IEEE Access
Multidisciplinary : Rapid Review : Open Access Journal

Received July 4, 2019, accepted July 26, 2019, date of publication August 2, 2019, date of current version August 30, 2019.

Digital Object Identifier 10.1109/ACCESS.2019.2932664

Further Generalized Hybrid Mappings and Common Attractive Points in CAT(0) Spaces: A New Iterative Process

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ABSTRACT There are many methods present in literature for finding attractive points for different mappings in various spaces. In this article, we aim to give an approximation method for the common attractive points (CAP) of further generalized hybrid mappings (FGHM) in CAT(0) spaces. We give the CAP of FGHM by using Picard-Mann iterative process generalized to the case of two mappings in framework of CAT(0) spaces. The results presented in this article, extend some known results of literature.

INDEX TERMS CAT(0) space, attractive points, further generalized hybrid mappings, Picard-Mann iterative process.

I. INTRODUCTION

The notion of attractive points was presented by Takahashi and Takeuchi in [2] to dispose of the speculation of convexity and closedness as utilized in Baillon's nonlinear ergodic theorem in the setting of Hilbert spaces (H) [3]. They proved the existence of attractive points in H without using convexity.

Suppose C is a nonempty subset of H. The set of attractive points and fixed points for $T : C \to H$ are defined as follow:

- $A(T) = \{u \in H : \|u - Tv\| \leq \|u - v\|, \forall v \in H\}$;
- $F(T) = \{u \in H : Tu = u\}$.

Recall that, a mapping $T : C \to C$ is nonexpanisive if $\|u - Tv\| \leq \|u - v\|$ for all $u, v \in C$.

Now we define hybrid mapping.

Definition 1: Let $C \subset H$, then $T : C \to H$ is called hybrid if

$$3\|Tu - Tv\|^2 \leq \|u - v\|^2 + \|Tu - v\|^2 + \|Tv - u\|^2, \quad \forall u, v \in C.$$

In 2010, Kocourek *et al.* [4] present another class of nonlinear mapping called generalized hybrid mappings (GHM) which is bigger class than the class of nonexpansive mapping. In 2012, Takahashi *et al.* [5] found more extensive class of nonlinear mapping called normally generalized hybrid mapping (NGHM) which contain the class of generalized hybrid (GH) and the class of contractive mappings.

The definition of NGHM is given below:

Definition 2: Let $C \subset H$, then $T : C \to H$ is normally generalized hybrid mapping (NGHM) if there exist $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that

- $\alpha + \beta + \gamma + \delta \geq 0$;
- $\alpha + \beta > 0$ or $\alpha + \gamma > 0$ and
- $\alpha\|Tu - Tv\|^2 + \beta\|u - Tv\| + \gamma\|Tu - v\| + \delta\|u - v\| \leq 0, \forall u, v \in C$

This mapping T is also called as $(\alpha, \beta, \gamma, \delta)$-NGHM.

"Widely more generalized hybrid mappings (WMGHM)" is a class of mapping in Hilbert spaces, due to Kawasaki and Takahashi [6] and has been studied in [7].

Definition 3: Let $C \subset H$, which is closed and convex, then $T : C \to C$ is called FGHM if for any $\alpha, \beta, \gamma, \delta, \varepsilon \in \mathbb{R}$, we have

$$\alpha\|Tu - Tv\|^2 + \beta\|u - Tv\|^2 + \gamma\|Tu - v\|^2 + \delta\|u - v\|^2 + \varepsilon\|u - Tu\|^2 \leq 0, \quad \forall u, v \in C. \quad (1)$$

It has been proved that the class of WMGHM contains the class of NGHM. Later, Khan [1] introduced the concept of further generalized mappings and CTP. He approximate CTP of FGHM by utilizing Picard-Mann iterative process for two mappings in H without closeness condition on C.

The associate editor coordinating the review of this article and approving it for publication was Ho Ching Iu.

For more detail in the direction of attractive points we refer [4], [6]–[13].

Researcher are always interested in developing approximation method for fixed points and attractive points, for example: Pakkaranang *et al.* [18] presented Proximal point algorithms involving fixed point iteration for nonexpansive mappings in CAT(0) spaces. Strong convergence of modified viscosity implicit approximation methods for asymptotically nonexpansive mappings in complete CAT(0) spaces has been proved in [19]. Proximal point algorithms for solving convex minimization problem and common fixed points problem of asymptotically quasi-nonexpansive mappings in CAT(0) spaces has been discussed in [20]. In [21], Kumam et al. gave convergence analysis of modified Picard-S hybrid iterative algorithms for total asymptotically nonexpansive mappings in CAT(0) spaces. In this article, we approximate CAP of FGHM by using Picard-Mann iterative process for two mappings in framework of CAT(0) spaces. The results presented in this paper are extension of many existing results.

II. PRELIMINARIES

Consider a metric space (M, d) and x_1, x_2 are two fixed elements of M with $d(x_1, x_2) = l$. An isometry α from $[0, l] \subset \mathbb{R}$ to M is the geodesic path from from x_1 to x_2 such that $\alpha^1(0) = x_1$, $\alpha(l) = x_2$, and $d(\alpha(a), \alpha(b)) = |a - b|$ for all $a, b \in [0, l]$. The geodesic segment is the image c of α, which is also refereed as a joining x_1 and x_2. The M is called geodesic metric space if any two points of M are joined by geodesic segment. The M is called D-geodesic space, if any two points of M with distance lesser than D are joined by a geodesic, where D is any positive constant. If this condition is satisfied in a convex set, then that convex set is called D-convex. Consider M_k be the simple connected, complete, 2 dimensional space of curvature k,(k is a constant). The diameter D_k of the space M_k for ($k \geq 0$) can be defined as $D_k = \frac{\pi}{\sqrt{k}}$; $k > 0$ and $D_k = \infty$; $k = 0$. A triangle $\triangle(x_1, x_2, x_3)$ in a geodesic metric is combination of three points x_1, x_2, and x_3 in M and a geodesic segment between each pair of vertices. For $\triangle(x_1, x_2, x_3)$ in a geodesic space M satisfying

$$d(x_1, x_2) + d(x_2, x_3) + d(x_3, x_1) < D_k,$$

we have points $\overline{x}_1, \overline{x}_2, \overline{x}_3 \in M_k$ with $d(x_1, x_2) = d_k(\overline{x}_1, \overline{x}_2)$, $d(x_2, x_3) = d_k(\overline{x}_2, \overline{x}_3)$ and $d(x_3, x_1) = d_k(\overline{x}_3, \overline{x}_1)$ where d_k is the metric defined on M_k. The triangle having vertices $\overline{x}_1, \overline{x}_2, \overline{x}_3 \in M_k$ is known as a comparison triangle $\triangle(x_1, x_2, x_3)$ in X with $d(x_1, x_2)+d(x_2, x_3)+d(x_3, x_1) < D_k$ satisfies CAT(k) inequality if, for any $u, v \in \triangle(x_1, x_2, x_3)$ and for their comparison points $\overline{u}, \overline{v} \in \triangle(\overline{x}_1, \overline{x}_2, \overline{x}_3)$, we have $d(u, v) \leq d(\overline{u}, \overline{v})$.

Definition 4: The M is called CAT(k) space if

- for non positive k, M is a geodesic metric space, with the property that its geodesic triangles fulfill the CAT(k) inequality;
- for positive k, M is D_k-geodesic and any geodesic triangle $\triangle(x_1, x_2, x_3)$ in M with $d(x_1, x_2) + d(x_2, x_3) + d(x_3, x_1) < 2D_k$ satisfy the CAT(k) inequality.

Remark 1: In a CAT(0) space M if $x_1, x_2, x_3 \in M$, then the CAT(0) inequality implies

$$d^2\left(x_1, \frac{x_2 \oplus x_3}{2}\right) \leq \frac{1}{2}d^2(x_1, x_2)+\frac{1}{2}d^2(x_1, x_3)-\frac{1}{4}d^2(x_2, x_3), \tag{2}$$

which is known as the (CN) inequality given by Bruhat and Tits [14].

Remark 2: The (CN) inequality was extended by Dhompongsa and Panyanak [15], as;

$$d^2(x_3, \alpha x_1 \oplus (1-\alpha)x_2) \leq \alpha d^2(x_3, x_1) + (1-\alpha)d^2(x_3, x_2) - \alpha(1-\alpha)d^2(x_1, x_2), \tag{3}$$

which is known as (CN)inequality and $\alpha \in (0, 1)$.*

For a geodesic space M, the following statements are equivalent:

- M is a CAT(0) space;
- M satisfy the (CN) inequality;
- M satisfy the (CN*) inequality.

Example 1: Example of CAT(0) spaces The following are examples of CAT(0) spaces;

1) Euclidean space $\mathbb{R}^n$;
2) *Hilbert spaces;*
3) *Simply connected Riemannian manifolds of nonpositive sectional curvature;*
4) *Hyperbolic spaces;*
5) *Trees;*
6) *Hilbert ball.*

Remark 3: Complete CAT(0) spaces are also known as Hadamard spaces (see [16]).

Berg *et al.* [17] proposed the idea of quasilinearization as follow: Each pair $(u, v) \in M \times M$, denoted by $\overrightarrow{uv}$ and call it a vector. Then, quasilinearization is a map

$$\langle ., . \rangle : (M \times M) \times (M \times M) \longrightarrow \mathbb{R}$$

defined as

$$\langle \overrightarrow{uv}, \overrightarrow{wt} \rangle = \frac{1}{2}(d^2(u, t)+d^2(v, w)-d^2(u, w)-d^2(v, t)). \tag{4}$$

It can be observed easily that $\langle \overrightarrow{uv}, \overrightarrow{wt} \rangle = \langle \overrightarrow{wt}, \overrightarrow{uv} \rangle$, $\langle \overrightarrow{uv}, \overrightarrow{wt} \rangle = -\langle \overrightarrow{vu}, \overrightarrow{wt} \rangle$ and $\langle \overrightarrow{uk}, \overrightarrow{wt} \rangle + \langle \overrightarrow{ku}, \overrightarrow{wt} \rangle = \langle \overrightarrow{uv}, \overrightarrow{wt} \rangle$ for all $u, v, w, t \in M$. The M satisfies Cauchy-Schwarz inequality if

$$\langle \overrightarrow{uv}, \overrightarrow{wt} \rangle \leq d(u, v)d(u, w)$$

for all $u, v, w, t \in X$.

The CAT(0) is the geodesically connected metric space which satisfy Cauchy-Schwarz inequality.

From now to onward in this paper, consider H be a complete CAT(0) space, C be a non-empty closed convex subset of a complete CAT(0) space M and $T : C \to C$ be a mapping. The metric projection $P_C : M \to C$ is defined as

$$u = P_C(x) \Longleftrightarrow \inf\{d(y,x) : y \in C\}, \quad \forall x \in M$$

The idea of CAP for two mapping T_1 and T_2 is defined as:

$$CAP(T_1,T_2) = \{w \in X : \max(d(T_1x,w), d(T_2x,w)) \leq d(x,w), \quad for \;\; x \in X\}$$

Remark 4: For every point $w \in CAP(T_1,T_2)$*, then* $w \in A(T_1)$ *as well as* $w \in T_2$.

Lemma 1: If $A(T) \neq 0$, *then* $F(T) = \emptyset$.

Lemma 2: $A(T)$ *is a closed and convex subset of* M.

Lemma 3: For a quasi-nonexpensive mapping T*, we have* $A(T) \cap C = F(T)$.

Recall that for every $C \subset H$, there exists a metric projection $P_C : H \to C$. That is, for each point $x \in H$, there is a unique element $P_C x \in C$ such that $d(x, P_C x) \leq d(x,y)$ for all $y \in C$.

Lemma 4: Let $P_C : C \to H$ *be a metric projection. Let* $\{x_n\}$ *be a sequence in* H*. If* $d(x_{n+1},y) \leq d(x_n,y), \forall y \in C$, *then* $\{P_C x_n\}$ *converges strongly to some* $y_0 \in Y$

Mann iterative process for two mappings as in CAT(0) is as follows:

$$\begin{cases} x_1 = x \in C, \\ y_n = (1-\alpha_n)x_n \oplus \alpha_n T x_n, \quad \alpha_n \in (0,1) \end{cases} \tag{5}$$

and Picard-Mann iterative hybrid process for two mappings as in CAT(0) is as follows:

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = T y_n, \\ y_n = (1-\alpha_n)x_n \oplus \alpha_n T x_n, \quad \alpha_n \in (0,1) \end{cases} \tag{6}$$

III. MAIN RESULTS

In this section, we present our main results.

Lemma 5: Let $T_1, T_2 : C \to C$ *be any two mappings. If* $\mathrm{CAP}(T_1,T_2) \neq \emptyset$, *then* $F(T_1) \cap F(T_2) \neq \emptyset$. *In particular, if* $w \in \mathrm{CAP}(T_1,T_2)$, *then* $P_C w \in F(T_1) \cap F(T_2)$.

Proof: Suppose $w \in \mathrm{CAP}(T_1,T_2)$, then $w \in A(T_1)$ and $w \in A(T_2)$. Thus by definition there exists a unique element $P_C w \in C$ such that $d(P_C w, w) \leq d(y,w)$ for all $y \in C$. Now $P_C w \in C$ implies that $d(P_C w, w) \leq d(T_2 P_C w, w)$. On the other hand, $w \in A(T_2)$, therefore $d(T_2 y, w) \leq d(y,w)$ for all $y \in C$ and, in particular, $d(T_2 P_C w, w) \leq d(P_C w, w) \leq d(T_2 P_C w, w)$ and $P_C \in F(T_2)$. Similarly, $P_C w \in F(T_1)$ and so $F(T_1) \cap F(T_2) \neq \emptyset$ and $P_C w \in F(T_1) \cap F(T_2)$. □

Lemma 6: Let $T_1, T_2 : C \to C$ *be two mappings. Then* $\mathrm{CAP}(T_1,T_2)$ *is a closed and convex subset of* X.

Lemma 7: Let $T_1, T_2 : C \to X$ *be two quasi-nonexpensive mapping. Then*

$$CAP(T_1,T_2) = F(T_1) \cap F(T_2).$$

Proof: Let $w \in \mathrm{CAP}(T_1,T_2) \cap C$, then by definition, $\max(d(T_1x,w), d(T_2x,w))$ for all $x \in C$. In particular, choosing $x = w \in C$, we obtain

$$\max(d(T_1x,w), d(T_2x,w)) \leq 0.$$

That is $w \in F(T_1) \cap F(T_2)$. Conversely, since $w \in F(T_1) \cap F(T_2)$ and $T_1, T_2 : C \to H$ are quai-nonexpensive mappings, we have $d(T_1x,w) \leq d(x.w), d(T_2x,w) \leq d(x,w)$ for all $x \in C$. This implies that $\max(d(T_1x,w), d(T_2x,w)) \leq d(x,w)$ for all $x \in C$. Clearly, $w \in C$. Hence $CAP(T_1,T_2) = F(T_1) \cap F(T_2)$. □

Theorem 1: Let $T_1, T_2 : C \to C$ *be any two FGHM which satisfy* $\alpha + \beta + \gamma \geq 0$ *and* $\varepsilon \geq 0$ *and either* $\alpha + \beta > 0$ or $\alpha + \gamma > 0$. *Then* $\mathrm{CAP}(T_1,T_2) \neq \emptyset$ *iff there exists* $w \in C$ *such that both* $\{T_1^n w, n = 0,1,\ldots\}$ *and* $\{T_2^n w, n = 0,1,\ldots\}$ *are bounded.*

Proof: Suppose that $\mathrm{CAP}(T_1,T_2) \neq \emptyset$ and $w \in \mathrm{CAP}(T_1,T_2)$. Then, by definition $\max(d(T_1x,w), d(T_2x,w)) \leq d(x,w)$ for all $x \in C$.This mean that

$$\max(d(T_1^{n+1}x,w) \leq d(T_1^n x,w)),$$

and $\max(d(T_2^{n+1}x,w) \leq d(T_2^n x,w))$ for all $x \in C$. That is, both $\{T_1^n w, n = 0,1,\ldots\}$ and $\{T_2^n w, n = 0,1,\ldots\}$ are bounded.

On the other hand, suppose that for all $w \in C$ such that, both $\{T_1^n w, n = 0,1,\ldots\}$ and $\{T_2^n w, n = 0,1,\ldots\}$ are bounded. Suppose that

$$\max(d(T_1x,w), d(T_2x,w)) \leq d(T_2x,w).$$

From long computation one can find that there exists $p \in H$ such that $d^2(T_2x,p) \leq d^2(x,p)$. This mean that $p \in A(T_2)$. However, by our supposition on maximum, we get $d^2(T_1x,p) \leq d^2(x,p)$. Thus $\mathrm{CAP}(T_1,T_2) \neq \emptyset$.

In case, $\max(d(T_1x,w), d(T_2x,w)) \leq d(x,w)$, we can get the result by interchanging the role of T_1 and T_2. □

Theorem 2: Let $T_1, T_2 : C \to X$ *be two FGHM defined as*

$$\alpha d^2(T_2x,T_2y) + \beta d^2(x,T_2y) + \gamma d^2(T_2x,y) + \delta d^2(x,y) + \varepsilon d^2(x,T_2x) \leq 0, \quad \forall x,y \in C \tag{7}$$

satisfying $\alpha + \beta + \gamma + \delta \geq 0$, $\varepsilon \geq 0$ *and, either* $\alpha + \beta > 0$ *or* $\alpha + \gamma > 0$. *Let* $CAP(T_1,T_2) \neq \emptyset$. *If* $\{x_n\}$ *is defined by*

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = T_1 y_n, \\ y_n = (1-\alpha_n)x_n \oplus \alpha_n T_2 x_n, \quad \alpha_n \in (0,1) \end{cases} \tag{8}$$

where $\{\alpha_n\}$ *is a sequence in* $(0,1)$ *with* $\alpha_n(1-\alpha_n) > 0$ *then,* $\{x_n\}$ *converges weakly to a point* $q \in CAP(T_1,T_2)$. *Furthermore,* $q = \lim_{n\to\infty} Px_n$, *where,* P *is a projection of* H *onto* $CAP(T_1,T_2)$

Proof: Consider $w \in \mathrm{CAP}(T_1,T_2)$. Then, by using (8)

$$\begin{aligned} d^2(y_n,w) &= d((1-\alpha_n)x_n \oplus \alpha_n T_2 x_n, w) \\ &\leq (1-\alpha_n)d^2(x_n,w) + \alpha_n d^2(T_2 x_n, w) \\ &\leq (1-\alpha_n)d^2(x_n,w) + \alpha_n d^2(x_n,w) \\ &= d^2(x_n,w) \end{aligned}$$

and

$$\begin{aligned} d^2(x_{n+1}, w) &= d^2(T_1 y_n, w) \\ &\leq d^2(y_n, w) \\ &\leq d^2(x_n, w) \end{aligned}$$

Therefore, we can have

$$d^2(x_{n+1}, w) \leq d^2(x_n, w). \tag{9}$$

Thus $\lim_{n\to\infty} d^2(x_n, w)$ exists and so $\{x_n\}$ must be bounded. Since H is complete CAT(0) space, so

$$\begin{aligned} d^2(x_{n+1}, w) &= d^2(T_1 y_n, w) \\ &\leq d^2(y_n, w) \\ &\leq d^2((1-\alpha_n)x_n \oplus \alpha_n T_2 x_n, w) \\ &= (1-\alpha_n)d^2(x_n, w) + \alpha_n d^2(T_2 x_n, w) \\ &\quad -\alpha_n(1-\alpha_n)d^2(T_2 x_n, x_n) \\ &= (1-\alpha_n)d^2(x_n, w) + \alpha_n d^2(x_n, w) \\ &\quad -\alpha_n(1-\alpha_n)d^2(T_2 x_n, x_n) \\ &= d^2(x_n, w) - \alpha_n(1-\alpha_n)d^2(T_2 x_n, x_n). \end{aligned}$$

This implies that

$$\alpha_n(1-\alpha_n)d^2(x_{n+1}, w) \leq d^2(x_n, w) - d^2(x_{n+1}, w)$$

Now, by using $\liminf \alpha_n(1-\alpha_n) > 0$ and the above proved reality that $\lim_{n\to\infty} d^2(x_n, w)$ exists, we get

$$\lim_{n\to\infty} d^2(T_2 x_n, x_n) = 0.$$

The boundedness of the sequence $\{x_n\}$ has also been proved in above lines, so we have subsequences $\{x_{n_j}\}$ with

$$x_{n_j} \rightharpoonup q \in C.$$

Since $T_2 : C \to C$ is a FGM, therefore for every $y \in C$, we have

$$\begin{aligned} \alpha d^2(T_2 x_{n_j}, T_2 y) + \beta d^2(x_{n_j}, T_2 y) + \gamma d^2(T_2 x_{n_j}, y) \\ + \delta d^2(x_{n_j}, y) + \varepsilon d^2(x_{n_j}, T_2 x_{n_j}) \leq 0, \end{aligned}$$

and

$$\begin{aligned} \alpha \left(d^2(T_2 x_{n_j}, x_{n_j}) + d^2(x_{n_j}, T_2 y)\right) + 2\langle \overrightarrow{T_2 x_{n_j} x_{n_j}}, \overrightarrow{x_{n_j} T_2 y}\rangle \\ + \beta d^2(x_{n_j}, T_2 y) + \gamma d^2(T_2 x_{n_j}, y) \\ + \delta d^2(x_{n_j}, y) + \varepsilon d^2(x_{n_j}, T_2 x_{n_j}) \leq 0. \end{aligned}$$

Making use of Banench limits μ, we get

$$(\alpha+\beta)\mu_n d^2(T_2 x_{n_j}, T_2 y) + (\gamma+\delta)\mu_n d^2(x_{n_j}, y) \leq 0.$$

This yield that

$$\begin{aligned} (\alpha+\beta)\mu_n \left[d^2(T_2 x_{n_j}, T_2 y) + d^2(y, T_2 y) + 2\langle \overrightarrow{x_{n_j} y}, \overrightarrow{y T_2 y}\rangle\right] \\ + (\gamma+\delta)\mu_n d^2(x_{n_j}, T_2 y) \leq 0. \end{aligned}$$

Thus

$$\begin{aligned} (\alpha+\beta+\gamma+\delta)\mu_n d^2(x_{n_j}, y) \\ + (\alpha+\beta)d^2(y, T_2 y) + 2(\alpha+\beta)\langle \overrightarrow{x_{n_j} y}, \overrightarrow{y T_2 y}\rangle \leq 0. \end{aligned}$$

But $\alpha+\beta+\gamma+\delta \geq 0$, so,

$$(\alpha+\beta)\mu_n d^2(y, T_2 y) + 2(\alpha+\beta)\langle \overrightarrow{x_{n_j} y}, \overrightarrow{y T_2 y}\rangle \leq 0.$$

Since $x_{n_j} \rightharpoonup p$, therefore

$$(\alpha+\beta)\mu_n d^2(y, T_2 y) + 2(\alpha+\beta)\langle \overrightarrow{p y}, \overrightarrow{y T_2 y}\rangle \leq 0.$$

In $CAT(0)$ we have quasi-linearization

$$\langle \overrightarrow{ab}, \overrightarrow{cd}\rangle = \frac{1}{2}(d^2(a, d) + d^2(b, c) - d^2(a, c) - d^2(b, d)) \tag{10}$$

We get

$$\begin{aligned} &(\alpha+\beta)d^2(y, T_2 y) \\ &\quad + (\alpha+\beta)\left[d^2(p, T_2 y) + d^2(y, y) - d^2(p, y) - d^2(y, T_2 y)\right] \leq 0. \end{aligned}$$

This implies that $(\alpha+\beta)\left[d^2(p, T_2 y) - d(p, y)\right] \leq 0$. As $(\alpha+\beta) > 0$ and

$$d^2(p, T_2 y) - d(p, y) \leq 0.$$

In the same way, we have $d^2(p, T_1 y) - d^2(p, y) \leq 0$ and we get $p \in \mathrm{CAP}(T_1, T_2)$. Next, we will prove that $\{x_n\} \rightharpoonup p$ by proving that any two subsequences of $\{x_n\}$ converges weakly to a same limit p. Let $\{x_{n_j}\} \rightharpoonup p_1$ and $\{x_{n_k}\} \rightharpoonup p_2$. By what we have just proved, $p_1, p_2 \in \mathrm{CAP}(T_1, T_2)$, and from the first step of prove, we deduce that

$$\lim_{n\to\infty} \left(d^2(x_n, p_1) - d^2(x_n, p_2)\right)$$

exists, call it l. Now using (10)again,

$$2\langle \overrightarrow{x_n 0}, \overrightarrow{p_1 p_2}\rangle = d^2(x_n, p_1) + d^2(p_2, 0) - d^2(x_n, p_2) - d^2(p_1, 0).$$

This gives

$$d^2(x_n, p_1) - d^2(x_n, p_2) = 2\langle \overrightarrow{x_n 0}, \overrightarrow{p_2 p_1}\rangle - d^2(p_2, 0) + d^2(p_1, 0).$$

Thus

$$d^2(x_{n_j}, p_1) - d^2(x_{n_j}, p_2) = 2\langle \overrightarrow{x_{n_j} 0}, \overrightarrow{p_2 p_1}\rangle - d^2(p_2, 0) + d^2(p_1, 0).$$

and

$$\begin{aligned} d^2(x_{n_k}, p_1) - d^2(x_{n_k}, p_2) \\ = 2\langle \overrightarrow{x_{n_k} 0}, \overrightarrow{p_2 p_1}\rangle - d^2(p_2, 0) + d^2(p_1, 0). \end{aligned}$$

Now, taking weak limit on the above two equations and making use $\{x_{n_j}\} \rightharpoonup p_1$ and $\{x_{n_k}\} \rightharpoonup p_2$, we get

$$\begin{aligned} l &= 2\langle \overrightarrow{p_1 0}, \overrightarrow{p_2 p_1}\rangle - d^2(p_2, 0) + d^2(p_1, 0). \\ l &= 2\langle \overrightarrow{p_2 0}, \overrightarrow{p_2 p_1}\rangle - d^2(p_2, 0) + d^2(p_1, 0). \end{aligned}$$

subtracting, we get $\langle \overrightarrow{p_2 p_1}, \overrightarrow{p_1 p_2}\rangle = 0$ and hence $p_1 = p_2$. In turn $x_n \rightharpoonup p \in \mathrm{CAP}(T_1, T_2)$. Finally, we show that $p = \lim_{n\to\infty} P x_n$, where P is the projection of X onto $\mathrm{CAP}(T_1, T_2)$.

Now from (2.1) it follows that $d(x_{n+1}, w) = d(x_n, w)$ for all $w \in \mathrm{CAP}(T_1, T_2)$. Since $\mathrm{CAP}(T_1, T_2)$ is closed and convex by Lemma 6, applying Lemma 4, $\lim_{n\to\infty} Px_n = p$ for some $p \in \mathrm{CAP}(T_1, T_2)$. For projections, we know that $\langle \overrightarrow{x_n 0}, \overrightarrow{p_1 p_2}\rangle$ for all $w \in \mathrm{CAP}(T_1, T_2)$. Therefore, $\langle \overrightarrow{x_n Px_n}, \overrightarrow{Px_n w}\rangle$ for all $w \in \mathrm{CAP}(T_1, T_2)$ and, in particular, $\langle \overrightarrow{qp}, \overrightarrow{pq}\rangle$. Hence, $q = p = \lim_{n\to\infty} Px_n$. □

The following corollaries can be obtained immediately form the above theorem and are new interesting results in CAT(0) spaces As mentioned earlier that the process (6) is faster and independent from many existing approximation processes, so our following results has their our value.

Corollary 1: Suppose that M, C, T_2 and $\alpha, \beta, \gamma, \delta$ be the same as in the Theorem 2. Consider $A(T_2) \neq \emptyset$. If $\{x_n\}$ is be a sequence of iterates defined in 5, where $\{\alpha_n\}$ is a monotonically increasing sequence in the interval $(0, 1)$ such that $\liminf \alpha_n(1-\alpha_n) > 0$, then the sequence $\{x_n\}$ converges weakly to $p \in A(T_2)$. Furthermore, $p = \lim_{n\to\infty} Px_n$, where P is the projection of H onto $A(T_2)$.

Corollary 2: Suppose that M, C, T_2 and $\alpha, \beta, \gamma, \delta$ be the same as in the Theorem 2. Consider $A(T_2) \neq \emptyset$. If $\{x_n\}$ is be a sequence of iterates defined in 6, where $\{\alpha_n\}$ is a monotonically increasing sequence in the interval $(0, 1)$ such that $\liminf \alpha_n(1-\alpha_n) > 0$, then the sequence $\{x_n\}$ converges weakly to $p \in A(T_2)$. Furthermore, $p = \lim_{n\to\infty} Px_n$, where P is the projection of H onto $A(T_2)$.

If we take C closed in Theorem 2, then we get the following Theorem:

Theorem 3: Let T_1, T_2 are two FGHM defined as (2.10) which satisfy $\alpha+\beta+\gamma+\delta \geq 0$, $\varepsilon \geq 0$ and either $\alpha+\beta > 0$ or $\alpha+\gamma > 0$. Let CAP$(T_1, T_2) \neq \emptyset$. If $\{x_n\}$ is defined by

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = T_1 y_n, \\ y_n = (1-\alpha_n)x_n + \alpha_n T_2 x_n, \quad \alpha_n \in (0,1) \end{cases} \tag{11}$$

where $\{\alpha_n^1\}$ is a sequence in $(0, 1)$ with $\alpha_n^1(1-\alpha_n^1) > 0$ then, $\{x_n\}$ converges weakly to a point $P_C q \in F(T_1) \cap F(T_2)$, where, $q \in H$ and $P_C : X \to C$ is metric projection.

Now, we give a numerical example to support our results.

Example 2: Let $X = R$ be a usual metric space with the metric d, which is also an Hadamard space, and $C = (-1, 1)$. We see that C is a convex subset of X. Define a mapping $T : C \to C$ by

$$T_1(x) = \begin{cases} \dfrac{1-x}{2}, & x \in (-1, 0]; \\ \dfrac{1+x}{2}, & (0, 1). \end{cases}$$

and

$$T_2(x) = \begin{cases} \dfrac{1-x}{3}, & x \in (-1, 0]; \\ \dfrac{1+x}{3}, & (0, 1). \end{cases}$$

TABLE 1. Iterates of new iterative scheme for initial guess -0.5.

n	x_n	$\lvert x_n - x_{n-1}\rvert$
1	-0.50000000	
2	0.58333333	1.13888883
3	0.55718949	0.0261438
4	0.77854974	0.22136025
5	0.83864378	0.06009404
6	0.85242929	0.01378551
7	0.85279187	0.00036258
8	0.85011661	8.4689203e-1
9	0.84725461	2.862e-3
10	0.84481774	2.43687e-3
11	0.84283554	1.9822e-3
12	0.84122232	0.00161322
13	0.83989198	0.00133034
14	0.83877775	0.00111423
15	0.83783126	0.00094649
16	0.83701714	0.00081412
17	0.83630926	0.00070788
18	0.83529582	0.00101344
19	0.83468811	0.00060771
20	0.83421899	0.00046912
21	0.83381504	0.00040395
22	0.83345376	0.00036128
23	0.83312591	0.00032785
24	0.83282638	0.00029953
25	0.83255142	0.00029953

for all $x \in C$. It is easy to see that both T_1 and T_2 are further generalized hybrid mapping with $a = 2, \beta = \gamma = -1, \delta = \epsilon = 0$ and $A(T) = [1, \infty)$. Let $\alpha_n = \frac{4n}{5n+7}$ for all $n \in \mathbb{N}$.

IV. CONCLUSION

In this paper, we introduced a new approximation method of CAP for the class of FGHM in CAT(0) spaces. We proved convergence of our proposed method and gave an example to validate our method. Our results are more generalized and interesting from many results existing in literature [22].

ACKNOWLEDGMENT

The authors are would like thankful to both reviewers for valuable suggestions and comments. The present form of the paper would not be possible without their suggestions.

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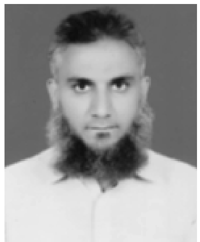

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