Series representations for densities functions of a family of distributions—Application to sums of independent random variables

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Abstract
Series representations for several density functions are obtained as mixtures of generalized gamma distributions with discrete mass probability weights, by using the exponential expansion and the binomial theorem. Based on these results, approximations based on mixtures of generalized gamma distributions are proposed to approximate the distribution of the sum of independent random variables, which may not be identically distributed. The applicability of the proposed approximations are illustrated for the sum of independent Rayleigh random variables, the sum of independent gamma random variables, and the sum of independent Weibull random variables. Numerical studies are presented to assess the precision of these approximations.

Keywords: binomial theorem, exponential expansion, generalized gamma distribution, mixtures

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1. Introduction

The exponential power series and the binomial expansion[1,2] are two important results in mathematics and physics. In this work, it is shown how these results may be used to obtain new series representations for density functions of well-known distributions in statistics. These new representations may be used in problems related to the algebra of random variables. The motivation for this work has its origin in some familiar expansions; for example, if one considers a gamma random variable Y with scale parameter λ and shape parameter r, denoted by $Y \sim \Gamma(r, \lambda)$, its density is given by

$$f_Y(y) = \frac{1}{\Gamma(r)\lambda^r} y^{r-1} \exp\left\{-\frac{y}{\lambda}\right\}, \quad y > 0. \tag{1}$$

As it is shown in the study of Marques,[3] for $\delta > 0$ and $\delta \neq \lambda$, the density in (1) may be represented as follows:

$$f_Y(y) = \sum_{j=0}^{\infty} p_j f_{X_j}(y),$$

which is the density of a mixture of gamma distributions, $X_j \sim \Gamma(r + j, \delta)$, with weights given in expression 1 of the study of Marques.[3] Note that when $\frac{\lambda}{\delta} < 1$, the weights are given by the mass probability function of a discrete negative binomial distribution. In the above representations, the parameter $\delta \neq \lambda$ can be chosen such that if more than one gamma random variable is observed, gamma series representations, for all the gamma random variables, with all the gamma distributions involved in the series representations with the same scale parameter may be considered. This is a convenient feature that can be used to obtain an exact representation for the sum of independent gamma random variables.[3] In the same reference, using the exponential expansion together with the binomial expansion, the authors developed a series gamma representation for the logbeta distribution with all the gamma distributions in the series having the same scale parameter. This latter result was used to obtain exact representations for the product of independent beta random variables and for the sum of independent logbeta random variables.

In this work, the generalized gamma distribution[4] is addressed, and series representations for this distribution are developed, as well as for its particular cases. The generalized gamma distribution may be applied in different applied fields of research (see, for example, the previous studies;[5-10] some of the applications will be detailed ahead). In addition, some of its particular cases play an important role in different fields of statistics, such as, for example, the Weibull distribution in survival analysis and in extreme value theory, the gamma distribution in insurance claims, rainfall and in Bayesian statistics as a conjugate prior for the exponential distribution, and finally the Rayleigh distribution in engineering and physical sciences. It is shown that a generalized gamma distribution can be represented as an infinite mixture of generalized gamma distributions, with specific parameters, and with weights given by the mass probability function of a negative binomial distribution. This series representation applies to all the particular cases of the generalized gamma distribution. Then, using the obtained representation as a basis, finite mixtures of generalized gamma distributions are considered for approximating the distribution of the sum of independent Rayleigh, Weibull, and gamma random variables. These approximations are based on a two-step method of moments and may easily be used since they are based on a finite mixture. The practical utility of these results is illustrated in engineering problems.

This paper is organized as follows: in Section 2, it is shown that the density of a generalized gamma distribution may be represented as a gamma-series expansion or as a mixture of generalized gamma distributions with negative binomial weights. The particular cases of this distribution are then also analyzed. In Section 3, simple approximations are obtained for the sum of independent Rayleigh, gamma, and Weibull random variables. Still in Section 3, numerical studies are developed in order to illustrate the precision of the approximations proposed, together with examples of application to illustrate the practical importance of the results provided. Finally, Section 4, is dedicated to the discussion and conclusions.

2. Series Representations for the Generalized Gamma Distribution and for its Particular Cases

The generalized gamma distribution[4] has several well-known distributions as particular cases such as the gamma, Rayleigh, and Weibull distributions (for more particular cases see, for example, the studies of Coelho and Arnold[6] and Crooks[11]). Because of its flexibility, this distribution has several applications in different areas of research. In the study of Aalo,[5] the generalized gamma distribution was used to characterize both multipath and shadow fading processes in wireless communication systems; in the study of Smirnov,[8] it was used as an approximation for the real line shape of a scintillation detector, and in the studies of Zaninetti,[9,10] it was used to model the luminosity function of galaxies (see the studies of Coelho and Arnold and Marques and Loingeville[6,7] for other examples of application). A random variable X has a generalized gamma distribution if its density function is given by

$$f(x) = \frac{\gamma e^{-\left(\frac{x-\mu}{\beta}\right)^{\gamma}} \left(\frac{x-\mu}{\beta}\right)^{\alpha\gamma-1}}{\beta\Gamma(\alpha)}, \tag{2}$$

with $\alpha > 0, \beta > 0, \gamma > 0$ and $\mu \in \mathbb{R}$ and is denoted by $X \sim G\Gamma(\alpha, \beta, \gamma, \mu)$. The h-th moment of X is given by

$$E(X^h) = \frac{\beta^h \Gamma\left(\alpha + \frac{h}{\gamma}\right)}{\Gamma(\alpha)}.$$

The characteristic function of X is not known, but since X has moments of any order, the following expansion is considered

$$\Phi_X(t) = \sum_{n=0}^{\infty} \frac{(\mathrm{i}t)^n}{n!} \frac{\beta^n \Gamma\left(\alpha + \frac{n}{\gamma}\right)}{\Gamma(\alpha)},$$

where $\Phi_X(.)$ denotes the characteristic function of X, $\mathrm{i} = \sqrt{-1}$ and $t \in \mathbb{R}$.

Other representations or extensions of the generalized gamma distribution exist in the literature; see, for example, the studies of Bourguignon et al, Cordeiro et al, and Nadarajah and Gupta.[12-14]

The following theorem shows that the density of a generalized gamma distribution may be represented as follows: (a) an infinite gamma-series expansion or (b) a mixture of generalized gamma distributions.

Theorem 2.1. *Let $X \sim G\Gamma(\alpha, \beta, \gamma, \mu)$, then, for a given positive real λ, the density of X has the following series representations:*

(a)

$$f_X(x) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} p_{n,k} f_{\Gamma(\gamma(n-k)+k+\alpha\gamma,\lambda)}(x-\mu),\ x > 0,$$

with the coefficients $p_{n,k}$ given by

$$p_{n,k} = \binom{n}{k}\left(-\left(\frac{1}{\beta}\right)^{\gamma}\right)^{n-k}\left(\frac{1}{\lambda}\right)^{k} \frac{\gamma\Gamma(\gamma(n-k)+k+\alpha\gamma)\lambda^{\gamma(n-k)+k+\alpha\gamma}}{\beta^{\alpha\gamma}\Gamma(\alpha)n!},$$

and where $f_{\Gamma(\gamma(n-k)+k+\alpha\gamma,\lambda)}(.)$, for a given k and n, denotes the density function of a gamma distribution with shape parameter $\gamma(n-k)+k+\alpha\gamma$ and scale parameter λ.

(b) for $\lambda < \beta$,

$$f_X(x) = \sum_{n=0}^{\infty} f_{NB\left(\alpha,\left(\frac{\beta}{\lambda}\right)^{-\gamma}\right)}(n) f_{G\Gamma(n+\alpha,\lambda,\gamma,\mu)}(x),\ x > 0, \tag{3}$$

where $f_{NB\left(\alpha,\left(\frac{\beta}{\lambda}\right)^{-\gamma}\right)}(.)$ is the mass probability function of a negative binomial distribution with parameters α and $\left(\frac{\beta}{\lambda}\right)^{-\gamma}$ and, for a given n, $f_{G\Gamma(n+\alpha,\lambda,\gamma,\mu)}(.)$ is the density function of a generalized gamma distribution with parameters $n+\alpha$, λ, γ, μ.

Proof.

(a) For a given $\lambda > 0$, it is written that

$$\begin{aligned} f(x) &= \frac{\gamma \exp\left\{-\left(\frac{x-\mu}{\beta}\right)^{\gamma}\right\}\left(\frac{x-\mu}{\beta}\right)^{\alpha\gamma-1}}{\beta\Gamma(\alpha)} \frac{\exp\left\{\left(\frac{x-\mu}{\beta}\right)^{\gamma} - \frac{x-\mu}{\lambda}\right\}}{\exp\left\{\left(\frac{x-\mu}{\beta}\right)^{\gamma} - \frac{x-\mu}{\lambda}\right\}} \\ &= \frac{\gamma}{\beta\Gamma(\alpha)}\left(\frac{x-\mu}{\beta}\right)^{\alpha\gamma-1} \exp\left\{-\left(\frac{x-\mu}{\beta}\right)^{\gamma} + \frac{x-\mu}{\lambda}\right\} \exp\left\{-\frac{x-\mu}{\lambda}\right\}, \end{aligned}$$

then applying the exponential expansion on the factor $\exp\left\{-\left(\frac{x-\mu}{\beta}\right)^{\gamma} + \frac{x-\mu}{\lambda}\right\}$, it is obtained that

$$\begin{aligned} f(x) &= \frac{\gamma}{\beta\Gamma(\alpha)}\left(\frac{x-\mu}{\beta}\right)^{\alpha\gamma-1}\left\{\sum_{n=0}^{\infty}\frac{1}{n!}\left(-\left(\frac{x-\mu}{\beta}\right)^{\gamma} + \frac{x-\mu}{\lambda}\right)^{n}\right\} \exp\left\{-\frac{x-\mu}{\lambda}\right\} \\ &= \sum_{n=0}^{\infty}\left(-\left(\frac{x-\mu}{\beta}\right)^{\gamma} + \frac{x-\mu}{\lambda}\right)^{n} \frac{\gamma}{\beta\Gamma(\alpha)n!}\left(\frac{x-\mu}{\beta}\right)^{\alpha\gamma-1} \exp\left\{-\frac{x-\mu}{\lambda}\right\}. \end{aligned}$$

Using the binomial expansion, it follows that

$$f(x) = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\left(-\left(\frac{x-\mu}{\beta}\right)^{\gamma}\right)^{n-k}\left(\frac{x-\mu}{\lambda}\right)^{k}\frac{\gamma}{\beta\Gamma(\alpha)n!}\left(\frac{x-\mu}{\beta}\right)^{\alpha\gamma-1}\exp\left\{-\frac{x-\mu}{\lambda}\right\}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\left(-\left(\frac{1}{\beta}\right)^{\gamma}\right)^{n-k}\left(\frac{1}{\lambda}\right)^{k}\frac{\gamma}{\beta^{\alpha\gamma}\Gamma(\alpha)n!}(x-\mu)^{\gamma(n-k)+k+\alpha\gamma-1}\exp\left\{-\frac{x-\mu}{\lambda}\right\}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\underbrace{\binom{n}{k}\left(-\left(\frac{1}{\beta}\right)^{\gamma}\right)^{n-k}\left(\frac{1}{\lambda}\right)^{k}\frac{\gamma\Gamma(\gamma(n-k)+k+\alpha\gamma)\lambda^{\gamma(n-k)+k+\alpha\gamma}}{\beta^{\alpha\gamma}\Gamma(\alpha)n!}}_{p_{n,k}}$$

$$\times f_{\Gamma(\gamma(n-k)+k+\alpha\gamma,\lambda)}(x-\mu)$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n}p_{n,k}f_{\Gamma(\gamma(n-k)+k+\alpha\gamma,\lambda)}(x-\mu).$$

(b) following a similar procedure to that used before, the density of X may be represented as follows

$$f_X(x) = \frac{\gamma\exp\left\{-\left(\frac{x-\mu}{\beta}\right)^{\gamma}\right\}\left(\frac{x-\mu}{\beta}\right)^{\alpha\gamma-1}}{\beta\Gamma(\alpha)}\frac{\exp\left\{\left(\frac{x-\mu}{\beta}\right)^{\gamma}-\left(\frac{x-\mu}{\lambda}\right)^{\gamma}\right\}}{\exp\left\{\left(\frac{x-\mu}{\beta}\right)^{\gamma}-\left(\frac{x-\mu}{\lambda}\right)^{\gamma}\right\}}$$

$$= \frac{\gamma}{\beta\Gamma(\alpha)}\left(\frac{x-\mu}{\beta}\right)^{\alpha\gamma-1}\exp\left\{-\left(\frac{x-\mu}{\beta}\right)^{\gamma}+\left(\frac{x-\mu}{\lambda}\right)^{\gamma}\right\}\exp\left\{-\left(\frac{x-\mu}{\lambda}\right)^{\gamma}\right\}.$$

using the exponential expansion, it is obtained that

$$f_X(x) = \sum_{n=0}^{\infty}\frac{1}{n!}\left(-\left(\frac{x-\mu}{\beta}\right)^{\gamma}+\left(\frac{x-\mu}{\lambda}\right)^{\gamma}\right)^{n}\frac{\gamma}{\beta\Gamma(\alpha)}\left(\frac{x-\mu}{\beta}\right)^{\alpha\gamma-1}\exp\left\{-\left(\frac{x-\mu}{\lambda}\right)^{\gamma}\right\}.$$

Now, the expression is written in terms of mixtures of generalized gamma distributions as follows

$$f_X(x) = \sum_{n=0}^{\infty}\frac{1}{n!}\left(-\left(\frac{1}{\beta}\right)^{\gamma}+\left(\frac{1}{\lambda}\right)^{\gamma}\right)^{n}\frac{\gamma\lambda^{(n+\alpha)\gamma-1}}{\beta^{\alpha\gamma}\Gamma(\alpha)}\left(\frac{x-\mu}{\lambda}\right)^{(n+\alpha)\gamma-1}\exp\left\{-\left(\frac{x-\mu}{\lambda}\right)^{\gamma}\right\},$$

which after some further simplifications may be written as follows

$$f_X(x) = \sum_{n=0}^{\infty}f_{\mathrm{NB}\left(\alpha,\left(\frac{\beta}{\lambda}\right)^{-\gamma}\right)}(n)f_{G\Gamma(n+\alpha,\lambda,\gamma,\mu)}(x).$$

□

Theorem 2.1 (a) shows that the density function of a generalized gamma random variable may be written as a gamma-series expansion where all the gamma distributions $\Gamma(\gamma(n-k)+k+\alpha\gamma,\lambda)$ have the same scale parameter λ. This interesting representation has one limitation in that it does not correspond to a mixture of distributions, since the weights do not sum to 1. Therefore, advantage cannot be taken of mixtures properties to address further results such as the sum of independent random variables.

Concerning Theorem 2.1 (b), it is pointed out that (a) the representation in (3) shows that a generalized gamma distribution is an infinite mixture of generalized gamma distributions with weights given by the mass probability function of a negative binomial distribution; a random variable Y has a negative binomial distribution, denoted by $Y \sim \mathrm{N}B(n,p)$ with $n \in \mathbb{R}$ and success probability p, if its mass probability function is given by[15, Chapter 7]

$$(1-p)^{k}p^{n}\binom{k+n-1}{n-1}, k \geq 0,$$

(b) note that the negative binomial distribution is still defined when the first parameter is not an integer, as it may be the case of α, (c) in this representation, the value of λ may be chosen and thus may take different values, and (d) if one has two generalized gamma distributions, mixture representations for both distributions having the same parameter λ may be considered. Although not explored in this work, the notes in points (c) and (d) together with the mixtures properties may be used to obtain further results on the sum of independent generalized gamma random variables.

As already mentioned, the generalized gamma distribution has as particular cases several well-known distributions; therefore, the results in Theorem 2.1 also applies to all its particular cases. In what follows, three corollaries of Theorem 2.1 are presented, addressing the distributions considered in Section 3, for establishing the notation and to point out interesting properties that will be further explored.

Corollary 2.2. *Let X be a random variable following a gamma distribution with shape parameter $\alpha > 0$ and scale parameter $\beta > 0$, denoted by $X \sim \Gamma(\alpha, \beta)$, and with density function*

$$f_X(x) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}} x^{\alpha-1} \exp\left\{-\frac{x}{\beta}\right\}, x > 0.$$

Then, for $\lambda > 0$ such that $\lambda < \beta$, the density function of X may be given by

$$f_X(x) = \sum_{n=0}^{\infty} f_{NB\left(\alpha, \frac{\lambda}{\beta}\right)}(n) f_{G\Gamma(\alpha+n,\lambda,1,0)}(x), x > 0, \tag{4}$$

where $f_{NB\left(\alpha, \frac{\beta}{\lambda}\right)}(.)$ is the mass probability function of a negative binomial distribution with s parameters α and $\frac{\beta}{\lambda}$, and for a given n, $f_{G\Gamma(\alpha+n,\lambda,1,0)}(.)$ is the density function of a generalized gamma distribution with parameters $\alpha + n$, λ, 1 and 0.

Note that the generalized gamma distribution with parameters $G\Gamma(\alpha + n, \lambda, 1, 0)$ it is a gamma distribution with parameters $\Gamma(\alpha + n, \lambda)$. The above result was used in the study of Marques[3] to address the sum of independent gamma random variables without noticing that the weights corresponded to that of the mass probability function of a negative binomial distribution. The importance of this result, as already mentioned in the introduction, is that if more than one gamma random variable is observed, mixture representations may be considered, for all of the variables, where the gamma distributions in the mixtures have the same scale parameter λ.

Corollary 2.3. *Let X be a random variable following a Rayleigh distribution with parameter $\sigma > 0$, denoted by $X \sim$ Rayleigh(σ), and with density function*

$$f_X(x) = \frac{x \exp\left\{-\frac{x^2}{2\sigma^2}\right\}}{\sigma^2}, \quad x > 0.$$

Then, for $\lambda > 0$ such that $\lambda < \sqrt{2}\sigma$ the density function of X may be given by

$$f_X(x) = \sum_{n=0}^{\infty} f_{Geo\left(\left(\frac{\lambda}{\sqrt{2}\sigma}\right)^2\right)}(n) f_{G\Gamma(n+1,\lambda,2,0)}(x), x > 0, \tag{5}$$

where $f_{Geo\left(\left(\frac{\lambda}{\sqrt{2}\sigma}\right)^2\right)}(.)$ is the mass probability function of a Geometric distribution with parameter $\left(\frac{\lambda}{\sqrt{2}\sigma}\right)^2$ and, for a given n, $f_{G\Gamma(n+1,\lambda,2,0)}(.)$ is the density function of a generalized gamma distribution with parameters $n + 1$, λ, 2, 0.

Finally, the Weibull distribution is considered.

Corollary 2.4. *Let X be a random variable following a Weibull distribution with parameters $\alpha > 0$ and $\beta > 0$, denoted by $X \sim$ Weibull(α, β), and with density function*

$$f_X(x) = \frac{\alpha \exp\left\{-\left(\frac{x}{\beta}\right)^{\alpha}\right\}\left(\frac{x}{\beta}\right)^{\alpha-1}}{\beta}, x > 0.$$

Then, for $\lambda > 0$ such that $\lambda < \beta$ the density function of X may be given by

$$f_X(x) = \sum_{n=0}^{\infty} f_{Geo\left(\left(\frac{\beta}{\lambda}\right)^{-\gamma}\right)}(n) f_{G\Gamma(n+1,\lambda,\gamma,0)}(x), x > 0, \tag{6}$$

where $f_{Geo\left(\left(\frac{\beta}{\lambda}\right)^{-\gamma}\right)}(.)$ is the mass probability function of a geometric distribution with parameter $\left(\frac{\beta}{\lambda}\right)^{-\gamma}$, and for a given n, $f_{G\Gamma(n+1,\lambda,\gamma,0)}(.)$ is the density function of a generalized gamma distribution with parameters $n + 1$, λ, γ, 0.

In the particular cases considered in Corollaries 2.2, 2.3, and 2.4, the shifted version of the positive distributions can also be considered.

Clearly, other particular cases may be addressed; for other examples see the studies of Coelho and Arnold, Crooks, and Yacoub.[6,11,16]

3. Application to the Sum of Independent Random Variables

The general representation of several distributions in terms of mixtures of generalized gamma distributions suggests that this kind of representation may be an efficient tool to approximate the distribution of sums or linear combinations of positive variables. As examples, in the next subsections, the sum of independent Rayleigh variables, the sum of independent gamma random variables, and the sum of independent Weibull random variables (not necessarily identically distributed) are considered. Thus, for all cases considered and motivated by the previous results, approximations for the sum of the independent random variables addressed in Subsections 3.1 to 3.3 are considered as mixtures of generalized gamma distributions. More precisely, consider $X_1, \ldots, X_p$ independent random variables; a mixture of gamma distributions are proposed for approximating the distribution of $Z = \sum_{i=1}^{p} X_i$, with X_i being independent distributed as in Subsections 3.1 to 3.3 and not necessarily identically distributed. The corresponding approximating density and cumulative distribution functions are defined respectively by

$$f_{\tilde{Z}}(x) = \sum_{n=0}^{m} \pi_n f_{G\Gamma(\alpha+\frac{n}{\ell},\beta,\gamma,0)}(x) \tag{7}$$

and

$$F_{\tilde{Z}}(x) = \sum_{n=0}^{m} \pi_n F_{G\Gamma(\alpha+\frac{n}{\ell},\beta,\gamma,0)}(x), \tag{8}$$

where $\tilde{Z}$ is a random variable with the distribution corresponding to density and cumulative distribution functions in (7) and (8), π_n are the weights, and $f_{G\Gamma(\alpha+\frac{n}{\ell},\beta,\gamma,0)}(.)$ and $F_{G\Gamma(\alpha+\frac{n}{\ell},\beta,\gamma,0)}(.)$ are the probability density and cumulative distribution functions of a generalized gamma distribution with parameters $\alpha + n/\ell$, β, γ, and 0. The parameter ℓ is a tuning parameter that may help to improve the precision of this approximation, and m is the number of exact moments matched by the approximating distribution. The parameters and weights in (7) and (8) will be determined using the following algorithm:

Step 1	the values of α, β and γ are determined in such a way that the first three moments of the approximating distribution are the same as the first exact three moments .
	↓
Step 2	for a starting value of ℓ, for a given integer m and values of α, β and γ defined as in Step 1.; the first m weights in the mixture are determined in such a way that the first m moments of the approximating distribution are equal to the first exact m moments; the last weight is equal to $\pi_m = 1 - \sum_{n=0}^{m-1} \pi_n$.
	↓
Step 3	this process is repeated to adjust the tuning parameter ℓ.

Using this procedure, it is possible to control the precision of the approximation by increasing or decreasing the value of m or by adjusting the tuning parameter ℓ.

In order to implement this procedure, the first m exact moments need to be determined. When the characteristic function of each X_i is known, this will be achieved computationally and numerically using the characteristic function of Z which is

$$\Phi_Z(t) = \prod_{i=1}^{p} \Phi_{X_i}(t)$$

by

$$E(Z^h) = \mathrm{i}^{-h} \left. \frac{\partial^h \Phi_Z(t)}{\partial t^h} \right|_{t=0}.$$

When the characteristic function of X_i, $i = 1, \ldots, p$, does not have an explicit expression, if the random variable has moments of any order, the following representation for Φ_{X_i} may always be considered:

$$\Phi_{X_i}(t) = \sum_{n=0}^{\infty} \frac{(\mathrm{i}t)^n}{n!} E(X_i^n).$$

However, using this expression, the characteristic function of Z is obtained as products of infinite sums, which is difficult to use in practice. Therefore, when the use of characteristic functions is not possible, the exact moments may be determined using the multinomial expansion, through the following expression:

$$E(Z^h) = \sum_{h_1=0}^{h} \sum_{h_2=0}^{h_1} \cdots \sum_{h_{p-1}=0}^{h_{p-2}} \binom{h}{h_1} \binom{h_1}{h_2} \cdots \binom{h_{p-2}}{h_{p-1}} E\left(X_1^{h-h_1}\right) E\left(X_2^{h_1-h_2}\right) \cdots E\left(X_p^{h_{p-1}}\right). \tag{9}$$

In order to apply a matching moments technique, the first m moments of the approximating distribution are also required. The h-th moment of a random variable with a generalized gamma distribution with parameters $\alpha + n/\ell, \beta, \gamma$, and 0, denoted by $G\Gamma(\alpha + \frac{n}{\ell}, \beta, \gamma, 0)$, is given by

$$\frac{\beta^h \Gamma\left(\alpha + \frac{n}{\ell} + \frac{h}{\gamma}\right)}{\Gamma\left(\alpha + \frac{n}{\ell}\right)}.$$

thus, using the mixtures properties, the hth moment of the $\tilde{Z}$ is

$$E\left(\tilde{Z}^h\right) = \sum_{n=0}^{m} \pi_n \frac{\beta^h \Gamma\left(\alpha + \frac{n}{\ell} + \frac{h}{\gamma}\right)}{\Gamma\left(\alpha + \frac{n}{\ell}\right)}.$$

Then, for Step 1, the values of α, β, and γ are determined as solutions of the system of equations

$$\mathrm{i}^{-h} \left. \frac{\partial^h \Phi_Z(t)}{\partial t^h} \right|_{t=0} = \frac{\beta^h \Gamma\left(\alpha + \frac{h}{\gamma}\right)}{\Gamma(\alpha)} \quad \text{, for } h = 1, 2, 3. \tag{10}$$

For a starting value of ℓ, fixed values of α, β, and γ and for a given integer m, the first m weights in the mixture are determined as solutions of the system of equations

$$\mathrm{i}^{-h} \left. \frac{\partial^h \Phi_Z(t)}{\partial t^h} \right|_{t=0} = \sum_{n=0}^{m} \pi_n \frac{\beta^h \Gamma\left(\alpha + \frac{n}{\ell} + \frac{h}{\gamma}\right)}{\Gamma\left(\alpha + \frac{n}{\ell}\right)} \quad \text{, for } h = 1, \cdots, m, \tag{11}$$

the last weight being equal to

$$\pi_m = 1 - \sum_{n=0}^{m-1} \pi_n.$$

Note that, although the system of equations in (11) (used to address Step 2.) is easily solved, for obtaining the solutions of the system of equations in (10) (used for solving Step 1.), initial values may have to be provided for α, β, and γ. If these choices are inadequate, the process may take some time to converge, or the solution may not make sense, in the sense that, for example, negative values for the parameters are returned. Thus, the initial values may be chosen by following some prior knowledge of the distribution or by chance attempts. In more complex scenarios, a small simulation experiment may be performed where the starting values are defined as those values that result in a better fit of the generalized gamma distribution to the simulated data. This point will be also addressed in the discussion section.

In the next subsections, the computations were done using software Mathematica 10.0.

3.1. Sums of independent Rayleigh random variables

As a first example, the sum of independent Rayleigh random variables is addressed. This distribution is widely applied in problems relating to wireless communications as pointed out in the studies of Divsalar and Simon, Hu and Beaulieu, Nadarajah, and Marcum.[17-20] In the studies of Hu and Beaulieu and Nadarajah,[18,19] it is stated that there is no close-form expression for the exact distribution of the sum of independent Rayleigh random variables. In the study of Hitczenko,[21] upper bounds on the tail probability were established for the linear combination of Rayleigh distributions, and in the study of Karagiannidis et al,[22] using the Meijer G-function,[23] an upper bound for the cumulative distribution function was obtained. In the study of Hu and Beaulieu,[18] closed-form approximations were developed for the sum of independent Rayleigh random variables when these are normalized, that is transformed into Rayleigh random variables with parameters $\sigma = 1$. In this subsection, a more general setting with no restrictions on the parameters is considered.

Consider p independent Rayleigh random variables, $X_i \sim \text{Rayleigh}(\sigma_i), i = 1, \ldots, p$. The interest is in the distribution of

$$Z = \sum_{i=1}^{p} X_i.$$

Clearly, since $\delta_i X_i \sim \text{Rayleigh}(\delta_i \sigma_i)$, for $\delta_i > 0$ and $i = 1, \ldots, p$, the linear combination of independent Rayleigh random variables reduces to the sum of $Y_i = \delta_i X_i \sim \text{Rayleigh}(\delta_i \sigma_i)$, $\delta_i > 0, i = 1, \ldots, p$.

The characteristic function of X_i is known and given by

$$\Phi_{X_i}(t) = 1 + \sqrt{\frac{\pi}{2}}\sigma_i t \exp\left\{-\frac{1}{2}\sigma_i^2 t^2\right\}\left(-\text{Erfi}\left(\frac{\sigma_i t}{\sqrt{2}}\right) + \mathrm{i}\right), t \in \mathbb{R},$$

with $\mathrm{i} = \sqrt{-1}$, and where Erfi(.) is the imaginary error function. Thus, the characteristic function of $Z = \sum_{i=1}^{p} X_i$ is given by

$$\Phi_Z(t) = \prod_{i=1}^{p} \Phi_{X_i}(t) = \prod_{i=1}^{p} \left\{1 + \sqrt{\frac{\pi}{2}}\sigma_i t \exp\left\{-\frac{1}{2}\sigma_i^2 t^2\right\}\left(-\text{Erfi}\left(\frac{\sigma_i t}{\sqrt{2}}\right) + \mathrm{i}\right)\right\}, t \in \mathbb{R}. \tag{12}$$

As already mentioned, using this expression, the h-th moment of Z may be obtained computationally and numerically by

$$E(Z^h) = \mathrm{i}^{-h} \left.\frac{\partial^h \Phi_Z(t)}{\partial t^h}\right|_{t=0}.$$

Following the three steps highlighted in the beginning of Section 3, approximations for the cumulative distribution or density functions of the sum of independent Rayleigh random variables are obtained as mixtures of generalized gamma distributions.

For illustrating the performance of the proposed approximation, four scenarios are considered:

Scenario 1: $\boldsymbol{\sigma} = \left\{2, \frac{1}{3}\right\}$;
Scenario 2: $\boldsymbol{\sigma} = \left\{2, \frac{1}{3}, \frac{17}{4}\right\}$;
Scenario 3: $\boldsymbol{\sigma} = \{3, 3, 3, 3\}$;
Scenario 4: $\boldsymbol{\sigma} = \left\{2, 3, \frac{1}{3}, \frac{5}{4}, 10\right\}$.

In Tables 1 to 4, the approximating cumulative distribution function in (8) has been computed for the exact quantiles q_δ with $\delta = 0.05, 0.1, 0.5, 0.90, 0.95$. In all scenarios considered, the tuning parameter ℓ was set equal to 2. The exact quantiles were determined using the bisection method with the numerical inversion of Φ_Z in (12) obtained using the inversion formulas in the study of Gil-Pelaez.[24] However, note that the proposed approximations are simple mixtures of

Table 1. Scenario 1—computed values of the approximating cumulative distribution in (8) for the exact quantiles q_δ with δ =0.05, 0.1, 0.5, 0.90, 0.95 for α =1.637, β =2.376, γ =1.759, and ℓ =2

m	$q_{0.05}$	$q_{0.1}$	$q_{0.5}$	$q_{0.90}$	$q_{0.95}$
2	0.0521	0.0995	0.4984	0.9014	0.9504
4	0.0504	0.0981	0.5011	0.8997	0.9492
6	0.0493	0.0987	0.5009	0.8998	0.9502
8	0.0498	0.0997	0.4999	0.9001	0.9500
10	0.0499	0.0998	0.4999	0.9000	0.9500
15	0.0500	0.1000	0.5000	0.9000	0.9500

generalized gamma distributions and as such easy to use in practice, avoiding this way the use of other procedures such as the one already described, or the use of simulations.

Results in Tables 1 to 4 suggest approximations with a high degree of precision. This precision improves with the number of moments matched. Note the high degree of precision revealed in Table 3 for Scenario 3; in this case, all the variables are independent and identically distributed. In Figure 1, plots of the probability density function of Z (solid line) and of the approximating density function (dashed line) are presented for (a) Scenario 1 and $m = 4$, (b) Scenario 2 and $m = 2$, (c) Scenario 3 and $m = 2$, and (4) Scenario 4 and $m = 4$.

As an example of application, in the study of Hu and Beaulieu,[18] the authors mentioned that "In several practical wireless communication applications, there is a need for the accurate computation of the cumulative distribution function (CDF) and the probability density function (PDF) of the sum of L statistically independent Rayleigh random variables (RV's)." and also that "such sums occur in the measurement of signal-to-noise ratio for handoff and in the evaluation of equal gain combining systems when determining the error probability or the outage probability." For L statistical independent Rayleigh random variables, R_i, $i = 1, \dots, L$, with $R_i \sim$ Rayleigh(σ_i), the authors first considered the transformation $Y_i = R_i/\sigma_i$, $i = 1, \dots, L$, which gives the normalized random variables, $Y_i \sim$ Rayleigh(1), but then, they considered the

Table 2. Scenario 2—computed values of the approximating cumulative distribution in (8) for the exact quantiles q_δ with δ =0.05, 0.1, 0.5, 0.90, 0.95 for α =2.580, β =4.926, γ =1.687, and ℓ =2

m	$q_{0.05}$	$q_{0.1}$	$q_{0.5}$	$q_{0.90}$	$q_{0.95}$
2	0.0511	0.0999	0.4985	0.9012	0.9504
4	0.0500	0.0993	0.5005	0.8999	0.9496
6	0.0499	0.0999	0.5001	0.9000	0.9500
8	0.0500	0.1000	0.5000	0.9000	0.9500
10	0.0500	0.1000	0.5000	0.9000	0.9500
15	0.0500	0.1000	0.5000	0.9000	0.9500

Table 3. Scenario 3—computed values of the approximating cumulative distribution in (8) for the exact quantiles q_δ with δ =0.05, 0.1, 0.5, 0.90, 0.95 for α =4.426, β =6.884, γ =1.838, and ℓ =2

m	$q_{0.05}$	$q_{0.1}$	$q_{0.5}$	$q_{0.90}$	$q_{0.95}$
2	0.0500	0.1000	0.5000	0.9000	0.9500
4	0.0500	0.1000	0.5000	0.9000	0.9500
6	0.0500	0.1000	0.5000	0.9000	0.9500
8	0.0500	0.1000	0.5000	0.9000	0.9500
10	0.0500	0.1000	0.5000	0.9000	0.9500
15	0.0500	0.1000	0.5000	0.9000	0.9500

Table 4. Scenario 4—computed values of the approximating cumulative distribution in (8) for the exact quantiles q_δ with δ =0.05, 0.1, 0.5, 0.90, 0.95 for α =4.266, β =7.785, γ =1.441, and ℓ =2

m	$q_{0.05}$	$q_{0.1}$	$q_{0.5}$	$q_{0.90}$	$q_{0.95}$
2	0.0522	0.0994	0.4974	0.9023	0.9508
4	0.0502	0.0987	0.5008	0.8999	0.9493
6	0.0503	0.0999	0.5002	0.8999	0.9500
8	0.0500	0.0998	0.5003	0.9000	0.9501
10	0.0500	0.1000	0.5000	0.9000	0.9500
15	0.0500	0.1000	0.5000	0.9000	0.9500

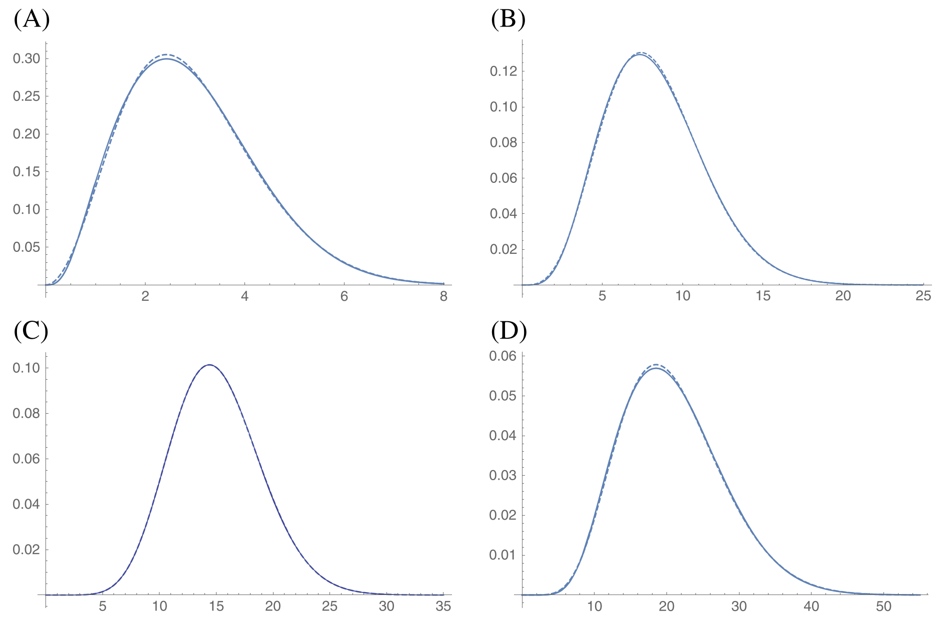

Figure 1. Plots of the probability density function of Z (solid line) and of the approximating density function (dashed line) for A, Scenario 1 and $m = 4$, B, Scenario II and $m = 2$, C, Scenario 3 and $m = 2$, and D, Scenario 4 and $m = 4$ [Colour figure can be viewed at wileyonlinelibrary.com]

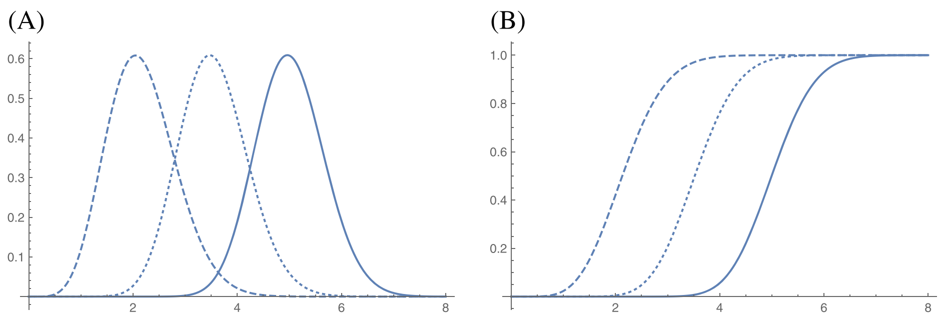

Figure 2. Plots of the approximating probability density A, and cumulative distribution B, functions of Z for $L = 3$ (dashed line), $L = 8$ (dotted line), and $L = 16$ (solid line) [Colour figure can be viewed at wileyonlinelibrary.com]

random variables $X_i = Y_i/\sqrt{L}$ with distribution $X_i \sim \text{Rayleigh}(1/\sqrt{L})$. Thus, the study focused on the distribution of the sum of L independent and identically distributed Rayleigh random variables

$$Z = \sum_{i=1}^{L} X_i.$$

The independent and identically distributed case can be easily addressed using the approach proposed in this work, and the high precision of the approximations have already been observed in Table 3. In Figure 2, the approximating density and cumulative distribution functions of Z are presented when $L = 3, 8, 16$. Note that the cases $L = 3$ and $L = 16$ were also considered in the study of Hu and Beaulieu.[18] In all cases, $\ell = 1$ and $m = 2$ were considered. The empirical density function is not presented because it would be indistinguishable.

3.2. Sums of independent gamma random variables

Following the procedure described in the beginning of this section, as a second example, the sum of independent gamma random variables will be addressed. The following results may also be applied to the linear combination of gamma random variables since, for a positive δ, if $X \sim \Gamma(\alpha, \beta)$ then $\delta X \sim \Gamma(\alpha, \delta\beta)$. There are several results available in the literature for the sum of independent gamma random variables, of which, only the most relevant are mentioned. A first result, for the sum of independent exponential random variables, is given in exercises 12 and 13 of the study of Feller.[25] In the study of Amari and Misra[26] and also in the study of Coelho,[27] the authors developed results for the sum of integer gamma random variables. The most interesting result is given in the study of Moschopoulos,[28] for the general case, where using an inversion of the moment generating function, it is shown that the sum of independent gamma random variables may be represented as an infinite mixture of gamma distributions, all with the same rate parameter. The practical use of this result may be limited by the fact that it is an infinite mixture representation. Interestingly, the same result may be obtained using the procedure described in the study of Marques.[3] A new approximation for the distribution of the sum of independent gamma random variables is proposed. Consider p independent gamma random variables, $X_i \sim \Gamma(\alpha_i, \beta_i), i = 1, \dots, p$. The interest is in the distribution of

$$Z = \sum_{i=1}^{p} X_i.$$

The characteristic function the X_i is given by

$$\Phi_{X_i}(t) = (1 - \mathrm{i}\beta_i t)^{-\alpha_i}, t \in \mathbb{R}.$$

Thus, the characteristic function of Z is given by

$$\Phi_Z(t) = \prod_{i=1}^{p} \Phi_{X_i}(t) = \prod_{i=1}^{p} (1 - \mathrm{i}\beta_i t)^{-\alpha_i}, t \in \mathbb{R}. \tag{13}$$

Since the expression of the characteristic function is known, the hth moment of Z may be obtained as in the previous subsection as follows

$$E(Z^h) = \mathrm{i}^{-h} \left. \frac{\partial^h \Phi_Z(t)}{\partial t^h} \right|_{t=0}.$$

For the case of the sum of independent gamma random variables, the following four scenarios are considered:

Scenario 1: $\boldsymbol{\alpha} = \left\{2, \frac{1}{3}\right\}$ and $\boldsymbol{\beta} = \{3, 10\}$;
Scenario 2: $\boldsymbol{\alpha} = \left\{2, \frac{1}{3}, 10\right\}$ and $\boldsymbol{\beta} = \left\{\frac{1}{3}, 4, \frac{5}{4}\right\}$;
Scenario 3: $\boldsymbol{\alpha} = \{3, 3, 3, 3\}$ and $\boldsymbol{\beta} = \left\{\frac{1}{2}, 5, 8, \frac{5}{4}\right\}$;

Table 5. Scenario 1—computed values of the approximating cumulative distribution in (8) for the exact quantiles q_δ with $\delta = 0.05, 0.1, 0.5, 0.90, 0.95$ for $\alpha = 10.362$, $\beta = 0.0312$, $\gamma = 0.4212$, and $\ell = 1$

m	$q_{0.05}$	$q_{0.1}$	$q_{0.5}$	$q_{0.90}$	$q_{0.95}$
2	0.0413	0.0929	0.5070	0.8978	0.9487
4	0.0414	0.0929	0.5069	0.8978	0.9487
6	0.0477	0.0989	0.5019	0.8992	0.9498
8	0.0512	0.1010	0.4997	0.8999	0.9499
10	0.0507	0.1010	0.4997	0.9000	0.9499
15	0.0505	0.1007	0.4998	0.9000	0.9500

Table 6. Scenario 2—computed values of the approximating cumulative distribution in (8) for the exact quantiles q_δ with $\delta = 0.05, 0.1, 0.5, 0.90, 0.95$ for $\alpha = 17.406$, $\beta = 0.3525$, $\gamma = 0.7721$, and $\ell = 1/2$

m	$q_{0.05}$	$q_{0.1}$	$q_{0.5}$	$q_{0.90}$	$q_{0.95}$
2	0.0517	0.1026	0.4996	0.8975	0.9491
4	0.0502	0.1003	0.4995	0.9003	0.9504
6	0.0501	0.1001	0.4998	0.9001	0.9500
8	0.0501	0.1001	0.4999	0.9000	0.9499
10	0.0500	0.1000	0.5000	0.9000	0.9500
15	0.0500	0.1000	0.5000	0.9000	0.9500

Table 7. Scenario 3—computed values of the approximating c umulative distribution i n (8) f or the e xact quantiles q_δ with δ =0.05, 0.1, 0.5, 0.90, 0.95 f or α =13.786, β =1.1576, γ =0.7238, and ℓ =2

m	$q_{0.05}$	$q_{0.1}$	$q_{0.5}$	$q_{0.90}$	$q_{0.95}$
2	0.0505	0.1001	0.4994	0.9004	0.9502
4	0.0500	0.0998	0.5000	0.9000	0.9500
6	0.0500	0.1000	0.5000	0.9000	0.9500
8	0.0500	0.1000	0.5000	0.9000	0.9500
10	0.0500	0.1000	0.5000	0.9000	0.9500
15	0.0500	0.1000	0.5000	0.9000	0.9500

Table 8. Scenario 4—computed values of the approximating c umulative distribution i n (8) f or the e xact quantiles q_δ with δ =0.05, 0.1, 0.5, 0.90, 0.95 f or α =13.154, β =1.9591, γ =0.8765, and ℓ =1

m	$q_{0.05}$	$q_{0.1}$	$q_{0.5}$	$q_{0.90}$	$q_{0.95}$
2	0.0500	0.1000	0.5000	0.9000	0.9500
4	0.0500	0.1000	0.5000	0.9000	0.9500
6	0.0500	0.1000	0.5000	0.9000	0.9500
8	0.0500	0.1000	0.5000	0.9000	0.9500
10	0.0500	0.1000	0.5000	0.9000	0.9500
15	0.0500	0.1000	0.5000	0.9000	0.9500

Scenario 4: $\boldsymbol{\alpha} = \left\{\frac{1}{2}, 3, \frac{4}{5}, 6, \frac{7}{8}\right\}$ and $\boldsymbol{\beta} = \{1, 2, 3, 4, 5\}$.

In Tables 5 to 8, the values of the approximating cumulative distribution function in (8) have been computed for the exact quantiles q_δ with $\delta = 0.05, 0.1, 0.5, 0.90, 0.95$. Similar to the previous subsection, the exact quantiles values were obtained through numerical inversion of Φ_Z in (13) using the inversion formulas in the study of Gil-Pelaez[24] and the bisection method.

From Tables 5 to 8, the high precision of these approximations is observed, which improves as the number of moments matched is increased. In Figure 3, plots of the probability density function of Z (solid line) and of the approximating density function (dashed line) are presented for (a) Scenario 1, $m = 6$ and $\ell = 1$, (b) Scenario 2, $m = 2$ and $\ell = 1/2$, (c) Scenario 3, $m = 2$ and $\ell = 2$, and (d) Scenario 4, $m = 2$ and $\ell = 1$.

In the study of Ansari et al,[29] the authors considered an application of the sum of independent gamma random variables to the performance analysis of diversity combining receivers operating over Nakagami-m^* fading channels. The authors refer that in a Nakagami multipath fading channel, $\gamma^* = |\alpha^*|^2$ follows a gamma distribution; see the study of Ansari et al[29] for details. More precisely, the density of γ^* is given by

$$p_{\gamma^*}(x) = \left(\frac{m^*}{\Omega}\right)^{m^*} \frac{x^{m^*-1}}{\Gamma(m^*)} \exp\left\{-\frac{m^*}{\Omega}x\right\},$$

where $m^* > 0$ is the Nakagami-m^* multipath fading parameter and $\Omega > 0$ is the mean of the local power. Using the notation established in Corollary 2.2, $\gamma^* \sim \Gamma(m^*, \Omega/m^*)$. The authors developed representations in terms of Meijer-G and on the Fox H-functions, for the density and cumulative distribution functions of

$$Z = \sum_{i=1}^{L} \gamma_i^*,$$

with $\gamma_i^* \sim \Gamma(m_i^*, \Omega_i/m_i^*)$, independent but not necessarily identical. In Figure 4, the scenarios in figure 1 of the study of Ansari et al[29] are considered, and the plots for the approximating density and cumulative distribution functions of Z are presented, for $m = 6$ and $\ell = 1$, when

(a) $\boldsymbol{m}^* = \{0.6, 1.1, 2\}$ and $\Omega = \{1, 1, 1\}$;

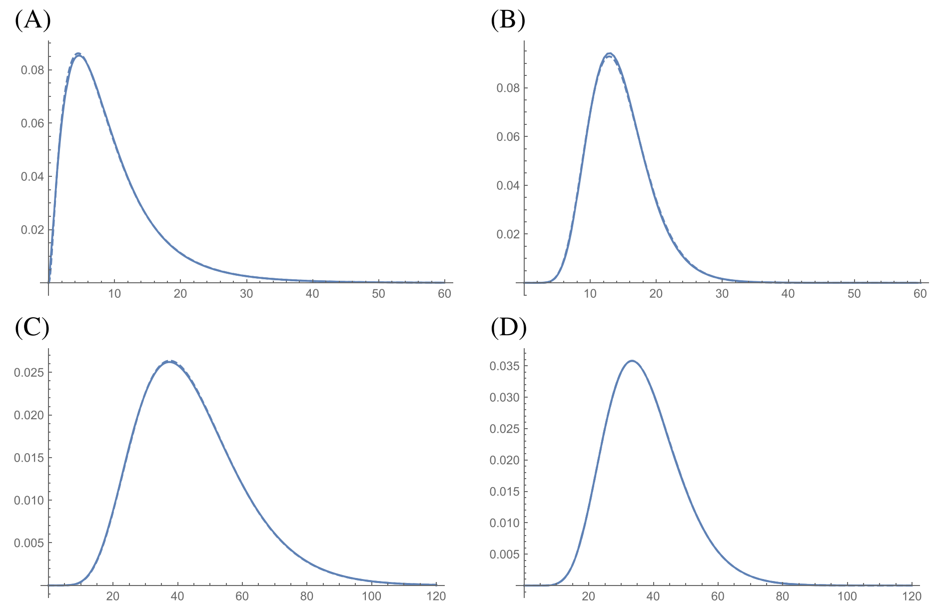

Figure 3. Plots of the smooth empirical probability density function of Z (solid line) evaluated from 10 000 000 simulated values of Z, a n d of the approximating density function (dashed line) for A, Scenario 1, $m = 2$ a n d $\ell = 3$ B, Scenario 2, $m = 2$ a n d $\ell = 3$, C, Scenario 3, $m = 6$ a n d $\ell = 3$, and D, Scenario 4, $m = 2$ a n d $\ell = 3$ [Colour figure can be viewed at wileyonlinelibrary.com]

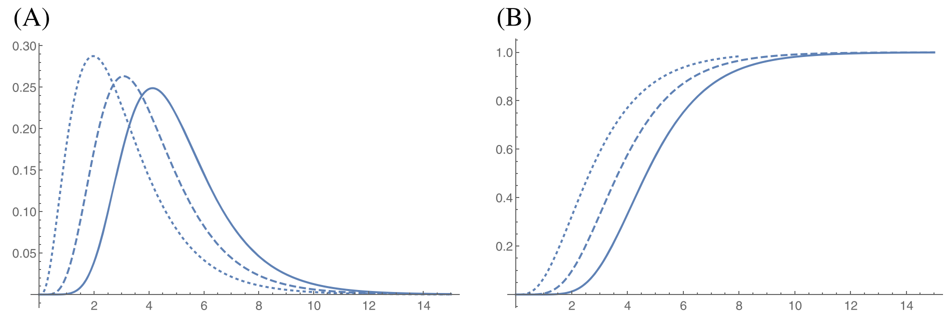

Figure 4. Plots of the approximating probability density A, and cumulative distribution B, functions of Z for scenarios (a) dotted line, (b) dashed line, and (c) solid line [Colour figure can be viewed at wileyonlinelibrary.com]

(b) $\boldsymbol{m}^* = \{0.6, 1.1, 2, 3.4\}$ and $\Omega = \{1, 1, 1, 1\}$;
(c) $\boldsymbol{m}^* = \{0.6, 1.1, 2, 3.4, 4.5\}$ and $\Omega = \{1, 1, 1, 1, 1\}$.

3.3. Sum of independent Weibull random variables

The last case considered is that of the sum of independent Weibull random variables. The distribution of this sum is very important, for example, in problems related to wireless communications.[30,31] The Weibull distribution is also a particular case of the well-known generalized extreme value distribution, which also include the Gumbel and Fréchet distributions. Thus, the distribution of the sum of independent Weibull random variables is also relevant for problems related to extremes. Finding an exact representation or approximation for the distribution of the sum of independent Weibull random variables is a difficult problem; for example, in the study of Nadarajah,[19] it is stated that "Unfortunately, no results (not even approximations) have been known for sums of Weibull random variables. It is expected that this review could help to motivate some work for this case." Recently, in the study of Filho and Yacoub,[30] approximations were developed for the independent and identically distributed case. For the nonidentically distributed case, in the study of Yilmaz and Alouini,[31] the authors give an infinite series representation that does not naturally lead to a straight forward implementation. For the sum of independent Weibull random variables, it is proposed that the approximating cumulative and density functions in (8) and (7) are obtained using the procedure described in Steps 1 to 3 at the beginning of this section. Consider

p independent Weibull random variables, $X_i \sim \text{Weibull}(\alpha_i, \beta_i), i = 1, \dots, p$ and the random variable

$$Z = \sum_{i=1}^{p} X_i.$$

The characteristic function of a Weibull random variable has no closed-form expression; thus, the exact moments can not be determined as in the previous subsections. Therefore, the exact moments for the sum of independent Weibull random variables are determined using expression (9). This expression was also used in the study of Filho and Yacoub.[30]

For illustrating the performance of the proposed approximation, four scenarios are considered:

Scenario 1: $\boldsymbol{\alpha} = \{2, 3\}$ and $\boldsymbol{\beta} = \left\{\frac{4}{5}, 6\right\}$;
Scenario 2: $\boldsymbol{\alpha} = \{6, 4, 2\}$ and $\boldsymbol{\beta} = \{1, 3, 5\}$;
Scenario 3: $\boldsymbol{\alpha} = \left\{\frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right\}$ and $\boldsymbol{\beta} = \{1, 1, 1, 1\}$;
Scenario 4: $\boldsymbol{\alpha} = \left\{\frac{18}{3}, \frac{12}{7}, 4, \frac{8}{3}, \frac{31}{3}\right\}$ and $\boldsymbol{\beta} = \left\{\frac{1}{5}, \frac{2}{3}, 10, \frac{7}{5}, \frac{20}{8}\right\}$.

For the sum of independent Weibull random variables, it was not possible to plot the exact probability density function nor to determine its exact quantiles. Therefore, 10 000 000 values of Z were simulated, and the corresponding smooth empirical density of Z plotted. The simulated data was also used to calculate the empirical quantiles $\tilde{q}_\delta$ with $\delta = 0.05, 0.1, 0.5, 0.90, 0.95$. Thus, in Tables 9 to 12, the values of the approximating cumulative distribution function in (8) were computed for the empirical quantiles $\tilde{q}_\delta$ with $\delta = 0.05, 0.1, 0.5, 0.90, 0.95$. Note that the results in Tables 9 to 12, are obtained using the empirical quantiles and therefore may not be accurate to a high degree since the empirical quantiles may only have two or three decimal places equal to the exact ones.

In Tables 9 to 12, the same features are observed as that already described in the previous subsections for the sum of independent Rayleigh random variables and for the sum of independent gamma random variables. However, note that (a) the approximating cumulative distribution function is evaluated in the empirical quantiles obtained from 10 000 000 simulated values of Z, which may have only two or three exact decimal places and (b) there are cases where this approximation may not give very accurate results; these cases may be easily identified as the cases where it may be difficult to complete Step 1 and solve the consequent system of equations in (10).

Table 9. Scenario 1—computed values of the approximating cumulative distribution in (8) for the empirical quantiles $\tilde{q}_\delta$ with $\delta = 0.05, 0.1, 0.5, 0.90, 0.95$ for $\alpha = 1.3342$, $\beta = 5.9760$, $\gamma = 2.8263$, and $\ell = 3$

m	$\tilde{q}_{0.05}$	$\tilde{q}_{0.1}$	$\tilde{q}_{0.5}$	$\tilde{q}_{0.90}$	$\tilde{q}_{0.95}$
2	0.0500	0.0988	0.5003	0.9005	0.9499
4	0.0494	0.0990	0.5011	0.8994	0.9496
6	0.0498	0.1000	0.4998	0.9002	0.9501
8	0.0500	0.1000	0.5000	0.9000	0.9500
10	0.0500	0.1000	0.5000	0.9000	0.9500
15	0.0500	0.1000	0.5000	0.9000	0.9500

Table 10. Scenario 2—computed values of the approximating cumulative distribution in (8) for the empirical quantiles $\tilde{q}_\delta$ with $\delta = 0.05, 0.1, 0.5, 0.90, 0.95$ for $\alpha = 3.2860$, $\beta = 0.2906$, $\gamma = 1.8545$, and $\ell = 1$

m	$\tilde{q}_{0.05}$	$\tilde{q}_{0.1}$	$\tilde{q}_{0.5}$	$\tilde{q}_{0.90}$	$\tilde{q}_{0.95}$
2	0.0517	0.1026	0.4996	0.8975	0.9491
4	0.0502	0.1003	0.4995	0.9003	0.9504
6	0.0501	0.1001	0.4998	0.9001	0.9500
8	0.0501	0.1001	0.4999	0.9000	0.9499
10	0.0500	0.1000	0.5000	0.9000	0.9500
15	0.0500	0.1000	0.5000	0.9000	0.9500

Table 11. Scenario 3—computed values of the approximating cumulative distribution in (8) for the empirical quantiles $\tilde{q}_\delta$ with $\delta =$ 0.05, 0.1, 0.5, 0.90, 0.95 for $\alpha = 13.786$, $\beta = 1.1576$, $\gamma = 0.7238$, and $\ell = 2$

m	$\tilde{q}_{0.05}$	$\tilde{q}_{0.1}$	$\tilde{q}_{0.5}$	$\tilde{q}_{0.90}$	$\tilde{q}_{0.95}$
2	0.0505	0.1001	0.4994	0.9004	0.9502
4	0.0500	0.0998	0.5000	0.9000	0.9500
6	0.0500	0.1000	0.5000	0.9000	0.9500
8	0.0500	0.1000	0.5000	0.9000	0.9500
10	0.0500	0.1000	0.5000	0.9000	0.9500
15	0.0500	0.1000	0.5000	0.9000	0.9500

Table 12. Scenario 4—computed values of the approximating cumulative distribution in (8) for the empirical quantiles $\tilde{q}_\delta$ with $\delta =$ 0.05, 0.1, 0.5, 0.90, 0.95 for $\alpha = 2.1785$, $\beta = 11.414$, $\gamma = 3.6669$, and $\ell = 3$

m	$\tilde{q}_{0.05}$	$\tilde{q}_{0.1}$	$\tilde{q}_{0.5}$	$\tilde{q}_{0.90}$	$\tilde{q}_{0.95}$
2	0.0489	0.0971	0.5018	0.9005	0.9494
4	0.0487	0.0985	0.5019	0.8986	0.9491
6	0.0498	0.0999	0.4997	0.9000	0.9501
8	0.0499	0.1002	0.4994	0.9001	0.9499
10	0.0500	0.1001	0.4999	0.8999	0.9500
15	0.0500	0.1000	0.4999	0.8999	0.9500

In Figure 5, plots of the smooth empirical probability density function of Z (solid line) are presented, evaluated from 10 000 000 simulated values of Z and of the approximating density function (dashed line) for (a) Scenario 1, $m = 2, \ell = 3$ (b) Scenario 2, $m = 2$ and $\ell = 3$, (c) Scenario 3, $m = 6$ and $\ell = 3$, and (d) Scenario 4, $m = 2$ and $\ell = 3$.

Although the presented figures may intuit less precision of approximations for $q_{0.5}$ or $\tilde{q}_{0.5}$ quantiles, the results in the tables show that the differences on the precision of the approximations do not seem to be substantial.

As an illustration of the applicability of the results provided, we consider the application in the study of Yilmaz and Alouini.[31] In this work, the authors mentioned that "Sum of Weibull random variables (RVs) is naturally of prime importance in wireless communications and related areas." In the study of Yilmaz and Alouini,[31] the Weibull distribution was used to describe the amplitude of the received power from a wireless channel. More precisely, the authors define that a random variable P_l is a channel power Weibull random variable if its density is given by

$$f_{P_l}(x) = \xi_l \left(\frac{\theta_l}{\Omega_l}\right)^{\xi_l} x^{\xi_l - 1} \exp\left\{-\left(\frac{\theta_l}{\Omega_l}x\right)^{\xi}\right\},$$

for $x \geq 0$, where $\Omega_l > 0, \xi_l \geq 1$ are designated as the average power and the shape parameters respectively, and where $\theta_l = \Gamma(1 + 1/2\xi_l)$ is designated as the power exponent coefficient.[31] Using the notation established in Corollary 2.4, it follows that

$$P_l \sim \text{Weibull}\left(\xi_l, \frac{\Omega_l}{\theta_l}\right).$$

As already mentioned, in Yilmaz and Alouini,[31] the results presented for the sum of independent Weibull random variables

$$Z = \sum_{l=1}^{L} P_l$$

are based on infinite series involving hypergeometric functions that may not be easy to use. As an illustration, the same scenarios as in figure 2 of the study of Yilmaz and Alouini[31] are considered, with the exception of case $\Omega = 1$ and $\xi = 1$, which only considers one single Weibull random variable. Thus, in Figure 6, for the following scenarios

a. $\Omega = \{1, 1\}$ and $\xi = \{1, 2\}$,

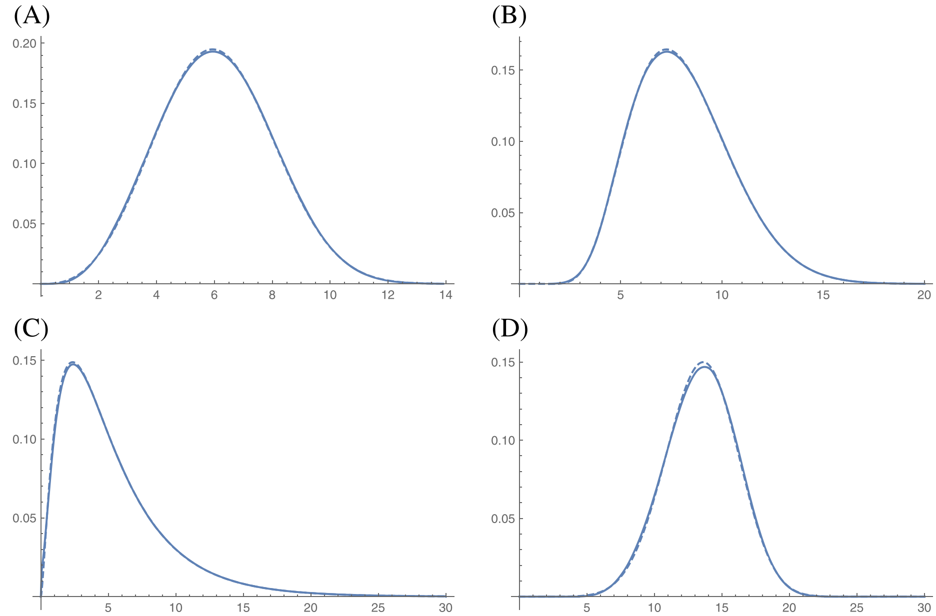

Figure 5. Plots of the probability density function of Z (solid line) and of the approximating density function (dashed line) for A, Scenario 1, $m = 6$ and $\ell = 1$ B, Scenario 2, $m = 2$ and $\ell = 1/2$, C, Scenario 3, $m = 2$ and $\ell = 2$, and D, Scenario 4, $m = 2$ and $\ell = 1$ [Colour figure can be viewed at wileyonlinelibrary.com]

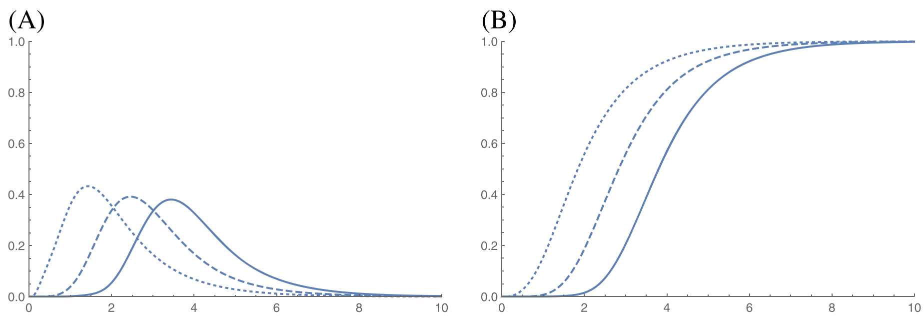

Figure 6. Plots of the approximating probability density A, and cumulative distribution B, functions of Z for scenarios (a) dotted line, (b) dashed line, and (c) solid line [Colour figure can be viewed at wileyonlinelibrary.com]

b. $\Omega = \{1, 1, 1\}$ and $\xi = \{1, 2, 3\}$,
c. $\Omega = \{1, 1, 1, 1\}$ and $\xi = \{1, 2, 3, 4\}$,

the plots of the approximating density and cumulative distribution functions are presented, when $m = 6$ and $\ell = 1$.

4. Discussion and Conclusions

It is shown that a generalized gamma distribution may be represented (a) as a gamma-series expansion where all the gamma distributions in the series have the same scale parameter and (b) as a mixture of generalized gamma distributions. These results provide interesting insights about the distribution of a generalized gamma distribution and of its particular cases. Motivated by these representations, a general method for approximating the distribution of a sum of independent variables belonging to the generalized gamma family is proposed, which is based on mixtures of generalized gamma distributions. The methodology was illustrated for the Rayleigh, gamma, and Weibull distributions in Subsections 3.1 to 3.3, for specific scenarios. In these scenarios, the approximating probability density and cumulative distribution functions displayed accurate results. These approximations may be improved by increasing the number of moments matched or by adjusting the tuning parameter. There may be some cases where the solution of the system of equations (10), in Step 1 of the procedure, may be difficult to obtain. This is mainly due (a) to the fact that the moments of the generalized gamma distribution are expressed in terms of gamma functions, which may present some computational problems or (b) to scenarios where it may be more difficult to fit a generalized gamma distribution. The system of equations in Step 2 (11) is very easy to solve since it involves only linear equations. In Step 3, the tuning parameter is defined. Although a measure of the impact of the tuning parameter on the precision of the approximation is not provided, this parameter is very important in the improvement of the approximation. After completing Steps 1 to 3, the approximating density and cumulative distribution functions can be easily used in practice since they are simple mixtures of generalized gamma distributions. The computation time of p values or quantiles is nearly zero. The authors aim, in the future, to simplify the proposed methodology, to develop user-friendly computational tools and to make the code available online for any interested user.

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