EXISTENCE OF A DENSITY OF THE 2DIM STOCHASTIC NAVIER STOKES EQUATION DRIVEN BY LÉVY PROCESSES OR FRACTIONAL BROWNIAN MOTION

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Abstract. In this article we are interested in the regularity properties of the probability measure induced by the solution process of the Lévy noise or a fractional Brownian motion driven Navier Stokes Equation on the two-dimensional torus $\mathbb{T}$. We mainly investigate under which conditions on the characteristic measure of the Lévy process or the Hurst parameter of the fractal Brownian motion the law of the projection of $u(t)$ onto any finite dimensional $F \subset L^2(\mathbb{T})$ is absolutely continuous with respect to the Lebesgue measure on F.

1. Introduction

We consider the Navier-Stokes equations (NSEs) subjected to the periodic boundary condition on the torus

$$\begin{cases} \partial_t u(t) - \nu\Delta u(t) + u(t)\cdot\nabla u(t) + \nabla \mathfrak{p}(t) = \dot{\Xi}(t), \\ \nabla\cdot u(t) = 0, \\ u(0) = u_0, \end{cases} \tag{1}$$

where u and $\mathfrak{p}$ are unknown vector field and scalar periodic functions in the space variable representing the fluid velocity and the pressure, respectively. We assume that we are given an initial velocity u_0. The perturbation $\dot{\Xi}$ denotes, roughly speaking, the Radon-Nikodym derivative with respect to the time variable of a Lévy process $\Xi = L$ or a fractional Brownian motion $\Xi = B^H$. In the case when Ξ is a Wiener noise the above system has been the subject of intensive mathematical studies since the pioneering work of Bensoussan and Temam. The analysis of the qualitative properties and long-time behaviour of its solutions has generated several significant results, see for instance [6, 9, 16, 18, 22], to cite a few results. Particularly, when

$$\Xi \;=\; \sum_{j=1}^{\infty} b_j\beta_j e_j, \tag{2}$$

where $(b_j)_{j\in\mathbb{N}}$ is a sequence of non-negative numbers, $(\beta_j)_{j\in\mathbb{N}}$ is a sequence of independent, identically distributed real-valued Brownian motions and $(e_j)_{j\in\mathbb{N}}$ is an orthonormal basis of the space of square integrable, periodic and divergence free functions with mean zero, the authors in [10], [2] and [24] proved the existence of densities for the laws of finite dimensional functionals of its solutions. In these papers, different methods are used to prove the existence of such densities, for instance in [10] a method based on Girsanov theorem is used, and the Malliavin calculus is used in [24]. In [2] a method based on controllability of (1) in finite-dimensional projections and an abstract result on an image of a decomposable measure under analytic mappings is used. This method does not use the Gaussian structure of the noise as the methods in [10] and [24]. In this paper, we are mainly interested in proving the existence of densities for the laws of finite-dimensional analytic functionals of the solution of (1) when the driving noise Ξ is a Lévy noise or a fractional Brownian motion. For this purpose, we extend the results in [2] to our framework. Although we closely follow the approach in [2] the extension of the result therein to our setting is not trivial. The proof in [2] relies very much on the natural decomposability of the driving noise law in a Hilbert space $\mathscr{H}$ which is not naturally satisfied by a Lévy process or a fractional Brownian. In fact, even if the

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Lévy noise (or fractional Brownian motion) Ξ has a decomposition as in (2), which is one of the main assumptions in [2], it is not known whether there exists a Hilbert space $\mathscr{H}$ on which the law of Ξ on $\mathscr{H}$ is decomposable. In order to overcome this difficulty we prove, by using wavelet analysis and the decomposability of measure on Banach space introduced in [12], that there exists a Banach space $\mathscr{H}$ with Schauder basis on which the law of Ξ is decomposable. With this result at hand and using the solid controllability of (1) we can prove the existence of densities for the laws of finite-dimensional projection of the solutions of (1).

Further works related to our result is, for instance, the work of Dong et.a. [13]. Let X be the solution of a possible infinite dimensional stochastic differential equation on a Hilbert space H driven by a pure Lévy process. The investigate under which conditions on the Lévy measure and the Markovian semigroup generated by the process is smoothing in space using the Kolmogorov equation. In [27], the authors prove exponential convergence to the invariant measure where the underlying process is again solution of a possible infinite dimensional stochastic (partial) differential equation on a Hilbert space H driven by an α stable process. Here, the Harris Theorem is used to get the result. In [1] the authors prove the strong Feller property and exponential mixing for the 3 dimensional Navier-Stokes equation under different degeneracy assumptions on the driving process. Here, the main tool is the Malliavin calculus and Hörmander's Theorem.

In the next section we will fix the notation and present some preliminary results. Section 3 is devoted to the statement and the proof of our main result which will be applied to the stochastic 2D Navier-Stokes equations in the torus. In Appendix A and Appendix B, we present and prove several results related to the wavelet expansion of Lévy noise and fractional Brownian motion, respectively. In Appendix C we establish a zero-one law result, which is crucial for the proof of the main result, for decomposable measures.

2. Notations, Hypotheses and preliminary results

For a separable Banach space E we denote by $\mathcal{B}(E)$ its Borel σ–algebra. For a finite dimensional subspace E_0 of E we denote by E_1 the complemented subspace of E, i.e. the subspace of E such that $E = E_0 \oplus E_1$, *i.e.*, $E_1 = E_0^\perp$. For two subspaces E_1 and E_2 of E, $E_1 \oplus E_2$ is defined by $E_1 \oplus E_2 := \{z = e_1 + e_2 : e_1 \in E_1, e_2 \in E_2\}$. Furthermore, for $A \subset E$ and $y \in E_1$ we put

$$A_{(E_0,E_1)}(y) = \{x \in E_0 : x + y \in A\}.$$

Let μ be a probability measure on $(E, \mathcal{B}(E))$ and E_0 and E_1 as above. We define a probability measure μ_{E_0} on $(E_0, \mathcal{B}(E_0))$ by

$$\mu_{E_0} : \mathcal{B}(E_0) \ni A \mapsto \mu(A + E_1) \in [0, 1].$$

For a subspace $\tilde{E}_0 \subset E_1$ we set

$$\mu_{(\tilde{E}_0,E_1)} : \mathcal{B}(\tilde{E}_0) \ni A \mapsto \mu(A + E_1) \in [0, 1].$$

Since E_0 is finite dimensional, we can denote by Leb_{E_0} the measure defined by

$$\mathrm{Leb}_{E_0} : \mathcal{B}(E_0) \ni U \mapsto \mu_{E_0}(U) := \mathrm{Leb}_{\mathbb{R}^n}(\iota^{-1}(U)),$$

where ι is the isomorphism $\iota : E_0 \to \mathbb{R}^n$, $n = \dim(E_0)$. Besides, let us define the projection operator π_{E_j}, $j = 0, 1$, given by $\pi_{E_j} x = y$, where y is the unique element with $y + y_1 = x$, $y_1 \in E_j^\perp$. We can now introduce the following definition.

Definition 2.1. *Let $(\Omega_1, \mathcal{A}_1)$ and $(\Omega_2, \mathcal{A}_2)$ be measurable spaces. A map $\kappa : \Omega_1 \times \mathcal{A}_2 \to [0, 1]$ is called a probability kernel (from Ω_1 to Ω_2) if*

(1) $\Omega_1 \ni \omega \mapsto \kappa(\omega_1, A_2)$ is measurable for any $A_2 \in \mathcal{A}_2$;
(2) $\mathcal{A}_2 \ni A_2 \mapsto \kappa(\omega_1, A_2)$ is a probability measure for any $\omega_1 \in \Omega_1$;

In case of a finite dimensional subspace and its complement, desintegration gives the existence of a probability kernel.

Remark 2.1. *Let $\{F_n : n \in \mathbb{N}\}$ be a family of mutually disjoint, linearly independent, and closed finite dimensional subspaces of E, i.e. $F_j \cap F_k = \{0\}$, $j \neq k$. We set $G_n := F_1 \oplus \cdots \oplus F_n$, $G^n := (F_1 \oplus \cdots \oplus F_n)^\perp$, and μ_{G^n} the probability measure induced on G^n by the projection $P_{G^n}(y) := \{z \in G^n : \exists x \in G_n : x + z = y\}$. Then, by desintegration (see e.g. Theorem 6.4* [20, p.108]*) we know that for any $n \in \mathbb{N}$ there exists a probability kernel*

$$l_n : G^n \times \mathcal{B}(F_1 \oplus \cdots \oplus F_n) \to \mathbb{R}_0^+,$$

such that we have for all $A \in \mathcal{B}(E)$

$$\mu(A) = \int_{G^n} \int_{A_n(y)} l_n(\mathbf{y}, dx) \mu_{G^n}(d\mathbf{y}),$$

where $A_n(\mathbf{y}) = A_{(F_1 \oplus \cdots \oplus F_n, G^n)}(\mathbf{y})$. In particular, we can decompose the measure μ into the quadruple $\{F_n, G^n, l_n\}_{n=1}^\infty$.

Hereafter we fix a separable Banach space E with Schauder basis $\{e_n : n \in \mathbb{N}\}$ and we set

$$F_n = \{\lambda e_n : \lambda \in \mathbb{R}\}. \tag{3}$$

We also set

$$G_n = F_0 \oplus F_1 \oplus \ldots \oplus F_n \text{ and } G^n = G_n^\perp, \tag{4}$$

along which we consider a probability kernel

$$l_n : G^n \times \mathcal{B}(G_n) \to [0, 1].$$

The projection onto any nontrivial subspace $F \subset E$ is denoted by π_F. Having fixed these notations we now proceed to the statement of our assumptions.

Analysing Theorem 2.2 of [2], one can easily verify that following assumption is essential.

Assumption 2.1. *Let $\mu \in \mathcal{P}(E)$ be a decomposable measure with decomposition $\{F_n, G^n, l_n\}_{n=0}^\infty$. We assume that for any $n \in \mathbb{N}$ there exists a positive function $\rho_n : G^n \times G_n \to \mathbb{R}_0^+$ such that μ_{G^n}–a.s. we have for all $U \in \mathcal{B}(G_n)$*

$$l_n(\mathbf{y}, U) = \int_U \rho_n(\mathbf{y}, x)\, dx.$$

Assumption 2.1 is often difficult to verify. Hence we formulate the next assumption which is stronger but easier to check than the above. In fact, we prove in Lemma C.1 that the following assumption, i.e. Assumption 2.2, implies Assumption 2.1.

Assumption 2.2. *Let $\mu \in \mathcal{P}(E)$ be a decomposable measure with decomposition*

$$\{F_n, G^n, l_n\}_{n=0}^\infty$$

constructed by a Schauder basis $\{e_n : n \in \mathbb{N}\}$ with $F_n := span\{e_1, \ldots, e_n\}$ such that μ_{G_n} is absolutely continuous with respect to the Lebesgue measure Leb_{G_n}.

Our third condition is given in the following next lines.

Assumption 2.3. *Let $\mu \in \mathcal{P}(E)$ and let $\{F_n, G^n, l_n\}_{n=0}^\infty$ be a decomposition of μ. There exists a point*

$$Y = \sum_{j=1}^\infty y_j e_j \in E, \tag{5}$$

such that

(1) for any $n \in \mathbb{N}$ and $\delta > 0$[1]

$$\mu_{G_n}(B_{G_n}(\pi_{G_n} Y, \delta)) > 0.$$

[1]For a Banach space E we denote by $B_E(y, \delta)$ the ball centered at y with radius δ, i.e. $B_E(y, \delta) = \{x \in E : |x - y|_E \leq \delta\}$.

(2) for all numbers $N \in \mathbb{N}$ there exists a $R_N > 0$ such that for all $x_0 \in B_{G_N}(\pi_{G_N} Y, R_N)$, and all $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\mu\left(\{x \in E \mid |x - x_0|_E \leq \epsilon\}\right) \geq \delta.$$

In order to clarify the role of the above assumption we shall introduce the following definition.

Definition 2.2. *We call a set $A \in \mathcal{B}^\mu(E)$ a finite zero one μ–set if and only if for all $n \in \mathbb{N}$*

$$\mu_{G^n}\left(\{y \in G^n : \mu_n(A_n(y)) = 0 \text{ or } 1\}\right) = 1,$$

where $A_n(y) = A_{(F_1 \oplus \cdots \oplus F_n, G^n)}(y)$.

Let $F_\infty = \cup_{n \in \mathbb{N}} \{F_0 + F_1 + \cdots + F_n\}$. Now, let us present the generalization of Theorem 4 in [12], respective [2, Theorem 1.6], whose proof requires that the measure μ is decomposable and has a finite second moment (see [2, Property (P), page 402] for the precise statement).

Theorem 2.1. *Let $f : X \to \mathbb{R}$ be an analytic function and let $\mu \in \mathcal{P}(X)$ be a decomposable measure with density satisfying Assumption 2.1 and Assumption 2.3. Let $\mathcal{N}_f \subset E$ be defined by*

$$\mathcal{N}_f := \{x \in E : f(x) = 0\}.$$

Then, we have

$$\mu(\mathcal{N}_f) = 0 \text{ or } 1.$$

Furthermore, if f is not identical zero, then $\mu(\mathcal{N}_f) = 0$.

Proof of Theorem 2.1: Let $\mathcal{N}_f := \{x \in G : f(x) = 0\}$. Since f is analytic, for all $n \in \mathbb{N}$ for any $y \in G^n$ the function $f_y(x) := f(y + x)$ is also analytic. Therefore, either $\mathrm{Leb}_{G_n}(\mathcal{N}_f^n(y)) = 0$ or $\mathrm{Leb}_{G_n}(E \setminus \mathcal{N}_f^n(y)) = 0$, where $\mathcal{N}_f^n(y) = \{x \in G_n : x + y \in \mathcal{N}_f\}$. Thus, $\mathcal{N}_f$ is a finite zero–one μ set, and there exists a set $\tilde{\mathcal{N}}_f \in \mathcal{B}(E)$ such that $\tilde{\mathcal{N}}_f + F_{(\infty)} = \tilde{\mathcal{N}}_f$ and $\mu(\tilde{\mathcal{N}}_f) = \mu(\mathcal{N}_f)$.

To prove the second part we assume that $f \not\equiv 0$. We will show that $\mu(\mathcal{N}_f^c) > 0$. For this purpose let $n \in \mathbb{N}$ be fixed and set $f_n := f_{|G_n}$ and $Y_n := \pi_{G_n} Y$ where Y is the point from Assumption 2.3. Observe that f_n is analytic and therefore

$$\mathrm{Leb}_{G_n}\Big(G_n \setminus \{x \in G_n : f(x) \neq 0\}\Big) = 0.$$

We now distinguish two cases: $f_n(Y_n) \neq 0$ and $f_n(Y_n) = 0$. In the first case, *i.e.*, $f_n(Y_n) \neq 0$ we observe that by the continuity of f_n there exists a number $\delta > 0$ such that $f(x) \neq 0$ for all $x \in B_E(Y_n, \delta)$. Therefore, and taking into account item (2) of Assumption 2.3 we conclude that $\mu(\mathcal{N}_f^c) > 0$. To treat the second case, *i.e*, $f_n(Y_n) = 0$, we first notice that, since f_n is analytic, we have

$$\mathrm{Leb}_{G_n}\Big(\{x \in G_n : f(x) = 0\}\Big) = 0.$$

This implies that for any $\epsilon > 0$ one can find an element $x_0 \in G_n$ such that $|x_0 - Y_n|_E \leq \epsilon$ and $f(x_0) \neq 0$. Since f is continuous we can find a number $\delta > 0$ such that $f(x) \neq 0$ for all $x \in B_E(x_0, \delta)$. Item (2) of Assumption 2.3 with $\epsilon = \frac{R}{2}$ yields that $\mu(B_E(x_0, \delta) > 0)$ from which it follows that $\mu(\mathcal{N}_f^c) > 0$. $\square$

The above theorem will, as in [2, Theorem 2.2], be used to prove the existence of the density of law of the finite projection on finite dimensional space of the solution of a stochastic evolution equation driven by Lévy noise and fractional Brownian motion.

3. The main result

In this section we consider an abstract stochastic evolution equation in a separable Banach space $\mathscr{E}$ given by

$$\left\{\begin{array}{rcl} du(t) + \mathcal{L}u(t)\,dt + B(u(t), u(t)) &=& \dot{\Xi}(t), \quad t > 0, \\ u(0) &=& u_0 \in \mathscr{E}, \end{array}\right. \tag{6}$$

where the driving noise Ξ is either a Lévy process or a fractional Brownian motion, $\mathcal{L}: D(\mathcal{L}) \to \mathscr{E}$ and $B: \mathscr{E} \times \mathscr{E} \to \mathscr{E}$ is a densely defined bilinear operator taking values in $\mathscr{E}$. We assume that the above equation is uniquely solvable in $\mathscr{E}$ and we denote the solution starting from $u_0 \in \mathscr{E}$ at time $t = 0$ by $\{u(t, u_0) : t \geq 0\}$.

In order to formulate the main result of this section we need to introduce few concepts from the control theory. For this aim, let $U \subset \mathscr{E}$ be a separable Banach space, let $r \geq 1$ be a fixed number and let us consider the following control problem

$$\left\{ \begin{array}{rcl} \dot{u}(t) + \mathcal{L}u(t) + B(u(t), u(t)) & = & v(t), \quad t > 0, \\ u(0) & = & u_0 \in \mathscr{E}, \end{array} \right. \tag{7}$$

where $v \in L^r(0, T; U)$ is the control and U is the control space (the trajectories of our noise will be basically belong to $L^r(0, T; U)$). For a fixed time $T > 0$ we denote by

$$\mathcal{R}_T : \mathscr{E} \times L^r(0, T; U) \quad \to \quad \mathscr{E} \tag{8}$$

the so called solution operator that takes each function $v \in L^r(0, T; U)$ and initial condition $u_0 \in \mathscr{E}$ to the solution $u(T, u_0)$ of the system (7).

Definition 3.1. *A system is controllable in time $T > 0$ for a finite dimensional subspace $F \subset \mathscr{E}$ if and only if*

$$\pi_F \mathcal{R}_T(u_0, L^r(0, T; U)) \supset F$$

for any $u_0 \in \mathscr{E}$.

Definition 3.2. *A system is solid controllable in time $T > 0$ for a finite dimensional subspace $F \subset \mathscr{E}$, if and only if for any $R > 0$ and any $u_0 \in \mathscr{E}$, there exists an $\epsilon > 0$ and a compact set $K_\epsilon \subset L^r(0, T; U)$ such that for any function $\Phi : K_\epsilon \to F$ satisfying*

$$\sup_{x \in K_\epsilon} |\Phi(x) - \pi_F \mathcal{R}_T(u_0, x)|_F \leq \epsilon,$$

we have

$$\Phi(K_\epsilon) \supset B_F(R).$$

In the approach, the control is replaced by the stochastic process. In case the system is perturbed by a d-dimensional Wiener process W with covariance I, we will consider the following system

$$\left\{ \begin{array}{rcl} du(t) + \mathcal{L}u(t)\, dt + B(u(t), u(t))\, dt & = & dW(t), \quad t > 0, \\ u(0) & = & u_0 \in \mathscr{E}. \end{array} \right. \tag{9}$$

Let $\gamma > \frac{1}{2}$ let us define for a function $\phi \in H_2^\gamma([0, T]; \mathbb{R}^d)$ the random variable $W(\phi)$ by

$$\int_0^T \phi(s)\, dW(s).$$

Since $C_b^{(0)}([0, T]; \mathbb{R}^d) \hookrightarrow H_2^\gamma([0, T]; \mathbb{R}^d)$ the mapping above is well defined. The Wiener process induces a cylindrical measure on $H_2^{-\gamma}([0, T]; \mathbb{R}^d)$, which is the dual space of $H_2^\gamma([0, T]; \mathbb{R}^d)$. In particular, for $\phi_1, \ldots, \phi_m \in H_2^\gamma([0, T]; \mathbb{R}^d)$ and $C \in \mathcal{B}(\mathbb{R}^d)$, let

$$\mu(\{\phi' \in H_2^{-\gamma}([0, T]; \mathbb{R}^d) : (\phi'(\phi_1), \ldots, \phi'(\phi_m) \in \mathbb{R}^m \in C\}) = \mathbb{P}\left((W(\phi_1), \ldots, W(\phi_m)) \in C\right).$$

Since $\mathbb{E}W(\phi_1)W(\phi_2) = \langle \phi_1, \phi_2 \rangle_{L^2}$ for $\phi_1, \phi_2 \in H_2^\gamma([0, T]; \mathbb{R}^d)$, is is straight forward that the cylindrical measure μ on is ($\mathfrak{R}$-)tight on $H_2^{-\gamma}([0, T]; \mathbb{R}^d)$ and therefore a probability measure on $H_2^{-\gamma}([0, T]; \mathbb{R}^d)$[2]. Besides, each finite dimensional subspace F of $L^r(0, T; U)$ with $U = \mathbb{R}^d$ is a finite dimensional subspace of $H_2^{-\gamma}([0, T]; \mathbb{R}^d)$. In this way, in case the integral $\int_0^T \phi(s)\, d\Xi(s)$ is well defined for $\phi \in E$, where $\phi \in E \subset \mathcal{S}([0, T])$ is a deterministic function, Ξ induces a measure on E'. With this preliminary works the following general result can be shown.

[2] see Mushtari [**?**, Section 1.6].

Theorem 3.1. *Let E be a separable Banach space with Schauder basis $\{e_n : n \in \mathbb{N}\}$. Let F be a finite dimensional subspace of $\mathscr{E}$. We assume that the embedding $\dot{\Xi}$ on E is decomposable on E with the decomposition $\{F_n, G^n, l_n\}_{n=0}^{\infty}$ with $F_n \subset L^r(0,T;U)$, where the notation used in (3) and (4) is enforced, and satisfies Assumption 2.1 and Assumption 2.3. For a fixed number $T > 0$ we also assume that*

- *(**A**1) the solution operator $\mathcal{R}_T$ defined in (8) which is generated by the system (7) is analytic,*
- *(**A**2) and for any finite dimensional space $F \subset \mathscr{E}$, the system (7) is solid controllable in time T for the finite dimensional space F.*

Then, for any $u_0 \in \mathscr{E}$ and for any finite dimensional subspace $F \subset \mathscr{E}$ there exists a density $\rho : F \to \mathbb{R}_0^+$ such that

$$\mathbb{E}1_{\mathcal{O}}(\pi_F u(T, u_0)) = \int_{\mathcal{O}} \rho(x)\, Leb_F(dx).$$

Proof. Let us fix a finite dimensional subspace F of $\mathscr{E}$ and consider the operator

$$f : \mathscr{E} \times X \ni (u_0, \xi) \mapsto \pi_F \mathcal{R}_T(u_0, \xi) \in F,$$

where $X = L^1(0,T;U)$, u solves equation (6), and $\mathcal{R}_T$ is defined in (8).

The proof of our theorem will follow from the applicability of [2, Theorem 2.2]. Thus we need to check if all the assumptions of [2, Theorem 2.2] are all satisfied. For this aim it is sufficient to prove the two claims below.

Claim 1. There exists a finite dimensional subspace G_m of X such that for any $u_0 \in \mathscr{E}$, there exists a ball $B_0 \subset G_m$ and a ball $B_F \subset F$ such that

$$f(u_0, B_0) \supset B_F.$$

To prove this claim we fix a large number $R > 0$ such that $u_0 \in B_{\mathscr{E}}(R)$. By the definition of solid controllability, we know that there exists an $\epsilon > 0$ and a compact set $K_\epsilon \subset \mathscr{E}$ such that, any function $\Phi : K_\epsilon \to \mathscr{E}$ satisfying

$$\sup_{y \in K_\epsilon} |\Phi(y) - \pi_F \mathcal{R}_T(u_0, y)|_F \le \epsilon,$$

satisfies

$$\Phi(K_\epsilon) \supset \{y \in F : |y|_F \le R\}.$$

Fix $u_0 \in B_{\mathscr{E}}(R)$, $\epsilon > 0$ and the corresponding compact set K_ϵ. Since the operator

$$\mathcal{R}_T(u_0, \cdot) : X \to \mathscr{E}$$

is continuous, it is uniformly continuous on K_ϵ, and, hence, there exists a $\delta_0 > 0$ such that

$$|\mathcal{R}_T(u_0, y_1) - \mathcal{R}_T(u_0, y_2)|_{\mathscr{E}} \le \epsilon, \quad \forall\, y_1, y_2 \in K_\epsilon \text{ with } |y_1 - y_2| \le \delta_0\,.$$

Since the function system $\{e_n : n \in \mathbb{N}\}$ is a Schauder basis of X, it follows that $\cup_{m \in \mathbb{N}} F_m$ is a dense subset in X. In particular, since K_ϵ is compact, for any $\delta > 0$, there exists a number m such that

$$\sup_{y \in K_\epsilon} \|y - \pi_{G_m} y\|_X \le \delta.$$

Let $m \in \mathbb{N}$ be sufficiently large such that

$$\sup_{y \in K_\epsilon} \|y - \pi_{G_m} y\|_X \le \delta_0,$$

Let us define

$$\Phi : K_\epsilon \to \mathscr{E}$$

by

$$\Phi(y) = \pi_F(\mathcal{R}_T(u_0, \pi_{G_m} y)).$$

From the consideration above, it follows that

$$\sup_{y \in K_\epsilon} |\Phi(y) - \pi_F \mathcal{R}_T(u_0, y)|_F \le \epsilon.$$

Hence, by the solid controllability

$$\Phi(K_\epsilon) \subset \{y \in F : |y|_F \leq R\}.$$

In particular, since $\pi_{G_m} K_\epsilon$ is a bounded set of G_m, there exists a number $R_1 > 0$ such that $\{y \in G_m : |y| \leq R_1\} \supset \pi_{G_m} K_\epsilon$. Setting $B_F := \{y \in F : |y|_F \leq R\}$ and $B_1 := \{y \in G_m : |y| \leq R_1\}$ we have

$$\mathcal{R}_T(u_0, B_1) \supset B_F,$$

which proves **Claim** 1.$\square$

Claim 2.The measure μ on E satisfies Assumption 2.1.

Claim 2 is easy to prove. Thanks to Lemma C.1 the probability measure satisfies Assumption 2.2, from which **Claim** 2. follows

$\square$

$\square$

4. Application to the 2D stochastic Navier-Stokes

Throughout this section $\mathbb{T}$ denotes the 2D torus, $L^p(\mathbb{T})$ and $W^m_p(\mathbb{T})$ will respectively denote the usual Lebesgue space of p-integrable functions and Sobolev spaces.

Let $\mathcal{V}$ be the set of periodic, divergence free and infinitely differentiable function with zero mean. In what follows, we denote by $\mathscr{H}$ and $\mathbf{V}$ the closures of $\mathcal{V}$ in $L^2(\mathbb{T})$ and $W^{1,2}(\mathbb{T})$, respectively. We endow the space $\mathscr{H}$ with the L^2-scalar product denoted by $(\cdot,\cdot)$ and the usual L^2-norm denoted by $|\cdot|$. The space $\mathbf{V}$ is equipped with the gradient norm $\|\cdot\| = |\nabla \cdot|$. We also set

$$D(\mathcal{L}) = [\mathscr{H}^2(\mathbb{T})]^2 \cap \mathbf{V}, \quad \mathcal{L}\mathbf{v} = -\Pi\Delta\mathbf{v}, \quad \mathbf{v} \in D(\mathcal{L}),$$

where Π is the orthogonal projection from $L^2(\mathbb{T})$ onto $\mathscr{H}$. It is well-known that the Stokes operator $\mathcal{L}$ is positive self-adjoint with compact resolvent and its eigenfunctions $\{e_1, e_2, \ldots\}$, with eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \ldots$, form an orthonormal basis of $\mathscr{H}$. It is also well-known that $\mathbf{V} = D(\mathcal{L}^{\frac{1}{2}})$, see [17, Appendix A.1 of Chapter II]. Furthermore, we see from [31, Chapter II, Section 1.2] and [17, Appendix A.3 of Chapter II] that one can define a continuous bilinear map B from $\mathbf{V} \times \mathbf{V}$ with values in $\mathbf{V}^*$ such that

$$\langle B(\mathbf{u},\mathbf{v}), \mathbf{w}\rangle = \int_{\mathbb{T}} [\mathbf{u}(z)\cdot\nabla\mathbf{v}(z)]\cdot\mathbf{w}(z)dz \quad \text{for any } \mathbf{u},\mathbf{v},\mathbf{w} \in \mathbf{V}, \tag{10}$$

$$\langle B(\mathbf{u},\mathbf{v}), \mathbf{v}\rangle = 0, \quad \text{for any } \mathbf{u},\mathbf{v} \in \mathbf{V}, \tag{11}$$

$$|\langle B(\mathbf{u},\mathbf{v}), \mathbf{w}\rangle| \leq C_0\|\mathbf{u}\|_{L^4}\|\mathbf{v}\|_{L^4}\|\mathbf{w}\|, \text{ for } \mathbf{u},\, \mathbf{v} \in L^4(\mathbb{T}),\, \mathbf{w} \in \mathbf{V}. \tag{12}$$

With all these notations the Navier-Stokes equations (1) can be written in the abstract form

$$\begin{cases} \dot{u}(t) + \kappa\mathcal{L}u(t) + B(u(t),u(t)) = \dot{\Xi}(t), \\ \qquad\qquad\qquad\qquad u(0) = u_0 \in \mathscr{H}, \end{cases} \tag{13}$$

where for the sake of simplicity we assume that $\Pi\dot{\Xi} = \dot{\Xi}$. The positive number $\kappa > 0$ denotes the viscosity. Before characterizing the noise entering our system, we introduce the trigonometric basis in $\mathscr{H}$ by elements in $\mathbb{Z}^2$. Namely, we write $j = (j_1, j_2) \in \mathbb{Z}^2$ and set

$$\begin{aligned} e_j(x) &= \sin(jx)j^\perp \quad \text{for } j_1 > 0 \text{ or } j_1 = 0, j_2 > 0, \\ e_j(x) &= \cos(jx)j^\perp \quad \text{for } j_1 < 0 \text{ or } j_1 = 0, j_2 < 0, \\ e_0^1(x) &= (1,0), \quad e_0^2(x) = (0,1), \end{aligned}$$

where $j^\perp = (-j_2, j_1)$. The family $\mathcal{E} = \{e_0^i, e_j, i = 1,2, j \in \mathbb{Z}^2 \setminus \{0\}\}$ is a complete set of eigenfunctions for the Stokes operator which forms an orthonormal basis in $\mathscr{H}$.

For any symmetric set $\mathcal{K} \subset \mathbb{Z}^2$ containing $(0,0)$ we write $\mathcal{K}^0 = \mathcal{K}$ and define $\mathcal{K}^i$ with $i \geq 1$ as the union of $\mathcal{K}^{i-1}$ and the family of vectors $l \in \mathbb{Z}^2$ for which there are $m, n \in \mathcal{K}^{i-1}$ such that $l = m + n$, $|m| \neq |n|$, and $|m \wedge n| \neq 0$, where $m \wedge n = m_1 n_2 - m_2 n_1$.

Definition 4.1. *A symmetric subset* $\mathcal{K} \subset \mathbb{Z}^2$ *containing* $(0,0)$ *is saturating, if and only if* $\cup_{i\in\mathbf{N}}\mathcal{K}^{i-1} = \mathbb{Z}^2$.

Throughout we set $d = \dim \mathcal{K}$ and denote by $\mathscr{H}_d$ the finite dimensional subspace of $\mathscr{H}$ spanned by the eigenvectors $\{e_j; j \in \mathcal{K}\}$. The driving process is either

$$\Xi(t) = \sum_{j\in\mathcal{K}} e_j l_j(t), \quad t \geq 0, \tag{14}$$

where $\{l_j : j \in \mathcal{K}\}$ is a family of mutually independent Lévy processes with Lévy measure ν over a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, or

$$\Xi(t) = \sum_{j\in\mathcal{K}} e_j \beta_j^H(t), \quad t \geq 0, \tag{15}$$

where $\{\beta_j^H : j \in \mathcal{K}\}$ is a family of identical distributed and mutual independent fractal Brownian motions with Hurst parameter $H \in (\frac{1}{2}, 1)$ over a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The existence of a unique solution $u = \{u(t) : t \geq 0\}$ to (13) follows from the results in [7] for example for the case of pure jump Lévy process, and from [23] for case of fractional Brownian motion perturbation.

We can now state the main results of this section. However, before let us introduce the following definition (compare [28, p. 13]). A measurable function $l : [0, \infty) \to \mathbb{R}$ is slow varying at ∞, iff for any $x > 0$

$$\lim_{t\to\infty} \frac{l(tx)}{l(t)} = 1.$$

We start with the following theorem which treats the case of Navier-Stokes equations driven by a Lévy process.

Theorem 4.1. *Let* $\mathcal{K}$ *be a saturating set and assume that the process* Ξ *entering the system* (6) *is defined by* (14)*. We also assume that the Lévy measures* ν_j, $j = 1, \ldots, d$, *are symmetric and equivalent to the Lebesgue measure on* $\mathbb{R} \setminus \{0\}$*, and satisfy*

$$\int_{|z|\leq 1} |z|^p \nu_j(dz) < \infty, \tag{16}$$

for some $p \in (1, 2)$*. In addition, we assume that there exists a number* $\alpha \in (0, 2]$ *such that*

$$\nu_j(\mathbb{R} \setminus [-\epsilon, \epsilon]) \sim \epsilon^{-\alpha} l(\epsilon) \quad as \quad \epsilon \to 0, \tag{17}$$

for some slowly varying function l*. Let* $u = \{u(t, u_0) : t \geq 0, u_0 \in \mathscr{H}\}$ *be the unique solution of system* (6)*. Then for any finite dimensional subspace* $F \subset \mathscr{H}$*, for all initial conditions* $u_0 \in \mathscr{H}$*, there exists a density* $\rho_{u_0} : F \to \mathbb{R}_0^+$ *such that*

$$\mathbb{E}1_{\mathcal{O}}(\pi_F u(T, u_0)) = \int_{\mathcal{O}} \rho_{u_0}(x)\, Leb_F(dx).$$

In addition, for any sequence $\{u_n : n \in \mathbb{N}\}$ *with* $u_n \to u_0 \in \mathscr{H}$ *as* $n \to \infty$*, we have*

$$\int_F |\rho_{u_0}(x) - \rho_{u_n}(x)| dx \longrightarrow 0 \;\; as \;\; n \to \infty.$$

Remark 4.1. *In fact, given* $\alpha \in (0, 2)$ *such that* (17) *holds, it is straightforward to see that* (16) *is satisfied for any* $p > \alpha$*. On the other hand, if* $p > 0$ *satisfies* (16)*, then this does not imply that there exists a number* $\alpha \in (0, 2)$ *such that* (17) *holds.*

Proof. For simplicity, let us assume $T = 1$. In addition, since in our example the underlying function space $\mathscr{E}$ is a Hilbert space, we denote instead of $\mathscr{E}$ the underlying function space by $\mathscr{H}$. In addition, the control space U is given by $\mathscr{H}_d$. As in the previous section (see (8)) we consider the solution operator

$$\mathcal{R}_T : \mathscr{H} \times L^2(0, T; \mathscr{H}_d) \;\; \to \;\; \mathscr{H}, \tag{18}$$

which takes each function $v \in L^2(0,T;\mathscr{H}_d)$ and initial condition $u_0 \in \mathscr{H}$ to the solution $u(T,u_0)$ of the control system (7) associated to the Navier-Stokes equations. It is proved in [2, Proposition A.2], see also [21], that the operator $\mathcal{R}_T$ is analytic. It is also known from [2, Proposition A.5], see also [3], that the system (7) for the Navier-Stokes is solid controllable in time T for any finite dimensional space $F \subset \mathscr{H}$. In particular, Assumptions (A1) and (A2) of Theorem 3.1 are satisfied. Hereafter we respectively identify $\mathscr{H}_d$ and F to $\mathbb{R}^d$ and $\mathbb{R}^{\dim F}$.

Let $p \in (1,2)$ such that (16) is satisfied. Let p' be the conjugate exponent to p and $s < \frac{1}{p} - 1$. For each $j \in \mathcal{K}$ let ξ_j be the map defined by

$$\xi_j : B^s_{p',p'}([0,1],\mathbb{R}) \ni \phi \mapsto \xi_j(\phi) = \int_0^1 \phi(\tau)\, dl_j(\tau) \in L^0(\Omega;\mathbb{R}),$$

and let μ_j be the cylindrical measure μ_j on $B^s_{p,p}([0,1],\mathbb{R})$ defined by

$$\mu_j\left(\left\{x \in B^s_{p,p}([0,1]) : (x(\phi_1),\dots,x(\phi_n)) \in C\right\}\right) := \mathbb{P}\left((\xi(\phi_1),\dots,\xi(\phi_n)) \in C\right),\, C \in \mathcal{B}(\mathbb{R}^n),$$

where $n \in \mathbb{N}$, $\phi_1,\dots,\phi_n \in \mathcal{S}(\mathbb{R})$. In Proposition A.1 (see also A.2) we show that the cylindrical measure is actually a Radon measure on $B^s_{p,p}([0,1])$.

From the results of Section A we infer that the probability measure μ_j on $B^s_{p,p}([0,1],\mathbb{R})$ is decomposable with decomposition $\{F_n, G^n, l_n\}_{n=0}^\infty$, where F_n and l_n are respectively defined by $F_0 = V_0$, $F_n = W_n$, $n \geq 2$, where V_0 and W_n are defined in (19), the space G^n is defined as in Definition 2.1. The existence of l_n is given by Lemma A.3. Lemma A.7 gives that the decomposition $\{F_n, G^n, l_n\}_{n=0}^\infty$ satisfies Assumption 2.1 and Assumption 2.3. In addition, we know by Triebel [32, Theorem 1.58] that the wavelets chosen to construct $\{F_n, G^n, l_n\}_{n=0}^\infty$ form an unconditionally basis of $B^s_{p,p}([0,1],\mathbb{R})$.

With these observation in mind, it is not difficult to check that the product measure $\mu = \otimes_{j\in\mathcal{K}}\mu_j$ satisfies Assumption 2.2 and Assumption 2.3 on the Banach space $E := B^s_{p,p}([0,1],\mathbb{R}^d)$ where $d = \dim(\mathcal{K})$. Now, the proof of the theorem easily follows from an application of Theorem 3.1. □

We now proceed to the statement and the proof of the above theorem when the random process entering the system is a fractional Brownian motion given by (15).

Theorem 4.2. *Let $\mathcal{K}$ be a saturating set and assume that the noise Ξ is a fractional Brownian motion defined by* (15) *with Hurst parameter $H \in (\frac{1}{2},1)$. Let $u = \{u(t,u_0) : t \geq 0, u_0 \in H\}$ be the unique solution of system* (6) *with initial condition u_0. Then, for any finite dimensional space $F \subset \mathscr{H}$ and initial condition $u_0 \in \mathscr{H}$, there exists a density $\rho_{u_0} : F \to \mathbb{R}_0^+$ such that*

$$\mathbb{E}1_{\mathcal{O}}(\pi_F u(T,u_0)) = \int_{\mathcal{O}} \rho_{u_0}(x)\, Leb_F(dx).$$

In addition, for any sequence $\{u_n : n \in \mathbb{N}\}$ with $u_n \to u_0 \in \mathscr{H}$ as $n \to \infty$, we have

$$\int_F |\rho_{u_0}(x) - \rho_{u_n}(x)| dx \longrightarrow 0 \text{ as } n \to \infty.$$

Proof. Let $\mathcal{R}_T : \mathscr{H} \times L^2(0,T;\mathscr{H}_d) \to \mathscr{H}$ be the solution operator defined by (18) in the proof of Theorem 4.1. It satisfies the properties enumerated in the proof of Theorem 4.1. Hereafter we respectively identify $\mathscr{H}_d$ and F to $\mathbb{R}^d$ and $\mathbb{R}^{\dim F}$. Let $s \in (-\frac{1}{2}, H-1)$. For each $j \in \mathcal{K}$ let ξ_j be the map defined by

$$\xi_j : B^{-s}_{2,2}([0,1]) \ni \phi \mapsto \xi_j(\phi) = \int_0^1 \phi(\tau)\, d\beta_j^H(\tau) \in L^2(\Omega;\mathbb{R}),$$

and μ_j be the cylindrical measure on $B^s_{2,2}([0,1],\mathbb{R})$ defined by

$$\mu_j\left(\left\{x \in B^{-s}_{2,2}([0,1]) : (x(\phi_1),\dots,x(\phi_n)) \in C\right\}\right) := \mathbb{P}\left((\xi(\phi_1),\dots,\xi(\phi_n)) \in C\right),\, C \in \mathcal{B}(\mathbb{R}^n),$$

where $n \in \mathbb{N}$, $\phi_1,\dots,\phi_n \in \mathcal{S}(\mathbb{R})$. From the results of Section B we infer that the cylindrical measure μ_j on $B^s_{2,2}([0,1],\mathbb{R})$ is actually a probability measure and is decomposable with decomposition $\{F_n, G^n, l_n\}_{n=0}^\infty$, where F_n and l_n are respectively defined by $F_0 = V_0$, $F_n = W_n$, $n \geq 2$, where V_0

and W_n are defined in (35). With F_n in mind we define G^n as in Definition 2.1. We also infer from Lemma A.7 that for each j the probability measure μ_j satisfies Assumptions 2.2 and 2.3. We now easily complete the proof by using a similar argument as in the proof of Theorem 4.1. $\square$

Appendix A. The Lévy Noise and its Wavelet Expansion

In this section we assume that we are given a real-valued Lévy process ℓ with σ-additive Lévy measure ν on $\mathbb{R}\backslash\{0\}$ satisfying (16), *i.e.*

$$\int_{|z|\leq 1} |z|^p \nu(dz) < \infty,$$

for some $p \in (1,2)$.

Before continuing let us recall shortly the definition of the Besov spaces as given in [**?**, Definition 2, pp. 7-8]. First we choose a function $\psi \in \mathcal{S}(\mathbb{R}^d)$ such that $0 \leq \psi(x) \leq 1$, $x \in \mathbb{R}^d$ and

$$\psi(x) = \begin{cases} 1, & \text{if} \quad |x| \leq 1, \\ 0 & \text{if} \quad |x| \geq \frac{3}{2}. \end{cases}$$

Then put

$$\begin{cases} \phi_0(x) &=& \psi(x),\ x \in \mathbb{R}^d, \\ \phi_1(x) &=& \psi(\frac{x}{2}) - \psi(x),\ x \in \mathbb{R}^d, \\ \phi_j(x) &=& \phi_1(2^{-j+1}x),\ x \in \mathbb{R}^d, \quad j = 2, 3, \dots . \end{cases}$$

We will use the definition of the Fourier transform $\mathcal{F} = \mathcal{F}^{+1}$ and its inverse $\mathcal{F}^{-1}$ as in [**?**, p. 6]. In particular, with $\langle \cdot, \cdot \rangle$ being the scalar product in $\mathbb{R}^d$, we put

$$(\mathcal{F}^{\pm 1} f)(\xi) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{\mp i\langle x,\xi\rangle} f(x)\, dx, \quad f \in \mathcal{S}(\mathbb{R}^d), \xi \in \mathbb{R}^d.$$

With the choice of $\phi = \{\phi_j\}_{j=0}^\infty$ as above and $\mathcal{F}$ and $\mathcal{F}^{-1}$ being the Fourier and the inverse Fourier transformations (acting in the space $\mathcal{S}'(\mathbb{R}^d)$ of Schwartz distributions) we have the following definition.

Definition A.1. *Let $s \in \mathbb{R}$, $0 < p \leq \infty$ and and $f \in \mathcal{S}'(\mathbb{R}^d)$. If $0 < q < \infty$ we put*

$$|f|_{B^s_{p,q}} = \left(\sum_{j=0}^\infty 2^{sjq} \left| \mathcal{F}^{-1} \left[\phi_j \mathcal{F} f \right] \right|^q_{L^p} \right)^{\frac{1}{q}} = \| \left(2^{sj} \left| \mathcal{F}^{-1} \left[\phi_j \mathcal{F} f \right] \right|_{L^p} \right)_{j\in\mathbb{N}} \|_{l^q}.$$

If $q = \infty$ we put

$$|f|_{B^s_{p,\infty}} = \sup_{j\in\mathbb{N}} 2^{sj} \left| \mathcal{F}^{-1} \left[\phi_j \mathcal{F} f \right] \right|_{L^p} = \| \left(2^{sj} \left| \mathcal{F}^{-1} \left[\phi_j \mathcal{F} f \right] \right|_{L^p} \right)_{j\in\mathbb{N}} \|_{l^\infty}.$$

We denote by $B^s_{p,q}(\mathbb{R}^d)$ the space of all $f \in \mathcal{S}'(\mathbb{R}^d)$ for which $|f|_{B^s_{p,q}}$ is finite.

Besov spaces can also be described by e.g. wavelets. In our case, we have chosen as space discretization the multiresolution analysis generated by Daubechies wavelets. To start with let us first define what a multiresoultion analysis is. A multiresolution analysis is a sequence of subspaces $\{V_n : n \in \mathbb{Z}\}$ satisfying the following properties

- $V_n \subset V_{n+1}$, $n \in \mathbb{Z}$;
- $\cup_{n\in\mathbb{Z}} V_n = L^2(\mathbb{R}^d)$;
- $\cap_{n\in\mathbb{Z}} V_n = \{0\}$;
- if $f \in V_n$, then $f(2\cdot) \in V_{n+1}$;
- if $f \in V_n$, then $f(\cdot - k) \in V_n$, $k \in \mathbb{Z}$;

In addition, there exists a, so called, scaling function ϕ such that V_n is generated by translating and scaling the scaling function, i.e.

$$\phi_{j,k} := 2^{-\frac{j}{2}} \phi(2^j t + k)$$

and

$$V_n := \mathrm{span}\{\phi_{j,k} : j = 1, \ldots, n,\ k \in J_j^\phi\}. \tag{19}$$

Also, there exists a so called mother wavelet ψ, such that $V_n = V_{n-1} \oplus W_{n-1}$ with

$$W_n := \mathrm{span}\{\psi_{n,k} : k \in J_n^\psi\} \tag{20}$$

and

$$\psi_{j,k} := 2^{-\frac{j}{2}} \phi(2^j t + k), \quad j \in \mathbb{N},\ k \in J_j.$$

Observe, that due to the scaling, there exists coefficients $\{p_k : k \in \mathbb{Z}\}$ and $\{h_k : k \in \mathbb{Z}\}$ such that

$$\phi(x) = \sum_{k\in\mathbb{Z}} p_k \phi(2x + k), \quad x \in \mathbb{R}. \tag{21}$$

and

$$\psi(x) = \sum_{k\in\mathbb{Z}} h_k \phi(2x + k), \quad x \in \mathbb{R}. \tag{22}$$

In case of Daubechie wavelets, the number of coefficients p_k, where $p_k \neq 0$ is finite. In addition, the scaling function and the mother wavelet of Daubechie wavelets are compactly supported in space, which was the reason for us to chose Daubechies wavelets. In addition, the number of coefficients which are not equal to zero, i.e. $\{k : p_k \neq 0\}$, is finite. If the Daubechies wavelet is sufficiently smooth, i.e. the order of the Daubechies wavelet k satisfies $k > \max(s, (1-\frac{1}{p})_+ - s)$, the multiresolution analysis forms an unconditional basis in $B^s_{p,p}([0,1])$ (see [32, Theorem 1.58]). To keep this section and the article short we refer the reader for an introduction to wavelets to [8, 19] or [32].

Note that for $s \in \mathbb{R}$ the Daubechies wavelets of order k, with $k > \max(s, (1 - \frac{1}{p})_+ - s)$, form an unconditional basis of $B^s_{p,p}([0,1])$. In particular, for each element $f \in B^s_{p,p}([0,1])$ there exists a unique sequence

$$\{\lambda_{j,k} : j \in \mathbb{N}, k \in J_j^\psi\}$$

such that f can be written as

$$f = \sum_{j\in\mathbb{N}} \sum_{k\in J_j^\psi} \lambda_{j,k}\psi_{j,k} + \lambda_0 \phi. \tag{23}$$

Note that since we are considering the process on the time interval $[0,1]$, we only need to sum over J_j^ψ. We also note that $|J_j^\psi| \sim 2^j$.

In the next paragraph, we will construct the probability measure induced by a Lévy process which will be represented as an integral with respect to a Poisson random measure. This representation is motivated in one hand by the fact that the use of Poisson random measure simplifies many calculation. In other hand the Poisson random measure framework seems more general. We refer to [4], [26, Chapters 6-8] and [30, Chapter 4] for a precise connection between Poisson random measures and Lévy processes and stochastic integration with respect to them.

Over a probability space $\mathfrak{A} = (\Omega, \mathcal{F}, \mathbb{P})$, we consider a time homogenous Poisson random measure η on $\mathbb{R}$ with symmetric intensity measure ν as above.

Proposition A.1. *The Poisson random measure η over a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ induces a Radon probability measure μ on $B^s_{p,p}([0,1])$ with $s < \frac{1}{p} - 1$.*

Proof. We will start the proof with removing jumps of size bigger than $\epsilon \in (0,1]$ and let ϵ converge to 0. For this purpose we take an arbitrary constant $\epsilon \in (0,1]$ and define a Poisson random measure η_ϵ by

$$\begin{array}{rcl} \eta_\epsilon : \mathcal{B}(\mathbb{R}) \times \mathcal{B}([0,1]) & \to & \bar{\mathbb{N}}, \\ (A \times I) & \mapsto & \eta(A \cap (\mathbb{R} \setminus [-\epsilon, \epsilon]) \times I). \end{array}$$

First we will show that the family $\{\eta_\epsilon : \epsilon \in (0,1]\}$ induces a family of cylindrical measures on $C_b([0,1])'$. Here, it is important that the to the Poisson random measure η_ϵ corresponding Lévy process can be written as a sum over finitely many jumps at random jump times. To be more precise, let ν_ϵ be defined by $\nu_\epsilon(A) = \nu(A \cap (\mathbb{R} \setminus [-\epsilon,\epsilon]))$, $\rho_\epsilon = \nu(\mathbb{R} \setminus [-\epsilon,\epsilon])$, let N_ϵ be a Poisson distributed random variable with parameter ρ_ϵ, $\{\tau_n^\epsilon : n = 1,\dots,N\}$ be a family of independent uniform distributed random variables on $[0,1]$, and $\{Y_n : n = 1,\dots,N\}$ be a family of independent, $\nu_\epsilon/\rho_\epsilon$ distributed random variables. Denoting δ_x the Dirac distribution concentrated at x, the Poisson random measure η_ϵ can be written as

$$\eta_\epsilon(A \times I) = \sum_{n=1}^{N_\epsilon} \delta_{\tau_n}(I)\, \delta_{Y_n}(A), \quad A \in \mathcal{B}(\mathbb{R}^d),\ I \in \mathcal{B}([0,1]).$$

In addition, for any $f \in C_b([0,1])$ the mapping

$$\xi_\epsilon(f) := \int_0^1 \int_{\mathbb{R}} f(s) z \eta_\epsilon(dz,ds) = \sum_{n=1}^{N_\epsilon} f(\tau_n) Y_n$$

is well defined.

Let us define the random variables

$$\zeta_{j,k}^\epsilon = \int_0^1 \int_{\mathbb{R}} \psi_{j,k}(\tau)\, z\eta_\epsilon(dz,d\tau), \quad j \in \mathbb{N},\ k \in J_j,$$

and

$$a_0^\epsilon = \int_0^1 \int_{\mathbb{R}} \phi_{0,0}(\tau)\, z\eta_\epsilon(dz,d\tau).$$

Since the mother wavelet ψ and the scaling function ϕ are supposed to be continuous, the families $\{\zeta_{j,k}^\epsilon : j \in \mathbb{N},\ k \in J_j\} \cup \{a_0^\epsilon\}$ of random variables over $\mathfrak{A}$ are well defined. In addition, by the definition of ζ^ϵ and a_0 and the fact that the multiresolution analysis is a Schauder basis in $B_{p,p}^s([0,1])$, and $\delta \in B_{p,p}^s([0,1])$ (see [29, Remark 3, p. 34]), we infer that ξ_ϵ admits a wavelet series representation as in (23).

Note that for any $C \in \mathcal{B}(\mathbb{R})$,

$$\mu_\epsilon(\{x \in B_{p,p}^s([0,1]) : x(\psi_{j,k}) \in C\}) = \mathbb{P}(\zeta_{j,k}^\epsilon \in C).$$

Later on we will need the following proposition which will be proved at the end of the current proof.

Proposition A.2. *Let ν be a Lévy measure satisfying* (16) *for some $p \in [1,2)$ and $\epsilon \in (0,1]$. Let*

$$\xi_\epsilon := \sum_{j=1}^\infty \sum_{k \in J_j^\psi} \zeta_{k,j}^\epsilon \psi_{j,k} + a_0^\epsilon \phi_{0,0}.$$

Then,

(1) for any $s < \frac{1}{p} - 1$, there exists a constant $C > 0$ such that

$$\mathbb{E}\left[|\xi_\epsilon|_{B_{p,p}^s}^p\right] \le C.$$

(2) for any $s < \frac{1}{p} - 1$, there exists a constant $C > 0$ such that for any $\epsilon_1, \epsilon_2 \in (0,1]$ we have

$$\mathbb{E}\left[|\xi_{\epsilon_1} - \xi_{\epsilon_2}|_{B_{p,p}^s}^p\right] \le C \min(\epsilon_1,\epsilon_2)^{2-p}.$$

By the choice of s and p, we have $B_{p',p'}^{-s}([0,1]) \hookrightarrow C_b([0,1])$ and η is a finite measure. Secondly, the mappings ξ_ϵ induces a family of cylindrical measures μ_ϵ on $B_{p,p}^s([0,1])$ defined by

$$\mu_\epsilon\left(\left\{x \in B_{p,p}^s([0,1]) : (x(\phi_1),\dots,x(\phi_n)) \in C\right\}\right) := \mathbb{P}\left((\xi_\epsilon(\phi_1),\dots,\xi_\epsilon(\phi_n)) \in C\right),$$

$\phi_1,\cdots,\phi_n \in (B_{p,p}^s([0,1]))' = B_{p',p'}^{-s}([0,1])$, and $C \in \mathcal{B}(\mathbb{R}^n)$.

We will now show that the family of cylindrical measures $\{\mu_\epsilon : \epsilon\in(0,1]\}$ has a limit in the set of probability measures on $B^s_{p,p}([0,1])$. In fact, the family of probability measures μ_ϵ is tight on $B^s_{p,p}([0,1])$. To show this claim we fix a constant $s_0\in(s,\frac1p-1)$. We firstly note that the embedding $B^{s_0}_{p,p}([0,1])\hookrightarrow B^s_{p,p}([0,1])$ is compact. Secondly, the Chebyscheff inequality and Proposition A.2 give that for any $\delta>0$ we can find a compact $K_\delta:=\{x\in B^s_{p,p}([0,1]) : |x|_{B^{s_0}_{p,p}}\le \delta^{-1/p}\}$ such that

$$\mathbb{P}\left(\xi_\epsilon\not\in K_\delta\right)\le \delta\,\mathbb{E}\left[|\xi_\epsilon|^p_{B^{s_0}_{p,p}}\right]\le C\delta.$$

It follows that the family of probability measures $\{\mu_\epsilon:\epsilon\in(0,1]\}$ is tight on $B^s_{p,p}([0,1])$. It even follows from Proposition A.2 that the sequence $\{\mu_{\epsilon_n}:\epsilon=\frac1n\}$ forms a Cauchy sequence and the limit μ is unique. Therefore, there exists a unique cylindrical measure μ on $B^s_{p,p}([0,1])$. Since there exists a constant $C>0$ such that for all $\epsilon>0$ we have $\mathbb{E}[|\xi_\epsilon|^p_{B^s_{p,p}}]\le C$, it follows from the Lebesgue Dominated Convergence Theorem that $\mathbb{E}[|\xi|^p_{B^s_{p,p}}]\le C$. Hence, μ is also a Radon probability measure on $B^s_{p,p}([0,1])$.

Now we shall consider the general case in which ν is assumed to satisfy (16) for some $p\in(1,2)$. For this purpose we consider the Poisson random measures η_1 and η_2 defined by

$$\mathcal{B}([0,\infty))\times\mathcal{B}(\mathbb{R})\ni(I\times A)\mapsto\eta_1(I\times A):=\eta(I\times A\cap[-1,1])$$

and

$$\mathcal{B}([0,\infty))\times\mathcal{B}(\mathbb{R})\ni(I\times A)\mapsto\eta_2(I\times A):=\eta(I\times A\cap\mathbb{R}\setminus[-1,1]),$$

respectively. Since $A\cap[-1,1]\cap A\cap\mathbb{R}\setminus[-1,1]=\emptyset$, the Poisson random measures η_1 and η_2 are independent. Hence, the two families of coefficients in the wavelet expansion η_1 and η_2 are independent too. In addition from the first part of the proof η_1 induces a Radon probability measure on $B^s_{p,p}([0,1])$. Since the process

$$L^2_t:=\int_0^t\int_{\mathbb{R}} z\eta_2(dz,ds)$$

can be written as a finite sum over jumps happen at random times within the interval $[0,1]$, $\dot L^2_t$ consist of a sum over finitely many Dirac distributions. Since any Dirac distributions belong to $B^s_{p,p}([0,1])$, $\dot L^2_t$ is an element of $B^s_{p,p}([0,1])$ and induces a probability measure on $B^s_{p,p}([0,1])$. Hence, η itself induces a Radon probability measure on $B^s_{p,p}([0,1])$. □

Proof of Proposition A.2: We recall that $\int|z|^p\nu(dz)<\infty$ for some $p\in(1,2)$. By the definition of the norm we get

$$\mathbb{E}|\xi_\epsilon|^p_{B^s_{p,p}}\quad\sim\quad\mathbb{E}\sum_{j=1}^\infty 2^{j(s-\frac1p)p}\sum_{k\in J^\psi_j}\left|\zeta^\epsilon_{k,j}\right|^p 2^{j\frac p2}.$$

Since

$$\mathbb{E}|\zeta^\epsilon_{j,k}|^p\quad\le\quad C_\nu\int_0^1|\psi_{j,k}(s)|^p\,ds=2^{\frac{jp}{2}}2^{-j},$$

we infer that there exists a constant $C>0$ such that

$$\mathbb{E}|\xi_\epsilon|^p_{B^s_{p,p}}\quad\le C\quad\sum_{j=1}^\infty 2^{j(ps-1)}2^j2^{j(\frac p2-1)}2^{j\frac p2}\le C\sum_{j=1}^\infty 2^{j(ps+\frac p2-1+\frac p2)},$$

which is finite for $s<\frac1p-1$.

□

Let us denote the Radon probability measure induced by η on $B^s_{p,p}([0,1])$ by μ and let us define the mapping

$$\xi: B^{-s}_{p',p'}([0,1])\ni\phi\mapsto\xi(\phi)=\int_0^1\int_{\mathbb{R}}\phi(\tau)\,z\,\eta(dz,d\tau).\tag{24}$$

This mapping is well defined thanks to the above calculation.

We are now interested in the properties of the decomposition of μ by the multiresolution analysis. In particular, we will show that for any $n \in \mathbb{N}$, the probability measure μ_{G_n} is equivalent to the Lebesgue measure.

Let us put

$$G_n = V_n, \quad n \in \mathbb{N}. \tag{25}$$

We firstly note that since $V_{n+1} = W_n \otimes W_{n-1} \otimes \cdots \otimes W_1 \otimes V_0$, given the coefficients $\{\zeta_{j,k} : j = 1, \dots, n,\ k \in J_j^\psi\} \cup \{a_0\}$, one knows the coefficient of $\phi_{n+1,k}$. For $k \in J_n^\phi$ let us denote $\gamma_{n,k}$ the coefficients of $\phi_{n,k}$. In particular, we have

$$\gamma_{n,k} := \int_0^1 \int_{\mathbb{R}} \phi_{n,k}(t) z \eta(dz, dt),$$

which implies that

$$\pi_{G_n} \xi = \sum_{k \in J_n^\phi} \gamma_{n,k} \phi_{n,k}.$$

Let us now denote by $\mathbf{z^n}$ and $\mathbf{g^n}$ the random vectors $(\zeta_{n,0}, \zeta_{n,1}, \dots, \zeta_{n,|J_n^\phi|})$ and $(\gamma_{n,1}, \gamma_{n,2}, \dots, \gamma_{n,|J_n^\psi|})$, respectively. Finally, for a function $f : [0,1] \to \mathbb{R}$ we write

$$\xi(f) := \int_0^1 \int_{\mathbb{R}} f(s) z \, \eta(dz, ds).$$

Lemma A.1. *Let $f : [0,1] \to \mathbb{R}$ be a function such that there exists constants $\delta > 0$ and $t_1, t_2 \in [0,1]$, $t_1 < t_2$ such that $|f(t)| \geq \delta$ for all $t \in [t_1, t_2]$. Then*

(1) $\operatorname{supp}(\xi(f)) = \mathbb{R}$*;*

(2) the law of $\xi(f)$ is absolutely continuous with respect to the Lebesgue measure.

Proof. Let us define the following Lévy measure

$$\nu_{t_1,t_2} : \mathcal{B}(\mathbb{R}) \ni B \mapsto \int_{t_1}^{t_2} \int_{\mathbb{R}} 1_B(f(t)z) \nu(dz) \, dt.$$

Then $\xi(f 1_{[t_1,t_2]})$ is an infinite divisible random variable. Item (i) follows from [30, Corollary 24.4]. Item (ii) follows from [30, Theorem 27.7]. $\square$

Lemma A.2. *For any $n \geq 1$, the measure*

$$\mathcal{B}(\mathbb{R}^{|J_{n+1}^\phi|}) \ni U \mapsto \mathbb{P}\left(\mathbf{g}^{n+1} \in U\right) \tag{26}$$

is equivalent to the Lebesgue measure on $\mathbb{R}^{|J_{n+1}^\phi|}$.

Proof. This follows from the fact that the scaling function ϕ has compact support and, therefore, that for all $k = \min(J_{n+1}^\phi), \dots, \max(J_{n+1}^\phi) - 1$ the set

$$\operatorname{supp}(\phi_{n+1,k}) \cup \operatorname{supp}(\phi_{n+1,k+1}) \setminus (\operatorname{supp}(\phi_{n+1,k}) \cup \operatorname{supp}(\phi_{n+1,k+1}))$$

has non–empty interior. Let us write $\phi_{n+1,k+1} = f_1 + f_2$ with $\operatorname{supp}(f_1) \cap \operatorname{supp}(\phi_{n+1,k}) = \emptyset$, $\operatorname{supp}(f_1)$ is an interval $[a,b]$, $\{s : f_2(s) > 0\} \cap [a,b] = \emptyset$, and f_1 is bounded away from zero. Then $\xi(f_1)$ and $\xi(f_2)$ are independent, and so are $\xi(f_1)$ and $\xi(\phi_{n+1,k})$. In addition, by Lemma A.1 the law of $\xi(f_1)$ is equivalent to the Lebesgue measure. Hence, from [30, Lemma 27.1-(iii)] it follows that the law of the sum of the random variables $\xi(f_1)$ and $\xi(f_2 + \phi_{n+1,k+1})$ is also equivalent to the Lebesgue measure. Now, one easily prove the assertion by an induction starting at $k = \min(J_{n+1}^\phi)$. $\square$

Lemma A.3. *For each $U \in \mathcal{B}(G_n)$ and $\mathbf{y} \in \mathbb{R}^{|J_n^\psi|}$, the conditioned measure*

$$\mathcal{B}(|J_n^\psi|) \ni U \mapsto l_n(\mathbf{y}, U) = \mathbb{P}\left(\mathbf{z}^n \in U \mid \mathbf{g}^n = \mathbf{y}\right) \tag{27}$$

is equivalent to the Lebesgue measure.

Proof. First, note that given the scaling function ϕ there exists coefficients $\{p_j : j = 1, \ldots, u\}$, where u is the order of the Daubechies wavelet, such that (see (22))

$$\phi(x) = \sum_{j=1}^{u} p_j \phi(2x + j), \quad x \in \mathbb{R}. \tag{28}$$

In addition, we have the following representation

$$\phi(x) = \sum_{j=1}^{u} (-1)^j p_j \psi(2x + j), \quad x \in \mathbb{R}. \tag{29}$$

Because of the orthogonality of the wavelet basis we additionally have that

$$\sum_{j=1}^{k} p_j \bar{p}_{j+2l} = \begin{cases} \frac{1}{\sqrt{2}} & \text{for } l = j, \\ 0 & \text{for } l \neq j. \end{cases} \tag{30}$$

Let us now consider the mapping

$$\mathcal{I} : V_{n+1} \ni f \mapsto (f_{n+1,1}, \ldots, f_{n+1,2^{n+1}}) \in \mathbb{R}^{|J^{\phi}_{n+1}|},$$

where $f_{n+1,k} = \phi_{n+1,k}(f)$. It is not difficult to show that $\mathcal{I}$ is an isomorphism from V_{n+1} onto $\mathbb{R}^{|J^{\phi}_{n+1}|}$. We note that since $V_{n+1} = V_n \otimes W_n$, it follows from (28) that there exists a linear mapping $T : V_{n+1} \to V_n$ which induces a mapping

$$\mathcal{T} : \mathbb{R}^{|J^{\phi}_{n+1}|} \to \mathbb{R}^{|J^{\phi}_{n}|}.$$

We can also define a mapping $\mathcal{S} : V_{n+1} \to W_n$ by $\mathcal{S}\mathbf{x} := \pi_{W_n}(I - \mathcal{T})\mathbf{x}$. As above we can also find a linear mapping $S : V_{n+1} \to W_n$ inducing a mapping

$$\mathcal{S} : \mathbb{R}^{|J^{\phi}_{n+1}|} \to \mathbb{R}^{|J^{\psi}_{n}|}.$$

Since $V_{n+1} = V_n \otimes W_n$, we have $\mathcal{I}^{-1}\mathrm{ker}(\mathcal{S}) = V_n$ and $\mathcal{I}^{-1}\mathrm{ker}(\mathcal{T}) = W_n$. Hence, from the Bayes formula we infer that

$$\mathbb{P}\left(\zeta = \mathbf{x} \mid \gamma = \mathbf{y}\right) = \frac{\mathbb{P}\left(\pi_{W_n}\mathcal{I}^{-1}\mathcal{S}^{-1}\mathbf{x} + \pi_{V_n}\mathcal{I}^{-1}\mathcal{S}^{-1}\mathbf{y}\right)}{\mathbb{P}\left(\mathcal{I}^{-1}\mathcal{S}^{-1}\mathbf{y}\right)},$$

for any $\mathbf{x} \in \mathbb{R}^{|J^{\psi}_{n}|}$ and $\mathbf{y} \in \mathbb{R}^{|J^{\phi}_{n}|}$. By Lemma A.2 $\mathbb{P}\left(\mathcal{I}^{-1}\mathcal{S}^{-1}\mathbf{y}\right) > 0$ and $\mathbb{P}\left(\pi_{W_n}\mathcal{I}^{-1}\mathcal{S}^{-1}\mathbf{x} + \pi_{V_n}\mathcal{I}^{-1}\mathcal{S}^{-1}\mathbf{y}\right) > 0$. In particular, there exists a density

$$h_n(\mathbf{x}, \mathbf{y}) = \mathbb{P}\left(\zeta = \mathbf{x} \mid \gamma = \mathbf{y}\right),$$

such that

$$l_n(\mathbf{y}, U) = \int_U h_n(\mathbf{x}, \mathbf{y})\, d\mathbf{x},$$

and $h_n(\mathbf{x}, \mathbf{y}) > 0$ for all $\mathbf{x} \in \mathbb{R}^{|J^{\psi}_{n}|}$ and $\mathbf{y} \in \mathbb{R}^{|J^{\phi}_{n}|}$. □

In order to verify Assumption 2.3 for a point Y we will show in the following Lemma that for all $n \in \mathbb{N}$, $\pi_{G_n}0$ belongs to the support of the measure μ. If this holds, we can set $Y = 0$.

Lemma A.4. *Let $\alpha \in (0, 2)$, $1 \le p < \alpha$ and $s < \frac{1}{p} - 1$. Let ν be a σ–finite symmetric measure on $\mathbb{R} \setminus \{0\}$ such that there exists a number $\alpha \in (0, 2]$ such that*

$$\nu(\mathbb{R} \setminus [-\epsilon, \epsilon]) \sim \epsilon^{-\alpha} l(\epsilon) \quad as \quad \epsilon \to 0,$$

for some slow varying function l. Let η be the to ν corresponding Poisson random measure over the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let μ be the from ξ defined in (24) on $B^s_{p,p}(\mathbb{R})$ induced probability measure. Then for any $\epsilon > 0$,

$$\mu\left(\{x \in B^s_{p,p}(0,1) : |x|^p_{B^s_{p,p}} \le \epsilon\}\right) > 0.$$

Proof. Let L be given by

$$L(t) := \int_0^t \int_{\mathbb{R}} z\, \eta(dz, ds), \quad t \in [0,1]. \tag{31}$$

From [5], Example 2.2 we know that

$$-\log \mathbb{P}\left(\sup_{0\le t\le 1} |L(t)| \le \epsilon \right) \sim K\epsilon^{\alpha},$$

hence, for any $\tilde\epsilon > 0$ there exists a $\delta > 0$ such that

$$\mathbb{P}\left(\sup_{0\le t\le 1} |L(t)| \le \tilde\epsilon \right) \ge \delta.$$

Let $\tilde\epsilon > 0$ be a constant to be chosen later and let us set

$$\Omega_{\tilde\epsilon} := \left\{ \sup_{0\le t\le 1} |L(t)| \le \tilde\epsilon \right\}.$$

Then,

$$\begin{aligned} \mathbb{P}\left(|\xi|^p_{B^s_{p,p}} \le \epsilon \right) &= \mathbb{P}\left(|\xi|^p_{B^s_{p,p}} \le \epsilon \mid \Omega_{\tilde\epsilon} \right) \mathbb{P}(\Omega_{\tilde\epsilon}) + \mathbb{P}\left(|\xi|^p_{B^s_{p,p}} \le \epsilon \mid \Omega \setminus \Omega_{\tilde\epsilon} \right) \mathbb{P}(\omega \setminus \Omega_{\tilde\epsilon}) \\ &\ge \mathbb{P}\left(|\xi|^p_{B^s_{p,p}} \le \epsilon \mid \Omega_{\tilde\epsilon} \right) \mathbb{P}(\Omega_{\tilde\epsilon}) \ge \delta \mathbb{P}\left(|\xi|^p_{B^s_{p,p}} \le \epsilon \mid \Omega_{\tilde\epsilon} \right). \end{aligned}$$

Note that on $\Omega_{\tilde\epsilon}$ the jump size of the process is less than $2\tilde\epsilon$. Hence

$$\begin{aligned} \mathbb{E}\left[|\zeta_{j,k}|^p \mid \Omega_{\tilde\epsilon} \right] &\le \mathbb{E} \int_0^1 \int_{\mathbb{R}} \psi_{j,k}(s) 1_{|z|\le 2\tilde\epsilon} \eta(dz, ds) \\ &\le \frac{(2\tilde\epsilon)^{p-\alpha}}{p-\alpha} \int_0^1 |\psi_{j,k}(s)|^p\, ds \le C_p \frac{(2\tilde\epsilon)^{p-\alpha}}{p-\alpha}\, 2^{(\frac{p}{2}-1)j}, \end{aligned} \tag{32}$$

and

$$\begin{aligned} \mathbb{E}\left[|\xi|^p_{B^s_{p,p}} \mid \Omega_{\tilde\epsilon} \right] &\le \mathbb{E}\left[\sum_{j=0}^{\infty} 2^{j(s-\frac{1}{p})p} \sum_{k\in J^\psi_j} |\zeta_{j,k}|^p 2^{\frac{jp}{2}} \mid \Omega_{\tilde\epsilon} \right] \\ &\le C_p \frac{(2\tilde\epsilon)^{p-\alpha}}{p-\alpha} \sum_{j=0}^{\infty} 2^{j(s-\frac{1}{p})p} \sum_{k\in J^\psi_j} 2^{(\frac{p}{2}-1)j} 2^{\frac{jp}{2}} \le \tilde C_p \frac{(2\tilde\epsilon)^{p-\alpha}}{p-\alpha}. \end{aligned}$$

From these calculations we infer that

$$\begin{aligned} \mathbb{P}\left(|\xi|^p_{B^s_{p,p}} \le \epsilon \mid \Omega_{\tilde\epsilon} \right) &= 1 - \mathbb{P}\left(|\xi|^p_{B^s_{p,p}} > \epsilon \mid \Omega_{\tilde\epsilon} \right) \\ \ge 1 - \frac{\mathbb{E}|\xi|^p_{B^s_{p,p}}}{\epsilon} \ge 1 - \tilde C_p \frac{(2\tilde\epsilon)^{p-\alpha}}{\epsilon(p-\alpha)}. \end{aligned}$$

Now, choosing $\tilde\epsilon$ such that

$$\tilde C_p \frac{(2\tilde\epsilon)^{p-\alpha}}{\epsilon(p-\alpha)} = \frac{1}{2},$$

we infer that

$$\mathbb{P}\left(|\xi|^p_{B^s_{p,p}} \le \epsilon \mid \Omega_{\tilde\epsilon} \right) \ge \frac{1}{2},$$

from which the assertion follows.

□

For any $\mathcal{D} \in \mathcal{B}(B^s_{p,p}([0,1])$ we define the conditional probability $\mu(\cdot \mid \mathcal{D})$ by

$$\mathcal{B}(B^s_{p,p}([0,1])) \ni U \mapsto \mu(U \mid \mathcal{D}) := \begin{cases} \frac{\mu(U\cap\mathcal{D})}{\mu(\mathcal{D})} & \text{if } \mu(\mathcal{D}) > 0, \\ 1 & \text{if } \mu(\mathcal{D}) = 0. \end{cases}$$

Lemma A.5. *Let $\alpha \in (0,2)$, $1 \le p < \alpha$ and $s < \frac{1}{p} - 1$. Let ν be a σ–finite symmetric measure on $\mathbb{R} \setminus \{0\}$ such that there exists a number $\alpha \in (0,2]$ such that*

$$\nu(\mathbb{R} \setminus [-\epsilon, \epsilon]) \sim \epsilon^{-\alpha} l(\epsilon) \quad as \quad \epsilon \to 0,$$

for some slow varying function l.

Let η also be the Poisson random measure, over the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, associated to the Lévy measure ν. Let μ be the probability measure on $B^s_{p,p}([0,1])$ induced by the mapping ξ defined in (24). Then, for any $R > 0$, $x \in B_E(R,0)$ and $\epsilon > 0$ there exist $n \in \mathbb{N}$ and some $\delta > 0$ such that

$$\mu\left(\{x \in B^s_{p,p}([0,1]) : |\pi_{G^n} x|^p_{B^s_{p,p}} \le \epsilon\}\right) > \delta.$$

Proof. From Lemma A.4 we infer that there exists a constant $\delta > 0$ such that for

$$\mathbb{P}\left(\mathcal{D}_\Omega\right) \ge \delta,$$

where $\mathcal{D}_\Omega := \left\{\sup_{0 \le t \le 1} |L(t)| \le 1\right\}$ and $L = \{L(t) : t \in [0,1]\}$ is defined in (31), Observe that the set $\mathcal{D} := \xi(\mathcal{D}_\Omega)$ satisfies $\mu(\mathcal{D}) \ge \delta$. Thus,

$$\begin{aligned} &\mu\left(\{x \in B^s_{p,p}([0,1]) : |\pi_{G^n} x|^p_{B^s_{p,p}} \le \epsilon\}\right) \\ \ge\ & \mu\left(\{x \in B^s_{p,p} : |\pi_{G^n} \xi|^p_{B^s_{p,p}} \le \epsilon\} \mid \mathcal{D}\right) \mu\left(\mathcal{D}\right) \ge \delta \cdot \mu\left(\{x \in B^s_{p,p} : |\pi_{G^n} \xi|^p_{B^s_{p,p}} \le \epsilon\} \mid \mathcal{D}\right). \end{aligned}$$

Now, from (32) we infer that

$$\begin{aligned} \mathbb{E}\left[|\pi_{G^n} \xi|^p_{B^s_{p,p}} 1_{\Omega_{\tilde\epsilon}}\right] &\le \mathbb{E}\left[1_{\Omega_{\tilde\epsilon}} \sum_{j=n+1}^\infty 2^{j(s-\frac{1}{p})p} \sum_{k \in J^\psi_j} |\zeta_{j,k}|^p 2^{\frac{jp}{2}}\right] \\ &\le C_p \frac{(2\tilde\epsilon)^{p-\alpha}}{p-\alpha} \sum_{j=n+1}^\infty 2^{j(sp+p-1)} \le \tilde C_p 2^{n(sp+p-1)} \sum_{j=0}^\infty 2^{j(sp+p-1)} \le \hat C_p 2^{n(sp+p-1)}. \end{aligned}$$

Therefore,

$$\begin{aligned} \mathbb{P}\left(|\pi_{G^n} \xi|^p_{B^s_{p,p}} \le \epsilon \mid \Omega_{\tilde\epsilon}\right) &= 1 - \mathbb{P}\left(|\pi_{G^n} \xi|^p_{B^s_{p,p}} > \epsilon \mid \Omega_{\tilde\epsilon}\right) \\ &\ge 1 - \frac{\mathbb{E}|\pi_{G^n} \xi|^p_{B^s_{p,p}}}{\epsilon} \ge 1 - \hat C_p 2^{n(sp+p-1)}/\epsilon. \end{aligned}$$

For any $\kappa < 1$ there exists a number $n \in \mathbb{N}$ sufficiently large, such that

$$\hat C_p 2^{n(sp+p-1)}/\epsilon \le 1 - \kappa.$$

which gives the assertion.

□

Lemma A.6. *Let $\alpha \in (0,2)$, $1 \le p < \alpha$ and $s < \frac{1}{p} - 1$. Let ν be a σ–finite symmetric measure on $\mathbb{R} \setminus \{0\}$ such that there exists a number $\alpha \in (0,2]$ such that*

$$\nu_j(\mathbb{R} \setminus [-\epsilon, \epsilon]) \sim \epsilon^{-\alpha} l(\epsilon) \quad as \quad \epsilon \to 0,$$

for some slow varying function l.

Then, for all $N \in \mathbb{N}$, $x_0 \in G_N$, and all $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\mu\left(\left\{x \in B^s_{p,p}([0,1]) \mid |x - x_0|_{B^s_{p,p}} \le \epsilon\right\}\right) \ge \delta.$$

Proof. Let $\epsilon > 0$ be a fixed constant and $s < s_0 < \frac{1}{p} - 1$. From Lemma A.5 we deduce that there exist $n_0 \in \mathbb{N}$ and $\delta_2 > 0$ such that

$$\mu\left(\left\{x \in B^s_{p,p}([0,1]) \mid |\pi_{G^{n_0}} x|_{B^{s_0}_{p,p}} \le \frac{\epsilon}{4}\right\}\right) \ge \delta_2.$$

Then

$$\begin{aligned}&\mu\left(\left\{x\in B^s_{p,p}([0,1])\mid |x_0-x|_{B^s_{p,p}}\le\epsilon\right\}\right)\\ &\ge\ \mu\left(\left\{x\in B^s_{p,p}([0,1])\mid |x_0-\pi_{G_{n_0}}x|_{B^s_{p,p}}\le\frac{\epsilon}{4}\right\}\cap\left\{x\in B^s_{p,p}([0,1])\mid |\pi_{G^{n_0}}x|_{B^{s_0}_{p,p}}\le\frac{\epsilon}{4}\right\}\right).\end{aligned}$$

We now set $A_{n_0}=\left\{x\in B^s_{p,p}([0,1])\mid |x_0-\pi_{G_{n_0}}x|_{B^s_{p,p}}\le\frac{\epsilon}{4}\right\}$ and observe that for $\gamma=\delta_2/2>0$ there exists a closed set $C_\gamma\subset G^{n_0}$ such that $\mu^{n_0}(G^{n_0}\setminus C_\gamma)\le\gamma$ and the function

$$C_\gamma\ni\mathbf{y}\mapsto l_{n_0}(\mathbf{y},A_{n_0})\in[0,1]$$

is continuous. Furthermore, since for all $\mathbf{y}\in G^{n_0}$ μ a.s. $l_n(\mathbf{y},\cdot)$ is equivalent to the Lebesgue measure and $\mathrm{Leb}_{G_{n_0}}(A_{n_0})>0$, we have $l_{n_0}(\mathbf{y},A_{n_0})>0$. Since the embedding $B^{s_0}_{p,p}([0,1])\hookrightarrow B^s_{p,p}([0,1])$ is compact,

$$C_{n_0}=\left\{x\in B^s_{p,p}([0,1])\mid |\pi_{G^{n_0}}x|_{B^{s_0}_{p,p}}\le\frac{\epsilon}{4}\right\}\cap C_\gamma$$

is a compact subset of G^{n_0} and there exists a $\delta_3>0$ such that for all $\mathbf{y}\in C_{n_0}\cap C_\gamma$, $l_{n_0}(\mathbf{y},A_{n_0})\ge\delta_3$. From the above consideration we now infer that

$$\begin{aligned}\mu\left(\left\{x\in B^s_{p,p}([0,1])\mid |x-x_0|\le\epsilon\right\}\right)&\ge\int_{\{|\pi_{G^{n_0}}x|_{B^{s_0}_{p,p}}\le\frac{\epsilon}{4}\}\cap C_\gamma} l_{n_0}(\mathbf{y},A_{n_0})\,\mu^{n_0}(d\mathbf{y})\\ &\ge\ \delta_3\mu^{n_0}\left(\left\{|\pi_{G^{n_0}}x|_{B^{s_0}_{p,p}}\le\tfrac{\epsilon}{4}\right\}\cap C_\gamma\right)\\ &\ge\ \delta_3\left(1-\mu^{n_0}\left(\left(G^{n_0}\setminus\left\{|\pi_{G^{n_0}}x|_{B^{s_0}_{p,p}}\le\tfrac{\epsilon}{4}\right\}\right)\cup(G^{n_0}\setminus C_\gamma)\right)\right)\\ &\ge\ \delta_3\left(1-(1-\delta_2+\gamma)\right)=\delta_3\frac{\delta_2}{2}.\end{aligned}$$

□

The above discussion is summarized in the following lemma.

Lemma A.7. *Let η be a time homogeneous Poisson random measure on $\mathbb{R}$ over a probability space $(\Omega,\mathcal{F},\mathbb{P})$. We assume that the Lévy measure ν associated to η is symmetric, σ-additive, absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}\setminus\{0\}$. In addition, we assume, that there exists some $p\in(1,2)$ with*

$$\int_{|z|\le1}|z|^p\nu(dz)<\infty$$

and there exists a number $\alpha\in(0,2]$ such that

$$\nu(\mathbb{R}\setminus[-\epsilon,\epsilon])\sim\epsilon^{-\alpha}l(\epsilon)\quad as\quad\epsilon\to0,$$

for some slow varying function l.

Let $\{\phi_{j,k}: j\in\mathbb{N}: k=1,\dots,2^j\}$ be the wavelet basis in $B^s_{p,p}([0,1])$ described in Section A. Then, the measure μ induced by the map ξ defined by (24) on $B^s_{p,p}([0,1])$ is decomposable with decomposition $\{F_n,G^n,l_n\}_{n=0}^\infty$ satisfying Assumption 2.2 and Assumption 2.3. Here the spaces F_n are defined by $F_0=V_0$, $F_n=W_n$, $n\ge2$, V_0 and W_n are defined in (19), and l_n is defined in (27).

Proof. The decomposability follows from the fact the wavelet basis described in section A is a Schauder basis of $B^s_{p,p}([0,1])$. Assumption 2.3-(1) follows from choosing $Y=(0,0,\dots)$ and from the fact that $\mathbb{P}\left(\pi_{G_{j+1}}x\in\cdot\mid\pi_{G_j}x=y\right)$ is equivalent to the Lebesgue measure and that for any $y\in\mathbb{R}$ we have (see Lemma A.3)

$$\mathbb{P}\left(\pi_{G_{j+1}}x\in\cdot\mid\pi_{G_j}x=y\right)>0.$$

Using an induction argument one can easily show that for any open set in G_n $\mu_{G_n}(\mathcal{O})>0$ from which it follows that μ_{G_n} is absolutely continuous with respect to Leb_{G_n}. Finally, Assumption 2.3-(2) follows from Lemma A.6.

□

Appendix B. The Fractional Brownian Noise and its Wavelet Expansion

Let $B^H = \{B^H(t) : t \geq 0\}$ a the fractional Brownian motion with Hurst parameter $H \in (\frac{1}{2}, 1)$. Let us fix $s \in (-\frac{1}{2}, H-1)$ and consider the mapping

$$\xi^H : \mathcal{S}([0,1]) \ni \phi \mapsto \xi^H(\phi) = \int_0^1 \phi(t)\, dB^H(t).$$

For all $n \in \mathbb{N}$, $\phi_1, \ldots, \phi_n \in \mathcal{S}(\mathbb{R})$, $C \in \mathcal{B}(\mathbb{R}^n)$ we set

$$\mu\left(\left\{x \in \mathcal{S}'([0,1]) : (x(\phi_1), \ldots, x(\phi_n)) \in C\right\}\right) := \mathbb{P}\left((\xi(\phi_1), \ldots, \xi(\phi_n)) \in C\right). \tag{33}$$

We will firstly show that this measure μ is a Radon measure on $B^s_{2,2}([0,1])$. For this aim, let us consider the Haar wavelet ψ defined by

$$\psi(t) := \begin{cases} 1; & \text{for } t \in [0, \frac{1}{2}), \\ -1; & \text{for } t \in [\frac{1}{2}, 1], \\ 0 & \text{elsewhere}, \end{cases}$$

and the scaling function ϕ defined by

$$\phi(t) := \begin{cases} 1; & \text{for } t \in [0,1], \\ 0; & \text{elsewhere}. \end{cases}$$

Also, we set

$$\psi_{j,k} := 2^{-\frac{j}{2}}\psi(2^j t + k) \text{ and } \phi_{j,k} := 2^{-\frac{j}{2}}\phi(2^j t + k), j = 1, \ldots, n,\ k = 1, \ldots, 2^j, \tag{34}$$

to which we associate the multiresolution analysis

$$V_n := \operatorname{span}\{\phi_{j,k} : j = 1, \ldots, n,\ k = 1, \ldots, 2^j\}, \quad W_n := \operatorname{span}\{\psi_{n,k} : k = 1, \ldots, 2^n\}. \tag{35}$$

The Haar wavelet is an unconditional basis in $L^p([0,1])$ with $1 < p < \infty$, a basis in $B^s_{p,q}([0,1])$ for $1 < p < \infty$ and $\frac{1}{p} - 1 < s < \frac{1}{p}$, and a basis for $B^s_{p,p}([0,1])$, $\frac{1}{2} < p \leq 1$ and $\frac{1}{p} - 1 < s < 1$ (see Triebel [32, Theorem 1.58]).

Now, let $F_0 := V_0$, $F_n = W_n$, $n \in \mathbb{N}$, and $G_n := F_0 \oplus F_1 \oplus \cdots \oplus F_n$, and $\mathcal{F}_n := \sigma(F_0 \oplus F_1 \oplus \ldots \oplus F_n)$. Let us denote the projection of ξ onto G_n by P_n and onto W_n by Q_n. For the time being let us assume that the Radon-Nikodym derivative of the fractional Brownian motion belongs to $B^s_{2,2}([0,1])$. Since the Haar wavelets are a basis of $E := B^s_{2,2}([0,1])$, then for each element $x \in E$ there exists a unique sequence $\{\lambda_{j,k} : j \in \mathbb{N}, k = 1, \ldots, 2^j\}$ such that

$$x = \sum_{j \in \mathbb{N}} \sum_{k=1}^{2^n} \lambda_{j,k}\psi_{j,k} + \lambda_0 \phi.$$

Observe also that

$$\xi(t) := \sum_{j=1}^{\infty} \sum_{k=1}^{2^j} \zeta_{j,k}\psi_{j,k}(t) + a_0\phi(t), \quad t \in [0,1].$$

where $\{\zeta_{j,i} : j \in \mathbb{N}, i = 1, \ldots, 2^j\}$ is a family of random variables defined by

$$\zeta_{j,k} \stackrel{d}{=} \int_0^1 \psi_{j,k}(s)\, dB^H(s),$$

and

$$a_0 \stackrel{d}{=} \int_0^1 \phi_{0,0}(s)\, dB^H(s).$$

In fact, given the coefficients $\{\zeta_{j,k} : j = 1, \ldots, n, k = 1, \ldots, 2^j\} \cup \{a_0\}$, one know the coefficient of $\phi_{n,k}$, $k = 1, \ldots, 2^n$. Since G_n consists of all functions $f : [0,1) \to \mathbb{R}$ that are constant on the intervals $[2^{-n}k, 2^{-n}(k+1))$, $k = 1, \dots, 2^n - 1$, there exists random coefficients $\gamma_{n,k}$, $k = 1, \dots, 2^n - 1$ such that

$$P_n\xi = \sum_{k=0}^{2^n-1} \gamma_{n,k}\phi_{n,k}.$$

It is now easy to see show that

$$\gamma_{n,k} := \int_0^1 \phi_{n,k}(t) dB^H(t).$$

Since for two functions $\phi, \psi : [0,1] \to \mathbb{R}$, the random variables $\xi^H(\phi)$ and $\xi^H(\psi)$ are Gaussian distributed with covariance

$$\mathbb{E}\left[\xi^H(\phi)\, \xi^H(\psi)\right] = \int_0^1 \int_0^1 \phi(s)\phi(t)\, |t-s|^{2H-2}\, dt\, ds,$$

straightforward calculations gives for $l \neq k$

$$\begin{aligned}
\mathbb{E}\left[\xi^H(\psi_{j,k})\, \xi^H(\psi_{j,l})\right] &= 2^j \int_{2^{-j}k}^{2^{-j}(k+1)} \int_{2^{-j}l}^{2^{-j}(l+1)} \psi_{j,k}(s)\psi_{j,l}(t)|t-s|^{2H-2}\, dt\, ds \\
&= 2^j \frac{1}{2H-1} \int_{2^{-j}k}^{2^{-j}(k+1)} \left[(t-2^{-j}l)^{2H-1} - (t-2^{-j}(l+1))\right]\, dt \\
&= 2^j \frac{1}{2H-1}\frac{1}{2H} \left\{ \left[(2^{-j}k - 2^{-j}l)^{2H-1} \right.\right. \\
&\quad \left.\left. - (2^{-j}k - 2^{-j}(l+1))\right] - \left[(2^{-j}(k+1) - 2^{-j}l)^{2H-1} - (2^{-j}(k+1) - 2^{-j}(l+1))\right] \right\} \\
&= 2^j \frac{1}{2H-1}\frac{1}{2H} \left\{ \left[2(2^{-j}k - 2^{-j}l)^{2H-1} \right.\right. \\
&\quad \left.\left. - (2^{-j}k - 2^{-j}(l+1)) - (2^{-j}(k+1) - 2^{-j}l)^{2H-1}\right] \right\} \sim 2^{-j}|k-j|^{2H-1}2^{-j}.
\end{aligned}$$

Hence

$$\mathbb{E}\zeta_{j,k}\zeta_{j,l} \sim 2^{-j}|k-j|^{2H-1}.$$

One can also easily prove that for $l = k$

$$\begin{aligned}
&\mathbb{E}\left[\xi^H(\psi_{j,k})\, \xi^H(\psi_{j,k})\right] \\
&\quad = 2^j \int_{2^{-j}k}^{2^{-j}(k+1)} \int_{2^{-j}k}^{2^{-j}(k+1)} \psi_{j,k}(s)\psi_{j,l}(t)|t-s|^{2H-2}\, dt\, ds \\
&\quad = 2^j \frac{1}{2H-1}\frac{1}{2H} \left\{ \left[(2^{-j}k - 2^{-j}l)^{2H-1} \right.\right. \sim 2^{1-2Hj}.
\end{aligned}$$

Using these estimates we can prove the following proposition.

Proposition B.1. *For $H \in (\frac{1}{2}, 1)$ and $-\frac{1}{2} < s < H - 1$ we have $\xi^H \in L^2(\Omega; B^s_{2,2}([0,1]))$.*

Proof. The proof is the result of the following straightforward calculation

$$\mathbb{E}|\xi|^2_{B^s_{2,2}} = \mathbb{E}\sum_{j=0}^{\infty} 2^{2sj} \sum_{k=0}^{2^j} \mathbb{E}\,\zeta_{j,k} \lesssim \sum_{j=0}^{\infty} 2^{2sj}2^j2^{1-2Hj} \lesssim \sum_{j=0}^{\infty} 2^{j(2s+2-2H)}2^j2^{1-2Hj}$$

Now, the sum is finite if $s + 1 - H < 0$. □

Remark B.1. *If $H \in (\frac{1}{2}, 1)$ one can find a number $s \in (-\frac{1}{2}, H-1)$ such that $\xi^H \in L^2(\Omega; B^s_{2,2}([0,1]))$. Since all coefficients of $\phi_{n,k}$ and $\psi_{n,k}$ are Gaussian distributed, their law are equivalent with respect to the Lebesgue measure. Now, since the Haar basis is a Schauder basis in $B^s_{2,2}([0,1])$, Assumption 2.2 is satisfied. By the same arguments as used in the proof of Lemma A.6, one can show that Assumption 2.3 is also satisfied.*

Lemma B.1. *Let B^H be a fractional Bownian motion with Hurst parameter $H > \frac{1}{2}$ and μ the probability measure on $B^s_{2,2}([0,1])$ defined by* (33). *Let $\{\phi_{j,k} : j \in \mathbb{N} : k = 1, \dots, 2^j\}$ be the wavelet basis in $B^s_{2,2}([0,1])$ described in* (34). *Then, the measure μ is decomposable with decomposition satisfying Assumption 2.2 and Assumption 2.3.*

Appendix C. Zero One Laws for decomposable measures with density

In this Section we generalize the Theorem 4 of [12] to decomposable measures with decomposition as defined in Definition 2.1. We will also identify the conditions under which a measure satisfies Assumption 2.1 and Assumption 2.2.

Throughout this section E denotes and arbitrary a separable Banach space and $\mathcal{B}(E)$ the $\sigma-$algebra generated by its open sets. Let μ be a measure on $(E, \mathcal{B}(E))$ and F and G be two subsets of E such that $E = F \oplus G$. Then, there is a probability measure

$$\mu_{(F,G)} : \mathcal{B}(F) \ni A \mapsto \mu(A + G) \in [0,1].$$

For $A \subset E$ and $y \in G$ let $A_{(F,G)}(y) = \{x \in F : x + y \in A\}$.

As mentioned in the introduction the concept of decomposability can be extended to the notion of decomposability we introduced in Definition 2.1.

Example C.1. *Let E be a separable Banach space and $\{e_n : n \in \mathbb{N}\}$ be a Schauder basis and $F_n := \{\lambda e_n : \lambda \in \mathbb{R}\}$. For each element $x \in E$ there exists a unique sequence $\{a_n : n \in \mathbb{N}\}$ in $\mathbb{R}$ such that $x = \sum_{n\in\mathbf{N}} a_n e_n$. Let $G_n := F_1 \oplus \cdots \oplus F_n$, $G^n = G_n^\perp$,*

$$\pi_{G_n} : E \ni x \mapsto a_1 e_2 + \cdots + a_n e_n \to G_n$$

be a projection from E onto $F_1 \oplus \cdots \oplus F_n$ and

$$\pi_{G^n} : E \ni x \mapsto \sum_{j\in\mathbf{N}} a_{j+n} e_{j+n} \in G^n.$$

Then, the probability measure of each E–valued random variable is decomposable in the sense of Definition 2.1. This can be shown by the following consideration. From the Radon-Nikodym Theorem (see [20, Theorem 6.3]*) for any E–valued random variable X there exists a probability kernel*

$$l_n : G^n \times \mathcal{B}(F_1 \oplus \cdots \oplus F_n) \to [0,1],$$

such that

(1) $\mathbb{P}(\pi_{G_n} X \in U \mid \pi_{G^n} X = y) = l_n(y, U)$ for all $U \in \mathcal{B}(F_1 \oplus \cdots \oplus F_n)$;
(2) for each $U \in \mathcal{B}(F_1 \oplus \cdots \oplus F_n)$ the mapping

$$G^n \ni y \mapsto l_n(y, U)$$

is $\mathcal{B}(G^n)$–measurable

To simplify the notation let us denote $\mu_{(G_n,G^n)}$ by μ_n and $\mu_{(G^n,G_n)}$ by μ^n. Note that given a decomposition (F_n, G_n, l_n) of μ it is essential that the kernel l_n has a density with respect to the Lebesgue measure on G_n which, as we will show in the next Lemma, follows from the absolute continuity of μ_n with respect to Leb_{G_n} for any $n \in \mathbb{N}$.

Lemma C.1. *Let E be a separable Banach space and $\{e_n : n \in \mathbb{N}\}$ be a Schauder basis and $F_n := \{\lambda e_n : \lambda \in \mathbb{R}\}$, $G_n := F_1 \oplus \cdots \oplus F_n$. Let us assume that for all $n \in \mathbb{N}$ the measure μ_{G_n} is absolutely continuous with respect to Leb_{G_n}. Then for any $n \in \mathbb{N}$,*

$$\mu^n(\{y \in G^n : l_n(y, \cdot) \text{ is abs. continuous with respect to } Leb_{G_n}\}) = 1.$$

In particular, for any $U \in \mathcal{B}(G_n)$ with $\mu_n(U) = 0$, we have

$$\mu^n(\{y \in G^n : l_n(\mathbf{y}, U) = 0\}) = 1.$$

Proof. Fix $U \in \mathcal{B}(G_n)$ with $\mu_{G_n}(U) = 0$. We will show that $\mu^n(\{\mathbf{y} \in G^n : l_n(y, U) > 0\}) = 0$. From [25, Theorem 4.1] we infer that for all $\epsilon > 0$ there exists a subset $C^\epsilon_{n,U} \subset G^n$ such that $\mu^n(G^n \setminus C^\epsilon_{n,U}) \le \epsilon$ and

$$l_n(\cdot, A) : C^\epsilon_{n,U} \ni \mathbf{y} \mapsto l_n(\mathbf{y}, U) \in [0, 1],$$

is continuous. Now let us set

$$G^*_\epsilon = \{\mathbf{y} \in G^n \cap C^\epsilon_{n,U} : l_n(\mathbf{y}, U) \ge \epsilon\}.$$

Since $l_n(\cdot, U)\big|_{C^\epsilon_{n,A}}$ is continuous and the sets $[\epsilon, 1]$ and $C^\epsilon_{n,U}$ are closed, the set G^*_ϵ is closed. Hence,

$$0 = \mu_{G_n}(U) = \mu(U + G^n) = \int_{G^n} l_n(\mathbf{y}, U)\, \mu^n(d\mathbf{y}).$$

Since $G^n \supset G^*_\epsilon$, we additionally have

$$0 = \mu_n(U) = \mu(U + G_n) = \int_{G^n} l_n(\mathbf{y}, U)\, \mu^n(d\mathbf{y}) \ge \int_{G^*_\epsilon} l_n(\mathbf{y}, U)\, \mu^n(d\mathbf{y}).$$

By the definition of the set G^*_ϵ we have

$$0 \ge \epsilon\, \mu^n(G^*_\epsilon).$$

Since $\epsilon > 0$, we have $\mu^n(G^*_\epsilon) = 0$. Now, from the closedness of G^*_ϵ and the regularity of the measureμ_{G^n} we infer that

$$\mu^n(\{\mathbf{y} \in G^n : l_n(\mathbf{y}, U) > 0\}) = \lim_{\epsilon \to 0} \mu_{G^n}(G^*_\epsilon) = 0.$$

□

Lemma C.2. *Let μ be a decomposable finite measure on E with decomposition $\{F_n, G^n, l_n\}_{n=1}^\infty$. Let us assume that μ_{G_n} is absolutely continuous with respect to Leb_{G_n}. Then, for any $U \in \mathcal{F}_n$ satisfying $\mu_{G_n}(U) = 0$ we have*

$$\mu_{G^n}(\{\mathbf{y} \in G^n : l_n(\mathbf{y}, U) = 0\}) = 1.$$

Proof. Let $n \in \mathbb{N}$ and $U \in \mathcal{F}_n$ such that $\mu_{G_n}(U) = 0$. We will show that $\mu_{G^n}(\{\mathbf{y} \in G^n : l_n(\mathbf{y}, U) > 0\}) = 0$.

Firstly, note that by the Radon-Nikodym Theorem the mapping

$$l_n : C^\epsilon_{n,U} \ni y \mapsto l_n(y, U) \in [0, 1],$$

is measurable. Hence, from [25, Theorem 4.1] we infer that for all $\epsilon > 0$ there exists a closed subset $C^\epsilon_{n,U}$ of G^n such that $\mu^n(G^n \setminus C^\epsilon_{n,U}) \le \epsilon$ and the function

$$l_n : C^\epsilon_{n,U} \ni \mathbf{y} \mapsto l_n(\mathbf{y}, U) \in [0, 1],$$

is continuous. Secondly, let us set

$$G^*_\epsilon = \{\mathbf{y} \in C^\epsilon_{n,U} : l_n(\mathbf{y}, U) \ge \epsilon\}.$$

From the continuity of $l_n(\cdot, U)\big|_{C^\epsilon_{n,U}}$ and the fact that the sets $[\epsilon, 1]$ and $C^\epsilon_{n,U}$ are closed we conclude the set G^*_ϵ is also closed. Next, thanks to the definition of μ_{G_n} we obtain that

$$0 = \mu_{G_n}(U) = \mu(U + G^n) = \int_{G^n} l_n(\mathbf{y}, U)\, \mu_{G^n}(d\mathbf{y}).$$

Furthermore, because $G^*_\epsilon \subset C^\epsilon_{n,U}$ we also have

$$0 = \int_{G^n} l_n(\mathbf{y}, U)\, \mu_{G^n}(d\mathbf{y}) \ge \int_{G^*_\epsilon} l_n(\mathbf{y}, U)\, \mu_{G^n}(d\mathbf{y}).$$

Invoking now the definition of the set G^*_ϵ we obtain

$$0 \ge \epsilon \mu_{G^n}(G^*_\epsilon).$$

Since $\epsilon > 0$, we have $\mu_{G^n}(G^*_\epsilon) = 0$. From the closedness of G^*_ϵ and the regularity of the measure μ^n we infer that

$$\mu_{G^n}(\{\mathbf{y} \in G^n : l_n(\mathbf{y}, U) > 0\}) = \lim_{\epsilon\to 0} \mu_{G^n}(G^*_\epsilon) = 0.$$

Therefore,

$$\mu_{G^n}(\{\mathbf{y} \in G^n : l_n(\mathbf{y}, U) = 0\}) = 1.$$

□

Corollary C.1. *Let E be a separable Banach space and $\{e_n : n \in \mathbb{N}\}$ be a Schauder basis. Put $F_n := \{\lambda e_n : \lambda \in \mathbb{R}\}$ and $G_n := F_1 \oplus \cdots \oplus F_n$. Let us assume that for all $n \in \mathbb{N}$ μ_{G_n} is absolutely continuous with respect to the Leb_{G_n}. Then for any $n \in \mathbb{N}$ there exists a function $h_n : G^n \times F_1 \oplus \cdots \oplus F_n \to \mathbb{R}^+_0$ such that μ_{G^n}–a.s.*

$$l_n(\mathbf{y}, U) = \int_U h_n(\mathbf{y}, x)\mu_{G_n}(dx).$$

Proof. From

$$\mu_{G^n}(\{\mathbf{y} \in G^n : l_n(\mathbf{y}, U) = 0\}) = 1, \text{ for any } U \in \mathcal{B}(G_n),$$

follows the corollary's assertion. Indeed the above identity implies the existence of a Radon-Nikodyn derivative. In particular, it holds that

$$\mu_{G^n}\Big(\{\mathbf{y} \in G^n : \text{ there exists a mapping } h_n(\mathbf{y}, \cdot) : G_n \to \mathbb{R} \\ \text{such that } l_n(\mathbf{y}, U) = \int_U h_n(\mathbf{y}, x)\mu_{G_n}(dx)\}\Big) = 1.$$

□

Definition C.1. *We call a set $U \in \mathcal{B}^\mu(E)$ a finite zero one μ–set if and only if for all $n \in \mathbb{N}$*

$$\mu_{G^n} \{\mathbf{y} \in G^n : \mu_{G_n}(U_n(\mathbf{y})) = 0 \text{ or } 1\} = 1,$$

where $U_n(\mathbf{y}) = U_{(F_1\oplus\cdots\oplus F_n, G^n)}(\mathbf{y})$.

Let us now present the generalization of Theorem 4 in [12].

Theorem C.1. *Let $\{F_n, G^n, l_n\}_{n=1}^\infty$ be a decomposition for μ such that for any $n \in \mathbb{N}$ μ_{G_n} is absolutely continuous with respect to $Leb_{F_1\oplus\cdots\oplus F_n}$. Let $F_\infty = \cup_{n\in\mathbb{N}}\{F_1 + F_2 + \cdots + F_n\}$. If U is a finite zero one μ measurable subset of E, then there exists $B \in \mathcal{B}(E)$ such that $B + F_\infty = B$ and $\mu(B) = \mu(U)$.*

Proof. The proof is very similar to the proof of [12, Theorem 4]. Let us assume $U \in \mathcal{B}(E)$. For fix $n \in \mathbb{N}$ we set $U^n = \{y \in G^n : \mu_{G_n}(U_n(y)) = 1\}$,

$$G_n = F_1 \oplus F_2 \oplus \cdots \oplus F_n = \text{linear span of} \cup_{k=1}^n F_k,$$

$B_n = G_n + U^n$ and $B = \liminf_{n\to\infty} B_n = \cup_{n=1}^\infty \{\cap_{m\geq n} B_m\}$. For the time being let us assume that

$$\mu(U) = \mu(B_n). \tag{36}$$

Then,

- $\mu(U \triangle B_n) = 0$ for all $n \in \mathbb{N}$,
- and $\mu(U) = \mu(B_n) \geq \mu(\cap_{m\geq n} B_m) \geq \mu(U)$,
- $\mu(B) = \lim_{n\to\infty} \mu(\cap_{m\geq n} B_m)$.

Since μ is regular we additionally have that

$$\mu(B) = \lim_{n\to\infty} \mu(\cap_{m\geq n} B_m) \geq \lim_{n\to\infty} \mu(B_n) = \mu(U),$$

from which the assertion of Theorem C.1 follows.

Now it remains to prove (36). To this end, observe first that because of Lemma C.1 the kernel l_n is μ^n–a.s. absolutely continuous on G_n. Hence, by the Radon-Nikodym Theorem for μ^n–a.s. there exists a probability kernel

$$h_n : G^n \times G_n \to \mathbb{R}^+_0,$$

such that

$$\mu(U) = \int_{G^n} \int_{U_n(y)} h_n(\mathbf{y}, x)\, \mu_{G_n}(dx) \mu_{G^n}(d\mathbf{y}).$$

Then, by using $B_n = G_n \oplus U_n$ we obtain that

$$\begin{aligned}
\mu(U) &= \int_{G^n} \int_{G_n} 1_U(x+\mathbf{y}) h_n(y,x)\, \mu_{G_n}(dx)\, \mu_{G^n}(d\mathbf{y}) \\
&= \int_{G^n} \int_{G_n} 1_{U_n(\mathbf{y})\oplus U^n}(x+y) h_n(\mathbf{y},x)\, \mu_{G_n}(dx)\, \mu_{G^n}(d\mathbf{y}) \\
&= \int_{G^n} \int_{G_n} 1_{(U_n(\mathbf{y})\oplus U^n)\cap B_n}(x+\mathbf{y}) h_n(\mathbf{y},x)\, \mu_{G_n}(dx)\, \mu_{G^n}(d\mathbf{y}) \\
&= \int_{G^n} \int_{G_n} 1_{(U_n(\mathbf{y})\cap G_n)\oplus U^n}(x+\mathbf{y}) h_n(\mathbf{y},x)\, \mu_{G_n}(dx)\, \mu_{G^n}(d\mathbf{y}) \\
&= \int_{U^n} l_n(\mathbf{y}, U_n(\mathbf{y}) \cap G_n)\, \mu_{G_n}(dx)\, \mu_{G^n}(d\mathbf{y}) \\
&= \int_{U^n} l_n(\mathbf{y}, G_n)\, d\mu_{G_n}(x) = \mu(B_n).
\end{aligned}$$

□

Appendix D. Besov spaces and their properties

Let us recall the definition of the Besov spaces as given in [**?**, Definition 2, pp. 7-8]. First we choose a function $\psi \in \mathcal{S}(\mathbb{R}^d)$ such that $0 \le \psi(x) \le 1$, $x \in \mathbb{R}^d$ and

$$\psi(x) = \begin{cases} 1, & \text{if} \quad |x| \le 1, \\ 0 & \text{if} \quad |x| \ge \frac{3}{2}. \end{cases}$$

Then put

$$\begin{cases} \phi_0(x) &= \psi(x),\ x \in \mathbb{R}^d, \\ \phi_1(x) &= \psi(\frac{x}{2}) - \psi(x),\ x \in \mathbb{R}^d, \\ \phi_j(x) &= \phi_1(2^{-j+1}x),\ x \in \mathbb{R}^d, \quad j = 2, 3, \dots. \end{cases}$$

We will use the definition of the Fourier transform $\mathcal{F} = \mathcal{F}^{+1}$ and its inverse $\mathcal{F}^{-1}$ as in [**?**, p. 6]. In particular, with $\langle \cdot, \cdot \rangle$ being the scalar product in $\mathbb{R}^d$, we put

$$(\mathcal{F}^{\pm 1} f)(\xi) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{\mp i \langle x, \xi \rangle} f(x)\, dx, \quad f \in \mathcal{S}(\mathbb{R}^d), \xi \in \mathbb{R}^d.$$

With the choice of $\phi = \{\phi_j\}_{j=0}^\infty$ as above and $\mathcal{F}$ and $\mathcal{F}^{-1}$ being the Fourier and the inverse Fourier transformations (acting in the space $\mathcal{S}'(\mathbb{R}^d)$ of Schwartz distributions) we have the following definition.

Definition D.1. *Let $s \in \mathbb{R}$, $0 < p \le \infty$ and and $f \in \mathcal{S}'(\mathbb{R}^d)$. If $0 < q < \infty$ we put*

$$|f|_{B^s_{p,q}} = \left(\sum_{j=0}^\infty 2^{sjq} \left| \mathcal{F}^{-1} [\phi_j \mathcal{F} f] \right|^q_{L^p} \right)^{\frac{1}{q}} = \| \left(2^{sj} \left| \mathcal{F}^{-1} [\phi_j \mathcal{F} f] \right|_{L^p} \right)_{j \in \mathbb{N}} \|_{l^q}.$$

If $q = \infty$ we put

$$|f|_{B^s_{p,\infty}} = \sup_{j \in \mathbb{N}} 2^{sj} \left| \mathcal{F}^{-1} [\phi_j \mathcal{F} f] \right|_{L^p} = \| \left(2^{sj} \left| \mathcal{F}^{-1} [\phi_j \mathcal{F} f] \right|_{L^p} \right)_{j \in \mathbb{N}} \|_{l^\infty}.$$

We denote by $B^s_{p,q}(\mathbb{R}^d)$ the space of all $f \in \mathcal{S}'(\mathbb{R}^d)$ for which $|f|_{B^s_{p,q}}$ is finite.

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