# The irreducible characters of the Sylow p-subgroups of the Chevalley groups $D_6(p^f)$ and $E_6(p^f)$


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## Abstract

We parametrize the set of irreducible characters of the Sylow p-subgroups of the Chevalley groups $D_6(q)$ and $E_6(q)$, for an arbitrary power $q$ of any prime $p$. In particular, we establish that the parametrization is uniform for $p \geq 3$in type $D_6$ and for $p \geq 5$ in type $E_6$, while the prime 2 in type $D_6$ and the primes 2, 3 in type $E_6$ yield character degrees of the form $q^m/p^i$ which force a departure from the generic situations. Also for the first time in our analysis we see a family of irreducible characters of a classical group of degree $q^m/p^i$ where $i > 1$which occurs in type $D_6$.



**Keywords**
Irreducible characters
Sylow subgroups
Nonabelian cores
Bad primes


## 1. Introduction

Let $q$ be a power of a prime $p$, and let $G$ be a finite group of Lie type over $\mathbb{F}_q$ and $U$ be a Sylow $p$-subgroup of $G$. Let $B := N_G(U)$ and let $P$ be a parabolic subgroup of $G$ with $U \leq B \leq P \leq G$. We denote by $\ell$ a prime distinct from $p$.

Harish-Chandra theory for $\ell$-modular representations of general finite groups of Lie type was initiated by Hiss (1991, 1993) and continued in Geck et al. (1996b). The theory suggests that the representation theory of parabolic subgroups $P$ of $G$ as above has strong influence on the representation theory of $G$, in particular towards a determination of its decomposition numbers. This is further evidenced by work of Gruber and Hiss (1997) for classical groups at primes $\ell$. More recently this has been taken further in Dudas and Malle (2015) and Dudas and Malle (2016) for some non-linear classes of primes. For small rank groups the calculation of decomposition numbers in Himstedt (2011), Himstedt and Noeske (2014) and Himstedt and Noeske (2015) made strong use of the representation theory of parabolic subgroups along with induction/restriction methods to compute decomposition numbers. Most recently the third author Paolini (2018) was able to compute most decomposition numbers of the groups $D_4(2^f)$ using the generic character table of $UD_4(2^f)$ which was computed in Goodwin et al. (2017); here and throughout $UY_r(q)$ denotes a Sylow $p$-subgroup of the group $Y_r(q)$ of type Y and rank $r$ defined over $\mathbb{F}_q$.

Our present work is a first step in computing the character tables of $UD_6(q)$ and $UE_6(q)$ which are intended to aid with the determination of the decomposition numbers of all finite groups of Lie type up to rank 8, in particular of those of exceptional type. A parametrization of the irreducible characters of each of the groups $UY_r(q)$ in the case where $p$ is at least the Coxeter number of $Y_r(q)$ and $r \leq 8$ (except $Y_r = E_8$) is provided in Goodwin et al. (2016b) by parametrizing of the coadjoint orbits of $UY_r(q)$ and then applying the Kirillov orbit method. The task of parametrizing the irreducible characters of $UY_r(q)$ for small primes, especially bad primes, presents much more complication. Our focus in the sequel is to consider $UY_r(p^f)$ for arbitrary $f \in \mathbb{Z}_{\geq 1}$, but taking $p$ strictly less than the Coxeter number of $Y_r(q)$.

The elements of $\mathrm{Irr}(U)$ where $G \cong F_4(q)$ with $q$ odd, in particular when $q = 3^f$ is a power of a bad prime, were parametrized in Goodwin et al. (2016a) by means of a recursive procedure, relying on basic character correspondences, which leads to a natural construction of characters via induction from linear characters of certain subgroups. Indeed the terminal points of our algorithm are certain subquotients of $U$, which we call cores. To construct the members of a family one starts with a core $Q$ with centre $Z$, and with a subset $\mathrm{Irr}(Q)_Z$ of $\mathrm{Irr}(Q)$ of characters which lie over the centre in such a way as to not contain any root subgroup of $Z$ in their kernels. For each element of $\mathrm{Irr}(Q)_Z$ one can trace back through the algorithm, and write down a unique character of $U$.

If $Q = Z$ is abelian, then we can simply extract the family and its parameters. We have been able to completely automate this part of the algorithm. If $Q \neq Z$, then we need to determine $\mathrm{Irr}(Q)_Z$, which in the situation of $UF_4(q)$ involved a manageable amount of case analysis. While the number of nonabelian cores for $UF_4(q)$ is 6, this number increases to 105 for $UE_6(q)$ and to several millions for $UE_8(q)$. We recall from Goodwin et al. (2017) and Himstedt et al. (2016) that the characters of $UE_6(q)$ are naturally partitioned into 833 families which are indexed by the antichains in the poset of positive roots. To each family we apply our algorithm, which naturally splits each family into collections of subfamilies. For type $E_8$ the first partition already leads to 25080 families. Thus it becomes clear that the generic character tables of the groups $UY_r(q)$ are best processed in a machine-readable format and ideally in a format that can be incorporated into Geck et al. (1996a), the computer algebra system which provides a platform for calculations with the generic character tables of finite groups of Lie type. Our main theorem thus takes the following form:

**Main Theorem.** *Let $q$ be a power of a prime $p$, let $G$ be a finite Chevalley group over $\mathbb{F}_q$ of type $D_6$ or $E_6$, and let $U$ be a Sylow $p$-subgroup of $G$. Then the irreducible characters of $U$ are completely parametrized. Each character can be obtained as an induced character of a linear character of a certain determined subgroup. In particular, if $v := q - 1$, we have*

1. $|\mathrm{Irr}(UD_6(q))| = \begin{cases} p_1(v), & \text{if } q \text{ is odd}, \\ p_1(v) + 3v^4(v^4 + 18v^3 + 63v^2 + 58v + 9), & \text{if } q = 2^f, \end{cases}$

2. $|\mathrm{Irr}(UE_6(q))| = \begin{cases} p_2(v), & \text{if } \gcd(q,6) = 1, \\ p_2(v) + v^6(v^2 + 6v + 12), & \text{if } q = 3^f, \\ p_2(v) + 3v^4(2v^4 + 26v^3 + 103v^2 + 317v + 45), & \text{if } q = 2^f, \end{cases}$

*where $p_1(v)$ and $p_2(v)$ are polynomial expressions in $v$ as in Tables 5 and 7 respectively.*

The parametrization is given as follows. For each character $\chi \in \mathrm{Irr}(U)$, we have been able to store electronically the subgroup $V$ and the values of the linear character $\lambda \in \mathrm{Irr}(V)$ such that $\chi = \mathrm{Ind}_V^U(\lambda)$ as in the Main Theorem, in terms of the root datum of $G$. The character $\lambda$ is, in turn, determined as the inflation of a linear character $\mu$ of an abelian quotient $\bar{V}$ of $V$. The labels in Tables 3 and 4 are given in terms of products of root subgroups or their diagonal subgroups defining the subquotient $\bar{V}$ of $U$. The degree of each character is then easily determined as $[U : V]$. We recall that the expressions of $p_1(v)$ and $p_2(v)$ had already been determined in Goodwin et al. (2014, Section 4) by means of a parametrization of the conjugacy classes of $\mathrm{UD}_6(q)$ when $p \geq 3$ and of $\mathrm{UE}_6(q)$ when $p \geq 5$ respectively.

We collect here further consequences of the parametrization in the above theorem. When $p$ is a bad prime for $\mathrm{E}_6$, then $\mathrm{Irr}(\mathrm{UE}_6(q))$ possesses characters of degree $q^i/2$ for $3 \leq i \leq 15$ if $p = 2$ and of degree $q^7/3$ if $p = 3$, whereas if $p = 2$ then $\mathrm{Irr}(\mathrm{UD}_6(q))$ possesses elements of degree $q^i/2$ for $3 \leq i \leq 11$, and also a family of characters of degree $q^{10}/4$ which is obtained by induction from the family $\mathcal{F}_3^{4,p=2}$ in Table 3 of characters of degree $q^6/4$; other irreducible character degrees are all powers of $q$. The numbers of characters of fixed degree are given in Tables 5 to 9. We easily check in these cases the validity of the generalization of Isaacs (2007, Conjecture B) in types different from A, which in turn refines (Lehrer, 1974, Conjecture 6.3), namely the numbers of irreducible characters of $U$ of fixed degree can always be expressed as polynomial expressions in $v$ with non-negative integral coefficients if $p$ is good. One also immediately deduces by the records in Table 8 for $\mathrm{Irr}(\mathrm{UE}_6(3^f))$ that an extension of such statement to bad primes would not hold; this is the only such instance for the groups $\mathrm{UY}_r(q)$ with Y of simply laced type and $r \leq 6$, namely in this case $\mathrm{UY}_r(q)$ is a natural quotient of $U$, and a parametrization for $\mathrm{Irr}(\mathrm{UY}_r(q))$ is obtained via the labels determined in our theorem. The actual complete list of families is available on the webpage of the third author Le et al. (2018) in both tabular and machine-readable format.

The obstruction to automating the parametrization of $\mathrm{Irr}(U)$ is the nonabelian cores mentioned above. Thus our focus in this paper is on nonabelian cores with a view towards automating these calculations as well. The total number of families of nonabelian cores that we have to consider is 27 for $\mathrm{D}_6$ and 105 for $\mathrm{E}_6$. Fortunately several cores are isomorphic, which reduces our problem to 7 isomorphism types of cores for $\mathrm{D}_6$ and 16 for $\mathrm{E}_6$ which are easily separated by a set of three invariants; this is proved in Section 3. Also certain cores are isomorphic to ones that we have seen in Goodwin et al. (2016a), which simplifies our work even further. In Section 4 we begin by proving a variant of our reduction lemma which serves as a foundational tool of our analysis of nonabelian cores. Also in this section we introduce the concept of a generalized root group which allows us to consider transversals which are well suited for our character correspondences, and the concept of a "circle quattern". In fact it is the latter concept which we believe will be crucial in automating the analysis of nonabelian cores. We illustrate all of this in our analysis of the nonabelian cores of $\mathrm{UD}_6(q)$ and $\mathrm{UE}_6(q)$ in Section 5. Here computer calculations are used extensively in order to find reasonable candidates for arms and legs as explained in Section 4, to compute stabilizers of central character extensions, and to obtain the branching determined by such stabilizers. The solutions of the equations which arise in this way and the determination of the corresponding labels for the characters in $\mathrm{Irr}(U)$ are essentially the only part which has been solved by hand. We collect the results of our analysis of the nonabelian cores in Tables 3 and 4.

To finish, we remark that for groups of rank higher than 6 the three invariants mentioned above are not strong enough to separate cores into isomorphism types, and we illustrate this with an example in $\mathrm{UE}_7(q)$. Also we remark here that for $\mathrm{UE}_8(q)$ the cardinality of the set of invariants of nonabelian cores is in the neighborhood of $2 \cdot 10^5$, and that the number of isomorphism types is around $4 \cdot 10^5$; again making clear the need for automation.

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## 2. Preliminaries

### *2.1. Characters of finite groups*

Let $G$ be a finite group. For each $g, h \in G$, we write $g^h := h^{-1}gh$ (respectively ${}^h g := hgh^{-1}$) for the right (respectively left) conjugation in $G$. The centre of $G$ is denoted by $Z(G)$. We usually denote by $\chi$ an irreducible character afforded by some representation. All characters considered in this work are ordinary. We denote by $\ker(\chi)$ the kernel of a character $\chi$, and by $Z(\chi)$ the centre of $\chi$. Moreover, we denote by $\mathrm{Irr}(G)$ the set of irreducible characters of $G$.

We recall some further notation and results on characters contained in Isaacs (1994). If $N \trianglelefteq G$, and $\chi \in \mathrm{Irr}(G/N)$, we denote by $\mathrm{Inf}_N^G(\chi)$ the inflation of $\chi$ to $G$. For $H \leq G$ and $\eta \in \mathrm{Irr}(H)$, we denote by $\mathrm{Ind}_H^G(\eta)$, or shortly $\eta^G$, the induction of the character $\eta$ from $H$ to $G$, and we define

$$\mathrm{Irr}(G \mid \eta) := \{\chi \in \mathrm{Irr}(G) \mid \langle \chi, \eta^G \rangle \neq 0\} = \{\chi \in \mathrm{Irr}(G) \mid \langle \chi|_H, \eta \rangle \neq 0\},$$

where $\langle\ ,\ \rangle$ is the usual scalar product of characters. Let $\chi_1$ and $\chi_2$ be two characters of $G$. The character $\chi_1 \otimes \chi_2$ denotes the tensor product of $\chi_1$ and $\chi_2$. If $H \leq G$, and $\chi \in \mathrm{Irr}(G)$ and $\psi \in \mathrm{Irr}(H)$, then $(\chi|_H \otimes \psi)^G = \chi \otimes \psi^G$. Let $\eta \in \mathrm{Irr}(N)$ with $N \trianglelefteq G$. For $g \in G$, we denote by ${}^g\eta$ the irreducible character of $N$ such that ${}^g\eta(x) := \eta(x^g)$ for every $x \in N$. The group $G$ naturally acts on $\mathrm{Irr}(N)$ by conjugation. Let us define the inertia subgroup of $\eta$ in $G$ by $I_G(\eta) := \{g \in G \mid {}^g\eta = \eta\}$. Then

$$\mathrm{Ind}_{I_G(\eta)}^G : \mathrm{Irr}(I_G(\eta) \mid \eta) \longrightarrow \mathrm{Irr}(G \mid \eta)$$

is a bijection of irreducible characters.

We also recall two useful facts from Goodwin et al. (2016a, §2.1). Let $N$ be a normal subgroup of $G$. For each subgroup $H$ of $G$ containing $N$, and each $\chi \in \mathrm{Irr}(H/N)$, we have that

$$\mathrm{Inf}_{G/N}^G \mathrm{Ind}_{H/N}^{G/N} \chi = \mathrm{Ind}_H^G \mathrm{Inf}_{H/N}^H \chi.$$

Moreover, let us assume that there exists $Z \leq Z(G)$ with $Z \cap N = 1$. If $\lambda \in \mathrm{Irr}(Z)$, then

$$\mathrm{Inf}_{G/N}^G : \mathrm{Irr}(G/N \mid \lambda) \longrightarrow \mathrm{Irr}(G \mid \mathrm{Inf}_Z^{ZN}(\lambda))$$

is a bijective map.

We finish by describing the set $\mathrm{Irr}(\mathbb{F}_q)$. Let us define $\phi : \mathbb{F}_q \to \mathbb{C}^\times$ by $\phi(t) := e^{2\pi i \,\mathrm{Tr}(t)/p}$ for all $t \in \mathbb{F}_q$, where $\mathrm{Tr} : \mathbb{F}_q \to \mathbb{F}_p \cong \mathbb{Z}_p$ is the trace map of the field extension $\mathbb{F}_q \mid \mathbb{F}_p$. It is easy to check that $\phi \in \mathrm{Irr}(\mathbb{F}_q)^\times$, and in fact $\phi_b := \phi \circ m_b \in \mathrm{Irr}(\mathbb{F}_q)$ for every $b \in \mathbb{F}_q$, where $m_b : \mathbb{F}_q \to \mathbb{F}_q$ denotes the multiplication by $b$ in $\mathbb{F}_q$. Notice that $\phi_{b_1} = \phi_{b_2}$ implies $b_1 = b_2$ for $b_1, b_2 \in \mathbb{F}_q$. Hence $\mathrm{Irr}(\mathbb{F}_q) = \{\phi_b \mid b \in \mathbb{F}_q\}$. Moreover, it is easy to see that if $a \in \mathbb{F}_q^\times$, then $\ker(\phi_a) = \{a^{p-1}t^p - t \mid t \in \mathbb{F}_q\}$.

### *2.2. Simple algebraic groups and Frobenius morphisms*

We refer to Digne and Michel (1991) and Malle and Testerman (2011) for basic properties and definitions of finite reductive groups. Let $q := p^f$, where $p$ is a prime and $f \in \mathbb{Z}_{>0}$. Let $\mathbb{F}_q$ be a general finite field with $q$ elements, and let $k := \overline{\mathbb{F}}_p$ be an algebraic closure of $\mathbb{F}_q$. We denote by $\mathbf{G}$ a simple algebraic group over the field $k$.

Let $F : \mathbf{G} \to \mathbf{G}$ be a standard Frobenius morphism. Let $\mathbf{T}$ be a maximal torus of $\mathbf{G}$ such that $F(\mathbf{T}) = \mathbf{T}$, and let $\mathbf{B}$ be a Borel subgroup of $\mathbf{G}$ containing $\mathbf{T}$ such that $F(\mathbf{B}) = \mathbf{B}$. Let $\mathbf{U}$ be the unipotent radical of $\mathbf{B}$. Here, $\mathbf{B} = N_{\mathbf{G}}(\mathbf{U}) = \mathbf{TU}$. From now on, we fix such $F$-stable subgroups $\mathbf{T}$, $\mathbf{U}$ and $\mathbf{B}$. The group $G := \mathbf{G}^F$ of fixed points of $\mathbf{G}$ under $F$ is a finite reductive group. Further, we set $B := \mathbf{B}^F$, $T := \mathbf{T}^F$, and $U := \mathbf{U}^F$. Here, we have $B = N_G(U) = T \ltimes U$, and $U$ is a Sylow $p$-subgroup of $G$. All subgroups $B$, $T$, and $U$ are fixed for the rest of the work.

Let $\Phi$ denote the root system associated to $\mathbf{G}$ with respect to $\mathbf{T}$, and let $\Pi := \{\alpha_1, \dots, \alpha_r\}$ be the set of simple roots of $\Phi$, where $r$ is the rank of $\Phi$. Let $\Phi^+ \subseteq \Phi$ denote the set of positive roots in $\Phi$, and let $m$ be the number of positive roots. We fix a total ordering on $\Phi^+ = \{\alpha_1, \dots, \alpha_m\}$ by refining the partial order on $\Phi^+$, defined by $\alpha < \beta$ if $\beta - \alpha$ is a sum of simple roots; this agrees with the ordering in GAP (2016). If $\Phi$ is of type Y and rank $r$, we sometimes denote $U$ more explicitly by $\mathrm{UY}_r(q)$.

For each $\alpha \in \Phi^+$, there exist an $F$-stable subgroup $\mathbf{U}_\alpha \subseteq \mathbf{U}$ and an isomorphism $x_\alpha : k \to \mathbf{U}_\alpha$, such that

$$\mathbf{U}_\alpha := \{x_\alpha(t) \mid t \in k\} \cong (k, +), \qquad \text{and} \qquad X_\alpha := \mathbf{U}_\alpha^F = \{x_\alpha(t) \mid t \in \mathbb{F}_q\} \cong (\mathbb{F}_q, +).$$

The subgroup $X_\alpha$ of $G$ is called a *root subgroup*, and an element of the form $x_\alpha(t)$ is called a *root element*. We often abbreviate and write $X_i$ for $X_{\alpha_i}$ and $x_i$ for $x_{\alpha_i}$. The group $U$ is the product of all root subgroups labelled by positive roots, and each element of $U$ can be uniquely written as a product $x_1(t_1) \cdots x_m(t_m)$ for some $t_1, \cdots, t_m \in \mathbb{F}_q$. A presentation for $U$ is given by the Chevalley relations

$$[x_\alpha(s), x_\beta(r)] = \prod_{i,j \in \mathbb{Z}_{>0} \mid i\alpha + j\beta \in \Phi^+} x_{i\alpha + j\beta}(c_{i,j}^{\alpha,\beta}(-r)^j s^i) \tag{1}$$

for every $r, s \in \mathbb{F}_q$ and $\alpha, \beta \in \Phi^+$, and for some $c_{i,j}^{\alpha,\beta} \in \mathbb{Z} \setminus \{0\}$, called *Lie structure constants*. As proved in Carter (1989, Section 5.2), the parametrizations of the root subgroups can be chosen so that the structure constants $c_{i,j}^{\alpha,\beta}$ are always $\pm 1$, $\pm 2$, $\pm 3$, where $\pm 2$ occurs only for $G$ of types $\mathrm{B}_r$, $\mathrm{C}_r$, $\mathrm{F}_4$ or $\mathrm{G}_2$, and $\pm 3$ only occurs for $G$ of type $\mathrm{G}_2$. The signs are determined by fixing the ones corresponding to the so-called extraspecial pairs of roots; our choice agrees with the records in the computer algebra system (Bosma et al., 1997).

We finally recall the definition of bad and very bad primes. The prime $p$ is *bad* for $G$ if it divides one of the coefficients of the longest root of $\Phi^+$ in its decomposition as sum of simple roots. We say that $p$ is *very bad* for $G$ if $p$ divides one of the constants $c_{i,j}^{\alpha,\beta}$ in Equation (1). As the name suggests, very bad primes turn out to be bad primes. A prime is *good* for $G$ if it is not bad. The very bad primes are the prime 2 in types $\mathrm{B}_r$, $\mathrm{C}_r$, $\mathrm{F}_4$ and $\mathrm{G}_2$, and the prime 3 in $\mathrm{G}_2$. The bad primes which are not very bad are the prime 2 in types $\mathrm{D}_r$, $\mathrm{E}_6$, $\mathrm{E}_7$ and $\mathrm{E}_8$, the prime 3 in types $\mathrm{F}_4$, $\mathrm{E}_6$, $\mathrm{E}_7$ and $\mathrm{E}_8$, and the prime 5 in type $\mathrm{E}_8$.

### *2.3. Quattern groups*

We now recall some properties that link the structure of $\Phi^+$ with that of $U$. If $\mathcal{A} = \{\alpha_{i_1}, \dots, \alpha_{i_k}\}$ is a subset of $\Phi^+$ where $i_1 < \cdots < i_k$, we define

$$X_{\mathcal{A}} := \prod_{j=1}^{k} X_{\alpha_{i_j}}.$$

This is in general not always a subgroup, but it will be in all cases of our interest.

We recall some definitions and properties from Himstedt et al. (2016). We say that $\mathcal{P}$ is a *pattern* in $\Phi^+$ if $\alpha, \beta \in \mathcal{P}$ and $\alpha + \beta \in \Phi^+$ imply $\alpha + \beta \in \mathcal{P}$. Patterns are also known as *closed subsets* of $\Phi^+$, see for example (Malle and Testerman, 2011, Definition 13.2). It is easy to check, with no restrictions on the prime $p$, that if $\mathcal{P}$ is a pattern, then $X_{\mathcal{P}}$ is a subgroup of $U$. If $p$ is not a very bad prime for $\Phi^+$, the converse also holds. For very bad primes the converse does not hold in general. For example, if $p=2$ and $\alpha_2$ is the simple short root in type $\mathrm{B}_2$, then $X_{\alpha_2}X_{\alpha_1+\alpha_2}$ is a subgroup of $\mathrm{UB}_2(2^f)$, but $\{\alpha_2, \alpha_1+\alpha_2\}$ is not a pattern.

Let $\mathcal{K}, \mathcal{P}$ be two patterns with $\mathcal{K} \subseteq \mathcal{P}$. We say that $\mathcal{K}$ is *normal* in $\mathcal{P}$, denoted as $\mathcal{K} \trianglelefteq \mathcal{P}$, if for all $\delta \in \mathcal{K}$ and $\beta \in \mathcal{P}$, $\delta+\beta \in \mathcal{P}$ implies $\delta+\beta \in \mathcal{K}$. A subset $\mathcal{S} \subseteq \Phi^+$ is called a *quattern* if $\mathcal{S} = \mathcal{P} \setminus \mathcal{K}$ for some pattern $\mathcal{P}$ and $\mathcal{K} \trianglelefteq \mathcal{P}$. If $\mathcal{K} \trianglelefteq \mathcal{P}$, then we have that $X_{\mathcal{K}} \trianglelefteq X_{\mathcal{P}}$. If $p$ is not a very bad prime for $\Phi^+$, then $X_{\mathcal{K}} \trianglelefteq X_{\mathcal{P}}$ for two patterns $\mathcal{K} \subseteq \mathcal{P}$ also implies $\mathcal{K} \trianglelefteq \mathcal{P}$. Again, this is not true in type $\mathrm{B}_2$ when $p=2$, namely $X_{\alpha_1+\alpha_2} \trianglelefteq \mathrm{UB}_2(2^f) = X_{\Phi^+}$, but $\{\alpha_1+\alpha_2\}$ is not a normal pattern in $\Phi^+$.

*From now on, and for the rest of the work, we assume that $p$ is not a very bad prime for $G$.* Under this assumption, the definitions and statements in the sequel determine a consistent dictionary from commutator relations of root elements in subquotients of $U$ to additive relations in quatterns of $\Phi^+$, and vice versa.

Given a quattern $\mathcal{S} \subseteq \Phi^+$ such that $\mathcal{S} = \mathcal{P} \setminus \mathcal{K}$, we define the *quattern group* $X_{\mathcal{S}}$ associated to $\mathcal{S}$ by

$$X_{\mathcal{S}} := X_{\mathcal{P}}/X_{\mathcal{K}}.$$

This subquotient of $U$ is well-defined, in the sense that if $\mathcal{S} = \mathcal{P}' \setminus \mathcal{K}'$ for $\mathcal{P}'$ a quattern and $\mathcal{K}' \trianglelefteq \mathcal{P}'$, then $X_{\mathcal{S}} \cong X_{\mathcal{S}'}$.

If $\mathcal{S}$ is a quattern, we define

$$\mathcal{Z}(\mathcal{S}) := \{\gamma \in \mathcal{S} \mid \gamma+\alpha \notin \mathcal{S} \text{ for all } \alpha \in \mathcal{S}\}$$

the set of central roots in $\mathcal{S}$, and

$$\mathcal{D}(\mathcal{S}) := \{\gamma \in \mathcal{Z}(\mathcal{S}) \mid \alpha+\beta \neq \gamma \text{ for all } \alpha, \beta \in \mathcal{S}\}$$

the set of roots parametrizing the root subgroups which are direct factors in $X_{\mathcal{S}}$. We have

$$Z(X_{\mathcal{S}}) = X_{\mathcal{Z}(\mathcal{S})} \qquad \text{and} \qquad X_{\mathcal{S}} = X_{\mathcal{S}\setminus\mathcal{D}(\mathcal{S})} \times X_{\mathcal{D}(\mathcal{S})}.$$

We define the set of irreducible characters of $X_{\mathcal{S}}$ with central root support $\mathcal{Z} \subseteq \mathcal{Z}(\mathcal{S})$ by

$$\mathrm{Irr}(X_{\mathcal{S}})_{\mathcal{Z}} := \{\chi \in \mathrm{Irr}(X_{\mathcal{S}}) \mid X_\alpha \not\subseteq \ker(\chi) \text{ for all } \alpha \in \mathcal{Z}\}.$$

Hence we have

$$\mathrm{Irr}(X_{\mathcal{S}})_{\mathcal{Z}} = \bigsqcup_{\lambda \in \mathrm{Irr}(X_{\mathcal{Z}})_{\mathcal{Z}}} \mathrm{Irr}(X_{\mathcal{S}} \mid \lambda), \tag{2}$$

and it is easy to see that

$$\sum_{\chi \in \mathrm{Irr}(X_{\mathcal{S}})_{\mathcal{Z}}} \chi(1)^2 = q^{|\mathcal{S}\setminus\mathcal{Z}|}(q-1)^{|\mathcal{Z}|}. \tag{3}$$

The importance of studying quatterns comes from the fact that we can partition $\mathrm{Irr}(U)$ into families of irreducible characters of quattern groups $X_{\mathcal{S}}$ with central root support a certain $\mathcal{Z} \subseteq \mathcal{Z}(\mathcal{S})$. More precisely, let $\Sigma$ denote an antichain of $\Phi^+$, that is, a subset of $\Phi^+$ such that

$$\alpha, \beta \in \Sigma \text{ and } \alpha \neq \beta \Longrightarrow \alpha \not\leq \beta \text{ and } \beta \not\leq \alpha.$$

The subset $\mathcal{K}_\Sigma$ defined by

$$\mathcal{K}_\Sigma := \{\beta \in \Phi^+ \mid \beta \not\leq \gamma \text{ for all } \gamma \in \Sigma\}$$

is a normal subset of $\Phi^+$. We define the *standard quattern* $\mathcal{S}_\Sigma$ associated to $\Sigma$ by $\mathcal{S}_\Sigma := \Phi^+ \setminus \mathcal{K}_\Sigma$. Notice that $\Sigma = \mathcal{Z}(\mathcal{S}_\Sigma)$. Finally, we define

$$\mathrm{Irr}(U)_\Sigma := \{\mathrm{Inf}^U_{X_{\mathcal{S}_\Sigma}}(\chi) \mid \chi \in \mathrm{Irr}(X_{\mathcal{S}_\Sigma})_\Sigma\}.$$

We can now state the partition of irreducible characters previously announced.

**Proposition 1** *(Himstedt et al. (2016), Proposition 5.16). We have that*

$$\mathrm{Irr}(U) = \bigsqcup_{\Sigma \text{ antichain in } \Phi^+} \mathrm{Irr}(U)_\Sigma.$$

*2.4. Cores and the Reduction algorithm*

In order to describe the sets of the form $\mathrm{Irr}(U)_\Sigma$ for an antichain $\Sigma$, we use the following procedure, explained in Goodwin et al. (2016a, Section 3), and implemented in GAP (2016). Our goal is to reduce from the study of some $\mathrm{Irr}(X_\mathcal{S})_\mathcal{Z}$, with $\mathcal{Z} \subseteq \mathcal{Z}(\mathcal{S})$, to the study of $\mathrm{Irr}(X_{\mathcal{S}'})_{\mathcal{Z}'}$, with $\mathcal{Z}' \subseteq \mathcal{Z}(\mathcal{S}')$, such that $|\mathcal{S}'| \lneq |\mathcal{S}|$.

We recall the following result.

**Proposition 2** *(Goodwin et al. (2016a), Lemma 3.1). Let $\mathcal{S} := \mathcal{P} \setminus \mathcal{K}$ be a quattern, let $\mathcal{Z} \subseteq \mathcal{Z}(\mathcal{S})$ and let $\gamma \in \mathcal{Z}$. Suppose that there exist $\delta, \beta \in \mathcal{S} \setminus \{\gamma\}$, such that:*

(i) $\delta + \beta = \gamma$,
(ii) $\alpha + \alpha' \neq \beta$ *for every* $\alpha, \alpha' \in \mathcal{S}$, *and*
(iii) $\delta + \alpha \notin \mathcal{S}$ *for every* $\alpha \in \mathcal{S} \setminus \{\beta\}$.

*Let $\mathcal{P}' := \mathcal{P} \setminus \{\beta\}$ and $\mathcal{K}' := \mathcal{K} \cup \{\delta\}$. Then we have that $\mathcal{S}' := \mathcal{P}' \setminus \mathcal{K}'$ is a quattern with $X_{\mathcal{S}'} \cong X_{\mathcal{P}'}/X_{\mathcal{K}'}$, and the map*

$$\begin{aligned}\mathrm{Irr}(X_{\mathcal{S}'})_\mathcal{Z} &\to \mathrm{Irr}(X_\mathcal{S})_\mathcal{Z}\\ \chi &\mapsto \mathrm{Ind}^\beta \, \mathrm{Inf}_\delta \, \chi\end{aligned}$$

*is a bijection of irreducible characters.*

The Reduction algorithm has been presented in Goodwin et al. (2016a, Algorithm 3.3) by applying repeatedly Proposition 2 to $\mathcal{S}_\Sigma$. We summarize it in this section. In particular, we obtain tuples of the form $\mathfrak{C} = (\mathcal{S}, \mathcal{Z}, \mathcal{A}, \mathcal{L}, \mathcal{K})$ of positive roots, called *cores*, and sets $\mathfrak{O}_1$, $\mathfrak{O}_2$ containing tuples of this form, such that we have a bijection

$$\mathrm{Irr}(U)_\Sigma \longleftrightarrow \bigsqcup_{\mathfrak{C} \in \mathfrak{O}_1} \mathrm{Irr}(X_\mathcal{S})_\mathcal{Z} \sqcup \bigsqcup_{\mathfrak{C} \in \mathfrak{O}_2} \mathrm{Irr}(X_\mathcal{S})_\mathcal{Z}.$$

The quattern group $X_\mathcal{S}$ is abelian if and only if $\mathfrak{C} \in \mathfrak{O}_1$, in which case we call $\mathfrak{C}$ an *abelian core*; if $\mathfrak{C} \in \mathfrak{O}_2$, we call $\mathfrak{C}$ a *nonabelian core*. In the sequel we sometimes drop the whole notation $(\mathcal{S}, \mathcal{Z}, \mathcal{A}, \mathcal{L}, \mathcal{K})$ for $\mathfrak{C}$, and just refer to the pair $(\mathcal{S}, \mathcal{Z})$ or to the quattern $\mathcal{S}$.

The abelian and nonabelian cores depend on two kind of choices in $\Phi^+$, namely the ordering of the simple roots and the choice of maximal/minimal roots in $\mathcal{R}(\mathcal{S})$ and $\mathcal{Z}(\mathcal{S}) \setminus (\mathcal{Z} \cup \mathcal{D}(\mathcal{S}))$ in Step 2 and Step 3 of the algorithm respectively. The number of nonabelian cores does actually change with different choices in $\Phi^+$, but the difference is small, and the behavior of the nonabelian cores and the insight required for their study remain the same in the examined cases.

The reduction procedure is as follows. At each step of the procedure, we examine a tuple $(\mathcal{S}, \mathcal{Z}, \mathcal{A}, \mathcal{L}, \mathcal{K})$, where the set $\mathcal{S}$ is a quattern with $\mathcal{Z} \subseteq \mathcal{Z}(\mathcal{S})$, the set $\mathcal{A}$ (respectively $\mathcal{L}$) keeps a record of the roots of the form $\beta$ (respectively $\delta$) at each step of the application of Proposition 2, and the set $\mathcal{K}$ keeps a record of the roots indexing root subgroups in the associated quattern group. The output of this procedure is the sets $\mathfrak{O}_1$ and $\mathfrak{O}_2$. We use in the sequel the notation $\mathrm{Ind}^\beta$, $\mathrm{Inf}_\delta$ and $\mathrm{Ind}^\mathcal{A}$, $\mathrm{Inf}_\mathcal{K}$ defined in Goodwin et al. (2016a, §2.3).

**Table 1**
The number of nonabelian cores in $U$, when $G$ is of rank 7 or less and $p$ is not a very bad prime for $G$.

| $Y_{r\leq 3}$ | $B_4$ | $C_4$ | $D_4$ | $F_4$ | $B_5$ | $C_5$ | $D_5$ | $B_6$ | $C_6$ | $D_6$ | $E_6$ | $B_7$ | $C_7$ | $D_7$ | $E_7$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 0 | 1 | 6 | 7 | 1 | 6 | 36 | 16 | 27 | 105 | 245 | 129 | 160 | 3401 |

**Setup.** We initialize by putting $\mathcal{S}=\mathcal{S}_\Sigma$, $\mathcal{Z}=\mathcal{Z}(\mathcal{S}_\Sigma)$, and $\mathcal{A}=\mathcal{L}=\mathcal{K}=\emptyset$ and $\mathfrak{O}_1=\mathfrak{O}_2=\emptyset$.

Let us now assume that $\mathfrak{C}=(\mathcal{S},\mathcal{Z},\mathcal{A},\mathcal{L},\mathcal{K})$ is constructed and taken into examination at this step of the procedure.

**Step 1.** Let us assume that $\mathcal{S}=\mathcal{Z}(\mathcal{S})$. Then $X_\mathcal{S}$ is abelian, and we can easily parametrize $\mathrm{Irr}(X_\mathcal{S})_\mathcal{Z}$. Namely, if $\mathcal{Z}=\{\alpha_{i_1},\dots,\alpha_{i_m}\}$ and $\mathcal{S}\setminus\mathcal{Z}=\{\alpha_{j_1},\dots,\alpha_{j_n}\}$, then

$$\mathrm{Irr}(X_\mathcal{S})_\mathcal{Z}=\left\{\chi^{\underline{a}}_{\underline{b}}\mid a=(a_{i_1},\dots,a_{i_m})\in(\mathbb{F}_q^\times)^m, b=(b_{j_1},\dots,b_{j_n})\in(\mathbb{F}_q^\times)^n\right\},$$

with $\chi^{\underline{a}}_{\underline{b}}=\mathrm{Ind}^\mathcal{A}\,\mathrm{Inf}_\mathcal{K}\,\lambda^{\underline{a}}_{\underline{b}}$ as explained after (Goodwin et al., 2016a, Lemma 3.5). We add the element $\mathfrak{C}$ to the set $\mathfrak{O}_1$.

**Step 2.** Let $\mathcal{S}\neq\mathcal{Z}(\mathcal{S})$, and let $\mathcal{R}(\mathcal{S})$ be the set of pairs of the form $(\beta,\delta)$ satisfying the assumptions of Proposition 2. Assume $\mathcal{R}(\mathcal{S})\neq\emptyset$. We choose one particular element $(\beta,\delta)\in\mathcal{R}(\mathcal{S})$, namely we choose $\delta$ to be maximal with respect to the linear ordering fixed on $\Phi^+$, and if $(\beta_1,\delta),\dots,(\beta_s,\delta)$ are in $\mathcal{R}(\mathcal{S})$, we choose $\beta_i$ minimal with respect to the linear ordering on $\Phi^+$. Let us put $\mathfrak{C}':=(\mathcal{S}',\mathcal{Z}',\mathcal{A}',\mathcal{L}',\mathcal{K}')$, with

$$\mathcal{S}'=\mathcal{S}\setminus\{\beta,\delta\},\quad \mathcal{Z}'=\mathcal{Z},\quad \mathcal{A}'=\mathcal{A}\cup\{\beta\},\quad \mathcal{L}'=\mathcal{L}\cup\{\delta\},\quad \mathcal{K}'=\mathcal{K}\cup\{\delta\}.$$

Then we have that

$$\mathrm{Ind}^\beta\,\mathrm{Inf}_\delta:\mathrm{Irr}(X_{\mathcal{S}'})_\mathcal{Z}\longrightarrow\mathrm{Irr}(X_\mathcal{S})_\mathcal{Z}$$

is a bijection of irreducible characters. We continue by going back to Step 1 with $\mathfrak{C}'$ in place of $\mathfrak{C}$.

**Step 3.** Let $\mathfrak{C}$ be such that $\mathcal{S}\neq\mathcal{Z}(\mathcal{S})$ and $\mathcal{R}(\mathcal{S})=\emptyset$. Assume that $\mathcal{Z}(\mathcal{S})\setminus(\mathcal{Z}\cup\mathcal{D}(\mathcal{S}))\neq\emptyset$, and let $\gamma$ be its maximal element with respect to the usual linear ordering. Then we have that

$$\mathrm{Irr}(X_\mathcal{S})_\mathcal{Z}=\mathrm{Irr}(X_{\mathcal{S}\setminus\{\gamma\}})_\mathcal{Z}\sqcup\mathrm{Irr}(X_\mathcal{S})_{\mathcal{Z}\cup\{\gamma\}}.$$

Correspondingly, we continue by going back to Step 1 with each of the tuples

$$\mathfrak{C}':=(\mathcal{S}\setminus\{\gamma\},\mathcal{Z},\mathcal{A},\mathcal{L},\mathcal{K}\cup\{\gamma\})\quad\text{and}\quad\mathfrak{C}'':=(\mathcal{S},\mathcal{Z}\cup\{\gamma\},\mathcal{A},\mathcal{L},\mathcal{K}).$$

**Step 4.** Let $\mathcal{S}$ be such that $\mathcal{S}\neq\mathcal{Z}(\mathcal{S})$, $\mathcal{R}(\mathcal{S})=\emptyset$ and $\mathcal{Z}(\mathcal{S})\setminus(\mathcal{Z}\cup\mathcal{D}(\mathcal{S}))=\emptyset$. Then $X_\mathcal{S}$ is not abelian and it cannot be reduced further using Proposition 2. We add $\mathfrak{C}$ to $\mathfrak{O}_2$. The set $\mathrm{Irr}(X_\mathcal{S})$ has to be investigated with different methods.

This algorithm has been implemented in GAP (2016) for all groups of rank 7 or less. The numbers of nonabelian cores in each case are recorded in Table 1. The convention for the choice of $(\beta,\delta)$ as in Step 2 is slightly different from the one in Goodwin et al. (2016a), hence there are some different numbers of nonabelian cores for ranks 5 or higher.

We notice that if $\mathcal{D}(\mathcal{S})\neq\emptyset$, then

$$X_\mathcal{S}=X_{\mathcal{S}\setminus\mathcal{D}(\mathcal{S})}\times X_{\mathcal{D}(\mathcal{S})},\quad\text{hence}\quad\mathrm{Irr}(X_\mathcal{S})_\mathcal{Z}=\mathrm{Irr}(X_{\mathcal{S}\setminus\mathcal{D}(\mathcal{S})})_{\mathcal{Z}\setminus\mathcal{D}(\mathcal{S})}\times\mathrm{Irr}(X_{\mathcal{D}(\mathcal{S})})_{\mathcal{Z}\cap\mathcal{D}(\mathcal{S})},$$

and $\mathrm{Irr}(X_{\mathcal{D}(\mathcal{S})})_{\mathcal{Z}\cap\mathcal{D}(\mathcal{S})}$ is easily parametrized, as $X_{\mathcal{D}(\mathcal{S})}$ is a direct product of its root subgroups. Then we assume in the sequel that we have a record of the set $\mathcal{D}(\mathcal{S})$, and by slight abuse we identify $\mathcal{S}$ with $\mathcal{S}\setminus\mathcal{D}(\mathcal{S})$ and $\mathcal{Z}$ with $\mathcal{Z}\setminus\mathcal{D}(\mathcal{S})$.

**Table 2**
The numbers of $[z, m, c]$-cores in types $D_6$ and $E_6$.

| $D_6$ | |
|---|---|
| $[z, m, c]$ | # |
| [3, 9, 6] | 7 |
| [3, 10, 9] | 15 |
| [4, 18, 18] | 1 |
| [4, 21, 28] | 1 |
| [4, 24, 43] | 1 |
| [5, 18, 18] | 1 |
| [6, 19, 20] | 1 |

| $E_6$ | | | |
|---|---|---|---|
| $[z, m, c]$ | # | $[z, m, c]$ | # |
| [3, 9, 6] | 24 | [3, 10, 9] | 45 |
| [4, 8, 4] | 11 | [5, 10, 5] | 1 |
| [5, 12, 8] | 2 | [5, 15, 11] | 3 |
| [5, 16, 15] | 1 | [5, 20, 25] | 1 |
| [5, 21, 30] | 1 | [6, 12, 6] | 5 |
| [6, 13, 7] | 1 | [6, 14, 8] | 3 |
| [6, 15, 12] | 2 | [6, 16, 12] | 1 |
| [6, 17, 17] | 1 | [7, 15, 9] | 3 |

## 3. Isomorphism of nonabelian cores

The behavior of the nonabelian cores is determined by the relations in the underlying quattern $\mathcal{S}$. We want to record some invariants associated to such quatterns.

**Definition 3.** Let $\mathfrak{C} = (\mathcal{S}, \mathcal{Z}, \mathcal{A}, \mathcal{L}, \mathcal{K})$ be a nonabelian core. We say that $\mathfrak{C}$ is a $[z, m, c]$-*core* if

- $|\mathcal{Z}| = z$,
- $|\mathcal{S}| = m$, and
- there are $c$ triples $(i, j, k)$, with $i < j$ and $\alpha_i, \alpha_j, \alpha_k \in \mathcal{S}$, such that $\alpha_i + \alpha_j = \alpha_k$.

The triple $[z, m, c]$ associated to $\mathfrak{C}$ is called the *form* of the nonabelian core $\mathfrak{C}$.

The focus of this section is to show that a triple $[z, m, c]$ uniquely determines the isomorphism type of a core $\mathfrak{C}$ in simply laced type when the rank of $G$ is 6 or less.

**Theorem 4.** *Let $\Phi^+ = D_6$ or $\Phi^+ = E_6$, and let $\mathcal{S}$, $\mathcal{S}'$ be two quatterns of $\Phi^+$ corresponding to nonabelian cores of the form $[z, m, c]$. Then we have that $X_{\mathcal{S}} \cong X_{\mathcal{S}'}$.*

The above theorem is proved computationally, following the procedure explained below. We start by recording in Table 2 the number of occurrences of each form of nonabelian cores in types $D_6$ and $E_6$. Notice that in these cases, for each $z$ and $m$ there exists a unique $c$ such that $[z, m, c]$ is a nonabelian core, then by Theorem 4 the knowledge of $|\mathcal{S}|$ and $|\mathcal{Z}|$ tells apart the isomorphism type of $X_{\mathcal{S}}$.

We recall the lower central series of $X_{\mathcal{S}}$. For all $k \in \mathbb{Z}_{>0}$, the $k$-th member of the lower central series is denoted by $X_{\mathcal{S}}^{(k)}$, recursively defined by $X_{\mathcal{S}}^{(1)} := X_{\mathcal{S}}$ and $X_{\mathcal{S}}^{(k+1)} := [X_{\mathcal{S}}, X_{\mathcal{S}}^{(k)}]$. Notice that $X_{\mathcal{S}}^{(k)}$ is always a quattern group, and $X_{\mathcal{S}}^{(k+1)} < X_{\mathcal{S}}^{(k)}$ whenever $X_{\mathcal{S}}^{(k)}$ is nontrivial. We denote by $d$ the *nilpotency class* of $X_{\mathcal{S}}$, that is the unique positive integer such that $X_{\mathcal{S}}^{(d)} \neq 1$ and $X_{\mathcal{S}}^{(d+1)} = 1$.

The isomorphism test for cores is mainly based on the concept of local height, which is defined as follows.

**Definition 5.** Let $\mathfrak{C} = (\mathcal{S}, \mathcal{Z}, \mathcal{A}, \mathcal{L}, \mathcal{K})$ be a core. A root $\alpha \in \mathcal{S}$ is said to have *local height* $k$ if $X_\alpha \subseteq X_{\mathcal{S}}^{(k)}$ and $X_\alpha \nsubseteq X_{\mathcal{S}}^{(k+1)}$.

The roots in $\mathcal{S}$ are then partitioned into their local height classes as $[\mathcal{S}_1, \dots, \mathcal{S}_d]$, where $\mathcal{S}_k$ denotes the set of all roots of local height $k$ in $\mathcal{S}$ for every $k = 1, \dots, d$. For any two cores $\mathfrak{C}$ and $\mathfrak{C}'$, it is clear that their quattern groups $X_{\mathcal{S}}$ and $X_{\mathcal{S}'}$ are not isomorphic if there exists a $k \geq 1$ such that $|\mathcal{S}_k| \neq |\mathcal{S}'_k|$.

Under the assumption that $\Phi^+$ is of simply laced type, for a quattern $\mathcal{S}$ we have that $X_{\mathcal{S}}^{(k+1)} = [X_{\mathcal{S}}, X_{\mathcal{S}}^{(k)}]$ if and only if $\mathcal{S}_{k+1} = \mathcal{S} + \mathcal{S}_k$; hence every $\delta \in \mathcal{S}_{k+1}$ can be written as $\alpha + \beta$, with $\alpha \in \mathcal{S}_k$ and $\beta \in \mathcal{S}_1$. We say that a root $\alpha \in \mathcal{S}_k$ is a *lower bound* of a root $\delta \in \mathcal{S}_{k+1}$ if there exists a root $\beta \in \mathcal{S}_1$

such that $\alpha+\beta=\delta$. The set $\mathcal{S}$ naturally inherits a poset structure from $\Phi^+$. The suprema of $\mathcal{S}$ are the elements of $\mathcal{Z}(\mathcal{S})$. Notice that if $\mathcal{S}$ is a disconnected poset with $\mathcal{S}=\mathcal{S}_a \sqcup \mathcal{S}_b$, then $X_{\mathcal{S}} = X_{\mathcal{S}_a} \times X_{\mathcal{S}_b}$; without loss of generality, we assume from now on that $\mathcal{S}$ is connected.

To show that the quattern groups of two cores of the form $[z,m,c]$ corresponding to $\mathcal{S}$ and $\mathcal{S}'$ are isomorphic, we proceed as follows.

(a) We find a *poset isomorphism* between $\mathcal{S}$ and $\mathcal{S}'$, that is, a bijection $\rho:\mathcal{S}\to\mathcal{S}'$ such that $\alpha\le\beta \Leftrightarrow \rho(\alpha)\le\rho(\beta)$ for every $\alpha,\beta\in\mathcal{S}$. Moreover, we require that if $\alpha,\beta\in\mathcal{S}$ are such that $\alpha+\beta\in\mathcal{S}'$, then $\rho(\alpha+\beta)=\rho(\alpha)+\rho(\beta)$, and that $\alpha+\beta=\gamma \Leftrightarrow \rho(\alpha)+\rho(\beta)=\rho(\gamma)$. If such a map $\rho$ exists, we go to step (b).

(b) Let $\rho$ be a poset isomorphism between $\mathcal{S}$ and $\mathcal{S}'$ as in (a). We try to lift $\rho$ to a group homomorphism $\varphi: X_{\mathcal{S}} \to X_{\mathcal{S}'}$ by checking the compatibility of the signs in the commutator relations between root elements.

Let $\mathcal{S}=[\mathcal{S}_1,\dots,\mathcal{S}_d]$ and $\mathcal{S}'=[\mathcal{S}'_1,\dots,\mathcal{S}'_d]$ be two quatterns corresponding to a nonabelian core of the form $[z,m,c]$, with $|\mathcal{S}_k|=|\mathcal{S}'_k|$ for all $k=1,\dots,d$. For constructing a poset isomorphism $\rho$ and lifting it up as a group isomorphism $\varphi$, we use the following algorithm.

**Setup and base step.** For (a), we start with a setup of roots at the first local height layers of $\mathcal{S}_1$ and $\mathcal{S}'_1$, i.e. we choose a bijection $\rho$ from $\mathcal{S}_1$ to $\mathcal{S}'_1$.

For (b), we set $\varphi(x_\alpha(t)) := x_{\rho(\alpha)}(\pm t)$ for all $\alpha\in\mathcal{S}_1$ and all $t\in\mathbb{F}_q$. This gives a setting for the first local height layer. Notice that the chosen sign '$+$' works in types $D_6$ and $E_6$, instead of trying every choice of the signs '$+$' and '$-$'.

**Iterative step.** Assume that we constructed the $k$-th local height layer map, and $\mathcal{S}_{k+1}$ and $\mathcal{S}'_{k+1}$ are nonempty. We construct the maps $\rho$ for $(k+1)$-th local height layers, and $\varphi$ for root groups at $(k+1)$-th local height layers.

For (a), let $\delta\in\mathcal{S}_{k+1}$, where $\rho(\delta)$ is yet to be defined. We find $\alpha\in\mathcal{S}_k$ and $\beta\in\mathcal{S}_1$ such that $\delta=\alpha+\beta$. If $\rho(\alpha)+\rho(\beta)\notin\mathcal{S}'_{k+1}$, then this construction ends here, and we return *no solution* for the choice of $\rho$ from the first local height layer. If $\rho(\alpha)+\rho(\beta)\in\mathcal{S}'_{k+1}$, then we define $\rho(\delta):=\rho(\alpha)+\rho(\beta)\in\mathcal{S}'_{k+1}$, and proceed further.

For (b), we set $\varphi(x_\delta(t)) := x_{\rho(\delta)}(\epsilon_{\alpha,\beta}t)$ for every $t\in\mathbb{F}_q$, where $\epsilon_{\alpha,\beta}$ is determined as follows. If $\varphi(x_\alpha(t)) = x_{\rho(\alpha)}(\epsilon_1 t)$, $\varphi(x_\beta(t)) = x_{\rho(\beta)}(\epsilon_2 t)$ and $[x_\alpha(1), x_\beta(t)] = x_\delta(\epsilon_3 t)$ for all $t\in\mathbb{F}_q$, then $\epsilon_{\alpha,\beta} := \epsilon_1\epsilon_2\epsilon_3$. Notice that in types $D_6$ and $E_6$ we have $\epsilon_1,\epsilon_2,\epsilon_3\in\{\pm1\}$.

We check the compatibility of this setting (that is, that the extension of $\varphi$ on $X_\delta$ is well-defined) by checking all other pairs $(\alpha',\beta')\in\mathcal{S}_{k-i}\times\mathcal{S}_{1+i}$ for $i=1,\dots,k-1$ such that $\alpha'+\beta'=\delta$. If the value $\epsilon_{\alpha,\beta}$ is unique, i.e. there is no pair $(\alpha',\beta')$ giving $\epsilon_{\alpha',\beta'}\neq\epsilon_{\alpha,\beta}$, then the mapping $x_\delta(t)\mapsto x_{\rho(\delta)}(\epsilon_{\alpha,\beta}t)$ is well-defined. Otherwise, this extension to $X_\delta$ is not well-defined, the construction ends here, and we return *no solution* for this choice of $\rho$.

Notice that when $p=2$, each choice for $\epsilon_{\alpha,\beta}$ is valid, thus the extension is always well-defined whenever we find an extension of $\rho$ such that $\rho(\delta)=\rho(\alpha)+\rho(\beta)\in\mathcal{S}'_{k+1}$ as above.

If there is any other root in $\mathcal{S}_{k+1}$, then we go back to the iterative step.

**Output.** Assume that we constructed the $k$-local height layer map, and $\mathcal{S}_{k+1}=\mathcal{S}'_{k+1}=\emptyset$. We obtain the required group isomorphism $\varphi$ from $X_{\mathcal{S}}$ onto $X_{\mathcal{S}'}$.

By using GAP (2016), we apply the isomorphism test to types $D_6$ and $E_6$, and we check that every two nonabelian cores corresponding to the same triple $[z,m,c]$ are in fact isomorphic. As previously remarked, in types $D_6$ and $E_6$ we only need the setup $\varphi(x_\alpha(t)) := x_{\rho(\alpha)}(t)$ for all $t\in\mathbb{F}_q$ in the base step, i.e. we do not have to check the negative sign choices. A possible explanation for this behavior lies in the fact that we have a small number of isomorphism types of nonabelian cores in rank 6 or less. However, Theorem 4 does not generalize to higher rank, as we demonstrate with an example.

**Example 6.** There exists a $[3,9,6]$-core $\mathfrak{C}$ in type $E_7$ such that

- $\mathcal{S}=\{\alpha_1,\alpha_5,\alpha_{14},\alpha_{17},\alpha_{20},\alpha_{21},\alpha_{22},\alpha_{26},\alpha_{37}\}$,
- $\mathcal{Z}=\{\alpha_{21},\alpha_{26},\alpha_{37}\}$,

- $\mathcal{A} = \{\alpha_2, \alpha_3, \alpha_6, \alpha_7, \alpha_8, \alpha_{10}, \alpha_{12}, \alpha_{13}, \alpha_{15}, \alpha_{24}, \alpha_{29}, \alpha_{31}, \alpha_{35}, \alpha_{36}\}$ and $\mathcal{L} = \{\alpha_{18}, \alpha_{19}, \alpha_{23}, \alpha_{25}, \alpha_{27}, \alpha_{28}, \alpha_{30}, \alpha_{34}, \alpha_{39}, \alpha_{40}, \alpha_{41}, \alpha_{42}, \alpha_{44}, \alpha_{45}\}$.

Using the methods described in §4.1, we obtain that $\mathrm{Irr}(X_{\mathcal{S}})_{\mathcal{Z}(\mathcal{S})}$ consists of $q^2(q-1)^3$ irreducible characters of degree $q^2$ for every prime $p$. On the other hand, we find in $\mathrm{E}_7$ another nonabelian $[3,9,6]$-core whose underlying quattern group $X_{\mathcal{S}'}$ is conjugate to the quattern group arising from the unique nonabelian core lying inside the natural quotient $\mathrm{UD}_4(q)$ of $\mathrm{UE}_7(q)$; this has been studied in Himstedt et al. (2011, §4). As recorded in Table 3, we have that $\mathrm{Irr}(X_{\mathcal{S}'})_{\mathcal{Z}(\mathcal{S}')}$ consists instead of $(q-1)^3$ irreducible characters of degree $q^3$ if $p \geq 3$.

On the one hand, under the action of the Weyl group of $\mathrm{E}_7$ on the root system we get that $X_{\mathcal{S}}$ is isomorphic to the quattern group corresponding to $\Phi^+ \setminus \{\alpha_1, \alpha_3, \alpha_5 = \alpha_1 + \alpha_3\}$ in the root system of type $\mathrm{D}_4$. On the other hand, it is noted that the $[3,9,6]$-cores we obtain in types $\mathrm{D}_6$ and $\mathrm{E}_6$ are conjugate to the quattern $\{\alpha_1, \dots, \alpha_{10}\} \setminus \{\alpha_3\}$ in type $\mathrm{D}_4$, which does correspond to the only nonabelian core in type $\mathrm{D}_4$.

## 4. A reduction process for nonabelian cores

### 4.1. A reduction lemma

A method for the study of nonabelian cores is presented in Goodwin et al. (2016a, §4.2). In this subsection we present a slight variation of the setup and the method, in order to have a direct algorithmic application to the study of the corresponding quatterns.

Throughout the rest of this subsection, we assume that $V$ is a finite group, $H$ is a subgroup of $V$ with fixed transversal $X$ in $V$, and $Y, Z$ are subgroups of $V$, such that

(i) $Z \subseteq Z(V)$,
(ii) $Y \subseteq Z(H)$,
(iii) $Z \cap Y = 1$, and
(iv) $[X, Y] \subseteq Z$.

Moreover, we fix $\lambda \in \mathrm{Irr}(Z)$, and we define

$$X' := \{x \in X \mid \lambda([x,y]) = 1 \text{ for all } y \in Y\}, \quad Y' := \{y \in Y \mid \lambda([x,y]) = 1 \text{ for all } x \in X\},$$

and $\hat{\lambda} := \mathrm{Inf}_Z^{YZ} \lambda$. As $[Y, H] = 1$, we have $[Y, V] = [Y, HX] = [Y, X] \subseteq Z$, hence $YZ \trianglelefteq V$ and $V$ acts on $\mathrm{Irr}(YZ)$.

**Lemma 7.** *We have $I_V(\hat{\lambda}) = HX'$.*

**Proof.** Let $h \in H$ and $x \in X$. For every $y \in Y$ and $z \in Z$, we have that

$$^{hx}\hat{\lambda}(yz) = \hat{\lambda}(y^{hx}z) = \hat{\lambda}(y^x z) = \hat{\lambda}(y[y,x]z) = \hat{\lambda}(yz)\lambda([y,x]).$$

Then $^{hx}\hat{\lambda} = \hat{\lambda}$ if and only if $\lambda([x,y]) = 1$ for every $y \in Y$, that is $x \in X'$. □

Let us define $H' := HX'$, and let us fix a transversal $\tilde{X}$ of $H'$ in $V$. For every $\tilde{x} \in \tilde{X}$, let us put $\psi_{\tilde{x}} := (^{\tilde{x}}\hat{\lambda})|_Y \in \mathrm{Irr}(Y)$. It is easy to check that $\psi_{\tilde{x}}(y) = \hat{\lambda}([y, \tilde{x}])$ for every $y \in Y$. Let us put $W_{\tilde{X}} := \{\psi_{\tilde{x}} \mid \tilde{x} \in \tilde{X}\}$.

**Lemma 8.** *$W_{\tilde{X}}$ is a subgroup of $\mathrm{Irr}(Y)$, and*

$$|\tilde{X}| = |W_{\tilde{X}}| = |Y : Y'|.$$

**Proof.** Let $\tilde{x}_1, \tilde{x}_2 \in \tilde{X}$. Notice that since $[Y, HX'] = \ker\lambda$, then if $\tilde{x}_1\tilde{x}_2 = \tilde{x}_3 h x'$ with $\tilde{x}_3 \in \tilde{X}$ then $\lambda([y, \tilde{x}_1\tilde{x}_2]) = \lambda([y, \tilde{x}_3])$ for every $y \in Y$. Hence if $\psi_{\tilde{x}_1}, \psi_{\tilde{x}_2} \in W_{\tilde{X}}$, then

$$(\psi_{\tilde{x}_1}\psi_{\tilde{x}_2})(y) = ({}^{\tilde{x}_1}\hat{\lambda}^{\tilde{x}_2}\hat{\lambda})(y) = {}^{\tilde{x}_1}\hat{\lambda}(y)^{\tilde{x}_2}\hat{\lambda}(y) = \hat{\lambda}([y, \tilde{x}_1][y, \tilde{x}_2]) = \hat{\lambda}([y, \tilde{x}_1\tilde{x}_2]) = \hat{\lambda}([y, \tilde{x}_3]),$$

and a similar computation yields $\psi_{\tilde{x}^{-1}} \in W_{\tilde{X}}$ if $\psi_{\tilde{x}} \in W_{\tilde{X}}$. Hence $W_{\tilde{X}}$ is a subgroup of $\mathrm{Irr}(Y)$. Analogously one checks that $\psi_{\tilde{x}_1} \neq \psi_{\tilde{x}_2}$ if $\tilde{x}_1 \neq \tilde{x}_2$, hence $|\tilde{X}| = |W_{\tilde{X}}|$. Finally, notice that

$$Y' = \bigcap_{\tilde{x}\in\tilde{X}} \ker(\psi_{\tilde{x}}) = \bigcap_{\eta\in W_{\tilde{X}}} \ker(\eta),$$

hence $|W_{\tilde{X}}| = |Y : Y'|$. $\square$

From now on we need an extra assumption on $Y'$, namely

(v) $Y'$ has a complement $\tilde{Y}$ in $Y$.

**Proposition 9.** *We have a bijection*

$$\mathrm{Ind}_{H'}^{V}\,\mathrm{Inf}_{H'/\tilde{Y}}^{H'} : \mathrm{Irr}(H'/\tilde{Y} \mid \lambda) \longrightarrow \mathrm{Irr}(V \mid \lambda). \tag{4}$$

**Proof.** Let $\tilde{\lambda} := \mathrm{Inf}_Z^{Z\tilde{Y}}\lambda$. Lemma 7 yields $I_V(\tilde{\lambda}) = HX'$ and ${}^{\tilde{x}_1}\tilde{\lambda} \neq {}^{\tilde{x}_1}\tilde{\lambda}$ if $\tilde{x}_1 \neq \tilde{x}_2$. Hence

$$\mathrm{Irr}(Z\tilde{Y} \mid \lambda) = \{{}^{\tilde{x}}\tilde{\lambda} \mid \tilde{x} \in \tilde{X}\}. \tag{5}$$

Since $Z\tilde{Y} \trianglelefteq V$, by Clifford's theory we have a bijection

$$\mathrm{Ind}_{H'}^{V} : \mathrm{Irr}(H' \mid \tilde{\lambda}) \longrightarrow \mathrm{Irr}(V \mid \tilde{\lambda}).$$

By identifying $\mathrm{Irr}(H'/\tilde{Y})$ with $\{\eta \in \mathrm{Irr}(H') \mid \tilde{Y} \subseteq \ker(\eta)\}$, the above yields the bijection

$$\mathrm{Ind}_{H'}^{V}\,\mathrm{Inf}_{H'/\tilde{Y}}^{H'} : \mathrm{Irr}(H'/\tilde{Y} \mid \lambda) \longrightarrow \mathrm{Irr}(V \mid \tilde{\lambda}).$$

We have that $\mathrm{Irr}(V \mid \tilde{\lambda}) = \mathrm{Irr}(V \mid \lambda) \cap \mathrm{Irr}(V \mid 1_{\tilde{Y}}) \subseteq \mathrm{Irr}(V \mid \lambda)$.

The claim is then proved if we show $\mathrm{Irr}(V \mid \lambda) \subseteq \mathrm{Irr}(V \mid \tilde{\lambda})$. If $\chi \in \mathrm{Irr}(V \mid \lambda)$, we have $\langle \chi|_{Z\tilde{Y}}, \lambda_{Z\tilde{Y}}\rangle \neq 0$. Let then $\eta \in \mathrm{Irr}(Z\tilde{Y} \mid \lambda)$ such that $\chi \in \mathrm{Irr}(V \mid \eta)$. Then

$$\tilde{\lambda} \in \{{}^{\tilde{x}}\tilde{\lambda} \mid \tilde{x} \in \tilde{X}\} = \{{}^{\tilde{x}}\eta \mid \tilde{x} \in \tilde{X}\} \subseteq \{{}^{g}\eta \mid g \in V\} = \{\mu \in \mathrm{Irr}(Z\tilde{Y}) \mid \langle \chi|_{Z\tilde{Y}}, \mu\rangle \neq 0\},$$

where the first equality holds by Equation (5), and the second equality holds by Clifford's theory. Hence $0 \neq \langle \chi|_{Z\tilde{Y}}, \tilde{\lambda}\rangle$, that is, $\chi \in \mathrm{Irr}(V \mid \tilde{\lambda})$. $\square$

**Definition 10.** The $X$ and $Y$ defined at the start of this section are called *candidate for an arm* and *candidate for a leg* respectively, and $\tilde{X}$ and $\tilde{Y}$ are called *arm* and *leg* respectively.

The terminology of arms and legs is motivated by the case $U = \mathrm{UA}_r(q)$, as remarked in Himstedt et al. (2016, Section 6).

Let $\overline{V} := H'/(\tilde{Y}\ker\lambda)$. We observe that $Y' \subseteq Z(\overline{V})$. Before stating a consequence of Proposition 9, we introduce some notation that is frequently used in the sequel.

**Definition 11.** Let $\mathcal{S}$ be a quattern, and let $\{\alpha_{i_1}, \ldots, \alpha_{i_m}\}$ be a quattern contained in $\mathcal{S}$ such that $X_{\{\alpha_{i_1},\ldots,\alpha_{i_m}\}}$ is abelian. For fixed $c_1, \ldots, c_m \in \mathbb{F}_q^{\times}$, we define

$$x_{i_1,\ldots,i_m}^{c_1,\ldots,c_m}(t) := x_{i_1}(c_1 t)\cdots x_{i_m}(c_m t) \quad \text{for all } t \in \mathbb{F}_q.$$

Moreover, we put

$$X_{i_1,\dots,i_m}^{c_1,\dots,c_m} := \{x_{i_1,\dots,i_m}^{c_1,\dots,c_m}(t) \mid t \in \mathbb{F}_q\}.$$

We usually drop the explicit labels $c_1, \dots, c_m \in \mathbb{F}_q^\times$ and we write $x_{i_1,\dots,i_m}(t)$ instead of $x_{i_1,\dots,i_m}^{c_1,\dots,c_m}(t)$, and $X_{i_1,\dots,i_m}$ instead of $X_{i_1,\dots,i_m}^{c_1,\dots,c_m}$; the choice of $c_1, \dots, c_m$ will be made explicit when needed. Notice that as $\mathcal{Z}(\{\alpha_{i_1}, \dots, \alpha_{i_m}\}) = \{\alpha_{i_1}, \dots, \alpha_{i_m}\}$, we have that $X_{i_1,\dots,i_m} \cong (\mathbb{F}_q, +)$. Moreover, if $\underline{i} := (i_1, \dots, i_m)$ with $1 \le i_1 < \cdots < i_m \le |\mathcal{S}|$, we denote by $x_{\underline{i}}(t)$ the element $x_{i_1,\dots,i_m}(t)$, similarly for $X_{\underline{i}}$.

**Corollary 12.** *Assume that $Y' = X_{\underline{i}_1} \times \cdots \times X_{\underline{i}_s}$, where $\underline{i}_1, \dots, \underline{i}_s$ are lexicographically ordered. For every $\underline{b} = (b_1, \dots, b_s) \in \mathbb{F}_q^s$, let*

$$K_{\underline{b}} := \prod_{j=1,\dots,s \mid b_j=0} X_{\underline{i}_j} \quad \text{and} \quad V_{\underline{b}} := \overline{V}/K_{\underline{b}},$$

*and let $\mu_{\underline{b}} \in \mathrm{Irr}(Y'/K_{\underline{b}})$ be such that $\mu_{\underline{b}}(x_{\underline{i}_j}(t)) = \phi(b_j t)$ for every $j = 1, \dots, s$. Then the map*

$$\Psi : \bigsqcup_{\underline{b} \in \mathbb{F}_q^s} \mathrm{Irr}(V_{\underline{b}} \mid \lambda \otimes \mu_{\underline{b}}) \longrightarrow \mathrm{Irr}(V \mid \lambda)$$

*with $\Psi(\chi) = \mathrm{Inf}_{\bar{V}}^{V} \mathrm{Inf}_{V_{\underline{b}}}^{\bar{V}}(\chi)$ if $\chi \in \mathrm{Irr}(V_{\underline{b}} \mid \lambda \otimes \mu_{\underline{b}})$, is a bijective map.*

**Proof.** Observe that for any $\chi \in \mathrm{Irr}(\bar{V})$, we have

$$\langle \chi|_Z, \lambda \rangle = \langle (\chi|_{Y'Z})|_Z, \lambda \rangle = \langle \chi|_{Y'Z}, \lambda^{Y'Z} \rangle = \sum_{\mu \in \mathrm{Irr}(Y')} \langle \chi|_{Y'Z}, \lambda \otimes \mu \rangle.$$

Hence $\mathrm{Irr}(\bar{V}|\lambda) = \bigsqcup_{\mu \in \mathrm{Irr}(Y')} \mathrm{Irr}(\bar{V} \mid \lambda \otimes \mu)$. The claim follows since inflation over $\tilde{Y} \ker(\lambda)$ gives the bijection $\mathrm{Irr}(\bar{V}|\lambda) \to \mathrm{Irr}(V|\lambda)$ as in Proposition 9, and the bijection $\bigsqcup_{\underline{b} \in \mathbb{F}_q^s} \mathrm{Irr}(V_{\underline{b}} \mid \lambda \otimes \mu_{\underline{b}}) \to \bigsqcup_{\mu \in \mathrm{Irr}(Y')} \mathrm{Irr}(\bar{V} \mid \lambda \otimes \mu)$ is given by partitioning $\mathrm{Irr}(Y')$ into characters with determined root kernel, namely each of the $K_{\underline{b}}$ for $\underline{b} \in \mathbb{F}_q^s$, and by inflating over each of the $K_b$. □

For example, let us suppose that $Y'$ is a diagonal subgroup of $X_{\mathcal{J}}$ isomorphic to $\mathbb{F}_q$, that is $s = 1$. Then

$$\mathrm{Irr}(V \mid \lambda) \cong \mathrm{Irr}(\overline{V}/Y' \mid \lambda) \sqcup \bigsqcup_{\mu \in \mathrm{Irr}(Y') \setminus \{1_{Y'}\}} \mathrm{Irr}(\overline{V} \mid \lambda \otimes \mu).$$

We are interested in applying Proposition 9 and Corollary 12 to the setting of quattern groups. Given a quattern in rank 6 or less, the validity of the assumptions of the following result is easy to check by using GAP (2016). We recall the assumptions (i)–(v) before Proposition 9.

**Corollary 13.** *Let $\mathcal{S} = \mathcal{P} \setminus \mathcal{K}$ be a quattern. Assume that there exist subsets $\mathcal{Z}$, $\mathcal{I}$ and $\mathcal{J}$ of $\mathcal{S}$, such that*

(0) *$\mathcal{S} \setminus \mathcal{I}$ is a quattern,*
(i) *$\mathcal{Z} \subseteq \mathcal{Z}(\mathcal{S})$,*
(ii) *$\mathcal{J} \subseteq \mathcal{Z}(\mathcal{S} \setminus \mathcal{I})$,*
(iii) *$\mathcal{J} \cap \mathcal{Z} = \emptyset$, and*
(iv) *$\alpha \in \mathcal{I}, \beta \in \mathcal{J}, \alpha + \beta \in \mathcal{S} \Rightarrow \alpha + \beta \in \mathcal{Z}$.*

*Let us put $Z = X_{\mathcal{Z}}$, $X = X_{\mathcal{I}}$, $Y = X_{\mathcal{J}}$ and $H = X_{\mathcal{S} \setminus \mathcal{I}}$, and define $X'$, $Y'$ and $H'$ as in Proposition 9 and $\overline{V}$ as in Corollary 12. Then we have a bijection*

$$\mathrm{Ind}_{\tilde{H}}^{X_{\mathcal{S}}} \mathrm{Inf}_{\overline{V}}^{H'} : \mathrm{Irr}(\overline{V} \mid \lambda) \longrightarrow \mathrm{Irr}(X_{\mathcal{S}} \mid \lambda).$$

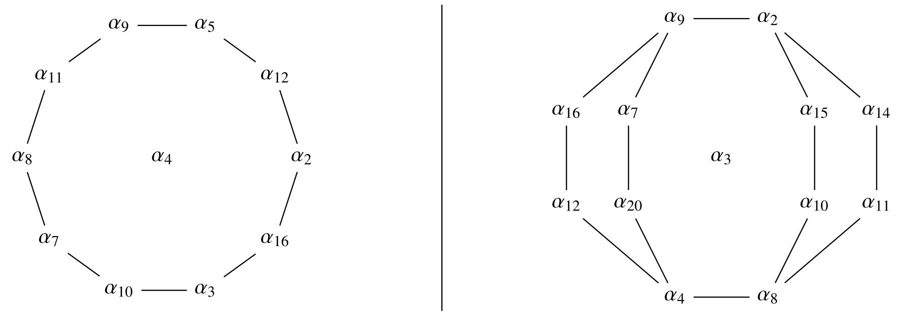


**Fig. 1.** The graphs of the core of the form [5, 16, 15] in $D_6$ and of the core of the form [6, 19, 20] in $E_6$.

**Proof.** By §2.3, it is easy to check that (0) is equivalent to $H$ being a subgroup of $X_{\mathcal{S}}$ and each of (i) to (iv) is equivalent to the corresponding assumption in Proposition 9. Moreover, (v) is clear as $Y$ is elementary abelian. □

**Remark 14.** Let us assume that $\mathcal{S} = \mathcal{Z} \cup \mathcal{I} \cup \mathcal{J}$, such that assumption (ii) of Corollary 13 is satisfied. Then $\mathcal{S} \setminus \mathcal{I}$ is automatically a quattern.

*4.2. Graphs of nonabelian cores*

Let us fix a nonabelian core $\mathfrak{C}$ corresponding to $\mathcal{S}$ and $\mathcal{Z}$. In order to check the assumptions of Corollary 13, we define a graph associated to $\mathfrak{C}$.

**Definition 15.** Let $\mathfrak{C}$ be a nonabelian core corresponding to $\mathcal{S}$ and $\mathcal{Z}$. We say that $\alpha, \beta \in \mathcal{S}$ are *$\mathcal{Z}$-connected*, or just *connected*, if $\alpha + \beta = \gamma$ with $\gamma \in \mathcal{Z}$.

With this definition, we regard $\mathfrak{C}$ as a graph whose vertices are the elements of $\mathcal{S} \setminus \mathcal{Z}$, and there is an edge between $\alpha$ and $\beta$ if and only if $\alpha$ and $\beta$ are $\mathcal{Z}$-connected. We then have the usual notion of *connected components*. We say that the *heart* $\mathcal{H}$ of the core $\mathfrak{C}$ is the set of roots $\alpha \in \mathcal{S} \setminus \mathcal{Z}$ such that $\{\alpha\}$ is a connected component on its own. If $\mathcal{H} = \emptyset$, we say that the underlying core $\mathfrak{C}$ is a *heartless core*. Otherwise, we call $\mathfrak{C}$ a *core with a heart*.

We now define some important cycles in $\mathfrak{C}$, whose analysis allows us to have a systematic procedure to reduce to the study of irreducible characters of smaller subquotients of $X_{\mathcal{S}}$.

**Definition 16.** We say that $\mathcal{C} := \{\beta_1, \dots, \beta_s\} \subseteq \mathcal{S}$ with $\beta_1, \dots, \beta_s$ distinct is a *circle* in $\mathcal{S}$ if $\beta_i$ is connected to $\beta_{i+1}$ for $i = 1, \dots, s-1$, and $\beta_s$ is connected to $\beta_1$.

The goal of the rest of this section is to construct *unique* $\mathcal{I}$ and $\mathcal{J}$ satisfying the assumptions of Corollary 13 for each nonabelian core $\mathfrak{C}$.

**Remark 17.** Let us assume that $\mathcal{S}$ contains just a *single* circle $\mathcal{C}$. Let $\mathcal{C} = \{\beta_1 \dots, \beta_s\}$ in the notation of Definition 16. We start by describing a construction of such $\mathcal{I}$ and $\mathcal{J}$ in this particular case. Let us define $\beta_0 := \beta_s$ and $\beta_{s+1} := \beta_1$. We first define $\delta_1$ to be the minimal root in $\mathcal{C}$ with respect to the usual linear ordering on roots. If $\delta_1 = \beta_{j_1}$, then we choose $\delta_2$ to be the maximum of $\beta_{j_1-1}$ and $\beta_{j_1+1}$.

Now we assume that $\delta_i$ is constructed for $2 \le i \le s-1$. Then $\delta_i = \beta_{j_i}$ for some $\beta_{j_i}$, hence $\delta_{i-1} \in \{\beta_{j_i-1}, \beta_{j_i+1}\}$. If $\delta_{i-1} = \beta_{j_i-1}$, we define $\delta_{i+1} := \beta_{j_i+1}$. Vice versa if $\delta_{i-1} = \beta_{j_i+1}$, we define $\delta_{i+1} := \beta_{j_i-1}$. Notice that $\delta_s$ is connected to $\delta_1$. If $s = 2m$ is even, then we put $\mathcal{I} := \{\delta_1, \delta_3, \dots, \delta_{2m-1}\}$ and $\mathcal{J} := \{\delta_2, \delta_4, \dots, \delta_{2m}\}$. If $s = 2m+1$ is odd, then we put $\mathcal{I} := \{\delta_1, \delta_3, \dots, \delta_{2m-1}, \delta_{2m+1}\}$ and $\mathcal{J} := \{\delta_2, \delta_4, \dots, \delta_{2m}\}$.

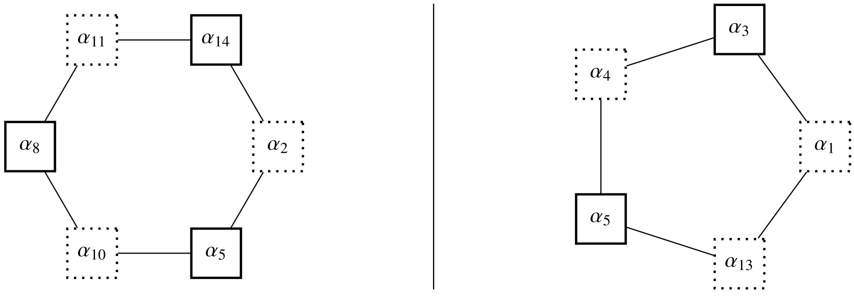


**Fig. 2.** The $\mathcal{I}$ and $\mathcal{J}$ constructed in Remark 17, corresponding to roots in a dotted box and roots in a straight box respectively, in the cases of the circles corresponding to the $[3, 9, 6]$-core of $D_6$ and to the $[5, 10, 5]$-core of $E_6$.

We now check the conditions of Corollary 13. Assumptions (i), (iii) and (iv) clearly hold. If two roots in $\mathcal{J}$ were connected to each other, we would have a smaller circle $\mathcal{C}'$ in $\mathcal{C}$, which contradicts the assumption of $\mathcal{C}$ being the unique circle in $\mathcal{S}$. Hence $\mathcal{Z}(\mathcal{J}) = \mathcal{J}$, which implies $\mathcal{J} \subseteq \mathcal{Z}(\mathcal{S} \setminus \mathcal{I})$. Then (ii) is satisfied, and by Remark 14 we have that (0) is also satisfied. Therefore, $\mathcal{I}$ and $\mathcal{J}$ satisfy all the assumptions of the corollary.

We now construct $\mathcal{I}$ and $\mathcal{J}$ for types $D_6$ and $E_6$ in the case when two or more circles occur in $\mathcal{S}$. The number of circles of each nonabelian core in rank 6 or less is relatively small, and we check by GAP (2016) that the $\mathcal{I}$ and $\mathcal{J}$ obtained as follows satisfy the assumptions of Corollary 13 for all nonabelian cores, except the $[4, 24, 43]$-core in type $D_6$. This core is examined in full details in §5.2. For several nonabelian cores in type $E_7$, the $\mathcal{I}$ and $\mathcal{J}$ constructed in this way do not satisfy the conditions of Corollary 13. One of the aims of subsequent work is to refine the following construction for higher numbers of circles in a nonabelian core.

**Setup.** We collect all the distinct circles $\mathcal{C}_1, \ldots, \mathcal{C}_t$ in $\mathcal{S}$, ordered such that

- if $|\mathcal{C}_i|$ is even and $|\mathcal{C}_j|$ is odd, then $i < j$;
- if $|\mathcal{C}_i| \neq |\mathcal{C}_j|$ have the same parity, then $i < j$ if and only if $|\mathcal{C}_i| > |\mathcal{C}_j|$; and
- if $|\mathcal{C}_i| = |\mathcal{C}_j|$, then $i < j$ if and only if $\min\{k \mid \alpha_k \in \mathcal{C}_i \setminus \mathcal{C}_j\} < \min\{k \mid \alpha_k \in \mathcal{C}_j \setminus \mathcal{C}_i\}$.

As circles are determined just by the relations among roots in $\mathcal{S}$, the sets $\mathcal{C}_1, \ldots, \mathcal{C}_t$ can easily be determined by using GAP (2016). We decide to look first at the circles with even cardinality because the construction of arms and legs of smaller even circles is often compatible with the one of bigger even circles, while in general we have less compatibility among odd circles.

**Base step.** We start by looking at $\mathcal{C}_1$. We define $\mathcal{I}_1$ and $\mathcal{J}_1$ as we determined $\mathcal{I}$ and $\mathcal{J}$ in Remark 17, namely we decide the minimum root $\delta$ of $\mathcal{C}_1$ to be in $\mathcal{I}_1$, and we alternate adjoining the remaining roots of $\mathcal{C}_1$ into $\mathcal{J}_1$ and $\mathcal{I}_1$ in the direction of the maximum neighbor of $\delta$ in $\mathcal{C}_1$.

**Iterative step.** Let us suppose that $\mathcal{I}_k$ and $\mathcal{J}_k$ have been constructed with $k < t$. Then there exists another circle $\mathcal{C}_{k+1}$ after $\mathcal{C}_k$ in the ordering previously fixed. We now define two sets $\mathcal{I}(\mathcal{C}_{k+1})$ and $\mathcal{J}(\mathcal{C}_{k+1})$ such that $\mathcal{I}(\mathcal{C}_{k+1}) \cup \mathcal{J}(\mathcal{C}_{k+1}) = \mathcal{C}_{k+1}$, and then we take advantage of them to construct $\mathcal{I}_{k+1}$ and $\mathcal{J}_{k+1}$.

If $\mathcal{C}_{k+1} \cap \mathcal{I}_k \neq \emptyset$, then we construct $\mathcal{I}(\mathcal{C}_{k+1})$ and $\mathcal{J}(\mathcal{C}_{k+1})$ exactly as we constructed $\mathcal{I}$ and $\mathcal{J}$ in Remark 17 respectively. If $\mathcal{C}_{k+1} \cap \mathcal{I}_k = \emptyset$ and $\mathcal{C}_{k+1} \cap \mathcal{J}_k \neq \emptyset$, then we let $\mathcal{C}_{k+1} = \{\beta_1 \ldots, \beta_s\}$ be as in Definition 16, with $\beta_0 := \beta_s$ and $\beta_{s+1} := \beta_1$. We let $\delta_1$ be the maximum root in $\mathcal{C}_{k+1} \cap \mathcal{J}_k$ ; we have $\delta_1 = \beta_{j_1}$ for some $j_1 \in \{1, \ldots, s\}$. Then we let $\delta_2$ be the minimum of the roots $\beta_{j_1-1}$ and $\beta_{j_1+1}$. If $2 \leq i \leq s-1$ and $\delta_i$ is constructed such that $\delta_i = \beta_{j_i}$, we define $\delta_{i+1}$ to be $\beta_{j_i+1}$ in the case $\delta_{i-1} = \beta_{j_i-1}$, and $\beta_{j_i-1}$ in the case $\delta_{i-1} = \beta_{j_i+1}$. If $s = 2m$ then we put $\mathcal{J}(\mathcal{C}_{k+1}) := \{\delta_1, \delta_3, \ldots, \delta_{2m-1}\}$ and $\mathcal{I}(\mathcal{C}_{k+1}) := \{\delta_2, \delta_4, \ldots, \delta_{2m}\}$; otherwise $|\mathcal{C}| = 2m+1$ and we put $\mathcal{J}(\mathcal{C}_{k+1}) := \{\delta_1, \delta_3, \ldots, \delta_{2m-1}\}$ and $\mathcal{I}(\mathcal{C}_{k+1}) := \{\delta_{2m+1}\} \cup \{\delta_2, \delta_4, \ldots, \delta_{2m}\}$. If $\mathcal{C}_{k+1} \cap \mathcal{I}_k = \mathcal{C}_{k+1} \cap \mathcal{J}_k = \emptyset$, again we proceed in the same way as Remark 17 to construct $\mathcal{I}(\mathcal{C}_{k+1})$ and $\mathcal{J}(\mathcal{C}_{k+1})$. Finally, we define $\mathcal{I}_{k+1} := \mathcal{I}_k \cup \mathcal{I}(\mathcal{C}_{k+1})$ and $\mathcal{J}_{k+1} := (\mathcal{J}_k \cup \mathcal{J}(\mathcal{C}_{k+1})) \setminus \mathcal{I}_{k+1}$.

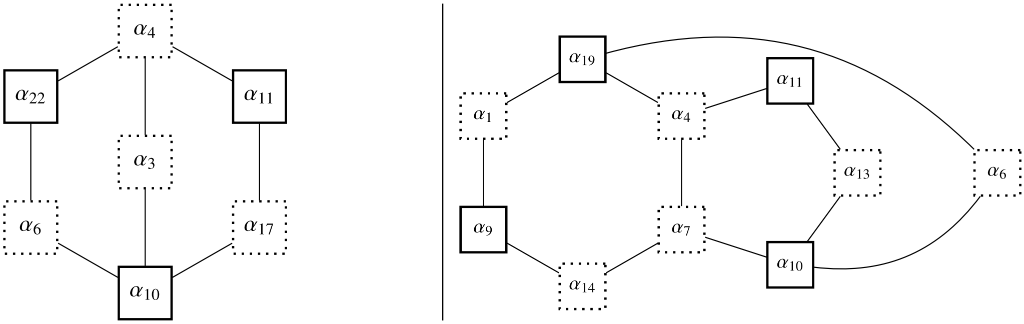


**Fig. 3.** The construction of $\mathcal{I}$ and $\mathcal{J}$ for the two nonabelian cores of the form $[5, 12, 8]$ and $[6, 16, 12]$ of $E_6$, represented as in Fig. 2.

**Output.** The sets $\mathcal{I}_t$ and $\mathcal{J}_t$ are constructed. We define $\mathcal{I} := \mathcal{I}_t$ and $\mathcal{J} := \mathcal{J}_t$. These sets turn out to satisfy our desired properties.

**Lemma 18.** *Let $\mathfrak{C}$ be a nonabelian core of* $D_6$ *not of the form* $[4, 24, 43]$ *or a nonabelian core of* $E_6$*, corresponding to $\mathcal{S}$ and $\mathcal{Z}$. Then the sets $\mathcal{I}$ and $\mathcal{J}$ constructed as above satisfy the assumptions of Corollary 13.*

**Proof.** This is a straightforward check, see the corresponding detailed computations and the GAP4 functions in Le et al. (2018). □

An explicit form of the sets $\mathcal{I}$ and $\mathcal{J}$ for each nonabelian core in types $D_6$ and $E_6$ is provided in Le et al. (2018). Through the construction pointed out above, we can reduce to smaller subquotients of $X_{\mathcal{S}}$. The analysis of $\mathrm{Irr}(X_{\mathcal{S}})$ for a heartless core is straightforward, once the sets $X'$ and $Y'$ are explicitly known, as such subquotients turn out to be abelian, except in the case of the $[6, 16, 12]$-core of $E_6$; the study of heartless cores is explained in §5.1. The analysis of nonabelian cores with a heart, detailed in §5.2, is more complicated. In particular, the case of the $[4, 24, 43]$-core of $D_6$ is not covered by Lemma 18; we treat it by applying directly Proposition 9.

**Remark 19.** Let $\Phi$ be a root system of type $D_6$ or $E_6$. Let $\mathcal{I} = \{i_1, \dots, i_m\}$, $\mathcal{J} = \{j_1, \dots, j_\ell\}$ and $\mathcal{Z}$ satisfy the assumptions of Corollary 13. The equation $\lambda([y, x]) = 1$, for $x = x_{i_1}(t_{i_1}) \cdots x_{i_r}(t_{i_r}) \in X$ and $y = x_{j_1}(s_{j_1}) \cdots x_{j_\ell}(s_{j_\ell}) \in Y$, where $t_{i_1}, \dots, t_{i_r}, s_{j_1}, \dots, s_{j_\ell}$ are unknown variables over $\mathbb{F}_q$, can be rewritten as

$$\sum_{h=1}^{\ell} \sum_{k=1}^{m} d_{h,k} s_{j_h} t_{i_k} = 0, \tag{6}$$

which just contains linear terms in the $s_{j_h}$'s and the $t_{i_k}$'s, where each constant $d_{h,k} \in \mathbb{F}_q$ depends on $\mathcal{S}$ and the choice of the extraspecial pairs in $U$. In particular, if $\alpha_{i_h} + \alpha_{j_k} \notin \mathcal{S}$ then $d_{h,k} = 0$.

It is then easy to work out explicitly $X'$ (respectively $Y'$), namely by finding the values of $t_{i_1}, \dots, t_{i_m} \in \mathbb{F}_q$ (respectively $s_{j_1}, \dots, s_{j_\ell} \in \mathbb{F}_q$) such that Equation (6) holds for every $s_{j_1}, \dots, s_{j_\ell} \in \mathbb{F}_q$ (respectively for every $t_{i_1}, \dots, t_{i_m} \in \mathbb{F}_q$).

## 5. Parametrization of $\mathrm{Irr}(\mathrm{UD}_6(q))$ and $\mathrm{Irr}(\mathrm{UE}_6(q))$

We now describe the parametrization of the sets $\mathrm{Irr}(X_{\mathcal{S}})_{\mathcal{Z}}$ that arise from nonabelian cores in types $D_6$ and $E_6$. By Theorem 4, it is enough to consider just one quattern arising from each $[z, m, c]$-core in Table 2. Applying Propositions 1 and 2, we can then obtain the corresponding parametrization of characters in $\mathrm{Irr}(U)$ using the information stored in $\mathcal{A}$ and $\mathcal{L}$ and the record of roots in direct products.

**Table 3**
Parametrization of nonabelian cores in $\mathrm{UD}_6(q)$ for every $p$.

| Form | Family | Label | Number | Degree |
|---|---|---|---|---|
| [3, 9, 6] | $\mathcal{F}_1^{p\neq 2}$ | $\chi^{a_{18},a_{19},a_{24}}$ | $(q-1)^3$ | $q^3$ |
| | $\mathcal{F}_1^{p=2}$ | $\chi^{a_{18},a_{19},a_{24}}_{b_{2,10,11},b_{8,14,15}}$ | $q^2(q-1)^3$ | $q^2$ |
| [3, 10, 9] | $\mathcal{F}_2^{p\neq 2}$ | $\chi^{a_{12},a_{27},a_{28}}_{b_2}$ | $q(q-1)^3$ | $q^3$ |
| | $\mathcal{F}_2^{1,p=2}$ | $\chi^{a_{12},a_{27},a_{28}}$ | $(q-1)^3$ | $q^3$ |
| | $\mathcal{F}_2^{2,p=2}$ | $\chi^{a_{7,8,26},a_{12},a_{27},a_{28}}_{d_2,d_{1,3,24}}$ | $4(q-1)^4$ | $q^3/2$ |
| [4, 18, 18] | $\mathcal{F}_3^{p\neq 2}$ | $\chi^{a_{16},a_{21},a_{22},a_{28}}_{b_2,b_4}$ | $q^2(q-1)^4$ | $q^6$ |
| | $\mathcal{F}_3^{1,p=2}$ | $\chi^{a_{16},a_{21},a_{22},a_{28}}$ | $(q-1)^4$ | $q^6$ |
| | $\mathcal{F}_3^{2,p=2}$ | $\chi^{a_{7,18,19},a_{16},a_{21},a_{22},a_{28}}_{d_{1,14,15},d_2}$ | $4(q-1)^5$ | $q^6/2$ |
| | $\mathcal{F}_3^{3,p=2}$ | $\chi^{a_{10,11,17},a_{16},a_{21},a_{22},a_{28}}_{d_4,d_{5,6,12}}$ | $4(q-1)^5$ | $q^6/2$ |
| | $\mathcal{F}_3^{4,p=2}$ | $\chi^{a_{7,18,19},a_{10,11,17},a_{16},a_{21},a_{22},a_{28}}_{d_{1,14,15},d_2,d_4,d_{5,6,12}}$ | $16(q-1)^6$ | $q^6/4$ |
| [4, 21, 28] | $\mathcal{F}_4^{p\neq 2}$ | $\chi^{a_{20},a_{21},a_{22},a_{26}}_{b_{3,13},b_8,b_9}$ | $q^3(q-1)^4$ | $q^7$ |
| | $\mathcal{F}_4^{1,p=2}$ | $\chi^{a_{20},a_{21},a_{22},a_{26}}_{b_{3,13}}$ | $q(q-1)^4$ | $q^7$ |
| | $\mathcal{F}_4^{2,p=2}$ | $\chi^{a_{12,18,19},a_{20},a_{21},a_{22},a_{26}}_{b_{3,9,13},d_{1,10,11},d_8}$ | $4q(q-1)^5$ | $q^7/2$ |
| | $\mathcal{F}_4^{3,p=2}$ | $\chi^{a_{14,15,17},a_{20},a_{21},a_{22},a_{26}}_{b_{3,8,13},d_{5,6,7},d_9}$ | $4q(q-1)^5$ | $q^7/2$ |
| | $\mathcal{F}_4^{4,p=2}$ | $\chi^{a_{12,18,19},a_{14,15,17},a_{20},a_{21},a_{22},a_{26}}_{d_{1,5,6,7,10,11},b_{3,8,9,13},d_{8,9}}$ | $4q(q-1)^6$ | $q^7/2$ |
| [4, 24, 43] | $\mathcal{F}_5^{1,p\neq 2}$ | $\chi^{a_{13},a_{21},a_{22},a_{23},a_{24}}_{b_3}$ | $q(q-1)^5$ | $q^9$ |
| | $\mathcal{F}_5^{2,p\neq 2}$ | $\chi^{a_{8,9},a_{21},a_{22},a_{23},a_{24}}$ | $(q-1)^5$ | $q^9$ |
| [4, 24, 43] | $\mathcal{F}_5^{3,p\neq 2}$ | $\chi^{a_{21},a_{22},a_{23},a_{24}}_{b_{2,4},b_3}$ | $q^2(q-1)^4$ | $q^8$ |
| | $\mathcal{F}_5^{1,p=2}$ | $\chi^{a_{17,18,19},a_{21},a_{22},a_{23},a_{24}}_{d_{1,5,6},b_3,d_{13}}$ | $4q(q-1)^5$ | $q^9/2$ |
| | $\mathcal{F}_5^{2,p=2}$ | $\chi^{a_{8,9},a_{17,18,19},a_{21},a_{22},a_{23},a_{24}}_{b_{2,4,7,10,11},b_{12,14,15}}$ | $q^2(q-1)^5$ | $q^8$ |
| | $\mathcal{F}_5^{3,p=2}$ | $\chi^{a_{21},a_{22},a_{23},a_{24}}_{b_{2,4}}$ | $q(q-1)^4$ | $q^8$ |
| | $\mathcal{F}_5^{4,p=2}$ | $\chi^{a_{12,14,15},a_{21},a_{22},a_{23},a_{24}}_{b_{2,4},d_3,d_{7,11}}$ | $4q(q-1)^5$ | $q^8/2$ |
| [5, 18, 18] | $\mathcal{F}_6^{p\neq 2}$ | $\chi^{a_{17},a_{18},a_{19},a_{24},a_{25}}_{b_3}$ | $q(q-1)^5$ | $q^6$ |
| | $\mathcal{F}_6^{1,p=2}$ | $\chi^{c^*_{8,14,15},a_{17},a_{18},a_{19},a_{24},a_{25}}_{b_{2,4,7,10,11,16},b_{9,12,20}}$ | $q^2(q-1)^6$ | $q^5$ |
| | $\mathcal{F}_6^{2,p=2}$ | $\chi^{a_{17},a_{18},a_{19},a_{24},a_{25}}_{b_{2,4,7,10,11,16}}$ | $q(q-1)^5$ | $q^5$ |
| | $\mathcal{F}_6^{3,p=2}$ | $\chi^{c^*_{9,12,20},a_{17},a_{18},a_{19},a_{24},a_{25}}_{b_{2,4,7,10,11,16},d_{2,4,7,10,11,16},d_3}$ | $4q(q-1)^6$ | $q^5/2$ |
| [6, 19, 20] | $\mathcal{F}_7^{1,p\neq 2}$ | $\chi^{a_{13},a_{17},a_{18},a_{19},a_{24},a^*_{25}}_{b_3}$ | $q(q-1)^5(q-2)$ | $q^6$ |
| | $\mathcal{F}_7^{2,p\neq 2}$ | $\chi^{a_{8,9,12,14,15,20},a_{13},a_{17},a_{18},a_{19},a_{24}}$ | $(q-1)^6$ | $q^6$ |
| | $\mathcal{F}_7^{3,p\neq 2}$ | $\chi^{a_{13},a_{17},a_{18},a_{19},a_{24}}_{b_{2,4,7,10,11,16},b_3}$ | $q^2(q-1)^5$ | $q^5$ |
| | $\mathcal{F}_7^{p=2}$ | $\chi^{a_{13},a_{17},a_{18},a_{19},a_{24},a_{25}}_{b_3}$ | $q(q-1)^6$ | $q^6$ |

The degrees of the irreducible characters and the numbers of irreducible characters of fixed degree arising from a nonabelian $[z, m, c]$-core are collected in Table 3 for $\mathrm{UD}_6(q)$ and in Table 4 for $\mathrm{UE}_6(q)$, along with the labels of the characters in each $\mathrm{Irr}(X_{\mathcal{S}})_{\mathcal{Z}}$. We notice that the parametrization is uniform for $p \geq 3$ in type $\mathrm{D}_6$, and for $p \geq 5$ in type $\mathrm{E}_6$. For $q = 2^f$ in type $\mathrm{D}_6$, and for $q = 2^f$ or $q = 3^f$ in type $\mathrm{E}_6$, the parametrization is more complicated.

Let us put $v := q - 1$, and let $\mathcal{S}$ and $\mathcal{Z}$ correspond to a nonabelian core $\mathfrak{C}$. The number $|\mathrm{Irr}(X_{\mathcal{S}})_{\mathcal{Z}}|$ may not always be expressed as a polynomial in $v$ with nonnegative coefficients, even when $p$ is a good prime, as in the case of $\mathfrak{C}$ of the form $[7, 15, 9]$ in type $\mathrm{E}_6$ (see Table 4), where $|\mathrm{Irr}(X_{\mathcal{S}})_{\mathcal{Z}}| = v^6(v^2 + 2v - 2)$ for corresponding $\mathcal{S}$ and $\mathcal{Z}$. Nevertheless, we notice that for a good prime $p$ in both cases $U = \mathrm{UD}_6(q)$ and $U = \mathrm{UE}_6(q)$, the character degrees of $U$ are powers of $q$, and the numbers $k(U, q^d)$ of irreducible characters of $U$ of degree $q^d$ are in $\mathbb{Z}[v]$ for every power $q$ of $p$ and every $d \in \mathbb{Z}_{\geq 0}$.

We collect the numbers of irreducible characters of each fixed degree of $\mathrm{UD}_6(q)$ in Tables 5 and 6 when $p \geq 3$ and $p = 2$ respectively, and of $\mathrm{UE}_6(q)$ in Tables 7, 8 and 9 when $p \geq 5$, $p = 3$ and $p = 2$ respectively. We notice that we have fractional degrees of the form $q^3/2, \dots, q^{11}/2$ and $q^{10}/4$ in $\mathrm{UD}_6(2^f)$, $q^3/2, \dots, q^{15}/2$ in $\mathrm{UE}_6(2^f)$, and $q^7/3$ in $\mathrm{UE}_6(3^f)$. Observe that the numbers $k(U, D)$ of irreducible characters of $U$ of fixed degree $D$ can always be expressed as polynomials in $v$ with nonnegative coefficients; such coefficients are in fact integers, except in the cases

$$k(\mathrm{UE}_6(q), q^7),\ k(\mathrm{UE}_6(q), q^7/3) \ \in\ \mathbb{Z}[v/2] \setminus \mathbb{Z}[v].$$

When $p$ is a good prime, the formulas for each $k(U, q^d)$ coincide with the expressions obtained in Goodwin et al. (2016b, Table 3) when $p$ is at least the Coxeter number of $G$.

We recall the algorithm in §2.3. If a core is abelian, then through our computer records of $\mathcal{A}, \mathcal{L}$ and $\mathcal{K}$, each of the characters $\chi_{\underline{b}}^{\underline{a}} = \mathrm{Ind}^{\mathcal{A}} \mathrm{Inf}_{\mathcal{K}} \lambda_{\underline{b}}^{\underline{a}}$ can be constructed. The elements $\underline{a} = (a_{i_1}, \dots, a_{i_n})$ and $\underline{b} = (b_{j_1}, \dots, b_{j_m})$ correspond to character values in $\mathbb{F}_q^\times$ (respectively $\mathbb{F}_q$) on the root subgroups $X_{\alpha_{i_1}}, \dots, X_{\alpha_{i_m}}$ (respectively $X_{\alpha_{j_1}}, \dots, X_{\alpha_{j_n}}$). In fact, the values of each $\chi_{\underline{b}}^{\underline{a}}$ can be in principle explicitly determined, see Goodwin et al. (2017). In the case of nonabelian cores, we still get our characters parametrized as $\mathrm{Ind}^{\mathcal{A}} \mathrm{Inf}_{\mathcal{K}} \psi$ with $\psi \in \mathrm{Irr}(X_{\mathcal{S}})_{\mathcal{Z}}$, but in this case $X_{\mathcal{S}}$ is not abelian. Here we are also able to record electronically the form of the abelian subquotients that yield characters of $U$ by inflation and induction, see the examples worked out in Le et al. (2018). The labels of the characters, which again correspond to character values in such subquotients, are though more elaborated, since diagonal subgroups of products of root groups are involved.

It is enough to provide a parametrization and labels for one representative of each isomorphism class of nonabelian cores. In fact, let $\mathcal{S} = \{\alpha_{i_1}, \dots, \alpha_{i_r}\}$ and $\mathcal{S}' = \{\alpha_{j_1}, \dots, \alpha_{j_r}\}$ be quatterns corresponding to the same isomorphism class of cores. Then the bijection $\rho : \mathcal{S} \to \mathcal{S}'$ yields a bijection $\sigma : \{i_1, \dots, i_r\} \to \{j_1, \dots, j_r\}$. Once we determine a parametrization for the core corresponding to $\mathcal{S}$, the parametrization of the core corresponding to $\mathcal{S}'$ is given just by replacing each index $i$ with $\sigma(i)$, and by possibly re-ordering these labels in increasing index order.

As in §2.4, the label $\underline{a}$ (respectively $\underline{b}$) corresponds to a tuple of elements of $\mathbb{F}_q^\times$ (respectively $\mathbb{F}_q$). The meaning of $\underline{a}^*$ in a label of the form $\underline{a}, \underline{a}^*$ is that $\underline{a}_j^* \in \mathbb{F}_q^\times \setminus \{f_j(\underline{a})\}$ for every $1 \leq j \leq \ell$, where $\ell$ is a positive integer and each $f_j(\underline{a})$ is a nonzero expression depending on $\underline{a}$; these are explicitly determined in each case. A label of the form $c^*$ corresponds to a more involved expression indexed by $\mathbb{F}_q^\times$, detailed in the case-by-case analysis. Finally, labels of the form $d$ index elements of a subset of $(\mathbb{F}_q, +)$ isomorphic to $(\mathbb{F}_p, +)$, and the labels $e^1$ and $e^2$ correspond respectively to $(q+1)/2$ and $(q-1)/2$ elements in $\mathbb{F}_q$ when $q = 3^f$.

### 5.1. *Heartless cores*

Recall that the heart of a nonabelian core consists of the roots $\alpha$ such that $\{\alpha\}$ is a connected component in the associated graph defined in §4.2. Among the cores of $\mathrm{D}_6$ (respectively $\mathrm{E}_6$) listed in Table 3 (respectively Table 4), the following forms are heartless,

$$[3,9,6],\quad [4,8,4],\quad [5,10,5],\quad [5,12,8],\quad [5,15,11],$$

$$[6,12,6],\quad [6,13,7],\quad [6,14,8],\quad [6,16,12],\quad [7,15,9].$$

In Proposition 20 we study in detail the $[6,16,12]$-core, whose analysis departs from the uniform treatment of the remaining heartless cores which we discuss first.

Now let $(\mathcal{S}, \mathcal{Z}, \mathcal{A}, \mathcal{L}, \mathcal{K})$ be a nonabelian core of one of the forms listed above, but not $[6,16,12]$. Then we have that $\mathcal{S} = \mathcal{Z} \cup \mathcal{I} \cup \mathcal{J}$, for $\mathcal{I} = \{i_1, \dots, i_k\}$ and $\mathcal{J} = \{j_1, \dots, j_h\}$ as defined in Section 4, and the study of Equation (6) yields $X' = 1$ or $X' = \{x_{i_1,\dots,i_k}(t) \mid t \in \mathbb{F}_q\}$, and $Y' = 1$ or $Y' = \{x_{j_1,\dots,j_h}(s) \mid s \in$

**Table 4**
Parametrization of nonabelian cores in $UE_6(q)$ for every $p$.

| Form | Family | Label | Number | Degree |
|---|---|---|---|---|
| [3, 9, 6] | $\mathcal{F}_1^{p\neq 2}$ | $\chi^{a_{23},a_{29},a_{31}}$ | $(q-1)^3$ | $q^3$ |
| | $\mathcal{F}_1^{p=2}$ | $\chi^{a_{23},a_{29},a_{31}}_{b_{7,11,19},b_{12,16,24}}$ | $q^2(q-1)^3$ | $q^2$ |
| [3, 10, 9] | $\mathcal{F}_2^{p\neq 2}$ | $\chi^{a_{23},a_{29},a_{31}}_{b_4}$ | $q(q-1)^3$ | $q^3$ |
| | $\mathcal{F}_2^{1,p=2}$ | $\chi^{a_{23},a_{29},a_{31}}$ | $(q-1)^3$ | $q^3$ |
| | $\mathcal{F}_2^{2,p=2}$ | $\chi^{a_{12,16,24},a_{23},a_{29},a_{31}}_{d_4,d_{7,11,19}}$ | $4(q-1)^4$ | $q^3/2$ |
| [4, 8, 4] | $\mathcal{F}_3^1$ | $\chi^{a_8,a_{12},a_{14},a_{18}^*}$ | $(q-1)^3(q-2)$ | $q^2$ |
| | $\mathcal{F}_3^2$ | $\chi^{a_8,a_{12},a_{14}}_{b_{2,7},b_{4,10}}$ | $q^2(q-1)^3$ | $q$ |
| [5, 10, 5] | $\mathcal{F}_4$ | $\chi^{a_7,a_9,a_{10},a_{17},a_{19}}_{b_{1,4,13}}$ | $q(q-1)^5$ | $q^2$ |
| [5, 12, 8] | $\mathcal{F}_5$ | $\chi^{a_9,a_{15},a_{16},a_{26},a_{27}}_{b_{3,4,6,17}}$ | $q(q-1)^5$ | $q^3$ |
| [5, 15, 11] | $\mathcal{F}_6^{p\neq 3}$ | $\chi^{a_{12},a_{16},a_{22},a_{24},a_{25}}$ | $(q-1)^5$ | $q^5$ |
| | $\mathcal{F}_6^{p=3}$ | $\chi^{a_{12},a_{16},a_{22},a_{24},a_{25}}_{b_{1,4,6,13,14},b_{7,9,10,11,19}}$ | $q^2(q-1)^5$ | $q^4$ |
| [5, 16, 15] | $\mathcal{F}_7^{p\neq 2}$ | $\chi^{a_{15},a_{17},a_{18},a_{20},a_{21}}_{b_4}$ | $q(q-1)^5$ | $q^5$ |
| | $\mathcal{F}_7^{1,p=2}$ | $\chi^{a_{15},a_{17},a_{18},a_{20},a_{21}}$ | $(q-1)^5$ | $q^5$ |
| | $\mathcal{F}_7^{2,p=2}$ | $\chi^{a_{8,9,10,12,16},a_{15},a_{17},a_{18},a_{20},a_{21}}_{d_4,d_{2,3,5,7,11}}$ | $4(q-1)^6$ | $q^5/2$ |
| [5, 20, 25] | $\mathcal{F}_8^{p\neq 3}$ | $\chi^{a_{17},a_{18},a_{20},a_{21},a_{24}}_{b_{2,3,5}}$ | $q(q-1)^5$ | $q^7$ |
| | $\mathcal{F}_8^{1,p=3}$ | $\chi^{a_{7,11,13,14,15},a_{17},a_{18},a_{20},a_{21},a_{24}}$ | $(q-1)^6$ | $q^7$ |
| | $\mathcal{F}_8^{2,p=3}$ | $\chi^{a_{17},a_{18},a_{20},a_{21},a_{24}}_{b_{1,6,8,9,10,12,16},b_{2,3,5}}$ | $q^2(q-1)^5$ | $q^6$ |
| [5, 21, 30] | $\mathcal{F}_9^{p\neq 3}$ | $\chi^{a_{17},a_{18},a_{19},a_{20},a_{21}}_{b_4,b_{8,9,10}}$ | $q^2(q-1)^5$ | $q^7$ |
| | $\mathcal{F}_9^{1,p=3}$ | $\chi^{a_{12,13,14,15,16},a_{17},a_{18},a_{19},a_{20},a_{21}}_{b_4}$ | $q(q-1)^6$ | $q^7$ |
| [5, 21, 30] | $\mathcal{F}_9^{2,p=3}$ | $\chi^{e^1_{8,9,10},a_{17},a_{18},a_{19},a_{20},a_{21}}$ | $(q-1)^5(q+1)/2$ | $q^7$ |
| | $\mathcal{F}_9^{3,p=3}$ | $\chi^{e^2_{8,9,10},a_{17},a_{18},a_{19},a_{20},a_{21}}_{d_{1,2,3,5,6,7,11},d_4}$ | $9(q-1)^6/2$ | $q^7/3$ |
| [6, 12, 6] | $\mathcal{F}_{10}^1$ | $\chi^{a_8,a_{10},a_{12},a_{15},a_{23},a_{25}^*}$ | $(q-1)^5(q-2)$ | $q^3$ |
| | $\mathcal{F}_{10}^2$ | $\chi^{a_8,a_{10},a_{12},a_{15},a_{23}}_{b_{1,2,5},b_{4,9,21}}$ | $q^2(q-1)^5$ | $q^2$ |
| [6, 13, 7] | $\mathcal{F}_{11}$ | $\chi^{a_7,a_{11},a_{18},a_{19},a_{26},a_{28}}_{b_{1,5,14,24}}$ | $q(q-1)^6$ | $q^3$ |
| [6, 14, 8] | $\mathcal{F}_{12}^1$ | $\chi^{a_{12},a_{13},a_{15},a_{16},a_{20},a_{22}^*}$ | $(q-1)^5(q-2)$ | $q^4$ |
| | $\mathcal{F}_{12}^2$ | $\chi^{a_{12},a_{13},a_{15},a_{16},a_{20}}_{b_{3,6,7,11},b_{4,8,10,14}}$ | $q^2(q-1)^5$ | $q^3$ |
| [6, 15, 12] | $\mathcal{F}_{13}^{p\neq 2}$ | $\chi^{a_8,a_9,a_{15},a_{20},a_{22},a_{23}}_{b_{4,10}}$ | $q(q-1)^6$ | $q^4$ |
| | $\mathcal{F}_{13}^{1,p=2}$ | $\chi^{a_8,a_9,a_{15},a_{20},a_{22},a_{23}}$ | $(q-1)^6$ | $q^4$ |
| | $\mathcal{F}_{13}^{2,p=2}$ | $\chi^{a_8,a_9,c^*_{14,16,18},a_{15},a_{20},a_{22},a_{23}}_{d_{2,3,6,7},d_{4,10}}$ | $4(q-1)^7$ | $q^4/2$ |
| [6, 16, 12] | $\mathcal{F}_{14}^{p\neq 3}$ | $\chi^{a_{12},a_{16},a_{18},a_{22},a_{24},a_{25}}$ | $(q-1)^6$ | $q^5$ |
| | $\mathcal{F}_{14}^{p=3}$ | $\chi^{a_{12},a_{16},a_{18},a_{22},a_{24},a_{25}}_{b_{1,6,7,14},b_{4,6,7,13}}$ | $q^2(q-1)^6$ | $q^4$ |
| [6, 17, 17] | $\mathcal{F}_{15}^{1,p\neq 2}$ | $\chi^{a_{13},a_{14},a_{15},a_{17},a_{20},a_{23}^*}_{b_4}$ | $q(q-1)^5(q-2)$ | $q^5$ |
| | $\mathcal{F}_{15}^{2,p\neq 2}$ | $\chi^{a_{8,9,10,12,16},a_{13},a_{14},a_{15},a_{17},a_{20}}$ | $(q-1)^6$ | $q^5$ |
| | $\mathcal{F}_{15}^{3,p\neq 2}$ | $\chi^{a_{13},a_{14},a_{15},a_{17},a_{20}}_{b_{2,3,5,7,11},b_4}$ | $q^2(q-1)^5$ | $q^4$ |
| | $\mathcal{F}_{15}^{1,p=2}$ | $\chi^{a_{8,9,10,12,16},a_{13},a_{14},a_{15},a_{17},a_{20}}$ | $(q-1)^6$ | $q^5$ |
| | $\mathcal{F}_{15}^{2,p=2}$ | $\chi^{a_{13},a_{14},a_{15},a_{17},a_{20},a_{23}^*}$ | $(q-1)^5(q-2)$ | $q^5$ |
| | $\mathcal{F}_{15}^{3,p=2}$ | $\chi^{a_{8,9,10,12,16},a_{13},a_{14},a_{15},a_{17},a_{20},a_{23}^*}_{d_{2,3,5,7,11},d_4}$ | $4(q-1)^6(q-2)$ | $q^5/2$ |
| | $\mathcal{F}_{15}^{4,p=2}$ | $\chi^{a_{13},a_{14},a_{15},a_{17},a_{20}}_{b_{2,3,5,7,11},b_4}$ | $q^2(q-1)^5$ | $q^4$ |
| [7, 15, 9] | $\mathcal{F}_{16}^1$ | $\chi^{a_9,a_{12}a_{13},a_{15},a_{16},a_{20}^*,a_{22}^*}$ | $(q-1)^5(q-2)^2$ | $q^4$ |
| | $\mathcal{F}_{16}^2$ | $\chi^{a_9,a_{12}a_{13},a_{15},a_{16},a_{22}}$ | $(q-1)^6$ | $q^4$ |
| | $\mathcal{F}_{16}^3$ | $\chi^{a_9,a_{12}a_{13},a_{15},a_{16},a_{20}^*}_{b_{3,6,7,11},b_{4,8,10,14}}$ | $q^2(q-1)^5(q-2)$ | $q^3$ |

**Table 5**
The numbers of irreducible characters of $\mathrm{UD}_6(q)$ of fixed degree for $q=p^d$, $p\geq 3$, where $v=q-1$.

| $D$ | $k(\mathrm{UD}_6(p^d),D)$, $p\geq 3$ |
|---|---|
| $1$ | $v^6+6v^5+15v^4+20v^3+15v^2+6v+1$ |
| $q$ | $v^7+9v^6+31v^5+54v^4+51v^3+25v^2+5v$ |
| $q^2$ | $v^8+9v^7+38v^6+89v^5+119v^4+89v^3+34v^2+5v$ |
| $q^3$ | $v^8+15v^7+72v^6+165v^5+201v^4+130v^3+40v^2+4v$ |
| $q^4$ | $3v^8+31v^7+124v^6+246v^5+260v^4+145v^3+39v^2+4v$ |
| $q^5$ | $v^{10}+10v^9+46v^8+135v^7+280v^6+393v^5+339v^4+163v^3+36v^2+2v$ |
| $q^6$ | $2v^9+18v^8+77v^7+200v^6+317v^5+288v^4+138v^3+30v^2+2v$ |
| $q^7$ | $5v^8+43v^7+154v^6+282v^5+270v^4+128v^3+25v^2+v$ |
| $q^8$ | $3v^8+31v^7+122v^6+227v^5+208v^4+89v^3+15v^2+v$ |
| $q^9$ | $v^9+9v^8+41v^7+113v^6+181v^5+152v^4+61v^3+8v^2$ |
| $q^{10}$ | $v^8+8v^7+31v^6+62v^5+61v^4+27v^3+5v^2$ |
| $q^{11}$ | $2v^7+12v^6+29v^5+32v^4+15v^3+2v^2$ |
| $q^{12}$ | $v^6+4v^5+6v^4+4v^3+v^2$ |
| $k(\mathrm{UD}_6(q))=v^{10}+13v^9+87v^8+393v^7+1157v^6+2032v^5+2005v^4+1060v^3+275v^2+30v+1$ | |

**Table 6**
The numbers of irreducible characters of $\mathrm{UD}_6(q)$ of fixed degree for $q=2^d$, where $v=q-1$.

| $D$ | $k(\mathrm{UD}_6(2^d),D)$ |
|---|---|
| $1$ | $v^6+6v^5+15v^4+20v^3+15v^2+6v+1$ |
| $q$ | $v^7+9v^6+31v^5+54v^4+51v^3+25v^2+5v$ |
| $q^2$ | $v^8+9v^7+38v^6+89v^5+119v^4+89v^3+34v^2+5v$ |
| $q^3/2$ | $4v^6+8v^5+4v^4$ |
| $q^3$ | $v^8+15v^7+71v^6+163v^5+200v^4+130v^3+40v^2+4v$ |
| $q^4/2$ | $4v^7+16v^6+16v^5+4v^4$ |
| $q^4$ | $4v^8+35v^7+128v^6+247v^5+260v^4+145v^3+39v^2+4v$ |
| $q^5/2$ | $4v^7+16v^6+20v^5+8v^4$ |
| $q^5$ | $v^{10}+10v^9+46v^8+135v^7+278v^6+388v^5+337v^4+163v^3+36v^2+2v$ |
| $q^6/2$ | $8v^7+28v^6+28v^5+8v^4$ |
| $q^6$ | $2v^9+18v^8+76v^7+196v^6+312v^5+286v^4+138v^3+30v^2+2v$ |
| $q^7/2$ | $4v^7+24v^6+32v^5+12v^4$ |
| $q^7$ | $6v^8+47v^7+157v^6+280v^5+268v^4+128v^3+25v^2+v$ |
| $q^8/2$ | $8v^7+36v^6+36v^5+8v^4$ |
| $q^8$ | $4v^8+35v^7+122v^6+221v^5+205v^4+89v^3+15v^2+v$ |
| $q^9/2$ | $12v^7+40v^6+36v^5+8v^4$ |
| $q^9$ | $v^9+9v^8+38v^7+102v^6+168v^5+149v^4+61v^3+8v^2$ |
| $q^{10}/4$ | $16v^6$ |
| $q^{10}/2$ | $8v^7+20v^6+28v^5+4v^4$ |
| $q^{10}$ | $v^8+6v^7+25v^6+55v^5+60v^4+27v^3+5v^2$ |
| $q^{11}/2$ | $8v^6+12v^5+4v^4$ |
| $q^{11}$ | $2v^7+10v^6+26v^5+31v^4+15v^3+2v^2$ |
| $q^{12}$ | $v^6+4v^5+6v^4+4v^3+v^2$ |
| $k(\mathrm{UD}_6(q))=v^{10}+13v^9+90v^8+447v^7+1346v^6+2206v^5+2050v^4+1060v^3+275v^2+30v+1$ | |

$\mathbb{F}_q\}$ for some $x_{i_1,\dots,i_k}(t)$ and $x_{j_1,\dots,j_h}(s)$ as in Definition 11. Hence by Proposition 9 and Equation (2), if we put $Z=X_{\mathcal{Z}}/(\ker\lambda)$ then we have that

$$\mathrm{Ind}_{X'X_{\mathcal{J}}X_{\mathcal{Z}}}^{X_{\mathcal{S}}}\mathrm{Inf}_{X'Y'Z}^{X'X_{\mathcal{J}}X_{\mathcal{Z}}}:\bigsqcup_{\lambda\in\mathrm{Irr}(X_{\mathcal{Z}})}\mathrm{Irr}(X'Y'Z\mid\lambda)\longrightarrow\mathrm{Irr}(X_{\mathcal{S}})_{\mathcal{Z}}$$

**Table 7**
The numbers of irreducible characters of $\mathrm{UE}_6(q)$ of fixed degree for $q = p^d$, $p \geq 5$, where $v = q - 1$.

| $D$ | $k(\mathrm{UE}_6(p^d), D)$, $p \geq 5$ |
|---|---|
| 1 | $v^6 + 6v^5 + 15v^4 + 20v^3 + 15v^2 + 6v + 1$ |
| $q$ | $v^7 + 9v^6 + 31v^5 + 54v^4 + 51v^3 + 25v^2 + 5v$ |
| $q^2$ | $5v^7 + 34v^6 + 93v^5 + 130v^4 + 97v^3 + 36v^2 + 5v$ |
| $q^3$ | $v^9 + 9v^8 + 42v^7 + 123v^6 + 223v^5 + 240v^4 + 145v^3 + 44v^2 + 5v$ |
| $q^4$ | $5v^8 + 42v^7 + 155v^6 + 300v^5 + 316v^4 + 176v^3 + 46v^2 + 4v$ |
| $q^5$ | $2v^9 + 23v^8 + 118v^7 + 327v^6 + 518v^5 + 462v^4 + 219v^3 + 48v^2 + 3v$ |
| $q^6$ | $14v^8 + 113v^7 + 367v^6 + 602v^5 + 523v^4 + 231v^3 + 45v^2 + 3v$ |
| $q^7$ | $v^{11} + 11v^{10} + 57v^9 + 186v^8 + 433v^7 + 730v^6 + 826v^5 + 560v^4 + 204v^3 + 36v^2 + 2v$ |
| $q^8$ | $v^{10} + 10v^9 + 51v^8 + 173v^7 + 396v^6 + 558v^5 + 444v^4 + 183v^3 + 31v^2 + v$ |
| $q^9$ | $3v^9 + 30v^8 + 144v^7 + 385v^6 + 575v^5 + 455v^4 + 177v^3 + 28v^2 + v$ |
| $q^{10}$ | $12v^8 + 95v^7 + 304v^6 + 480v^5 + 375v^4 + 131v^3 + 16v^2 + v$ |
| $q^{11}$ | $2v^9 + 21v^8 + 97v^7 + 243v^6 + 334v^5 + 233v^4 + 71v^3 + 10v^2$ |
| $q^{12}$ | $2v^8 + 20v^7 + 76v^6 + 139v^5 + 124v^4 + 49v^3 + 6v^2$ |
| $q^{13}$ | $3v^7 + 24v^6 + 63v^5 + 68v^4 + 28v^3 + 3v^2$ |
| $q^{14}$ | $4v^6 + 19v^5 + 27v^4 + 12v^3 + v^2$ |
| $q^{15}$ | $3v^5 + 8v^4 + 5v^3$ |
| $q^{16}$ | $v^4 + v^3$ |
| $k(\mathrm{UE}_6(q)) = v^{11} + 12v^{10} + 75v^9 + 353v^8 + 1286v^7 + 3178v^6 + 4770v^5 + 4035v^4 + 1800v^3 + 390v^2 + 36v + 1$ | |

**Table 8**
The numbers of irreducible characters of $\mathrm{UE}_6(q)$ of fixed degree for $q = 3^d$, where $v = q - 1$.

| $D$ | $k(\mathrm{UE}_6(3^d), D)$ |
|---|---|
| 1 | $v^6 + 6v^5 + 15v^4 + 20v^3 + 15v^2 + 6v + 1$ |
| $q$ | $v^7 + 9v^6 + 31v^5 + 54v^4 + 51v^3 + 25v^2 + 5v$ |
| $q^2$ | $5v^7 + 34v^6 + 93v^5 + 130v^4 + 97v^3 + 36v^2 + 5v$ |
| $q^3$ | $v^9 + 9v^8 + 42v^7 + 123v^6 + 223v^5 + 240v^4 + 145v^3 + 44v^2 + 5v$ |
| $q^4$ | $5v^8 + 42v^7 + 155v^6 + 300v^5 + 316v^4 + 176v^3 + 46v^2 + 4v$ |
| $q^5$ | $2v^9 + 23v^8 + 118v^7 + 327v^6 + 518v^5 + 462v^4 + 219v^3 + 48v^2 + 3v$ |
| $q^6$ | $14v^8 + 113v^7 + 367v^6 + 602v^5 + 523v^4 + 231v^3 + 45v^2 + 3v$ |
| $q^7/3$ | $9v^6/2$ |
| $q^7$ | $v^{11} + 11v^{10} + 57v^9 + 186v^8 + 434v^7 + 1463v^6/2 + 827v^5 + 560v^4 + 204v^3 + 36v^2 + 2v$ |
| $q^8$ | $v^{10} + 10v^9 + 52v^8 + 178v^7 + 403v^6 + 560v^5 + 444v^4 + 183v^3 + 31v^2 + v$ |
| $q^9$ | $3v^9 + 30v^8 + 144v^7 + 384v^6 + 572v^5 + 455v^4 + 177v^3 + 28v^2 + v$ |
| $q^{10}$ | $12v^8 + 95v^7 + 304v^6 + 480v^5 + 375v^4 + 131v^3 + 16v^2 + v$ |
| $q^{11}$ | $2v^9 + 21v^8 + 97v^7 + 243v^6 + 334v^5 + 233v^4 + 71v^3 + 10v^2$ |
| $q^{12}$ | $2v^8 + 20v^7 + 76v^6 + 139v^5 + 124v^4 + 49v^3 + 6v^2$ |
| $q^{13}$ | $3v^7 + 24v^6 + 63v^5 + 68v^4 + 28v^3 + 3v^2$ |
| $q^{14}$ | $4v^6 + 19v^5 + 27v^4 + 12v^3 + v^2$ |
| $q^{15}$ | $3v^5 + 8v^4 + 5v^3$ |
| $q^{16}$ | $v^4 + v^3$ |
| $k(\mathrm{UE}_6(q)) = v^{11} + 12v^{10} + 75v^9 + 354v^8 + 1292v^7 + 3190v^6 + 4770v^5 + 4035v^4 + 1800v^3 + 390v^2 + 36v + 1$ | |

is a bijective map.

If $|X'| = q^s$ and $|Y'| = q^t$, put $\delta := \log_q(|X'||Y'|) = s + t$.

- If the sizes of $|X'|$ and $|Y'|$ do not depend on the values of $\lambda$ on $\mathrm{Irr}(X_{\mathcal{Z}})$, then we have that

$$\mathrm{Irr}(X_{\mathcal{S}})_{\mathcal{Z}} = \{\chi^{\underline{a}}_{\underline{b}} \mid \underline{b} \in \mathbb{F}_q^{\delta}, \underline{a} \in (\mathbb{F}_q^{\times})^{|\mathcal{Z}|}\},$$

**Table 9**
The numbers of irreducible characters of $\mathrm{UE}_6(q)$ of fixed degree for $q = 2^d$, where $v = q - 1$.

| $D$ | $k(\mathrm{UE}_6(2^d), D)$ |
|---|---|
| $1$ | $v^6 + 6v^5 + 15v^4 + 20v^3 + 15v^2 + 6v + 1$ |
| $q$ | $v^7 + 9v^6 + 31v^5 + 54v^4 + 51v^3 + 25v^2 + 5v$ |
| $q^2$ | $5v^7 + 34v^6 + 93v^5 + 130v^4 + 97v^3 + 36v^2 + 5v$ |
| $q^3/2$ | $4v^6 + 8v^5 + 4v^4$ |
| $q^3$ | $v^9 + 9v^8 + 42v^7 + 122v^6 + 221v^5 + 239v^4 + 145v^3 + 44v^2 + 5v$ |
| $q^4/2$ | $8v^6 + 16v^5 + 8v^4$ |
| $q^4$ | $5v^8 + 44v^7 + 159v^6 + 302v^5 + 316v^4 + 176v^3 + 46v^2 + 4v$ |
| $q^5/2$ | $12v^6 + 24v^5 + 12v^4$ |
| $q^5$ | $2v^9 + 24v^8 + 123v^7 + 333v^6 + 517v^5 + 459v^4 + 219v^3 + 48v^2 + 3v$ |
| $q^6/2$ | $16v^6 + 32v^5 + 16v^4$ |
| $q^6$ | $14v^8 + 115v^7 + 368v^6 + 597v^5 + 519v^4 + 231v^3 + 45v^2 + 3v$ |
| $q^7/2$ | $24v^7 + 92v^6 + 92v^5 + 20v^4$ |
| $q^7$ | $v^{11} + 11v^{10} + 57v^9 + 188v^8 + 437v^7 + 723v^6 + 811v^5 + 555v^4 + 204v^3 + 36v^2 + 2v$ |
| $q^8/2$ | $4v^7 + 28v^6 + 44v^5 + 20v^4$ |
| $q^8$ | $v^{10} + 10v^9 + 51v^8 + 176v^7 + 399v^6 + 553v^5 + 441v^4 + 183v^3 + 31v^2 + v$ |
| $q^9/2$ | $8v^7 + 44v^6 + 56v^5 + 20v^4$ |
| $q^9$ | $3v^9 + 32v^8 + 154v^7 + 398v^6 + 577v^5 + 452v^4 + 177v^3 + 28v^2 + v$ |
| $q^{10}/2$ | $4v^7 + 28v^6 + 44v^5 + 20v^4$ |
| $q^{10}$ | $13v^8 + 102v^7 + 314v^6 + 479v^5 + 370v^4 + 131v^3 + 16v^2 + v$ |
| $q^{11}/2$ | $4v^7 + 36v^6 + 56v^5 + 20v^4$ |
| $q^{11}$ | $2v^9 + 21v^8 + 98v^7 + 239v^6 + 320v^5 + 224v^4 + 71v^3 + 10v^2$ |
| $q^{12}/2$ | $12v^6 + 28v^5 + 16v^4$ |
| $q^{12}$ | $2v^8 + 20v^7 + 74v^6 + 132v^5 + 119v^4 + 49v^3 + 6v^2$ |
| $q^{13}/2$ | $8v^6 + 24v^5 + 12v^4$ |
| $q^{13}$ | $3v^7 + 22v^6 + 57v^5 + 64v^4 + 28v^3 + 3v^2$ |
| $q^{14}/2$ | $8v^5 + 8v^4$ |
| $q^{14}$ | $4v^6 + 17v^5 + 25v^4 + 12v^3 + v^2$ |
| $q^{15}/2$ | $4v^4$ |
| $q^{15}$ | $3v^5 + 7v^4 + 5v^3$ |
| $q^{16}$ | $v^4 + v^3$ |
| $k(\mathrm{UE}_6(q)) = v^{11} + 12v^{10} + 75v^9 + 359v^8 + 1364v^7 + 3487v^6 + 5148v^5 + 4170v^4 + 1800v^3 + 390v^2 + 36v + 1$ | |

where $\underline{a}$ is indexed by root indices of $X_{\mathcal{Z}}$, and $\underline{b}$ is indexed by the $i_1, \dots, i_k$ or $j_1, \dots, j_h$ in the cases when $X' \neq 1$ or $Y' \neq 1$.

- In the case when $|X'|$ and $|Y'|$ do depend on the values of $\lambda$, then we get a branching which results in a decomposition of $\mathrm{Irr}(X_{\mathcal{S}})_{\mathcal{Z}}$ as a union of families of the following form,

$$\{\chi_{\underline{b}}^{\underline{a},\underline{a}^*} \mid \underline{b} \in \mathbb{F}_q^{\delta}, \underline{a} \in (\mathbb{F}_q^{\times})^{|\mathcal{Z}|-\ell}, \underline{a}^* \in S\},$$

where $1 \leq \ell \leq |\mathcal{Z}|$ is an integer, and

$$S := (\mathbb{F}_q^{\times} \setminus \{f_1(\underline{a})\}) \times \cdots \times (\mathbb{F}_q^{\times} \setminus \{f_\ell(\underline{a})\})$$

for fractional polynomial expressions $f_1, \dots, f_\ell$ that are explicitly determined. Notice that in the case of $\mathrm{D}_6$ and $\mathrm{E}_6$ we always have that $\ell \in \{0, 1\}$ except in the case of a $[7, 15, 8]$-core, where we have that $\ell = 2$.

The successive stabilizers in a study of a nonabelian core are given in terms of the solutions of an equation of the form $\phi(P)=0$ with $P$ some polynomial expression in several variables. If $P$ is linear in all variables, as in the case of Equation (6) corresponding to the first stabilizer in the core examination, then we can find solutions and labels via our programs in GAP4. If this is not the case, then the number of such solutions does depend on the values of the character $\lambda$, and we obtain families of curves in $\mathbb{F}_q^s$ for some $s \in \mathbb{Z}_{\geq 1}$. Describing such curves is a challenge in computational algebra. Here lies essentially the only part of the computations that one has to perform by hand.

The cores of the form [3, 9, 6] in both $D_6$ and $E_6$ are isomorphic to one of the nonabelian cores of $F_4$ determined in Goodwin et al. (2016a, §4.3). As done in Goodwin et al. (2016a), we include no further details here for the straightforward analysis of heartless cores, and we refer to Le et al. (2018) for information on the sets $\mathcal{S}$, $\mathcal{Z}$, $\mathcal{A}$ and $\mathcal{L}$ and the form of Equation (6) and other equations defining stabilizers in each case. The labels of the sets $\mathrm{Irr}(X_{\mathcal{S}})_{\mathcal{Z}}$ are collected in Tables 3 and 4.

Recall that there is a unique nonabelian core of $E_6$ of the form [6, 16, 12]. In this case,

- $\mathcal{S}=\{\alpha_1,\alpha_4,\alpha_6,\alpha_7,\alpha_9,\alpha_{10},\alpha_{11},\alpha_{12},\alpha_{13},\alpha_{14},\alpha_{16},\alpha_{18},\alpha_{19},\alpha_{22},\alpha_{24},\alpha_{25}\}$,
- $\mathcal{Z}=\{\alpha_{12},\alpha_{16},\alpha_{18},\alpha_{22},\alpha_{24},\alpha_{25}\}$,
- $\mathcal{A}=\{\alpha_2,\alpha_3,\alpha_5,\alpha_{15}\}$ and $\mathcal{L}=\{\alpha_8,\alpha_{17},\alpha_{20},\alpha_{21}\}$,
- $\mathcal{I}=\{\alpha_1,\alpha_4,\alpha_6,\alpha_7,\alpha_{13},\alpha_{14}\}$ and $\mathcal{J}=\{\alpha_9,\alpha_{10},\alpha_{11},\alpha_{19}\}$.

Our analysis differs from the previous cases in that $X'$ is not a subgroup, and $|X'|=q^2$. The graph structure of $\mathfrak{C}$, represented in Fig. 3, is more complicated than in the case of the other heartless cores, as we have 7 circles of both parities. This case can be examined in a similar way by applying Proposition 9 after the first reduction. We end this subsection by including the computational details in this case.

**Proposition 20.** *The irreducible characters corresponding to the* [6, 16, 12]*-core in type* $E_6$ *are parametrized as follows:*

- *If $p \neq 3$, then $\mathrm{Irr}(X_{\mathcal{S}})_{\mathcal{Z}}=\mathcal{F}_{14}^{p\neq 3}$ consists of $(q-1)^6$ characters of degree $q^5$.*
- *If $p=3$, then $\mathrm{Irr}(X_{\mathcal{S}})_{\mathcal{Z}}=\mathcal{F}_{14}^{p=3}$ consists of $q^2(q-1)^6$ characters of degree $q^4$.*

*The labels of the characters in $\mathcal{F}_{14}^{p\neq 3}$ and $\mathcal{F}_{14}^{p=3}$ are collected in Table 4.*

**Proof.** The form of Equation (6) is

$$s_1(a_{22}t_{19}+a_{12}t_9)+s_4(a_{16}t_{11}+a_{24}t_{19})+s_6(-a_{16}t_{10}-a_{25}t_{19})+s_7(a_{18}t_{10}) \\ +s_{13}(a_{24}t_{10}+a_{25}t_{11})+s_{14}(a_{24}t_9)=0.$$

We have $X'=X_1'X_2'$ with $X_1':=\{x_{1,6,7,14}(t_1)\mid t_1\in\mathbb{F}_q\}$ and $X_2':=\{x_{4,6,7,13}(t_2)\mid t_2\in\mathbb{F}_q\}$, and $Y'=1$, where

$$x_{1,6,7,14}(t_1):=x_1(a_{18}a_{24}a_{25}t_1)x_6(a_{18}a_{22}a_{24}t_1)x_7(a_{16}a_{22}a_{24}t_1)x_{14}(-a_{12}a_{18}a_{25}t_1)$$

and

$$x_{4,6,7,13}(t_2):=x_4(a_{18}a_{25}t_2)x_6(a_{18}a_{24}t_2)x_7(2a_{16}a_{24}t_2)x_{13}(-a_{16}a_{18}t_2).$$

We notice that each of $X_1'$ and $X_2'$ are subgroups, but we have

$$[x_{1,6,7,14}(t_1),x_{4,6,7,13}(t_2)]=x_{12}(a_{16}a_{18}a_{22}a_{24}a_{25}t_1t_2)x_{16}(2a_{12}a_{18}a_{22}a_{24}a_{25}t_1t_2),$$

hence

$$\lambda([x_{1,6,7,14}(t_1),x_{4,6,7,13}(t_2)])=\phi(3a_{12}a_{16}a_{18}a_{22}a_{24}a_{25}t_1t_2)$$

and $X'$ is not necessarily a subgroup of $X_{\mathcal{S}}$.

If $p \neq 3$, then we can apply again Proposition 9 with arm $X'_1$ and leg $X'_2$, reducing to the abelian subquotient $X_{\mathcal{Z}}/(\ker\lambda)$. This gives the family $\mathcal{F}_{14}^{p\neq 3}$ in Table 4.

If $p = 3$, then $X'$ and $X'X_{\mathcal{Z}}$ are abelian subgroups of $X_{\mathcal{S}}/(\ker\lambda)$. In this case, we obtain the family $\mathcal{F}_{14}^{p=3}$ in Table 4, which concludes our analysis. □

### 5.2. *Cores with a heart*

We now want to investigate the cores of the form

$$[3,10,9],\ [4,18,18],\ [4,21,28],\ [4,24,43],\ [5,18,18],\ [6,19,20]$$

in type $D_6$, and of the form

$$[3,10,9],\ [5,16,15],\ [5,20,25],\ [5,21,30],\ [6,15,12],\ [6,17,17]$$

in type $E_6$. To study them, we apply repeatedly Proposition 9 and Corollary 12.

We recall the structure of the graph of $\mathfrak{C}$ in each case. The cores of the form $[3,10,9]$ have a single circle, and a heart of size 1. The graphs of the cores of the form $[4,18,18], [4,21,28]$ and $[5,18,18]$ have two connected components, and each of them is a hexagon. Their hearts have sizes 2, 5 and 1 respectively. The graph of the $[6,19,20]$-core contains 6 circles, as Fig. 1 shows, and its heart has size 1. The graph of $[5,16,15]$-core, as in Fig. 1, has three circles, as well as the graphs of the $[5,20,25]$-core and the $[5,21,30]$-core, and their hearts have sizes 1, 5 and 6 respectively. The heart of the latter nonabelian core is the biggest among all nonabelian cores in rank 6 or less, which makes it one of the most complicated to study. We just refer to Le and Magaard (2015, Section 3) in the sequel, where the study of the $[5,21,30]$-core of $E_6$ has been carried out thoroughly. The graph of $[6,15,12]$-core has two connected components, namely its unique circle, and its heart of size 1. Finally, we find 3 circles in the $[6,17,17]$-core; here $|\mathcal{H}| = 1$.

The cores of the form $[3,10,9]$ in types $D_6$ and $E_6$ are isomorphic to the only $[3,10,9]$-core in type $F_4$. The additional complication in the analysis of a core with a heart, as in Goodwin et al. (2016a, §4.3), lies in the determination of a certain non-linear polynomial over $\mathbb{F}_q$, which arises from the action of the root subgroups indexed by the heart on a suitable subquotient of $X_{\mathcal{S}}$, and of the solutions in $\mathbb{F}_q$ of an equation depending on such a polynomial and the function $\phi : \mathbb{F}_q \to \mathbb{C}^\times$ defined in §2.1. The typical situation is that we get a polynomial of degree $p$ when $p$ is a bad prime for $G$. If $p = 2$, then the situation can be easily described.

**Remark 21.** Let $q = 2^f$, and let us consider the following expression in $\mathbb{F}_q$,

$$f(s,t) = \phi(st(b + at))$$

for every $b, a \in \mathbb{F}_q$. Let us define

$$Z_1 := \{s \in \mathbb{F}_q \mid f(s,t) = 1 \text{ for all } t \in \mathbb{F}_q\}, \quad Z_2 := \{t \in \mathbb{F}_q \mid f(s,t) = 1 \text{ for all } s \in \mathbb{F}_q\}.$$

It is easy to see that

- If $b = a = 0$, then $Z_1 = Z_2 = \mathbb{F}_q$.
- If ($b \neq 0$ and $a = 0$) or ($b = 0$ and $a \neq 0$), then $Z_1 = Z_2 = \{0\}$.
- If $b \neq 0$ and $a \neq 0$, then $Z_1 = \{0, a/b^2\}$ and $Z_2 = \{0, b/a\}$.

When $p = 3$ is a bad prime for $E_6$, the polynomial arising from the above investigation is of degree 3 just in the case of the core of the form $[5,21,30]$, which gives rise to irreducible character degrees $q^7/3$ in $UE_6(3^f)$; as previously remarked, the study of this core is detailed in Le and Magaard (2015, Section 3). We include below the analysis just for three nonabelian cores with a heart. We discuss first the core of the form $[4,18,18]$ in $D_6$, which for $p = 2$ gives rise to the only examples of irreducible characters of $\mathrm{UY}_r(q)$ of the form $q^m/p^i$ with $i \geq 2$ when Y is of simply laced type and $r \leq 6$. We then include full details for the $[4, 24, 43]$-core in type $\mathrm{D}_6$ and the $[5, 20, 25]$-core in type $\mathrm{E}_6$; one notices the different behavior of the bad primes $p = 2$ in type $\mathrm{D}_6$ and $p = 3$ in type $\mathrm{E}_6$ respectively. We decide to include every step in their study since these two cores, along with the $[5, 21, 30]$-core, seem to be the most difficult cases to examine. The other cases are investigated in a similar manner; the computations are available in Le et al. (2018). All character labels for each nonabelian core with a heart are also collected in Tables 3 and 4.

We use the following notation for a core $\mathfrak{C}$ corresponding to $\mathcal{S}$ and $\mathcal{Z}$. We construct subquotients $V^n_{\underline{b}}$, and $H^n_{\underline{b}}$, $X^n_{\underline{b}}$, $Y^n_{\underline{b}}$, $X'^n_{\underline{b}}$, $Y'^n_{\underline{b}}$ and the character $\lambda^{\underline{b}}$ in the following way. The index $n$ corresponds to the $n$-th application of Proposition 9 (possibly trivial if we just enlarge the kernel of a central character from step $n-1$ to step $n$), and $\underline{b}$ denotes a certain tuple with entries in $\mathbb{F}_q$, which corresponds to the value of a central character.

We initialize $\underline{b} = \emptyset$ the empty tuple and $V^0_\emptyset = X_{\mathcal{S}}$, and $\lambda^\emptyset = \lambda$ a central character. Now assume $V^{n-1}_{\hat{\underline{b}}}$ is constructed for $n \geq 1$. We assume that Proposition 9 applies to $V = V^{n-1}_{\hat{\underline{b}}}$ and let $H =: H^n_{\hat{\underline{b}}}$, $X =: X^n_{\hat{\underline{b}}}$, $Y =: Y^n_{\hat{\underline{b}}}$, $X' =: X'^n_{\hat{\underline{b}}}$, $Y' =: Y'^n_{\hat{\underline{b}}}$. Finally, for every $\tilde{\underline{b}} = (\tilde{b_1}, \ldots, \tilde{b_s}) \in \mathbb{F}_q^s$ and $\mu_{\tilde{\underline{b}}}$ as in Corollary 12, we define $\underline{b}$ as the concatenation of $\hat{\underline{b}}$ and $\tilde{\underline{b}}$ and $\lambda^{\underline{b}} := \lambda^{\hat{\underline{b}}} \otimes \mu_{\tilde{\underline{b}}}$, and we construct $V^n_{\underline{b}} := (V^{n-1}_{\hat{\underline{b}}})_{\tilde{\underline{b}}}$. By convention, in the sequel we omit the top index 1, and we drop the symbol $\emptyset$ when it occurs.

We expand in the rest of this section the computations for the $[4, 18, 18]$- and the $[4, 24, 43]$-cores in type $\mathrm{D}_6$, and for the $[5, 20, 25]$-core in type $\mathrm{E}_6$. The computations for the other nonabelian cores in types $\mathrm{D}_6$ and $\mathrm{E}_6$, which happen to be easier, are collected in Le et al. (2018). We recall that our computer programs help out finding candidates for arms and legs as in §4.2. Such computer programs are also used to compute commutators of products of root elements and finding the form of the equations involving $\lambda$ and $\phi$. The branching for the values of the $a_i$'s and the solutions of non-linear equations in several variables remain a computational challenge, and constitute the part of the computations that has been studied by a case-by-case check.

We start by studying the unique core of the form $[4, 18, 18]$ in type $\mathrm{D}_6$. In this case,

- $\mathcal{S} = \{\alpha_1, \alpha_2, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_{10}, \alpha_{11}, \alpha_{12}, \alpha_{14}, \alpha_{15}, \alpha_{16}, \alpha_{17}, \alpha_{18}, \alpha_{19}, \alpha_{21}, \alpha_{22}, \alpha_{28}\}$,
- $\mathcal{Z} = \{\alpha_{16}, \alpha_{21}, \alpha_{22}, \alpha_{28}\}$,
- $\mathcal{A} = \{\alpha_3, \alpha_8, \alpha_9, \alpha_{13}\}$ and $\mathcal{L} = \{\alpha_{20}, \alpha_{23}, \alpha_{24}, \alpha_{26}\}$,
- $\mathcal{I} = \{\alpha_1, \alpha_5, \alpha_6, \alpha_{12}, \alpha_{14}, \alpha_{15}\}$ and $\mathcal{J} = \{\alpha_7, \alpha_{10}, \alpha_{11}, \alpha_{17}, \alpha_{18}, \alpha_{19}\}$.

**Proposition 22.** *The irreducible characters corresponding to the* $[4, 18, 18]$*-core in type* $\mathrm{D}_6$ *are parametrized as follows:*

- *If* $p \neq 2$*, then* $\mathrm{Irr}(X_{\mathcal{S}})_{\mathcal{Z}} = \mathcal{F}_3^{p\neq 2}$ *consists of* $q^2(q-1)^4$ *characters of degree* $q^6$.
- *If* $p = 2$*, then*

$$\mathrm{Irr}(X_{\mathcal{S}})_{\mathcal{Z}} = \mathcal{F}_3^{1,p=2} \sqcup \mathcal{F}_3^{2,p=2} \sqcup \mathcal{F}_3^{3,p=2} \sqcup \mathcal{F}_4^{4,p=2},$$

  *where*
  - $\mathcal{F}_3^{1,p=2}$ *consists of* $(q-1)^4$ *characters of degree* $q^6$,
  - $\mathcal{F}_3^{2,p=2}$ *and* $\mathcal{F}_3^{3,p=2}$ *consist each of* $4(q-1)^5$ *characters of degree* $q^6/2$*, and*
  - $\mathcal{F}_3^{4,p=2}$ *consists of* $16(q-1)^6$ *characters of degree* $q^6/4$.

*The labels of the characters in* $\mathcal{F}_3^{p\neq 2}$ *and in* $\mathcal{F}_3^{1,p=2}, \ldots, \mathcal{F}_3^{4,p=2}$ *are collected in Table 3.*

**Proof.** The form of Equation (6) is

$$s_7(a_{21}t_{14} + a_{22}t_{15}) + s_{10}(-a_{16}t_6 - a_{21}t_{12}) + s_{11}(-a_{16}t_5 - a_{22}t_{12}) + s_{17}(a_{21}t_5 + a_{22}t_6) +$$

$$+ s_{18}(-a_{21}t_1 - a_{28}t_{15}) + s_{19}(-a_{22}t_1 - a_{28}t_{14}) = 0.$$

If $p \neq 2$, then $X' = Y' = 1$, and $\overline{V} = X_2X_4Z/(\ker\lambda)$ is abelian. We obtain the family $\mathcal{F}_3^{p\neq 2}$ in Table 3.

If $p = 2$, we have that $X' := X_1'X_2'$ and $Y_1'Y_2'$, where

$$X_1' := \{x_{1,14,15}(t_1) \mid t_1 \in \mathbb{F}_q\} \quad \text{and} \quad X_1' := \{x_{5,6,12}(t_2) \mid t_2 \in \mathbb{F}_q\},$$
$$Y_1' := \{x_{7,18,19}(s_1) \mid s_1 \in \mathbb{F}_q\} \quad \text{and} \quad Y_2' := \{x_{10,11,17}(s_2) \mid s_2 \in \mathbb{F}_q\},$$

and for every $s_1, s_2, t_1, t_2 \in \mathbb{F}_q$,

$$x_{1,14,15}(t_1) := x_1(a_{28}t_1)x_{14}(a_{22}t_1)x_{15}(a_{21}t_1) \quad \text{and}$$
$$x_{5,6,12}(t_2) := x_5(a_{22}t_2)x_6(a_{21}t_2)x_{12}(a_{16}t_2)$$
$$x_{7,18,19}(s_1) := x_7(a_{28}s_1)x_{18}(a_{22}s_1)x_{19}(a_{21}s_1) \quad \text{and}$$
$$x_{10,11,17}(s_2) := x_{10}(a_{22}s_2)x_{11}(a_{21}s_2)x_{17}(a_{16}s_2).$$

Notice that $X'$ is a subgroup of $\overline{V}$. We extend $\lambda$ to $\lambda' = \lambda^{c_{7,18,19},c_{10,11,17}}$ for every $c_{7,18,19}, c_{10,11,17} \in \mathbb{F}_q$. In $\overline{V}$, we have that $[X_1', X_4] = [X_2', X_2] = 1$, and that

$$[x_2(s_2)x_4(s_4), x_{1,14,15}(t_1)x_{5,6,12}(t_2)] = x_{7,18,19}(s_2t_1)x_{10,11,17}(s_4t_2)x_{16}(a_{21}a_{22}s_4t_2^2) \cdot$$
$$\cdot x_{21}(a_{22}a_{28}s_2t_1^2 + a_{16}a_{22}s_4t_2^2)x_{22}(a_{21}a_{28}s_2t_1^2 + a_{16}a_{21}s_4t_2^2)x_{28}(a_{21}a_{22}s_2t_1^2).$$

We then want to apply Proposition 9 with $X'$ as a candidate for an arm, and $X_2X_4$ as a candidate for a leg. We apply $\lambda$ to the above, and we use Remark 21 study the equation

$$\phi(s_2t_1(c_{7,18,19} + a_{21}a_{22}a_{28}t_1) + s_4t_2(c_{10,11,17} + a_{16}a_{21}a_{22}t_2)) = 1.$$

If $c_{7,18,19} = 0$ and $c_{10,11,17} = 0$, then $X'^2_{(0,0)} = Y'^2_{(0,0)} = 1$ and $V^2_{(0,0)}$ is abelian. This gives the family $\mathcal{F}_3^{1,p=2}$ in Table 3.

If $a_{7,18,19} := c_{7,18,19} \neq 0$ and $c_{10,11,17} = 0$, then

$$X'^2_{(a_{7,18,19},0)} := \{1, x_{1,14,15}(a_{7,18,19}/(a_{21}a_{22}a_{28}))\} \text{ and}$$
$$Y'^2_{(a_{7,18,19},0)} := \{1, x_2(a_{21}a_{22}a_{28}/(a_{7,18,19}^2))\},$$

and $V^2_{(a_{7,18,19},0)}$ is abelian. This gives the family $\mathcal{F}_3^{2,p=2}$ in Table 3.

If $c_{7,18,19} = 0$ and $a_{10,11,17} := c_{10,11,17} \neq 0$, then

$$X'^2_{(0,a_{10,11,17})} := \{1, x_{5,6,12}(a_{10,11,17}/(a_{16}a_{21}a_{22}))\} \text{ and}$$
$$Y'^2_{(0,a_{10,11,17})} := \{1, x_4(a_{16}a_{21}a_{22}/(a_{10,11,17}^2))\},$$

and $V^2_{(0,a_{10,11,17})}$ is abelian. This gives the family $\mathcal{F}_3^{3,p=2}$ in Table 3.

Finally, if $a_{7,18,19} := c_{7,18,19} \neq 0$ and $a_{10,11,17} := c_{10,11,17} \neq 0$, then we have that $X'^2_{(a_{7,18,19},a_{10,11,17})} = X'^2_{(a_{7,18,19},0)}X'^2_{(0,a_{10,11,17})}$ and $Y'^2_{(a_{7,18,19},a_{10,11,17})} = Y'^2_{(a_{7,18,19},0)}Y'^2_{(0,a_{10,11,17})}$, and $V^2_{(a_{7,18,19},a_{10,11,17})}$ is abelian. This yields the family $\mathcal{F}_3^{4,p=2}$ in Table 3.

We observe that

$$(q^6)^2|\mathcal{F}_3^{1,p=2}| + (q^6/2)^2|\mathcal{F}_3^{2,p=2}| + (q^6/2)^2|\mathcal{F}_3^{3,p=2}| + (q^6/4)^2|\mathcal{F}_3^{4,p=2}| = q^{14}(q-1)^4,$$

and since $|\mathcal{S} \setminus \mathcal{Z}| = 14$ and $|\mathcal{Z}| = 4$, Equation (3) then yields

$$\mathrm{Irr}(X_{\mathcal{S}})_{\mathcal{Z}} = \mathcal{F}_3^{1,p=2} \sqcup \mathcal{F}_3^{2,p=2} \sqcup \mathcal{F}_3^{3,p=2} \sqcup \mathcal{F}_4^{1,p=2},$$

which is our second claim. □

We now study the unique core of the form $[4, 24, 43]$ in type $\mathrm{D}_6$. In this case,

- $\mathcal{S} = \{\alpha_1, \dots, \alpha_{24}\}$,
- $\mathcal{Z} = \{\alpha_{21}, \alpha_{22}, \alpha_{23}, \alpha_{24}\}$,
- $\mathcal{A} = \mathcal{L} = \emptyset$,
- $\mathcal{I} = \{\alpha_1, \alpha_5, \alpha_6\}$ and $\mathcal{J} = \{\alpha_{17}, \alpha_{18}, \alpha_{19}\}$.

**Proposition 23.** *The irreducible characters corresponding to the $[4, 24, 43]$-core in type $\mathrm{D}_6$ are parametrized as follows:*

- *If $p \neq 2$, then*
$$\mathrm{Irr}(X_{\mathcal{S}})_{\mathcal{Z}} = \mathcal{F}_5^{1,p\neq 2} \sqcup \mathcal{F}_5^{2,p\neq 2} \sqcup \mathcal{F}_5^{3,p\neq 2},$$
*where*
  - $\mathcal{F}_5^{1,p\neq 2}$ *consists of $q(q-1)^5$ characters of degree $q^9$,*
  - $\mathcal{F}_5^{2,p\neq 2}$ *consists of $(q-1)^5$ characters of degree $q^9$, and*
  - $\mathcal{F}_5^{3,p\neq 2}$ *consists of $q^2(q-1)^4$ characters of degree $q^8$.*
- *If $p = 2$, then*
$$\mathrm{Irr}(X_{\mathcal{S}})_{\mathcal{Z}} = \mathcal{F}_5^{1,p=2} \sqcup \mathcal{F}_5^{2,p=2} \sqcup \mathcal{F}_5^{3,p=2} \sqcup \mathcal{F}_5^{4,p=2},$$
*where*
  - $\mathcal{F}_5^{1,p=2}$ *consists of $4q(q-1)^5$ characters of degree $q^9/2$,*
  - $\mathcal{F}_5^{2,p=2}$ *consists of $q^2(q-1)^5$ characters of degree $q^8$,*
  - $\mathcal{F}_5^{3,p=2}$ *consists of $q(q-1)^4$ characters of degree $q^8$, and*
  - $\mathcal{F}_5^{4,p=2}$ *consists of $4q(q-1)^5$ characters of degree $q^8/2$.*

*The labels of the characters in $\mathcal{F}_5^{1,p\neq 2}, \dots, \mathcal{F}_5^{3,p\neq 2}$ and in $\mathcal{F}_5^{1,p=2}, \dots, \mathcal{F}_5^{4,p=2}$ are collected in Table 3.*

**Proof.** The form of Equation (6) is

$$s_{17}(a_{21}t_5 + a_{22}t_6) + s_{18}(-a_{21}t_1 - a_{23}t_6) + s_{19}(-a_{22}t_1 - a_{23}t_5) = 0.$$

Let $p \neq 2$. Then $X' = Y' = 1$, and $\overline{V} = X_2X_3X_4X_7 \cdots X_{16}X_{20}Z/(\ker\lambda)$. Notice that $X_2 \cap [\overline{V}, \overline{V}] = X_4 \cap [\overline{V}, \overline{V}] = 1$, and $[X_i, X_{20}] \neq 1$ just for $i = 2, 4$. Then we can take $X_2X_4$ for a candidate of an arm and $X_{20}$ for a candidate of a leg. We have

$$[x_{20}(s_{20}), x_2(t_2)x_4(t_4)] = x_{23}(-s_{20}t_2)x_{24}(-s_{20}t_4).$$

Hence we apply Proposition 9 with $X'^2 = X_{2,4} = \{x_{2,4}(t) \mid t \in \mathbb{F}_q\}$ and $Y'^2 = 1$, reducing to $V^2 = X_{2,4}X_3X_7 \cdots X_{16}Z/(\ker\lambda)$; here, we have

$$X_{2,4} := \{x_{2,4}(t) \mid t \in \mathbb{F}_q\} \quad \text{where} \quad x_{2,4}(t) := x_2(a_{24}t)x_4(-a_{23}t).$$

We have that $X_{12}X_{14}X_{15}$ is a subgroup of $V^2$, and that

$$[X_{12}X_{14}X_{15}, X_i] \neq 1 \implies i \in \{7, 10, 11\} \text{ and } X_i \cap [V^2, V^2] = 1 \text{ for } i \in \{7, 10, 11\}. \tag{7}$$

We then apply Proposition 9 with $X_7X_{10}X_{11}$ and $X_{12}X_{14}X_{15}$ as candidates for an arm and a leg respectively, reducing to studying the equation

$$s_{12}(a_{21}t_{10} + a_{22}t_{11}) + s_{14}(-a_{21}t_7 - a_{24}t_{11}) + s_{15}(-a_{22}t_7 - a_{24}t_{10}) = 0. \tag{8}$$

As $p \neq 2$, we have that $X'^3 = Y'^3 = 1$. We reduce to $V^3 = X_{2,4}X_3X_8X_9X_{13}X_{16}Z/(\ker\lambda)$.

We observe that in $V^3$ we have that if $i=8,9$, then $[X_{2,4},X_i]=X_{13}$ and $[X_k,X_i]\neq 1$ just for $k=16$, and that $X_{2,4}\cap[V^3,V^3]=1=X_{16}\cap[V^3,V^3]$. Moreover, we notice that $X_{13}$ is central in $V^3$; we extend $\lambda$ to $\lambda^{c_{13}}$ in the usual way for every $c_{13}\in\mathbb{F}_q$. If $a_{13}:=c_{13}\neq 0$, we apply Proposition 9 with $X_{2,4}X_{16}$ as a candidate for an arm, and $X_8X_9$ as a candidate for a leg. We have that

$$\lambda([x_{2,4}(t)x_{16}(t_{16}),x_8(s_8)x_9(s_9)])=\phi(a_{13}t(a_{24}s_9+a_{23}s_8)+t_{16}(a_{24}s_9-a_{23}s_8)). \tag{9}$$

We get that $X'^4_{(a_{13})}=Y'^4_{(a_{13})}=1$, and $V^4_{(a_{13})}=X_3X_{13}Z/(\ker\lambda)$ is abelian. We obtain the family $\mathcal{F}_5^{1,p\neq 2}$ in Table 3.

Let us now assume that $c_{13}=0$. We examine $V^3_{(0)}$, and we notice that in this case $[X_{2,4},X_i]=1$ if $i=8,9$. Hence we apply Proposition 9 with $X_8X_9$ as candidate for a leg, and $X_{16}$ as candidate for an arm. We get the expression as in Equation (9) by replacing $a_{13}$ with 0. We obtain $X'^4_{(0)}=X_{8,9}:=\{x_{8,9}(t)\mid t\in\mathbb{F}_q\}$ and $Y'^4_{(0)}=1$. Here, we have $x_{8,9}(t):=x_8(a_{24}t)x_9(a_{23}t)$ for every $t\in\mathbb{F}_q$. Notice that $X_{8,9}$ is central in $V'^4_{(0)}=X_{2,4}X_3X_{8,9}Z/(\ker\lambda)$; we denote by $\lambda''=\lambda'^{c_{8,9}}$ the usual extension of $\lambda'$ to $X_{8,9}$ for every $c_{8,9}\in\mathbb{F}_q$.

If $a_{8,9}:=c_{8,9}\neq 0$, then we have

$$\lambda([x_{2,4}(s),x_3(t)])=\lambda(x_{8,9}(st))=\phi(a_{8,9}st).$$

Proposition 9 applies again, with arm $X_{2,4}$ and leg $X_3$, and we reduce to $V^5_{(0,a_{8,9})}=X_{8,9}Z/(\ker\lambda')$. We get the family $\mathcal{F}_5^{2,p\neq 2}$ in Table 3.

Finally, if $a_{8,9}:=c_{8,9}\neq 0$ then $V^5_{(0,0)}:=X_{2,4}X_3Z/(\ker\lambda')$ is abelian; this gives the family $\mathcal{F}_5^{3,p\neq 2}$ in Table 3.

As done at the end of Proposition 22, the claim in the case $p\neq 2$ follows by a counting argument and Equation (3).

Let us now assume $p=2$. In this case, we have

$$X':=\{x_{1,5,6}(t)\mid t\in\mathbb{F}_q\}\qquad\text{and}\qquad Y':=\{x_{17,18,19}(s)\mid s\in\mathbb{F}_q\},$$

where for every $s,t\in\mathbb{F}_q$,

$$x_{1,5,6}(t):=x_1(a_{23}t)x_5(a_{22}t)x_6(a_{21}t)\qquad\text{and}\qquad x_{17,18,19}(s):=x_{17}(a_{23}s)x_{18}(a_{22}s)x_{19}(a_{21}s),$$

and $\overline{V}=X_2X_3X_4X_7\cdots X_{16}X_{20}X'Y'Z/(\ker\lambda)$. In a similar way to the case $p\neq 2$ after computing $X'$ and $Y'$, we notice that we can apply Proposition 9 with $X'^2=X_{2,4}$ and $Y'^2=1$. We reduce to $V^2=X_{2,4}X_3X_7\cdots X_{16}X'Y'Z/(\ker\lambda)$. We notice that $Y'$ is central in $V^2$; let us denote by $\lambda':=\lambda^{c_{17,18,19}}$ the usual extension of $\lambda$.

Suppose $a_{17,18,19}:=c_{17,18,19}\neq 0$. In the group $V$, we have

$$[X_i,X_{13}]\neq 1\Rightarrow i\in\{1,5,6\}\quad\text{and}\quad X_i\cap[V,V]=1\text{ for }i\in\{1,5,6\}.$$

We can then apply Proposition 9 with $X_{1,5,6}$ as a candidate for an arm, and $X_{13}$ as a candidate for a leg. In $V^2$, we have

$$[x_{13}(s_{13}),x_{1,5,6}(t)]=x_{17,18,19}(s_{13}t)x_{21}(a_{22}a_{23}s_{13}t^2)x_{22}(a_{21}a_{23}s_{13}t^2)x_{23}(a_{21}a_{22}s_{13}t^2),$$

hence applying $\lambda'$ we obtain the following equation,

$$\phi(a_{17,18,19}s_{13}t+a_{21}a_{22}a_{23}s_{13}t^2)=1. \tag{10}$$

We have that

$$X'^3_{(a_{17,18,19})}=\{1,x_{1,5,6}(a_{17,18,19}/(a_{21}a_{22}a_{23}))\}\text{ and }Y'^3_{(a_{17,18,19})}:=\{1,x_{13}(a_{21}a_{22}a_{23}/(c^2_{17,18,19}))\},$$

and $V^3_{(a_{17,18,19})}=X_{2,4}X_3X_7\cdots X_{12}X_{14}X_{15}X_{16}X'^3_{(a_{17,18,19})}Y'^3_{(a_{17,18,19})}Y'Z/(\ker\lambda')$. In this subquotient, we have that $[X_{2,4},X_{12}X_{14}X_{15}]\cap X_{17,18,19}\neq 0$, that $X_{2,4}\cap[V^3_{(a_{17,18,19})},V^3_{(a_{17,18,19})}]$=1, and that Equation

(7) holds. Moreover, recall that in $V$ we have that if $k \in \{7, 10, 11\}$, then $[X_i, X_j] \cap X_k \neq 1$ implies $i \in \{2, 4\}$ or $j \in \{2, 4\}$. We can then take $X_{2,4}X_7X_{10}X_{11}$ and $X_{12}X_{14}X_{15}$ as candidates for an arm and a leg respectively. We get the equation

$$\lambda([x_{12}(s_{12})x_{14}(s_{14})x_{15}(s_{15}), x_{2,4}(t)x_7(t_7)x_{10}(t_{10})x_{11}(t_{11})]) = \lambda(x_{17}(a_{23}s_{12}t_1)x_{18}(a_{24}s_{14}t_1)\cdot$$
$$\cdot x_{19}(a_{24}s_{15}t_1))\phi(s_{12}(a_{21}t_{10} + a_{22}t_{11}) + s_{14}(a_{21}t_7 + a_{24}t_{11}) + s_{15}(a_{22}t_7 + a_{24}t_{10})) = 1.$$

We get that $X'^4_{(a_{17,18,19})} = X_{7,10,11} := \{x_{7,10,11}(t) \mid t \in \mathbb{F}_q\}$ and $Y'^4_{(a_{17,18,19})} = 1$, where for every $t \in \mathbb{F}_q$ we have $x_{7,10,11}(t) := x_7(a_{24}t)x_{10}(a_{22}t)x_{11}(a_{21}t)$, and

$$V^4_{(a_{17,18,19})} = X_3X_{7,10,11}X_8X_9X_{16}X'^3_{(a_{17,18,19})}Y'^3_{(a_{17,18,19})}Y'Z/(\ker\lambda').$$

Notice that $X'^4_{(a_{17,18,19})}X_{7,10,11}$ is a subgroup of $V_{a_{17,18,19}}$, and that $X_3$ is there a direct product factor. Observe then that $[X_8, X_9] = [X_{16}, X_{7,10,11}] = 1$, and that

$$\lambda([x_8(s_8)x_9(s_9), x_{7,10,11}(t)x_{16}(t_{16})]) = \lambda(x_{17}(a_{24}s_9t)x_{18}(a_{22}s_8t)x_{19}(a_{21}s_8t))$$
$$\times \phi(a_{23}s_8t_{16} + a_{24}s_9t_{16}).$$

As $a_{17,18,19} \neq 0$, applying Proposition 9 with arm $X_{7,10,11}X_{16}$ and leg $X_{7,10,11}X_{16}$ yields $X'^5_{(a_{17,18,19})} = Y'^5_{(a_{17,18,19})} = 1$, and the subquotient $V^5_{(a_{17,18,19})} = X_3X'^3_{(a_{17,18,19})}Y'^3_{(a_{17,18,19})}Y'Z/(\ker\lambda')$ of $V$ is abelian. We obtain the family $\mathcal{F}_5^{1,p=2}$ in Table 3.

Let us now assume $c_{17,18,19} = 0$. As done for $c_{17,18,19} \neq 0$, we take $X_{1,5,6}$ and $X_{13}$ as candidates for an arm and a leg respectively, but as we have no $a_{17,18,19}$ term in Equation (10) we now get $X'^3_{(0)} = Y'^3_{(0)} = 1$ and $V^3_{(0)} = X_{2,4}X_3X_7\cdots X_{12}X_{14}X_{15}X_{16}Z/(\ker\lambda')$. Notice that in this subquotient we have $[X_{2,4}, X_j] = 1$ for $j = 8, 9, 16$, and that $[X_{16}, X_i] \neq 1$ implies $i \in \{8, 9\}$. We can apply Proposition 9 with $X_{16}$ as a candidate for an arm, and $X_8X_9$ as a candidate for a leg. We have that

$$\lambda([x_8(s_8)x_9(s_9), x_{16}(t_{16})]) = \lambda(x_{23}(s_8t_{16})x_{24}(s_9t_{16})) = \phi(t_{16}(a_{23}s_8 + a_{24}s_9)).$$

We then get $X'^4_{(0)} = 1$ and $Y'^4_{(0)} = \{x_{8,9}(s) \mid s \in \mathbb{F}_q\}$, where $x_{8,9}(s) = x_8(a_{24}s)x_9(a_{23}s)$ for every $s \in \mathbb{F}_q$, and

$$V^4_{(0)} = X_{2,4}X_3X_7X_{8,9}X_{10}X_{11}X_{12}X_{14}X_{15}Z/(\ker\lambda').$$

Now we observe that (7) holds with $V^4_{(0)}$ in place of $V^2$, as $X_{20}$ and $X_{17}X_{18}X_{19}$ are contained in $\ker\lambda'$. We take $X_7X_{10}X_{11}$ as a candidate for an arm and $X_{12}X_{14}X_{15}$ as a candidate for a leg. Equation (8) yields in this case

$$X'^5_{(0)} := X_{7,10,11} = \{x_{7,10,11}(t) \mid t \in \mathbb{F}_q\} \quad \text{and} \quad Y'^5_{(0)} := X_{12,14,15} = \{x_{12,14,15}(s) \mid s \in \mathbb{F}_q\},$$

and $V^5_{(0)} = X_{2,4}X_3X_{8,9}X_{7,10,11}X_{12,14,15}Z/(\ker\lambda')$. Here, for $s, t \in \mathbb{F}_q$ we have

$$x_{7,10,11}(t) = x_7(a_{24}t)x_{10}(a_{22}t)x_{11}(a_{21}t) \quad \text{and} \quad x_{12,14,15}(s) = x_{12}(a_{24}s)x_{14}(a_{22}s)x_{15}(a_{21}s).$$

Finally, we observe that $X_{8,9}$ and $X_{12,14,15}$ are central in $V^5_{(0)}$; we extend $\lambda'$ to $\lambda'' := \lambda'^{c_{8,9},c_{12,14,15}}$ in the usual way. Observe that $[X_{2,4}, X_{7,10,11}] = 1$. We can then take $X_{2,4}X_{7,10,11}$ and $X_3$ as candidates for an arm and a leg respectively. We study

$$\lambda([x_3(s_3), x_{2,4}(t_1)x_{7,10,11}(t_2)]) = \phi(s_3(c_{8,9}t_1 + c_{12,14,15}t_2 + a_{21}a_{22}a_{24}t_2^2)) = 1.$$

If $a_{8,9} := c_{8,9} \neq 0$ and $b_{12,14,15} := c_{12,14,15}$ is arbitrary in $\mathbb{F}_q$, we have that

$$X'^6_{(0,a_{8,9},b_{12,14,15})} = \{x_{2,4}((b_{12,14,15}t + a_{21}a_{22}a_{24}t^2)/(a_{8,9}^2))x_{7,10,11}(t) \mid t \in \mathbb{F}_q\} \text{ and}$$
$$Y'^6_{(0,a_{8,9},b_{12,14,15})} = 1,$$

and $V^6_{(0,a_{8,9},b_{12,14,15})} = X'^6_{(0,a_{8,9},b_{12,14,15})} X_{8,9} Y'^5_{(0)} Z/(\ker\lambda'')$ is abelian; this gives the family $\mathcal{F}_5^{2,p=2}$ in Table 3.

If $c_{8,9} = 0$ and $a_{12,14,15} := c_{12,14,15} \neq 0$, we have that

$$X'^6_{(0,0,a_{12,14,15})} = X_{2,4}\{1, x_{7,10,11}(c_{12,14,15}/(a_{21}a_{22}a_{24}))\}$$

and

$$Y'^6_{(0,0,a_{12,14,15})} = \{1, x_3(a_{21}a_{22}a_{24}/(c^2_{12,14,15}))\},$$

and $V^6_{(0,0,a_{12,14,15})} = X'^6_{(0,0,a_{12,14,15})} Y'^6_{(0,0,a_{12,14,15})} Y'^5_{(0)} Z/(\ker\lambda'')$ is abelian; we obtain the family $\mathcal{F}_5^{4,p=2}$ in Table 3.

If $c_{8,9} = c_{12,14,15} = 0$, we have that

$$X'^6_{(0,0,0)} = X_{2,4} \quad \text{and} \quad Y'^6_{(0,0,a_{12,14,15})} = 1,$$

and $V^6_{(0,0,0)} = X_{2,4}Z/(\ker\lambda'')$ is abelian. This yields the family $\mathcal{F}_5^{3,p=2}$ in Table 3.

As done for the case $p \neq 2$, we check that

$$\mathrm{Irr}(X_{\mathcal{S}})_{\mathcal{Z}} = \mathcal{F}_5^{1,p=2} \sqcup \mathcal{F}_5^{2,p=2} \sqcup \mathcal{F}_5^{3,p=2} \sqcup \mathcal{F}_5^{4,p=2}$$

by apply the counting argument and Equation (3). This concludes our analysis. □

Finally, we study the unique core of the form $[5, 20, 25]$ in type $\mathrm{E}_6$. In this case,

- $\mathcal{S} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_5, \alpha_6, \alpha_7, \alpha_8, \alpha_9, \alpha_{10}, \alpha_{11}, \alpha_{12}, \alpha_{13}, \alpha_{14}, \alpha_{15}, \alpha_{16}, \alpha_{17}, \alpha_{18}, \alpha_{20}, \alpha_{21}, \alpha_{24}\}$,
- $\mathcal{Z} = \{\alpha_{17}, \alpha_{18}, \alpha_{20}, \alpha_{21}, \alpha_{24}\}$,
- $\mathcal{A} = \{\alpha_4\}$ and $\mathcal{L} = \{\alpha_{19}\}$,
- $\mathcal{I} = \{\alpha_1, \alpha_6, \alpha_8, \alpha_9, \alpha_{10}\}$ and $\mathcal{J} = \{\alpha_7, \alpha_{11}, \alpha_{13}, \alpha_{14}, \alpha_{15}\}$.

**Proposition 24.** *The irreducible characters corresponding to the* $[5, 20, 25]$*-core in type* $\mathrm{E}_6$ *are parametrized as follows:*

- *If* $p \neq 3$, *then* $\mathrm{Irr}(X_{\mathcal{S}})_{\mathcal{Z}} = \mathcal{F}_8^{p\neq 3}$ *consists of* $q(q-1)^5$ *characters of degree* $q^7$.
- *If* $p = 3$, *then*

$$\mathrm{Irr}(X_{\mathcal{S}})_{\mathcal{Z}} = \mathcal{F}_8^{1,p=3} \sqcup \mathcal{F}_8^{2,p=3},$$

*where*
  - $\mathcal{F}_8^{1,p=3}$ *consists of* $(q-1)^6$ *characters of degree* $q^7$, *and*
  - $\mathcal{F}_8^{2,p=3}$ *consists of* $q^2(q-1)^5$ *characters of degree* $q^6$.

*The labels of the characters in* $\mathcal{F}_8^{p\neq 3}$ *and in* $\mathcal{F}_8^{1,p=3}$, $\mathcal{F}_8^{2,p=3}$ *are collected in Table 4.*

**Proof.** The form of Equation (6) is

$$s_7(a_{17}t_8 + a_{18}t_{10}) + s_{11}(-a_{20}t_8 - a_{21}t_9) + s_{13}(-a_{17}t_1 + a_{24}t_{10}) +$$
$$+ s_{14}(a_{20}t_6 + a_{24}t_9) + s_{15}(-a_{18}t_1 + a_{21}t_6 + a_{24}t_8) = 0.$$

Let $p \neq 3$. Then $X' = Y' = 1$, and $\overline{V} = X_2X_3X_5X_{12}X_{16}Z/(\ker\lambda)$. Observe that in $\overline{V}$ the pairs of root subgroups that give nontrivial commutator brackets are exactly the following,

$$[X_2, X_{12}] = X_{17}, \quad [X_2, X_{16}] = X_{20}, \quad [X_3, X_{16}] = X_{21}, \quad [X_5, X_{12}] = X_{18}. \tag{11}$$

We apply Proposition 9 with $X_2X_3X_5$ as a candidate for an arm, and $X_{12}X_{16}$ as a candidate for a leg. We have

$$[x_2(t_2)x_3(t_3)x_5(t_5), x_{12}(s_{12})x_{16}(s_{16})] = x_{17}(s_{12}t_2)x_{18}(-s_{12}t_5)x_{20}(s_{16}t_2)x_{21}(s_{16}t_3),$$

hence

$$\lambda([x_2(t_2)x_3(t_3)x_5(t_5), x_{12}(s_{12})]) = \phi(s_{12}(a_{17}t_2 - a_{18}t_5) + s_{16}(a_{20}t_2 + a_{21}t_3)).$$

We get $X'^2 = \{x_{2,3,5}(t) \mid t \in \mathbb{F}_q\}$ and $Y'^2 = 1$, where

$$x_{2,3,5}(t) := x_2(a_{18}a_{21}t)x_3(-a_{18}a_{20}t)x_5(a_{17}a_{21}t)$$

for every $t \in \mathbb{F}_q$. As $V^2 = X''Z/(\ker\lambda)$ is abelian, we get the family $\mathcal{F}_8^{p\neq 3}$ in Table 4.

Let us now assume that $p = 3$. Then we have

$$X' := \{x_{1,6,8,9,10}(t) \mid t \in \mathbb{F}_q\} \qquad \text{and} \qquad Y' := \{x_{7,11,13,14,15}(s) \mid s \in \mathbb{F}_q\},$$

where for every $s, t \in \mathbb{F}_q$,

$$x_{1,6,8,9,10}(t) := x_1(a_{21}a_{24}t)x_6(-a_{18}a_{24}t)x_8(-a_{18}a_{21}t)x_9(a_{18}a_{20}t)x_{10}(a_{17}a_{21}t)$$

and

$$x_{7,11,13,14,15}(s) := x_7(a_{20}a_{24}s)x_{11}(-a_{17}a_{24}s)x_{13}(-a_{18}a_{20}s)x_{14}(-a_{17}a_{21}s)x_{15}(a_{17}a_{20}s),$$

and $\overline{V} = X_2X_3X_5X_{12}X_{16}X'Y'Z/(\ker\lambda)$. We extend $\lambda$ to $\lambda' = \lambda^{c_{7,11,13,14,15}}$, $c_{7,11,13,14,15} \in \mathbb{F}_q$.

Notice that $X'$ is a subgroup of $\overline{V}$. Moreover, the nontrivial commutator relations in $\overline{V}$ are as in Equation (11), plus $[X', X_i] \neq 1$ if and only if $i \in \{2, 3, 5\}$, in which case such a commutator lies inside $Y'$. In this case, Proposition 9 applies with $X'X_{12}X_{16}$ as a candidate for an arm and $X_2X_3X_5$ as a candidate for a leg. We study the equation

$$\begin{aligned}
\lambda([x_{1,6,8,9,10}(t)x_{12}(t_{12})x_{16}(t_{16}), x_2(s_2)x_3(s_3)x_5(s_5)]) &= \lambda(x_7(-a_{21}a_{24}s_3t)x_{11}(-a_{18}a_{24}s_5t))\cdot \\
&\cdot \lambda(x_{13}(a_{18}a_{20}s_2t - a_{18}a_{21}s_3t)x_{14}(a_{18}a_{21}s_5t + a_{17}a_{21}s_2t)x_{15}(-a_{18}a_{20}s_5t + a_{17}a_{21}s_3t))\cdot \\
&\cdot \phi(s_2(a_{17}t_{12} + a_{20}t_{16} + a_{17}a_{18}a_{20}a_{21}a_{24}t^2) + s_3(a_{21}t_{16} - a_{17}a_{18}a_{21}^2a_{24}t^2))\cdot \\
&\cdot \phi(s_5(-a_{18}t_{12} - a_{18}^2a_{20}a_{21}a_{24}t^2)) = 1.
\end{aligned}$$

If $a_{7,11,13,14,15} := c_{7,11,13,14,15} \neq 0$, then we have that $X'^2_{(a_{7,11,13,14,15})} = Y'^2_{(a_{7,11,13,14,15})} = 1$, and $V^2_{(a_{7,11,13,14,15})} = Y'Z/(\ker\lambda')$ is abelian. We get the family $\mathcal{F}_8^{1,p=3}$ in Table 4.

If $c_{7,11,13,14,15} = 0$, then we have

$$\begin{aligned}
X'^{,2}_{(0)} &= \{x_{1,6,8,9,10}(t)x_{12}(a_{18}a_{20}a_{21}a_{24}t^2)x_{16}(a_{17}a_{18}a_{21}a_{24}t^2) \mid t \in \mathbb{F}_q\}, \\
Y'^{,2}_{(0)} &= \{x_2(a_{18}a_{21}s)x_3(-a_{18}a_{20}s)x_5(a_{17}a_{21}s) \mid s \in \mathbb{F}_q\},
\end{aligned}$$

and $V^2_{(0)} = X'^{,2}_{(0)}Y'^{,2}_{(0)}Z/(\ker\lambda')$ is abelian. This yields the family $\mathcal{F}_8^{2,p=3}$ in Table 4.

The claim now follows by Equation (3) as done in Propositions 22 and 23. □

## References


Bosma, W., Cannon, J., Playoust, C., 1997. The Magma algebra system. I. The user language. J. Symb. Comput. 24, 235–265.

Carter, R.W., 1989. Simple Groups of Lie Type, Vol. 22. John Wiley & Sons.

Digne, F., Michel, J., 1991. Representations of Finite Groups of Lie Type, Vol. 21. Cambridge University Press.

Dudas, O., Malle, G., 2015. Decomposition matrices for low-rank unitary groups. Proc. Lond. Math. Soc. 110 (6), 1517–1557.

Dudas, O., Malle, G., 2016. Decomposition matrices for exceptional groups at $d = 4$. J. Pure Appl. Algebra 220 (3), 1096–1121.

Geck, M., Hiss, G., Lübeck, F., Malle, G., Pfeiffer, G., 1996a. CHEVIE – A system for computing and processing generic character tables for finite groups of Lie type, Weyl groups and Hecke algebras. Appl. Algebra Eng. Commun. Comput. 7, 175–210.

Geck, M., Hiss, G., Malle, G., 1996b. Towards a classification of the irreducible representations in non-defining characteristic a finite group of Lie type. Math. Z. 221, 353–386.

Goodwin, S.M., Le, T., Magaard, K., 2017. The generic character table of a Sylow $p$-subgroup of a finite Chevalley group of type $D_4$. Commun. Algebra 45, 5158–5179.

Goodwin, S.M., Le, T., Magaard, K., Paolini, A., 2016a. Constructing characters of Sylow p-subgroups of finite Chevalley groups. J. Algebra 468, 395–439.

Goodwin, S.M., Mosch, P., Röhrle, G., 2014. Calculating conjugacy classes in Sylow $p$-subgroups of finite Chevalley groups of rank six and seven. LMS J. Comput. Math. 17 (1), 109–122.

Goodwin, S.M., Mosch, P., Röhrle, G., 2016b. On the coadjoint orbits of maximal unipotent subgroups of reductive groups. Transform. Groups 21 (2), 399–426.

Gruber, J., Hiss, G., 1997. Decomposition numbers of finite classical groups for linear primes. J. Reine Angew. Math. 485, 55–92.

Hiss, G., 1991. Decomposition Numbers of Finite Groups of Lie Type in Non-defining Characteristic. Progr. Math., vol. 95. Birkhäuser, Basel (Bielefeld, pp. 405–418.

Hiss, G., 1993. Harish-Chandra series of Brauer characters in a finite group with a split $BN$-pair. J. Lond. Math. Soc. 48 (2), 219–228.

Himstedt, F., 2011. On the decomposition numbers of the Ree groups ${}^2F_4(q^2)$ in non-defining characteristic. J. Algebra 325 (1), 364–403.

Himstedt, F., Le, T., Magaard, K., 2011. Characters of the Sylow $p$–subgroups of the Chevalley groups $D_4(p^n)$. J. Algebra 332 (1), 414–427.

Himstedt, F., Le, T., Magaard, K., 2016. On the characters of the Sylow $p$-subgroups of untwisted Chevalley groups $Y_n(p^a)$. LMS J. Comput. Math. 19, 303–359.

Himstedt, F., Noeske, F., 2014. Decomposition numbers of $SO_7(q)$ and $Sp_6(q)$. J. Algebra 413, 15–40.

Himstedt, F., Noeske, F., 2015. Restricting unipotent characters in special orthogonal groups. LMS J. Comput. Math. 18 (1), 456–488.

Isaacs, I.M., 1994. Character Theory of Finite Groups. Dover Books on Mathematics. New York.

Isaacs, I.M., 2007. Counting characters of upper triangular groups. J. Algebra 315, 698–719.

Le, T., Magaard, K., 2015. On the character degrees of Sylow $p$-subgroups of Chevalley groups $G(p^f)$ of type $E$. Forum Math. 27 (1), 1–55.

Le, T., Magaard, K., Paolini, A., 2018. Computational details for the analysis of the nonabelian cores in types $D_6$ and $E_6$. http://www.alessandropaolini.com/d6e6.html.

Lehrer, G., 1974. Discrete series and the unipotent subgroup. Compos. Math. 28, 9–19.

Malle, G., Testerman, D., 2011. Linear Algebraic Groups and Finite Groups of Lie Type, Vol. 133. Cambridge University Press.

Paolini, A., 2018. On the decomposition numbers of $SO_8^+(2^f)$. J. Pure Appl. Algebra 222, 3982–4003.

The GAP Group, 2016. GAP – groups, algorithms, and programming, version 4.8.6. http://www.gap-system.org.