it is sufficient to show that

$$\mathcal{I}[\mu(\tau, \cdot)] - \mathcal{I}[\mu_0] \leq -C_1\tau \text{ for all } \tau \in [\tau_1, \tau_2].$$

We prove this result as follows:

Fix $\tau \in [\tau_1, \tau_2]$, where τ_1, τ_2 are to be defined later. Fix $x' \in D_{x_1}$, where $D_{x_1} = \{z \in \mathbb{R}^{d-1} : (x_1, z) \in D\}$. Consider the interval $[a, b]$, where

$$a := \max\{x_1 \in \mathbb{R} : \mu_0(x_1, x') = \max_{x\in\mathbb{R}} \mu_0(x, x')\}$$

and

$$b := \min\{x_1 \in [a, a + \tfrac{R}{2}] : \mu_0(x_1, x') = \min_{x\in[a,a+\frac{R}{2}]} \mu_0(x, x')\}.$$

From the definition of a and b, we have that $0 < b - a \leq \frac{R}{2}$ and $\mu_0(a, x') > \mu_0(x, x') > \mu_0(b, x')$ for all $x \in (a, b)$.

For $\alpha > 0$, define $\mathcal{H}^\alpha := \{(x', h) \in D_{x_1} \times (0, \infty) : |U^h_{x'} \cap [a, b]| > \alpha\}$. By continuity of μ_0, we can choose α sufficiently small so that $\mathcal{H}^\alpha$ has positive measure.

By [21, Proposition 2.8], for any $h_0 > 0$, $\mu(\tau, \cdot)$ satisfies (4.18) and (4.19) for all $\tau \in [\tau_1, \tau_2]$, since $\tau_1, \tau_2 \leq \delta_1$. Hence, we may choose $h_0 := \mu_0(\frac{a+b}{2}, x')$ and define

$$B_1 = \{(x', h) \in \mathcal{H}^\alpha : h \geq h_0\}$$

and

$$B_2 = \{(x', h) \in \mathcal{H}^\alpha : h < h_0\}.$$

By our choice of h_0 and continuity of μ_0 we have that B_1 and B_2 have positive measure.

Now, consider $(x', h_1) \in B_1$ and $(y', h_2) \in B_2$ where $0 < ||x' - y'|| < 1$. Let $U^{h_1}_{x'}$ and $U^{h_2}_{y'}$ consist of either a finite or infinite union of disjoint open intervals. Then, there exist intervals $I(c^1_{k*}, r^1_{k*})$ and $I(c^2_{\ell*}, r^2_{\ell*})$ in $U^{h_1}_{x'}$ and $U^{h_2}_{y'}$, respectively, such that $|I(c^1_{k*}, r^1_{k*}) \cap [a, b]| > \beta$ and $|I(c^2_{\ell*}, r^2_{\ell*}) \cap [a, b]| > \beta$ for some $\beta \in (0, \alpha]$.

We show that there exists some τ_1, τ_2 with $0 \leq \tau_1 < \tau_2 \leq \delta_0$ such that $\mu_1 := \chi_{M^{v(h_1)\tau_0}(I(c^1_{k*}, r^1_{k*}))}$ and $\mu_2 := \chi_{M^{v(h_2)\tau_0}(I(c^2_{\ell*}, r^2_{\ell*}))}$ satisfy the assumptions of Lemma 4.14.2 for all $\tau_0 \in [\tau_1, \tau_2]$.

We choose $H = \{x_1 = \tilde{x}\}$ for some $\tilde{x} \in D_{x'} = \{x_1 \in \mathbb{R} : (x_1, x') \in D\}$ and let $I(c^1_{k*}, r^1_{k^*}) =: I(\tilde{x} + c_1, r_1)$ and $I(c^2_{k*}, r^2_{k^*}) =: I(\tilde{x} + c_2, r_2)$. We consider the following two cases: $c_1 = c_2$ and $c_1 \neq c_2$.

Case 1: $c_1 = c_2$

We choose $\tilde{x}$ such that $|c_1| = |\tilde{x} - c^1_{k*}| = \frac{1}{4} \min\{R, \delta_0, \tau^*, \frac{C_1}{C_2}\}$ where τ^* is the value at which $M^{v(h_2)\tau^*}(I(c^2_{k*}, r^2_{k^*}))$ shares a common endpoint with a neighbouring interval (if one exists). Since μ_0 is not symmetric decreasing about any hyperplane with normal vector e, we have that μ_0 is not symmetric decreasing about $H = \{x_1 = \tilde{x}\}$.

Now, let $\tau_1 = |c_1|$. We claim that the assumptions of Lemma 4.14.2 hold at $\tau = \tau_1$. Indeed, we have that

$$M^{v(h_1)\tau_1}(I(\tilde{x} + c_1, r_1)) = I(\tilde{x} + c_1 - v(h_1)\tau_1 \operatorname{sgn} c_1, r_1)$$

and

$$M^{v(h_2)\tau_1}(I(\tilde{x} + c_2, r_2)) = I(\tilde{x} + c_2 - v(h_2)\tau_1 \operatorname{sgn} c_2, r_2).$$

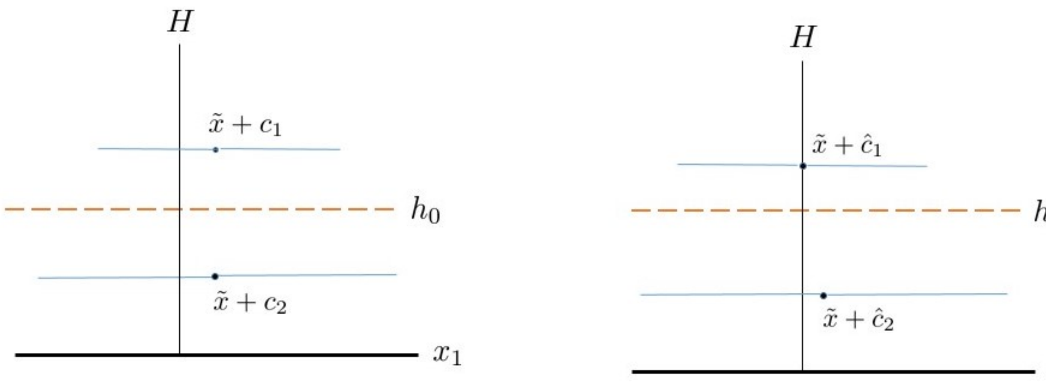

Figure 4.3: Illustration of case 1 where $c_1 = c_2 > 0$ and x' is fixed. On the left, $\tau = 0$ and on the right, $\tau = \tau_1$.

Set $\hat{c}_1 := c_1 - v(h_1)\tau_1 \operatorname{sgn} c_1$ and $\hat{c}_2 := c_2 - v(h_2)\tau_1 \operatorname{sgn} c_2$. Then, $\hat{c}_1 = 0$, since $v(h_1) = 1$. If $c_2 > 0$, then $\hat{c}_2 = |c_2| - v(h_2)|c_2| > 0$ and if $c_2 < 0$, then $\hat{c}_2 = -|c_2| + v(h_2)|c_2| < 0$. This follows from the fact that $v(h_2) = (\frac{h_2}{h_0})^{m-1} < 1$. Hence, $\operatorname{sgn} \hat{c}_1 \neq \operatorname{sgn} \hat{c}_2$.

Furthermore, since $h_1 > h_2$ implies $r_1 < r_2$, by continuity of μ_0, and since $|c_1| \leq \frac{R}{4}$, we have that

$$|\hat{c}_2 - \hat{c}_1| = |\hat{c}_2| = |c_2 - v(h_2)\tau_1 \operatorname{sgn} c_2| \leq |c_2| + v(h_2)|c_1| < 2|c_1| \leq \frac{R}{2} < r_1 + r_2 + R. \quad (4.22)$$

Also, since $[\tilde{x} + c_1 + r_1, \tilde{x} + c_2 + r_2] \subseteq [a, b]$ and $b - a \leq \frac{R}{2}$, it follows that

$$r_2 - r_1 = (\tilde{x} + c_2 + r_2) - (\tilde{x} + c_1 + r_1) \leq \tfrac{R}{2} < |\hat{c}_2 - \hat{c}_1| + R,$$

as required.

It remains to be shown that the assumptions of Lemma 4.14.2 continue to hold for all $\tau \in [\tau_1, \tau_2]$, where $\tau_2 := \min\{\tau_1 + |\hat{c}_2|, \frac{\tau^*}{2}, \delta_0\}$.

Now, we have that

$$M^{v(h_1)\tau_2}(I(\tilde{x} + c_1, r_1)) = I(\tilde{x}, r_1)$$

and

$$M^{v(h_2)\tau_2}(I(\tilde{x} + c_2, r_2)) = I(\tilde{x} + c_2 - v(h_2)\tau_2 \operatorname{sgn} c_2, r_2).$$

Set $\tilde{c}_1 := c_1 - v(h_1)\tau_2 \operatorname{sgn} c_1$ and $\tilde{c}_2 := c_2 - v(h_2)\tau_2 \operatorname{sgn} c_2$. Then $\tilde{c}_1 = 0$. Furthermore, if $\hat{c}_2 > 0$ then $c_2 > 0$ and so $\tilde{c}_2 = c_2 - v(h_2)\tau_2 \geq |\hat{c}_2| - v(h_2)|\hat{c}_2| > 0$ and if $\hat{c}_2 < 0$ then $c_2 < 0$ and so $\tilde{c}_2 = c_2 + v(h_2)\tau_2 \leq -|\hat{c}_2| + v(h_2)|\hat{c}_2| < 0$. Hence, $\operatorname{sgn} \tilde{c}_1 \neq \operatorname{sgn} \tilde{c}_2$.

Additionally, since $|\hat{c}_2| \leq \frac{R}{2}$ by (4.22), we have that

$$|\tilde{c}_2 - \tilde{c}_1| = |\tilde{c}_2| \leq |c_2| + v(h_2)\tau_2 \leq |c_2| + v(h_2)(|c_2| + |\hat{c}_2|) < r_2 + r_1 + R$$

and

$$|r_2 - r_1| \leq \tfrac{R}{2} < |\tilde{c}_2 - \tilde{c}_1| + R,$$

as required.

Case 2: $c_1 \neq c_2$

Suppose $c_1 \neq c_2$. We choose $\tilde{x} = c^1_{k^*}$ such that $c_1 = 0$ and $\operatorname{sgn} c_1 \neq \operatorname{sgn} c_2$. Now, let $\tau_1 = 0$ and $\tau_2 = \frac{1}{2}\min\{|c_2|, \tau^*, \delta_0, R\} > 0$. Then we have that $M^{v(h_1)\tau_2}(I(\tilde{x} + c_1, r_1)) = I(\tilde{x} + \tilde{c}_1, r_1)$ and $M^{v(h_2)\tau_2}(I(\tilde{x} + c_2, r_2)) = I(\tilde{x} + \tilde{c}_2, r_2)$ where $\tilde{c}_1 = 0$ and $\tilde{c}_2 = c_2 - v(h_2)\tau_2 \operatorname{sgn} c_2$.

If $c_2 > 0$, then $\tilde{c}_2 = c_2 - v(h_2)\tau_2 \geq |c_2| - v(h_2)|c_2| > 0$.
If $c_2 < 0$, then $\tilde{c}_2 = c_2 + v(h_2)\tau_2 \leq -|c_2| + v(h_2)|c_2| < 0$.

Hence, $\operatorname{sgn} \tilde{c}_1 \neq \operatorname{sgn} \tilde{c}_2$. Furthermore, by continuity of μ_0, we have that $a \leq \tilde{x} + c_1 + r_1 = \tilde{x} + \tilde{c}_1 + r_1 < \tilde{x} + c_2 + r_2 \leq b$.

Now, if $c_2 > 0$, then $a \leq \tilde{x} + \tilde{c}_1 + r_1 = \tilde{x} + r_1 < \tilde{x} + \tilde{c}_2 + r_1 < \tilde{x} + \tilde{c}_2 + r_2 < b$. Hence, since $b - a \leq \frac{R}{2}$, we have that $\tilde{x} + \tilde{c}_2 + r_2 - (\tilde{x} + \tilde{c}_1 + r_1) \leq \frac{R}{2}$ and so

$$\tilde{c}_2 - \tilde{c}_1 \leq r_1 - r_2 + \tfrac{R}{2} < r_1 + r_2 + R.$$

If $c_2 < 0$, then $a \leq \tilde{x} + \tilde{c}_1 + r_1 = \tilde{x} + r_1 < \tilde{x} + \tilde{c}_2 + r_2 = \tilde{x} + c_2 + v(h_2)\tau_2 + r_2 \leq \tilde{x} + c_2 + v(h_2)\frac{R}{2} + r_2 \leq b + v(h_2)\frac{R}{2}$. Hence, $b + v(h_2)\frac{R}{2} - a \leq \frac{R}{2} + v(h_2)\frac{R}{2} < R$ and so $\tilde{x} + \tilde{c}_2 + r_2 - (\tilde{x} + \tilde{c}_1 + r_1) < R$, which yields

$$\tilde{c}_2 - \tilde{c}_1 < r_1 + r_2 + R.$$

Furthermore, for both $c_2 > 0$ and $c_2 < 0$, we see that

$$r_2 - r_1 < \tilde{c}_1 - \tilde{c}_2 + R \leq |\tilde{c}_2 - \tilde{c}_1| + R,$$

as required.

Hence, for both cases 1 and 2 it follows that $\chi_{M^{v(h_1)\tau_0}(I(c^1_{k^*}, r^1_{k^*}))}$ and $\chi_{M^{v(h_2)\tau_0}(I(c^2_{k^*}, r^2_{k^*}))}$ satisfy the asumptions of Lemma 4.14.2 for all $\tau_0 \in [\tau_1, \tau_2]$.

Thus, for s sufficiently small and $\tau = v(h)\tau_0 + s$, we have that

$$\begin{aligned} \frac{d^+}{d\tau} I(v(h)\tau_0) &= \frac{d^+}{ds} I_K[\chi_{M^{v(h_1)\tau_0}(U^{h_1}_{x'})}, \chi_{M^{v(h_2)\tau_0}(U^{h_2}_{y'})}](s)|_{s=0} \\ &\geq \frac{d^+}{ds} I_K[\chi_{M^{v(h_1)\tau_0}(I(c^1_{k^*}, r^1_{k^*}))}, \chi_{M^{v(h_2)\tau_0}(I(c^2_{k^*}, r^2_{k^*}))}](s)|_{s=0} \\ &\geq c > 0, \end{aligned} \tag{4.23}$$

for all $\tau_0 \in [\tau_1, \tau_2]$, by Lemma 4.14 and 4.15.

Now, define a 1-D kernel $K_l(z) = -\frac{1}{2}\omega(\sqrt{z^2+l^2})$. For any $l > 0$, $K_l \in C^1(\mathbb{R})$ is even with $K_l(z) < 0$ for all $0 < z < R$, where $R := \sqrt{1-l^2}$, $l^2 < 1$, and $K_l(z) = 0$ for all $z \geq R$. Then,

$$\begin{aligned}
\mathcal{I}[\tilde{S}^\tau \mu_0] &= \frac{1}{2}\int_D \int_D \tilde{S}^\tau \mu_0(x)\tilde{S}^\tau \mu_0(y) W(x-y)\ dydx \\
&= \frac{1}{2}\int_{D^2}\int_{(\mathbb{R}^+)^2} \chi_{M^{v(h_1)\tau}(U_{x'}^{h_1})}(x_1)\chi_{M^{v(h_2)\tau}(U_{y'}^{h_2})}(y_1) W(x-y)\ dh_1 dh_2 dy dx \\
&= -\int_{D^2}\int_{(\mathbb{R}^+)^2} \chi_{M^{v(h_1)\tau}(U_{x'}^{h_1})}(x_1)\chi_{M^{v(h_2)\tau}(U_{y'}^{h_2})}(y_1) K_{||x'-y'||}(|x_1-y_1|)\ dh_1 dh_2 dy dx.
\end{aligned}$$

This follows from the definition of $\tilde{S}^\tau \mu_0$ and since $W(x-y) = \omega(||x-y||) = \omega(((x_1-y_1)^2 + ... + (x_d-y_d)^2)^{1/2}) = -2K_{||x'-y'||}(|x_1-y_1|)$, where $0 < ||x'-y'|| < 1$.

Now, using the definition given in (4.10), we have that

$$\mathcal{I}[\tilde{S}^\tau \mu_0] = -\int_{D_{x_1}}\int_{D_{y_1}}\int_{(\mathbb{R}^+)^2} I_{K_{||x'-y'||}}[\chi_{U_{x'}^{h_1}}, \chi_{U_{y'}^{h_2}}](v(h)\tau)\ dh_1 dh_2 dy' dx'.$$

Taking the right derivative, we obtain

$$\begin{aligned}
-\frac{d^+}{d\tau}\mathcal{I}[\tilde{S}^\tau \mu_0] &= \int_{D_{x_1}}\int_{D_{y_1}}\int_{(\mathbb{R}^+)^2} \frac{d^+}{d\tau} I_{K_{||x'-y'||}}[\chi_{U_{x'}^{h_1}}, \chi_{U_{y'}^{h_2}}](v(h)\tau)\ dh_1 dh_2 dy' dx' \\
&\geq \int_{B_1}\int_{B_2} \frac{d^+}{d\tau} I_{K_{||x'-y'||}}[\chi_{U_{x'}^{h_1}}, \chi_{U_{y'}^{h_2}}](v(h)\tau)\ dy' dh_2 dx' dh_1,
\end{aligned}$$

by Lemma 4.15. Now, for any $(x', h_1) \in B_1$ and $(y', h_2) \in B_2$, by (4.23), we have that

$$-\frac{d^+}{d\tau}\mathcal{I}[\tilde{S}^\tau \mu_0] \geq |B_1||B_2|c =: C_1 > 0,$$

for all $\tau \in [\tau_1, \tau_2]$. Hence, by the fundamental theorem of calculus and using the fact that $\tilde{S}^0 \mu_0 = \mu_0$, it follows that

$$\mathcal{I}[\tilde{S}^\tau \mu_0] - \mathcal{I}[\mu_0] \leq -C_1\tau,$$

for all $\tau \in [\tau_1, \tau_2]$. Hence, we have that

$$\mathcal{E}[\mu(\tau,\cdot)] - \mathcal{E}[\mu_0] \leq -C_1\tau \quad \text{and} \quad |\mathcal{E}[\mu(\tau,\cdot)] - \mathcal{E}[\mu_0]| \leq C_2\tau^2$$

for all $\tau \in [\tau_1, \tau_2]$ (since $\tau_2 \leq \delta_0$). Now, by definition, $\tau_1 < \dfrac{C_1}{2C_2}$. But, since

$$-C_2\tau_1^2 \leq \mathcal{E}[\mu(\tau,\cdot)] - \mathcal{E}[\mu_0] \leq -C_1\tau_1,$$

it must be that $\tau_1 \geq \dfrac{C_1}{C_2}$, a contradiction. □

We show in Theorem 4.16 below that the compactness of each connected component of $\operatorname{supp}\rho_s$ follows from Theorem 4.13.1.

Theorem 4.16. *Let $m > 2$. Assume (W1), (W2), and (W4) hold. If ρ_s is a stationary state of (1.1), then each connected component of* $\operatorname{supp}\rho_s$ *is compact.*

Proof. We consider an arbitrary connected component D of $\operatorname{supp}\rho_s$. We assume, for a contradiction, that D is not compact. Since D is closed, it must be that D is unbounded. Therefore, since ρ_s is radially symmetric we have that $D = \mathbb{R}^d$. The proof then follows exactly the same as that of Theorem 4.7.

□

Since each connected component D of the support of ρ_s is compact by Theorem 4.16, we have that $\rho_s|_{\partial D} = 0$. Hence, as in the statement of Theorem 4.13.2, we consider $x \in \text{interior}\,(D)$ and $y \in \text{interior}\,(\operatorname{supp}\rho_s \setminus D)$. Note that, since D and $\text{closure}(\operatorname{supp}\rho_s \setminus D)$ are disjoint by assumption and are compact, we have that $\min\{||x-y|| : x \in D,\ y \in \text{closure}(\operatorname{supp}\rho_s \setminus D)\} > 0$.

Proof of Theorem 4.13.2. Set $\rho_s = \mu_0 = \mu(0,\cdot)$. We consider $\operatorname{supp}\mu_0$ made up of two connected components, D_1 and D_2. Suppose for a contradiction that there exists some $x^* = (x_1^*, x'^*) \in \text{interior}\, D_1$ and $y^* = (y_1^*, y'^*) \in \text{interior}\, D_2$ such that $||x^* - y^*|| < 1$. Then, $|x_1^* - y_1^*| < \sqrt{1 - ||x'^* - y'^*||^2} = R$, where $||x'^* - y'^*||^2 < 1$.

Set $x_1^* = \tilde{x} + c_1 + r_1$ and $y_1^* = \tilde{x} + c_2 - r_2$ where $\tilde{x}$ is chosen such that $c_1 < 0$ and $c_2 > 0$. We claim that $\chi_{I(\tilde{x}+c_1,r_1)}$ and $\chi_{I(\tilde{x}+c_2,r_2)}$ satisfy the assumptions of Lemma 4.14.2. This can be seen from the fact that $|x_1^* - y_1^*| < R$ implies $|c_2 - r_2 - c_1 - r_1| < R$, which gives $||c_2 - c_1| - |r_2 + r_1|| < R$, yielding $|c_2 - c_1| < r_1 + r_2 + R$ and $|r_2 - r_1| < |c_2 - c_1| + R$.

Furthemore, taking $\tau_m := \frac{1}{2}\min\{|c_1|, |c_2|, |x_1^* - y_1^*|, \delta_0\} > 0$, with δ_0 as defined in [21, Theorem 2.2], we have that $M^{\tau_m}(I(\tilde{x} + c_1, r_1)) = I(\tilde{x} + \tilde{c}_1, r_1)$ and $M^{\tau_m}(I(\tilde{x} + c_2, r_2)) = I(\tilde{x} + \tilde{c}_2, r_2)$ where

$$\tilde{c}_1 = c_1 - \tau_m \operatorname{sgn} c_1 = c_1 + \tau_m < -|c_1| + |c_1| = 0,$$

and

$$\tilde{c}_2 = c_2 - \tau_m \operatorname{sgn} c_2 = c_2 - \tau_m > |c_2| - |c_2| = 0.$$

Hence, $\operatorname{sgn}\tilde{c}_1 \neq \operatorname{sgn}\tilde{c}_2$.

Also, since $[\tilde{x} + \tilde{c}_1 + r_1, \tilde{x} + \tilde{c}_2 - r_2] \subset [\tilde{x} + c_1 + r_1, \tilde{x} + c_2 - r_2]$, we have that $|\tilde{c}_2 - \tilde{c}_1 - r_2 - r_1| < |c_2 - c_1 - r_2 - r_1| < R$, yielding $|\tilde{c}_2 - \tilde{c}_1| < r_1 + r_2 + R$ and $|r_2 - r_1| < |\tilde{c}_2 - \tilde{c}_1| + R$, as required.

Furthermore, setting $h_0 = \max\mu_0$, we have that $\chi_{M^{v(h)\tau_m}(I(\tilde{x}+c_1,r_1))}$ and $\chi_{M^{v(h)\tau_m}(I(\tilde{x}+c_2,r_2))}$ satisfy the assumptions of Lemma 4.14.2. for all $\tau \in [0, \tau_m]$, since $v(h) < 1$ for $h < h_0$.

Now, fix $(x', h_1) \in D_1^{x_1} \times (0, \infty)$ and $(y', h_2) \in D_2^{y_1} \times (0, \infty)$, where $D_i^{z_1} = \{z' \in \mathbb{R}^{d-1} : (z_1, z') \in D_i\}$, for $i = 1, 2$ and $z_1 \in \mathbb{R}$. We define the set

$$B := \{(x', h_1) \times (y', h_2) : |\alpha - \beta| < R,\ \alpha \in \text{interior}\, D_1^{x'},\ \beta \in \text{interior}\, D_2^{y'}\},$$

where $D_i^{z'} = \{z_1 \in \mathbb{R} : (z_1, z') \in D_i\}$, for $i = 1, 2$ and $z' \in \mathbb{R}^{d-1}$. We know by assumption that B is non-empty. Furthermore, since B is an open set, it has positive measure. Hence, using the same approach as in the proof of Theorem 4.13.1, we find that

$$\begin{aligned} -\frac{d^+}{d\tau}\mathcal{I}[\tilde{S}^\tau \mu_0] &= \int_{(\mathbb{R}^{d-1})^2} \int_{(\mathbb{R}^+)^2} \frac{d^+}{d\tau} I_{K_{||x'-y'||}}[\chi_{U_{x'}^{h_1}}, \chi_{U_{y'}^{h_2}}](v(h)\tau)\ dh_1 dh_2 dx' dy' \\ &\geq c|B| > 0 \ \text{ for all } \ \tau \in [0, \tau_m], \end{aligned}$$

by Lemma 4.14 and 4.15. Thus, there exists a $C > 0$ such that

$$\mathcal{E}[\tilde{S}^\tau \mu_0] - \mathcal{E}[\mu_0] < -C\tau < 0 \tag{4.24}$$

for all $\tau \in (0, \tau_m]$. However; using the fact that μ_0 is a stationary state of (1.1) we have that

$$|\mathcal{E}[\mu(\tau, \cdot)] - \mathcal{E}[\mu_0]| \leq C_2 \tau^2, \tag{4.25}$$

for all $\tau \in [0, \tau_m]$, as discussed in the proof of Theorem 4.13.1. Combining Inequality (4.24) and (4.25) and taking $\tau = \dfrac{C}{2C_2}$, we see that

$$-\frac{C^2}{4C_2} \leq \mathcal{E}[\mu(\tau, \cdot)] - \mathcal{E}[\mu_0] < -\frac{C^2}{2C_2},$$

a contradiction. □

Hence, from the theorems in this chapter, we have provided a proof for Theorem 4.3.

5

Mass-independent boundedness of stationary states

In this chapter, we prove that for $m > 2$ and for attractive kernels with both compact and infinite support, stationary states of Equation (1.1) possess a mass-independent upper-bound. That is, regardless of the size of the support of W, there exists an upper-bound of any stationary solution to (1.1) that does not depend on the initial condition. Furthermore, we provide numerical simulations that confirm our analytical results on stationary states given in both the current and the previous chapter.

5.1 A theoretical upper-bound

Theorem 5.1. *Let $m > 2$. If ρ_s is a stationary state of (1.1) with W satisfying assumptions (W1), (W2), and (W4), then*

$$\rho_s(x) \leq \rho_s^* := \left(\frac{m-1}{m\varepsilon}||W||_{L^1(\mathbb{R}^d)}\right)^{1/(m-2)} \text{ for all } x \in \mathbb{R}^d.$$

Proof. Since ρ_s is a stationary state of (1.1),

$$\frac{m\varepsilon}{m-1}\rho_s^{m-1}(x) + (W * \rho_s)(x) = C_j \text{ for all } x \in D_j,$$

where C_j may be different on each connected component D_j of $\operatorname{supp}\rho_s$. Now, let $z \in \partial D_j$. Then,

$$\frac{m\varepsilon}{m-1}\rho_s^{m-1}(x) + (W * \rho_s)(x) = \frac{m\varepsilon}{m-1}\rho_s^{m-1}(z) + (W * \rho_s)(z) \text{ for all } x \in D_j.$$

Since $\rho_s \in C(\mathbb{R}^d)$ by Theorem 4.11 and ρ_s is compactly supported by Theorem 4.7, we have, for $z \in \partial D_j$, that $\rho_s(z) = 0$. Hence, for any $x \in D_j$,

$$\frac{m\varepsilon}{m-1}\rho_s^{m-1}(x) + (W * \rho_s)(x) = (W * \rho_s)(z).$$

Furthermore, by $(W4)$, we have that

$$(W * \rho_s)(z) = \int_{\mathbb{R}^d} W(z-y)\rho_s(y)\ dy \leq 0.$$

Therefore, for all $x \in D_j$,

$$\frac{m\varepsilon}{m-1}\rho_s^{m-1}(x) \leq -(W * \rho_s)(x) \leq ||W||_{L^1(\mathbb{R}^d)}||\rho_s||_{L^\infty(\mathbb{R}^d)},$$

which implies that

$$\rho_s^{m-1}(x) \leq \frac{m-1}{m\varepsilon}||W||_{L^1(\mathbb{R}^d)}||\rho_s||_{L^\infty(\mathbb{R}^d)}, \tag{5.1}$$

for all $x \in D_j$. Since D_j is an arbitrary connected component of $\operatorname{supp}\rho_s$ and the upper-bound of ρ_s^{m-1} is independent of D_j, then Inequality (5.1) holds for any $D_j \subseteq \operatorname{supp}\rho_s$ and, thus, for any $x \in \mathbb{R}^d$. This yields

$$||\rho_s||_{L^\infty(\mathbb{R}^d)} \leq \left(\frac{m-1}{m\varepsilon}||W||_{L^1(\mathbb{R}^d)}\right)^{1/(m-2)}$$

as required. □

It is quite interesting that this boundedness property can be extended to the case of ω' strictly positive on $(0,\infty)$. More precisely, the following theorem holds true.

Theorem 5.2. *Let $m > 2$. If ρ_s is a stationary state of (1.1) with W satisfying assumptions (W1)-(W3), then*

$$\rho_s(x) \leq \rho_s^* := \left(\frac{m-1}{m\varepsilon}||W||_{L^1(\mathbb{R}^d)}\right)^{1/(m-2)} \quad \textit{for all } x \in \mathbb{R}^d.$$

Proof. Since ρ_s is a stationary state of (1.1) and is radially decreasing up to a translation on $\mathbb{R}^d$, by [21, Theorem 2.2], we have that

$$\frac{m\varepsilon}{m-1}\rho_s^{m-1}(x) + (W * \rho_s)(x) = C = \text{const}, \ \text{for all } x \in \operatorname{supp}\rho_s.$$

Since $\rho_s \in C(\mathbb{R}^d)$ by [21, Lemma 2.3] and ρ_s is compactly supported by [26, Lemma 3.2], we have, for $z \in \partial(\operatorname{supp}\rho_s)$, that $\rho_s(z) = 0$. Hence,

$$\frac{m\varepsilon}{m-1}\rho_s^{m-1}(x) + (W * \rho_s)(x) = (W * \rho_s)(z) < 0.$$

This follows from the fact that ω is strictly increasing on $(0,\infty)$ and $\lim_{r\to\infty}\omega(r) = 0$. Therefore,

$$\frac{m\varepsilon}{m-1}\rho_s^{m-1}(x) < -(W * \rho_s)(x) \leq ||W||_{L^1(\mathbb{R}^d)}||\rho_s||_{L^\infty(\mathbb{R}^d)},$$

for all $x \in \operatorname{supp}\rho_s$. Hence,

$$||\rho_s||_{L^\infty(\mathbb{R}^d)} \leq \left(\frac{m-1}{m\varepsilon}||W||_{L^1(\mathbb{R}^d)}\right)^{1/(m-2)},$$

as required.

□

5.2 Numerical simulations

In this section we present numerical simulations for Equation (1.1) to illustrate the results in Theorems 4.13, 5.1, and 5.2. We consider two examples of attractive kernels, one with bounded and the other with unbounded support. We show that in both cases the numerical solutions converge towards stationary states that are radially decreasing up to a translation and compactly supported.

Furthermore, we demonstrate that when $m > 2$ the stationary states have an upper-bound independent of the mass. In particular, we are able to confirm numerically that $m = 2$ is a threshold such that for $m > 2$ there exists a mass-independent upper-bound for the density, while for $m \leq 2$, the maximal density increases without bound as more mass is added to the system. In addition, we see that for $m > 2$ and sufficiently large mass, the stationary states are approximately constant in the interior of their support. In this case, we show that, for any $W \in L^1(\mathbb{R}^d)$, we can approximate the maximum height of the density by minimizing the energy functional. We consider numerical results in one dimension for both Examples 1 and 2. For the initial data, we use a characteristic function, defined on a symmetric real interval. That is, $\rho_0 = \chi_{[-a,a]}$, for some $a > 0$.

In addition, we provide simulations in two-dimensions for the particular case of W compactly supported. We demonstrate that for this choice of interaction kernel, we can obtain stationary solutions which exhibit pattern formation. For these experiments in $2D$, we vary the choice of initial data in order to illustrate the range of patterns that can be obtained. For our numerical computations, we set $dx = 0.4$ and $dt = dx$, where dx and dt denote the sizes of the spatial and time steps, respectively. Unless specified otherwise, for all numerical results, we have set $\varepsilon = 1$.

Example 1 Consider

$$W(x) = \begin{cases} -5e^{1/(|x|^2-1)}, & |x| < 1 \\ 0, & |x| \geq 1. \end{cases}$$

We can clearly see from Figures 5.1, 5.2, 5.3, and 5.4 that the existence of a mass-independent upper-bound is dependent on the value of the diffusion exponent m. More precisely, we see in Figure 5.1 that, for $m = 2.1$ and sufficiently large mass, the maximal density of the stationary state remains constant as the mass of the population is increased. That is, for $M \geq 40$, the maximal density remains at a value of $\max \rho_s \approx 2.443$, with only the support increasing as the mass of the initial data is increased. The same behaviour occurs in Figure 5.2, where the maximal value that the density obtains for any mass is $\max \rho_s \approx 1.19$.

Furthermore, our numerical results agree with the upper-bound ρ_s^* of ρ_s, derived in Theorem 5.1. Here, $\rho_s^* \approx 3.942$ for $m = 2.1$ and $\rho_s^* \approx 1.726$ for $m = 2.5$, where $||W||_{L^1}$ is computed numerically using the same spatial step size as the computations for ρ_s.

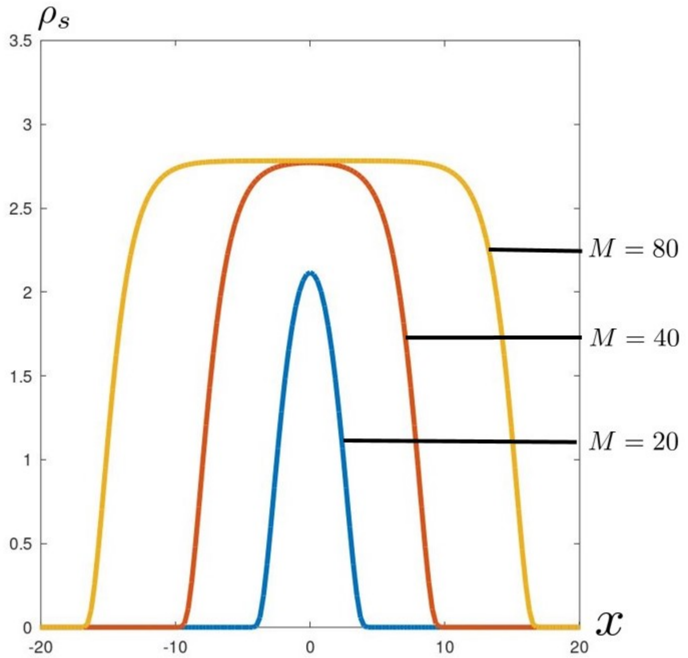

Figure 5.1: Example 1: Stationary solutions of Equation (1.1), where $m = 2.1$. For $M \geq 40$ we have $\max \rho_s \approx 2.443$.

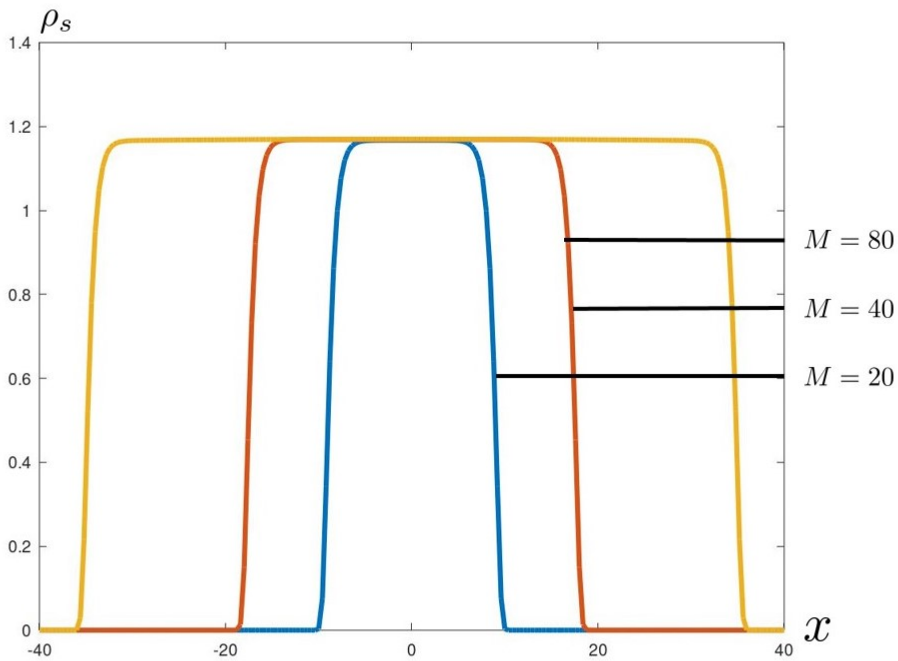

Figure 5.2: Example 1: Stationary solutions of Equation (1.1), where $m = 2.5$. For $M \geq 20$ we have $\max \rho_s \approx 1.19$.

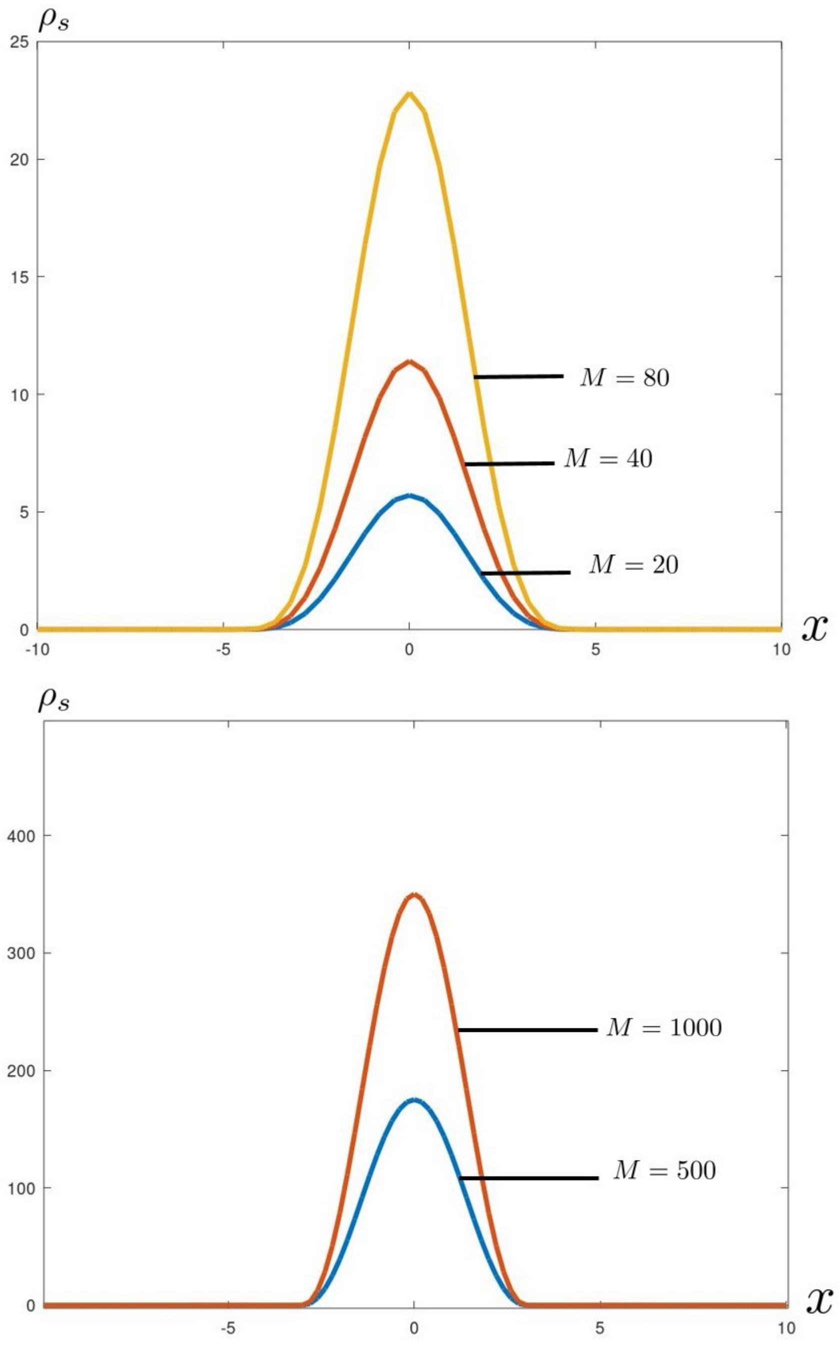

Figure 5.3: Example 1: Stationary solutions of Equation (1.1), where $m = 2$. The maximum density continues to increase with the mass, while the support remains constant.

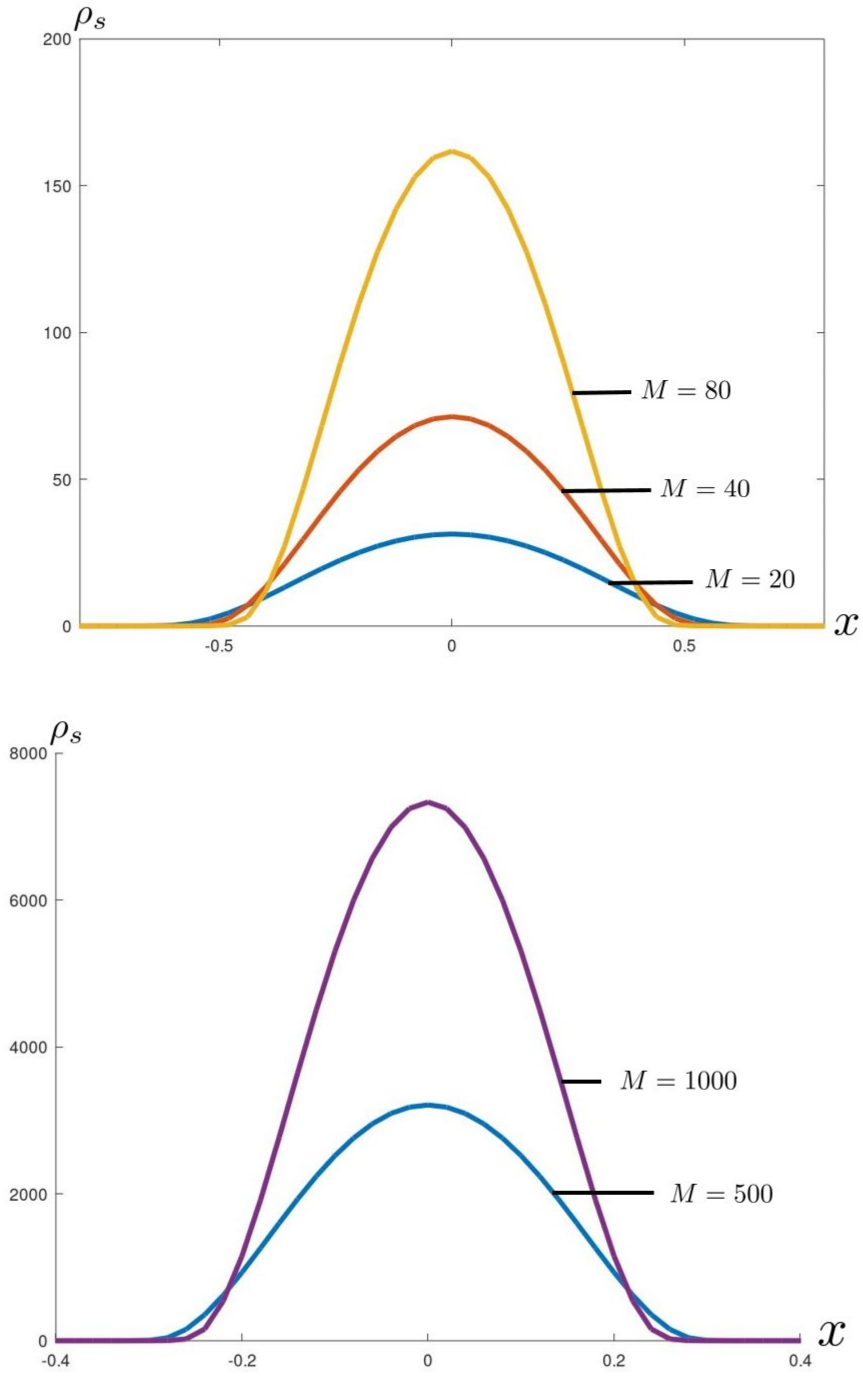

Figure 5.4: Example 1: Stationary solutions of Equation (1.1) where $m = 1.5$. The height of the density increases and the support decreases as the mass is increased.

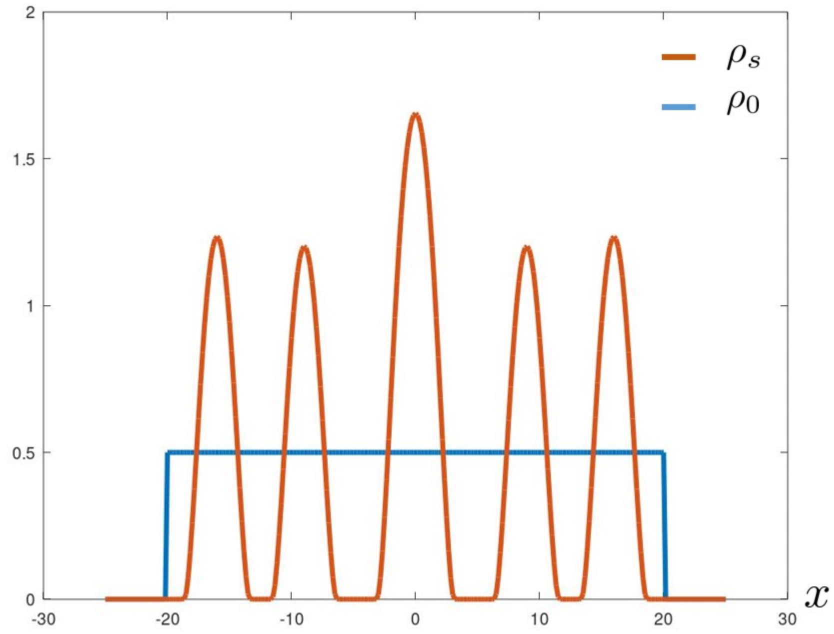

Figure 5.5: Example 1: Stationary solution of Equation (1.1), where $m = 2.1$, depicting the formation of multiple connected components.

In contrast, in Figures 5.3 and 5.4 we see that, for $m = 2$ and $m = 1.5$, respectively, the maximal density continues to grow with the mass of the initial data, implying the absence of a mass-independent upper-bound. Using the same mass as in Figures 5.1 and 5.2, we obtain convergence towards stationary solutions that do not reach a plateau where the internal density is approximately constant, as is the case for $m > 2$. Furthermore, for stationary states with mass that is well above that of the stationary states depicted in Figures 5.1 and 5.2, the maximum height of the density continues to grow with the mass, suggesting that the maximum height is dependent on the mass for any $M > 0$. It is also interesting to note that for $m = 2$, the support of the stationary state remains constant as the mass is increased, while for $m = 1.5$, as the mass increases, the size of the support decreases.

The emergence of multiple clumps can be seen in Figure 5.5. Indeed, we see that, for initial data with sufficiently large support, the support of the stationary solution is made up of multiple connected components. Figure 5.6 gives another example of a stationary state with multiple connected components. We choose a randomly distributed initial condition with mass large enough such that some of the swarms in the stationary state reach their preferred maximum density with approximately constant interior. In Figure 5.7, we provide a close up view of the two swarms in Figure 5.6 with the shortest proximity, that is, the second and third swarms from the left. We see that the distance between these two swarms is no smaller than the radius of $\operatorname{supp} W$, confirming the analytical result given in Theorem 4.13.2.

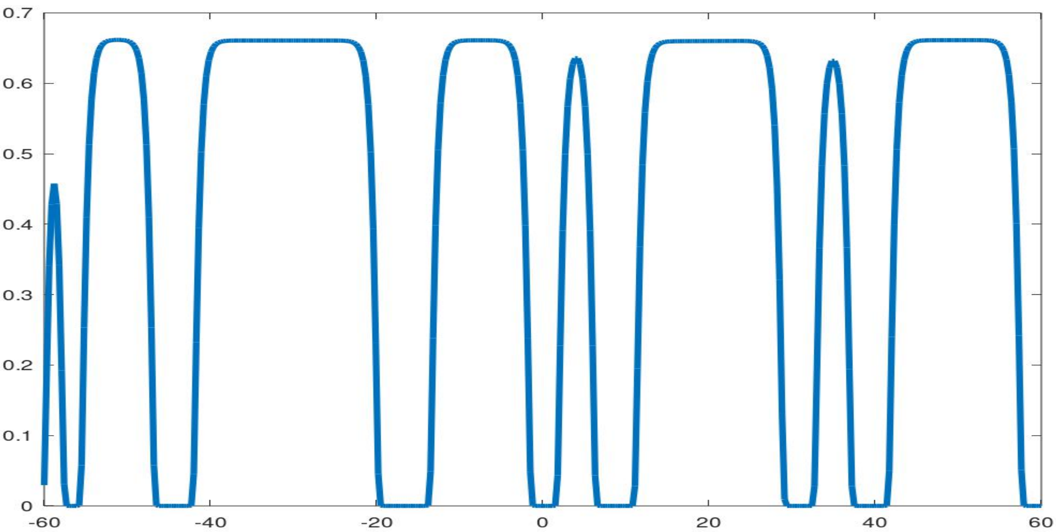

Figure 5.6: Stationary solution of Equation (1.1), where $m = 3$ and $W(x) = -\max(1-|x|^2, 0)$. The initial condition is randomly distributed with mass $M = 55$. The mass of the initial condition is large enough to allow for some swarms to reach their preferred maximum density, where their interior is approximately constant.

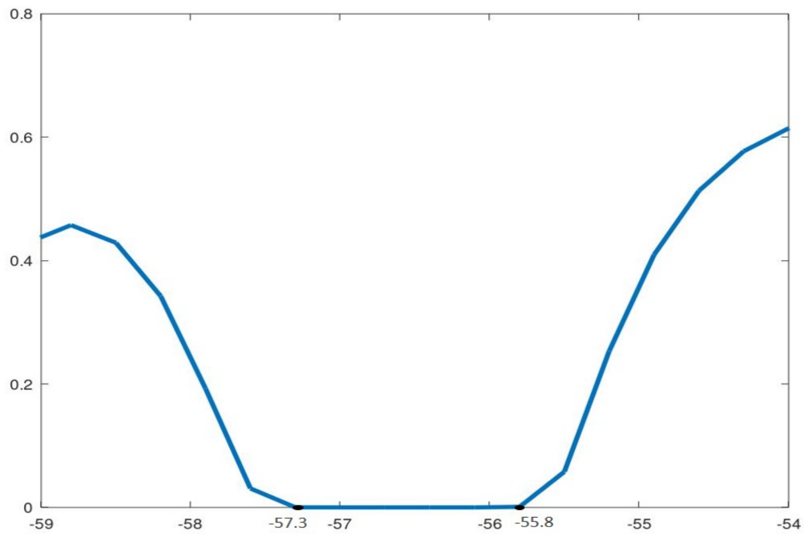

Figure 5.7: Close up of the two closest swarms of the stationary state depicted in Figure 5.6, that is, the second and third swarms from the left. The distance between these two swarms is not smaller than the radius of $\operatorname{supp} W$, confirming the analytical result given in Theorem 4.13.2.

Finally, we comment on the fact that, in all figures for Example 1, the stationary states are compactly supported and radially decreasing up to a translation on each connected component of their support, agreeing with the analytical results given in Section 4.2.

Example 2 Consider

$$W(x) = -e^{-|x|},\ x \in \mathbb{R}.$$

In Example 2, we consider the case of ω with unbounded support. Similarly to Example 1, we see in Figures 5.8, 5.9, 5.10, and 5.11 that $m = 2$ acts as a threshold value separating stationary states that possess a mass-independent upper-bound with those that do not. In Figures 5.8 and 5.9, where $m = 2.1$ and $m = 2.5$, respectively, we obtain stationary states that are bounded above independent of the mass of the initial data with the internal density approximately constant, while in Figures 5.10 and 5.11, where $m = 2$ and $m = 1.5$, respectively, we obtain stationary states whose height continues to increase regardless of the mass of the initial data. Hence, for W with either bounded or unbounded support we obtain the same dichotomy of behaviour as we vary m.

As in Example 1, our numerical results have an upper-bound less than the theoretical upper-bound ρ_s^* of ρ_s, derived in Theorem 5.1. Indeed, in our numerical simulation we obtain a mass-independent upper-bound for $m = 2.1$ and $m = 2.5$, respectively, given by, $\max \rho_s \approx 1.1$ and $\max \rho_s \approx 1.0117$. Comparing this to our analytical result, where $\rho_s^* \approx 1.59$ for $m = 2.1$ and $\rho_s^* \approx 1.44$ for $m = 2.5$, we see that our numerical upper-bound remains below our analytical upper-bound, regardless of the mass.

Furthermore, we obtain stationary solutions that are compactly supported and are radially decreasing up to a translation, agreeing with the analytical results given in [21] and [26]. Moreover, we see in Figure 5.12 that, using the same initial data as in Figure 5.5, Example 1, we obtain a stationary state whose support consists of a single component. This is expected, since the attractive kernel W has infinite support. Hence, we are able to obtain different stationary solutions by varying the size of the support of W.

In particular, attractive kernels with compact support can be considered advantageous over kernels with infinite support, since they allow for the formation of patterns, as can be seen in Figures 5.13, 5.14, and 5.15. In addition, compactly supported interaction kernels can be considered as more realistic with regard to modelling physical and biological aggregations, as the sensing mechanisms of agents in nature do not have an infinite range.

We note that, for both Examples 1 and 2, our numerical simulations of the stationary solutions are in line with the analytical result given in [31, Corollary 2.3] where it is proved that, for the particular case of the Newtonian kernel, where $m > 2 - \frac{2}{d}$, and for two stationary states ρ_1 and ρ_2, with masses $M_1 > M_2$, the following properties hold:

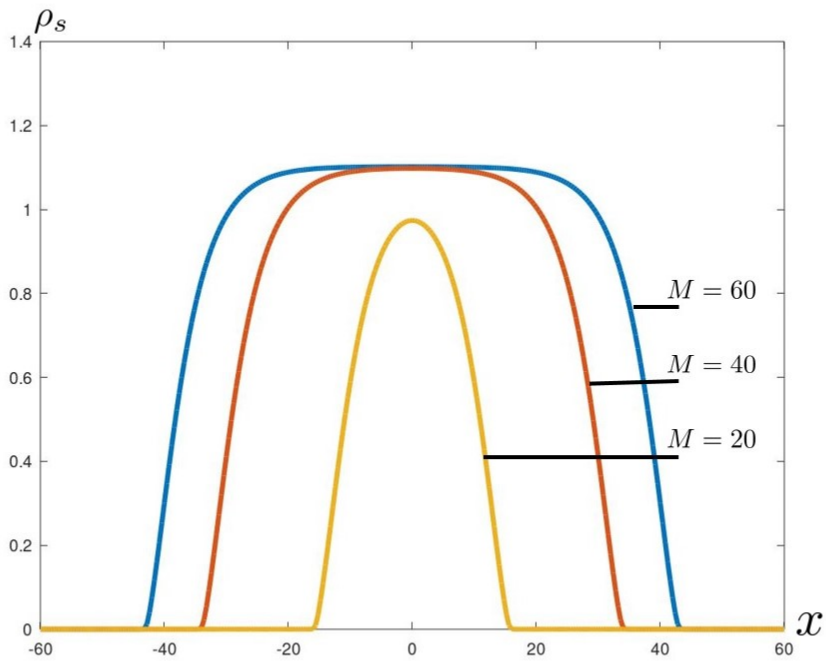

Figure 5.8: Example 2: Stationary solutions of Equation (1.1), where $m = 2.1$. For $M \geq 40$ we have $\max \rho_s \approx 1.1$.

(a) If $m > 2$, then ρ_1 has a bigger support and bigger height than ρ_2.

(b) If $m = 2$, then ρ_1 and ρ_2 have the same size support.

(c) If $2 - \frac{2}{d} < m < 2$ then ρ_1 has a smaller support and greater height than ρ_2.

We remark that Property (a) holds unless M_1 and M_2 are large enough so that both stationary states reach their preferred maximum density. In this case, both stationary states will have the same height and ρ_1 will have a larger support than ρ_2.

Additionally, we see that, in both examples, our numerical results satisfy the properties of conservation of mass and the preservation of positivity of the solution, as expected. Furthermore, when we make the domain large enough to accommodate the mass of the initial condition, we observe that the centre of mass is conserved as the solution evolves. In particular, for the case of W with unbounded support, the centre of mass acts as a centre for the radially decreasing stationary solution.

By our numerical simulations for both Examples, we see that, for sufficiently large mass, the stationary states are approximately constant in the interior of their support. This is in agreement with the numerical results obtained in [42], where stationary states corresponding to a special case of Equation (1.1), where $m = 3$, are considered. In particular, the authors derive an approximation for the stationary states using the energy functional (3.1) for the case when the mass is large enough such that the stationary states are approximately constant inside their support. We

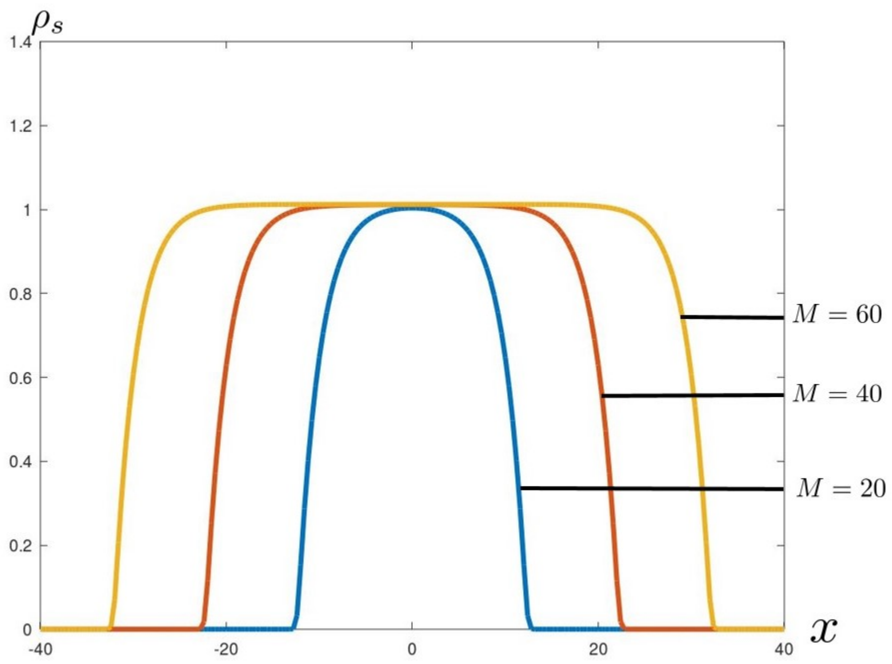

Figure 5.9: Example 2: Stationary solutions of Equation (1.1), where $m = 2.5$. For $M \geq 20$ we have $\max \rho_s \approx 1.0117$.

extend this approximation to stationary states of Equation (1.1) with diffusion coefficient $m > 2$ as follows:

We consider the energy functional (3.1) and assume that the mass is large enough such that the stationary states are approximately constant inside their support. Then,

$$\begin{aligned}
\mathcal{E}[\rho_s] &= \frac{\varepsilon}{m-1}\int_{\mathbb{R}^d} \rho_s^m \, dx + \frac{1}{2}\int_{\mathbb{R}^d} \rho_s (W * \rho_s) \, dx \\
&\approx \frac{\varepsilon}{m-1}\rho_s^m \int_{\operatorname{supp}\rho_s} dx - \frac{\rho_s^2}{2}||W||_{L^1(\mathbb{R}^d)} \int_{\operatorname{supp}\rho_s} dx \\
&= \frac{\varepsilon}{m-1}\rho_s^m |\operatorname{supp}\rho_s|_d - \frac{\rho_s^2}{2}||W||_{L^1(\mathbb{R}^d)} |\operatorname{supp}\rho_s|_d.
\end{aligned}$$

Now, since we assume ρ_s is approximately constant in its support, we have that

$$M = \int_{\operatorname{supp}\rho_s} \rho_s \, dx \approx \rho_s |\operatorname{supp}\rho_s|_d.$$

Hence, $|\operatorname{supp}\rho_s|_d \approx \frac{M}{\rho_s}$, and so

$$\mathcal{E}[\rho_s] \approx \frac{\varepsilon}{m-1}\rho_s^{m-1} M - \frac{\rho_s}{2}||W||_{L^1(\mathbb{R}^d)} M.$$

Now, since ρ_s is a stationary state of (1.1), it is a stationary point of the energy functional, by Theorem 4.12. Therefore,

$$\varepsilon \rho_s^{m-2} M - \frac{M}{2}||W||_{L^1(\mathbb{R}^d)} \approx 0,$$

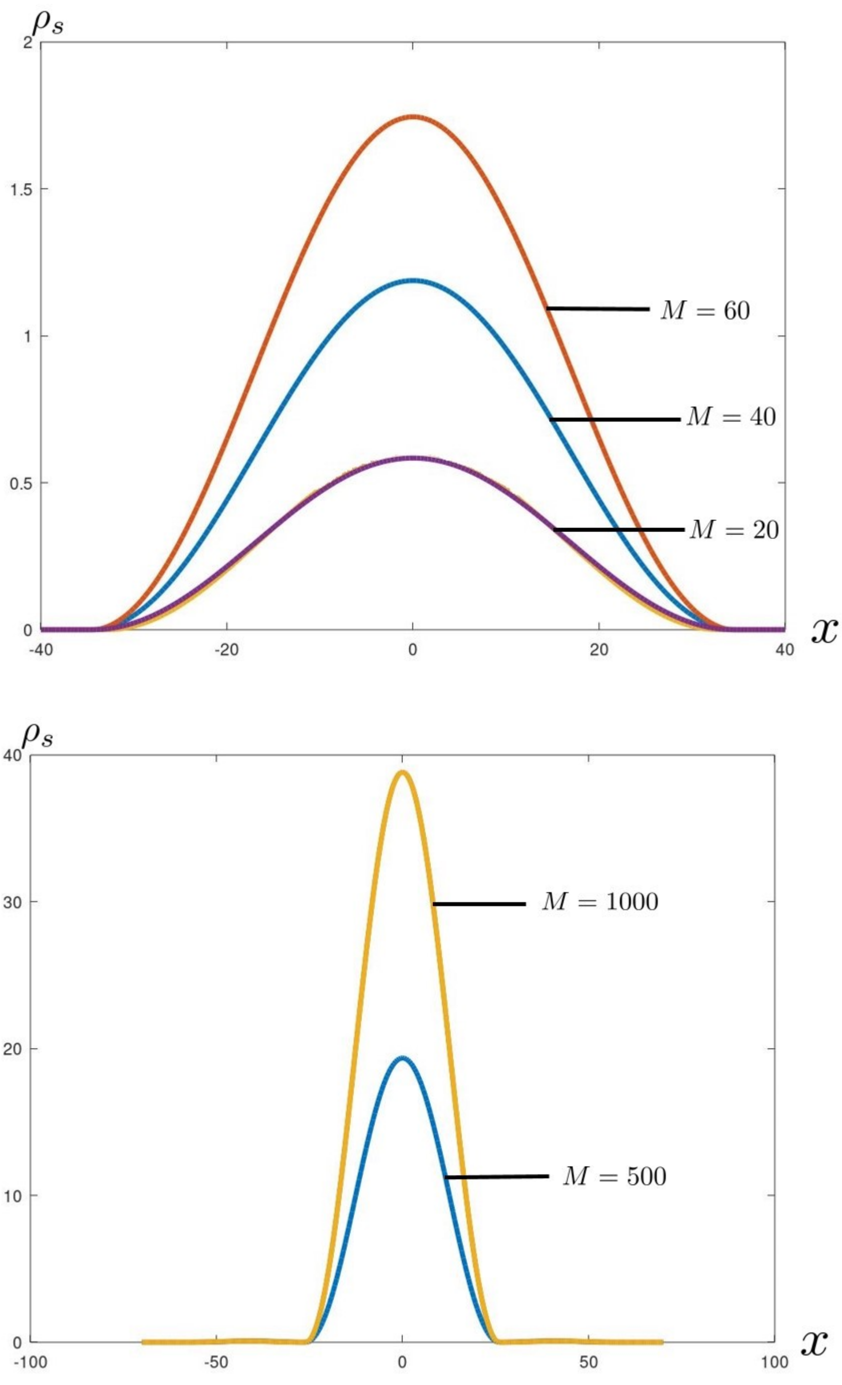

Figure 5.10: Example 2: Stationary solutions of Equation (1.1), where $m = 2$. The maximum density continues to increase with the mass, while the support remains constant.

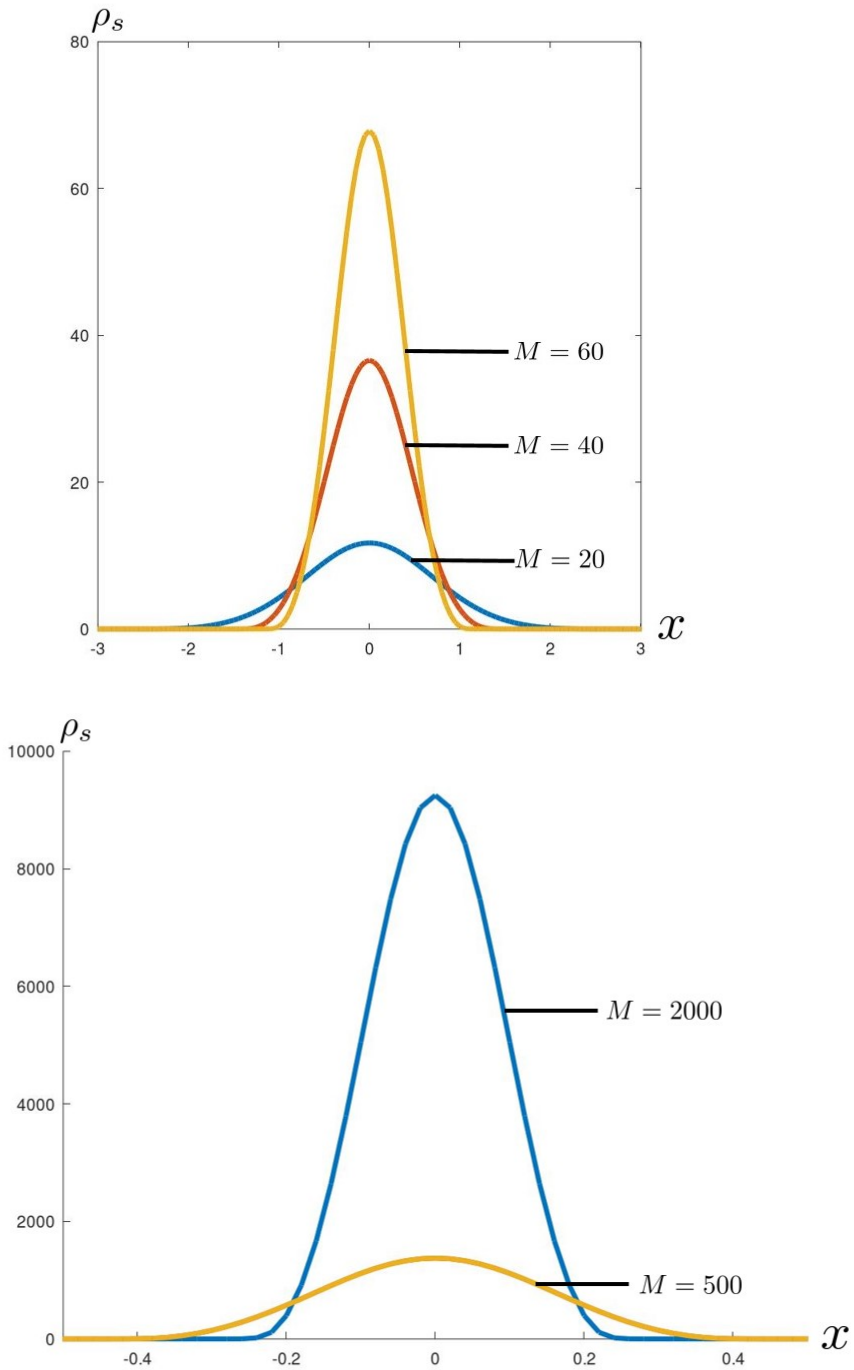

Figure 5.11: Example 2: Stationary solutions of Equation (1.1) where $m = 1.5$. The height of the density increases and the support decreases as the mass is increased.

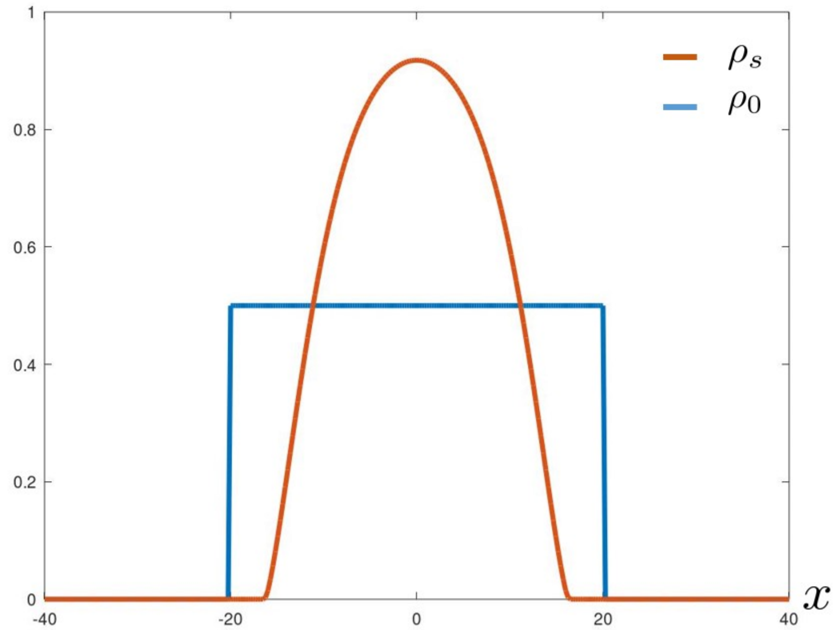

Figure 5.12: Example 2: Stationary solution of Equation (1.1), where $m = 2.1$. The support of ρ_s consists of a single component, contrasting the stationary state with the same initial data given in Figure 5.5.

yielding

$$\rho_s \approx \left(\frac{||W||_{L^1(\mathbb{R}^d)}}{2\varepsilon} \right)^{\frac{1}{m-2}}.$$

in supp ρ_s. Denoting the above approximation by ρ_E, we find that, for Example 1, $\rho_E \approx 2.476$ if $m = 2.1$ and $\rho_E \approx 1.2$ if $m = 2.5$, where we have used the trapezoidal rule, with spatial step size $dx = 0.4$, to approximate the integral of W, obtaining $||W||_{L^1(\mathbb{R}^d)} \approx 2.1898$. Additionally, for Example 2, we find that $\rho_E \approx 1$ for both $m = 2.1$ and $m = 2.5$.

In Table 5.1, a summary of the results obtained in this section regarding the mass-independent upper-bound of stationary states of (1.1) for $m = 2.1$ and $m = 2.5$ is given. In addition, we include results for cases $m = 3$ and $m = 3.5$. We note that the numerical solution for $m = 2.1$ and $m = 2.5$ goes above ρ_E, implying that ρ_E is not an upper-bound for stationary states of (1.1). However, we conjecture that this is a numerical error as the numerical stationary state moves above ρ_E when approaching the threshold value $m = 2$.

We also note that $\lim_{m\to\infty} \left(\frac{m-1}{m\varepsilon}||W||_{L^1(\mathbb{R}^d)}\right)^{\frac{1}{m-2}} = 1$ and $\lim_{m\to\infty} \left(\frac{||W||_{L^1(\mathbb{R}^d)}}{2\varepsilon}\right)^{\frac{1}{m-2}} = 1$, showing that our theoretical upper-bound ρ_s^* and the energy approximated upper-bound ρ_E agree in the limit of large m.

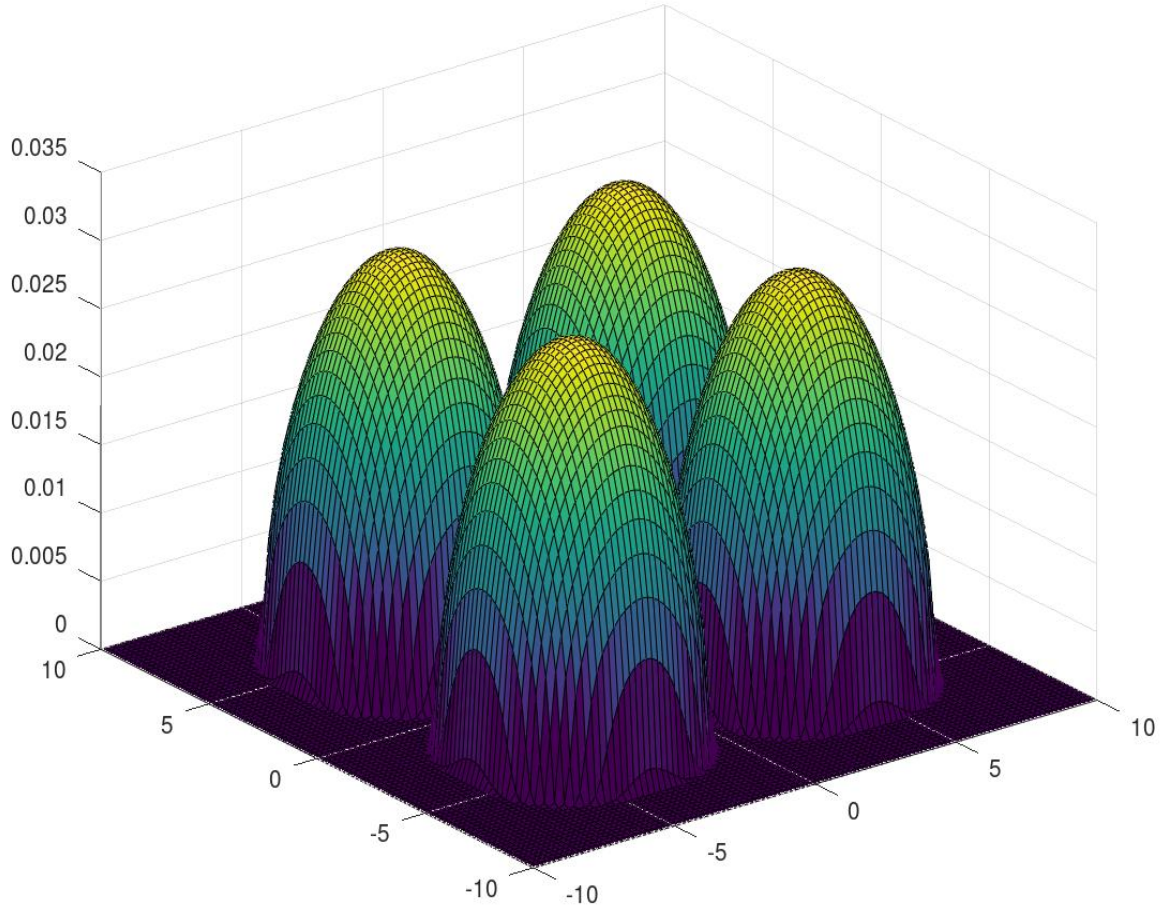

Figure 5.13: Pattern formation of a stationary solution of Equation (1.1), where $m = 3$ and $\omega(r) = -5e^{1/(r^2-1)}$ for $0 < r < 1$ and $\omega(r) = 0$ for $r \geq 1$. The initial condition is made up of a linear combination of step functions of the same height that are evenly distributed throughout the domain.

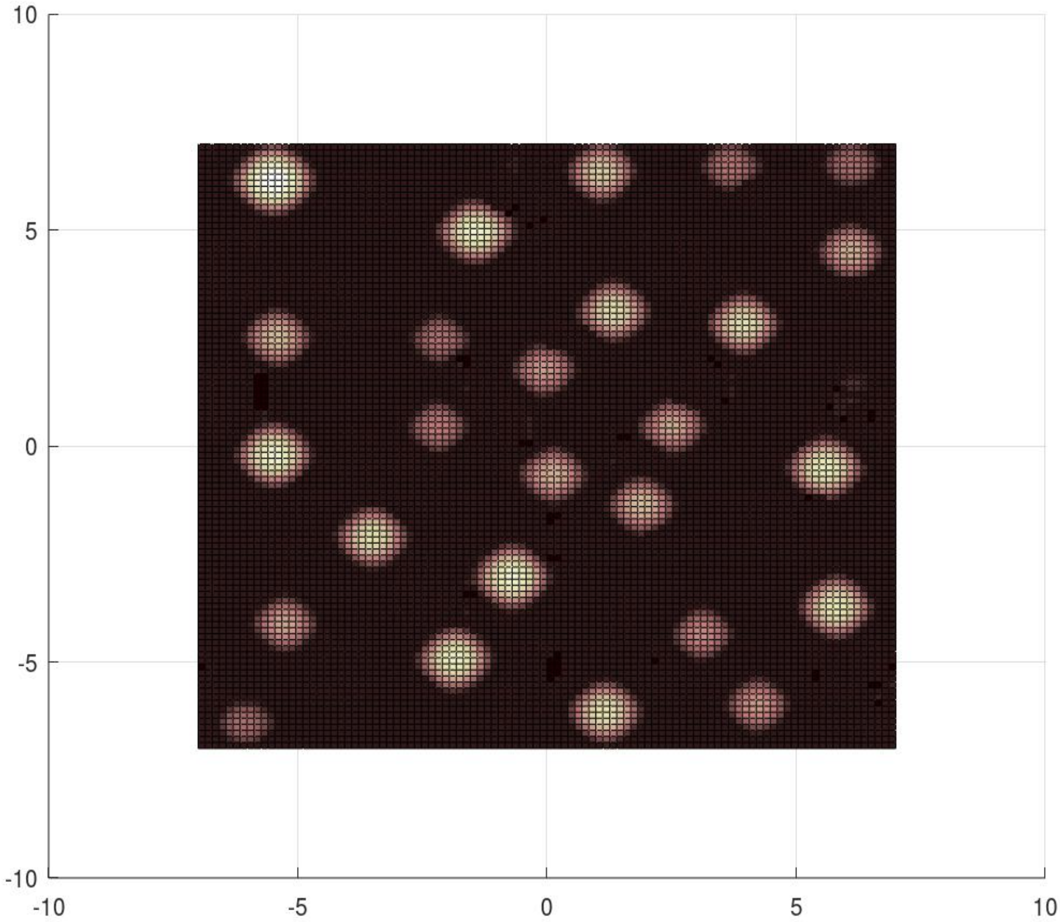

Figure 5.14: Pattern formation of a stationary solution of Equation (1.1), where $m = 3$ and $\omega(r) = -5e^{1/(r^2-1)}$ for $0 < r < 1$ and $\omega(r) = 0$ for $r \geq 1$. The initial condition is randomly distributed.

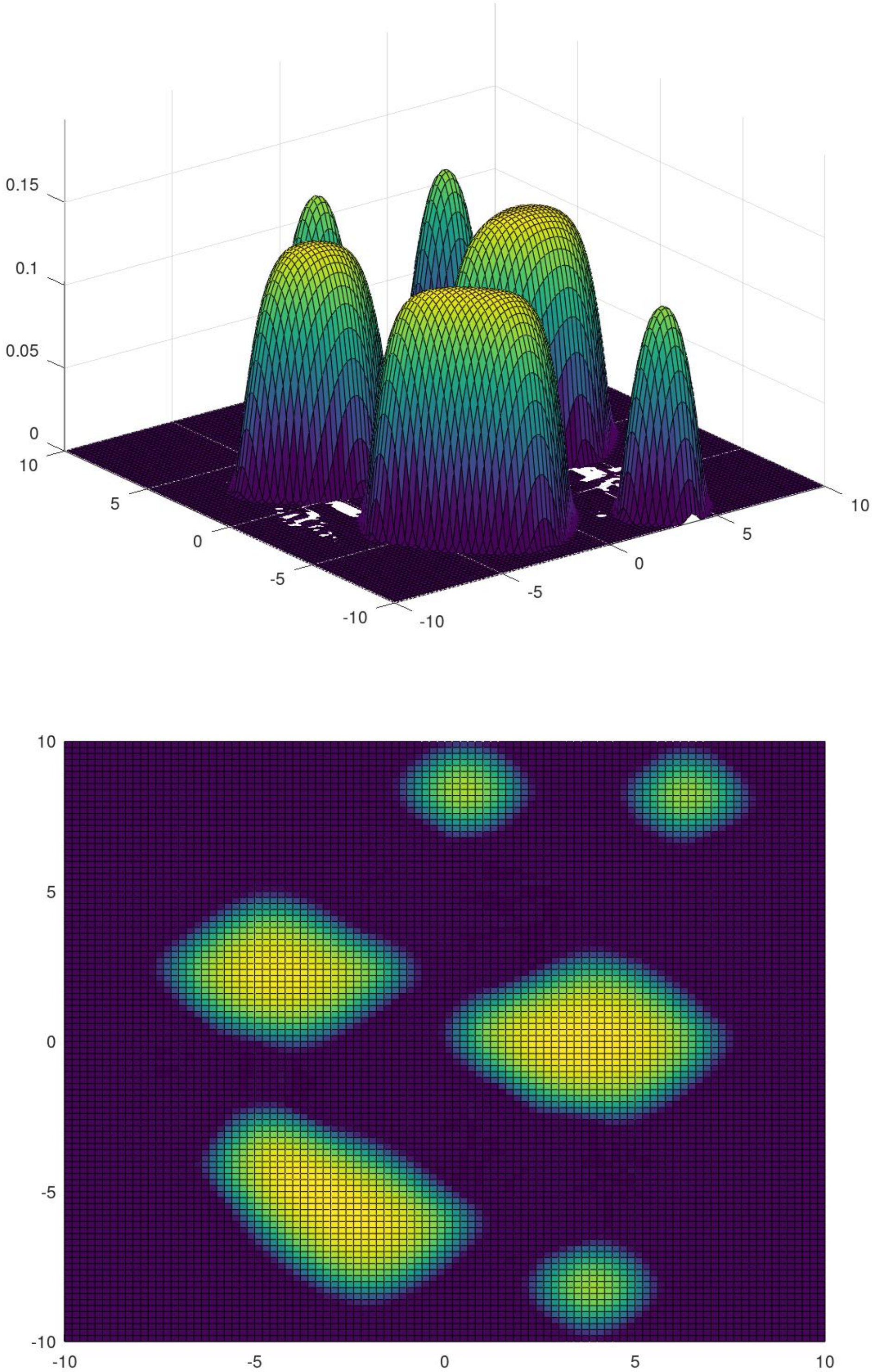

Figure 5.15: Pattern formation of a stationary solution of Equation (1.1), where $m = 3$, $\varepsilon = 5$ and $W = -(1 - ||x||^2)_+$. We choose a randomly distributed initial condition with mass large enough to allow for some swarms to reach their preferred maximum density, where their interior is approximately constant.

Table 5.1: Approximation of the maximal value of ρ_s for Equation (1.1).

Example 1			
	ρ_s^*	numerical max ρ_s	ρ_E
m=2.1	3.942	2.443	2.476
m=2.5	1.726	1.19	1.2
m=3	1.333	1.0925	1.0949
m=3.5	1.333	1.0589	1.0623

Example 2			
	ρ_s^*	numerical max ρ_s	ρ_E
m=2.1	1.59	1.1	1
m=2.5	1.44	1.0117	1
m=3	1.333	0.99933	1
m=3.5	1.27	0.99742	1

5.3 The aggregation-diffusion equation with a time dependent attractive kernel

In this section, we consider the case where the attractive kernel W is dependent on both space and time. In particular, we provide numerical results in two dimensions of Equation (1.1) with the addition of an exogenous force $V(x,t)$. That is, we consider the equation given by

$$\partial_t \rho = \varepsilon \Delta \rho^m + \nabla \cdot (\rho \nabla (W * \rho + V)). \tag{5.2}$$

In many biological swarms, such as locusts or fruit flies, the aggregation of the species does not occur constantly over time [43]. For example, the species will aggregate to find a potential mate, but then disperse to forage for food. Hence, it is noteworthy to consider how the behaviour of the swarm evolves when the strength of the aggregation is dependent on time. For our numerical simulations, we consider two choices of attractive kernels, given by Examples 1 and 2 below.

Example 1

$$W(x,t) = \begin{cases} -\frac{1}{2\pi} e^{-||x||}, & \text{for } t \in A \\ 0, & \text{otherwise} \end{cases}$$

Example 2

$$W(x,t) = \begin{cases} -(1 - ||x||^2)_+, & \text{for } t \in A \\ 0, & \text{otherwise} \end{cases}$$

In both examples, we denote A to be some union of disjoint intervals.

We first consider solutions to Equation (5.2) where W is given by Example 1, that is, where W has infinite support. We choose the exogenous force $V(x,t)$ to be a cluster of four Gaussian functions situated at the bottom left corner of the domain, which disappears when $t \in A$, that is, when the attractive kernel is "switched on". In a biological setting, we make this choice of V in order to model the presence of a food source. However; other choices of V can be considered to model other exogenous forces such as wind, gravity, or light sources.

We pick the initial condition to be a single Gaussian function located away from the exogenous force V, depicted in Figure 5.16 (a). In (b), we see the time evolution of the solution as a result of the attractive kernel being switched off and the exogenous force being switched on. We see that the population breaks away from a single swarm and moves towards the food source. As a result of the diffusion term and the limited food source, some of the population remains close to the site of the initial aggregation.

After a certain time, the aggregation term is switched on and the exogenous force is switched off. As a result, two separate swarms are formed, as illustrated in (c). As time passes, we see in (d) that the two swarms merge into a single swarm. This is a consequence of the attractive kernel having infinite support. We comment that the state in which the solution is comprised of two separate swarms is long-lived. This meta-stable behaviour is a well-known attribute of solutions to non-linear aggregation diffusion equations where the attractive kernel decays quickly over space, as shown in [18] and references therein. A video providing an example of this metastable behaviour in the dynamics of Equation (1.1) may be found in [12].

Lastly, we note that the final swarm in (d) is in a different location to the initial condition, attributed to the presence of an exogenous force. Indeed, in the absence of the exogenous force, the final state of the swarm would be in the same location as the initial condition in (a). Hence, the exogenous force allows for the swarm to travel in space over time.

Figure 5.17 illustrates the evolution of the solution of Equation (5.2), where W is given by Example 2, that is, where W is compactly supported, and where $V(x,t)$ is a random distribution in space and vanishes when $t \in A$. We pick the initial condition to be a single Gaussian function situated in the centre of the domain. In (a), we see the time evolution of the solution as a result of the attractive kernel being switched off and the exogenous force being switched on. The population breaks away from a single swarm and moves towards the randomly distributed food source.

When the aggregation term is switched back on and the exogenous force is switched off, multiple swarms of varying size are formed, as illustrated in (b). Because of the metastability behaviour, we terminate the simulation before the stationary state is reached; however, from our analytical results given in Chapter 4, we expect that the solution will converge in time towards a stationary state where each swarm has a compact and circular support. We refer to [12] for a video representation of the dynamics of Equation (5.2) for both Examples 1 and 2.

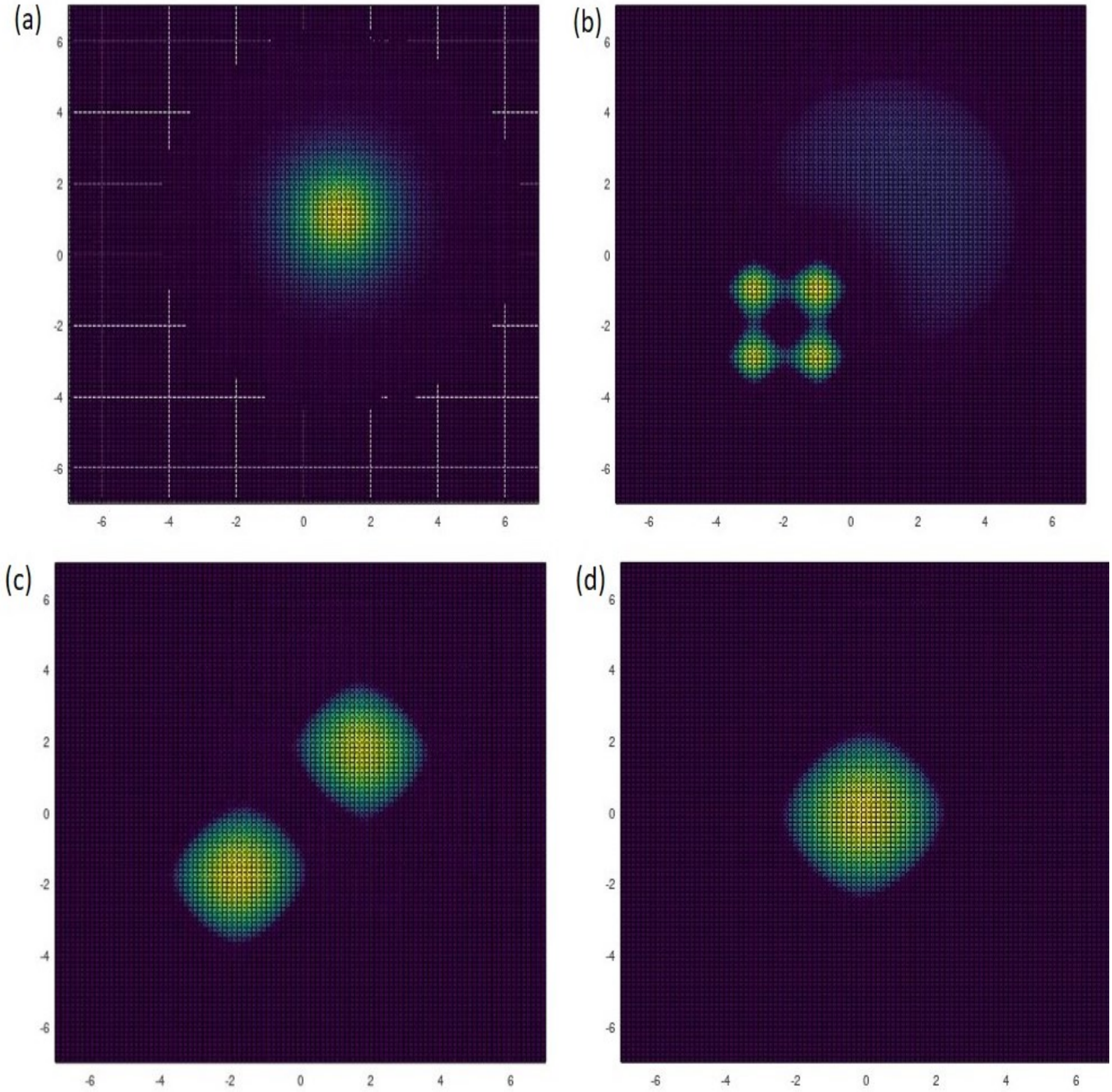

Figure 5.16: Simulation of aggregations of a population over time where the time-dependent attractive kernel W of Equation (1.1) has infinite support and the model incorporates a space and time-dependent exogenous force V. (a) Depiction of the initial condition given by a Gaussian function. (b) The attractive kernel is switched off and the exogenous force, modelling food sources is switched on. The population breaks away from a single swarm and moves towards the food source. (c) Two separate swarms are formed when the aggregation term is switched on and the exogenous force is switched off. (d) The two swarms merge into a single swarm, which has a different location to the initial condition.

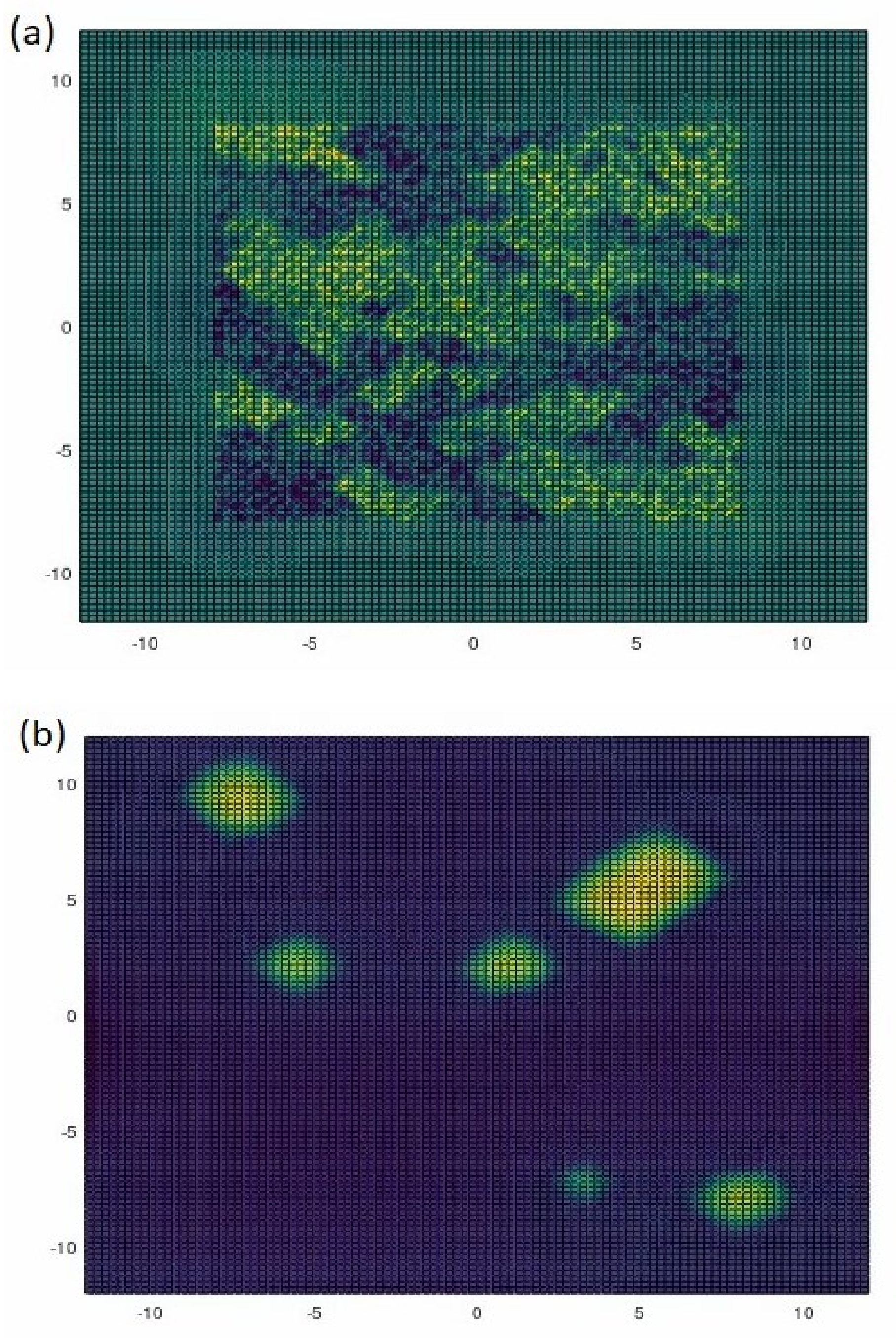

Figure 5.17: Simulation of aggregations of a population over time where the time-dependent attractive kernel W of Equation (1.1) is compactly supported and the model incorporates a space and time-dependent exogenous force V. (a) The attractive kernel is switched off and the exogenous force, modelling food sources, randomly distributed over the domain, is switched on. The population breaks away from a single swarm and moves towards the food sources. (b) Multiple swarms of varying size are formed when the aggregation term is switched on and the exogenous force is switched off. We expect the swarms to converge towards a stationary state where each swarm has a compact and circular support.

5.3.1 Numerical procedures used

For all numerical results in one dimension, an implicit-explicit (IMEX) scheme, presented in [16], is used. This scheme is based on the explicit scheme devised in [19]. The convective term is approximated in space using the same second-order upwind scheme as presented in [19], while the diffusive term is approximated in space by second order central finite differences. The discrete convolution is evaluated using a fast Fourier transform algorithm [28].

In our simulations, we use the forward Euler's method to discretize time. The diffusive term is discretized implicitly in time while the the convective term is disctretized explicitly in time, resulting in the Euler IMEX method, given by

$$\rho^{n+1} = \rho^n + dt(\mathcal{C}(\rho^n) + \mathcal{D}(\rho^{n+1})),$$

where $\mathcal{C}(\rho)$ and $\mathcal{D}(\rho)$ are the spatial discretizations of the convective and diffusive terms, respectively. The resulting nonlinear problem is solved using Newton's method.

The benefit of the diffusive term being handled implicitly is that less restriction is needed for the CFL condition, compared to explicit schemes. In addition, under a suitable CFL condition, it is proved in [16] that this Euler IMEX scheme is positivity-preserving.

For all numerical simulations in two dimensions, we again use the Euler IMEX scheme presented in [16], as well as the Alternating-Direction implicit (ADI) method to handle the implicit in time diffusive term [34]. The ADI method is used here to allow for a good compromise between accuracy and speed.

Finally, we note that in all simulations presented in this dissertation, we have used periodic boundary conditions. Since the problem is posed on $\mathbb{R}$ or $\mathbb{R}^2$, periodic boundary conditions are convenient as they account for the fact that we are not interested in any boundary effects. For efficiency, in our 2D simulations we made use of the Sherman-Morrison formula, which treats our system of equations generated by our finite difference scheme as a tridiagonal system plus a correction [40].

6

Model of non-local aggregation and non-local repulsion

In this chapter, we investigate a model incorporating both non-local repulsion and non-local attraction. Similarly to Equation (1.1), we aim to model the behaviour of individuals as a result of long-range attraction and short-range repulsion; however, as opposed to modelling the repulsion using non-linear diffusion, we consider a non-local term where the kernel, denoted R, has a shorter range of interaction compared to the attractive kernel W. We consider the following non-local integro-differential equation given by

$$\partial_t \rho = \nabla \cdot (\rho \nabla (W * \rho + \epsilon R * ((R * \rho)^{m-2} \rho) + \frac{\epsilon}{m-1} (R * \rho)^{m-1})). \tag{6.1}$$

Recalling the property of the Dirac measure given in Section 2.1.4, if R is the Dirac measure, then

$$(R * \rho)(x) = \int_{\mathbb{R}^d} \rho(y) \; d\delta_x(y) = \rho(x).$$

For this choice of R, Equation (6.1) reduces to Equation (1.1). Hence, (6.1) can be considered as a generalization of (1.1).

It is proved in [20] that under certain regularity conditions on R and W, and for $m \geq 2$, we have global existence and uniqueness of weak solutions to Equation (6.1), where test functions are taken in C_0^∞ so that weak solutions of (6.1) lie in the dual space of C_0^∞.

We note that Equation (6.1) looks rather complicated. The motivation to have it in such a form is the fact that the corresponding energy functional is conveniently given by

$$\mathcal{E}[\rho] = \frac{\varepsilon}{m-1} \int_{\mathbb{R}^d} \rho (R * \rho)^{m-1} \; dx + \frac{1}{2} \int_{\mathbb{R}^d} \rho (W * \rho) \; dx,$$

as shown in [20]. Furthermore, the energy functional is a Lyapunov functional for the dynamics of (6.1). Indeed, suppose first that ρ is a classical solution of (6.1). Taking the derivative of $\mathcal{E}[\rho]$ with respect to time, we see that

$$\partial_t \mathcal{E}[\rho] = \int_{\mathbb{R}^d} \rho_t (W * \rho) \; dx + \frac{\varepsilon}{m-1} \int_{\mathbb{R}^d} \partial_t \Big[\rho (R * \rho)^{m-1} \Big] \; dx$$

where the first term on the right is obtained using the same argument given for the proof of the dissipation of (3.1) in Section 3.1. Now, considering the second term on the right, we see that

$$\int_{\mathbb{R}^d} \partial_t \Big[\rho(R*\rho)^{m-1}\Big] \; dx = \int_{\mathbb{R}^d} \rho_t(R*\rho)^{m-1} + (m-1)\rho(R*\rho)^{m-2}(R*\rho_t) \; dx.$$

Using the assumption that R is even and making a slight abuse of notation by suppressing the t dependence of ρ, we have that

$$\begin{aligned}\int_{\mathbb{R}^d} \rho(R*\rho)^{m-2}(R*\rho_t) \; dx &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \rho(x)(R*\rho)^{m-2}(x)\big[R(x-y)\rho_t(y)\big] \; dydx \\ &= \int_{\mathbb{R}^d} \rho_t(y) \int_{\mathbb{R}^d} R(y-x)\big[\rho(x)(R*\rho)^{m-2}(x)\big] \; dxdy \\ &= \int_{\mathbb{R}^d} \rho_t R * \big((R*\rho)^{m-2}\rho\big) \; dx,\end{aligned}$$

by change of variables. Hence,

$$\partial_t \mathcal{E}[\rho] = \int_{\mathbb{R}^d} \rho_t \Big(W*\rho + \frac{\varepsilon}{m-1}(R*\rho)^{m-1} + \varepsilon R * \big((R*\rho)^{m-2}\rho\big)\Big) \; dx.$$

Now, using the fact that ρ satisfies equation (6.1) and applying integration by parts, we find that

$$\partial_t \mathcal{E}[\rho] = -\int_{\mathbb{R}^d} \rho V^2 \; dx \leq 0,$$

where

$$V = \nabla\Big(W*\rho + \frac{\varepsilon}{m-1}(R*\rho)^{m-1} + \varepsilon R * \big((R*\rho)^{m-2}\rho\big)\Big).$$

This property of energy dissipation is extended in [20] to hold for weak solutions in the dual of C_0^∞. Hence, similarly to Equation (1.1) and its corresponding energy, the energy functional (6) may provide insights into stationary states of (6.1).

6.1 Stationary states for $m = 2$

Recall that for Equation (1.1), for the particular case of $m > 2$ and for any attractive kernel W satisfying $(W1)$, $(W2)$ and either $(W3)$ or $(W4)$, we obtain stationary states that are physically and biologically relevant. That is, they have an upper-bound that is independent of the mass of the initial condition, they are compactly supported with sharp edges, and their internal density is approximately constant for sufficiently large mass. Hence, it is natural to consider whether we obtain the same result for stationary solutions to Equation (6.1).

In this section, we consider the case when $m = 2$ so that Equation (6.1) reduces to

$$\partial_t \rho = \nabla \cdot (\rho(\nabla Q * \rho)) \tag{6.2}$$

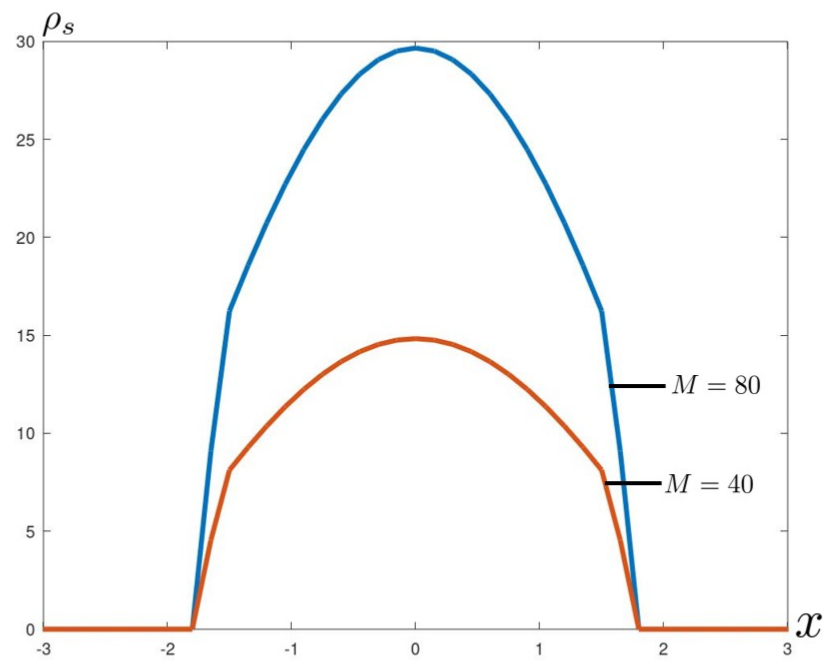

Figure 6.1: Stationary solution of Equation (6.2), where W given by (6.3) with $G = 0.4$ and $L = 4$. The initial condition is a step function centred at 0.

where $Q : \mathbb{R}^d \to \mathbb{R}$ is the interaction kernel incorporating both attractive and repulsive parts, given by $Q = W + 2\epsilon R$. Examples of attractive-repulsive interaction kernels found in the literature include the attractive-repulsive power-law kernels [24], given by

$$Q(x) = \frac{|x|^a}{a} - \frac{|x|^r}{r},$$

as well as the Morse potential,

$$Q(x) = -C_a e^{-|x|/\ell_a} + C_r e^{-|x|/\ell_r}.$$

Considerable research has been done in the existing literature on Equation (6.2) for Q equal to the Morse potential [8, 33]. In particular, for the special case of the Morse potential, given by

$$W(x) = -GLe^{-|x|/L} + e^{-|x|} \text{ where } L > 1, \tag{6.3}$$

the authors of [33] provide numerical results depicting solutions to (6.2) that either decay to zero, blow up in finite time or converge in time to a compactly supported stationary state.

More precisely, for the case when solutions exist globally in time, that is when $G \leq 1$, it is proved numerically in [33] that the value $GL^2 = 1$ acts as a threshold value. That is, for $GL^2 \geq 1$, solutions converge to compactly supported stationary states whose maximum height doubles by doubling the mass of the initial condition. These numerical results imply that stationary solutions in this regime do not have a mass-independent upper-bound and thus are not physically and biologically realistic. In contrast, for $GL^2 < 1$, solutions decay to zero on an unbounded domain, implying the absence of non-trivial stationary solutions. Hence, we may conclude that, for the case of $m = 2$, we do not obtain stationary solutions of (6.1) that are physically and biologically relevant.

6.2 Stationary states for $m > 2$

In this section, we obtain numerical results on the stationary solutions to Equation (6.1) for $m > 2$. We consider two choices of the interaction kernels R and W - one where both kernels have compact support and one where both have infinite support. We see that, just as for stationary solutions of Equation (1.1) for $m > 2$ and for both choices of W, numerical solutions of Equation (6.1) converge to stationary states that are compactly supported with steep edges and have a mass-independent upper-bound, as depicted in Figures 6.2 and 6.3. Figure 6.2 illustrates stationary states where R are W are compactly supported, while Figure 6.3 gives an illustration of stationary states where R and W have infinite support. In addition, we see that for large enough mass, the stationary states have a constant internal density. Hence, we obtain stationary states of Equation (6.1) with the same characteristics as those of Equation (1.1). Therefore, we can conclude that $m = 2$ is a threshold value such that for $m > 2$ we obtain stationary solutions that are physically and biologically realistic.

We note that for all numerical simulations of Equation (6.1), we have set $\varepsilon = 1$. Furthermore, we have used the upwind scheme presented in [19], where we have chosen $dx = 0.267$ and $dt = dx^7$.

Remark 6.1. We recall that Equation (6.1) is a generalization of Equation (1.1). In fact, it is proved in [20] that for $m > 2$, solutions of Equation (6.1) converge in the sense of distributions to solutions of Equation (1.1) as the interaction potential R is localized. More precisely, considering the repulsive kernel

$$R_r(x) = r^{-d}R(r^{-1}x),$$

and for all $r > 0$ where ρ_r is a weak solution to Equation (6.1) and ρ is a weak solution to Equation (1.1), it is proved that

$$\int_{\mathbb{R}^d} \rho_r(\cdot, t)\phi \, dx \to \int_{\mathbb{R}^d} \rho(\cdot, t)\phi \, dx$$

as $r \to 0$, for all $\phi \in C_0^\infty(\mathbb{R}^d)$ and for almost every $t \geq 0$.

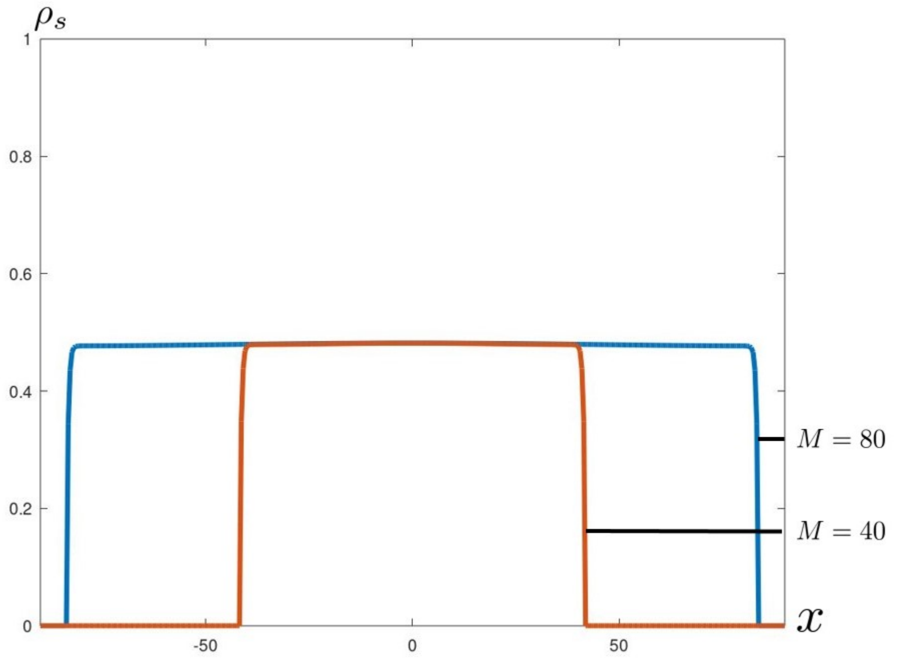

Figure 6.2: Stationary solutions to Equation (6.1) for $m = 3$. We choose $W = -(1 - |x|)_+$ and $R = 2^{-1}(1 - 2^{-1}|x|)_+$.

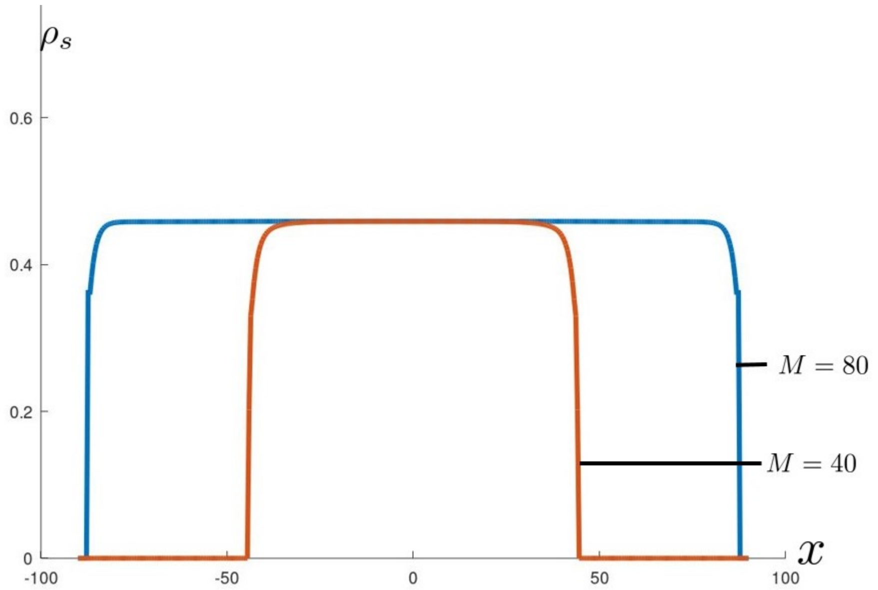

Figure 6.3: Stationary solutions to Equation (6.1) for $m = 3$. We choose $W = -\frac{1}{2}e^{-|x|}$ and $R = e^{-2|x|}$.

6.3 Approximation of stationary states using the energy

In this section, we derive an approximation for the stationary solutions to Equation (1.1) for $m > 2$ and when the mass of the swarm is sufficiently large so that stationary solutions have a constant internal density. Similarly to Equation (1.1), we can approximate the maximum height of these stationary states by minimizing the energy functional of Equation (6.1). Indeed, for ρ_s a stationary state of (6.1) with internal density approximately constant, we have that

$$\begin{aligned}
\mathcal{E}[\rho_s] &= \frac{\varepsilon}{m-1}\int_{\operatorname{supp}\rho_s} \rho_s (R*\rho_s)^{m-1}\,dx + \frac{1}{2}\int_{\operatorname{supp}\rho_s} \rho_s (W*\rho_s)\,dx \\
&\approx \frac{\varepsilon\rho_s^m}{m-1}||R||_{L^1(\mathbb{R}^d)}^{m-1}\int_{\operatorname{supp}\rho_s} dx - \frac{\rho_s^2}{2}||W||_{L^1(\mathbb{R}^d)}\int_{\operatorname{supp}\rho_s} dx \\
&= \frac{\varepsilon\rho_s^m}{m-1}||R||_{L^1(\mathbb{R}^d)}^{m-1}|\operatorname{supp}\rho_s|_d - \frac{\rho_s^2}{2}||W||_{L^1(\mathbb{R}^d)}|\operatorname{supp}\rho_s|_d.
\end{aligned}$$

Since $M = \int_{\mathbb{R}^d}\rho_s\,dx = \rho_s|\operatorname{supp}\rho_s|_d$, we have that $|\operatorname{supp}\rho_s|_d = \frac{M}{\rho_s}$. Hence,

$$\mathcal{E}[\rho_s] \approx \frac{\varepsilon M}{m-1}\rho_s^{m-1}||R||_{L^1(\mathbb{R}^d)}^{m-1} - \frac{M\rho_s}{2}||W||_{L^1(\mathbb{R}^d)}.$$

Using the fact that $\mathcal{E}$ is a Lyapunov functional for the evolution of (6.1), we have that $\varepsilon M\rho_s^{m-2}||R||_{L^1(\mathbb{R}^d)}^{m-1} \approx \frac{M}{2}||W||_{L^1(\mathbb{R}^d)}$, which yields

$$\rho_s \approx \left(\frac{||W||_{L^1}}{2\varepsilon||R||_{L^1}^{m-1}}\right)^{\frac{1}{m-2}} =: \rho_E,$$

in $\operatorname{supp}\rho_s$.

In Table 6.1, we summarize the results obtained for stationary states of (6.1) for varying m and for both examples of interaction kernels. In particular, we provide a comparison between the numerically obtained maximum height of stationary states and the mass-independent upper-bound ρ_E derived from the energy functional.

Table 6.1: Approximation of the maximal value of ρ_s for Equation (6.1).

R and W with compact support		
	numerical max ρ_s	ρ_E
m=2.5	0.25	0.25
m=3	0.498	0.5
m=3.5	0.61	0.63

R and W with infinite support		
	numerical max ρ_s	ρ_E
m=2.5	0.207	0.25
m=3	0.45	0.5
m=3.5	0.58	0.63

We expect that the numerical maximum of ρ_s will converge to 1 as $m \to \infty$, since

$$\lim_{m\to\infty} \left(\frac{||W||_{L^1}}{2\varepsilon ||R||_{L^1}^{m-1}} \right)^{\frac{1}{m-2}} = 1.$$

7

Conclusion and Future work

In this dissertation, we extend the theory of stationary solutions to Equation (1.1) to compactly supported attractive kernels. We prove that, for $m > 2$, stationary solutions are compactly supported and radially decreasing up to a translation on each connected component of their support. Furthermore, we prove that, if the support of the stationary solution is made up of more than one component, then the distance between any two connected components must be greater than the radius of the support of the attractive kernel. In addition, for $m > 2$ and for attractive kernels with either bounded or unbounded support, we prove that stationary solutions have a mass-independent upper-bound.

Additionally, we obtain numerical results that agree with our analytical results. We note that our analytical and numerical results agree with characteristics of physical and biological aggregations [38, 39, 42]. That is, the population aggregates to form a group whose spatial extent is bounded, the density has a mass-independent upper-bound and, for sufficiently large mass, the internal density of the population is approximately constant, implying a preferred inter-organism spacing that is independent of the mass of the population. More precisely, for the case of a domain that is large enough to accommodate the mass, increasing the population beyond a certain point does not cause over-crowding, but rather an increase in the support of the population.

Furthermore, we find that there is an advantage to consider attractive kernels with compact support over those with infinite support, as they allow for the formation of patterns, where multiple disjoint "clumps" are formed, in comparison to a single connected "clump" for the case of an attractive kernel with infinite support. We also consider the case where the attractive kernel is dependent on time and where a term modelling an exogenous force is incorporated into the model. When the attractive kernel has compact support, numerical results show the formation of multiple aggregations that can change location over time due to the presence of the exogenous force and the fact that the attractive kernel is time-dependent.

Finally, we consider a model, given by Equation (6.1), where the diffusion is replaced by a non-local repulsive term. Just as for stationary solutions to Equation (1.1), numerical results show that $m = 2$ acts as a threshold value such that for $m = 2$ stationary states of (6.1) either decay to zero or double in height as we double the mass of the initial condition, whereas, for $m > 2$, stationary states are bounded above independent of the mass of the initial condition. Moreover, for sufficiently large mass, the internal density is approximately constant. Lastly, we obtain an approximate upper-bound for stationary states of (6.1) for $m > 2$ by minimizing the equation's corresponding energy functional.

As a result, we can conclude that for both Equations (1.1) and (6.1), with $m > 2$ and no restriction on the support of the attractive kernel, we obtain stationary states with characteristics that are physically and biologically realistic. We also show that by considering an attractive kernel with compact support, solutions of Equation (1.1) form spatial patterns over time.

There are several lines of research arising from this work which can be pursued. For instance, an extension of the current theory can be made where Equations (1.1) and (6.1) are defined on a bounded domain. Interestingly, in [8] it is proved through numerical and analytical results that, for Equation (6.1) with $m = 2$ and when the domain of definition is bounded, solutions with sufficiently large mass will converge to a stationary state with internal density almost constant and with an accumulation of mass on the boundary in the form of delta-like functions. In addition, our own numerical results in one dimension show that, for Equation (6.1) with $m > 2$, we also obtain an accumulation of mass on the boundary of our domain when no-flux boundary conditions are used. This occurs when the mass of the initial data is larger than what the domain can accommodate to allow for stationary states to reach their preferred maximum density over the whole domain.

It is interesting to note that this accumulation of mass on the boundary does not occur for Equation (1.1), which results in stationary solutions that do not reach their preferred maximum density for the mass sufficiently large. Hence, the non-local repulsion may provide an advantage over non-linear diffusion in the modelling of collective behaviour on a bounded domain. This is because it is more biologically realistic that individuals would not tolerate a density higher than their preferred maximum, resulting in individuals on the periphery of the swarm being forced to accumulate on the boundary. However, to our knowledge, no theoretical analysis has been done regarding stationary states of Equation (6.1) for $m > 2$ on a bounded domain. Hence, this can be considered as a possible line of research for the future.

Additionally, since the uniqueness result for stationary solutions to Equation (1.1) where W has unbounded support relies on the radial symmetry property, one may expect that the result on radial symmetry given here may open a window to obtain uniqueness of stationary states on each connected component of the support for the case of compactly supported attractive kernels.

Bibliography

[1] Adioui, M., Arino, O. and El Saadi, N. (2005). A nonlocal model of phytoplankton aggregation. *Nonlinear Analysis: Real World Applications*, 6(4), pp. 593-607.

[2] de Barra, G. (1981). *Measure Theory and Integration.* Sawston: Woodhead Publishing, Cambridge.

[3] Bedrossian, J. (2011). Intermediate asymptotics for critical and supercritical aggregation equations and Patlak-Keller-Segel models. *Commun. Math. Sci.*, 9(4), pp. 1143-1161.

[4] Bedrossian, J. (2011). Global Minimizers for Free Energies of Subcritical Aggregation Equations with Degenerate Diffusion. *Appl. Math. Lett.*, 24(11), pp. 1927-1932.

[5] Bedrossian, J. and Rodriguez, N. (2014). Inhomogeneous Patlak-Keller-Segel models and Aggregation Equations with Nonlinear Diffusion in $\mathbb{R}^d$. *Discrete Contin. Dyn. Syst. Ser. B*, 19(5), pp. 1279-1309.

[6] Bedrossian, J., Rodrguez, N., and Bertozzi, A. (2011). Local and Global Well-Posedness for Aggregation Equations and Patlak-Keller-Segel Models with Degenerate Diffusion. *Nonlinearity*, 24(6), pp. 1683-1714.

[7] Bernoff, A.J. and Topaz, C.M. (2016). Biological Aggregation Driven by Social and Environmental Factors: A Nonlocal Model and Its Degenerate Cahn-Hilliard Approximation. *SIAM J. Appl. Dyn. Syst.*, 15(3), pp. 15281562.

[8] Bernoff, A.J. and Topaz, C.M. (2013). Nonlocal aggregation models: A primer of swarm equilibria. *SIAM Rev.*, 55(4), pp. 709-747.

[9] Biler, P. and Zienkiewicz, J. (2019). Blowing up radial solutions in the minimal Keller-Segel model of chemotaxis. *J. Evol. Equ.*, 19, pp. 71-90.

[10] Blanchet, A., Carrillo, J. A. and Laurenot, P. (2008). Critical mass for a Patlak-Keller-Segel model with degenerate diffusion in higher dimensions. *Calc. Var. Partial Differ. Equ.*, 35(2), pp. 133-168.

[11] Brezis, H. (2011). *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, New York: Springer.

[12] Bright, C.A. Nonlinear dynamics (webpage), http://linus.up.ac.za/academic/maths/ND/DAM/index.html

[13] Burger, M., Caffarelli, L. and Markowich P.A. (2014). The Keller-Segel model for Chemotaxis with Prevention of Overcrowding: Linear vs. Nonlinear Diffusion. *SIAM J. on Math. Analysis* , 38(4), pp. 1288-1315.

[14] Burger, M., Capasso, V. and Morale, D. (2007). On an aggregation model with long and short range interactions. *Nonlinear Analysis: Real World Applications*, 8, pp. 939-958.

[15] Burger, M., Di Francesco, M. and Dolak-Struss, Y. (2014). Partial differential equation models in the socio-economic sciences. *Philos Trans A Math Phys Eng Sci.* , 372, pp. 2028.

[16] Bürger, R., Inzunza, D., Mulet, P. and Villada, L.M. (2019). Implicit-explicit methods for a class of nonlinear nonlocal gradient flow equations modelling collective behaviour. *Appl. Num. Math.*, 144 , pp. 234-252.

[17] Calvez, V., Carrillo, J. A. and Hoffmann, F. (2017). Equilibria of homogeneous functionals in the fair-competition regime. *Nonlinear Anal.*, 159, pp. 85-128.

[18] Carrillo, J.A., Craig, K. and Yao, Y. (2019). *Aggregation-diffusion equations: dynamics, asymptotics, and singular limits.* Active Particles, Volume 2, Advances in Theory, Models, and Applications, Springer Nature Switzerland, pp.65-108.

[19] Carrillo, J.A., Chertock, A. and Huang, Y. (2014). A finite-volume method for nonlinear nonlocal equations with a gradient flow structure. *Commun. Comput. Phys.*, 17, pp. 233-258.

[20] Carrillo, J.A., Craig, K. and Patacchini, F.S. (2019). A blob method for diffusion, *Calc. Var.*, 58(53).

[21] Carrillo, J.A., Hittmeir, S., Volzone, B. and Yao, Y. (2019). Nonlinear aggregation-diffusion equations: radial symmetry and long time asymptotics. *Inventiones mathematicae*, 218 (3), pp. 889-977.

[22] Carrillo, J.A., Castorina, D. and Volzone, B. (2014). Ground States for Diffusion Dominated Free Energies with Logarithmic Interaction. *SIAM J. on Math. Analysis* .

[23] Carrillo, J.A., DiFrancesco, M., Figalli, A., Laurent, T. and Slepčev, D. (2014). Global-in-time weak measure solutions and finite-time aggregation for nonlocal interaction equations. *SIAM J. on Math. Analysis* .

[24] Carrillo, J. A. and Huang, Y. (2016). Explicit Equilibrium Solutions For the Aggregation Equation with Power-Law Potentials. *Kinetic and Related Models*, 10(1), pp. 171-192.

[25] Craig, K. and Topaloglu, I. (2019). Aggregation-Diffusion to Constrained Interaction: Minimizers and Gradient Flows in the Slow Diffusion Limit. *arXiv preprint arXiv:1806.07415v2.*

[26] Delgadino, M.G, Yan, X. and Yao, Y. (2019). Uniqueness and non-uniqueness of steady states of aggregation-diffusion equations. *arXiv preprint arXiv:1908.09782*

[27] Fraenkel, L.E. (2000). *An Introduction to Maximum Principles and Symmetry in Elliptic Problems.* Cambridge University Press.

[28] von zur Gathen, J. and Gerhard, J. (2003). *Modern Computer Algebra*. 2nd ed. Cambridge University Press.

[29] Kaib, G. (2017) Stationary states of an aggregation equation with degenerate diffusion and bounded attractive potential. *SIAM J. Math. Anal.*, 49(1), pp. 272-296.

[30] Kesavan, S. (2006). *Symmetrization and applications*. Series in Analysis, 3. Hackensack, NJ: World Scientific Publishing Co. Pte. Ltd.

[31] Kim, I. and Yao, Y. (2012). The Patlak-Keller-Segel model and its variations: properties of solutions via maximum principle. *SIAM J. Math. Anal.*, 44(4), pp. 568-602.

[32] Hörmander, L. (2003). *The Analysis of Linear Partial Differential Operators I*. Springer, Berlin Heidelberg.

[33] Leverentz, A.J., Topaz, C.M. and Bernoff, A.J. (2009). Asymptotic dynamics of attractive-repulsive swarms. *SIAM J. Appl. Dyn. Syst.*, 8, pp. 880-908.

[34] Li, Z., Qiao, Z. and Tang, T. (2018). *Numerical Solution of Differential Equations*. Cambridge University Press.

[35] Lieb, E. H. and Loss, M. (1997). *Analysis*. Graduate Studies in Mathematics, 14. Providence, RI: American Mathematical Society.

[36] Mogilner, A. and Edelstein-Keshet, L. (1999). A non-local model for a swarm. *J. Math. Biol.*, 38, pp. 534570.

[37] Mogilner, A., Edelstein-Keshet, L., Bent, L. and Spiros, A. (2003). Mutual interactions, potentials, and individual distance in a social aggregation. *J. Math. Biol.*, 47, pp. 353389.

[38] Parrish, J. K. and Edelstein-Keshet, L. (1999). Complexity, Pattern, and Evolutionary Trade-Offs in Animal Aggregation. *Science*, 284, pp. 99-101.

[39] Parrish, J. K. and Hamner, W.M. (1997). *Animal Groups in Three Dimensions*. Cambridge University Press.

[40] Press, W. H., Teukolsky, S.A., Vetterling, W.T. and Flannery, B.P. (2007). Numerical Recipes 3rd Edition: The Art of Scientific Computing. Cambridge University Press.

[41] Rudin, W. (1976). *Principles of Mathematical Analysis*, New York: McGraw-Hill, Inc.

[42] Topaz, C. M., Bertozzi, A.L. and Lewis, M.A. (2006). A nonlocal continuum model for biological aggregation. *Bull. Math. Biol.*, 68, pp. 16011623.

[43] Topaz, C. M., D'Orsogna, M. R., Edelstein-Keshet, L. and Bernoff, A.J. (2012). Locust Dynamics: Behavioral Phase Change and Swarming. *PLoS Comput Biol*, 8(8).

[44] Vázquez, J.L. (2007). *The Porous Medium Equation: Mathematical Theory*, Oxford University Press.

[45] Wittenburger, J.F. and Hunt, G.L. (1985). The adaptive significance of coloniality in birds. *Avian Biol.*, 8(1).