# Joinings and relative ergodic properties of W*-dynamical systems

by

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# Declaration

I, Malcolm Bruce King, declare that the dissertation, which I hereby submit for the degree Philosophiae Doctor at the University of Pretoria, is my own work and has not previously been submitted by me for a degree at this or any other tertiary institution.

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# Joinings and relative ergodic properties of W*-dynamical systems

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Degree: Philosophiae Doctor

## Summary

We prove a characterization of relative weak mixing in W*-dynamical systems in terms of a relatively independent joining. We then define a noncommutative version of relative discrete spectrum, show that it generalizes both the classical and noncommutative absolute cases and give examples.

Chapter 1 reviews the GNS construction for normal states, the related semicyclic representation on von Neumann algebras, Tomita-Takasaki theory and conditional expectations. This will allow us to define, in the tracial case, the basic construction of Vaughan Jones and its associated lifted trace. Dynamics is introduced in the form of automorphisms on von Neumann algebras, represented using the cyclic and separating vector and then extended to the basic construction.

In Chapter 2, after introducing a relative product system, we discuss relative weak mixing in the tracial case. We give an example of a relative weak mixing W*-dynamical system that is neither ergodic nor asymptotically abelian, before proving the aforementioned characterization.

Chapter 3 defines relative discrete spectrum as complementary to relative weak mixing. We motivate the definition using work from Chapter 2. We show that our definition generalizes the classical and absolute noncommutative case of isometric extensions and discrete spectrum, respectively. The first example is a skew product of a classical system with a noncommutative one. The second is a purely noncommutative example of a tensor product of a W*-dynamical system with a finite-dimensional one.

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# Contents

# Introduction

Ergodic theory has its origins in statistical mechanics, a subfield of physics (see, for instance, [TKS92]) and concerns itself with abstract dynamical systems. Classically this is a measure space together with a measure-preserving group action. There are a number of textbooks available on the topic, for instance, [Pet89], [Fur14] and [Gla03]. Ergodic theory has made an impact within mathematics itself. We mention, for example, the work of Furstenberg in number theory ([Fur14]).

Of particular interest is the study of the long term behaviour of systems encapsulated in two complementary properties, that of “mixing” and “compactness”. The former term describes systems whose orbits “fill up” the phase space, whereas the latter refers to those systems whose orbits are relatively compact.

In this thesis, we work in a noncommutative framework. The measure space is replaced by a von Neumann algebra with a normal faithful state and the dynamics is given by a state-preserving $*$-automorphism. We will examine weak mixing and discrete spectrum relative to a subsystem. In Chapter 2, we study relative weak mixing using a relative product system, defined using a special type of state called a relatively independent joining. This will allow us to obtain a new result which appeared in [DK19]. It is a noncommutative and relative version of the following classical characterization: a measure-preserving system is weak mixing if and only if the system formed by its Cartesian product with itself is ergodic.

The basic construction together with its trace, is an important tool throughout this thesis. Its use will lead us naturally to a definition of relative discrete spectrum, explored in Chapter 3.

We end the introduction with some remarks. Two indexes, one for symbols and the other for terms appear at the end of the thesis. With the exception of tensor products (which we refer the reader to [WO93, Appendix T], and [KR97b, Section 11.2]), Chapter 1 and Appendix A summarize what we require of the following fundamental topics:

- von Neumann algebras, their normal states and tracial weights;

- The GNS- and semicyclic representations;
- Tomita-Takesaki theory for states;
- Conditional expectations of von Neumann algebras;
- The basic construction and its trace;
- Direct integrals of Hilbert spaces and von Neumann algebras;
- Algebraic tensor products and von Neumann tensor products.

The contents of Appendix A is used after Remark 3.3.2. In contrast, the other items on the list are used throughout the thesis.

Sections are numbered in the form $a.b$ where $a$ is the chapter (when $a = A$ we are referring to the appendix). Results such as theorems and propositions are indexed in the form $a.b.c$ where $a.b$ is the section.

Lastly, for the non-expert, we recommend acquaintance with the basics of functional analysis and von Neumann algebras. As an example, [Kre78, Chapters 1-3] and the content of [Zhu93] (up to and including Chapter 18) should suffice.

# Chapter 1

# Background

This first chapter is a modification of the first two chapters of the author's MSc [Kin17]. Sections 1.1 to 1.4 are meant as a summary to establish notation, conventions and remind the reader of some terminology. We discuss a very important tool for this thesis, the basic construction and its trace in Section 1.5. Beginning with Section 1.6, the rest of the chapter discusses noncommutative dynamical systems theory.

## 1.1 Faithful Normal States and Cyclic and Separating Vectors

**Proposition 1.1.1.** *Let $\mu$ be a normal state on a von Neumann algebra $A$. Then there exists a triple $(H, \pi, \Omega)$ consisting of a Hilbert space $H$ carrying a normal (i.e. $\sigma$-weakly continuous) representation $\pi$ of $A$ and a distinguished cyclic vector $\Omega$ of the representation satisfying $\mu(x) = \langle \Omega, \pi(x)\Omega \rangle$ (in this thesis, we take all inner products to be linear in the second coordinate).*

**Proposition 1.1.2.** *([BR02, Proposition 2.5.6]) Suppose $A$ is a von Neumann algebra on a Hilbert space $H$. The following are all equivalent*

1. *$A$ is $\sigma$-finite (i.e all collections of mutually orthogonal projections have at most a countable cardinality).*
2. *there is a countable subset $K$ of $H$ which is separating for $A$ (i.e for any $a \in A$, $ax = 0$ for all $x \in K$ implies $a = 0$.*
3. *there exists a faithful normal state on $A$.*
4. *$A$ is isomorphic with a von Neumann algebra $\pi(A)$ which admits a cyclic and separating vector.*

We emphasize that there are two ways of viewing $A$ (which we will keep distinct in this thesis). One is as a subalgebra of $\mathcal{B}(H)$ (using the GNS representation). The other is as a dense set in $H = \overline{\pi(A)\Omega}$ via the map $a \mapsto \pi(a)\Omega$.

**Remark 1.1.3.** For the remainder of the work, let us reserve the symbols $A$ and $\mu$. We let $A$ be a von Neumann algebra admitting a distinguished normal faithful state $\mu$. We will assume that $A$ is already in its GNS representation (standard representation) acting on the Hilbert space $H$. We let $\Omega$ denote the distinguished cyclic and separating vector obtained from $\mu$.

## 1.2 The Semicyclic Representation

We describe a generalisation of the GNS construction where a normal semifinite tracial weight is used instead of a state. Details can be found in [KR97b] §7.5.

Let $\tau$ be a normal semifinite tracial weight ([KR97b] Definition 7.5.1, [BR02] Definition 2.7.12) on a von Neumann algebra $R$. We consider the quotient space $\mathcal{K}_\tau/K_\tau$, where

$$\mathcal{K}_\tau := \{x \in R \mid \tau(x^*x) < \infty\} \tag{1.2.1}$$

is a left $R$-ideal ([KR97a] Lemma 7.5.2) and $K_\tau := \{x \in R \mid \tau(x^*x) = 0\}$. The quotient map $\gamma_\tau : \mathcal{K}_\tau \to \mathcal{K}_\tau/K_\tau$ sends elements $x \in \mathcal{K}_\tau$ to elements $x + K_\tau$.

Though the tracial weight $\tau$ is only defined on the positive elements of $R$, it can be extended uniquely to a positive hermitian functional on $\mathcal{S}_\tau := \operatorname{span}\{a \in R^+ \mid \tau(a) < \infty\}$. Thus, because $\mathcal{K}_\tau^*\mathcal{K}_\tau = \mathcal{S}_\tau$ ([KR97b] Lemma 7.5.2) the expression $\tau(y^*x)$ makes sense for every $x, y \in \mathcal{K}_\tau$. Hence, $\mathcal{K}_\tau/K_\tau$ has the inner product

$$\langle x + K_\tau, y + K_\tau\rangle := \tau(y^*x), \qquad \text{for all } x, y \in \mathcal{K}_\tau. \tag{1.2.2}$$

Completing $\mathcal{K}_\tau/K_\tau$ in the norm $\|x + K_\tau\|_\tau := \sqrt{\langle x + K_\tau, x + K_\tau\rangle}$ yields a Hilbert space which we denote by $H_\tau$.

The elements of $R$ are represented as operators on $H_\tau$. We first define the action of $r \in R$ on $\mathcal{K}_\tau/K_\tau$

$$\pi_\tau(r)(x + K_\tau) := rx + K_\tau. \tag{1.2.3}$$

The arguments in the GNS construction -see for instance [KR97b] Theorem 4.5.2- give us a unique bounded linear extension with domain $H_\tau$, which we still denote by $\pi_\tau(r)$.

## 1.3 Tomita-Takasaki Theory and Modules

We discuss concepts related to the modular conjugation operator. Details can be found in [BR02, Section 2.5.2.],[KR97b], for instance. The book [Str81] presents the theory for more general weights. For a general introduction to the theory of unbounded linear operators we refer to [Kre78, Chapter 10]. For the polar decomposition for unbounded linear operators we refer the reader to [Sch12].

Consider

$$S_0 : A\Omega \to A\Omega : a\Omega \mapsto a^*\Omega. \tag{1.3.1}$$

Such a map is, in general, not continuous on its domain. However, $S_0$ is closable and its closure is an unbounded linear operator $S$ with domain $A\Omega$. Consequently, we can use the polar decomposition for unbounded operators to find a unique bounded linear anti-unitary operator $J$ and a unique, positive self-adjoint operator $\Delta$ such that $S = J\Delta^{\frac{1}{2}}$ ([KR97b, p. 598]). We call $J$ the *modular conjugation operator* and $\Delta$ the *modular operator*.

If $A' := \{x' \in \mathcal{B}(H) \mid \forall m \in A \quad x'm = mx'\}$ denotes the *commutant* of $A$ in $H$, the closure $T$ of the closable, densely defined linear operator $A'\Omega \to A'\Omega : a'\Omega \to a'^*\Omega$ satisfies $T = J\Delta^{-\frac{1}{2}}$.

We take note of some of the properties of $J$.

**Proposition 1.3.1.** *([BR02, Proposition 2.5.11, p.84], [KR97b, Proposition 9.2.3]) We have the following properties of $J$:*

*(a) For all $\xi, \eta \in H$ we have, $\langle J\xi, J\eta\rangle = \langle \eta, \xi\rangle$; in particular, $J$ is an isometry.*

*(b) $J = J^* = J^{-1}$, where $J^*$ is defined as follows: for every $x, y \in H$,*

$$\langle J^*x, y\rangle = \overline{\langle x, Jy\rangle} = \langle Jy, x\rangle.$$

*(c) $J\Omega = \Omega$.*

*(d) In the case when $\mu$ is a trace, we have that $S_0$ is continuous on $A\Omega$ and therefore can be extended to a bounded linear operator $S$ on $H$. Moreover, in this case, $J = S$.*

The following result is essential throughout the thesis.

**Theorem 1.3.2.** *(Tomita-Takasaki theorem [BR02, Theorem 2.5.14]) If $A$ has a cyclic and separating vector, $J$ is the modular conjugation and $\Delta$ the modular operator, then,*

$$JAJ = A',$$

*and for each* $t \in \mathbb{R}$, $\Delta^{it}$ *is a unitary operator and we have*

$$\Delta^{it} A \Delta^{-it} = A.$$

**Definition 1.3.3.** For each $t \in \mathbb{R}$, let

$$\sigma_t^\mu : A \to A : a \mapsto \Delta^{it} a \Delta^{it}.$$

We call the one-parameter group of automorphism $\{\sigma_t^\mu \mid t \in \mathbb{R}\}$ the *modular automorphism group associated with the pair* $(A, \mu)$.

In much of our work the state will be tracial, in which case $\sigma_t^\mu$ is trivial, i.e. $\sigma_t^\mu = \mathrm{id}_A$.

**Definition 1.3.4.** Let $B$ and $C$ be von Neumann algebras. By a *left-$B$-module* we mean a Hilbert space $K$ with a normal ($\sigma$-weakly) continuous representation $\pi : B \to \mathcal{B}(K)$. If $\pi$ satisfies $\pi(ab) = \pi(a)\pi(b)$, for all $a, b \in B$, we call $K$ a *right-$B$-module.* Call $K$ a *$B$-$C$-module* if $K$ is both a left-$B$-module and right-$C$-module with representations $\pi_l$ and $\pi_r$, respectively, and satisfies

$$\pi_l(b)\pi_r(c)x = \pi_r(c)\pi_l(b)x,$$

for all $x \in K$, $b \in B$ and $c \in C$. We sometimes refer to $\pi_l$ as a *left action* of $B$ on $K$. Similarly, $\pi_r$ is referred to as a *right action* of $C$ on $K$. We usually write $xc$ instead of $\pi_r(c)x$.

We now put

$$j : \mathcal{B}(H) \to \mathcal{B}(H) : a \mapsto Ja^*J. \tag{1.3.2}$$

Using this, we define

$$xa := j(a)x$$

for all $x \in H$ and $a \in A$, making $H$ a right-$A$-module (when we take scalar multiplication on the right as $x\lambda := \bar{\lambda}x$ for all $\lambda \in \mathbb{C}$). Of course, $H$ is already a left-$A$-module by the usual action of $A$ on $H$, so $H$ is in fact a bimodule, but, as will be seen in Section 1.5, it is the right module structure that will be of particular significance for us.

We now set up a "mirror image" $(A', \mu')$ of $(A, \mu)$. Later we will introduce a conditional expectation and dynamics into the picture and mirror that as well (Section 1.8).

Given our von Neumann algebra $(A, \mu)$, carry the state $\mu$ over to $A'$ in a natural way using $j$, by defining a state $\mu'$

$$\mu'(b) := \mu \circ j(b) = \langle \Omega, b\Omega \rangle\,, \tag{1.3.3}$$

for all $b \in A'$.

## 1.4 Conditional Expectations of von Neumann Algebras

**Definition 1.4.1.** ([Tak03b, Definition IX.4.1]) Let $\nu$ be a faithful normal state on a von Neumann algebra $B$ and $C$ be a von Neumann subalgebra of $B$. A linear map $\mathcal{E}$ of $B$ onto $C$ is called the conditional expectation of $B$ onto $C$ with respect to $\nu$ if the following conditions are satisfied:

1. $\|\mathcal{E}(r)\| \leq \|r\|$ for every $r \in B$;
2. $\mathcal{E}(r) = r$ for every $r \in C$;
3. $\nu = \nu \circ \mathcal{E}$.

We take note of the following properties of $\mathcal{E} : B \to C$:

**Proposition 1.4.2.** *([Tak03a, Theorem III.3.4]) Let $r \in B$ and $a, b \in C$. Then*

1. $\mathcal{E}(r^*r) \geq 0$;
2. $\mathcal{E}(arb) = a\mathcal{E}(r)b$;
3. $\mathcal{E}(r)^*\mathcal{E}(r) \leq \mathcal{E}(r^*r)$.

**Theorem 1.4.3.** *( [Tak03b, Theorem IX.4.2]) Let $\nu$ be a faithful normal state on a von Neumann algebra $B$, and let $C$ be a von Neumann subalgebra of $B$. The existence of a conditional expectation $\mathcal{E} : B \to C$ with respect to $\nu$, is equivalent to the global invariance*

$$\sigma_t^\nu(C) = C \qquad t \in \mathbb{R}$$

*of $C$ under the modular automorphism group. If this is the case, then $\mathcal{E}$ is normal and uniquely determined by $\nu$.*

Assume that $F \subseteq A$ is a von Neumann subalgebra of $A$ such that $1_F = 1_A$ and let $\lambda$ be the state on $F$ such that $\lambda = \mu|_F$.

Set

$$\tilde{F} := j(F) \subseteq A' \qquad \tilde{\lambda} := \mu'|_{\tilde{F}}. \tag{1.4.1}$$

We let

$$D : A \to F$$

be the unique conditional expectation from $A$ onto $F$ with respect to $\mu$. Then

$$D' := j \circ D \circ j : A' \to \tilde{F}$$

is the unique conditional expectation such that $\tilde{\lambda} \circ D' = \mu'$. Let

$$P : H \to H \tag{1.4.2}$$

be the projection of $H$ onto

$$H_F := \overline{F\Omega}. \tag{1.4.3}$$

We now take note of a number of results extracted from Section 10.2 of [Str81].

**Proposition 1.4.4.** *(part of [Str81, Section 10.2])*
*The conditional expectation $D : A \to F$ satisfies the equality*

$$PD(a) = D(a)P = PaP,$$

*for all $a \in A$.*

We take note of a special case of Proposition 1.4.4:

$$D(a)\Omega = P(a\Omega). \tag{1.4.4}$$

Similarly, we also have

$$D'(b)\Omega = Pb\Omega \tag{1.4.5}$$

for all $b \in A'$.

**Proposition 1.4.5.** *(Part of [Str81, Section 10.2])*

$$JH_F = H_F \qquad \textit{and} \qquad H_F = \overline{\tilde{F}\Omega}.$$

**Remark 1.4.6.** In the case when $\mu$ is tracial, we reserve the symbol $e_F$ to be the projection from $H = \overline{A\Omega}$ onto $\overline{F\Omega}$. We refer to $e_F$ as the *Jones projection.*

**Lemma 1.4.7** ([SS08] Lemma 3.6.2)**.** *Assume that $\mu$ is tracial. Let $a \in A$. The unique conditional expectation $D : A \to F$ from $A$ onto $F$ with respect to $\mu$ has the following property: for all $x \in H$,*

$$e_F a e_F x = D(a) e_F x = e_F D(a) x.$$

## 1.5 The Basic Construction $\langle A, e_F \rangle$

**Definition 1.5.1.** Assume that $\mu$ is tracial. We consider $\langle A, e_F \rangle$, the smallest von Neumann algebra in $\mathcal{B}(H)$ containing $A$ and $e_F$. We shall refer to $\langle A, e_F \rangle$ as the *basic construction.*

We set

$$Ae_F A := \text{span}\{xe_F y : x, y \in A\}.$$

The equality $JA'J = A$ leads us naturally to consider if $\langle A, e_F \rangle$ can be expressed similarly.

**Proposition 1.5.2.** *([SS08] Lemma 4.2.3 and part of [JS97] Proposition 3.1.2)*

*(a)* $e_F \in F'$.

*(b) The vector space* $Ae_F A$ *is dense in* $\langle A, e_F \rangle$ *in both the weak- and strong operator topologies.*

*(c)* $\langle A, e_F \rangle = JF'J = (JFJ)'$ *and* $\langle A, e_F \rangle' = JFJ$.

*(d)* $\langle A, e_F \rangle$ *is a semifinite von Neumann algebra ([KR97b, pp. 423-424]).*

It is a non-trivial fact that there exists a faithful semifinite normal tracial weight $\overline{\mu}$ on $\langle A, e_F \rangle$ satisfying

$$\overline{\mu}(xe_F y) = \mu(xy), \tag{1.5.1}$$

for all $x, y \in A$, referred to as the trace of $\langle A, e_F \rangle$ (see §4.2 in [SS08]). (Weights are normally only defined on positive elements, but in addition $\overline{\mu}$ is defined on $Ae_F A$). In [AET11] $\bar{\mu}$ is referred to as the *lifted trace.*

The following result is useful for verifying the lifted trace in examples:

**Theorem 1.5.3** (Part of [SS08] Theorem 4.3.11)**.** *Let* $\varphi$ *be a weight on* $\langle A, e_F \rangle$ *with* $\varphi = \overline{\mu}$ *on* $Ae_F A^+ := \{a \in Ae_F A \mid \exists\, b \in Ae_F A : a = b^* b\}$. *If* $\varphi$ *is normal, then* $\varphi = \bar{\mu}$.

There is a form of the lifted trace that we will find useful in Chapter 3:

**Lemma 1.5.4.** *([SS08, Lemma 4.3.4]) Let* $C$ *be a von Neumann subalgebra of a von Neumann algebra* $B$ *with a faithful finite normal trace* $\nu$. *Let* $\{v_i : v_i \in \langle B, e_C \rangle^+\}$ *be a set with index set* $\mathcal{I}$ *satisfying*

$$\sum_{i \in \mathcal{I}} v_i^* e_F v_i = 1. \tag{1.5.2}$$

*Then*

$$\widetilde{\mathrm{Tr}}(t) = \sum_{i\in\mathcal{I}} \langle Jv_i^*, tJv_i^*\Omega\rangle \tag{1.5.3}$$

*for all* $t \in \langle B, e_C\rangle$ *defines a weight on* $\langle B, e_C\rangle$. *Moreover,* (1.5.3) *also defines* $\widetilde{\mathrm{Tr}}$ *on* $Be_CB$, *and* $\bar{\nu} = \widetilde{\mathrm{Tr}}$ *on this space.*

The semicyclic Hilbert space is related to the lifted trace in a manner similar to what we see in the state case.

**Proposition 1.5.5.** *([SS08, Lemma 4.3.10]) The vector space* $Ae_FA$, *when viewed as a set of vectors of the form* $t+K_{\overline{\mu}}$, *is dense in the semicylic Hilbert space* $H_{\overline{\mu}}$ *in the* $\|\cdot\|_{\overline{\mu}}$*-norm.*

The following result explains why the basic construction is useful for our "mirrored" systems (Section 1.3):

**Proposition 1.5.6.** *([AET11] Lemma 3.4) Let* $V$ *be a closed subspace of* $H$. *Then* $V$ *is a right* $F$*-submodule, if and only if* $P_V \in \langle A, e_F\rangle$.

*Proof.* Simply note that, for all $a \in F$,

$$j(F)V \subseteq V \Leftrightarrow \forall\, a \in F \;\; P_V j(a) = j(a)P_V \Leftrightarrow P_V \in (JFJ)' = \langle A, e_F\rangle,$$

the last equality following from Proposition 1.5.2 (c). □

**Definition 1.5.7.** Suppose $V \subseteq H$ is a closed right-$F$-submodule with orthogonal projection $P_V : H \to V$. Let $\overline{\mu}$ be the lifted trace of $\langle A, e_F\rangle$. We say that $P_V$ has *finite lifted trace* if $\overline{\mu}(P_V) < \infty$.

## 1.6 W*-Dynamical Systems

Here we define the dynamical systems on von Neumann algebras that we are going to study, and also consider their Hilbert space representations.

**Definition 1.6.1.** Suppose $A$ is a von Neumann algebra with a distinguished faithful normal state $\mu$. Let $\alpha : A \to A$ be a $*$-automorphism (an algebra automorphism that preserves the adjoint ($\alpha(a^*) = \alpha(a)^*$) such that $\mu$ is $\alpha$-preserving ($\mu(\alpha(a)) = \mu(a)$ for all $a \in A$). We call the triple $\mathbf{A} = (A, \mu, \alpha)$ a *W*-dynamical system* (or just *system* when there is no confusion).

**Definition 1.6.2.** We call $\mathbf{F} = (F, \lambda, \varphi)$ a *subsystem* of $\mathbf{A}$ if $F$ is a von Neumann subalgebra of $A$ (containing the unit of $A$) such that $\mu|_F = \lambda$ and $\alpha|_F = \varphi$. If $F$ is globally invariant under the modular automorphism group associated to $\mu$ (i.e. $\sigma_t^\mu(F) = F$ for all $t \in \mathbb{R}$), then $\mathbf{F}$ is called a *modular subsystem* of $\mathbf{A}$.

Throughout Chapter 2, $\mathbf{F}$ will be a modular subsystem of $\mathbf{A}$. Note that if the state $\mu$ of the system $\mathbf{A}$ is a trace (i.e. $\mu(ab) = \mu(ba)$ for all $a, b \in A$), then all of its subsystems are modular. Much of our work, in Chapter 2, is for the case where $\mu$ is tracial. We assume that $\mu$ is tracial throughout Chapter 3.

In studying the structure of systems, the following type of system naturally appears.

**Definition 1.6.3.** Suppose $R$ is a von Neumann algebra with normal semifinite faithful tracial weight $\tau$. Let $\zeta : R \to R$ be a $*$-automorphism such that $\tau$ satisfies

$$\tau \circ \zeta = \tau \tag{1.6.1}$$

(on the set of all positive elements $a \in R$). We shall refer to the triple $(R, \tau, \zeta)$ as a *semifinite ($W^*$-dynamical) system.*

## 1.7 Representation of $W^*$-dynamics on $(A, \mu)$

We refer the reader to the author's MSc [Kin17] for any omitted proofs in the rest of the chapter. They are, for the most part, fairly standard and also discussed in the literature (see, for example, [Duv08] and [AET11]). Let $\mathbf{A}$ be a system and $\mathbf{F}$ a subsystem (not necessarily modular) of $\mathbf{A}$. We shall examine the unitary operator, denoted $U$, that arises from $\alpha$ acting on $A$. Afterwards, we shall see how $U$ can be used to extend the dynamics of $\alpha$ to $\langle A, e_F \rangle$.

As before, we assume that $A$ acts on its GNS Hilbert space $H$. The $*$-automorphism $\alpha : A \mapsto A$ induces a linear operator $U : H \to H$ defined (first on the dense subspace $A\Omega$) by

$$U(a\Omega) = \alpha(a)\Omega \qquad \text{for every } a \in A. \tag{1.7.1}$$

As $\Omega$ is separating, $U$ is well-defined.

**Proposition 1.7.1.** *The map $U : H \to H$ is a unitary operator.*

Let us discuss some properties of $U$.

We note that $U^* = U^{-1}$ behaves in the following manner:

$$\forall y \in A \qquad U^{-1}(y\Omega) = \alpha^{-1}(y)\Omega. \tag{1.7.2}$$

Note that our assumption that $\alpha(F) = F$ leads us to conclude that $\overline{F\Omega}$ is a reducing subspace for $U$ i.e. $U\overline{F\Omega} \subseteq \overline{F\Omega}$ and $U^*\overline{F\Omega} \subseteq \overline{F\Omega}$. Hence, from [Zhu93] Corollary 18.3, we have,

$$UP = PU, \tag{1.7.3}$$

with $P$ as in (1.4.2).

We also have, as a result of [BR02, Corollary 2.4.32],

$$JU = UJ. \tag{1.7.4}$$

We can express $\alpha$ in terms of $U$ :

$$\alpha^n(a) = U^n(a)U^{-n} \qquad \text{for all } n \in \mathbb{N}, \tag{1.7.5}$$

## 1.8 The System on the Commutant

Recall in Section 1.3 that we “mirrored” $(A, \mu)$ to obtain the von Neumann algebra $A'$ and the normal faithful state $\mu'$. We set,

$$\alpha'(b) := j \circ \alpha \circ j(b) = UbU^* \tag{1.8.1}$$

for all $b \in A'$, using (1.7.4). This defines the system

$$\mathbf{A}' := (A', \mu', \alpha').$$

We obtain a subsystem

$$\tilde{\mathbf{F}} = \left(\tilde{F}, \tilde{\lambda}, \tilde{\varphi}\right)$$

of $\mathbf{A}'$ using (1.4.1) and setting

$$\tilde{\varphi} := \alpha'|_{\tilde{F}}. \tag{1.8.2}$$

We also note that

$$D \circ \alpha = \alpha \circ D = \varphi \circ D, \qquad D' \circ \alpha' = \alpha' \circ D' = \varphi' \circ D' \tag{1.8.3}$$

since, in terms of $P$ from (1.4.2),

$$PU = UP,$$

as is easily verified from $\alpha(F) = F$.

## 1.9 The Dynamics on $\langle A, e_F \rangle$

Here we consider an important semifinite system that arises from a W*-dynamical system.

We shall now extend $\alpha$ to the basic construction $\langle A, e_F \rangle \subseteq \mathcal{B}(H)$, which we will denote by $\bar{\alpha}$ in the sequel. For every $x \in \langle A, e_F \rangle$, (cf. with (1.7.5))

$$\bar{\alpha}(x) := UxU^*. \tag{1.9.1}$$

In the special case where $x = e_F$, using (1.7.3), (1.9.1) becomes

$$\bar{\alpha}(e_F) = e_F. \tag{1.9.2}$$

It can be shown that $(\langle A, e_F \rangle, \bar{\mu}, \bar{\alpha})$ is a semifinite W*-dynamical system with

$$\bar{\alpha}^n(ae_F b) = U^n(ae_F b)U^{-n} = \alpha^n(a)e_F\alpha^n(b). \tag{1.9.3}$$

To prove $\bar{\mu} \circ \bar{\alpha} = \bar{\mu}$ is not elementary, requiring Theorem 1.5.3.

**Corollary 1.9.1.** *The $*$-automorphism $\bar{\alpha} : \langle A, e_F \rangle \to \langle A, e_F \rangle$ satisfies $\bar{\mu} \circ \bar{\alpha} = \overline{\alpha}$ i.e. $(\langle A, e_F \rangle, \bar{\mu}, \bar{\alpha})$ is a semifinite system.*

*Proof.* We will apply Theorem 1.5.3 to $\varphi = \bar{\mu} \circ \bar{\alpha}$. We have already remarked that $\bar{\mu}$ is normal (see the paragraph before (1.5.1)). As $\bar{\alpha}$ is a *-automorphism on a von Neumann algebra it is normal ([Con00] Proposition 46.6), hence, so is $\bar{\mu} \circ \bar{\alpha}$ . So we just need to check equality of $\bar{\mu} \circ \bar{\alpha}$ and $\bar{\alpha}$ on $Ae_F A^+$.

We show something stronger by showing agreement on the dense subspace $Ae_F A$. For all $a, b \in A$,

$$\begin{aligned} &\bar{\mu} \circ \bar{\alpha}(ae_F b) \\ &= \bar{\mu}(\alpha(a)e_F\alpha(b)) && \text{using (1.9.3)} \\ &= \mu(\alpha(ab)) \\ &= \mu(ab) = \bar{\mu}(ae_F b) && \text{using (1.5.1).} \end{aligned}$$

As $\bar{\mu}$ and $\bar{\alpha}$ are linear, $\bar{\mu} \circ \bar{\alpha} = \bar{\mu}$ on $Ae_F A$. □

## 1.10 Representation of Semifinite $W^*$-dynamics

Let $(R, \tau, \zeta)$ be a semifinite system. Just below, in a procedure similar to obtaining $U$, we can express, at least partially, the action of $\zeta$ on $R$ as a unitary $U_\zeta$ acting on $H_\tau$.

We use the notation in §1.2.

For every $x \in \mathcal{K}_\tau$ ((1.2.1)) define

$$U_\zeta : \mathcal{K}_\tau/K_\tau \ni x + K_\tau \mapsto \zeta(x) + K_\tau \in \mathcal{K}_\tau/K_\tau. \tag{1.10.1}$$

We note that $U_\zeta$ is well-defined, because using the fact that $\zeta$ is a $\tau$-preserving $*$-automorphism on $R$, we have $\tau(\zeta(x)^*\zeta(x)) = \tau(\zeta(x^*x)) = \tau(x^*x) < \infty$. This shows not only that $\zeta(x) \in \mathcal{K}_\tau$, but also that $U_\tau$ is isometric on $\mathcal{K}_\tau/K_\tau$, that is, $\|x + K_\tau\|_\tau = \|\zeta(x) + K_\tau\|_\tau$ for all $x \in \mathcal{K}_\tau$. Thus, $U_\zeta$ is a unitary operator on $H_\tau$.

Viewing $R$ in its semicyclic representation $\pi_\tau(R)$ on $H_\tau$ allows us to express the action of $\zeta$ on $R$ in the following manner.

**Proposition 1.10.1.** *Define $\zeta_\tau : \pi_\tau(R) \to \pi_\tau(R)$ by the prescription:*

$$\zeta_\tau(\pi_\tau(a)) := U_\zeta \pi_\tau(a) U_\zeta^* \qquad \text{for all } a \in R.$$

*Then,*

$$\zeta_\tau = \pi_\tau \circ \zeta \circ \pi_\tau^{-1}. \tag{1.10.2}$$

**Remark 1.10.2.** Of interest to us, of course, is the special case where we consider the semifinite system, $(\langle A, e_F\rangle, \bar{\mu}, \bar{\alpha})$. However, the slightly more abstract approach clarifies the analogous structures and procedures that arise in comparison with finite W*-dynamical systems (Definition 1.6.1). We let $\bar{H} := H_{\bar{\mu}}$ denote the semicyclic Hilbert space of $(\langle A, e_F\rangle, \bar{\mu})$ and $\bar{U} := U_{\bar{\mu}}$ the unitary defined in (1.10.1).

# Chapter 2

# Relative Weak Mixing

## 2.1 Introduction

This chapter is a modification of the paper [DK19].

We study relative weak mixing for W*-dynamical systems in terms of joinings. The main result is a characterization of relative weak mixing in terms of relative ergodicity of the relative product of the system with its mirror image on the commutant (in the cyclic representation). The relative product system is defined using the relatively independent joining obtained from the conditional expectation onto the von Neumann subalgebra relative to which we are working. Generalizing the classical case, the subalgebra in question is always taken to be globally invariant under the dynamics of the W*-dynamical system.

The proof involves a careful analysis of the interplay between the von Neumann algebra, its commutant, and the conditional expectation.

In classical ergodic theory it is well known that a dynamical system is weakly mixing if and only if its product with itself is ergodic. Our main result in this chapter is essentially noncommutative and relative version of this.

A noncommutative theory of joinings has been developed in [Duv08], [Duv10] and [Duv12], generalizing some aspects of the classical theory (see [Gla03] for a thorough treatment, and [Fur67] as well as [Rud79] for the origins). It included a study of weak mixing, relative ergodicity and compact subsystems. Subsequent work was done in [BCM17], which among other things developed various characterizations of joinings and also obtained a more complete theory for weak mixing, building on an approach to noncommutative joinings outlined in [KLP09, Section 5]. Also see [BCM16] for connected results. Earlier work related to noncommutative joinings appeared in [ST92], connected to entropy, and [Fid09a], regarding ergodic theorems.

An investigation of relative weak mixing is a natural next step in the development of the theory of noncommutative joinings. Relative weak mixing has already been studied and used very effectively in the noncommutative context in [Pop07] and [AET11], but not from a joining point of view.

In particular, the authors of [AET11] proved quite a remarkable structure theorem, namely that an asymptotically abelian W*-dynamical system is weakly mixing relative to the centre of the von Neumann algebra. This allowed them to apply classical ergodic results to the system on the centre, and then extend these results to the noncommutative system. They defined relative weak mixing in terms of a certain ergodic limit, which is the approach taken in this chapter as well. However, we adapt their definition to a form which is more convenient in the proof of our main result. The two definitions are nevertheless equivalent when the invariant state is tracial. To prove this, we make use of the lifted trace (Section 1.5).

Since systems which are not asymptotically abelian do occur, we do not assume asymptotic abelianness in this thesis.

Furthermore, systems can be weakly mixing relative to nontrivial subalgebras other than the centre. This includes cases where the von Neumann algebra of the system is a factor (i.e. when the centre is trivial). Therefore we work relative to more general von Neumann subalgebras.

In the classical case, relative weak mixing is often defined in terms of a relatively independent joining, or relative product, illustrating the importance of this characterization in the classical case. However, it is in many cases just stated for ergodic systems, since any system can be decomposed into ergodic parts. See for example [Fur77, Theorem 7.5], [Z$^+$76, Definition 7.9] and [Gla03, Definition 9.22]. But we note that in [FK78] and [Fur14, Definition 6.2], on the other hand, ergodicity is not assumed.

In the noncommutative case the assumption of ergodicity becomes problematic, as typically some form of asymptotic abelianness is required to do an ergodic decomposition. See for example [BR02, Subsection 4.3.1] for an exposition. Therefore we study the joining characterization of relative weak mixing without the assumption of ergodicity. In particular the proof of Theorem 2.4.2 has to deal with the difficulty of the system not being ergodic.

A number of other noncommutative relative ergodic properties have already been studied in the literature, for example in [DM14], building on ideas from [Fid09b], which was based in turn on variations of unique ergodicity as studied in [AD09]. Those properties, however, are more of a topological nature, rather than purely measure theoretic in origin, if one thinks in terms of classical ergodic theory, and the techniques involved are quite different from those in this chapter.

The required background on relatively independent joinings appears in Section 2.2. The definition of relative weak mixing is formulated in Section 2.3. Some relevant characterizations in terms of ergodic limits are then derived. A noncommutative example is subsequently presented to illustrate the points made above regarding asymptotic abelianness, the centre, and ergodicity. The main result of the chapter, and its proof, appear in Section 2.4.

## 2.2 Relatively independent joinings

We use the notation of Chapter 1, Sections 1.4 to 1.8.

**Definition 2.2.1.** Let $\mathbf{B} = (B, \nu, \beta)$ and $\mathbf{C} = (C, \sigma, \gamma)$ be systems. A *joining* of $\mathbf{B}$ and $\mathbf{C}$ is a state $\omega$ on the algebraic tensor product $B \odot C$ such that $\omega(b \otimes 1_C) = \nu(b)$, $\omega(1_B \otimes c) = \sigma(c)$ and $\omega \circ (\beta \odot \gamma) = \omega$ for all $b \in B$ and $c \in C$.

We can now construct the relatively independent joining of $\mathbf{A}$ and $\mathbf{A}'$ over $\mathbf{F}$ ([Duv12]):

Define the unital $*$-homomorphism

$$\delta : F \odot \tilde{F} \to \mathcal{B}(H),$$

to be the linear extension of $F \times \tilde{F} \to \mathcal{B}(H) : (a, b) \mapsto ab$. Defining the *diagonal* state

$$\Delta_\lambda : F \odot \tilde{F} \to \mathbb{C}$$

of $\lambda$ by

$$\Delta_\lambda(c) := \langle \Omega, \delta(c)\Omega \rangle$$

for all $c \in F \odot \tilde{F}$, allows us to define a state $\mu \odot_\lambda \mu'$ on $A \odot A'$ by

$$\mu \odot_\lambda \mu' := \Delta_\lambda \circ E \tag{2.2.1}$$

where

$$E := D \odot D'.$$

Using (1.4.4), (1.4.5) (1.7.5) and (1.8.1), note that $\mu \odot_\lambda \mu'$ is indeed a joining of $\mathbf{A}$ and $\mathbf{A}'$, with the property that $(\mu \odot_\lambda \mu')|_{F \odot \tilde{F}} = \Delta_\lambda$, and it is called the *relatively independent joining of* $\mathbf{A}$ *and* $\mathbf{A}'$ *over* $\mathbf{F}$. We also denote this joining by

$$\omega := \mu \odot_\lambda \mu'.$$

**Remark 2.2.2.** In the case of a state, the relatively independent joining fits in very naturally with the modular theory of von Neumann algebras:

Note firstly that similar to the fact that $\omega$ is a joining of $\mathbf{A}$ and $\mathbf{A}'$, we also have

$$\omega \circ (\sigma_t^\mu \odot (\sigma_t^\mu)') = \omega$$

where $\sigma_t^\mu$ denotes the modular group associated to $\mu$, and $(\sigma_t^\mu)'$ is defined analogously to $\alpha'$. This follows, since $D \circ \sigma_t^\mu = \sigma_t^\mu \circ D$ and $D' \circ \sigma_t^{\mu'} = \sigma_t^{\mu'} \circ D'$, and where we also note that $(\sigma_t^\mu)' = \sigma_{-t}^{\mu'}$. From the point of view of von Neumann algebras (i.e. noncommutative measure theory), this is a very natural property for a joining to have, and indeed in [BCM17, Definition 3.1] it is included as part of the definition of joinings more generally, even though here we have not required it in Definition 2.2.1.

Secondly, by [HT70, Lemma 1 of Section 1] (or see [Tak03b, Corollary VIII.1.4]) it follows that $\alpha^{-1} \circ \sigma_t^\mu \circ \alpha = \sigma_t^{\mu \circ \alpha} = \sigma_t^\mu$, so

$$\sigma_t^\mu \circ \alpha = \alpha \circ \sigma_t^\mu$$

and analogously for $\alpha'$ and $\sigma_t^{\mu'}$, again showing that the framework used here fits in very neatly with of modular theory.

We write

$$\mathbf{A} \odot_{\mathbf{F}} \mathbf{A}' := (A \odot A', \mu \odot_\lambda \mu', \alpha \odot \alpha')$$

and call $\mathbf{A} \odot_{\mathbf{F}} \mathbf{A}'$ the *relative product system* (of $\mathbf{A}$ and $\mathbf{A}'$ over $\mathbf{F}$). It is an example of a $*$-*dynamical system*, namely it consists of a state $\omega = \mu \odot_\lambda \mu'$ on a unital $*$-algebra $A \odot A'$, and a $*$-automorphism $\alpha \odot \alpha'$ of $A \odot A'$ such that $\omega \circ (\alpha \odot \alpha') = \omega$. However, this is typically not a W*-dynamical system as given by Definition 1.6.1.

The cyclic representation of $A \odot A'$ obtained from $\omega$ by the GNS construction will be denoted by $(H_\omega, \pi_\omega, \Omega_\omega)$. Since $\omega$ can be extended to a state on the maximal C*-algebraic tensor product $A \otimes_m A'$ (see for example [Duv10, Proposition 4.1]), we know that $\pi_\omega$ is a $*$-homomorphism from $A \odot A'$ into the bounded operators $\mathcal{B}(H_\omega)$. Let

$$\gamma_\omega : A \odot A' \to H_\omega : t \mapsto \pi_\omega(t)\Omega_\omega.$$

Furthermore, let $W$ denote the unitary representation of

$$\tau := \alpha \odot \alpha'$$

on $H_\omega$, i.e. it is defined as the extension of

$$W\gamma_\omega(t) := \gamma_\omega(\tau(t)) \tag{2.2.2}$$

for all $t \in A \odot A'$.

The cyclic representation obtained from $\omega$, allows us to construct cyclic representations $(H_\mu, \pi_\mu, \Omega_\omega)$ and $(H_{\mu'}, \pi_{\mu'}, \Omega_\omega)$ of $(A, \mu)$ and $(A', \mu')$ respectively, which are naturally embedded into $H_\omega$ (as in [Duv08, Construction 2.3]), by setting

$$H_\mu := \overline{\gamma_\omega(A \otimes 1)} \quad \text{and} \quad \pi_\mu(a) := \pi_\omega(a \otimes 1)|_{H_\mu}$$

for every $a \in A$, and similarly for $H_{\mu'}$ and $\pi_{\mu'}$.

The representation $(H_\mu, \pi_\mu, \Omega_\omega)$ is unitarily equivalent to our initial representation $(H, \mathrm{id}_A, \Omega)$ of $(A, \mu)$ (via the unitary obtained from $H \to H_\mu : a\Omega \mapsto \pi_\mu(a)\Omega_\omega$), but we make use of both representations later on. In terms of notation, whereas $a \in A$ is in the initial cyclic representation, we always write $\pi_\mu(a)$ when using the cyclic representation $(H_\mu, \pi_\mu, \Omega_\omega)$.

Now we consider cyclic representations of $(F, \lambda)$ and $(\tilde{F}, \tilde{\lambda})$:

Note that $(H_F, \delta, \Omega)$ is a cyclic representation of $(F \odot \tilde{F}, \Delta_\lambda)$, since $H_F = \overline{\delta(F \odot \tilde{F})\Omega}$ (as is easily verified using Proposition 1.4.5 and 1.3.1 (c)). However, $(\overline{\gamma_\omega(F \odot \tilde{F})}, \pi_\omega|_{F\odot\tilde{F}}, \Omega_\omega)$ is also a cyclic representation of $(F \odot \tilde{F}, \Delta_\lambda)$, so these two representations are unitarily equivalent via the unitary operator $V : H_F \to \overline{\gamma_\omega(F \odot \tilde{F})}$ defined as the extension of $\delta(t)\Omega \mapsto \gamma_\omega(t)$ for $t \in F \odot \tilde{F}$. Therefore

(2.2.3)
$$H_\lambda := \overline{\gamma_\omega(F \otimes 1)} = V\overline{\delta(F \otimes 1)\Omega} = VH_F = V\overline{\delta(1 \otimes \tilde{F})\Omega} = \overline{\gamma_\omega(1 \otimes \tilde{F})},$$

which means that $(F, \lambda)$ and $(\tilde{F}, \tilde{\lambda})$ are cyclicly represented on the same subspace $H_\lambda$ of $H_\omega$ by

$$\pi_\lambda(f) := \pi_\mu(f)|_{H_\lambda} \quad \text{and} \quad \pi_{\tilde{\lambda}}(\tilde{f}) := \pi_{\mu'}(\tilde{f})|_{H_\lambda}$$

for all $f \in F$ and $\tilde{f} \in \tilde{F}$.

## 2.3 Relative weak mixing

This section presents the definition and two closely related characterizations of relative weak mixing in terms of ergodic averages. These characterizations do not yet involve the relative independent joining. An example of relative weak mixing is also given.

In terms of the notation in the previous section, our main definition is the following:

**Definition 2.3.1.** We call a system **A** *weakly mixing relative to* the modular subsystem **F** if

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \lambda\left(|D(b\alpha^n(a))|^2\right) = 0 \tag{2.3.1}$$

for all $a, b \in A$ with $D(a) = D(b) = 0$.

In the classical case this is often also expressed by saying that **A** is a *weakly mixing extension* of **F**.

**Remark 2.3.2.** We recover the absolute case of weak mixing from this definition, by using $F = \mathbb{C}1_A$. Indeed, in this case we have $D(a) = \mu(a)1_A$ for all $a \in A$. Thus, (2.3.1) becomes

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} |\mu(ba^n(a))|^2 = 0,$$

or equivalently,

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} |\mu(ba^n(a))| = 0, \tag{2.3.2}$$

for all $a, b \in A$ such that $\mu(a) = \mu(b) = 0$.

The reason for this equivalence is that for any bounded sequence $(c_n)$ of non-negative real numbers, bounded by $c > 0$, say, we have

$$\frac{1}{N}\sum_{n=1}^{N} c_n^2 \leq \frac{c}{N}\sum_{n=1}^{N} c_n$$

and, using the Cauchy-Schwarz inequality,

$$\frac{1}{N}\sum_{n=1}^{N} c_n \leq \left(\frac{1}{N}\sum_{n=1}^{N} c_n^2\right)^{\frac{1}{2}}. \tag{2.3.3}$$

Therefore,

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} c_n^2 = 0 \Leftrightarrow \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} c_n = 0.$$

Condition (2.3.2) in turn is easily seen to be equivalent to the following:

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} |\mu(ba^n(a)) - \mu(b)\mu(a)| = 0$$

for all $a, b \in A$ (simply replace $a$ and $b$ by $a - \mu(a)$ and $b - \mu(b)$ respectively in (2.3.2)). This is the standard definition of weak mixing.

Our first simple characterization of relative weak mixing, which will also be used in the proof of this chapter's main theorem in the next section, is the following:

**Proposition 2.3.3.** *The system* $\mathbf{A}$ *is weakly mixing relative to* $\mathbf{F}$ *if and only if*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\lambda\left(\left|D\left(b\alpha^n(a)\right)-D(b)D(\alpha^n(a))\right|^2\right)=0 \tag{2.3.4}$$

*for all* $a,b\in A$.

*Proof.* Assume that $\mathbf{A}$ is weakly mixing relative to $\mathbf{F}$. For any $a,b\in A$, setting $a_0 := a - D(a)$ and $b_0 := b - D(b)$, we have $D(a_0) = D(b_0) = 0$ and

$$D(b_0\alpha^n(a_0)) = D\left(b\alpha^n(a)\right) - D(b)D(\alpha^n(a)).$$

Hence (2.3.4) follows from Definition 2.3.1. The converse is trivial by assuming either $D(a) = 0$ or $D(b) = 0$. □

This gives us variations of this characterization as well, for example, $\mathbf{A}$ is weakly mixing relative to $\mathbf{F}$ if and only if (2.3.1) holds for all $a,b\in A$ with $D(a) = 0$.

Next we are going to show that when $\mu$ is a trace, Definition 2.3.1 is equivalent to [AET11, Definition 3.7]. To do this, we use the basic construction in a similar way to how it was used in [AET11, Sections 3 and 4] to prove their structure theorem.

We need three lemmas which we present now. The first is just a slight variation of the calculations that appear at the beginning of the proof of [AET11, Proposition 3.8] (see also Chapter 1 Section 1.9):

**Lemma 2.3.4.** *Assume that* $\mu$ *is a trace. Let* $a,b\in A$. *Then*

$$\bar{\mu}(b^*e_Fb\bar{\alpha}^n(ae_Fa^*)) = \lambda(|D(b\alpha^n(a))|^2).$$

*Proof.* $\bar{\mu}(b^*e_Fb\bar{\alpha}^n(ae_Fa^*)) = \bar{\mu}(D(c)e_FD(c^*)) = \mu(D(c)D(c^*))$ in terms of $c := b\alpha^n(a)$. □

The following is a version of the van der Corput lemma:

**Lemma 2.3.5.** *[Tao09, Lemma 2.12.7] Let* $(v_n)$ *be a bounded sequence of vectors in a Hilbert space* $\mathfrak{H}$ *such that*

$$\lim_{M\to\infty}\frac{1}{M}\sum_{h=1}^{M}\left(\limsup_{N\to\infty}\left|\frac{1}{N}\sum_{n=1}^{N}\langle v_n, v_{n+h}\rangle\right|\right)=0. \tag{2.3.5}$$

*Then*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} v_n = 0.$$

Putting these two lemmas together, we obtain the following:

**Lemma 2.3.6.** *Assume $\mu$ is a trace. Let $a \in A$ satisfy*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \lambda(|D(a^*\alpha^n(a))|^2) = 0. \tag{2.3.6}$$

*Then, for all $b \in A$, we have*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \lambda(|D(b\alpha^n(a))|^2) = 0. \tag{2.3.7}$$

*Proof.* We use the semicyclic representation of Chapter 1 Section 1.2. Let $x := ae_Fa^*$ and $y := b^*e_Fb$. Then both $x$ and $y$ belong to $\mathcal{K}_{\bar{\mu}}$. For notational convenience, put $\hat{x} := x + K_{\bar{\mu}}$. Observe that $\bar{\mu}(y\bar{\alpha}^n(x)) \geq 0$ by Lemma 2.3.4. Then,

$$\begin{aligned}\frac{1}{N}\sum_{n=1}^{N}\bar{\mu}(y\bar{\alpha}^n(x))^2 &= \frac{1}{N}\sum_{n=1}^{N}\langle\hat{y},\bar{U}^n\hat{x}\rangle^2 = \left\langle \hat{y}, \frac{1}{N}\sum_{n=1}^{N}\langle\hat{y},\bar{U}^n\hat{x}\rangle\bar{U}^n\hat{x}\right\rangle \\ &\leq \|\bar{y}\|\left\|\frac{1}{N}\sum_{n=1}^{N}\langle\hat{y},\bar{U}^n\hat{x}\rangle\bar{U}^n\hat{x}\right\| \end{aligned} \tag{2.3.8}$$

Let $v_n := \langle\hat{y},\bar{U}^n\hat{x}\rangle\bar{U}^n\hat{x}$, for every $n \in \mathbb{N}$. Clearly, the sequence $(v_n)$ is bounded. We can estimate, for every $n, h \in \mathbb{N}$,

$$|\langle v_n, v_{n+h}\rangle| \leq \|\hat{x}\|^2 \|\hat{y}\|^2 \bar{\mu}(x\bar{\alpha}^h(x)).$$

This, together with Lemma 2.3.4 and our assumption (2.3.6), imply (2.3.5). Thus, from Lemma 2.3.5, we have $\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} v_n = 0$. Therefore, from (2.3.8), we obtain

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \bar{\mu}(y\bar{\alpha}^n(x))^2 = 0.$$

Consequently, from (2.3.3),

$$\lim_{N\to\infty} \sum_{n=1}^{N} \bar{\mu}(y\bar{\alpha}^n(x)) = 0.$$

Again by Lemma 2.3.4, we are done. □

This finally implies the following characterization of relative weak mixing (which in [AET11] was used as the definition):

**Proposition 2.3.7.** *Assume that $\mu$ is a trace. Then $\mathbf{A}$ is weakly mixing relative to the subsystem $\mathbf{F}$ if and only if*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \lambda(|D(a^*\alpha^n(a))|^2) = 0,$$

*for all $a \in A$ such that $D(a) = 0$.*

**Remark 2.3.8.** Essential in the commutative version of Lemma 2.3.6 proof (outlined in [Tao09, Exercise 2.14.1]), is a conditional version of the Cauchy-Schwarz inequality in terms of the conditional expectation $\mathbb{E}$:

$$|\mathbb{E}(\bar{f}g|Y)| \leq \|\mathbb{E}(|f|^2|Y)\|_{L^2(X|Y)}\|\mathbb{E}(|g|^2|Y)\|_{L^2(X|Y)}$$

where $f, g$ belong to the $L^\infty(Y)$-module

$$L^2(X|Y) = \{h \in L^2(X) : \mathbb{E}(|h|^2|Y) \in L^\infty(Y)\}$$

([Tao09, Section 2.13]). In the noncommutative case, however, our approach above allows us to simplify the argument and avoid some snags. We essentially used a noncommutative translation of the proof of the absolute case [Tao09, Corollary 2.12.8], but in terms of the basic construction, to prove Lemma 2.3.6.

Before we get to an example, we note a few simple general facts:

Firstly, $D(a) = 0$ for $a \in A$, if and only if $a$ is of the form $a = c - D(c)$ for some $c \in A$.

Secondly,

$$\lambda(D(\alpha^n(a^*)b^*)D(b\alpha^n(a))) = \|PbU^n a\Omega\|^2$$

for all $a, b \in A$, by a straightforward calculation. If, in addition $\lambda$ is a trace, then we have

$$\|PbU^n a\Omega\| = \left\|PU^n a^* U^{-n} b^*\Omega\right\| \tag{2.3.9}$$

for all $a, b \in A$, by a similar calculation for $\lambda(D(b\alpha^n(a))D(\alpha^n(a^*)b^*))$.

To show that relative weak mixing is indeed relevant in noncommutative W*-dynamical systems, in particular for non-ergodic systems which are not asymptotically abelian, we provide the following example:

**Example 2.3.9.** Let $G$ be any discrete group, and let $A$ be the group von Neumann algebra obtained from it. In other words, $A$ is the von Neumann algebra on $H = l^2(G)$ generated by the following set of unitary operators:

$$\{l(g) : g \in G\}$$

where $l$ is the left regular representation of $G$, i.e. the unitary representation of $G$ on $H$ with each $l(g) : H \to H$ given by

$$[l(g)f](h) = f(g^{-1}h)$$

for all $f \in H$ and $g, h \in G$. Equivalently,

$$l(g)\delta_h = \delta_{gh}$$

for all $g, h \in G$, where $\delta_g \in H$ is defined by $\delta_g(g) = 1$ and $\delta_g(h) = 0$ for $h \neq g$. Setting

$$\Omega := \delta_1$$

where $1 \in G$ denotes the identity of $G$, we can define a faithful normal trace $\mu$ on $A$ by

$$\mu(a) := \langle \Omega, a\Omega \rangle$$

for all $a \in A$. Then $(H, \mathrm{id}_A, \Omega)$ is the cyclic representation of $(A, \mu)$.

Given any automorphism $T$ of $G$, we define a unitary operator on $H$ by

$$Uf := f \circ T^{-1}$$

for all $f \in H$. From this we obtain a $*$-automorphism of $A$ by setting

$$\alpha(a) := UaU^*$$

for all $a \in A$, which satisfies $\alpha(l(g)) = l(T(g))$ for all $g \in G$.

Then $\mathbf{A} = (A, \mu, \alpha)$ is a system which we call the *dual system* of $(G, T)$. (See [Duv10, Section 3] for more background on this type of system in the context of quantum groups, W*-algebraic ergodic theory and joinings.)

Define a subsystem $\mathbf{F} = (F, \lambda, \varphi)$ of $\mathbf{A}$ by letting $F$ be the von Neumann subalgebra of $A$ generated by

$$\{l(g) : g \in K\}$$

where $K := \{g \in G : T^{\mathbb{N}}(g) \text{ is finite}\}$. Here

$$T^{\mathbb{N}}(g) := \{T(g), T^2(g), T^3(g), ...\}$$

is the *orbit* of $g$. Furthermore $\lambda := \mu|_F$ and $\varphi := \alpha|_A$.

We call **F** the *finite orbit subsystem* of **A**.

We can find $D$ explicitly in this case: The projection $P$ above is now the projection of $H$ onto the Hilbert subspace spanned by $\{\delta_g : g \in K\}$. Therefore we have

$$D(l(g)) = \begin{cases} l(g) \text{ for } g \in K \\ 0 \text{ for } g \notin K \end{cases} \tag{2.3.10}$$

for all $g \in G$.

Note that the unital $*$-algebra generated by $\{l(g) : g \in G\}$ is exactly $A_0 = \text{span}\{l(g) : g \in G\}$.

Suppose that for any $g, h \in G$ with $g \notin K$, it is true that

$$D(l(hT^n(g))) = 0 \tag{2.3.11}$$

for $n$ large enough, i.e. for $n > n_0$ for some $n_0$. Then, for any $c_0, b_0 \in A_0$, and $a_0 := c_0 - D(c_0)$, we have

$$Pb_0U^n a_0\Omega = 0$$

for $n$ large enough. Since $A_0$ is strongly dense in $A$, it follows that

$$\lim_{n\to\infty} Pb_0U^n a\Omega = 0$$

for all $a \in A$ such that $D(a) = 0$, by simply considering any $c \in A$ and some $c_0 \in A_0$ such that $\|c_0\Omega - c\Omega\| < \varepsilon$ for an $\varepsilon > 0$ of our choosing, and setting $a := c - D(c)$.

Since $\lambda$ is a trace, we can apply a similar argument to $\|Pb_0U^n a\Omega\| = \|PU^n a^* U^{-n} b_0^* \Omega\|$ (see (2.3.9)) to show that

$$\lim_{n\to\infty} PbU^n a\Omega = 0$$

and therefore

$$\lim_{n\to\infty} \lambda(D(\alpha^n(a^*)b^*)D(b\alpha^n(a))) = 0$$

for all $a, b \in A$ such that $D(a) = 0$. It follows easily from this that **A** is weakly mixing relative to **F**. The limit above could be interpreted as **A** having a stronger property, namely that **A** is "strongly mixing relative to **F**".

What remains is to show specific cases for which (2.3.11) holds and which illustrate the points made above about noncommutative systems.

A simple case is when $G$ is the free group on a countably infinite set of symbols $S$. We then consider any bijection $T : S \to S$ which has both finite and infinite orbits in $S$, say $T$ is a permutation when restricted to some finite non-empty subset, or to each of infinitely many finite non-empty subsets, while it shifts the remaining infinite subset of $S$. We obtain an automorphism $T$ of $G$ from this bijection. Then (2.3.11) follows from (2.3.10).

But at the same time, $F$ is then not trivial, i.e. $F$ strictly contains the subalgebra $\mathbb{C}1$, and is in general not abelian. In fact, $F$ is $*$-isomorphic to the group von Neumann algebra of the free group $K$ on the symbols with finite orbits. That $F \neq \mathbb{C}1$, also implies that $\mathbf{A}$ is not ergodic (see [Duv10, Theorem 3.4]). Furthermore,

$$\|[\alpha^n(l(g)), l(h)]\Omega\| = \sqrt{2}$$

if $T^n(g)h \neq hT^n(g)$, which is the case if $g$ and $h$ are in two separate orbits, or if $g = h$ has an infinite orbit. Hence $\mathbf{A}$ is not asymptotically abelian in the sense of [AET11, Definition 1.10]. Furthermore, $A$ is a factor.

We summarize the key conclusions from this example, as they concretely illustrate a number of remarks made in Section 2.1, motivating the remarks made at the beginning of the chapter:

**Proposition 2.3.10.** *Let $\mathbf{A}$ be the dual system of $(G, T)$, where $G$ is the free group on a countably infinite set of symbols $S$, and $T$ is an automorphism of $G$ induced by a bijection $T|_S : S \to S$ which has both finite and infinite orbits (the former on non-empty subsets of $S$). Then $\mathbf{A}$ is weakly mixing relative to its non-trivial finite orbit subsystem (which in general consists of a noncommutative von Neumann subalgebra), but $\mathbf{A}$ is neither ergodic, nor asymptotically abelian, and furthermore its von Neumann algebra $A$ is a factor.*

## 2.4 The joining characterization

This section presents the main result of the chapter, still using the notation from Section 2.2.

Let $H_\omega^W$ denote the fixed point space of $W$. The relative independent joining (or the relative product system) will connect to relative weak mixing via the following notion:

**Definition 2.4.1.** We say that $\mathbf{A} \odot_{\mathbf{F}} \mathbf{A}'$ *is ergodic relative to* the modular subsystem $\mathbf{F}$ of $\mathbf{A}$, if $H_\omega^W \subseteq H_\lambda$.

Our main goal in this chapter is to prove the following characterization of relative weak mixing:

**Theorem 2.4.2.** *Assume that $\mu$ is a trace. Then* $\mathbf{A}$ *is weakly mixing relative to* $\mathbf{F}$ *if and only if* $\mathbf{A} \odot_{\mathbf{F}} \mathbf{A}'$ *is ergodic relative to* $\mathbf{F}$.

The rest of this section is devoted to the proof of this theorem. We break the proof into a sequence of smaller results. Some of these are of independent interest (in particular Propositions 2.4.6, 2.4.10 and 2.4.11, and Remark 2.4.9), and do not require $\mu$ to be tracial.

Lemma 2.4.4 below uses the following result:

**Proposition 2.4.3.** *([Duv12, Proposition 3.6]) Suppose that $\vartheta$ is a joining of $(A, \mu)$ and $(A', \mu')$ such that $\vartheta|_{F \odot \tilde{F}} = \Delta_\lambda$. Then $\vartheta = \mu \odot_\lambda \mu'$ if and only if any of the following three equivalent conditions hold:*

1. $(H_\mu \ominus H_\lambda) \perp (H_{\mu'} \ominus H_\lambda)$
2. $(H_\mu \ominus H_\lambda) \perp H_{\mu'}$
3. $H_\mu \perp (H_{\mu'} \ominus H_\lambda)$.

The following lemma and proposition proves one direction of Theorem 2.4.2. In the classical case, this direction is also proven in [Fur14, Proposition 6.2] and [FK78, Lemma 1.3], but using different arguments.

**Lemma 2.4.4.** *Consider a modular subsystem* $\mathbf{F}$ *of the system* $\mathbf{A}$. *For any $a \in A$ with $D(a) = 0$ and any $b \in A'$, we have*

$$\pi_\omega(a \otimes b)\Omega_\omega \perp H_\lambda.$$

*Proof.* For any $c \in F$,

$$\begin{aligned}\langle \pi_\lambda(c)\Omega_\omega, \pi_\mu(a)\Omega_\omega \rangle &= \langle \Omega_\omega, \pi_\mu(c^* a)\Omega_\omega \rangle = \mu(c^* a) \\ &= \lambda(D(c^* a)) = \lambda(c^* D(a)) \\ &= 0.\end{aligned}$$

Hence, $\pi_\mu(a)\Omega_\omega \in H_\mu \ominus H_\lambda$. So $\pi_\mu(a)\Omega_\omega \perp H_{\mu'}$ by Proposition 2.4.3. On the other hand, $\pi_{\mu'}(b^* f)\Omega_\omega \in H_{\mu'}$ for any $f \in \tilde{F}$, so $\langle \pi_{\mu'}(b^* f)\Omega_\omega, \pi_\mu(a)\Omega_\omega \rangle = 0$. Therefore,

$$\begin{aligned}\langle \pi_{\tilde{\lambda}}(f)\Omega_\omega, \pi_\omega(a \otimes b)\Omega_\omega \rangle &= \langle \pi_\omega(1 \otimes b^*)\pi_{\mu'}(f)\Omega_\omega, \pi_\omega(a \otimes 1)\Omega_\omega \rangle \\ &= \langle \pi_{\mu'}(b^* f)\Omega_\omega, \pi_\mu(a)\Omega_\omega \rangle \\ &= 0,\end{aligned}$$

proving the lemma, since $\pi_{\tilde{\lambda}}(\tilde{F})\Omega_\omega$ is dense in $H_\lambda$. □

Using this lemma we can show one direction of Theorem 2.4.2:

**Proposition 2.4.5.** *Assume that $\mu$ is a trace and that $\mathbf{A} \odot_{\mathbf{F}} \mathbf{A}'$ is ergodic relative to $\mathbf{F}$. Then*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \lambda \left( |D\left(b\alpha^n(a)\right)|^2 \right) = 0$$

*for all $a, b \in A$ such that $D(a) = 0$ or $D(b) = 0$.*

*Proof.* Let $Q$ be the projection of $H_\omega$ onto the fixed point space $H_\omega^W$ of $W$. By the mean ergodic theorem we then have

$$\begin{aligned}\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \omega(\tau^n(s)t) &= \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \langle W^n \pi_\omega(s^*)\Omega_\omega, \pi_\omega(t)\Omega_\omega \rangle \\ &= \langle Q\pi_\omega(s^*)\Omega_\omega, \pi_\omega(t)\Omega_\omega \rangle\end{aligned}$$

for all $s, t \in A \odot A'$. This holds in particular for $s = a^* \otimes j(a)$ and $t = b^* \otimes j(b)$, where $a, b \in A$, and $D(a) = 0$ or $D(b) = 0$.

Suppose $D(a) = 0$ (the case $D(b) = 0$ is similar, by taking $Q$ to the other side in the inner product above). Then $\pi_\omega(s^*)\Omega_\omega \perp H_\omega^W$ by Lemma 2.4.4, so $Q\pi_\omega(s^*)\Omega_\omega = 0$. This means, by the definition of $\omega = \mu \odot_\lambda \mu'$ in (2.2.1), that

$$\begin{aligned}0 &= \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \langle \Omega, D(\alpha^n(a^*)b^*) D'(\alpha'^n(j(a))j(b))\Omega \rangle \\ &= \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \lambda(D(\alpha^n(a^*)b^*) D(b\alpha^n(a))),\end{aligned}$$

as required, since

$$D'(\alpha'^n(j(a))j(b))\Omega = PJ\alpha^n(a^*)b^*\Omega = D(b\alpha^n(a))\Omega,$$

where we have used the fact that $\mu$ is a trace (so $Jc\Omega = c^*\Omega$ for all $c \in A$). □

Next we consider the other direction of Theorem 2.4.2. We don't have a reference to a proof of the classical case of this direction. Our first step is the following:

**Proposition 2.4.6.** $\mathbf{A} \odot_{\mathbf{F}} \mathbf{A}'$ *is ergodic relative to* $\mathbf{F}$ *if and only if*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \omega(t\tau^n(s)) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \omega(E(t)\tau^n(E(s))) \tag{2.4.1}$$

*for all* $s, t \in A \odot A'$. *Both limits exist, whether* $\mathbf{A} \odot_{\mathbf{F}} \mathbf{A}'$ *is ergodic relative to* $\mathbf{F}$ *or not.*

*Proof.* Let $Q$ be the projection of $H_\omega$ onto the fixed point space $H_\omega^W$ of $W$. Let $R$ be the projection of $H_\omega$ onto $H_\lambda$.

By the mean ergodic theorem, for all $s, t \in A \odot A'$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \omega(t\tau^n(s)) = \langle \gamma_\omega(t^*), Q\gamma_\omega(s)\rangle$$

and

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \omega(E(t)\tau^n(E(s))) = \langle \gamma_\omega(E(t^*)), Q\gamma_\omega(E(s))\rangle .$$

Let $P_\mu$ be the projection of $H_\mu$ onto $H_\lambda$, and $P_{\mu'}$ the projection of $H_{\mu'}$ onto $H_\lambda$. Consider $s = a \otimes b$, where $a \in A$ and $b \in A'$. Then, because $\pi_{\mu'}(D'(b))\Omega_\omega \in H_\lambda$, we know by the construction of $D$ (Proposition 1.4.4) that

$$\begin{aligned}\gamma_\omega(E(s)) &= \pi_\mu(D(a))\pi_{\mu'}(D'(b))\Omega_\omega = \pi_\lambda(D(a))\pi_{\mu'}(D'(b))\Omega_\omega\\ &= P_\mu\pi_\mu(a)\pi_{\mu'}(D'(b))\Omega_\omega = P_\mu\pi_\mu(a)P_{\mu'}\pi_{\mu'}(b)\Omega_\omega\\ &= R\pi_\mu(a)R\pi_{\mu'}(b)\Omega_\omega.\end{aligned}$$

since $R|_{H_\mu} = P_\mu$ and $R|_{H_{\mu'}} = P_{\mu'}$.

For $y \in H_{\mu'} \ominus H_\lambda$ and $f \in F$, we have

$$\langle \pi_\lambda(f)\Omega_\omega, \pi_\omega(a\otimes 1)y\rangle = \langle \pi_\mu(a^*f)\Omega_\omega, y\rangle = 0,$$

since $\pi_\mu(a^*f)\Omega_\omega \in H_\mu \perp (H_{\mu'} \ominus H_\lambda)$ by Proposition 2.4.3. So $\pi_\omega(a\otimes 1)y \perp H_\lambda$, which means that

$$\gamma_\omega(E(s)) = R\pi_\omega(a\otimes 1)\pi_{\mu'}(b)\Omega_\omega = R\pi_\omega(a\otimes b)\Omega_\omega.$$

So

$$\gamma_\omega(E(s)) = R\gamma_\omega(s)$$

for all $s \in A \odot A'$. Hence,

$$\langle \gamma_\omega(E(t^*)), Q\gamma_\omega(E(s)) \rangle = \langle R\gamma_\omega(t^*), QR\gamma_\omega(s) \rangle$$

for all $s, t \in A \odot A'$.

Now, if $\mathbf{A} \odot_{\mathbf{F}} \mathbf{A}'$ is ergodic relative to $\mathbf{F}$, i.e. $Q \leq R$, it follows that

$$\langle \gamma_\omega(E(t^*)), Q\gamma_\omega(E(s)) \rangle = \langle \gamma_\omega(t^*), Q\gamma_\omega(s) \rangle$$

from which we see that (2.4.1) holds for all $s, t \in A \odot A'$.

Conversely, if (2.4.1) holds for all $s, t \in A \odot A'$, then we have

$$\langle R\gamma_\omega(t^*), QR\gamma_\omega(s) \rangle = \langle \gamma_\omega(t^*), Q\gamma_\omega(s) \rangle$$

for all $s, t \in A \odot A'$. It follows that $RQR = Q$, so $Q \leq R$, meaning $\mathbf{A} \odot_{\mathbf{F}} \mathbf{A}'$ is ergodic relative to $\mathbf{F}$. □

As a consequence of this proposition, we have the following lemma towards the proof of Theorem 2.4.2:

**Lemma 2.4.7.** *Assume that $\mu$ is a trace. Then $\mathbf{A} \odot_{\mathbf{F}} \mathbf{A}'$ is ergodic relative to $\mathbf{F}$ if and only if*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \lambda(|D(b\alpha^n(a))|^2) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \lambda(|D(b)D(\alpha^n(a))|^2) \tag{2.4.2}$$

*for all $a, b \in A$. Both limits exist, whether $\mathbf{A} \odot_{\mathbf{F}} \mathbf{A}'$ is ergodic relative to $\mathbf{F}$ or not.*

*Proof.* Suppose $\mathbf{A} \odot_{\mathbf{F}} \mathbf{A}'$ is ergodic relative to $\mathbf{F}$, then (2.4.1) holds. Applying it to $s = a \otimes c$ and $t = b \otimes d$, for $a, b \in A$ and $c, d \in A'$, we obtain

$$\begin{aligned}
&\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \omega([b\alpha^n(a)] \otimes [d\alpha'^n(c)]) \\
&= \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \omega([D(b)D(\alpha^n(a))] \otimes [D'(d)D'(\alpha'^n(c))]).
\end{aligned}$$

Using the definition of $\omega$, this is equivalent to

$$\begin{aligned}
&\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \langle \Omega, D(b\alpha^n(a))D'(d\alpha'^n(c))\Omega \rangle \\
&= \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \langle \Omega, D(b)D(\alpha^n(a))D'(d)D'(\alpha'^n(c))\Omega \rangle .
\end{aligned}$$

Setting $c = j(a^*) = JaJ$ and $d = j(b^*) = JbJ$, we have in particular

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \langle \Omega, D(b\alpha^n(a)) JD(b\alpha^n(a))\Omega\rangle$$
$$= \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \langle \Omega, D(b)D(\alpha^n(a)) JD(b)D(\alpha^n(a))\Omega\rangle\,.$$

Since $\mu$ is a trace, this is equivalent to

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \langle \Omega, D(b\alpha^n(a))D(\alpha^n(a^*)b^*)\Omega\rangle$$
$$= \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \langle \Omega, D(b)D(\alpha^n(a))D(\alpha^n(a^*))D(b^*)\Omega\rangle\,.$$

Since $\lambda$ is a trace, this is equivalent to (2.4.2).

Note that from the manipulations above we also see that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \lambda(|D(b\alpha^n(a))|^2) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \omega(t\tau^n(s))$$

and

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \lambda(|D(b)D(\alpha^n(a))|^2) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \omega(E(t)\tau^n(E(s))$$

exist by Proposition 2.4.6, whether $\mathbf{A} \odot_{\mathbf{F}} \mathbf{A}'$ is ergodic relative to $\mathbf{F}$ or not, where $s = a \otimes (JaJ)$ and $t = b \otimes (JbJ)$.

Now, suppose (2.4.2) holds, then we have by the equivalences above, that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \omega([b\alpha^n(a)] \otimes [JbJ\alpha'^n(JaJ)])$$
$$= \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \omega([D(b)D(\alpha^n(a))] \otimes [D'(JbJ)D'(\alpha'^n(JaJ))]),$$

i.e.

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \omega((b \otimes (JbJ))\tau^n(a \otimes (JaJ)))$$
$$= \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \omega(E(b \otimes (JbJ))\tau^n(E(a \otimes (JaJ)))).$$

Because of the polarization identity, applied in turn to the two appearances of the sesquilinear form $A \times A \ni (a,c) \mapsto a \otimes (JcJ)$ above (once inside $\tau^n$ and once outside), (2.4.1) then follows, so $\mathbf{A} \odot_{\mathbf{F}} \mathbf{A}'$ is ergodic relative to $\mathbf{F}$ by Proposition 2.4.6. □

In order to proceed, we need the notion of relative ergodicity for a system itself:

**Definition 2.4.8.** We say that $\mathbf{A}$ *is ergodic relative to* $\mathbf{F}$ if $H^U \subseteq H_F$, where $H^U$ is the fixed point space of $U : H \to H$, and $H_F = \overline{F\Omega}$.

This generalizes ergodicity of $\mathbf{A}$, which is the special case $H^U = \mathbb{C}\Omega$.

**Remark 2.4.9.** In [Duv12, Definition 4.1] an alternative condition was used instead of $H^U \subseteq H_F$ to define relative ergodicity, namely

$$A^\alpha \subseteq F,$$

where $A^\alpha := \{a \in A : \alpha(a) = a\}$. For our purposes here, Definition 2.4.8 is the more convenient definition, but the question nevertheless arises whether the two conditions are equivalent. From [Duv12, Proposition 4.2] we know that $H^U = \overline{A^\alpha\Omega}$, so if $A^\alpha \subseteq F$, then $H^U \subseteq H_F$. This fact is used in Proposition 2.4.10.

We do not need the converse. However, it does hold, since $\mathbf{F}$ is a modular subsystem, as we now explain. The conditional expectation $D$ is determined by

$$D(a)|_{H_F} = Pa|_{H_F}$$

for all $a \in A$ (Proposition 1.4.4). The subalgebra $A^\alpha$ is easily seen to be globally invariant under the modular group as well (see [Duv12, Proposition 4.2]), hence we also have a unique conditional expectation $D_{A^\alpha} : A \to A^\alpha$ such that $\mu \circ D_{A^\alpha} = \mu$, which is similarly determined by

$$D_{A^\alpha}(a)|_{H^U} = Qa|_{H^U}$$

where $Q$ is the projection of $H$ onto $H^U$. Assuming $H^U \subseteq H_F$, it follows that

$$D(D_{A^\alpha}(a))|_{H^U} = PD_{A^\alpha}(a)|_{H^U} = Qa|_{H^U} = D_{A^\alpha}(a)|_{H^U}$$

and therefore $D(D_{A^\alpha}(a)) = D_{A^\alpha}(a)$, since $\Omega \in H^U$ is separating for $A$. So, for $a \in A^\alpha$, we have

$$a = D_{A^\alpha}(a) = D(D_{A^\alpha}(a)) \in F$$

which means that $A^\alpha \subseteq F$.

To summarize: $\mathbf{A}$ is ergodic relative to $\mathbf{F}$, if and only if $A^\alpha \subseteq F$.

The following generalizes the standard fact that weak mixing implies ergodicity:

**Proposition 2.4.10.** *If* **A** *is weakly mixing relative to* **F**, *then* **A** *is ergodic relative to* **F**.

*Proof.* From Proposition 2.3.3, we have $\lambda(|D(ba) - D(b)D(a)|^2) = 0$ for $a \in A^\alpha$ and all $b \in A$. Since $\lambda$ is faithful, it follows that $D(b(a - D(a))) = D(ba) - D(b)D(a) = 0$. In particular, setting $b = (a - D(a))^*$, we conclude that $a = D(a) \in F$, since $\mu$ is faithful and $\lambda \circ D = \mu$. So $A^\alpha \subseteq F$, hence $H^U \subseteq H_F$ by the first part of Remark 2.4.9. □

Next we consider a version of Proposition 2.4.6 for a system itself.

**Proposition 2.4.11.** **A** *is ergodic relative to* **F** *if and only if*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mu(b\alpha^n(a)) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \lambda(D(b)\alpha^n(D(a))) \tag{2.4.3}$$

*for all* $a, b \in A$. *Both limits exist, whether* **A** *is ergodic relative to* **F** *or not.*

*Proof.* Essentially the same argument, using the mean ergodic theorem, as in the proof of Proposition 2.4.6, but with $Q$ now the projection of $H$ onto $H^U$, and with $R$ replaced by $P$. □

Using the last three results, we can now prove the remaining direction of Theorem 2.4.2:

**Proposition 2.4.12.** *Assume that* $\mu$ *is tracial and that* **A** *is weakly mixing relative to* **F**. *Then* $\mathbf{A} \odot_{\mathbf{F}} \mathbf{A}'$ *is ergodic relative to* **F**.

*Proof.* Note that for all $a, b \in A$,

$$\begin{aligned} \lambda\left(|D(b\alpha^n(a)) - D(b)D(\alpha^n(a))|^2\right) &= \lambda(|D(b\alpha^n(a))|^2) \\ &\quad - \lambda(D(\alpha^n(a^*)b^*)D(b)D(\alpha^n(a))) \\ &\quad - \lambda(D(\alpha^n(a^*))D(b^*)D(b\alpha^n(a))) \\ &\quad + \lambda(D(\alpha^n(a^*))D(b^*)D(b)D(\alpha^n(a))). \end{aligned}$$

Consider the second term and use the trace property of $\mu$:

$$\begin{aligned} \lambda(D(\alpha^n(a^*)b^*)D(b)D(\alpha^n(a))) &= \lambda(D(\alpha^n(a^*)b^*D(b)D(\alpha^n(a)))) \\ &= \mu(\alpha^n(a^*)b^*D(b)D(\alpha^n(a))) \\ &= \mu(b^*D(b)\alpha^n(D(a)a^*)). \end{aligned}$$

Since **A** is ergodic relative to **F** by Proposition 2.4.10, we now have by Proposition 2.4.11 that

$$\begin{aligned}
&\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\lambda(D(\alpha^n(a^*)b^*)D(b)D(\alpha^n(a))) \\
&= \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\lambda(D(b^*)D(b)\alpha^n(D(a)D(a^*))) \\
&= \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\lambda(|D(b)D(\alpha^n(a))|^2)
\end{aligned}$$

Similarly

$$\begin{aligned}
&\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\lambda(D(\alpha^n(a^*))D(b^*)D(b\alpha^n(a))) \\
&= \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\lambda(|D(b)D(\alpha^n(a))|^2)
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\lambda(D(\alpha^n(a^*))D(b^*)D(b)D(\alpha^n(a))) \\
&= \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\mu(D(b^*)D(b)\alpha^n(aD(a^*))) \\
&= \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\lambda(|D(b)D(\alpha^n(a))|^2)
\end{aligned}$$

Keep in mind that all these limits exist by Proposition 2.4.11. Then by Proposition 2.3.3,

$$\begin{aligned}
0 &= \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\lambda\left(|D\left(b\alpha^n(a)\right) - D(b)D(\alpha^n(a))|^2\right) \\
&= \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}[\lambda(|D(b\alpha^n(a))|^2) - \lambda(|D(b)D(\alpha^n(a))|^2)],
\end{aligned}$$

so

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\lambda(|D(b\alpha^n(a))|^2) = \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\lambda(|D(b)D(\alpha^n(a))|^2),$$

since both limits exist (see Lemma 2.4.7). By Lemma 2.4.7 we are done. □

This completes the proof of Theorem 2.4.2. To summarize: the one direction is given by Proposition 2.4.5, the other by Proposition 2.4.12.

To connect this to the structure theorem in [AET11], we mention the following: Suppose that we have an asymptotically abelian W*-dynamical system $\mathbf{A}$ with a tracial invariant state, as defined in [AET11, Definition 1.10]. According to [AET11, Theorem 1.14] (and Proposition 2.3.7), such a system is weakly mixing relative to the *central system* $\mathbf{C} := (A \cap A', \mu|_{A\cap A'}, \alpha|_{A\cap A'})$. Theorem 2.4.2 then shows that $\mathbf{A} \odot_{\mathbf{C}} \mathbf{A}'$ is ergodic relative to $\mathbf{C}$.

# Chapter 3

# Relative Discrete Spectrum

In the classical case, systems with discrete spectrum relative to a factor (in the ergodic, not operator algebraic sense), appear in the celebrated ergodic theoretic proof of Szermerédi's Theorem ([Fur77]). Such systems, referred to as isometric extensions, were originally defined in terms of a type of skew product, and are equivalent to the underlying $L^2$-space being spanned by generalized eigenfunctions. In the review paper [FK78], almost periodic functions are used to define compact extensions. These two approaches are equivalent. More precisely, [Rob16, Section 4] using, a slightly different definition of almost periodic functions (but equivalent to the five properties appearing in [Fur14, Theorem 6.13]) shows that the closed subspace spanned by the almost periodic functions coincide with the closed subspace spanned by the generalized eigenfunctions. In this chapter we do not deal at all with noncommutative analogues of compact extensions.

Besides the original papers of Furstenberg and Zimmer, the books [Gla03] and [Tao09] provide an exposition on isometric extensions and compact extensions respectively. The approach of [Gla03] is based around the original papers of Furstenberg and Zimmer. In contrast, [Tao09] uses Hilbert C*-modules.

The chapter consists of two main parts. The first gives our noncommutative definition of relative discrete spectrum, defined as a complementary concept to relative weak mixing (expressed in terms of a closed subspace of $H$). A subsequent discussion presents various attempts at finding a viable definition of relative discrete spectrum. Next, our definition is shown to not only be a noncommutative generalization of classical isometric extensions, but also generalizes the noncommutative version of discrete spectrum. The second part, consisting of Sections 3.3 and 3.4, we discuss two examples of relative discrete spectrum. The first example (Section 3.3) is a skew product of a commutative system with a noncommutative one. The second (Section 3.4) is a purely noncommutative example of the von Neumann tensor product of two noncommutative systems, where the second system is finite dimensional.

We end the chapter with some open problems.

Note that throughout this chapter we will be working only with traces.

## 3.1 The Semicyclic System Revisited

Recall that $(A, \mu, \alpha)$ is a system with subsystem $(F, \lambda, \varphi)$ (Section 1.6), $\bar{H}$ is the semicyclic Hilbert space obtained from $(\langle A, e_F\rangle, \bar{\mu})$ and $\bar{U}$ is the unitary representation of $\bar{\alpha}$ on $\bar{H}$ (Section 1.2 and Remark 1.10.2).

We now turn our attention to expressing the GNS representation of the relatively independent joining $\omega$ (Section 2.2) in terms of $\bar{H}$ which is convenient for our subsequent work. We will construct a natural unitary operator $R : H_\omega \to \bar{H}$. In the classical case, such a result appears in [Pet11, pp. 63-64].

**Proposition 3.1.1.** *Recall that $(A, \mu, \alpha)$ is a system with $(A, \mu)$ as in Remark 1.1.3. Let $\mu$ be tracial. We have a uniquely determined well-defined unitary operator*

$$R : H_\omega \to \bar{H}$$

*satisfying $R(\gamma_\omega(a \otimes j(b))) = \gamma_{\bar{\mu}}(ae_F b)$ for all $a, b \in A$.*

*Furthermore,*

$$\bar{U} = RWR^*.$$

*with $\bar{U}$ as in Remark 1.10.2 and $W$ in (2.2.2).*

*Proof.* Let $a, c \in A$ and $b, d \in A'$. Since $j$ is linear, we may define $R_0 : A \odot A' \to \langle A, e_F\rangle$ via the prescription

$$R_0(a \otimes b) := ae_F j(b).$$

From the universal property of $A \odot A'$, $R_0$ is well-defined and linear. Note that $R_0(A \otimes A') \subseteq \mathcal{K}_{\bar{\mu}}$ with $\mathcal{K}_{\bar{\mu}} = \{x \in \langle A, e_F\rangle : \bar{\mu}(x^* x) < \infty\}$ as in (1.2.1). Indeed, take $a \otimes b \in A \odot A'$ and set

$$c := R_0(a \otimes b) = ae_F j(b).$$

Then, just as in the proof of [KR97b, Lemma 7.5.2],

$$\begin{aligned}\bar{\mu}(c^* c) &= \bar{\mu}(j(b^*)e_F a^* a e_F j(b)) \\ &\leq \|a^* a\| \, \bar{\mu}(j(b^*)e_F j(b)) \\ &= \|a\|^2 \, \mu(j(b)^* j(b)) < \infty,\end{aligned}$$

since $x^*y^*yx \leq \|y^*y\|\, x^*x$ in a unital $C^*$-algebra, as $y^*y \leq \|y^*y\|$ from the spectral theorem. Thus $c \in \mathcal{K}_{\bar{\mu}}$. Hence, we can consider

$$R : \gamma_{\bar{\mu}}(A \odot A') \to \bar{H} : \gamma_\omega(t) \mapsto \gamma_{\bar{\mu}}(R_0(t)).$$

We need to show that $R$ is well-defined, and uniquely extends to a unitary operator $H_\omega \to \bar{H}$. Note that since $j(f)\Omega = Jf^*\Omega = f\Omega$ for all $f \in F$, we have

$$D'(b)\Omega = D(j(b))\Omega. \tag{3.1.1}$$

Thus, for all $a, c \in A$ and $b, d \in A'$,

$$\begin{aligned}\langle \gamma_{\bar{\mu}}(R_0(a \otimes b)), \gamma_{\bar{\mu}}(R_0(c \otimes d))\rangle_{\bar{\mu}} =& \langle \gamma_{\bar{\mu}}(ae_F j(b)), \gamma_{\bar{\mu}}(ce_F j(d))\rangle_{\bar{\mu}} \\ =& \bar{\mu}(j(b^*)e_F a^* ce_F j(d)) \\ =& \bar{\mu}(e_F a^* ce_F j(d) j(b^*) e_F) \\ =& \bar{\mu}(D(a^*c)e_F D(j(b^*d))) \\ =& \mu(D(a^*c)D(j(b^*d))) \\ =& \langle \Omega, D(a^*c)D'(b^*d)\Omega\rangle \\ =& \langle \Omega, \delta \circ (D \odot D')((a^*c) \otimes (b^*d))\Omega\rangle \\ =& \omega((a^*c) \otimes (b^*d)) = \omega((a \otimes b)^*(c \otimes d)) \\ =& \langle \gamma_\omega(a \otimes b), \gamma_\omega(c \otimes d)\rangle_\omega.\end{aligned}$$

So it follows that for all $s, t \in A \odot_F A'$,

$$\langle \gamma_{\bar{\mu}}(R_0(s)), \gamma_{\bar{\mu}}(R_0(t))\rangle_{\bar{\mu}} = \langle \gamma_\omega(s), \gamma_\omega(t)\rangle_\omega. \tag{3.1.2}$$

Thus, $R$ is well-defined (as $\gamma_\omega(t) = 0$ implies $\gamma_{\bar{\mu}}(R_0(t)) = 0$) and can be extended to an isometric linear operator, still denoted by $R$, from $H_\omega$ to $\bar{H}$. From Proposition 1.5.5, $\gamma_{\bar{\mu}}(Ae_F A)$ is dense in $\bar{H}$. It follows that $R\gamma_\omega(A \odot A') = \gamma_{\bar{\mu}}(R_0(A \odot A')) = \gamma_{\bar{\mu}}(Ae_F A)$ is dense in $\bar{H}$. Hence, $RH_\omega = \bar{H}$ and therefore $R$ is a unitary operator.

Now we carry over the dynamics of $H_\omega$ onto $\bar{H}$. Recall that the dynamics on $H_\omega$ is given by

$$W\gamma_\omega(a \otimes b) = \gamma_\omega(\alpha(a) \otimes \alpha'(b)) = \gamma_\omega((UaU^*) \otimes (UbU^*)).$$

For $a, b \in A$,

$$\begin{aligned}RWR^*(\gamma_{\bar{\mu}}(ae_F b)) =& RW\gamma_\omega(a \otimes j(b)) = R\gamma_\omega(\alpha(a) \otimes j(\alpha(b))) \\ =& \gamma_{\bar{\mu}}(\alpha(a)e_F\alpha(b)) = \gamma_{\bar{\mu}}(\bar{\alpha}(ae_F b)) \\ =& \bar{U}(\gamma_{\bar{\mu}}(ae_F b)),\end{aligned}$$

which implies that $\bar{U} = RWR^*$. □

Note that we can express the relatively independent joining in terms of $\bar{\mu}$ using $R$ :

$$\mu \odot_F \mu'(a \otimes b) = \bar{\mu}(e_F a e_F j(b)) = \bar{\mu}(D(a) e_F D(j(b)).$$

Indeed, We have

$$\begin{aligned}\mu \odot_F \mu'(t) =& \omega(1^* t) = \langle \gamma_\omega(1), \gamma_\omega(t) \rangle \\ =& \langle R(\gamma_\omega(1)), R(\gamma_\omega(t)) \rangle \\ =& \bar{\mu}(e_F R_0(t)) \qquad \text{since } R_0(1) = e_F.\end{aligned}$$

In particular, for $t = a \otimes b$ with $a \in A$ and $b \in A'$,

$$\mu \odot_F \mu'(a \otimes b) = \bar{\mu}(e_F a e_F j(b)).$$

From the definition of $\omega$,

$$\begin{aligned}\omega(a \otimes b) =& \langle \Omega, D(a) D(b) \Omega \rangle \\ =& \langle \Omega, D(a) j(D(b)) \Omega \rangle \\ =& \langle \Omega, D(a) D(j(b)) \Omega \rangle \\ =& \mu(D(a) D(j(b))) \\ =& \bar{\mu}(D(a) e_F D(b)),\end{aligned} \tag{3.1.3}$$

(3.1.3) follows from $j(c)\Omega = Jc^*\Omega = c\Omega$, for every $c \in A'$, using Proposition 1.3.1(d) and the fact that $J$ is also the modular conjugation operator associated to $\mu'$ ([BR02, Proposition 2.5.11].

If $H_\omega^W$ denotes the vector space of all fixed points of $W$, then

$$\bar{H}^{\bar{U}} := R H_\omega^W,$$

must be the fixed points of $\bar{U}$. We also have a copy of $H_\lambda$ in $\bar{H}$ :

$$\begin{aligned}\bar{H}_\lambda :=& R H_\lambda = R\overline{\gamma_\omega(1 \otimes \tilde{F})} \quad \text{from (2.2.3)} \\ =& \overline{R\gamma_\omega(1 \otimes \tilde{F})} \\ =& \overline{\gamma_{\bar{\mu}}[R_0(1 \otimes \tilde{F})]} \\ =& \overline{\gamma_{\bar{\mu}}(e_F F)}.\end{aligned} \tag{3.1.4}$$

## 3.2 Relative discrete spectrum

Having obtained our unitary equivalence $R$ in Proposition 3.1.1, we can equivalently restate relative ergodicity (Definition 2.4.1) from a "basic construction" point-of-view:

**Definition 3.2.1.** We say that $\mathbf{A} \odot_{\mathbf{F}} \mathbf{A}'$ is ergodic relative to a subsystem $\mathbf{F}$ of $\mathbf{A}$, if $\bar{H}^{\bar{U}} \subseteq \bar{H}_\lambda$.

We shall continue to use such a point-of-view to define relative discrete spectrum.

The inspiration for our noncommutative definition of relative discrete spectrum is based on the treatment appearing in [Gla03] of the original work of Furstenberg and Zimmer (see the notes on [Gla03, p. 193]). The $U$-$\bar{\mu}$-modules (Definition 3.2.3) play a role analogous to that of the finite rank modules appearing in [Gla03, Definition 9.2] and [Gla03, Definition 9.10]. However, unlike [Gla03], we do not use an analogue of generalized eigenfunctions. Instead we opt to directly use the $U$-$\bar{\mu}$-modules to define a subspace analogous to the vector space $\mathcal{E}(\mathbf{X}/\mathbf{Y})$ of all generalized eigenfunctions appearing in [Gla03, Definition 9.10].

**Definition 3.2.2.** Given a closed subspace $V$ of $H$, denote the projection of $H$ onto $V$ by $P_V$. We call $V$ a *right-$F$-submodule* (of $H$) if $VF \subseteq V$, i.e. if $xa \in V$ for all $x \in V$ and for all $a \in F$.

**Definition 3.2.3.** Suppose $V \subseteq H \ominus H_F$ (the orthogonal complement of $H_F$ in $H$) is a right-$F$-submodule. Call $V$ a *$U$-$\bar{\mu}$-module* if $V$ satisfies

$$\bar{\mu}(P_V) < \infty \quad \text{and} \quad UV = V.$$

**Definition 3.2.4.** By $\mathcal{E}_{A/F}$ denote the closed subspace of $H \ominus H_F$ spanned by all $U$-$\bar{\mu}$-modules.

We now want to capture the idea that relative weak mixing and relative discrete spectrum exist as complementary concepts ([Tao09, §12.4] presents this point of view in the commutative case). It is based on the following result, the one direction of which is proven in [AET11, Proposition 3.8], although they also mention that the other direction holds. We prove the latter using Theorem 2.4.2.

**Theorem 3.2.5.** *The system* $\mathbf{A}$ *is weakly mixing relative to* $\mathbf{F}$ *if and only if* $\mathcal{E}_{A/F} = \{0\}$.

*Proof.* Note that the statement of the theorem can be rephrased as follows: The system $\mathbf{A}$ is weakly mixing relative to $\mathbf{F}$ if and only if there are no non-trivial $U$-$\bar{\mu}$-modules.

That **A** is weakly mixing relative to **F** holds if there are no non-trivial $U$-$\bar{\mu}$-modules, follows from Definition 2.3.1, Proposition 2.3.7 and [AET11, Proposition 3.8]. We prove the converse as follows:

Assume there is a non-trivial $U$-$\bar{\mu}$-module $V$.Hence, $P_V \in \mathcal{K}_{\bar{\mu}}$ and we can set

$$x := \gamma_{\bar{\mu}}(P_V) \in \bar{H}.$$

As $UV = V$, we have $\bar{\alpha}(P_V) = UP_V U_\alpha^* = P_V$. Hence, $x \in \bar{H}^{\bar{U}}$, with $x \neq 0$, since $P_V \neq 0$ and $\bar{\mu}$ is faithful.

Since $P_V e_F = 0$,

$$\langle x, \gamma_{\bar{\mu}}(e_F a)\rangle_{\bar{\mu}} = \bar{\mu}(P_V^* e_F a) = 0,$$

for all $a \in F$. Hence, from (3.1.4), $x \perp \bar{H}_\lambda$, so $x \notin \bar{H}_\lambda$ (since $x \neq 0$) and thus $\bar{H}^{\bar{U}} \nsubseteq \bar{H}_\lambda$.

In other words, $\mathbf{A} \odot_{\mathbf{F}} \mathbf{A}'$ is not ergodic relative to **F**. By Theorem 2.4.2 we are done. □

Motivated by the above result, we give the main definition of this chapter:

**Definition 3.2.6.** We say that the system **A** has *discrete spectrum relative to* **F** if $\mathcal{E}_{A/F} = H \ominus H_F$. Alternative terminology for this is to say that **A** is an *isometric extension* of **F**.

Thus relative weak mixing and relative discrete spectrum correspond to the two extremes of $\mathcal{E}_{A/F}$, and are, in this sense, complementary.

In the remainder of this section we show that the classical definition of relative discrete spectrum as well as the absolute case of noncommutative discrete spectrum are special cases of this definition, confirming that it is a sensible definition in a noncommutative framework. We also make some remarks on other possible definitions and their deficiencies compared to Definition 3.2.6. In the next two sections we then study examples.

The following is easy to show since $e_F$ is a projection of finite lifted trace and $H_F$ is $U$-invariant as $\alpha(F) \subseteq F$, and is a convenient characterization of relative discrete spectrum:

**Proposition 3.2.7.** *A system* **A** *has discrete spectrum with respect to* **F** *if and only if the following condition holds:*

(†) *there exists a family $\mathcal{U}$ of $U$-invariant right-$F$-modules such that their corresponding projections are of finite lifted trace and*

$$\overline{\operatorname{span}\{x : x \in V, V \in \mathcal{U}\}} = H.$$

*Proof.* Suppose $\mathbf{A}$ has discrete spectrum relative to $\mathbf{F}$. Then there is a family $\mathcal{W}$ of $U$-invariant right-$F$-modules whose corresponding projections are of finite lifted trace and $\operatorname{span}\mathcal{W} = H \ominus H_F$, where we abuse notation and write $\operatorname{span}\mathcal{W} = \operatorname{span}\{x \in V : V \in \mathcal{W}\}$. Now $H_F$ is a right-$F$-module, since $e_F \in \langle A, e_F\rangle$ and $\langle A, e_F\rangle$ contains all the right-$F$-modules (Proposition 1.5.6). Since $\alpha(F) = F$, we have that $U(e_F(H)) = e_F(H)$. We also have $\bar{\mu}(e_F) = \bar{\mu}(1e_F1) = \mu(1{\cdot}1) = 1$, so that $e_F$ is of finite lifted trace. So we have (†) with $\mathcal{U} = \mathcal{W} \cup \{e_F\}$. Conversely if (†) holds, then we make the same observation that $e_F$ is a $U$-invariant projection whose image corresponds to a right-$F$-module and is of finite lifted trace to conclude that $\mathbf{A}$ has discrete spectrum relative to $\mathbf{F}$. □

**Remark 3.2.8.** There were a number of attempts from the author to define relative discrete spectrum in a way closer to the commutative case. However, each approach suffered from ambiguity or technical difficulty.

We outline these approaches in a more informal manner than is used in the rest of the thesis.

Let us remark that compact extensions (in the sense of [Tao09] are isometric extensions (in the sense of [Fur77]) as proved in the notes [ZK]. We do not know if compact extensions in the sense of [Tao09]) are compact (in the sense of [Fur14]) or isometric (in the sense of [Gla03]).

If one considers the absolute classical case, then compact systems are those whose orbits are relatively compact (the closure of each orbit is compact). Relative compactness can be characterised using zonotopes.

Along these lines, [Tao09, Section 2.13] uses the notion of an $L^\infty$-Hilbert module to define what he calls $L^2(X \mid Y)$. Here the measure preserving dynamical system $(X, \mathcal{X}, \mu, T)$ is an extension of $(Y, \mathcal{Y}, \nu, S)$. Functions in $L^2(X \mid Y)$ whose orbits are relatively compact (in a sense that we will not define here) are called *conditionally almost periodic*. Functions $f \in L^2(X \mid Y)$ which satisfy the following condition are referred to as *conditionally almost periodic in measure*: for all $\epsilon > 0$ there is a $E \in \mathcal{Y}$ satisfying $\nu(Y) \le \epsilon$ such that

$$f\chi_{(Y\setminus E)} \text{ is conditionally almost periodic,} \tag{3.2.1}$$

where, $\chi_E$ is the indicator function of the set $E$. [Tao09] calls $(X, \mathcal{X}, \mu, T)$ a compact extension of $(Y, \mathcal{Y}, \nu, S)$ if every function in $L^2(X \mid Y)$ is conditionally almost periodic in measure. Now it is not so clear what the noncommutative analogue of (3.2.1) would be. For instance, how do we interpret the existence of the noncommutative analogue of $\chi_{Y\setminus E}$? Should we, given $\epsilon > 0$, find a projection $p_\epsilon$ whose trace is less than $\epsilon$?

Rather than working with an $L^\infty$-Hilbert module perhaps we should remain in an $L^2$-space? We are in a metric space. So, in the absolute case, we can use total boundedness to characterise relative compactness of the orbits. It would therefore make sense to use an analogous form of total boundedness, in the relative case, to define almost periodic functions.

For instance,

**Definition 3.2.9.** (Definition 3.4.3 [Pet11]) Let $\Gamma$ be a countable group and let $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$ be an extension of a probability measure preserving action $\Gamma \curvearrowright (X, \mathcal{A}, \mu)$. A function $f \in L^2(X, \mathcal{B}, \mu)$ is almost periodic relative to $L^\infty(X, \mathcal{A}, \mu)$ if for all $\epsilon > 0$, there exist $g_1, g_2, \ldots, g_n \in L^\infty(X, \mathcal{B}, \mu)$ such that for all $\gamma \in \Gamma$ we have $k_1^\gamma, k_2^\gamma, \ldots, k_n^\gamma \in L^2(X, \mathcal{A}, \mu)$ such that

$$\left\| \sigma_\gamma(f) - \sum_{j=1}^{n} k_j^\gamma g_j \right\|_2 < \epsilon, \tag{3.2.2}$$

where $\sigma_\gamma$ denotes the group action of $\Gamma$ on $L^2(X, \mathcal{B}, \mu)$ via the Koopman representation.

Note that the functions $g_i$ appearing in (3.2.2) are purposely written on the right. We call $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$ a compact extension of $\Gamma \curvearrowright (X, \mathcal{A}, \mu)$ provided that every $f \in L^2(X, \mathcal{B}, \mu)$ is almost periodic with respect to $L^\infty(X, \mathcal{A}, \mu)$. This definition therefore has the advantage that we do not refer to conditional almost periodicity in measure.

We could propose the following definition for almost periodic functions in the noncommutative case:

**Definition 3.2.10.** Let $\mathbf{A}$ be a system with $\mathbf{F}$ a subsystem of $\mathbf{A}$. Call a vector $f \in H$ almost periodic if the following property is satisfied: For every $\epsilon > 0$, there is an $r \in \mathbb{N}$ and operators $h_1, h_2, \ldots, h_r, g_1, g_2, \ldots, g_r$ in $A$ such that for every $n \in \mathbb{Z}$ there are vectors $k_1^{(n)}, k_2^{(n)}, \ldots, k_r^{(n)} \in H$,

$$\left\| U^n(f) - \sum_{i=1}^{r} h_i j(g_i) k_i^{(n)} \right\| < \epsilon, \tag{3.2.3}$$

where $j$ is as in (1.3.2).

Such vectors $f$ in Definition 3.2.10 could be referred to as "two-sided" almost periodic vectors, since we have operators acting on the left and the right (via $j$). We could also have a "right-sided" version (in (3.2.3) replace $\sum_{i=1}^{r} h_i j(g_i) k_i^{(n)}$ with $\sum_{i=1}^{r} h_i k_i^{(n)}$). Similarly we could have a "left-sided" version. Thus we have ambiguity as to the "handedness" of the sums appearing in (3.2.3). Additionally, it is not clear how to select the operators $h_i$ and $g_i$.

We turned to [Gla03]'s approach to isometric extensions, in the classical case, via generalized eigenfunctions. Again, considering $\mathbf{X} = (X, \mathcal{X}, \mu, T)$ an extension of $\mathcal{Y} = (Y, \mathcal{Y}, \nu, S)$, if $\mathcal{E}(X/Y)$ denotes the vector space of all generalized eigenfunctions, call $\mathbf{X}$ an isometric extension of $\mathbf{Y}$ provided $L^2(X) = \overline{\mathcal{E}(X/Y)}$.

In an attempt to obtain a noncommutative analogue of generalized eigenfunction, we will need the the concept of "finite rank".

**Definition 3.2.11.** ([AET11, Definition 3.5])A left- (respectively right-) $F$-submodule $W$ of $H$ has *finite rank* if there are some $x_1, x_2, \ldots x_r \in W$ such that $W = \overline{\sum_{i=1}^{r} Fx_i}$ (respectively, $W = \overline{\sum_{i=1}^{r} j(F)x_i}$).

We would like to replicate the definition of "generalized eigenvector" in the noncommutative setting. However, we obtain a number of possible variations. Note, as before as in Definition 3.2.10, we have left-, right- and two-sided variants.

**Definition 3.2.12.** Let $v \in H$. Then call $v$

1. a point eigenvector if $v$ is contained in a $U$-invariant finite rank module;

2. an orbit eigenvector if the orbit of $v$ is contained in some finite rank submodule; equivalently, the submodule spanned by the orbit of $v$ is contained in some finite rank module.

Note that our Definition 3.2.6 eliminates the need to choose a type of eigenvector while Definition 3.2.4 allows us to make a choice of "handedness" due to Proposition 1.5.6.

We turn our attention to checking that our definition of relative discrete spectrum generalizes the classical notion of relative discrete spectrum, defined as follows (see [Gla03, Definition 9.10]):

**Definition 3.2.13.** Assume that $\mathbf{A}$ is a classical system, i.e. $A = L^\infty(\eta)$ for a standard probability space $(Y, \Sigma, \eta)$ (see Section A.1). A $F$-submodule $V$ of $H = L^2(\eta)$ is said to be of *finite rank* if there are $x_1, ..., x_n \in V$ such that

$$V = \overline{\left\{ \sum_{i=1}^{n} a_i x_i : a_1, ..., a_n \in F \right\}},$$

where $a_j x_j$ is simply pointwise multiplication of functions. We call $x \in H$ an *$F$-eigenvector* of $U$ if $x$ belongs to some $U$-invariant finite rank $F$-module. If $H$ is spanned by the $F$-eigenvectors of $U$, then we say that $\mathbf{A}$ has *relative discrete spectrum over* $\mathbf{F}$ *in the classical sense.*

This is indeed a special case of Definition 3.2.6 as is proved below in Proposition 3.2.14. The proof uses direct integral theory, as it is used in [AET11, Lemma 4.1]. This is why we assume that $(X, \mathcal{X}, \eta)$ is standard, as it ensures that $L^2(\eta)$ is separable (Corollary A.2.4).

**Proposition 3.2.14.** *Assume that* $\mathbf{A}$ *is a classical system, i.e.* $A = L^\infty(\eta)$ *for a standard probability space* $(X, \mathcal{X}, \eta)$ *and* $\alpha(f) = f \circ T$ *for some fixed invertible map* $T : X \to X$ *satisfying* $\eta(Z) = \eta(T^{-1}(Z))$ *for all* $Z \in \mathcal{X}$*. The system* $\mathbf{A}$ *has discrete spectrum relative to* $\mathbf{F}$ *(in the sense of Definition 3.2.6) if and only if it has relative discrete spectrum over* $\mathbf{F}$ *in the classical sense.*

*Proof.* ($\Rightarrow$) Assume that $\mathbf{A}$ has discrete spectrum relative to $\mathbf{F}$. The approach of the proof is to express any $U$-$\bar{\mu}$ module $V$ as the direct sum of finite rank modules, using ideas from the proof of [AET11, Lemma 4.1].

Using [KR97b, Theorem 14.2.1], since $F$ is commutative, we have a unitary operator $\Phi : H \to H_\oplus$ where $H_\oplus$ is a direct integral $H_\oplus = \int_Y^\oplus H_p \, \mathrm{d}\nu(p)$ of Hilbert spaces $H_p$ indexed by some standard probability space $(Y, \mathcal{Y}, \nu)$. Thus, in particular, any statement about a module $V$ in $H_\oplus$ has a corresponding statement about $\Phi^{-1}(V)$ in $H$.

The von Neumann algebra $F$ is then identified with the von Neumann algebra of all diagonalizable operators $\phi(F) := \Phi F \Phi^{-1} = \{M_f : f \in L^\infty(\nu)\}$. Each $M_f \in \mathcal{B}_\oplus$ acts on elements $x \in H_\oplus$ as multiplication operators via the equality $(M_f x)(p) = f(p)x(p)$ for almost all $p \in X$. Given any $U$-$\bar{\mu}$-module $V$, then as in the proof of [AET11, Lemma 4.1] we can write

$$\Phi(V) = \int_Y^\oplus V_p \, \mathrm{d}\nu(p),$$

for a measurable field of Hilbert subspaces $V_p \subseteq H_p$. (Details can be found in [Kin17, Section 4.6]).

We shall now express $\Phi V$ as a direct sum of $\phi(F)$-modules of finite rank. For each $n \in \mathbb{N} \cup \{\infty\}$ write

$$Y_n := \{p \in Y : \dim(H_p) = n\}.$$

Each $Y_n$ turns out to be measurable [KR97b, Remark 14.1.5]. Consider the projections $M_{\chi_{Y_n}}$ and define

$$V_n := \int_{Y_n} V_p \, \mathrm{d}\nu(p) = M_{\chi_{Y_n}}(V),$$

where $\chi_{Y_n}$ denote the indicator functions. Since $\bar{\mu}(V) < \infty$, $p \mapsto \dim(V_p)$ is integrable, so $\nu(Y_\infty) = 0$, hence $V_\infty = 0$. and the collection $\{Y_n : n \in \mathbb{N}\}$ satisfies $\nu(\cup_{n\in\mathbb{N}} Y_n) = 1$. It follows that $\Phi(V)$ can be identified with $\oplus_{n\geq 1} V_n$.

It is now straightforward to verify that each $\Phi^{-1}(V_n)$ is a $U$-$\bar{\mu}$-module. We have, for every $f \in F$,

$$f\phi^{-1}(M_{\chi_{Y_n}})(H) = \phi^{-1}(M_{\chi_{Y_n}})f(H) \subseteq \phi^{-1}(M_{\chi_{Y_n}})(H),$$

so that each $V_n$ is a right $\phi(F)$-module.

In a similar way to the proof of [AET11, Lemma 4.1], $\alpha$ induces dynamics on $Y$ leaving each $Y_n$ invariant, which in turn means that each $V_n$ is $U$-invariant. (We refer the interested reader to [Kin17, Proposition 4.7.6] for the details.)

By construction, $\dim(V_p) = n$ whenever $p \in Y_n$ and it follows that $\Phi^{-1}(V_n)$ is of finite rank.

Hence $\Phi(V)$ consists solely of $\phi(F)$-eigenvectors. It follows that $H \ominus H_F$ is spanned by $F$-eigenvectors. Since $H_F = \overline{F\Omega}$, $H_F$ is spanned by the $F$-eigenvector $\Omega$. Hence $H$ is spanned by $F$-eigenvectors as required.

($\Leftarrow$) We now prove the converse. Assume that $\mathbf{A}$ has relative discrete spectrum over $\mathbf{F}$ in the classical sense. The key idea that we use here is that the projection $P_V$ corresponding to a finite rank $F$-module $V$ satisfies $\bar{\mu}(P_V) < \infty$.

We have that $H = \overline{\operatorname{span}\mathcal{E}}$, where $\mathcal{E} = \{x_1, x_2, \ldots, x_n\}$ is the set of all $F$-eigenvectors of $U$.

Let $x \in \operatorname{span}\mathcal{E}$ and write

$$x = \sum_{i=1}^{n} t_i e_i$$

for some $t_i \in \mathbb{C}$. For each $i$ there is some $U$-invariant $F$-module $V_i$ of finite rank such that $e_i \in V_i$. Denote by $\mathcal{U}$, the collection of all $U$-invariant right-$F$-modules $V$ with $\bar{\mu}(P_V) < \infty$. It is here that we use our key idea: To prove the proposition, it is sufficient to show that all the modules $V_i$ have corresponding projections of finite lifted trace. Indeed, it will follow that $x \in \operatorname{span}\mathcal{U}$ and hence

$$H = \overline{\operatorname{span}\mathcal{E}} \subseteq \overline{\operatorname{span}\mathcal{U}} \subseteq H,$$

which is sufficient to show that $\mathbf{A}$ has discrete spectrum with respect to $\mathbf{F}$, by Proposition 3.2.7.

Consider then any finite rank $F$-module $V := \overline{\left\{\sum_{i=1}^{t} f_i v_i : f_i \in F\right\}}$.

We now give a description of $\phi(P_V)(p)H_p$ for almost all $p$. Put $w_i := \Phi v_i$ for each $i = 1, 2, \ldots, t$. Thus,

$$\phi(P_V)(H_\oplus) = \Phi(V) = \overline{\left\{ \sum_{i=1}^{t} M_{g_i} w_i : g_i \in L^\infty(Y) \right\}}.$$

Hence all vectors of the form $M_g w$ for $g \in L^\infty(Y)$ and $w \in \{w_i : i = 1, 2, \ldots, t\}$ form a dense spanning set for $\Phi(V)$ and thus, from [KR97b, Lemma 14.1.3], for almost all $p$,

$$\begin{aligned} \phi(P_V)(p)H_p &= \overline{\left\{ \sum_{i=1}^{t} g_i(p) w_i(p) : g_i \in L^\infty(Y) \right\}} \\ &= \operatorname{span}\{w_i(p) : i = 1, 2, \ldots, t\}. \end{aligned}$$

We thus have,

$$\bar{\mu}(P_V) = \int_Y \dim((\phi(P_V))(p)H_p) \, \mathrm{d}\nu(p)) \leq \int_Y t \, \mathrm{d}\nu(p) = t < \infty.$$

□

We now consider another special case of Definition 3.2.6 when $F = \mathbb{C}1$ and $\lambda = \mu|_F$. We take note that the basic construction, in this case, is given by $\langle A, e_F \rangle = JF'J = J\mathcal{B}(H)J = \mathcal{B}(H)$, using Proposition 1.5.2(c). Thus, since the trace on $\mathcal{B}(H)$ is unique up to nonzero scalar multiples, we may take $\bar{\mu}$ to be the canonical trace Tr on $\mathcal{B}(H)$. In particular, this means that our $U$-$\bar{\mu}$-modules are all the finite dimensional subspaces of $H$.

**Proposition 3.2.15.** *Let* $\mathbf{A} = (A, \mu, \alpha)$ *be a system and* $\mathbf{F}$ *be the trivial system i.e* $F = \mathbb{C}1$, $\lambda = \mu|_F$, *and* $\varphi = \alpha|_F$. *Then* $\mathbf{A}$ *has discrete spectrum relative to* $\mathbf{F}$, *if and only if* $\mathbf{A}$ *has discrete spectrum i.e.* $H$ *is spanned by the eigenvectors of* $U$.

*Proof.* Let $\mathcal{E}$ denote the set of all eigenvectors of $U$. Assume that $\mathbf{A}$ has discrete spectrum, that is, $\overline{\operatorname{span} \mathcal{E}} = H$. For any $x \in \mathcal{E} \cap (H \ominus H_F)$, let

$$S_x := \{tx : t \in \mathbb{C}\}.$$

Then it easy to verify that $S_x$ is a $U$-$\bar{\mu}$-module.

Indeed, $S_x$ is $U$-invariant: for every $t \in \mathbb{C}$, if $s$ is the eigenvalue of $x$, then $U(tx) = tU(x) = stx \in S_x$. Since $S_x$ is finite-dimensional, it is closed and $P_{S_x} \in \mathcal{B}(H) = \langle A, e_F \rangle$. Moreover,

$$H \ominus H_F = \overline{\operatorname{span}\{S_x : x \in \mathcal{E} \cap (H \ominus H_F)\}} \subseteq H \ominus H_F.$$

Thus, $\mathbf{A}$ has discrete spectrum relative to $\mathbf{F}$.

Conversely, assume that $\mathbf{A}$ has discrete spectrum relative to $\mathbf{F}$. Then, as remarked above, all $U$-$\bar{\mu}$-modules $V$ have finite dimension. As $U|_V$ is bijective onto $V$, and

$$\langle x, y\rangle = \langle Ux, Uy\rangle = \langle U|_V x, U|_V y\rangle,$$

for all $x, y \in V$, we have that $U|_V$ is a unitary operator on $V$. Thus, the normal operator $U|_V$ is unitarily diagonalizable: there exists an basis $\mathcal{E}_V$ for $V$ consisting solely of eigenvectors of $U|_V$. If $\mathcal{U}$ denotes the collection of all $U$-$\bar{\mu}$-modules consider

$$\mathcal{E} = \bigcup_{V\in\mathcal{U}} \mathcal{E}_V \cup \{\Omega\}.$$

We wish to verify that $H \subseteq \overline{\operatorname{span}\mathcal{E}}$. If $x \in H_F$ then $x = t\Omega$ for some $t \in \mathbb{C}$ and thus $x \in \overline{\operatorname{span}\mathcal{E}}$. If $x \in H \ominus H_F$, then $x \in \overline{\operatorname{span}(\mathcal{E}\backslash\{\Omega\})}$, by our assumption that $\mathbf{A}$ has discrete spectrum relative to $F$. Consequently, we can find a sequence, $(x_n) \subseteq \operatorname{span}(\mathcal{E}\backslash\{\Omega\})$, such that $x_n \to x$. For each $n \in \mathbb{N}$, there is a $k_n \in \mathbb{N}$ such that $x_n = \sum_{i=1}^{k_n} t_i^{(n)} v_i^{(n)}$ for some $t_i^{(n)} \in \mathbb{C}$ and some $v_i^{(n)} \in \mathcal{E}\backslash\{\Omega\}$.

As each $v_i^{(n)}$ is a finite linear combination of the eigenvectors of $U$ so too is each $x_n$. Thus, $x \in \overline{\operatorname{span}\mathcal{E}}$. As we already have $\overline{\operatorname{span}\mathcal{E}} \subseteq H$, it follows that $\mathbf{A}$ has discrete spectrum.

□

## 3.3 Skew Products

In order to show that the definition of relative discrete spectrum (Definition 3.2.6) is sensible, we still need to exhibit some examples. This is what we do in this section and the next.

In this section we focus on a skew product, starting with a classical system and extending it by a noncommutative one.

The following result will be useful for our examples:

**Proposition 3.3.1.** *Let $(B, \nu)$ and $(C, \sigma)$ be von Neumann algebras with faithful normal tracial states $\nu$ and $\sigma$, both in their GNS representations on the Hilbert spaces $H_\nu$ and $H_\sigma$, with cyclic vectors $\Omega_\nu$ and $\Omega_\sigma$, respectively. Consider the Hilbert space $H := H_\nu \otimes H_\sigma$, the von Neumann tensor product $A := B\bar{\otimes}C$ and the faithful normal trace $\mu := \nu\bar{\otimes}\sigma$ ([KR97b, Proposition*

*11.2.3]). Set $F := B \otimes 1$ with state $\lambda := \mu|_F$. Then $\langle A, e_F\rangle = B\bar{\otimes}\mathcal{B}(H_\sigma)$. The trace $\bar{\mu}$ of $\langle A, e_F\rangle$ is given by*

$$\bar{\mu}(t) = \sum_{i\in\mathcal{I}}\langle \Omega_\nu \otimes h_i, t(\Omega_\nu \otimes h_i)\rangle = \mu\bar{\otimes}\mathrm{Tr}(t), \tag{3.3.1}$$

*for all $t \in \langle A, e_F\rangle^+$, where $\{h_i : i \in \mathcal{I}\}$ is any orthonormal basis for $H_\sigma$, $J_\sigma$ is the modular conjugation for $(C, \sigma)$ and* Tr *is the canonical trace on $\mathcal{B}(H_\sigma)$.*

*Proof.* Let $J_\nu$,$J_\sigma$ and $J = J_\nu \oplus J_\sigma$ denote the modular conjugation operators associated to $\nu$, $\sigma$, and $\mu$, respectively. By Proposition 1.5.2 and [SZ79, Section 10.7 Lemma 1] we have

$$\begin{aligned} \langle A, e_F\rangle &= (J_\nu \otimes J_\sigma)(B'\bar{\otimes}\mathcal{B}(H_\sigma))(J_\nu \otimes J_\sigma) \\ &= (J_\nu B' J_\nu)\bar{\otimes}(J_\sigma\mathcal{B}(H_\sigma)J_\sigma) = B\bar{\otimes}\mathcal{B}(H_\sigma). \end{aligned} \tag{3.3.2}$$

We compute the lifted trace using Lemma 1.5.4. To do this, we need a set $\{v_i : i \in \mathcal{I}\}$ in $\langle A', e_F\rangle$ (indexed by some set $\mathcal{I}$) such that $\sum_i v_i^* e_F v_i = 1$ (see Remark 3.3.2 below). Let

$$v_i = 1 \otimes w_i$$

where, for all $z \in H_\sigma$,

$$w_i z := \langle l_i, z\rangle\Omega_\sigma$$

and $l_i := J_\sigma h_i$.

Note that,

$$\begin{aligned} \langle A', e_F\rangle &= \langle JAJ, Je_F J\rangle = J\langle A, e_F\rangle J \\ &= (J_\nu B J_\nu)\bar{\otimes}(J_\sigma B(H_\sigma)J_\sigma) \\ &= B'\bar{\otimes}\mathcal{B}(H_\sigma). \end{aligned}$$

So we have $v_i \in \langle A', e_F\rangle$.

In terms of the projection $P$ of $H_\sigma$ onto $\mathbb{C}\Omega_\sigma$ we have,

$$v_i^* e_F v_i = 1 \otimes w_i^* P w_i,$$

since $e_F = 1 \otimes P$, $H = H_\nu \otimes H_\sigma$ and $H_F = H_\nu \otimes (\mathbb{C}\Omega_\sigma)$. For each $i$, the linear operator $w_i^* P w_i$ is the projection of $H_\sigma$ onto $\mathbb{C}l_i$. Hence,

$$\sum_i v_i^* e_F v_i = 1. \tag{3.3.3}$$

Thus, applying (1.5.3) in Lemma 1.5.4 in terms of $\Omega = \Omega_\nu \otimes \Omega_\sigma$, for all $t \in \langle A, e_F \rangle^+$,

$$\begin{aligned}\bar{\mu}(t) &= \sum_i \langle J v_i^* \Omega, t J v_i^* \Omega \rangle \\ &= \sum_i \langle \Omega_\nu \otimes J_\sigma l_i, t(\Omega_\nu \otimes J_\sigma l_i) \rangle \\ &= \sum_i \langle \Omega_\nu \otimes h_i, t(\Omega_\nu \otimes h_i) \rangle.\end{aligned}$$

From the faithfulness of $\bar{\mu}$, the first equality of (3.3.1) and the trivialness of $\varphi_t^{\bar{\mu}}$, it follows from [Str81, Theorem 8.2] that the second equality of (3.3.1) is true.

□

**Remark 3.3.2.** The reference for Lemma 1.5.4, [SS08, Lemma 4.3.4], requires a net $(v_i)_{i \in \mathcal{I}}$ satisfying (3.3.3). However, we do not see $\mathcal{I}$ as a directed set being used, neither in the proof of [SS08, Lemma 4.3.4] nor in any results that [SS08, Lemma 4.3.4] depends on. In fact the only explicit requirement is that the $(v_i)$ form a maximal set such that $(v_i^* e_F v_i)$ are orthogonal projections ([SS08, p.61]).

We now turn to the skew product. Let $(X, \mathcal{X}, \rho)$ be a standard probability space with compact Hausdorff space $X$ and Borel measure $\rho$. We let $S : X \to X$ be an invertible map which is measure preserving with respect to $\rho$, that is,

$$\forall\, K \in \mathcal{X} \;\; \rho(K) = \rho(S^{-1}(K)).$$

We set

$$B := L^\infty(\rho),\ \Omega_\nu := 1,\ \nu(f) := \int_X f \, \mathrm{d}\rho \text{ and } \beta : B \to B : f \mapsto f \circ S.$$

Then $\mathbf{B}$ is a system if we view $B$ as operators acting on $L^2(\rho)$ via pointwise multiplication: for every $f \in L^\infty(\rho)$, we have an operator

$$M_f : L^2(\rho) \to L^2(\rho) : g \mapsto fg.$$

We let $(C, \sigma, \gamma)$ be a system such that $C$, in its GNS representation, acts on a separable Hilbert space $H_\sigma$. Denote the unitary representation of $\gamma$ on $H_\sigma$ by $U_\gamma$. Now put

$$A := B \bar{\otimes} C.$$

Then

$$(L^2(\rho) \otimes H_\sigma, \mathrm{id}_A, 1 \otimes \Omega_\sigma)$$

is the GNS triple for $A$ when we use the product state

$$\mu := \nu \bar{\otimes} \sigma.$$

Put

$$F := B \otimes 1$$

and let $\lambda := \mu|_F$.

We now construct the skew product dynamics $\alpha$ on $A$ using the theory of direct integrals (it will be convenient for us to use the approach contained in [Nie80] and [Tak03a, Section IV.8]; Appendix A contains a summary). Consider the space of $H_\sigma$-valued $\rho$-square integrable functions $L^2(\rho; H_\sigma)$. Then $L^\infty(\rho)$ is $*$-isomorphic to the von Neumann algebra $\mathcal{M}$ of all diagonalizable operators on $L^2(\rho; H_\sigma) \cong L^2(\rho) \otimes H_\sigma$ as multiplication operators on $L^2(\rho; H_\sigma)$ (Propositions A.2.7 and A.2.3). So, in effect, any $f \in L^\infty(\rho)$ is identified with $M_f \otimes 1$. Furthermore, $1 \otimes \Omega_\sigma$ is represented by $\Omega \in L^2(\rho, H_\sigma)$ given by $\Omega(p) = \Omega_\sigma$ for all $p \in X$. If we put $\mathcal{N}(p) = C$ for all $p \in X$, then from Corollary A.3.3 and its proof we have the isomorphism

$$\int_X^{\oplus} \mathcal{N}(p)\, \mathrm{d}\rho(p) \cong B \bar{\otimes} C.$$

The elements $a = \int_X^{\oplus} a(p)\, \mathrm{d}\rho(p)$ of $\int_X^{\oplus} \mathcal{N}(p)\, \mathrm{d}\rho(p)$ consist of decomposable operators with $a(p) \in \mathcal{B}(H_\sigma)$ for all $p \in X$, such that

$$\|a(\cdot)\| \in L^\infty(\rho),$$

and for any $z \in L^2(\rho; H_\sigma)$ the element $az \in L^2(\rho, H_\sigma)$ is given by

$$(az)(p) = a(p)z(p)$$

for all $p \in X$. Moreover, from [Tak03a, Theorem IV.8.18], we have $a(p) \in C$. Thus, we may represent each $a \in \int_X^{\oplus} C\, \mathrm{d}\rho$ by a map $a : X \to C : p \mapsto a(p)$.

Let

$$k : X \to \mathbb{Z}$$

be any measurable map. For $a \in \int_X^{\oplus} C\, \mathrm{d}\rho$, define for all $p \in X$,

$$\alpha(a)(p) := \gamma^{k(p)}(a(Sp)). \tag{3.3.4}$$

Then $\alpha$ is the skew product dynamics, where $k$ acts as the generator of a cocycle.

We verify that the map $p \mapsto \alpha(a)(p)$ indeed represents an element in $A$. For all $c \in C$ and $z \in H_\sigma$ the map

$$t : X \to H_\sigma : p \mapsto \gamma^{k(p)}(c)z \tag{3.3.5}$$

is measurable in terms of the Borel $\sigma$-algebra generated from the norm topology on $H_\sigma$. Indeed, the inverse image $t^{-1}(\{\gamma^{k(p)}(c)z\})$ is a countable collection of $p \mapsto k(p)$'s inverse images. However, the range of $k$ is $\mathbb{Z}_C$. So $p \mapsto k(p)$'s inverse images can be at most countable.

We can now use (3.3.5) to show that $\alpha(b\otimes c) \in \int_X^\oplus C\,\mathrm{d}\rho$. for every $b \in B$ and $c \in C$. If $x \in L^2(\rho; H_\sigma)$, we have

$$y(p) := \alpha(b \otimes c)(p) = \beta(b)(p)\gamma^{k(p)}(c)x(p).$$

Therefore, for every $z \in H_\sigma$,

$$p \mapsto \langle z, y(p)\rangle = \beta(b)(p)\langle \gamma^{k(p)}(c^*)z, x(p)\rangle.$$

is measurable from (3.3.5) and Proposition A.2.1. Since

$$\|\beta(b)(p)\| \leq \|\beta(b)\|_\infty = \|b\|_\infty \equiv \|b\|$$

and $\left\|\gamma^{k(p)}(c)x(p)\right\| \leq \|b\|\,\|x(p)\|$, we have

$$\|y(p)\| \leq \|b\|\,\|c\|\,\|x(p)\|,$$

so that $\|y(\cdot)\|$ is square-integrable and $y \in L^2(\rho; H_\sigma)$. Further, $p \mapsto \alpha(b \otimes c)(p)$ is essentially bounded since,

$$\|\alpha(b \otimes c)(p)\| \leq \|\beta(b)\|\,\gamma^{k(p)}(c) = \|b\|\,\|c\|.$$

Thus, $\alpha(b \otimes c)$ is a decomposable operator and since $\gamma^{k(p)}(c) \in B$, we have

$$\alpha(b \otimes c) \in \int_X^\oplus C\,\mathrm{d}\rho(p).$$

So $\alpha(B\bar{\otimes}C) \subseteq \int_X^\oplus C\,\mathrm{d}\rho(p)$. Similarly, if we define

$$\kappa(a)(p) = \gamma^{-k(p)}(a(S^{-1}p)),$$

for all $p \in X$, then $\kappa(a) \in \int_X^\oplus C\,\mathrm{d}\rho$ and $\kappa = \alpha^{-1}$. Thus, $A \subseteq \alpha^{-1}(A)$ and therefore $\alpha(A) = A$. It is then routine to check that $\alpha$ is a $*$-automorphism.

Indeed, for all $x, y \in \int_X^\oplus C\,\mathrm{d}\rho(p)$ and all $z \in \mathbb{C}$,

1. Additivity

$$\begin{aligned}\alpha(x+y)(p) =& \gamma^{k(p)}((x+y)(Sp)) = \gamma^{k(p)}((x)(Sp) + (y)(Sp)) \\ =& \gamma^{k(p)}(x(Sp)) + \gamma^{k(p)}(y(Sp)) = \alpha(x)(p) + \alpha(y)(p);\end{aligned}$$

2. Homogeneity

$$\begin{aligned}\alpha(zx)(p) =&\gamma^{k(p)}[(zx)(Sp)] = \gamma^{k(p)}[z(x)(Sp)]\\ =&z\gamma^{k(p)}[(x)(Sp)] = z\alpha(x)(p);\end{aligned}$$

3. Multiplicativity

$$\begin{aligned}\alpha(xy)(p) =&\gamma^{k(p)}(xy)(Sp) = \gamma^{k(p)}(x(Sp)y(Sp))\\ =&\gamma^{k(p)}(x(Sp))\gamma^{k(p)}(y(Sp)) = \alpha(x)(p)\alpha(y)(p);\end{aligned}$$

4. $*$-Preservation

$$\begin{aligned}\alpha(x^*)(p) =&\gamma^{k(p)}(x^*)(Sp) = \gamma^{k(p)}[(x)(Sp)]^*\\ =&[\gamma^{k(p)}x(Sp)]^* = [\alpha(x)(p)]^*.\end{aligned}$$

Since the cyclic vector of $A$, $\Omega = 1 \otimes \Omega_\sigma$ is identified with the map

$$X \to H_\sigma : p \mapsto 1(p)\Omega_\sigma = \Omega_\sigma,$$

(Proposition A.2.3) we have $\Omega(p) = \Omega(Sp) = \Omega_\sigma$.

It follows that the state $\mu$ is preserved by the dynamics:

$$\begin{aligned}\int_X \langle \Omega(p), [\gamma^{k(p)}(a(Sp))]\Omega(p)\rangle \,\mathrm{d}\rho(p) &= \int_X \langle \Omega(p), a(Sp)\Omega(p)\rangle \,\mathrm{d}\rho(p)\\ &= \int_X \langle \Omega(p), a(p)\Omega(p)\rangle \,\mathrm{d}\rho(p).\end{aligned}$$

We check that $F$ is preserved by $\varphi = \alpha|_F$ : for all $p \in X$,

$$\alpha(b \otimes 1)(p) = (b \circ S) \otimes 1. \tag{3.3.6}$$

We describe the unitary representation $U$ of $\alpha$. Note first that

$$\begin{aligned}(Ua\Omega)(p) &= (\alpha(a)\Omega)(p) = \alpha(a)(p)\Omega(p) = \gamma^{k(p)}(a(Sp))\Omega_\sigma\\ &= U_\gamma^{k(p)}(a(Sp)\Omega_\sigma) = U_\gamma^{k(p)}(a\Omega)(Sp).\end{aligned}$$

Let $x \in \int_X^\oplus H_\sigma \,\mathrm{d}\rho(p)$ and approximate $x$ by a sequence $(x_n) = (a_n\Omega)$ in $A\Omega$.

Since,

$$\begin{aligned}\|x_n - x\|^2 &= \int_X \|x_n(p) - x(p)\|^2 \,\mathrm{d}\rho(p)\\ &= \int_X \|x_n(Sp) - x(Sp)\|^2 \,\mathrm{d}\rho(p) \to 0 \quad \text{as } n \to \infty,\end{aligned}$$

it follows as in the proof of the completeness of $L^p$ spaces, there is a subsequence $(\|x_{n_i}(Sp) - x(Sp)\|)$ which tends to 0 except for $p$ in a null set $N_0 \subseteq X$.

Thus,

$$(Ux)(p) = \lim_i U_\gamma^{k(p)}(x_{n_i}(Sp)) = U_\gamma^{k(p)}(x(Sp)),$$

for all $p \in X \backslash N_0$. Without loss, we may define $Ux$ such that this holds for all $p \in X$. Then it follows that

$$(U^{-1}x)(p) = U_\gamma^{-k(S^{-1}p)} x(S^{-1}p). \tag{3.3.7}$$

Next we discuss a concrete example of $C$. The main points from this example are summarized in Proposition 3.3.4.

**Example 3.3.3.** Let $G$ be a countable group endowed with the discrete topology and let $T : G \to G$ be any group automorphism such that for each $g \in G$, the orbit of $g$, $T^{\mathbb{Z}}g := \{T^n g : n \in \mathbb{Z}\}$ is a finite set (we refer to $T^{\mathbb{Z}}g$ as a *finite orbit*). Consider the dual system on $C := \mathfrak{L}(G)$, the group von Neumann algebra on the group $G$ as in Example 2.3.9. Thus, $C$ is the von Neumann algebra on $\ell^2(G)$ generated by the following set of unitary operators:

$$\{l(g) : g \in G\} \tag{3.3.8}$$

where $l$ is the left regular representation of $G$, i.e. the unitary representation of $G$ on $\ell^2(G)$ with each $l(g) : \ell^2(G) \to \ell^2(G)$ given by

$$[l(g)f](h) = f(g^{-1}h)$$

for all $f \in \ell^2(G)$ and $g, h \in G$. Equivalently,

$$l(g)\delta_h = \delta_{gh}$$

for all $g, h \in G$, where $\delta_g \in \ell^2(G)$ is defined by $\delta_g(g) = 1$ and $\delta_g(h) = 0$ for $h \neq g$. Setting

$$\Omega_\sigma := \delta_1$$

where $1 \in G$ denotes the identity of $G$, we can define a faithful normal trace $\sigma$ on $B$ by

$$\sigma(a) := \langle \Omega_\sigma, a\Omega_\sigma \rangle$$

for all $a \in C$. It follows that $(\ell^2(G), \mathrm{id}_C, \Omega_\sigma)$ is the cyclic representation of $(C, \sigma)$.

We have a unitary $U_\gamma : \ell^2(G) \to \ell^2(G)$, defined by

$$U_\gamma(f) = f \circ T.$$

We define a $*$-automorphism $\gamma$ on $C$ by

$$\gamma(c) = U_\gamma c U_\gamma^*,$$

for all $c \in C$. Then, $(C, \sigma, \gamma)$ is a system.

Using Proposition 3.3.1, the basic construction is given by

$$\langle A, e_F \rangle = L^\infty(\rho) \bar{\otimes} \mathcal{B}(\ell^2(G)).$$

For each $g \in G$ let

$$R_g := \operatorname{span}(U_\gamma^{\mathbb{Z}} \delta_g)$$

and let $Q_g$ be the projection of $\ell^2(G)$ onto $R_g$. Set

$$V_g := L^2(\rho) \otimes R_g$$

and let $P_g = 1 \otimes Q_g$ be the projection of $H := L^2(\rho) \otimes \ell^2(G)$ onto $V_g$.

We have

$$\bar{\mu}(P_g) = \sum_{h \in G} \langle \Omega_\nu \otimes \delta_h, P_g(\Omega_\nu \otimes \delta_h) \rangle = \sum_{h \in G} \langle \delta_h, Q_g \delta_h \rangle = \dim(R_g) < \infty,$$

since all orbits are finite.

The $V_g$'s , for $g \neq 1$, span $H \ominus H_F = L^2(\rho) \otimes \Omega_\sigma^\perp$, since the $R_g$'s span $\Omega_\sigma^\perp$. As $R_g$ is spanned by an orbit, we have $U_\gamma R_g = R_g$. It follows that if $x \otimes y \in V_g$, then,

$$U(x \otimes y)(p) = U_\gamma^{k(p)}(x \otimes y)(Sp) = U_\gamma^{k(p)}(x(Sp)y) = x(Sp)U_\gamma^{k(p)} y \in R_g,$$

for all $p \in X$, since $x \otimes y$ is represented by $p \mapsto x(p)y$ in $\int_X^\oplus H_\sigma \, \mathrm{d}(\rho)$. Hence $U(x \otimes y) \in L^2(\rho) \otimes R_g$, so $UV_g \subset V_g$. Using (3.3.7), it similarly follows that $U^{-1}V_g \subseteq V_g$, so $UV_g = V_g$.

The $V_g$'s are trivially right-$F$-modules, since $F = L^\infty(\rho) \otimes 1$. Hence the $V_g$'s are indeed $U$-$\bar{\mu}$-modules which (when excluding $g = 1$) span $H \ominus H_F$ as required by Definition 3.2.6.

We briefly summarize:

**Proposition 3.3.4.** *Consider a dual system* $\mathbf{C}$ *generated from a discrete countable group* $G$*, with automorphism* $T : G \to G$ *producing only finite orbits, and a classical system* $\mathbf{B}$ *obtained from a standard probability space* $(X, \mathcal{X}, \rho)$ *and measure-preserving transformation* $S : X \to X$*. Form the system* $(B \bar{\otimes} C, \mu, \alpha)$ *with* $\mu$ *as a vector state from* $1 \otimes \delta_1$ *and dynamics given by equation (3.3.4). Then* $(B \bar{\otimes} C, \mu, \alpha)$ *has discrete spectrum relative to* $(B \otimes 1, \mu|_{B \otimes 1}, \alpha|_{B \otimes 1})$.

Taking $G$ to be a free group, with $T$ induced by a finite orbit bijection of the symbols, provides a concrete and non-trivial realization of $C$.

## 3.4 Finite Extensions

In this section we present a second example of relative discrete spectrum. In this case, unlike the previous section, we start with a noncommutative system and extend it by a finite dimensional noncommutative system (hence the name "finite extension").

**Definition 3.4.1.** Consider a system $\mathbf{B} = (B, \nu, \beta)$. Let $n \in \mathbb{N}$. Set $A = B \odot M_n(\mathbb{C})$ with faithful normal trace $\mu = \nu \bar{\otimes} \mathrm{tr}$, where $M_n(\mathbb{C})$ is the $n \times n$ matrices over $\mathbb{C}$ and tr is the normalized trace on $M_n(\mathbb{C})$. Suppose further that there is a $*$-automorphism $\alpha$ of $A$ such that $\alpha(b \otimes 1) = \beta(b) \otimes 1$. Represent $\mathbf{B}$ as the subsystem $\mathbf{F}$ of $\mathbf{A}$ given by $F = B \otimes 1$, $\lambda(b \otimes 1) = \nu(b)$ and $\varphi(b \otimes 1) = \beta(b) \otimes 1$. Then we refer to $\mathbf{A} = (A, \mu, \alpha)$ as a *finite extension of* $\mathbf{F}$. Equivalently, we say that $\mathbf{A}$ is a *finite extension* of $\mathbf{B}$.

Note that we can view $B \odot M_n(\mathbb{C})$ as all $n \times n$ matrices with entries in $B$.

There is a general reason why finite extensions are isometric extensions (Proposition 3.4.6): if the lifted trace is finite, we automatically have relative discrete spectrum, which we now show (Corollary 3.4.3).

**Proposition 3.4.2.** *Let* $\mathbf{A}$ *be a system with subsystem* $\mathbf{F}$. *Then the subspace* $H \ominus H_F$ *is a* $U$*-invariant right* $F$*-submodule.*

*Proof.* Consider $H \ominus H_F$ and its corresponding projection $1_A - e_F$. Since $1_A - e_F \in \langle A, e_F \rangle$, $H \ominus H_F$ is a right $F$-module using Proposition 1.5.6. Furthermore, since $\alpha(F) = F$, we have $UH_F = H_F$, and therefore multiplying by $U^*$ on both sides we obtain $U^* H_F = H_F$. Consequently, $\forall x \in H \ominus H_F$ $\forall y \in H_F$, using $U^* H_F = H_F$, we have

$$\langle Ux, y \rangle = \langle x, U^* y \rangle = 0, \tag{3.4.1}$$

so that $U(H \ominus H_F) \subseteq H \ominus H_F$. □

**Corollary 3.4.3.** *Suppose that* $\mathbf{A}$ *is a system with subsystem* $\mathbf{F}$ *and assume that the lifted trace* $\bar{\mu}$ *is finite, in the sense that for every* $x \in \langle A, e_F \rangle^+$, $\bar{\mu}(x) < \infty$. *Then* $\mathbf{A}$ *has discrete spectrum relative to* $\mathbf{F}$.

*Proof.* Since $\bar{\mu}(1_A - e_F) < \infty$, $H \ominus H_F$ is spanned by a $U$-$\bar{\mu}$-module, namely itself. □

Since the basic construction of a finite-dimensional von Neumann algebra is again finite-dimensional, the lifted trace is finite and we have:

**Corollary 3.4.4.** *Every system on a finite-dimensional von Neumann algebra has discrete spectrum relative to every subsystem.*

Another example follows from [JS97, Proposition 3.1.2]:

**Corollary 3.4.5.** *Suppose that both $A$ and $F$ are type $\mathrm{II}_1$ factors such that their index $[A : F]$ is finite ([JS97, p.29]). Then $\mathbf{A}$ has discrete spectrum relative to $\mathbf{F}$.*

Using Corollary 3.4.3, we are now in a position to prove the following:

**Proposition 3.4.6.** *If $\mathbf{A}$ is a finite extension of $\mathbf{F}$, then $\mathbf{A}$ has discrete spectrum relative to $\mathbf{F}$.*

*Proof.* Without loss of generality, assume that $(B, \nu)$ in Definition 3.4.1 is in its GNS representation $B \to \mathcal{B}(H_\nu)$ with cyclic vector $\Omega_\nu$. One can easily verify that the GNS triple for $M_n(\mathbb{C})$ is $(\mathbb{C}^n \odot \mathbb{C}^n, \pi_n, \Lambda)$, where $\pi_n : M_n(\mathbb{C}) \to M_n(\mathbb{C}) \odot M_n(\mathbb{C}) : c \mapsto c \otimes 1$, and $\Lambda = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} e_j \otimes e_j$ with $\{e_j\}$ an orthonormal basis for $\mathbb{C}^n$. Thus the GNS triple for $A = B \odot M_n(\mathbb{C})$ is given by $(H_\nu \odot \mathbb{C}^n \odot \mathbb{C}^n, \pi, \Omega)$, where $\Omega = \Omega_\nu \otimes \Lambda$, $\pi : B \odot M_n(\mathbb{C}) \to B \odot M_n(\mathbb{C}) \odot M_n(\mathbb{C}) : a \mapsto \mathrm{diag}_n(a)$, $\mathrm{diag}_n(a)$ being the $n \times n$ matrix with $a$'s all along the diagonal and zero elsewhere.

From Proposition 3.3.1,

$$\langle A, e_F \rangle = B \odot M_n(\mathbb{C}) \odot M_n(\mathbb{C}).$$

Let

$$\mathrm{Tr} := tr \odot tr$$

be the normalized trace on $M_n(\mathbb{C}) \odot M_n(\mathbb{C})$. Let $P_\Lambda$ be the projection

$$P_\Lambda : \mathbb{C}^n \odot \mathbb{C}^n \to \mathbb{C}^n \odot \mathbb{C}^n : a\Lambda \mapsto \mathrm{Tr}(a)\Lambda,$$

where $a \in M_n(\mathbb{C}) \odot M_n(\mathbb{C})$. One can verify directly by calculation that $P_\Lambda$ is a projection in $B(M_n(\mathbb{C}) \odot M_n(\mathbb{C}))$. If $1_B$ denotes the identity operator in $B$, then the operator $1_B \otimes P_\Lambda$ is the projection onto $(B \otimes 1)\Omega_A$, that is, $1_B \otimes P_\Lambda = e_F$.

By Proposition 3.3.1, the lifted trace of $\langle A, e_F \rangle$ is $\nu \odot \mathrm{Tr}$.

As $\bar{\mu}$ is finite, $\mathbf{A}$ has discrete spectrum relative to $\mathbf{F}$, by Corollary 3.4.3. □

**Example 3.4.7.** We give a concrete realization of a finite extension for which the dynamics is not compact nor a tensor product of the dynamics on the underlying algebras.

Let $B_1$ be the group von Neumann algebra generated from a free group $G$ on two symbols $c$ and $d$. Let $\nu_1$ the trace on $B_1$ (Example 3.3.3). The map $\beta_1 : B_1 \to B_1 : a \mapsto l(d)al(d)^*$ is a $*$-automorphism of $B_1$. Furthermore, using the fact that $\nu_1$ is a trace, $\nu_1(\beta_1(b_1)) = \nu_1(l(d)b_1l(d)^*) = \nu_1(l(d)^*l(d)b_1) = \nu_1(b_1)$. So $\mathbf{B_1} = (B_1, \nu_1, \beta_1)$ is a system.

We let $\mathbf{B_2} = (B_2, \nu_2, \beta_2)$ be any system.

Consider $B = B_1 \oplus B_2$ which we view as the set of all matrices of the form

$$\begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix}$$

for all $b_1 \in B_1$ and all $b_2 \in B_2$.

Let $s \in (0,1) \subseteq \mathbb{R}$ and put

$$\nu = s(\nu_1 \oplus 0) + (1-s)(0 \oplus \nu_2).$$

Then $\nu$ is a faithful normal state on $B$. So $\mathbf{B} = (B, \nu, \beta)$, with $\beta = \beta_1 \oplus \beta_2$, is a system.

Set $A = (B \odot M_2(\mathbb{C})$ and $\mu = \nu \odot \mathrm{tr})$. We now describe dynamics on $(A, \mu)$. For $w_i \in B$, let

$$W = \begin{bmatrix} w_1 & w_2 \\ w_3 & w_4 \end{bmatrix} \in A,$$

be a unitary and define $\alpha(a) := WaW^*$ for all $a \in B \odot M_2(\mathbb{C})$. Then $\alpha(A) \subseteq A$ and $\mu \circ \alpha = \mu$ as is verified from the unitarity of $W$.

From direct calculations, the requirements that $W$ satisfy $\alpha(b \otimes 1) = W \begin{bmatrix} b & 0 \\ 0 & b \end{bmatrix} W^* \in B \otimes 1$ for every $b \in B$, and that $\alpha(b \otimes 1) = \beta(b) \otimes 1$ yield

$$\beta(b) = w_1bw_1^* + w_2bw_2^* = w_3bw_3^* + w_4bw_4^*, \tag{3.4.2}$$

and

$$w_1bw_3^* + w_2bw^*w_4^* = w_3bw_1^* + w_4bw_2^* = 0$$

for all $b \in B$. The direct sum structure of $B$ will allow us to satisfy the latter requirement easily, while still giving nontrivial dynamics. This is done by setting

$$w_1 = v_1 \oplus 0 \qquad \text{and } w_4 = v_4 \oplus 0$$

for $v_1, v_4 \in B_1$, and

$$w_2 = 0 \oplus v_2 \qquad \text{and} w_3 = 0 \oplus v_3$$

for $v_2, v_3 \in B_2$. Then (3.4.2) reads

$$v_1 b_1 v_1^* \oplus v_2 b_2 v_2^* = v_4 b_1 v_4^* \oplus v_3 b_2 v_3^*,$$

for every $b = b_1 \oplus b_2 \in B$. The $v_i$ are unitary, since $W$ is. It follows that $v_4^* v_1 \in B_1'$ and $v_3^* v_2 \in B_2'$.

We now show that $\alpha$ is not a product of the $*$-automorphism $\beta$ and a $*$-automorphism on $M_2(\mathbb{C})$. By direct calculation, for every $m = \begin{bmatrix} m_1 & m_2 \\ m_3 & m_4 \end{bmatrix} \in M_2(\mathbb{C})$,

$$\alpha(1_B \otimes m) = \begin{bmatrix} m_1 1_{B_1} & 0 & m_2 v_1 v_4^* 1_{B_1} & 0 \\ 0 & m_4 1_{B_2} & 0 & m_3 v_2 v_3^* 1_{B_2} \\ v_4 v_1^* m_3 1_{B_1} & 0 & m_4 1_{B_1} & 0 \\ 0 & m_2 v_3 v_2^* 1_{B_2} & 0 & m_1 1_{B_2}. \end{bmatrix}$$

So, $\alpha(1_B \otimes m)$ is not of the form

$$1_B \otimes t = \begin{bmatrix} t_1 1_B & t_2 1_B \\ t_3 1_B & t_4 1_B \end{bmatrix}.$$

Thus, $\alpha(1_B \otimes M_2(\mathbb{C})) \not\subseteq 1_B \otimes M_2(\mathbb{C})$ and $\alpha$ cannot be a tensor product of dynamics on $B$ and $M_2(\mathbb{C})$, respectively, unless $B_1 = 0$ and $v_2 v_3^* = v_3 v_2^* = 1_{B_1}$, or $B_2 = 0$ and $v_1 v_4^* = v_4 v_1^* = 1_{B_1}$.

Now consider a specific case. Assume that $B_2 \neq 0$. Let $v_1 = v_4 := l(d)$. Then we show that $\mathbf{B}$ is not compact. If we consider the orbit $U_{\beta_1}^{\mathbb{Z}} \delta_c$ of $\delta_c$ under $U_{\beta_1}$

$$U_{\beta_1}^{\mathbb{Z}} \delta_c = \{\ldots, \delta_{d^{-2}cd^2}, \delta_{d^{-1}cd^1}, \delta_c, \delta_{dcd^{-1}}, \delta_{d^2cd^{-2}}, \delta_{d^3cd^{-3}}, \ldots\},$$

for all $m, n \in \mathbb{Z}$, $m \neq n$, then we have $d^m c d^{-m} \neq d^n c d^{-n}$, and

$$\|\delta_{d^m c d^{-m}} - \delta_{d^n c d^{-n}}\| = \sqrt{2}.$$

Hence, $U_{\beta_1}^{\mathbb{Z}} \delta_c$ cannot be totally bounded, so that, as we are in a metric space, the closure of $U_{\beta_1}^{\mathbb{Z}} \delta_c$ cannot be compact. It follows that $\mathbf{B}$ is not a compact system.

Thus, we have constructed a finite extension $\mathbf{A}$ of a non-compact system $\mathbf{B}$, such that $\alpha$ is not the product of the dynamics on $B$ with the dynamics on $M_2(\mathbb{C})$.

## 3.5 Some Considerations for Further Research

We collect a number of problems related to isometric extensions.

1. Equivalent Conditions for Relative Discrete Spectrum Are there any useful characterizations of relative discrete spectrum? Of the definitions appearing in Remark 3.2.8, which ones, if any, are equivalent to Definition 3.2.6?

2. A Decision Problem In order to use the definition of relative discrete spectrum (Definition 3.2.6), it would be useful to decide whether or not a given a projection $P_V \in \langle A, e_F \rangle$ satisfies $\bar{\mu}(P_V) < \infty$ when $\bar{\mu}$ is not finite (in the sense of Corollary 3.4.3.

3. Further Examples of Isometric Extensions We have seen in Section 3.4 that the existence of finite lifted traces automatically implies that the system is an isometric extension. The absolute case of discrete spectrum serves as an example of an isometric extension where the lifted trace is not finite. Can one find further examples?

4. Intermediate Subsystems Given a system $\mathbf{A}$ and a subsystem $\mathbf{F}$, one can define an *intermediate system* $\mathbf{B}$ of $\mathbf{A}$ and $\mathbf{F}$. By definition, $\mathbf{B}$ is a subsystem of $\mathbf{A}$, but also has $\mathbf{F}$ as a subsystem. It can be shown, that if $\mathbf{A}$ has discrete spectrum relative to $\mathbf{F}$, then $\mathbf{B}$ has discrete spectrum relative to $\mathbf{F}$. However, we do not know if $\mathbf{A}$ has discrete spectrum relative to $\mathbf{B}$.

# Appendix A

# Direct Integrals

For convenience, we summarize some basic definitions and results from [Nie80] (used in Section 3.3). This appendix contains more than what is needed in this thesis.

## A.1 Borel Spaces and Borel Functions

The use of the adjective "Borel" is meant to convey what, in modern terminology, would be associated with the word "measurable". However, we will be working with standard probability spaces (defined below). Thus, the $\sigma$-algebras of interest will be generated from a topology.

**Definition A.1.1.** ([Nie80, p.1] ) Let $X$ be a set. By a *Borel structure* on $X$ we mean a $\sigma$-algebra on $X$. The members of the Borel structure are referred to as *Borel sets* of $X$. A *Borel space* $(X, \mathcal{X})$ is a set $X$ together with a Borel structure $\mathcal{X}$. A *Borel function* is a function $f : X \to Y$ from a Borel space $X$ to a Borel space $Y$ such that for every $K \in \mathcal{Y}$, $f^{-1}(K) \in \mathcal{X}$. A *Borel measure* on $X$ is a countably additive $\sigma$-finite measure defined on the Borel sets of $X$ and takes values in $[0, \infty]$. We will say a property holds *$\mu$-almost everywhere* (abbreviated $\mu$-a.e) if it holds for all $p \in X$ except on a set $K \in \mathcal{X}$ with $\mu(K) = 0$.

Two Borel spaces $K$ and $L$ are *isomorphic* to each other if there is a bijective function $f : K \to L$ such that both $f$ and $f^{-1}$ are Borel functions.

A *Polish space* is a separable complete metric space with the Borel structure generated from the metric. Suppose that $(X, \mathcal{X}, \mu)$ is a probability space with a complete probability measure and $Y$ is isomorphic, differing with the possible exception of a null set, to a Polish space. We then say that $(X, \mathcal{X}, \mu)$ is a *standard probability space.* Thus, the measure $\mu$ is *standard* (there is a standard Borel space $Z \subseteq X$ such that $\mu(X \backslash Z) = 0$). Hence, $L^2(\mu)$ is separable (Corollary A.2.4).

## A.2 Hilbert Spaces and Operators

We can distinguish between two cases of direct integrals of Hilbert spaces. The first is when the elements of the direct integral are maps of the form

$$v : X \to H$$

where $H$ is a single Hilbert space. This is in contrast with the second case, where a countable family of Hilbert spaces replace $H$.

Consider a separable (possibly finite-dimensional) Hilbert space $H$. The Borel structure on $H$ is given by the norm topology. Let $\mu$ be a Borel measure on a Borel space $X$. Then

**Proposition A.2.1.** *([Nie80, p.15,p.24]) A function $v : X \to H$ is Borel if and only if for every $x \in H$ $p \mapsto \langle x, v(p) \rangle$ is a Borel function.*

**Definition A.2.2.** (Cf. [Nie80, Theorem 5.1]) By $L^2(\mu; H)$ denote the Hilbert space of all Borel functions $v : X \to H$ (where, as usual, we identify functions which agree $\mu$-a.e) such that $p \mapsto \|v(p)\|$ is $\mu$-square integrable with inner product

$$\langle v, w \rangle = \int_X \langle v(p), w(p) \rangle \, \mathrm{d}\mu(p).$$

As an aside, in [Nie80], the notation $v(\mu)$ is used to refer to functions in the normed space $L^2(\mu; H)$, while $v$ is used to refer to the function in the corresponding seminormed space. We shall not adopt this convention in this thesis.

For any $v \in L^2(\mu; H)$, we use the notation

$$v = \int_X^{\oplus} v(p) \, \mathrm{d}\mu(p).$$

Similarly, the notation $\int_X^{\oplus} H(p) \, \mathrm{d}\mu(p)$ may be used in place of $L^2(\mu; H)$ (in light of Definition A.2.10, one may think of $H(p) = H$ for all $p \in X$).

**Proposition A.2.3.** *([Nie80, Proposition 5.2]) There is a unique linear isometry of $L^2(\mu; H)$ onto $L^2(\mu) \otimes H$ mapping $p \mapsto f(p)x$ to $f \otimes x$ for all $f \in L^2(\mu)$ and $x \in H$.*

**Corollary A.2.4.** *([Nie80, Corollary 5.3]) If $\mu$ is standard, then $L^2(\mu; H)$ is separable.*

Let $\mathcal{B}(H)$ denote the normed space of all bounded linear operators on a Hilbert space $H$. Then a map $a : X \to \mathcal{B}(H)$ is Borel if and only if for all $x, y$ in $H$, the function $p \mapsto \langle x, a(p)y \rangle$ is Borel.

**Definition A.2.5.** ([Nie80, p.18]) A *decomposable operator* $a$ on $L^2(\mu; H)$ is a Borel function $a : X \to \mathcal{B}(H)$ such that

1. $p \mapsto \|a(p)\|$ belongs to $L^\infty(\mu)$ (we say that $a$ is $\mu$-*essentially bounded*);
2. for any $v \in L^2(\mu; H)$ there exists a function $av : L^2(\mu; H) \to L^2(\mu; H)$ defined by
$$(av)(p) = a(p)v(p)$$
and satisfies
$$\|(av)(p)\| \leq kv(p),$$
where $k = \operatorname{ess\,sup}_{p \in X} \|a(p)\|$.

In this case we write $a = \int_X^\oplus a(p) \, \mathrm{d}\mu(p)$.

**Definition A.2.6.** ([Nie80, p.18]) We call linear operators $a$ on $L^2(\mu; H)$ *diagonalizable* if they are of the form $\int_X^\oplus f(p)I \, \mathrm{d}\mu(p)$, where $I$ is the identity operator on $H$.

**Proposition A.2.7.** *([Nie80, pp.18-19]) The set of all diagonalizable operators on $L^2(\mu; H)$ is a von Neumann algebra $*$-isomorphic to $L^\infty(\mu)$.*

**Theorem A.2.8.** *([Nie80, Proposition 6.1] ) If $a$ is a $\mu$-essentially bounded Borel function from $X$ to $\mathcal{B}(H)$, then $a$ is a unitary (respectively, positive, a projection, a partial isometry) if and only if $a(p)$ is unitary (respectively, positive, a projection, a partial isometry) for $\mu$-a.e.*

We can characterise decomposable operators as follows

**Theorem A.2.9.** *([Nie80, Theorem 6.2] An operator on $L^2(\mu; H)$ is decomposble if and only if it commutes with every diagonalizable operator.*

We now turn to the second case of direct integrals of Hilbert spaces. The concept of coherence is important here. For each $n \in \mathbb{N}$, let $\ell_n^2$ denote the set consisting of all those elements in $\ell^2$ (the Hilbert space of all square-summable complex-valued sequences) whose values at $n+1$, $n+2, \ldots$ are zero. Then $\ell_0^2 \subseteq \ell_1^2 \subseteq \ldots \subseteq \ell^2$ and $\dim \ell_n^2 = n$ for all $n \in \mathbb{N} \cup \{\infty\}$

If $\{H(p) : p \in X\}$ is a family of separable Hilbert spaces, a *field of separable Hilbert spaces* is just a map $\mathcal{H} : X \to \{H(p) : p \in X\} : p \mapsto H(p)$. A *coherence* [Nie80, p.23] for a field of separable Hilbert spaces $\mathcal{H}$ is a map $r$ which maps every point $p \in X$ to an isometry $r(p) : H(p) \to \ell^2$ whose range is $\ell^2_{\dim H(p)}$. Defining $r$ in such a way removes the dependency on $p \in X$ when considering the the domain of the adjoint $r(p)^*$.

Similarly, we can have a coherence for a field of operator fields $p \mapsto a(p)$.

If, for each $n \in \mathbb{N} \cup \{\infty\}$, the sets $X_n = \{p \in X \mid \dim H(p) = n\}$ are measurable, we say that $\mathcal{H}$ is a *Borel field of Hilbert spaces.*

A map $v : X \to \{H(p) \mid p \in X\}$ such that $v(p) \in H(p)$ for each $p \in X$ is called a *vector field* over $\mathcal{H}$. An $r$-Borel vector field over $\mathcal{H}$ [Nie80, p.23] is a vector field over $\mathcal{H}$ for which

$$p \mapsto r(p)v(p)$$

from $X$ to $\ell^2$ is a Borel function.

We have a similar definition for an $r$-Borel operator field over $\mathcal{H}$ : call an operator field $p \mapsto a(p)$ $r$-Borel, provided the map $p \mapsto r(p)a(p)r(p)^*$ from $X \to \mathcal{B}(\ell^2)$ is Borel measurable.

**Definition A.2.10.** (see [Nie80, Theorem 7.1]) The Hilbert space

$$L^2(\mu; \mathcal{H}, r)$$

(also denoted by $\int_X^r H(p)\,\mathrm{d}\mu(p)$) consists of all those $r$-Borel measurable vector fields $v$ over $\mathcal{H}$ such that $p \mapsto \|v(p)\|$ belongs to $L^2(\mu)$. The inner product is given by

$$\langle v, w\rangle = \int_X \langle v(p), w(p)\rangle\,\mathrm{d}\mu(p).$$

We refer to $\int_X^r H(p)\,\mathrm{d}\mu(p)$ as a *direct integral of Hilbert spaces* of $\mathcal{H}$ with respect to $\mu$ and $r$. For an element $v$ of $\int_X^r H(p)\,\mathrm{d}\mu(p)$ write $v = \int_X^r v(p)\,\mathrm{d}\mu(p)$.

An $r$-Borel $\mu$-essentially bounded operator field $a$ over $\mathcal{H}$ induces a linear operator, denoted $\int_X^r a(p)\,\mathrm{d}\mu(p)$, satisfying

$$\int_X^r a(p)\,\mathrm{d}\mu(p) \int_X^r v(p)\,\mathrm{d}\mu(p) = \int_X^r a(p)v(p)\,\mathrm{d}\mu(p).$$

Such operators will be referred to as *decomposable.*

## A.3 von Neumann Algebras

A field of von Neumann algebras over a Borel field of Hilbert spaces $\mathcal{H}$ is just a map $p \mapsto A(p)$ which assigns to each $p \in X$ a von Neumann algebra $A(p) \subseteq \mathcal{B}(H(p))$.

Let $\mathrm{vN}(H)$ denote the collection of all von Neumann algebras defined on $H$. Call a field of von Neumann algebras $p \mapsto A(p)$ a *$r$-Borel field of von Neumann algebras* [Nie80, p.74] if for each $n \in \mathbb{N} \cup \{\infty\}$,

$$p \mapsto r(p)A(p)r(p)^*|_{\ell^2_n}$$

from $X_n$ to $\mathrm{vN}(\ell^2_n)$ is Borel. The Borel structure placed on $\mathrm{vN}(\ell^2_n)$ is the so-called *Effros-Borel structure* ([Nie80, p. 67]). Equivalently, $p \mapsto A(p)$ is $r$-Borel if and only if there exists a *$r$-Borel generating sequence* for $p \mapsto A(p)$ : a sequence $(a_n)$ of $r$-Borel operator fields over $\mathcal{H}$ such that, for all $p \in X$, $(a_n(p))$ generates the von Neumann algebra $A(p)$ ([Nie80, p.74]).

When $\mu$ is standard, the von Neumann algebra

$$\int_X^r A(p)\,\mathrm{d}\mu(p) \tag{A.3.1}$$

(called the *direct integral of von Neumann algebras* of $A(p)$) consists of all the decomposable operators $\int_X^r a(p)\,\mathrm{d}\mu(p)$ ([Nie80, Theorem 18.2, p.75]).

The vector space of all diagonalizable operators (denoted by $\mathcal{M}$ in [Nie80, p.51]) consists of all those linear operators on $\int_X^r H(p)\,\mathrm{d}\mu(p)$ of the form

$$\int_X^r f(p)I_p\,\mathrm{d}\mu(p),$$

where $f \in L^\infty(\mu)$ and $I_p$ is the identity operator on $H(p)$.

We note that when $r$ is a constant coherence (i.e. $r(p)$ is the same isometry for all $p$) then we replace $r$ with $\oplus$ in (A.3.1) writing

$$\int_X^\oplus A(p)\,\mathrm{d}\mu(p),$$

instead of $\int_X^r f(p)I_p\,\mathrm{d}\mu(p)$. (In this case, $L^2(\mu; H) = L^2(\mu; \mathcal{H}, r)$ with $\mathcal{H}(p) = H$ for all $p \in X$ and $\int_X^r a(p)\,\mathrm{d}\mu(p) = \int_X^\oplus a(p)\,\mathrm{d}\mu(p)$ for all decomposable operators $a$ acting on $L^2(\mu; H)$ ([Nie80, p.24]).)

The next two results tell us that a direct integral decomposition of a von Neumann algebra acting on a separable Hilbert space is always possible.

**Theorem A.3.1.** *([Nie80, Theorem 9.1]) Suppose that $B$ is an abelian von Neumann algebra acting on a separable Hilbert space $K$. Then there is* *a standard Borel space $Z$ and a Borel measure $\nu$ on $Z$ such that $B$ and $L^\infty(\nu)$ are $*$-isomorphic, and there is a Borel field of Hilbert space $\mathcal{H}$ on $Z$, a coherence $r$ for $\mathcal{H}$, and a linear isometry $I$ of $K$ onto $L^2(\nu; \mathcal{H}, r)$ such that $IBI^{-1}$ is the algebra of diagonalizable operators on $L^2(\nu; \mathcal{H}, r)$.*

**Theorem A.3.2.** *([Nie80, Theorem 19.4]) If $A$ is a von Neumann algebra on $\int_X^r H(p)\,\mathrm{d}\mu(p)$ whose centre contains $\mathcal{M}$, then there is an $r$-Borel field of von Neumann algebras $p \mapsto A(p)$ over $\mathcal{H}$ with $A = \int_X^r A(p)\,\mathrm{d}\mu(p)$. In particular, a given von Neumann algebra always has a direct integral decomposition with respect to a given subalgebra of its centre.*

Below is a key result used in Section 3.3:

**Corollary A.3.3.** *([Nie80, Corollary 19.9]) Let $p \mapsto A(p)$ be an $r$-Borel field of von Neumann algebras over $\mathcal{H}$. Suppose that there is a von Neumann algebra $A_0$ which acts on a separable Hilbert space and is $*$-isomorphic to $A(p)$ for almost all $p \in X$. Then $\int_X^\oplus A(p)\,\mathrm{d}\mu(p)$ and $A_0 \bar{\otimes} \mathcal{M}$ are $*$-isomorphic.*

**Remark A.3.4.** Books such as [KR97b] and [Dix81] provide different definitions for direct integrals of Hilbert spaces, decomposable operators and decomposable von Neumann algebras. These definitions "differ superficially" from the definitions presented here ([Nie80, p.43, p.82] -see also [KR97b, Definition 14.1.1, Definition 14.1.6, Lemma 14.1.23]).

# Index of Symbols

# Index of Terms

# Bibliography

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