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Research Article

On Best Proximity Results for a Generalized Modified Ishikawa's Iterative Scheme Driven by Perturbed 2-Cyclic Like-Contractive Self-Maps in Uniformly Convex Banach Spaces

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This paper proposes a generalized modified iterative scheme where the composed self-mapping driving can have distinct step-dependent composition order in both the auxiliary iterative equation and the main one integrated in Ishikawa's scheme. The self-mapping which drives the iterative scheme is a perturbed 2-cyclic one on the union of two sequences of nonempty closed subsets $\{A_n\}_{n=0}^{\infty}$ and $\{B_n\}_{n=0}^{\infty}$ of a uniformly convex Banach space. As a consequence of the perturbation, such a driving self-mapping can lose its cyclic contractive nature along the transients of the iterative process. These sequences can be, in general, distinct of the initial subsets due to either computational or unmodeled perturbations associated with the self-mapping calculations through the iterative process. It is assumed that the set-theoretic limits below of the sequences of sets $\{A_n\}_{n=0}^{\infty}$ and $\{B_n\}_{n=0}^{\infty}$ exist. The existence of fixed best proximity points in the set-theoretic limits of the sequences to which the iterated sequences converge is investigated in the case that the cyclic disposal exists under the asymptotic removal of the perturbations or under its convergence of the driving self-mapping to a limit contractive cyclic structure.

1. Introduction

The problem of existence of best proximity points in uniformly convex Banach spaces and in reflexive Banach spaces as well as the convergence of sequences built via cyclic contractions or cyclic φ-contractions to such points has been focused on and successfully solved in some classic pioneering works. See, for instance, [1–5].

A relevant attention has been recently devoted to the research of existence and uniqueness of fixed points of self-mappings as well as to the investigation of associated relevant properties like, for instance, stability of the iterations. The various related performed researches include the cases of strict contractive cyclic self-mappings and Meir-Keeler type cyclic contractions [3, 4, 6, 7]. Some contractive conditions and related properties under general contractive conditions including some ones of rational type have been also investigated. See, for instance, [8–10] and some of the references therein. The study of existence, uniqueness of best proximity points, and the convergence to them has been studied in [11–14] and some references therein. In [15–18], a close research is performed for proximal contractions. Fixed point theory has also been applied to the investigation of the stability of dynamic systems including the case of fractional modelling [19, 20] and references therein. See also [21] for some recent solvability methods in the fractional framework. On the other hand, some links of fractals structures and fixed point theory with some applications have been investigated in [22, 23]. In particular, collage and anticollage results for iterated function systems are proved in [23].

The basic objective of this paper is the presentation of a generalized modified Ishikawa's iterative equation which is driven by an auxiliary 2-cyclic self-mapping on the union of pairs of sequences of closed convex subsets of a uniformly convex Banach space. As a result, the iterative schemes also generate sequences which take alternated values on each subsequence of subsets in the cyclic disposal. The generalization of the modified Ishikawa's iterative scheme consists basically in the fact that the iteration powers of the auxiliary self-map can be modulated depending on the iteration step. Furthermore, the modulation powers are, in general, distinct in the main and the auxiliary equation of Ishikawa's iterative scheme. It is assumed that such a self-mapping is subject to computational and/or unmodeled errors while it satisfies a contractive-like cyclic condition. Such a condition is contractive in the absence of computational uncertainties. In the case when such sequences of subsets are monotonically nonincreasing with nonempty set-theoretic limits, the convergence of the sequences to best proximity points of the set-theoretic limits is proved. The paper is organized as follows. Section 2 develops a simple motivating example which emphasizes that an Ishikawa's scheme can stabilize the solution under certain computational errors of the auxiliary self-mapping even if this one loses its contractive nature. On the other hand, Section 3 formulates some preliminary results about distances under perturbations under perturbed cyclic maps satisfying extended contractive-like conditions which become contractive in the absence of errors. It is assumed, in general, that the sets involved in the cyclic disposal and their mutual distances can be also subject to point-dependent perturbations so that the self-mapping is defined on the union of pairs of sequences of subsets of a normed space. Section 4 gives a generalization of the modified Ishikawa's iterative scheme where the composition orders of the auxiliary self-map can be modulated along the iteration procedure. Afterwards, some relevant results on the contractive-like cyclic self-mappings of Section 3 are correspondingly reformulated for the sequences generated via the generalized modified Ishikawa's iterative procedure when driven by such an auxiliary cyclic self-mapping. Finally, Section 5 deals with the convergence of distances to best proximity points of the set-theoretic limits of the involved sequences of sets on which the cyclic self-mapping is defined.

2. Motivating Example

The following example emphasizes the feature that an iterative modified Ishikawa's-type scheme [24–26] can recover the asymptotic convergence properties and the equilibrium stability [27], in the case when certain computational perturbations on its driving self-mapping can lose its contractive (or asymptotic stability) properties. Now, assume real positive scalar sequences $\{x_n\}_{n=0}^{\infty}$ generated as follows by the linear discrete equation:

$$x_{n+1} = tx_n = kx_n + \tilde{k}_n(x_n)\, x_n; \quad n \geq 0 \tag{1}$$

for any given $x_0 \geq 0$, where $k \in [0,1)$ and $\{\tilde{k}_n(x)\}_{n=0}^{\infty} \subset \mathbf{R}_{0+}$ for any $x \in \mathbf{R}$. Note that

(i) if $\sup_{n\geq 0}\sup_{x\in\mathbf{R}_{0+}} \tilde{k}_n(x) < 1-k$ then the self-mapping $t : \mathbf{R}_{0+} \longrightarrow \mathbf{R}_{0+}$ is a strict contraction whose unique fixed point is $x = 0$ and all sequences $\{x_n\}_{n=0}^{\infty}(\subseteq \mathbf{R}_{0+}) \longrightarrow 0$ and are bounded for any given finite $x_0 \geq 0$,

(ii) if $\sup_{n\geq 0}\sup_{x\in\mathbf{R}_{0+}} \tilde{k}_n(x) \leq 1-k$ then $t : \mathbf{R}_{0+} \longrightarrow \mathbf{R}_{0+}$ is nonexpansive, $x = 0$ is a fixed point of $t : \mathbf{R}_{0+} \longrightarrow \mathbf{R}_{0+}$, and all sequences $\{x_n\}_{n=0}^{\infty}(\subseteq \mathbf{R}_{0+})$ are bounded for any given finite $x_0 \geq 0$,

(iii) if $\lim\inf_{n\longrightarrow\infty} \tilde{k}_n(x) > 1-k$, $\forall x \in \mathbf{R}_{0+}$ then $t : \mathbf{R}_{0+} \longrightarrow \mathbf{R}_{0+}$ is asymptotically expansive, $x = 0$ is still a fixed point of $t : \mathbf{R}_{0+} \longrightarrow \mathbf{R}_{0+}$ but any sequence $\{x_n\}_{n=0}^{\infty}$ diverges as $n \longrightarrow \infty$ if $x_0 \neq 0$ so that the only converging sequence to the fixed point is the trivial solution.

We can interpret this simple discussion in the following terms. We have at hand a "nominal" (i.e., disturbance-free) discrete one-dimensional linear time-varying positive difference equation $x_{n+1}^0 = t^0 x_n^0$; $n \geq 0$ under any arbitrary finite initial condition $x_0^0 \geq 0$. This nominal solution is globally asymptotically stable to its unique stable equilibrium point $x = 0$ which is also the unique fixed point of the strictly contractive mapping $t^0 : \mathbf{R}_{0+} \longrightarrow \mathbf{R}_{0+}$ which defines the iteration which generates the solution sequence. If we have additive (in general, solution-dependent) disturbance sequences $\{\tilde{k}_n(x_n)x_n\}_{n=0}^{\infty}$ which make the "current" solution to be defined by $x_{n+1} = tx_n$, $n \geq 0$ for any arbitrary finite initial condition $x_0 \geq 0$ then the above property of strictly contractive mapping and associated global asymptotic stability still holds if the disturbance is sufficiently small as under the conditions (i) which lead to $\{x_{n+1} - x_{n+1}^0\}_{n=0}^{\infty} \longrightarrow 0$. The mapping defining the current solution is guaranteed to be nonexpansive if the disturbance amount increases moderately. The solution is still globally (but nonasymptotically) stable since any solution sequence is bounded for any finite initial condition. See conditions (ii). However, if the disturbance is large enough exceeding a certain minimum threshold [see conditions (iii)] then the solution diverges and the difference equation is unstable since the mapping which defines it is asymptotically expansive.

It is now discussed the feature that if the Ishikawa iterative scheme is used then the conditions under which the asymptotic stability is kept leading to a convergence to the solution sequence of the same fixed point $x = 0$ are improved. The Ishikawa iterative scheme becomes for this case:

$$\begin{aligned} x_{n+1} = t_I x_n = \left(1 - \alpha_n + \alpha_n t + \alpha_n \beta_n t\,(t-1)\right) x_n = \Big(1 \\ - \alpha_n + \alpha_n \left(k + \tilde{k}_n(x_n)\right) \\ + \alpha_n \beta_n \left(k + \tilde{k}_n(x_n)\right)\left(k + \tilde{k}_n(x_n) - 1\right)\Big) x_n; \quad n \geq 0 \end{aligned} \tag{2}$$

for a given $x_0 \geq 0$. Note that

$$\begin{aligned} x_{n+1} - x_n = -\alpha_n \Big[1 - \left(k + \tilde{k}_n(x_n)\right)(1 - \beta_n) \\ - \beta_n \left(k + \tilde{k}_n(x_n)\right)^2\Big] x_n < 0; \quad n \geq 0, \ \text{all } x_n \neq 0 \end{aligned} \tag{3}$$

so that $\{x_n\}_{n=0}^{\infty}(\subseteq \mathbf{R}_{0+}) \longrightarrow 0$ and the solution is strictly monotonically decreasing if $\{\tilde{k}_n(x)\}_{n=0}^{\infty} \subset \mathbf{R}_{0+}$ provided that

$$\liminf_{n\longrightarrow\infty} \alpha_n > 0;$$

$$\limsup_{n\longrightarrow\infty}\left(\beta_n - \sup_{x\in\mathbf{R}_{0+}} \frac{1}{\left(k+\tilde{k}_n(x_n)\right)\left(k+\tilde{k}_n(x_n)-1\right)}\right) < 0 \quad (4)$$

so that there are conditions of asymptotic convergence of the iterative scheme to the zero fixed point of $t : \mathbf{R}_{0+} \longrightarrow \mathbf{R}_{0+}$ in some cases that conditions [(ii)-(iii)] fail for the iteration $x_{n+1} = tx_n$; $n \geq 0$ for given x_0. Furthermore, $\{t_I(x_n) - t^0(x_n^0)\}_{n=0}^{\infty} \longrightarrow 0$.

3. Preliminary Results on Distances in Iterated Sequences Built under Perturbed 2-Cyclic Self-Maps

This section gives some preliminary results related to distances between points of sequences generated with 2-cyclic self-maps subject to computational or unmodeled errors and 2-cyclic contractive-like constraints. The precise cyclic contractive nature might become lost due to such errors. It is assumed, in general, that the sets involved in the cyclic disposal and their mutual distances can be also subject to point-dependent perturbations so that the relevant feature is that one deals with pairs of sequences of subsets (rather than with two iteration-independent subsets) of a normed space when constructing the relevant sequences. In the sequel, we simply refer to 2-cyclic self-maps and 2-cyclic contractions as cyclic self-maps and cyclic contractions, respectively, since the discussion in this paper is always concerned with cyclic self-maps on the union of two sets.

Let $A^*(\neq \varnothing)$, $B^*(\neq \varnothing)$, A and B, fulfilling $A^* \subseteq A$ and $B^* \subseteq B$, subsets of a linear space and let $T : A \cup B \longrightarrow A \cup B$ be a mapping fulfilling $T(A^*) \subseteq B^*$, $T(B^*) \subseteq A^*$, $T(A) \subseteq B$, $T(B) \subseteq A$ which satisfies the subsequent condition:

$$\|Tx - Ty\| \leq \left(k_0 + \tilde{k}(x,y)\right)\|x-y\| + \left(1 - k_0 - \tilde{k}(x,y)\right)\widehat{D}(x,y); \quad \forall x, y \in A \cup B \quad (5)$$

where $k_0 \in [0,1)$, $\tilde{k}(x,y) \in [\tilde{k}_{00}, \tilde{k}_{10}] \subseteq [-k_0, \tilde{k}_{10}]$; $\forall x, y \in A \cup B$,

$$D = d(A,B) = \inf_{x\in A, y\in B}\|x-y\| \in [D_1, D_2] \subseteq [0, D_2], \quad (6)$$

and

$$\widehat{D}(x,y) = 0 \quad \text{if } x, y \in A \text{ or if } x, y \in B,$$
$$\widehat{D}(x,y) = D \quad \text{if } x \in A \text{ and } y \in B \text{ or if } x \in B \text{ and } y \in A. \quad (7)$$

Note that $D^* = d(A^*, B^*) = \inf_{x\in A^*, y\in B^*}\|x-y\| \geq D_2 \geq D$ since $A^* \subseteq A$ and $B^* \subseteq B$.

The amount $\tilde{k}(x,y)$ is a point-dependent uncertainty function which accounts for the computational perturbations through the self-mapping T on $A \cup B$.

Assume that $T^* : A^* \cup B^* \longrightarrow A^* \cup B^*$ is a nominal mapping fulfilling $T^*(A^*) \subseteq B^*, T^*(B^*) \subseteq A^*$, which satisfies the subsequent nominal cyclic contractive condition:

$$\|T^*x^* - T^*y^*\| \leq k_0\|x^* - y^*\| + (1-k_0)\widehat{D}^*(x^*, y^*); \quad \forall x^*, y^* \in A^* \cup B^* \quad (8)$$

and

$$\widehat{D}^*(x,y) = D^* \quad \text{if } x \in A^* \text{ and } y \in B^*,$$
$$\widehat{D}*(x,y) = 0 \quad \text{if either } x, y \in A^* \text{ or } x, y \in B^*. \quad (9)$$

It is not assumed, in general, that $T(A^*) = T(A)$ and $T(B^*) = T(B)$. The nominal cyclic contraction (8) implies (5) under a class of perturbations of the nominal self-mapping $T^* : A^* \cup B^* \longrightarrow A^* \cup B^*$ leading to a perturbed one $T : A \cup B \longrightarrow A \cup B$.

Proposition 1. *Assume that $T(A^*) \subseteq T(A)$ and $T(B^*) \subseteq T(B)$ and that*

$$\tilde{k}(x,y) \geq \frac{\|Tx - T^*x\| + \|Ty - T^*y\| + (1-k_0)\left(\widehat{D}^*(x,y) - \widehat{D}(x,y)\right)}{\|x-y\| - \widehat{D}(x,y)} \quad (10)$$

for any $x, y \in A^* \cup B^*$ such that $\|x - y\| \neq \widehat{D}(x,y)$. Then, the nominal cyclic contractive condition (8) implies condition (5) under perturbations subject to (10).

Proof. Take $x, y \in A^* \cup B^*$ and assume that (8) and (10) hold. Then,

$$\begin{aligned}\|Tx - Ty\| &\leq \|T^*x - T^*y\| + \|Tx - T^*x\| + \|Ty - T^*y\| \\ &\leq k_0\|x-y\| + (1-k_0)\widehat{D}^*(x,y) + \|Tx - T^*x\| + \|Ty - T^*y\|\end{aligned}$$

$$\begin{aligned}&\leq k_0\|x-y\|+(1-k_0)\widehat{D}^*(x,y)\\&\quad+\widetilde{k}(x,y)\left(\|x-y\|-\widehat{D}(x,y)\right)\\&\quad+(1-k_0)\left(\widehat{D}(x,y)-\widehat{D}^*(x,y)\right)\\&\leq\left(k_0+\widetilde{k}(x,y)\right)\|x-y\|\\&\quad+\left(1-k_0-\widetilde{k}(x,y)\right)\widehat{D}(x,y)\end{aligned} \tag{11}$$

Some technical properties on limiting upper-bounds derived from (5) are given in the next result. □

Proposition 2. *Assume that $T(A^*)\subseteq T(A)$ and $T(B^*)\subseteq T(B)$. Assume also that*

$$\begin{aligned}&\max_{0\leq j\leq n-1}\sup_{x\in A^*,y\in B^*}\widetilde{k}\left(T^jx,T^jy\right)\left\|T^jx-T^jy\right\|\\&\quad\leq m_{T\widetilde{k}}(0,n).\end{aligned} \tag{12}$$

Then,

$$\begin{aligned}&\limsup_{n\longrightarrow\infty}\left(\sup_{x\in A^*,y\in B^*}\|T^nx-T^ny\|\right.\\&\quad\left.-\frac{1}{1-k_0}\left(\left(1-\widetilde{k}_{00}-\widetilde{k}_{10}\right)D+m_{T\widetilde{k}}(0,n)\right)\right)\leq 0\end{aligned} \tag{13}$$

under condition (5). Furthermore, if $\widetilde{k}_{10}=-k_0$ and $\limsup_{\ell,n\longrightarrow\infty}m_{T\widetilde{k}}(\ell,\ell+n)=0$ then

$$\begin{aligned}&\limsup_{\ell,n\longrightarrow\infty}\left(\sup_{x\in T^\ell(A^*),y\in T^\ell(B^*)}\left\|T^{n+\ell}x-T^{n+\ell}y\right\|-\frac{D^*}{1-k_0}\right)\\&\quad\leq\limsup_{\ell,n\longrightarrow\infty}\left(\sup_{x\in T^\ell(A^*),y\in T^\ell(B^*)}\|T^nx-T^ny\|-\frac{D}{1-k_0}\right)\\&\quad\leq 0\end{aligned} \tag{14}$$

Proof. It follows from (5) that

$$\begin{aligned}\|T^nx-T^ny\|&\leq k_0^n\|x-y\|+\sum_{j=0}^{n-1}k_0^{n-j-1}\left[(1-k_0)D+\widetilde{k}\left(T^jx,T^jy\right)\left(\left\|T^jx-T^jy\right\|-\widehat{D}\left(T^jx,T^jy\right)\right)\right]\\&\leq k_0^n\|x-y\|+\frac{1-k_0^n}{1-k_0}\left[(1-k_0)D+\max_{0\leq j\leq n-1}\sup_{x\in A^*,y\in B^*}\left(\widetilde{k}\left(T^jx,T^jy\right)\left(\left\|T^jx-T^jy\right\|-\widehat{D}\left(T^jx,T^jy\right)\right)\right)\right]\\&\leq k_0^n\|x-y\|+D-\frac{k_1}{1-k_0}\max_{0\leq j\leq n-1}\sup_{x\in A^*,y\in B^*}\left(\widetilde{k}\left(T^jx,T^jy\right)\left(\left\|T^jx-T^jy\right\|-\widehat{D}\left(T^jx,T^jy\right)\right)\right);\end{aligned} \tag{15}$$

$\forall x\in A^*$, $\forall y\in B^*$. Since $\widetilde{k}(x,y)\in[\widetilde{k}_{10},\widetilde{k}_{20}]\subseteq[-k_0,\widetilde{k}_{20}]$ then $k_0D\geq-\widetilde{k}_{10}D\geq-\widetilde{k}(x,y)D\geq-\widetilde{k}_{20}D$; $\forall x,y\in A^*\cup B^*$. Since $\max_{0\leq j\leq n-1}\sup_{x\in A^*,y\in B^*}\widetilde{k}(T^jx,T^jy)\|T^jx-T^jy\|\leq m_{T\widetilde{k}}(0,n)$ and since $\max_{0\leq j\leq n-1}\sup_{x\in A^*,y\in B^*}\widehat{D}(T^jx,T^jy)=D$ one gets that (13) holds. On the other hand, note that if $k_1=-k_0$ and $\limsup_{n\longrightarrow\infty}m_{T\widetilde{k}}(n)=0$ then (14) holds. □

Now, assume that the existence of perturbation in the calculation of the sequences through T implies that the sets of the cyclic mapping depend on the iteration under the following constraints. Define the following nonempty sets $A_0=A^*$, $B_0=B^*$ $A_{n+1}=T(B_n)$, $B_{n+1}=T(A_n)$; $n\geq 0$, where $T^*(A^*)\subseteq B^*$, $T^*(B^*)\subseteq A^*$. The interpretation is that A^* and B^* are the nominal sets to which any initial value of a built sequence belongs and the self-mapping T on $\bigcup_{n\geq 0}(A_n\cup B_n)$ is a perturbation of the (perturbation-free) nominal cyclic self-mapping T^* on $A^*\cup B^*$. Assume that $D^*=D_0=d(A^*,B^*)$ and $D_n=\max[D_{nA},D_{nB}]$ for $n\geq 0$ with $D_{nA}=d(A_n,B_{n+1})$, $D_{nB}=d(B_n,A_{n+1})$ such that $\widetilde{D}_n=D_n-\overline{D}\geq-\overline{D}$ for $n\geq 0$ with $\overline{D}\geq 0$ being some constant set distance of interest for analysis such as $D^*=d(A^*,B^*)$ or $\limsup_{n\longrightarrow\infty}D_n$ or $\liminf_{n\longrightarrow\infty}D_n$, or eventually, $\lim_{n\longrightarrow\infty}D_n$ if both of them coincide. In the same way, we will define a nonnegative real amount $\widetilde{D}$ as a reference for the set distance error sequence $\{\widetilde{D}_n\}_{n=0}^{\infty}$ to obtain some further results.

Condition (5) is now modified as follows for any sequence $\{x_n\}_{n=0}^{\infty}$ with initial condition $x_0\in A^*\cup B^*$:

$$\begin{aligned}&\left\|T^{n+2}x_0-T^{n+1}x_0\right\|\\&\quad\leq\left(k_0+\widetilde{k}\left(T^{n+1}x_0,T^nx_0\right)\right)\left\|T^{n+1}x_0-T^nx_0\right\|\\&\qquad+\left(1-k_0-\widetilde{k}\left(T^{n+1}x_0,T^nx_0\right)\right)D_n\end{aligned} \tag{16}$$

for any $n\geq 0$ and any given $x_0\in A^*\cup B^*$, where $k_0\in[0,1)$, $\widetilde{k}(x,y)\in[\widetilde{k}_{10},\widetilde{k}_{20}]\subseteq[-k_0,\widetilde{k}_{20}]$; $\forall x,y\in A\cup B$.

The following result is concerned with the derivation of some asymptotic upper-bounds for the distances in-between consecutive values of the sequences generated through a cyclic self-mapping. Such a mapping is defined on the union of two sequences of subsets of a normed space under a contractive-like condition (which becomes cyclic contractive in the absence of computational and modelling errors). It is assumed that the distances in-between the pairs corresponding members of the two sequences of sets can vary along the iterative procedure.

Theorem 3. *Define the following nonempty sets in a normed space $(E, \|.\|)$ and associated set distances:*

$$A_0 = A^*,$$
$$B_0 = B^*,$$
$$A_{n+1} \subseteq T(B_n),$$
$$B_{n+1} \subseteq T(A_n); \quad n \geq 0, \tag{17}$$
$$D_n = \overline{D} + \widetilde{D}_n = \max[D_{nA}, D_{nB}];$$
$$D_{nA} = d(A_n, B_{n+1}),$$
$$D_{nB} = d(B_n, A_{n+1}); \quad n \geq 0,$$

for some set distance prefixed reference constant $\overline{D} \geq D^*$ with $D^* = d(A^*, B^*)$ and assume that $\widetilde{D}_n \in [-\widetilde{d}_{00}, \widetilde{d}_{10}] \in [-\overline{D}, 0]$; $n \geq 0$ for some $\widetilde{d}_{00} \in [0, \overline{D}]$ and $\widetilde{d}_{10} \geq 0$ (so that $D_n \geq 0$, $n \geq 0$).

Consider the nominal and perturbed cyclic self-mappings $T^* : A^* \cup B^* \longrightarrow A^* \cup B^*$ and T on $\bigcup_{n\geq 0}(A_n \times B_{n+1} \cup B_n \times A_{n+1})$, this last one subject to the condition:

$$\|Tx - Ty\| \leq \left(k_0 + \widetilde{k}(x, y)\right)\|x - y\| + \left(1 - k_0 - \widetilde{k}(x, y)\right) D_n; \quad \forall (x, y) \in A_n \times B_{n+1} \cup B_n \times A_{n+1}, \; n \geq 0 \tag{18}$$

where $k_0 \in [0, 1)$, $\widetilde{k}(x, y) \in [-\widetilde{k}_{00}, \widetilde{k}_{10}] \subseteq [-k_0, \widetilde{k}_{10}] \subseteq [-k_0, 1 - k_0]$ for some $\widetilde{k}_{00}, \widetilde{k}_{10} \in \mathbf{R}_{0+}$; $\forall (x, y) \in \bigcup_{n\geq 0}(A_n \times B_{n+1} \cup B_n \times A_{n+1})$, the nominal $T^* : A^* \cup B^* \longrightarrow A^* \cup B^*$ being subject to (18) with $\widetilde{k}(x, y) \equiv 0$; $\forall (x, y) \in A^* \times B^* \cup B^* \times A^*$ and $A_n = A^*$, $B_n = B^*$; $n \geq 0$.

Assume also that the sequence $\{M_n\}_{n=0}^{\infty} \subset \mathbf{R}$ is bounded, where

$$M_n = M_n(x_0) = (1 - k_0)\widetilde{d}_{10}\left(\sum_{j\in IN_+(n)} k_0^{n-j} - \sum_{j\in IN_-(n)} k_0^{n-j}\right) + \overline{D}\widetilde{k}_{00}\left(\sum_{j\in IN\widetilde{k}_-(n)} k_0^{n-j} - \sum_{j\in IN\widetilde{k}_+(n)} k_0^{n-j}\right) + \widetilde{k}_{00}\left(\sum_{j\in IN_+(n)\cap IN\widetilde{k}_-(n)} k_0^{n-j}\widetilde{d}_{10} - \sum_{j\in IN_+(n)\cap IN\widetilde{k}_+(n)} k_0^{n-j}\widetilde{d}_{00}\right) + \widetilde{k}_{10}\left(\sum_{j\in IN_-(n)\cap IN\widetilde{k}_+(n)} k_0^{n-j}\widetilde{d}_{00} - \sum_{j\in IN_-(n)\cap IN\widetilde{k}_-(n)} k_0^{n-j}\widetilde{d}_{10}\right), \quad n \geq 0 \tag{19}$$

where

$$IN_-(n) = \left\{j \geq 0 : j \leq n, \; \widetilde{D}_j \in \left[-\widetilde{d}_{10}, 0\right]\right\},$$
$$IN_+(n) = \left\{j \geq 0 : j \leq n, \; \widetilde{D}_j > 0\right\}; \quad n \geq 0$$
$$IN\widetilde{k}_+(x_0, n) = \left\{j \geq 0 : j \leq n, \widetilde{k}_j\left(T^{j+1}x_0, T^j x_0\right) > 0\right\} \tag{20}$$
$$IN\widetilde{k}_-(x_0, n) = \left\{j \geq 0 : j \leq n, \widetilde{k}_j\left(T^{j+1}x_0, T^j x_0\right) \leq 0\right\}; \quad n \geq 0$$

are indicator integer sets for nonpositive and positive incremental set distance while the members of the real weighting sequences $\{d_n\}_{n=0}^{\infty}$ and $\{\widetilde{d}_n\}_{n=0}^{\infty}$ are defined for all $n \geq 0$ by $d_n = D_n/\overline{D} > 0$ if $\overline{D} > 0$ and $d_n = 0$, otherwise. Then, the following properties hold:

(i) *$\{\|T^{n+2}x_0 - T^{n+1}x_0\|\}_{n=0}^{\infty}$ is bounded, an upper-bound being $((1 - k_0)/(1 - k_0 - \widetilde{k}_{10}))(\|Tx_0 - x_0\| + \max_{n\geq 0} M_n)$.*

(ii) $\limsup_{n\longrightarrow\infty} \sup_{x_0 \in A^* \cup B^*}$ *$(\|T^{n+2}x_0 - T^{n+1}x_0\| - \overline{D} - \sum_{j=0}^{n} k_0^{n-j}\widetilde{k}(T^{j+1}x_0, T^j x_0)(\|T^{j+1}x_0 - T^j x_0\| - d_j\overline{D})$.*

$$-(1 - k_0)\left(\sum_{j\in IN_+(n)} k_0^{n-j}\widetilde{d}_j - \sum_{j\in IN_-(n)} k_0^{n-j}\left|\widetilde{d}_j\right|\right)\Bigg) \leq 0 \tag{21}$$

(iii) *If $\{\widetilde{k}(T^{n+1}x_0, T^n x_0)\}_{n=0}^{\infty} \longrightarrow 0$ and $\{M_n(x_0)\}_{n=0}^{\infty} \longrightarrow 0$, $\forall x_0 \in A^* \cup B^*$, then*

$$\limsup_{n\longrightarrow\infty} \sup_{x_0 \in A^* \cup B^*} \left\|T^{n+2}x_0 - T^{n+1}x_0\right\| \leq \overline{D}. \tag{22}$$

(iv) *Assume that $\limsup_{n\longrightarrow\infty} \widetilde{k}(T^{n+1}x_0, T^n x_0) = \widetilde{k}_{\infty}(x_0) \in [-\widetilde{k}_{00}, \widetilde{k}_{10}]$, $\forall x_0 \in A^* \cup B^*$. Then,*

$$\limsup_{n\longrightarrow\infty} \left\|T^{n+2}x_0 - T^{n+1}x_0\right\| \leq \frac{(1 - k_0)\left(\overline{D} + M_{0\infty}(x_0)\right)}{1 - k_0 - \widetilde{k}_{\infty}(x_0)} \leq \frac{(1 - k_0)\left(\overline{D} + M_{\infty}(x_0)\right)}{1 - k_0 - \widetilde{k}_{10}}; \quad \forall x_0 \in A^* \cup B^* \tag{23}$$

where $M_\infty(x_0) = \limsup_{n\to\infty} M_n(x_0)$, *with* $M_n(x_0)$ *defined in (19),* $\forall x_0 \in A^* \cup B^*$, *and* $M_{0\infty}(x_0) = \limsup_{n\to\infty} M_{0n}(x_0)$, *with* $M_{0n}(x_0)$ *defined for all* $x_0 \in A^* \cup B^*$ *by*

$$\begin{aligned} M_{0n}(x_0) = {} & (1-k_0)\left(\sum_{j\in IN_+(n)} k_0^{n-j}\left|\widetilde{D}_j\right|\right) \\ & -(1-k_0)\left(\sum_{j\in IN_-(n)} k_0^{n-j}\left|\widetilde{D}_j\right|\right) \\ & -\sum_{j=0}^{n} k_0^{n-j}\widetilde{k}\left(T^{j+1}x_0, T^j x_0\right)\overline{D} \\ & -\sum_{j\in IN_+(n)} k_0^{n-j}\widetilde{k}\left(T^{j+1}x_0, T^j x_0\right)\left|\widetilde{D}_j\right| \\ & +\sum_{j\in IN_-(n)} k_0^{n-j}\widetilde{k}\left(T^{j+1}x_0, T^j x_0\right)\left|\widetilde{D}_j\right|; \\ & \forall x_0 \in A^* \cup B^* \end{aligned} \tag{24}$$

Proof. It follows from (16) that any sequence $\{T^n x_0\}_{n=0}^{\infty}$ with $x_0 \in A^* \cup B^*$ satisfies from (18) the condition:

$$\begin{aligned} & \left\|T^{n+2}x_0 - T^{n+1}x_0\right\| \le k_0^{n+1}\left\|Tx_0 - x_0\right\| \\ & + \sum_{j=0}^{n} k_0^{n-j}\widetilde{k}\left(T^{j+1}x_0, T^j x_0\right)\left\|T^{j+1}x_0 - T^j x_0\right\| \\ & + \sum_{j=0}^{n} k_0^{n-j}\left[\left(1-k_0-\widetilde{k}\left(T^{j+1}x_0, T^j x_0\right)\right)\left(\overline{D}+\widetilde{D}_j\right)\right] \\ & \le k_0^{n+1}\left\|Tx_0 - x_0\right\| + \left(1-k_0^{n+1}\right)\overline{D} + (1-k_0) \\ & \cdot\left(\sum_{j=0}^{n} k_0^{n-j}\widetilde{D}_j\right) + \sum_{j=0}^{n} k_0^{n-j}\widetilde{k}\left(T^{j+1}x_0, T^j x_0\right) \\ & \cdot\left[\left\|T^{j+1}x_0 - T^j x_0\right\| - D_j\right] \le k_0^{n+1}\left\|Tx_0 - x_0\right\| + \left(1\right. \\ & \left.- k_0^{n+1}\right)\overline{D} + \sum_{j=0}^{n} k_0^{n-j}\widetilde{k}\left(T^{j+1}x_0, T^j x_0\right)\left(\left\|T^{j+1}x_0\right.\right. \\ & \left.\left.- T^j x_0\right\| - d_j\overline{D}\right) + (1-k_0)\left(\sum_{j\in IN_+(n)} k_0^{n-j}\widetilde{d}_j\right. \\ & \left.- \sum_{j\in IN_-(n)} k_0^{n-j}\left|\widetilde{d}_j\right|\right) \le k_0^{n+1}\left\|Tx_0 - x_0\right\| + \left(1\right. \\ & \left.- k_0^{n+1}\right)\overline{D} + \frac{1-k_0^{n+1}}{1-k_0} \\ & \cdot \max_{0\le j\le n}\sup_{x_0\in A^*\cup B^*}\left(\widetilde{k}\left(T^{j+1}x_0, T^j x_0\right)\right. \\ & \left.\cdot\left(\left\|T^{j+1}x_0 - T^j x_0\right\|\right)\right) + (1-k_0) \\ & \cdot\left(\sum_{j\in IN_+(n)} k_0^{n-j}\left|\widetilde{D}_j\right|\right) - (1-k_0) \\ & \cdot\left(\sum_{j\in IN_-(n)} k_0^{n-j}\left|\widetilde{D}_j\right|\right) - \sum_{j=0}^{n} k_0^{n-j}\widetilde{k}\left(T^{j+1}x_0, T^j x_0\right) \\ & \cdot\overline{D} - \sum_{j\in IN_+(n)} k_0^{n-j}\widetilde{k}\left(T^{j+1}x_0, T^j x_0\right)\left|\widetilde{D}_j\right| \\ & + \sum_{j\in IN_-(n)} k_0^{n-j}\widetilde{k}\left(T^{j+1}x_0, T^j x_0\right)\left|\widetilde{D}_j\right| \le k_0^{n+1}\left\|Tx_0\right. \\ & \left.- x_0\right\| + \left(1-k_0^{n+1}\right)\overline{D} + \frac{1-k_0^{n+1}}{1-k_0} \\ & \cdot \max_{0\le j\le n}\sup_{x_0\in A^*\cup B^*}\left(\widetilde{k}\left(T^{j+1}x_0, T^j x_0\right)\right. \\ & \left.\cdot\left(\left\|T^{j+1}x_0 - T^j x_0\right\|\right)\right) + (1-k_0) \\ & \cdot\left(\sum_{j\in IN_+(n)} k_0^{n-j}\left|\widetilde{d}_j\right|\right) - (1-k_0) \\ & \cdot\left(\sum_{j\in IN_-(n)} k_0^{n-j}\left|\widetilde{d}_j\right|\right) \\ & - \sum_{j\in IN\widetilde{k}_+(x_0,n)} k_0^{n-j}\left|\widetilde{k}\left(T^{j+1}x_0, T^j x_0\right)\right|\overline{D} \\ & + \sum_{j\in IN\widetilde{k}_-(x_0,n)} k_0^{n-j}\left|\widetilde{k}\left(T^{j+1}x_0, T^j x_0\right)\right|\overline{D} \\ & - \sum_{j\in IN_+(n)\cap IN\widetilde{k}_+(x_0,n)} k_0^{n-j}\left|\widetilde{k}\left(T^{j+1}x_0, T^j x_0\right)\right|\left|\widetilde{d}_j\right| \\ & + \sum_{j\in IN_+(n)\cap IN\widetilde{k}_-(x_0,n)} k_0^{n-j}\left|\widetilde{k}\left(T^{j+1}x_0, T^j x_0\right)\right|\left|\widetilde{d}_j\right| \\ & + \sum_{j\in IN_-(n)\cap IN\widetilde{k}_+(x_0,n)} k_0^{n-j}\left|\widetilde{k}\left(T^{j+1}x_0, T^j x_0\right)\right|\left|\widetilde{d}_j\right| \\ & - \sum_{j\in IN_-(n)\cap IN\widetilde{k}_-(x_0,n)} k_0^{n-j}\left|\widetilde{k}\left(T^{j+1}x_0, T^j x_0\right)\right|\left|\widetilde{d}_j\right|; \\ & \forall x_0 \in A^* \cup B^* \end{aligned} \tag{25}$$

Note that

if $\widetilde{k}(x,y) > 0$

then $\left|\widetilde{k}(x,y)\right| \le \widetilde{k}_{10}$ and $-\left|\widetilde{k}(x,y)\right| \le \widetilde{k}_{00}$,

if $\widetilde{k}(x,y) \le 0$

then $\left|\widetilde{k}(x,y)\right| \le \widetilde{k}_{00}$ and $-\left|\widetilde{k}(x,y)\right| \le \widetilde{k}_{10}$,

$$\text{if } \tilde{d}_n > 0 \quad \text{then } \left|\tilde{d}_n\right| \le \tilde{d}_{10} \text{ and } -\left|\tilde{d}_n\right| \le \tilde{d}_{00},$$
$$\text{if } \tilde{d}_n \le 0 \quad \text{then } \left|\tilde{d}_n\right| \le \tilde{d}_{00} \text{ and } -\left|\tilde{d}_n\right| \le \tilde{d}_{10}. \tag{26}$$

Thus, one gets from (25) that

$$\begin{aligned}\left\|T^{n+2}x_0 - T^{n+1}x_0\right\| &\le k_0^{n+1}\left\|Tx_0 - x_0\right\| + \left(1 - k_0^{n+1}\right)\overline{D} \\ &+ \frac{1-k_0^{n+1}}{1-k_0}\max_{0\le j\le n}\sup_{x_0\in A^*\cup B^*}\left(\tilde{k}\left(T^{j+1}x_0, T^j x_0\right)\right. \\ &\cdot \left.\left(\left\|T^{j+1}x_0 - T^j x_0\right\|\right)\right) + (1 \\ &- k_0)\,\tilde{d}_{10}\left(\sum_{j\in IN_+(n)} k_0^{n-j} - \sum_{j\in IN_-(n)} k_0^{n-j}\right) \\ &+ \overline{D}\tilde{k}_{00}\left(\sum_{j\in IN\tilde{k}_-(n)} k_0^{n-j} - \sum_{j\in IN\tilde{k}_+(n)} k_0^{n-j}\right) \\ &+ \tilde{k}_{00}\left(\sum_{j\in IN_+(n)\cap IN\tilde{k}_-(n)} k_0^{n-j}\tilde{d}_{10}\right. \\ &- \left.\sum_{j\in IN_+(n)\cap IN\tilde{k}_+(n)} k_0^{n-j}\tilde{d}_{00}\right) \\ &+ \tilde{k}_{10}\left(\sum_{j\in IN_-(n)\cap IN\tilde{k}_+(n)} k_0^{n-j}\tilde{d}_{00}\right. \\ &- \left.\sum_{j\in IN_-(n)\cap IN\tilde{k}_-(n)} k_0^{n-j}\tilde{d}_{10}\right); \quad \forall x_0 \in A^*\cup B^*\end{aligned} \tag{27}$$

and one gets Property (iii) from (25) which also leads from (27) to

$$\begin{aligned}\limsup_{n\longrightarrow\infty}\sup_{x_0\in A^*\cup B^*}&\left(\left\|T^{n+2}x_0 - T^{n+1}x_0\right\| - \overline{D} - \frac{1}{1-k_0}\max_{0\le j\le n}\sup_{x_0\in A^*\cup B^*}\left(\tilde{k}\left(T^{j+1}x_0, T^j x_0\right)\left(\left\|T^{j+1}x_0 - T^j x_0\right\|\right)\right)\right. \\ &- (1-k_0)\,\tilde{d}_{10}\left(\sum_{j\in IN_+(n)} k_0^{n-j} - \sum_{j\in IN_-(n)} k_0^{n-j}\right) - \overline{D}\tilde{k}_{00}\left(\sum_{j\in IN\tilde{k}_-(x_0,n)} k_0^{n-j} - \sum_{j\in IN\tilde{k}_+(x_0,n)} k_0^{n-j}\right) \\ &- \tilde{k}_{00}\left(\sum_{j\in IN_+(n)\cap IN\tilde{k}_-(x_0,n)} k_0^{n-j}\tilde{d}_{10} - \sum_{j\in IN_+(n)\cap IN\tilde{k}_+(x_0,n)} k_0^{n-j}\tilde{d}_{00}\right) \\ &- \left.\tilde{k}_{10}\left(\sum_{j\in IN_-(n)\cap IN\tilde{k}_+(x_0,n)} k_0^{n-j}\tilde{d}_{00} - \sum_{j\in IN_-(n)\cap IN\tilde{k}_-(x_0,n)} k_0^{n-j}\tilde{d}_{10}\right)\right) \le 0\end{aligned} \tag{28}$$

$\forall x_0 \in A^*\cup B^*$. Proceed by contradiction to prove that $\{\|T^{n+2}x_0 - T^{n+1}x_0\|\}_{n=0}^{\infty}$ is bounded. Assume that $\{\|T^{n+2}x_0 - T^{n+1}x_0\|\}_{n=0}^{\infty}$ is unbounded, $\forall x_0 \in A^*\cup B^*$. Then, there is a subsequence $\{\|T^{n_k+2}x_0 - T^{n_k+1}x_0\|\}_{k=0}^{\infty}$ of it, with $\{n_k\} \subset \mathbf{Z}_{0+}$ being strictly increasing, which then diverges as $k \longrightarrow \infty$, and one gets from (27) that

$$\begin{aligned}\left\|T^{n_k+2}x_0 - T^{n_k+1}x_0\right\| &< \left\|T^{n_{k+1}+2}x_0 - T^{n_{k+1}+1}x_0\right\| \\ &\le k_0^{\min(n_{k+1}+1,n_k+1)}\left\|Tx_0 - x_0\right\| + \left(1\right. \\ &- \left.k_0^{\max(n_{k+1}+1,n_k+1)}\overline{D}\right) + \frac{1-k_0^{\max(n_{k+1}+1,n_k+1)}}{1-k_0} \\ &\times \max_{0\le j\le\max(n_{k+1},n_k+1)}\sup_{x_0\in A^*\cup B^*}\left(\tilde{k}\left(T^{j+1}x_0, T^j x_0\right)\right. \\ &\cdot \left.\left(\left\|T^{j+1}x_0 - T^j x_0\right\|\right)\right) + \max\left(M_{n_{k+1}}, M_{n_k+1}\right) \\ &\le \left\|T^{n_{k+1}+2}x_0 - T^{n_{k+1}+1}x_0\right\| \le k_0^{\min(n_{k+1}+1,n_k+1)}\left\|Tx_0\right. \\ &- \left.x_0\right\| + \left(1 - k_0^{\max(n_{k+1}+1,n_k+1)}\overline{D}\right) \\ &+ \frac{1-k_0^{\max(n_{k+1}+1,n_k+1)}}{1-k_0} \\ &\times \max_{0\le j\le\max(n_{k+1},n_k+2)}\sup_{x_0\in A^*\cup B^*}\left(\tilde{k}\left(T^{j+1}x_0, T^j x_0\right)\right. \\ &\cdot \left.\left(\left\|T^{j+1}x_0 - T^j x_0\right\|\right)\right) + \max\left(M_{n_{k+1}}, M_{n_k+2}\right)\end{aligned} \tag{29}$$

for $k \ge 0$, $\forall x_0 \in A^*\cup B^*$, where M_n for $n \ge 0$ is defined in (19). Thus,

$$\left(1-\frac{1-k_0^{\max(n_{k+1}+1,n_k+1)}}{1-k_0}\widetilde{k}_{10}\right)\left\|T^{n_k+2}x_0-T^{n_k+1}x_0\right\| \le k_0^{\min(n_{k+1}+1,n_k+1)}\left\|Tx_0-x_0\right\| + \left(1-k_0^{\max(n_{k+1}+1,n_k+1)}\overline{D}\right) + \max\left(M_{n_{k+1}},M_{n_k+2}\right), \quad k\ge 0;\ \forall x_0\in A^*\cup B^* \tag{30}$$

so that

$$\left(\frac{1-k_0-\widetilde{k}_{10}}{1-k_0}+\max\left[\left|o\left(k_0^{n_{k+1}+1}\right)\right|+\left|o\left(k_0^{n_k+1}\right)\right|\right]\right)\cdot\left\|T^{n_k+2}x_0-T^{n_k+1}x_0\right\|\le M_0<+\infty; \quad \forall x_0\in A^*\cup B^* \tag{31}$$

and $\{\|T^{n_k+2}x_0 - T^{n_k+1}x_0\|\}_{k=0}^{\infty}$ is bounded, a contradiction. Thus, $\{\|T^{n+2}x_0 - T^{n+1}x_0\|\}_{n=0}^{\infty}$ is bounded as claimed. Now, one gets from (27), (28), and (19) that

$$M_T = M_T(x_0) = \max_{n\ge 0}\sup_{x_0\in A^*\cup B^*}\left(\left\|T^{n+2}x_0-T^{n+1}x_0\right\|\right. \le \left\|Tx_0-x_0\right\|+\overline{D}+\frac{\widetilde{k}_{10}}{1-k_0}M_T+\max_{n\ge 0}M_n; \quad \forall x_0\in A^*\cup B^* \tag{32}$$

leading to $M_T \le \frac{1-k_0}{1-k_0-\widetilde{k}_{10}}\left(\left\|Tx_0-x_0\right\| + \max_{n\ge 0}M_n\right); \quad \forall x_0\in A^*\cup B^*$.

Property (i) is fully proved. Property (iii) is proved as follows. Take any integers $N\ge 0$, $n\ge N$, and $m\ge N+1$ so that one gets from (27)

$$\left\|T^{n+m+2}x_0-T^{n+m+1}x_0\right\| \le k_0^{n+1}\left\|T^m x_0-T^{m-1}x_0\right\|+\left(1-k_0^{m+m+1}\right)\overline{D} + \frac{1-k_0^{m+m+1}}{1-k_0}\varepsilon_{n+m}\left(x_0,\widetilde{k}\right)\left\|T^{j+m+1}x_0-T^{j+m}x_0\right\| + M_{n+m}\left(x_0\right) \tag{33}$$

$\forall x_0 \in A^*\cup B^*$ with $\varepsilon_{n+m}(x_0,\widetilde{k}) > \max(|\widetilde{k}(T^{n+1}x_0, T^n x_0)|, M_{n+m}(x_0))$ for all $n\ge N$ and $m\ge N+1$. Since the sequences $\{\widetilde{k}(T^{n+1}x_0,T^n x_0)\}_{n=0}^{\infty}\longrightarrow 0$ and $\{M_n(x_0)\}_{n=0}^{\infty}\longrightarrow 0$, they are Cauchy sequences, since convergent, so that $\{\varepsilon_{n+m}(x_0,\widetilde{k})\}_{n=0}^{\infty}\longrightarrow 0$ can be chosen as an upper-bounding vanishing sequence; that is, for any given $\varepsilon>0$, there is $N\ge 0$ such that $\varepsilon_{n+m}(x_0,\widetilde{k})\le\varepsilon$ for any integers $n\ge N$ and $m\ge N+1$. Furthermore, $\{\varepsilon_{n+m}(x_0,\widetilde{k})\|T^{j+m+1}x_0 - T^{j+m}x_0\|\}_{n=0}^{\infty}\longrightarrow 0$ for any given integers $m,j\ge 0$ since $\{\|T^{j+1}x_0 - T^j x_0\|\}_{n=0}^{\infty}$ is bounded. Then, it follows from (33) that $\limsup_{n\longrightarrow\infty}\sup_{x_0\in A^*\cup B^*}\|T^{n+2}x_0 - T^{n+1}x_0\|\le\overline{D}$. Property (iii) has been proved. Property (iv) follows from the given assumptions, (25) and (33) since

$$\left(1-\frac{\widetilde{k}_{10}}{1-k_0}\right)\left(\limsup_{n\longrightarrow\infty}\left\|T^{n+2}x_0-T^{n+1}x_0\right\|\right) \le\left(1-\frac{\widetilde{k}_{\infty}(x_0)}{1-k_0}\right)\left(\limsup_{n\longrightarrow\infty}\left\|T^{n+2}x_0-T^{n+1}x_0\right\|\right) \le\overline{D}+M_{0\infty}(x_0)\le\overline{D}+M_{\infty}(x_0); \quad \forall x_0\in A^*\cup B^*. \tag{34}$$

□

Note from Theorem 3(iv) that $M_{0\infty}(x_0)\ge -\overline{D}$, $\forall x_0\in A^*\cup B^*$.

Remark 4. Note that if the sets of the cyclic mapping are not uncertain along the iteration via T, then its mutual distance is identical to D^* along the iteration. If, furthermore, such a mapping is contractive with $\widetilde{k}(x_0)=0$, $\forall x_0\in A^*\cup B^*$, then, Theorem 3 (iv) yields $D^*=\lim_{n\longrightarrow\infty}\sup\|T^{n+2}x_0 - T^{n+1}x_0\|\le D^*$, $\forall x_0\in A^*\cup B^*$. Thus, there exists the limit $\lim_{n\longrightarrow\infty}\|T^{n+2}x_0 - T^{n+1}x_0\| = D^*$. As a result, if $A^*_{bp}\subseteq A^*$ and $B^*_{bp}\subseteq B^*$ are the nonempty sets of best proximity points of A^* to B^* and of B^* to A^*, respectively, it follows that for any $x_0\in A^*\cup B^*$, one has

$$\lim_{n\longrightarrow\infty}\left(T^{2n}x_0-z_{An}\right)=\lim_{n\longrightarrow\infty}\left(T^{2n+1}x_0-z_{Bn}\right)=0 \tag{35}$$

with $\{z_{An}\}_{n=0}^{\infty}\subseteq A^*_{bp}$ and $\{z_{Bn}\}_{n=0}^{\infty}\subseteq B^*_{bp}$ if $x_0\in A^*$ and $\{z_{An}\}_{n=0}^{\infty}\subseteq B^*_{bp}$ and $\{z_{Bn}\}_{n=0}^{\infty}\subseteq A^*_{bp}$ if $x_0\in B^*$.

Remark 5. The existence of $\lim_{n\longrightarrow\infty}\|T^{n+2}x_0 - T^{n+1}x_0\|$ is not guaranteed in the general uncertain case of Theorem 3. However, provided that $\liminf_{n\longrightarrow\infty}D_n = \underline{D}\ge 0$, then it follows from Theorem 3(ii) that

$$\underline{D}\le\liminf_{n\longrightarrow\infty}\inf_{x_0\in A^*\cup B^*}\left\|T^{n+2}x_0-T^{n+1}x_0\right\|\le\limsup_{n\longrightarrow\infty}\cdot\sup_{x_0\in A^*\cup B^*}\left\|T^{n+2}x_0-T^{n+1}x_0\right\|\le\overline{D}. \tag{36}$$

If $\{\widetilde{k}(T^{n+1}x_0,T^n x_0)\}_{n=0}^{\infty}\longrightarrow 0$ and $\{M_n(x_0)\}_{n=0}^{\infty}\longrightarrow 0$, $\forall x_0\in A^*\cup B^*$,

$$
\begin{aligned}
\underline{D} &\leq \liminf_{n\longrightarrow\infty}\left\|T^{n+2}x_0 - T^{n+1}x_0\right\| \\
&\leq \limsup_{n\longrightarrow\infty}\left\|T^{n+2}x_0 - T^{n+1}x_0\right\| \\
&\leq \frac{(1-k_0)\left(\overline{D}+M_{0\infty}(x_0)\right)}{1-k_0-\widetilde{k}_{\infty}(x_0)} \\
&\leq \frac{(1-k_0)\left(\overline{D}+M_{\infty}(x_0)\right)}{1-k_0-\widetilde{k}_{10}}; \quad \forall x_0 \in A^* \cup B^*
\end{aligned}
\tag{37}
$$

from Theorem 3 (iv) with $\overline{D} \geq \limsup_{n\longrightarrow\infty}\sup_{x_0\in A^*\cup B^*}((1-k_0-\widetilde{k}_{\infty}(x_0))/(1-k_0))\underline{D}-M_{0\infty}$. The above formulas quantify the bounds of the limiting reachable distances between consecutive points of the sequences calculated from the iterations performed via the self-mapping T on $\bigcup_{n\geq 0}(A_n \times B_{n+1} \cup B_n \times A_{n+1})$ with initial conditions in $A^* \cup B^*$.

Proposition 6. *Any sequence of* $\{T^n x_0\}_{n=0}^{\infty}$ *is bounded for any given* $x_0 \in A^* \cup B^*$ *under all the given general assumptions of Theorem 3 provided that* $\limsup_{n\longrightarrow\infty}\widetilde{k}(T^{n+1}x_0, T^n x_0) = \widetilde{k}_{\infty}(x_0) \in [-\widetilde{k}_{00}, \widetilde{k}_{10}]$.

Proof. If all the given general assumptions of Theorem 3 and, furthermore, $\limsup_{n\longrightarrow\infty}\widetilde{k}(T^{n+1}x_0, T^n x_0) = \widetilde{k}_{\infty}(x_0) \in [-\widetilde{k}_{00}, \widetilde{k}_{10}]$, then Theorem 3(iv) holds. Thus, one has $\limsup_{n\longrightarrow\infty}\|T^{n+2}x_0 - T^{n+1}x_0\| \leq L < +\infty, \forall x_0 \in A^* \cup B^*$, where $L = (1-k_0)(\overline{D}+M_{0\infty}(x_0))/(1-k_0-\widetilde{k}_{\infty}(x_0))$. Since it is assumed that $\widetilde{k}(x,y) \in [-\widetilde{k}_{00}, \widetilde{k}_{10}], \forall (x,y) \in \bigcup_{n\geq 0}(A_n \times B_{n+1} \cup B_n \times A_{n+1})$, then the above constraint also holds in the form $\limsup_{n\longrightarrow\infty}\|T^{n+2}y_0 - T^{n+1}x_0\| \leq L < +\infty, \forall (x_0, y_0) \in A^* \times B^*$ and $\forall (x_0, y_0) \in B^* \times A^*, \forall (x_0, y_0) \in A^* \times (\bigcup_{n\geq 0} B_n)$ and $\forall (x_0, y_0) \in B^* \times (\bigcup_{n\geq 0} A_n)$. So, if $y_0 = T^{2m+1}x_0$ for any given nonnegative integer m and any given $x_0 \in A^* \cup B^*$, one has that $\limsup_{n\longrightarrow\infty}\|T^{n+2}y_0 - T^{n+1}x_0\| \leq L < +\infty$. Proceed by contradiction by assuming that $\{T^n x_0\}_{n=0}^{\infty}$ is unbounded for some $x_0 \in A^* \cup B^*$. Then, for any given real positive constants M and ε and any given $x_0 \in A^* \cup B^*$, there exists some integer $N \geq 0$, infinitely many strictly sequences of positive integers $\{n_k\}_{k=0}^{\infty}$, and some sequences of real constants $\{\lambda_{0k}\}_{k=0}^{\infty}$, with $n_k \geq N$ and $\lambda_{0k} = \lambda_{0k}(x_0, \varepsilon, M, n_k) > 1$; $k \geq 0$, such that, for any arbitrary real sequence $\{\lambda_k\}_{k=0}^{\infty}$ satisfying $\lambda_k \geq \lambda_{0k}$, $k \geq 0$, one has that any unbounded sequence $\{T^{n_k}x_0\}_{k=0}^{\infty}$ of $\{T^n x_0\}_{n=0}^{\infty}$ satisfies

$$
\begin{aligned}
L+\varepsilon &\geq \left\|T^{n_{k+1}}x_0 - T^{n_k}x_0\right\| \\
&= \left\|T^{n_k+1}\left(T^{n_{k+1}-n_k-1}x_0\right) - T^{n_k}x_0\right\| \\
&\geq \left\|T^{n_{k+1}}x_0 - x_0\right\| - \left\|T^{n_k}x_0 - x_0\right\| \geq (\lambda_k - 1)M \\
&\geq (\lambda_{0k}-1)M
\end{aligned}
\tag{38}
$$

implying that $\lambda_k \in [\lambda_{0k}, 1+(1/M)(L+\varepsilon)]$, $k \geq 0$. But since $\{T^{n_k}x_0\}_{k=0}^{\infty}$ is unbounded, the sequence of integers $\{n_k\}_{k=0}^{\infty}$ can be chosen such that $n_{k+1}(n_k) > n_k$; $\{n_k\}_{k=0}^{\infty}$ satisfies that λ_k is large enough to satisfy $\lambda_k > 1+(1/M)(L+\varepsilon)$, hence a contradiction. Then, any subsequence of $\{T^n x_0\}_{n=0}^{\infty}$ is bounded for any given $x_0 \in A^* \cup B^*$, so $\{T^n x_0\}_{n=0}^{\infty}$ is bounded for any given $x_0 \in A^* \cup B^*$. □

4. Some Properties of Approximate Convergence of a Generalized Modified Ishikawa's Iterative Scheme Based on Cyclic Self-Mappings

A generalization of the modified Ishikawa's iteration in a normed real space $(E, \|.\|)$ is as follows:

$$
\begin{aligned}
y_n &= (1-\beta_n)x_n + \beta_n T^{n+m(n)}x_n \\
x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T^{n+q(n)}y_n
\end{aligned}
\tag{39}
$$

for integers $m(n) \geq 0, q(n) \geq 0$ and all $n \geq 0$, $\forall x_0 \in A^* \cup B^*$ under parameterizing sequences $\{\alpha_n\}_{n=0}^{\infty} \subset [0,1]$ and $\{\beta_n\}_{n=0}^{\infty} \subset [0,1]$ provided that T is an uncertain cyclic self-mapping defined on $\bigcup_{n\geq 0}(A_n \times B_{n+1} \cup B_n \times A_{n+1})$. The choice of the integer $m(n) \geq 0, q(n) \geq 0$, in general depending on n, is relevant for the allocation of the elements of the solution sequence $\{x_n\}_{n=0}^{\infty}$ in the subset sequences $\{A_n\}_{n=0}^{\infty}$ or $\{B_n\}_{n=0}^{\infty}$ depending on the integers $n+m(n), n+q(n)$ being even or odd. The subsequent auxiliary result will be then used by linking it to some of the results of Section 3.

Lemma 7. *The following properties hold when the generalized modified Ishikawa's iteration (39) is used:*

(i) *The subsequent incremental relations hold for each integer* $m(n) \geq 0, q(n) \geq 0$ *and* $n \geq 0$, $\forall x_0 \in A^* \cup B^*$:

$$
\begin{aligned}
\widetilde{x}_n &= x_{n+1} - x_n = \alpha_n\left(T^{n+q(n)}y_n - x_n\right) \\
&= \widetilde{x}_{n-1} \\
&\quad + \alpha_{n-1}\left(T^{n+q(n)}y_n - T^{n-1+q(n-1)}y_{n-1}\right) \\
&\quad + \widetilde{\alpha}_{n-1}\left(T^{n+q(n)}y_n - x_{n-1}\right) \\
y_n - x_{n+1} &= (\alpha_n - \beta_n)x_n \\
&\quad + \left(\beta_n T^{n+m(n)}x_n - \alpha_n T^{n+q(n)}y_n\right) \\
\widetilde{y}_n &= y_{n+1} - y_n \\
&= (1-\beta_n)\widetilde{x}_n - \widetilde{\beta}_n x_{n+1} \\
&\quad + \beta_n\left(T^{n+1+m(n+1)}x_{n+1} - T^{n+m(n)}x_n\right) \\
&\quad + \widetilde{\beta}_n T^{n+1+m(n+1)}x_{n+1}
\end{aligned}
\tag{40}
$$

where $\widetilde{\alpha}_n = \alpha_{n+1} - \alpha_n$ *and* $\widetilde{\beta}_n = \beta_{n+1} - \beta_n$; $n \geq 0$.

(ii) *If* $\{\alpha_n\}_{n=0}^{\infty} \longrightarrow \alpha$ *then* $\{\widetilde{x}_n - \alpha(T^{n+q(n)}y_n - x_n)\}_{n=0}^{\infty} \longrightarrow 0$; *that is,* $\{x_{n+1} - (1-\alpha)x_n + \alpha T^{n+q(n)}y_n\}_{n=0}^{\infty} \longrightarrow 0$, *equivalently,* $\{\widetilde{x}_n - \widetilde{x}_{n-1} + \alpha(T^{n+q(n)}y_n - T^{n-1+q(n-1)}y_{n-1})\}_{n=0}^{\infty} \longrightarrow 0$.

(iii) *If* $\{\beta_n\}_{n=0}^{\infty} \longrightarrow \beta$ *then* $\{(1-\beta)\widetilde{x}_n - \widetilde{y}_n - \beta(T^{n+1+m(n+1)}x_{n+1} - T^{n+m(n)}x_n)\}_{n=0}^{\infty} \longrightarrow 0$.

(iv) *If* $\{\alpha_n - \beta_n\}_{n=0}^{\infty} \longrightarrow 0$ *then* $\{y_n - x_{n+1} + \alpha_n T^{n+q(n)}y_n - \beta_n T^{n+m(n)}x_n\}_{n=0}^{\infty} \longrightarrow 0$.

If $\{\alpha_n\}_{n=0}^{\infty} \longrightarrow \alpha$ *and* $\{\beta_n\}_{n=0}^{\infty} \longrightarrow \beta$ *then* $\{y_n - x_{n+1} - (\alpha - \beta)x_n - (\beta T^{n+m(n)}x_n - \alpha T^{n+q(n)}y_n)\}_{n=0}^{\infty} \longrightarrow 0$ *and, in particular, if* $\alpha = \beta$ *then* $\{y_n - x_{n+1} + \alpha(T^{n+q(n)}y_n - T^{n+m(n)}x_n)\}_{n=0}^{\infty} \longrightarrow 0$.

(v) *If* $\{\alpha_n\}_{n=0}^{\infty} \longrightarrow \alpha$ *and* $\{T^{n+q(n)}y_n - x_n\}_{n=0}^{\infty} \longrightarrow L_{yx}$ *then*

$$\{x_{n+1} - x_n\}_{n=0}^{\infty} \longrightarrow \alpha L_{yx}. \tag{41}$$

If $\{\alpha_n\}_{n=0}^{\infty} \longrightarrow \alpha$ *and* $\{T^{n+q(n)}y_n - T^{n-1+q(n-1)}y_{n-1}\}_{n=0}^{\infty} \longrightarrow L_y$ *then*

$$\{x_{n+1} - 2x_n - x_{n-1}\}_{n=0}^{\infty} \longrightarrow \alpha L_y. \tag{42}$$

If $\{\beta_n\}_{n=0}^{\infty} \longrightarrow \beta$ *and* $\{T^{n+1+m(n+1)}x_{n+1} - T^{n+m(n)}x_n\}_{n=0}^{\infty} \longrightarrow L_x$ *then*

$$\{(1-\beta)(x_{n+1} - x_n) - (y_{n+1} - y_n)\}_{n=0}^{\infty} \longrightarrow \beta L_x. \tag{43}$$

If $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty} \longrightarrow \alpha$ *and* $\{T^{n+q(n)}y_n - T^{n+m(n)}x_n\}_{n=0}^{\infty} \longrightarrow L_{xy}$ *then* $\{y_n - x_{n+1}\}_{n=0}^{\infty} \longrightarrow \alpha L_{xy}$.

Proof. Property (i) follows from (39) through simple direct calculations. Properties (ii) to (iv) are a direct consequence of Property (i). Finally, Property (v) follows directly from Properties (ii) to (iv). □

Note that the limits L_x, L_y, L_{yx}, and L_{xy} might be, in general, dependent on x_0. Lemma 7 (v) can be reformulated in the case when L_x, L_y, L_{yx}, and L_{xy} are limit superiors or upper-bounds of the limit superiors rather than limits as follows.

Lemma 8. *The following properties hold when the generalized modified Ishikawa's iteration (39) is used:*

(i) *If* $\{\alpha_n\}_{n=0}^{\infty} \longrightarrow \alpha$ *and* $\limsup_{n\longrightarrow\infty} \sup_{x_0 \in A^* \cup B^*} (T^{n+q(n)}y_n - x_n) \le L_{yx}$ *then*

$$\limsup_{n\longrightarrow\infty} \sup_{x_0 \in A^* \cup B^*} \left(x_{n+1} - x_n - \alpha L_{yx}\right) \le 0. \tag{44}$$

(ii) *If* $\{\alpha_n\}_{n=0}^{\infty} \longrightarrow \alpha$ *and* $\limsup_{n\longrightarrow\infty} \sup_{x_0 \in A^* \cup B^*} (T^{n+q(n)}y_n - T^{n-1+q(n-1)}y_{n-1}) \le L_y$ *then*

$$\limsup_{n\longrightarrow\infty} \sup_{x_0 \in A^* \cup B^*} \left(x_{n+1} - 2x_n - x_{n-1} - \alpha L_y\right) \le 0. \tag{45}$$

(iii) *If* $\{\beta_n\}_{n=0}^{\infty} \longrightarrow \beta$ *and* $\limsup_{n\longrightarrow\infty} \sup_{x_0 \in A^* \cup B^*} (T^{n+1+m(n+1)}x_{n+1} - T^{n+m(n)}x_n) \le L_x$ *then*

$$\limsup_{n\longrightarrow\infty} \sup_{x_0 \in A^* \cup B^*} \left((1-\beta)(x_{n+1} - x_n) - (y_{n+1} - y_n) - \beta L_x\right) \le 0. \tag{46}$$

(iv) *If* $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty} \longrightarrow \alpha$ *and* $\limsup_{n\longrightarrow\infty} \sup_{x_0 \in A^* \cup B^*} (T^{n+q(n)}y_n - T^{n+m(n)}x_n) \le L_{xy}$ *then* $\limsup_{n\longrightarrow\infty} \sup_{x_0 \in A^* \cup B^*} (y_n - x_{n+1} - \alpha L_{xy}) \le 0$.

The subsequent result links Lemma 8 with Theorem 3.

Theorem 9. *Assume that all the general assumptions of Theorem 3 and the assumption* $\limsup_{n\longrightarrow\infty} \widetilde{k}(T^{n+1}x_0, T^n x_0) = \widetilde{k}_\infty(x_0) \in [-\widetilde{k}_{00}, \widetilde{k}_{10}]$ *of Theorem 3(iv) hold and, furthermore, that all the sets in the set sequences* $\{A_n\}_{n=0}^{\infty}$ *and* $\{B_n\}_{n=0}^{\infty}$ *in the normed space* $(E, \|.\|)$ *are nonempty and convex. Then, the following properties hold when the generalized modified Ishikawa's iteration (39) is used:*

(i) *There exist real sequences* $\{\alpha_{0n}\}_{n=0}^{\infty} (\subseteq [\alpha_{0m}, 1])$ *and* $\{\beta_{0n}\}_{n=0}^{\infty} (\subseteq [\beta_{0m}, 1])$, *for some* $\alpha_{0m}, \beta_{0m} \in [0,1]$, *such that the sequences* $\{y_n\}$ *and* $\{x_n\}$ *built from (39) by using any parameterizing sequences* $\{\alpha_n\}_{n=0}^{\infty} (\subseteq [0,1])$, $\{\beta_n\}_{n=0}^{\infty} (\subseteq [0,1])$ *being subject to* $\alpha_n \ge \alpha_{0n}$ *and* $\beta_n \ge \beta_{0n}$, $n \ge 0$, *are in* $\bigcup_{n\ge 0}(A_n \times B_{n+1} \cup B_n \times A_{n+1})$ *for any given* $x_0 \in A^* \cup B^*$.

(ii) *Assume, in addition, that* $q(n) = 2\ell(n) + 1 - n$ *for some arbitrary n-dependent integer* $\ell(n) \ge (n-1)/2$, $n \ge 0$, $\{\alpha_n\}_{n=0}^{\infty} \longrightarrow \alpha$. *Then, one gets*

$$\limsup_{n\longrightarrow\infty} \sup_{x_0 \in A^* \cup B^*} \left(\left\|x_{n+1} - x_n\right\| - \frac{\alpha(1-k_0)\left(\overline{D} + M_{0\infty}(x_0)\right)}{1 - k_0 - \widetilde{k}_\infty(x_0)}\right) \le 0. \tag{47}$$

(iii) *Assume, in addition, that* $m(n) = 2z(n) - n$ *and* $q(n) = 2\ell(n) + 1 - n$ *for some arbitrary n-dependent integers* $z(n) \ge n/2$, $n \ge 0$ *and* $\ell(n) \ge (n-1)/2$, $n \ge 0$, $\{\alpha_n\}_{n=0}^{\infty} \longrightarrow \alpha$, $\{\beta_n\}_{n=0}^{\infty} \longrightarrow \beta = \alpha$. *Then, one gets*

$$\limsup_{n\longrightarrow\infty} \sup_{x_0 \in A^* \cup B^*} \left(\left\|y_n - x_{n+1}\right\| - \frac{\alpha(1-k_0)\left(\overline{D} + M_{0\infty}(x_0)\right)}{1 - k_0 - \widetilde{k}_\infty(x_0)}\right) \le 0. \tag{48}$$

Proof. First note from (39) that if $\alpha_n = \beta_n = 1$ for all $n \ge 0$ then

$$y_n = T^{n+m(n)}x_n; \quad x_{n+1} = T^{n+q(n)}y_n = T^{2n+m(n)+q(n)}x_n; \quad n \ge 0 \tag{49}$$

for any $x_0 \in A^* \cup B^*$. Thus, one has

(a) if $n + q(n)$ is odd then y_n and x_{n+1} are either in some $A_{j(n)} \in \bigcup_{n=0}^{\infty} A_n$ and in some $B_{\ell(n+q(n))} \in \bigcup_{n=0}^{\infty} B_n$, respectively, or in some $B_{j(n)} \in \bigcup_{n=0}^{\infty} A_n$ and some $A_{\ell(n+q(n))} \in \bigcup_{n=0}^{\infty} B_n$. If $n+q(n)$ is even then x_{n+1} and y_n are in some $A_{j(n)} \in \bigcup_{n=0}^{\infty} A_n$ and in some $A_{\ell(n+q(n))} \in \bigcup_{n=0}^{\infty} A_n$, respectively, or in some $B_{j(n)} \in \bigcup_{n=0}^{\infty} B_n$ and in some $B_{\ell(n+q(n))} \in \bigcup_{n=0}^{\infty} B_n$, respectively.

(b) If $n + m(n)$ is odd then x_n and y_n are either in some $A_{j(n)} \in \bigcup_{n=0}^{\infty} A_n$ and in some $B_{\ell(n+m(n))} \in \bigcup_{n=0}^{\infty} A_n$, respectively, or in some $B_{j(n)} \in \bigcup_{n=0}^{\infty} B_n$ and in some

$A_{\ell(n+m(n))} \in \bigcup_{n=0}^{\infty} B_n$, respectively. If $n + m(n)$ is even then x_n and y_n are either in some $A_{j(n)} \in \bigcup_{n=0}^{\infty} A_n$ and in some $A_{\ell(n+m(n))} \in \bigcup_{n=0}^{\infty} A_n$, respectively, or in some $B_{j(n)} \in \bigcup_{n=0}^{\infty} B_n$ and in some $B_{\ell(n+m(n))} \in \bigcup_{n=0}^{\infty} B_n$, respectively.

(c) If $2n + m(n) + q(n)$ is odd then x_n and x_{n+1} are either in some $A_{j(n)} \in \bigcup_{n=0}^{\infty} A_n$ and in some $B_{j(2n+m+q)} \in \bigcup_{n=0}^{\infty} B_n$, respectively, or in some $B_{j(n)} \in \bigcup_{n=0}^{\infty} B_n$ and in some $A_{j(2n+m+q)} \in \bigcup_{n=0}^{\infty} A_n$, respectively. If $2n+m(n)+q(n)$ is even then x_n and x_{n+1} are either in some $A_{j(n)} \in \bigcup_{n=0}^{\infty} A_n$ and in some $A_{j(2n+m+q)} \in \bigcup_{n=0}^{\infty} A_n$, respectively, or in some $B_{j(n)} \in \bigcup_{n=0}^{\infty} B_n$ and in some $B_{j(2n+m+q)} \in \bigcup_{n=0}^{\infty} B_n$, respectively.

Then, $\{x_n\}, \{y_n\} \subset \bigcup_{n\geq 0}(A_n \times B_{n+1} \cup B_n \times A_{n+1})$ if $\alpha_n = \beta_n = 1$ for all $n \geq 0$ since $\{A_n\}_{n=0}^{\infty}$ and $\{B_n\}_{n=0}^{\infty}$ are convex for any given $x_0 \in A^* \cup B^*$. It turns out that there exist real sequences $\{\alpha_{0n}\}_{n=0}^{\infty}(\subseteq [\alpha_{0m}, 1])$ and $\{\beta_{0n}\}_{n=0}^{\infty}(\subseteq [\beta_{0m}, 1])$, for some $\alpha_{0m}, \beta_{0m} \in [0, 1]$, such that the sequences $\{y_n\}$ and $\{x_n\}$ built from (39) by using any parameterizing sequences $\{\alpha_n\}_{n=0}^{\infty}(\subseteq [0, 1])$, $\{\beta_n\}_{n=0}^{\infty}(\subseteq [0, 1])$ subject to $\alpha_n \geq \alpha_{0n}$ and $\beta_n \geq \beta_{0n}$, $n \geq 0$, are in $\bigcup_{n\geq 0}(A_n \times B_{n+1} \cup B_n \times A_{n+1})$ for any given $x_0 \in A^* \cup B^*$. Property (i) has been proved.

On the other hand, since $q(n) = 2\ell(n) + 1 - n$ for some arbitrary integer $\ell(n) \geq (n-1)/2$, $n+q(n)$ is odd. Then, $\{x_{n+1}\}$ and $\{x_n\}$ are in distinct convex unions $\bigcup_{n\geq 0} A_n$ and $\bigcup_{n\geq 0} B_n$. Thus, the result follows from Lemma 8 (i) and Theorem 3(iv). Property (ii) has been proved.

On the other hand, since $m(n) = 2z(n) - n$ and $q(n) = 2\ell(n) + 1 - n$ for some arbitrary integers $z(n) \geq n/2$ and $\ell(n) \geq (n-1)/2$, $n + m(n)$ is even and $n + q(n)$ is odd. Then, $\{x_{n+1}\}$ and $\{y_n\}$are in distinct convex unions $\bigcup_{n\geq 0} A_n$ and $\bigcup_{n\geq 0} B_n$. Thus, the result follows from Lemma 8 (iv) and Theorem 3(iv). Property (iii) has been proved. □

If the computational disturbances are asymptotically removed under the conditions of Theorem 3(iii), one gets the following results from Theorem 9 and Remark 5.

Corollary 10. *Assume that all the assumptions of Theorem 9 hold and, furthermore,* $\overline{D} \geq D/\alpha$, $\liminf_{n\to\infty} D_n \geq \underline{D}$, *and* $\{\tilde{k}(T^{n+1}x_0, T^n x_0)\}_{n=0}^{\infty} \longrightarrow 0$ *and* $\{M_n(x_0)\}_{n=0}^{\infty} \longrightarrow 0$, $\forall x_0 \in A^* \cup B^*$. *Then*

$$\begin{aligned}
\underline{D} &\leq \liminf_{n\to\infty} \inf_{x_0 \in A^*\cup B^*} \left(\left\|x_{n+1} - x_n\right\|\right) \\
&\leq \limsup_{n\to\infty} \sup_{x_0 \in A^*\cup B^*} \left(\left\|x_{n+1} - x_n\right\|\right) \leq \alpha\overline{D} \\
\underline{D} &\leq \liminf_{n\to\infty} \inf_{x_0 \in A^*\cup B^*} \left(\left\|x_{n+1} - y_n\right\|\right) \\
&\leq \limsup_{n\to\infty} \sup_{x_0 \in A^*\cup B^*} \left(\left\|x_{n+1} - y_n\right\|\right) \leq \alpha\overline{D}
\end{aligned} \tag{50}$$

if $m(n) = 2z(n) - n$ *and* $q(n) = 2\ell(n) + 1 - n$ *for some arbitrary n-dependent integers* $z(n) \geq n/2$ *and* $\ell(n) \geq (n-1)/2$, $n \geq 0$, $\{\alpha_n\}_{n=0}^{\infty} \longrightarrow \alpha$ *and* $\{\beta_n\}_{n=0}^{\infty} \longrightarrow \beta = \alpha$.

Proof. It follows from Theorem 3(iii), Remark 5, and Theorem 9. □

5. Generalized Modified Ishikawa's Iterative Scheme, Uncertain Cyclic Self-Mappings, and Best Proximity Points

This section relies on the study of further properties concerning the limit best positivity points under the generalized modified Ishikawa's iterative scheme studied in Section 4 being ran by the uncertain cyclic self-mapping of Section 3. Some basic results are given in this section about limit best proximity points and the convergence of sequences generated by cyclic self-maps of Sections 3-4 to them. It is assumed that the set-theoretic limits below of the sequences of sets $\{A_n\}_{n=0}^{\infty}$ and $\{B_n\}_{n=0}^{\infty}$ in the normed space $(E, \|.\|)$ exist:

$$\begin{aligned}
A_\infty &= \lim_{n\to\infty} \bigcup_{n\geq 1}\bigcap_{j\geq n}\left\{A_j\right\} = \liminf_{n\to\infty} \bigcup_{n\geq 1}\bigcap_{j\geq n}\left\{A_j\right\} \\
&= \limsup_{n\to\infty} \bigcap_{n\geq 1}\bigcup_{j\geq n}\left\{A_j\right\}, \\
B_\infty &= \lim_{n\to\infty} \bigcup_{n\geq 1}\bigcap_{j\geq n}\left\{B_j\right\} = \liminf_{n\to\infty} \bigcup_{n\geq 1}\bigcap_{j\geq n}\left\{B_j\right\} \\
&= \limsup_{n\to\infty} \bigcap_{n\geq 1}\bigcup_{j\geq n}\left\{B_j\right\}.
\end{aligned} \tag{51}$$

We denote $\{A_n\}_{n=0}^{\infty} \longrightarrow A_\infty$ and $\{B_n\}_{n=0}^{\infty} \longrightarrow B_\infty$ and the distance between the limit sets is $d_\infty = d(A_\infty, B_\infty) = d(clA_\infty, clB_\infty)$, the distance between points x and y in E being identified with the norm of $z = x - y$ in the linear space E. The sets A_∞ and B_∞ are said to be the set-theoretic limits of the respective sequences $\{A_n\}_{n=0}^{\infty}$ and $\{B_n\}_{n=0}^{\infty}$. It is well known that a set-theoretic limit is not guaranteed to be closed even if the involved set sequence consists of closed sets (in fact, note that the union of infinitely many closed sets is not necessarily closed). Consider a norm-induced distance $d : E \times E \longrightarrow \mathbf{R}_{0+}$ in $(E, \|.\|)$ defined by $d(x, y) = \|x - y\|$, $\forall x, y \in E$ such that for any nonempty subsets A and B of E, one has

$d(x, A) = \inf_{y\in A}\|x - y\|$, $\forall x \in E$ and $d(A, B) = \inf_{x\in A, y\in B}\|x - y\|$. Define

$$\begin{aligned}
P_{EA}(x) &= \{y \in E : d(x, y) = d(x, A)\}; \quad \forall x \in E, \\
P_A(x) &= \{y \in A : d(x, y) = d(x, A)\}; \quad \forall x \in E, \\
A_0 &= \{y \in A : d(x, y) = d(A, B)\}.
\end{aligned} \tag{52}$$

Then, $P_A(B_0) = \{y \in A : d(x, y) = d(A, B_0)\}; \forall x \in E$. Similarly we can define $B_0 = \{y \in B : d(x, y) = d(A, B)\}$ and $P_B(A_0) = \{y \in B : d(x, y) = d(B, A_0)\}; \forall x \in E$. See [1, 5]. The sets A_0 and B_0 are referred to as the sets of best proximity points (or best proximity sets) of A and B, respectively.

Lemma 11. *Let* $(X, \|.\|)$ *be a reflexive Banach space, let* $\{A_n\}_{n=0}^{\infty}$ *and* $\{B_n\}_{n=0}^{\infty}$ *be monotonically nonincreasing sequences of nonempty, closed, bounded, and convex subsets of X (i.e.,* $A_i \supseteq A_{i+1}, B_i \supseteq B_{i+1}; i \geq 0$*). Then, the set-theoretic limits* A_∞ *and* B_∞ *exist; i.e.,* $\{A_n\}_{n=0}^{\infty} \longrightarrow A_\infty$ *and* $\{B_n\}_{n=0}^{\infty} \longrightarrow B_\infty$, *and they are nonempty, closed, bounded, and convex sets, and the limit* *best proximity sets $A_{0\infty}$ and $B_{0\infty}$ are nonempty and satisfy $P_B(A_{0\infty}) \subseteq B_{0\infty}$ and $P_A(B_{0\infty}) \subseteq A_{0\infty}$.*

Proof. It follows that A_∞ and B_∞ exist, are given by $A_\infty = \bigcap_{n\geq 0} A_n$, $B_\infty = \bigcap_{n\geq 0} B_n$, from the identities (51), and are nonempty closed, bounded, and convex sets since $\{A_n\}_{n=0}^{\infty}$ and $\{B_n\}_{n=0}^{\infty}$ are monotonically nonincreasing sequences of nonempty, closed, bounded, and convex sets of a reflexive Banach space. Then, it follows from Lemma 2.1 ([1], see also [5]) that the sets of best proximity points $A_{0\infty}$ and $B_{0\infty}$ of the set-theoretic limits A_∞ and B_∞ are nonempty and satisfy $P_B(A_{0\infty}) \subseteq B_{0\infty}$ and $P_A(B_{0\infty}) \subseteq A_{0\infty}$. □

It turns out that if $\{A_n\}_{n=0}^{\infty}$ and $\{B_n\}_{n=0}^{\infty}$ are not monotonically nonincreasing sequences of nonempty, closed, bounded, and convex subsets of X, it is not guaranteed that the identities (51) hold and also that, even if they hold, so that the set-theoretic limits A_∞ and B_∞ exist, such sets are bounded, closed, and convex even if the members of the sequences of sets are bounded, closed, and convex. Note that the unions of infinitely many sets do not necessarily keep the properties of boundedness, closeness, and convexity of the elements of the sequences and such unions are invoked in the identities (51) provided that they hold. Therefore, the assumption that the limits A_∞ and B_∞ exist and are bounded, closed, and convex has to be made explicitly as addressed in the subsequent more general result than Lemma 11.

Lemma 12. *Let $(X, \|.\|)$ be a reflexive Banach space, let $\{A_n\}_{n=0}^{\infty}$ and $\{B_n\}_{n=0}^{\infty}$ be sequences of nonempty bounded convex subsets of X such that the set-theoretic limits A_∞ and B_∞ exist; i.e., the identities (51) hold. Assume that $A_\infty$and B_∞ are nonempty, bounded, and convex sets. Then, the limit best proximity sets $A_{0c\infty}$ and $B_{0c\infty}$ of the closures of the set-theoretic limits A_∞ and B_∞, that is, clA_∞ and clB_∞, are nonempty and satisfy $P_B(A_{0c\infty}) \subseteq B_{0c\infty}$ and $P_A(B_{0c\infty}) \subseteq A_{0c\infty}$.*

The conditions of Lemma 11 for one of the sequences of sets together with the less restrictive conditions of Lemma 12 for the other sequences lead to the subsequent result.

Lemma 13. *Let $(X, \|.\|)$ be a reflexive Banach space. Let $\{A_n\}_{n=0}^{\infty}$ be a monotonically nonincreasing sequence of nonempty, closed, bounded, and convex subsets of X. Let $\{B_n\}_{n=0}^{\infty}$ be a sequence of nonempty, closed, and convex subsets of X which satisfies the second identity of (51). Then, the nonempty set-theoretic limits A_∞ (being nonempty, closed, bounded, and convex) and B_∞ exist. Then, if B_∞ is nonempty, closed, and convex, then the limit best proximity sets $A_{0\infty}$ and $B_{0c\infty}$ are nonempty and satisfy $P_B(A_{0\infty}) \subseteq B_{0c\infty}$ and $P_A(B_{0c\infty}) \subseteq A_0\infty$.*

Proof. It follows from Lemmas 11 and 12 that A_∞ exists since $\{A_n\}_{n=0}^{\infty}$ is monotonically nonincreasing and it is nonempty, closed, bounded, and convex and B_∞ exists and it is nonempty, closed, bounded, and convex. □

Conditions of nonemptiness of the best proximity sets $A_{0\infty}$ and $B_{0\infty}$ are given in the next result.

Lemma 14. *Let $(E, \|.\|)$ be a normed space and $\{A_n\}_{n=0}^{\infty}$ and $\{B_n\}_{n=0}^{\infty}$ two sequences of sets of E. Then, the set-theoretic limits A_∞ and B_∞ of the sequences $\{A_n\}_{n=0}^{\infty}$ and $\{B_n\}_{n=0}^{\infty}$ exist and their sets of best proximity points $A_{0\infty}$ and $B_{0\infty}$ are nonempty if any of the following constraints hold:*

(**1**) $\{A_n\}_{n=0}^{\infty}$ is monotonically nonincreasing sequence of nonempty, closed, bounded, and convex subsets of E and $\{B_n\}_{n=0}^{\infty}$ is a sequence of nonempty subsets of E which satisfies the second identity of (51) with set-theoretic limit B_∞ being approximatively compact with respect to A_∞.

(**2**) $\{B_n\}_{n=0}^{\infty}$ is monotonically nonincreasing sequence of nonempty, closed, bounded, and convex subsets of E and $\{A_n\}_{n=0}^{\infty}$ is a sequence of nonempty subsets of E which satisfies the first identity of (51) with set-theoretic limit A_∞ being approximatively compact with respect to B_∞.

Proof. Since $\{A_n\}_{n=0}^{\infty}$ is a monotonically nonincreasing sequence of nonempty, closed, bounded, and convex subsets of E, then the set-theoretic limit A_∞ of $\{A_n\}_{n=0}^{\infty}$ exists; it is nonempty and compact. Since $\{B_n\}_{n=0}^{\infty} \longrightarrow B_\infty$ (i.e., the second identity of (51) is satisfied) and the set-theoretic limit B_∞ of $\{B_n\}_{n=0}^{\infty}$ is nonempty and approximatively compact with respect to A_∞ then any sequence $\{x_n\}_{n=0}^{\infty} \subset B_\infty$, such that $\{\|y-x_n\|\}_{n=0}^{\infty} \longrightarrow \inf_{\sigma\in B_\infty}\|y-\sigma\| = d(y, B_\infty)$ for $y \in A_\infty$ has a convergent subsequence $\{x_{nk}\}_{k=0}^{\infty}(\subset B_\infty) \longrightarrow z$, [1]. Then, $y \in A_\infty$ can be chosen such that the limit z of $\{x_{nk}\}_{k=0}^{\infty}$ is such that $\|y - z\| = d(A_\infty, B_\infty) = \inf_{\mu\in A_\infty, \nu\in B_\infty}\|\mu - \nu\|$. Therefore, $y \in A_{0\infty}$ and $z \in B_{0\infty}$ so that and $A_{0\infty}$ and $B_{0\infty}$ are nonempty. The result has been proved for the first set of constraints. The proof under the second set of constraint follows by duality. □

Auxiliary technical results to be then used are summarized in the result which follows.

Theorem 15. *Let $(X, \|.\|)$ be a uniformly convex Banach space, let $\{A_n\}_{n=0}^{\infty}$ and $\{B_n\}_{n=0}^{\infty}$ be monotonically nonincreasing sequences of nonempty, closed, and convex subsets of X. Let $\{x_n\}_{n=0}^{\infty}$ and $\{z_n\}_{n=0}^{\infty}$ be sequences in A_∞ and $\{y_n\}_{n=0}^{\infty}$, a sequence in B_∞. Then, the following properties hold:*

(i) *Assume that $\{\|z_n - y_n\|\}_{n=0}^{\infty} \longrightarrow d(A_\infty, B_\infty)$ and that for every $\varepsilon > 0$ there exists N_0 such that for all $m > n \geq N_0$, $\{\|x_m - y_n\|\}_{n=0}^{\infty} \leq d(A_\infty, B_\infty) + \varepsilon$. Then, for every $\varepsilon > 0$ there exists N_1 such that for all $m > n \geq N_1$, $\{\|x_m - y_n\|\}_{n=0}^{\infty} \leq \varepsilon$.*

(ii) *If $\{\|z_n-y_n\|\}_{n=0}^{\infty} \longrightarrow d(A_\infty, B_\infty)$ and $\{\|x_n-y_n\|\}_{n=0}^{\infty} \longrightarrow d(A_\infty, B_\infty)$ then $\{\|x_n - z_n\|\}_{n=0}^{\infty} \longrightarrow 0$.*

(iii) *If $\{\|x_n - y\|\}_{n=0}^{\infty} \longrightarrow d(A_\infty, B_\infty)$ for some $y \in B_\infty$ then $\{d(x_n, P_{A_\infty}(y))\}_{n=0}^{\infty} \longrightarrow 0$, $y \in B_{0\infty}$ and $\{d(x_n, P_{A_\infty}(B_{0\infty}))\}_{n=0}^{\infty} \longrightarrow 0$ and $\{d(x_n, A_{0\infty})\}_{n=0}^{\infty} \longrightarrow 0$.*

Proof. Since $(X, \|.\|)$ is a uniformly convex Banach space then it is reflexive. From Lemma 11, the set- theoretic limits A_∞ and B_∞ exist, i.e., $\{A_n\}_{n=0}^{\infty} \longrightarrow A_\infty$ and $\{B_n\}_{n=0}^{\infty} \longrightarrow B_\infty$, and they are nonempty, closed, and convex sets whose nonempty best proximity sets $A_{0\infty}$ and $B_{0\infty}$ satisfy $P_B(A_{0\infty}) \subseteq B_{0\infty}$, so that $B_{0\infty}$ is nonempty and $P_A(B_{0\infty}) \subseteq A_{0\infty}$ and $d(A_\infty, B_\infty) = d(A_{0\infty}, B_{0\infty})$. Now, Property (i), Property (ii), and Property

(iii) follow, respectively, from Lemma 3.7, Lemma 3.8, and Corollary 3.9 of [1]. □

Now, we address some convergence conditions of sequences generated by the cyclic self-mapping T on $\bigcup_{n\geq 0}(A_n \times B_{n+1} \cup B_n \times A_{n+1})$ under condition (18), which becomes contractive in the perturbation-free case, provided that some limiting conditions are fulfilled by the perturbations.

Theorem 16. *Let $(X, \|.\|)$ be a uniformly convex Banach space and consider the monotonically nonincreasing sequences $\{A_n\}_{n=0}^{\infty}$ and $\{B_n\}_{n=0}^{\infty}$ of nonempty, closed, and convex subsets of X and let T be a cyclic self-mapping on $\bigcup_{n\geq 0}(A_n \times B_{n+1} \cup B_n \times A_{n+1})$ being subject to all the general assumptions of Theorem 3 including the further assumption of Theorem 3 (iv).*

Then, the following properties hold:

***(i)** It follows that*

$$\begin{aligned} D_{\infty} &\leq \liminf_{n\longrightarrow\infty} \inf_{x_0\in A^*\cup B^*} \left\|T^{n+2}x_0 - T^{n+1}x_0\right\| \\ &\leq \limsup_{n\longrightarrow\infty} \sup_{x_0\in A^*\cup B^*} \left\|T^{n+2}x_0 - T^{n+1}x_0\right\| \leq L_T \end{aligned} \quad (53)$$

where $L_T = \sup_{x_0\in A^*\cup B^*}((1-k_0)(\overline{D} + M_{0\infty}(x_0))/(1 - k_0 - \widetilde{k}_{\infty}(x_0)))$ with $M_{\infty}(x_0) = \limsup_{n\longrightarrow\infty} M_n(x_0)$ and $M_{0\infty}(x_0) = \limsup_{n\longrightarrow\infty} M_{0n}(x_0)$, where $M_n(x_0)$ and $M_{0n}(x_0)$ are defined in (19) and (24), respectively, for all $n \geq 0$. Furthermore, the subsequent chain of inequalities is true:

$$\begin{aligned} L_T &\geq \overline{D} \geq D^* = d\left(A^*, B^*\right) = d\left(A_0, B_0\right) \geq D_{\infty} \\ &= d\left(A_{\infty}, B_{\infty}\right) \end{aligned} \quad (54)$$

***(ii)** Assume that*

$$\begin{aligned} &\overline{D} \\ &= \sup_{x_0\in A^*\cup B^*}\left(\frac{D_{\infty}\left(1-k_0-\widetilde{k}_{\infty}\left(x_0\right)\right)}{1-k_0} - M_{0\infty}\left(x_0\right)\right) \\ &\geq D^* \end{aligned} \quad (55)$$

and, furthermore, assume also that the assumptions of Theorem 9 (ii) hold with $\{\alpha_n\}_{n=0}^{\infty} \longrightarrow \alpha(=1)$ for the generalized modified Ishikawa's iterative scheme (39). Then, $\{\|x_{n+1} - x_n\|\}_{n=0}^{\infty} \longrightarrow D_{\infty}$ and $\{\|x_{2n+2} - x_{2n}\|\}_{n=0}^{\infty} \longrightarrow 0$ for any $x_0 \in A^* \cup B^*$. If, in addition, the assumptions of Theorem 9 (iii) hold with $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty} \longrightarrow \alpha(=\beta=1)$ then $\{\|x_{n+1} - y_n\|\}_{n=0}^{\infty} \longrightarrow D_{\infty}$ and $\{\|y_{2n+2} - y_{2n}\|\}_{n=0}^{\infty} \longrightarrow 0$ for any $x_0 \in A^* \cup B^*$.

Proof. One has from Theorem 3 (iv) that (53) holds.

$$\begin{aligned} D_{\infty} &\leq \liminf_{n\longrightarrow\infty} \inf_{x_0\in A^*\cup B^*} \left\|T^{n+2}x_0 - T^{n+1}x_0\right\| \\ &\leq \limsup_{n\longrightarrow\infty} \sup_{x_0\in A^*\cup B^*} \left\|T^{n+2}x_0 - T^{n+1}x_0\right\| \leq L_T; \\ &\qquad \forall x_0 \in A^* \cup B^*. \end{aligned} \quad (56)$$

Property (i) is proved as follows. The reference distance fulfils $\overline{D} \geq D^*$ by hypothesis of Theorem 3. $D^* = d(A^*, B^*) = d(A_0, B_0) \geq d(A_{\infty}, B_{\infty})$ since $\{A_n\}_{n=0}^{\infty}$ and $\{B_n\}_{n=0}^{\infty}$ are monotonically nonincreasing sequences under set inclusion of nonempty closed sets so that there exist the set- theoretic limits $A_{\infty} = \bigcap_{n\geq 0} A_n$ and $B_{\infty} = \bigcap_{n\geq 0} B_n$ which are nonempty and closed. The inequality $L_T \geq \overline{D}$ holds always for $\overline{D} \neq 0$ and it is now proved by contradiction for the case $\overline{D} \neq 0$. Assume that $\overline{D} \neq 0$ and $0 \leq L_T = \sup_{x_0\in A^*\cup B^*}((1 - k_0)(\overline{D} + M_{0\infty}(x_0))/(1 - k_0 - \widetilde{k}_{\infty}(x_0))) < \overline{D}$ and $\widetilde{k}_{\infty}(x_0) = \limsup_{n\longrightarrow\infty} \widetilde{k}(T^{n+1}x_0, T^n x_0) \in [-\widetilde{k}_{00}, \widetilde{k}_{10}]$, $\forall x_0 \in A^* \cup B^*$, since $\widetilde{k}_{\infty}(x_0) \in [0, 1 - k_0)$ for all $x_0 \in A^* \cup B^*$and $k_0 \in [0,1)$. Then, there exists some $x_0 \in A^* \cup B^*$ such that $0 \leq (1 - k_0)M_{0\infty} < -\widetilde{k}_{\infty}(x_0) \leq 0$ leading to the contradiction $0 < 0$. Thus $L_T \geq \overline{D}$ and Property (i) has been proved. The proof of Property (ii) follows directly from Theorem 3 (iv) under the hypotheses of Theorem 9 [(ii)-(iii)], by using the results of Theorem 15 [(ii)-(iii)] since the upper-bound of $\limsup_{n\longrightarrow\infty}\|T^{n+2}x_0 - T^{n+1}x_0\|$ becomes exactly a limit being equal to D_{∞}(see (53)). □

The following result is an "ad hoc" extension from Theorem 3.10 of [1] for this problem under the given results and the relevant related assumptions.

Theorem 17. *Let $(X, \|.\|)$ be a uniformly convex Banach space and consider the monotonically nonincreasing sequences $\{A_n\}_{n=0}^{\infty}$ and $\{B_n\}_{n=0}^{\infty}$ of nonempty, closed, and convex subsets of X and let T be a cyclic self-mapping on $\bigcup_{n\geq 0}(A_n \times B_{n+1} \cup B_n \times A_{n+1})$ being subject to all the general assumptions of Theorem 3 and the further assumption of Theorem 3 (iv). Assume also*

(a) $\overline{D} = \sup_{x_0\in A^*\cup B^*}(D_{\infty}(1 - k_0 - \widetilde{k}_{\infty}(x_0))/(1 - k_0) - M_{0\infty}(x_0)) \geq D^*$,

(b) the assumptions of Theorem 9 (ii) hold for the generalized modified Ishikawa's iterative scheme (39) with $\{\alpha_n\}_{n=0}^{\infty} \longrightarrow \alpha(=1)$.

Then,

(1) Any sequence $\{x_{2n}\}_{n=0}^{\infty}$ generated from the generalized modified Ishikawa's iterative scheme (39) for any given $x_0 \in A^*$ is convergent to $x_{A\infty}$ which is the unique best proximity point of A_{∞} (the set-theoretic limit of $\{A_n\}_{n=0}^{\infty}$). Furthermore, $\{x_{2n+1}\}_{n=0}^{\infty} \longrightarrow x_{B\infty}$ the set-theoretic limit of $\{B_n\}_{n=0}^{\infty}$).

(2) Any sequence $\{x_{2n}\}_{n=0}^{\infty}$ generated from the generalized modified Ishikawa's iterative scheme (39) for any given $x_0 \in B^*$ is convergent to $x_{B\infty}$ and $\{x_{2n+1}\}_{n=0}^{\infty} \longrightarrow x_{A\infty}$.

(3) $\{x_{2(n+m)} - x_{2n}\}_{n=0}^{\infty} \longrightarrow 0$ for any positive integer m and any given $x_0 \in A^* \cup B^*$.

Proof. Under the assumptions, the set-theoretic limits A_{∞} and B_{∞} of the sequences $\{A_n\}_{n=0}^{\infty}$ and $\{B_n\}_{n=0}^{\infty}$ exist with $d(A_{\infty}, B_{\infty}) = D_{\infty}$ and are nonempty, closed, and convex since they are the intersections of infinitely many subsets ordered in a monotonically nonincreasing sequence which are all nonempty, closed, and convex. Then, $\{\|x_{n+1} - x_n\|\}_{n=0}^{\infty} \longrightarrow D_{\infty}$ and $\{\|x_{2n+2} - x_{2n}\|\}_{n=0}^{\infty} \longrightarrow 0$ for any $x_0 \in A^* \cup B^*$ from Theorem 16 (ii), $\forall x_0 \in A^* \cup B^*$. If $x_{A\infty}$ and $x_{B\infty}$ are best proximity points then $\{\|x_{n+1} - x_n\|\}_{n=0}^{\infty} \longrightarrow \|x_{A\infty} -$

$x_{B\infty}\|$ and $\{\|x_{2(n+m)} - x_{2n}\|\}_{n=0}^{\infty} \longrightarrow 0$ for any given integer $m \geq 1$ and $x_{A\infty}$ and $x_{B\infty}$ are the unique best proximity points in A_∞ and B_∞ from the convexity of the set-theoretic limits to some of them all the sequences $\{x_{2n}\}_{n=0}^{\infty}$ depending on the initial point being in A^* or in B^*. □

Remark 18. Assume the hypotheses of Theorem 17 except that the sets of one of the sequences $\{A_n\}_{n=0}^{\infty}$ or $\{B_n\}_{n=0}^{\infty}$ are not convex. Then, the uniqueness of the best proximity point in the convex set-theoretic limit of one of the sequences is guaranteed and it is a limit of the subsequences (with either even or odd subscript), depending on the initial point allocation, of any generated subsequence. Since the self-mapping T is single-valued the best proximity point, the complementary subsequence (with either odd or even subscript) also converges to a best proximity point of the other eventually nonconvex set-theoretic limit even if such a set has more than one best proximity point.

Data Availability

The underlying data to support this study are included within the references.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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