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Tomita-Takesaki Theory in $\mathcal{B}(\mathscr{H})$ with Physical Applications

by

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Abstract

The main aim of this project is to develop a simple, yet mathematically rigorous, version of Tomita-Takesaki theory for the von Neumann algebra $\mathcal{B}(\mathscr{H})$ with a faithful normal state.

In Chapter 2 we formulate the theory in terms of tensor products. Even in this fairly general setup we can already attach physical interpretation to the modular objects Δ and J. Namely that, the former, the modular operator induces a unique modular automorphism group σ_t which in turn gives the time-evolution (dynamics) of some physical system. Whereas the modular conjugation implements a time-reversal.

Chapter 3 presents an alternative formulation of Tomita-Takesaki theory, unitarily equivalent to the first, but with the space of Hilbert-Schmidt operators as our preferred choice of Hilbert space.

To gain further insight into the theory, in Chapter 4, a certain simple physical system is explored. In particular, we look at how the system of an electron in a constant orthogonal magnetic field, together with the associated phenomenon of Landau levels, displays a modular structure in the sense of Tomita-Takesaki theory. In such a case, the algebra of observables and its commutant correspond to the two directions of the magnetic field.

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Contents

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General symbols

$\emptyset$: The empty set

$\mathbb{N}$: Natural numbers $\{1, 2, 3, ...\}$.

$\mathbb{R}$: Field of real numbers.

$\mathbb{C}$: Field of complex numbers.

δ_{jk} : The Kronecker delta

Symbols defined in the text

$\mathcal{A}, \mathcal{B}$: Typical (C*-) algebras.

$\mathcal{A}'$: The commutant of the algebra $\mathcal{A}$.

$\mathcal{A} \subset \mathcal{B}$: Means $x \in \mathcal{A} \implies x \in \mathcal{B}$.

$\mathcal{A} \cong \mathcal{B}$: The algebras $\mathcal{A}, \mathcal{B}$ are isomorphic.

$\mathcal{A} \odot \mathcal{B}, \mathcal{A} \otimes \mathcal{B}$: The algebraic (respectively, completion) tensor product of $\mathcal{A}$ and $\mathcal{B}$.

$\mathcal{B}(\mathscr{H}, \mathscr{K})$: The space of bounded linear operators $\mathscr{H} \to \mathscr{K}$.

$\mathcal{B}_1(\mathscr{H}, \mathscr{K})$: The space of trace-class operators $\mathscr{H} \to \mathscr{K}$.

$\mathcal{B}_2(\mathscr{H}, \mathscr{K})$: The space of Hilbert-Schmidt operators $\mathscr{H} \to \mathscr{K}$.

$\mathscr{D}(a)$: The domain of the operator a.

G : The tensor product Hilbert space $\mathscr{H} \otimes \mathscr{H}$.

$\mathscr{H}, \mathscr{K}$: Typical (complex) Hilbert spaces.

$\overline{\mathscr{H}}$: The conjugate Hilbert space of $\mathscr{H}$.

$\mathscr{H}_0$: The linear span of finitely many linear combitinations of the basis elements $\{e_j : j \in \mathbb{N}\}$.

H : The Hamiltonian operator.

$H_\perp, H_\parallel$: The perpendicular and parallel components of the Hamiltonian H respectively.

I : An ideal in an algebra.

J_0, J : The pre- and modular conjugation defined on $\mathscr{H}_0 \otimes \mathscr{H}_0$ and G, respectively.

$L^2(\mathbb{R}, dx)$	:	The function space over $\mathbb{R}$ with measure dx.
$\mathbb{P}_i$	:	The projection $\|e_i\rangle \langle e_i\|$, where the e_j are basis elements.
P	:	The momentum operator.
Q	:	The position operator.
S_0, S	:	The pre- and Tomita operators respectively.
T	:	Temperature.
Tr	:	The trace functional on an algebra $\mathcal{A}$, i.e. $Tr : \mathcal{A} \to \mathbb{C}$.
U	:	The unitary operator defined in Sec.4.5.
$\mathfrak{A}$	:	Typical von Neumann algebra.
$\mathfrak{A}_l, \mathfrak{A}_r$	:	The von Neumann algebras defined in Chap. 3.
W	:	The unitary transformation defined in Sec. 1.7.
X, Y	:	Typical vector (or normed) spaces.
a, b, c	:	Typical (bounded) linear operators
$\\|a\\|$	:	The operator norm of a.
$\tilde{a}$	:	The bounded linear extension of a.
$a \|_M$	:	The restriction of operator a onto the set M.
a^T	:	The transpose of matrix a.
a^*	:	The Hilbert-adjoint operator of a.
$a, a^\dagger$	:	The creation and annihilation operators.
$a \odot b, a \otimes b$	:	The algebraic (respectively, completion) tensor product of a and b.
$a \vee b$	:	The bounded linear operator on $\mathcal{B}_2(\mathscr{H})$ defined in Chap. 3.
$[\cdot, \cdot]$	:	The commutator on an algebra $\mathcal{A}$. That is, $[a, b] = ab - ba$ for all $a, b \in \mathcal{A}$.
$\langle \cdot, \cdot \rangle$	:	An inner product on a Hilbert space.
$id_{\mathcal{A}}$	:	The identity operator on the algebra $\mathcal{A}$.
$1, 1_{\mathcal{A}}$	:	The unit element of the algebra $\mathcal{A}$.
j	:	The *-anti-isomorphic map defined in Sec.2.4.

$\ker(a)$: The kernel of operator a.

$l^2, l^2(\mathbb{N}), l^2(\mathbb{N}^2)$: The Hilbert sequence spaces.

l_B : The magnetic length.

$|n\rangle, |n\rangle \otimes |m\rangle$: The quantum harmonic oscillator states on $\mathscr{H}$ and $\mathscr{H} \otimes \mathscr{H}$, respectively.

x, y, z : Typical elements of a Hilbert space.

x^* : The conjugate of the representation of x, i.e. $x^* := \sum_{j\in\mathbb{N}} e_j \bar{x}_j$ where $x = \sum_{j\in\mathbb{N}} e_j x_j$.

$\bar{x}$: The element $x \in \mathscr{H}$ view as an element of the conjugate Hilbert space $\overline{\mathscr{H}}$.

$x \otimes y$: The elementary tensor of x and y.

Δ_0, Δ : The pre- and modular operators defined on $\mathscr{H}_0 \otimes \mathscr{H}_0$ and G, respectively.

Δ^{it} : The unitary operator defined in Sec.2.3.

Δ' : The modular operator defined on the commutant.

Ω, Φ : The cyclic and separating vectors.

α, λ : Typical complex numbers.

$\bar{\alpha}$: The conjugate of a complex number α.

$\beta := 1/T$: The inverse temperature.

ι : The unitary map defined in Sec. 3.2.

μ : A faithful normal state on $\mathcal{A}$ given by $\mu(a) = Tr(\rho a)$.

ω, ψ, φ : Typical states on an algebra.

ω_B : The cyclotron frequency.

π : The (faithful) representation of $\mathcal{B}(\mathscr{H})$ onto $\mathcal{B}(G)$.

$\mathbf{\Pi}, \widetilde{\mathbf{\Pi}}$: The gauge-invariant momentum, and pseudo-momentum.

ρ : A density operator.

$\sigma_t, \sigma_t^{\Omega}, \sigma_t^{\mu}$: The modular group on $\mathcal{A}$.

σ_t' : The modular group on the commutant of $\mathcal{A}$.

Conventions

(i) Unless explicitly stated otherwise, any vector space or algebra will be over the field of complex numbers.

(ii) An inner product of an inner product space will always be taken to be linear in the second argument and conjugate linear in the first argument, as is conventional in physics.

Prologue

The *Tomita-Takesaki theory* (or *modular theory*) has been one of the most exciting subjects for operator algebras and for its many applications to mathematical physics [20]. The origins of the theory lie in two unpublished papers of M. Tomita whose manuscripts were distributed to the participants of the Baton Rouge conference in 1967.

At the same time Haag, Hugenholtz, and Winnink published their paper [15] on the description of thermodynamic equilibrium states using the *Kubo-Martin-Schwinger (KMS) boundary condition*. Probably Hugenholtz and Winnink were the first to realize the similarity between certain aspects of their approach and Tomita's theory and hence the importance of this then new mathematical theory for theoretical physics [6].

But general awareness of Tomita's theory only came subsequently through a slim volume published by M. Takesaki [31] in the Lecture Notes in Mathematics. In which Takesaki gave a careful and complete exposé of Tomita's work with many refinements and clarifications as well as several applications. Since then this theory came to be known as the Tomita-Takesaki theory.

The Tomita-Takesaki theory is one of the most important and useful developments in the history of operator algebras, giving a very precise and intimate connection between an algebra and its commutant, along with a one-parameter group of automorphisms [7]. This theory made possible the great advances in the 1970s by A. Connes et al. on the classification of factors.

Later on Connes remarked, concerning the relation between the KMS condition and the modular operator of Tomita-Takesaki theory discovered by Takesaki and Winnink, that it "remains one of the deepest points of contact between physics and pure mathematics" [10]. Whereas R. Haag described this connection as "a beautiful example of 'prestabilized harmony' between physics and mathematics" [14].

In this dissertation we will present a simple version of the Tomita-Takesaki modular theory for a von Neumann algebra $\mathcal{B}(\mathscr{H})$ with a faithful normal state. This is a special case of the more general situation of the theory (for example, see [8],[30]). This approach is close enough to the standard approach that it can also serve as an introduction to the more general theory.

The broad structure of this dissertation consists of three parts with clearly defined chapters, and is organized as follows:

- First, we give in Chapter 1 the basic mathematical background required to develop a simple version of the Tomita-Takesaki theory.

- The second part contains the main body of our project which is to formulate the Tomita-Takesaki theory. In Chapter 2 we give the formulation of the theory in terms of tensor products, the advantage of this approach lies in that we gain intuitive insight into the relation of algebra and its commutant as “mirror images” of each other.

An alternative approach to the theory that has appeared in the physics literature, centered around the Hilbert space of Hilbert-Schmidt operators, is presented in Chapter 3.

Even in this general framework we can already see how the modular automorphism group gives the dynamics. This also allows us to attach physical interpretation to the modular conjugation J as time reversal. This approach gives additional insight into Tomita-Takesaki theory, but the tensor product approach is the main focus of this dissertation.

- Finally to gain further insight into the theory, in Chapter 4 we look at the *Landau problem*- related to the motion of a charged particle on the flat xy-plane in the presence of a constant magnetic field along the z-axis, along with the related phenomenon of *Landau quantization*. We show how the algebra of observables associated with this physical system displays a modular structure in the sense of the Tomita-Takesaki theory, with the algebra and its commutant corresponding to the two orientations of the magnetic field.

Our tensor product approach differs from the standard textbook treatment of general Tomita-Takesaki theory, and gives a much more concrete representation of the theory than one can obtain in general. The tensor product approach also gives a clear indication that Tomita-Takesaki theory is closely related to entanglement, in particular in our case via the (entangled) cyclic vector Ω (for example, see [12]).

Furthermore, we use novel arguments to handle the unbounded operators in the theory, which allows us to avoid most of the subtleties and technical difficulties usually associated to unbounded operators.

The dissertation is mostly devoted to a mathematically rigorous development of Tomita-Takesaki theory. However, when treating some of the physics (in particular in Chapter 4 and the Appendix), we do not work completely rigorous.

Chapter 1

Background

Here we treat material needed in the rest of the dissertation. While much of it is standard, we also develop some tools and techniques specific to our goals, and which can not be found in standard texts, in particular Sections 1.5 and 1.8. Only separable Hilbert spaces over the complex field are considered in this dissertation.

1.1 Operators on Hilbert spaces

In this section we review the basic properties of linear and conjugate-linear operators on Hilbert spaces. We state results without proofs as they are readily available (see, for example [18], [22]).

Definition 1.1.1 (Notation for operators)**.** Let X and Y be complex vector spaces, and $a : \mathscr{D}(a) \to Y$ an operator, where $\mathscr{D}(a) \subset X$ is the domain of a. We write either $a(x)$ or ax to denote the image under a of an element $x \in \mathscr{D}(a)$.

1. a is *linear* if $a(\alpha x + y) = \alpha ax + ay$ for all $x, y \in \mathscr{D}(a), \alpha \in \mathbb{C}$.
2. a is *conjugate linear* if $a(\alpha x + y) = \bar{\alpha} ax + ay$ for all $x, y \in \mathscr{D}(a), \alpha \in \mathbb{C}$.
3. a is *injective* if $a(x) = a(y)$ implies $x = y$.
4. The *kernel* of a is $\ker(a) = \{x \in \mathscr{D}(a) : ax = 0\}$.
5. The *range* of a is $\mathscr{R}(a) = \{ax : x \in \mathscr{D}(a)\}$.
6. a is *surjective* if $\mathscr{R}(a) = Y$.
7. a is a *bijection* if it is both injective and surjective.

We use either id or id_X to denote the identity operator of a space X onto itself.

Definition 1.1.2 (Bounded operator)**.** Let X and Y be complex normed spaces. An operator $a : \mathscr{D}(a) \to Y$, where $\mathscr{D}(a) \subset X$, is said to be *bounded* if there is a real number γ such that for all $x \in \mathscr{D}(a)$,

$$\|a(x)\| \leq \gamma \|x\|. \tag{1.1.1}$$

An unbounded operator is an operator that is not bounded.

Remarks 1.1.3. The notions of boundedness and continuity are equivalent for operators. Typically it is easier to check boundedness. An immediate consequence of the definition is that (1.1.1) with $\gamma = \|a\|$ gives

$$\|ax\| \leq \|a\|\|x\| \tag{1.1.2}$$

which is used extensively.

Definition 1.1.4 (Isometries and isomorphisms)**.** Let X, Y be normed spaces, and let $a : \mathscr{D}(a) \subset X \to Y$ be linear.

1. If a preserves the norm, i.e.

   $$\forall x \in \mathscr{D}(a),\ \|ax\| = \|x\|,$$

   then a is called an *isometry* (hence bounded).

2. A bijective isometry a is called an *isomorphism*.

3. If there exists an isomorphism a from X into Y, then X is *isomorphic* with Y, and X and Y are called *isomorphic normed spaces*.

On occasion, we will deal with conjugate-linear isometries and isomorphisms, which are entirely analogous except that the operator a is conjugate-linear instead of linear.

Denote by $\mathcal{B}(X, Y)$ the collection of all bounded operators from X to Y, and write $\mathcal{B}(X)$ in place of $\mathcal{B}(X, X)$.

Example 1.1.5. If X, Y are normed spaces, then $\mathcal{B}(X, Y)$ is itself a normed space with the point-wise defined operations for addition and scalar multiplication, and norm the *operator norm*:

$$\|a\| = \sup_{x \neq 0} \frac{\|ax\|}{\|x\|} = \sup_{\|x\|=1} \|ax\|. \tag{1.1.3}$$

Moreover, if Y is a Banach space, then $\mathcal{B}(X, Y)$ is a Banach space.

In particular, $\mathcal{B}(\mathscr{H})$ is complete, where $\mathscr{H}$ is a Hilbert space.

Definition 1.1.6. Let $\{e_j : j \in \mathbb{N}\}$ be an orthonormal basis for a separable Hilbert space $\mathscr{H}$.

1. An operator $a \in \mathcal{B}(\mathscr{H})$ is *positive*, and written $a \geq 0$, if

$$\langle x, ax \rangle \geq 0 \ \forall x \in \mathscr{H}.$$

2. For any positive operator $a \in \mathcal{B}(\mathscr{H})$ we define

$$Tr(a) := \sum_{j=1}^{\infty} \langle e_j, ae_j \rangle .$$

   The number $Tr(a)$ is called the *trace of* a and is independent of the choice of orthonormal basis.

Definition 1.1.7 (Symmetric operator)**.** Let $a : \mathscr{D}(a) \to \mathscr{H}$ be a linear operator, which is densely defined on a Hilbert space $\mathscr{H}$, i.e. $\mathscr{D}(a)$ is dense in $\mathscr{H}$. Then a is called a *symmetric* linear operator if for all $x, y \in \mathscr{D}(a)$,

$$\langle ax, y \rangle = \langle x, ay \rangle . \tag{1.1.4}$$

It can be shown that a densely defined operator on a Hilbert space $\mathscr{H}$ is symmetric if and only if $\langle ax, x \rangle$ is real for all $x \in \mathscr{D}(a)$.

An unbounded operator a satisfying (1.1.4) cannot be defined on all of $\mathscr{H}$ (cf. [22, Thm. 10.1-1]):

Theorem 1.1.8 (Hellinger-Toeplitz)**.** *If a linear operator a is defined on all of a complex Hilbert space $\mathscr{H}$ and satisfies (1.1.4) for all $x, y \in \mathscr{H}$, then a is bounded.*

As a consequence of the above theorem, $\mathscr{D}(a) = \mathscr{H}$ is impossible for unbounded operators. This makes the problems of determining suitable domains and extensions become of prime importance.

Definition 1.1.9. An *extension* of $a : \mathscr{D}(a) \to \mathscr{H}$ to a set $M \supset \mathscr{D}(a)$ is an operator

$$\tilde{a} : M \to \mathscr{H} \ \text{ such that } \ \tilde{a}\,|_{\mathscr{D}(a)} = a,$$

i.e. $\tilde{a}x = ax$ for all $x \in \mathscr{D}(a)$. Call a an operator *on* $\mathscr{H}$ if $\mathscr{D}(a) = \mathscr{H}$, and an operator *in* $\mathscr{H}$ if $\mathscr{D}(a)$ lies in $\mathscr{H}$ but may not be all of $\mathscr{H}$.

Extensions that are usually of practical importance are those which preserve some basic property of the operator $a : \mathscr{D}(a) \to \mathscr{H}$, for instance linearity (if a is linear) or boundedness (if $\mathscr{D}(a)$ lies in $\mathscr{H}$ and a is bounded).

Theorem 1.1.10 (Bounded linear extension)**.** *Let $\mathscr{H}$ and $\mathscr{K}$ be Hilbert spaces, and $a : \mathscr{D}(a) \to \mathscr{K}$ a bounded linear operator, where $\mathscr{D}(a)$ lies in $\mathscr{H}$. Then a has a unique extension*

$$\tilde{a} : \overline{\mathscr{D}(a)} \to \mathscr{K}$$

where $\overline{\mathscr{D}(a)}$ is the closure of the domain of a. Moreover, $\tilde{a}$ is a bounded linear operator of norm

$$\|\tilde{a}\| = \|a\|.$$

If, in addition, a is isometric then $\tilde{a}$ is also isometric.

Remarks 1.1.11 (Conjugate linear operators)**.** Completely analogous to bounded linear operators, a bounded conjugate-linear operator has a unique bounded conjugate-linear extension. In particular, the bounded extension of an isometric conjugate-linear operator is itself isometric.

The Hilbert-adjoint of a densely-defined linear operator is given by:

Definition 1.1.12 (Hilbert-adjoint operator)**.** Let $a : \mathscr{D}(a) \to \mathscr{H}$ be a (possibly unbounded) densely-defined linear operator in a complex Hilbert space $\mathscr{H}$. Then the *Hilbert-adjoint operator* $a^* : \mathscr{D}(a^*) \to \mathscr{H}$ of a is defined as follows. The domain $\mathscr{D}(a^*)$ of a^* consists of all $y \in \mathscr{H}$ such that there is a $y \in \mathscr{H}$ satisfying

$$\langle ax, y\rangle = \langle x, y^*\rangle \tag{1.1.5}$$

for all $x \in \mathscr{D}(a)$. For each such $y \in \mathscr{D}(a^*)$ the Hilbert-adjoint operator a^* is then defined in terms of that y^* by

$$y^* = a^* y. \tag{1.1.6}$$

Remarks 1.1.13. Note that by definition of a^*, it follows that

$$\langle ax, y\rangle = \langle x, a^* y\rangle \tag{1.1.7}$$

for all $x \in \mathscr{D}(a), y \in \mathscr{D}(a^*)$, and $\mathscr{R}(a) = \mathscr{D}(a^*)$.

The Hilbert-adjoint of a conjugate-linear operator a is also conjugate-linear and satisfies

$$\langle x, a^* y\rangle = \langle y, ax\rangle = \overline{\langle ax, y\rangle} \tag{1.1.8}$$

for all $x \in \mathscr{D}(a)$ and $y \in \mathscr{D}(a^*)$.

Definition 1.1.14 (Self-adjoint linear operator)**.** Let $a : \mathscr{D}(a) \to \mathscr{H}$ be a linear operator, which is densely defined on a Hilbert space $\mathscr{H}$. Then a is called a *self-adjoint linear operator* if

$$a = a^*. \tag{1.1.9}$$

We have the following special terminology for isomorphisms on Hilbert spaces:

Definition 1.1.15 (Unitary operator)**.** Let $\mathscr{H}$ and $\mathscr{K}$ be Hilbert spaces, and $u : \mathscr{D}(u) \subset \mathscr{H} \to \mathscr{K}$ be a linear isometric isomorphism. Then u is called a *unitary operator* if

$$u^* u \mid_{\mathscr{D}(u)} = 1_{\mathscr{H}} \text{ and } uu^* \mid_{\mathscr{D}(u^*)} = 1_{\mathscr{K}}. \tag{1.1.10}$$

If such a u exists, $\mathscr{H}$ and $\mathscr{K}$ are said to be *unitarily equivalent* (or *isomorphic*).

A conjugate-linear isometric isomorphism for which (1.1.10) holds is called an *anti-unitary*, and in this case $\mathscr{H}$ and $\mathscr{K}$ are said to be *anti-unitarily equivalent* (or *anti-isomorphic*).

An isometry on an inner product space must preserve the inner product as well as the norm.

Proposition 1.1.16. *Let $\mathscr{H}$ be a Hilbert space. Then the following conditions are equivalent:*

1. *u is a unitary operator on $\mathscr{H}$.*
2. *u is surjective and preserves the inner product.*
3. *The range of u is dense in $\mathscr{H}$ and preserves the inner product.*

1.2 Infinite matrix representation of operators

Here we represent operators on a separable Hilbert space as infinite matrices with respect to a given orthonormal basis. This allows us to introduce the notion of transposition of an operator with respect to the basis for use in Tomita-Takesaki theory later on.

What we develop in this Section can be thought of as a simple generalization of the familiar case of linear operators and matrices on finite dimensional spaces of [18, Sec.2.9].

If $\mathscr{H}$ is a separable Hilbert space, then (by [18, Thm.3.6-4]) $\mathscr{H}$ has a *countable* orthonormal basis.

Definition 1.2.1. Let $\{e_j : j \in \mathbb{N}\}$ be a countable orthonormal basis for the Hilbert space $\mathscr{H}$. The *matrix representation* of $a \in \mathcal{B}(\mathscr{H})$ with respect to $\{e_j : j \in \mathbb{N}\}$ is the matrix $[a_{jk}]_{j,k\in\mathbb{N}}$ whose entries are given by

$$a_{jk} := \langle e_j, ae_k \rangle \tag{1.2.1}$$

for all $j, k \in \mathbb{N}$.

Remarks 1.2.2 (Connection between operators and their matrix representations)**.** Linear operators are matrix transformations and vice versa, this holds in the finite dimensional case as well as in infinite dimension. Since linear operators on finite dimensional normed spaces are trivially bounded, it follows that every finite matrix is a representation of some bounded linear operator. However, in the infinite dimensional case not every infinite matrix represents a bounded operator.

We state the following result from elementary functional analysis without proof.

Theorem 1.2.3 (Riesz-Fischer)**.** *Let $\{e_j : j \in \mathbb{N}\}$ be a countable orthonormal basis for the Hilbert space $\mathscr{H}$. The following statements are equivalent:*

1. *$\{e_j : j \in \mathbb{N}\}$ is total, i.e. if $\langle e_k, x\rangle = 0\ \forall k \in \mathbb{N}$ then $x = 0$.*

2. *$\{e_j : j \in \mathbb{N}\}$ is closed, i.e.*

$$\forall x \in \mathscr{H},\ x = \sum_{k=1}^{\infty} e_k \langle e_k, x\rangle . \tag{1.2.2}$$

3. *The linear span of the set $\{e_j : j \in \mathbb{N}\}$ is dense in $\mathscr{H}$, i.e. $\overline{span\{e_j : j \in \mathbb{N}\}} = \mathscr{H}$.*

4. *Parseval relation holds, i.e. $\forall x, y \in \mathscr{H},\ \langle x, y\rangle = \sum_{k=1}^{\infty} \langle x, e_k\rangle \langle e_k, y\rangle$.*

Proposition 1.2.4. *Let $[a_{jk}]_{j,k\in\mathbb{N}}$ be the matrix representation of $a \in \mathcal{B}(\mathscr{H})$ with respect to the orthonormal basis $\{e_j : j \in \mathbb{N}\}$ for $\mathscr{H}$. Then the action of $[a_{jk}]_{j,k\in\mathbb{N}}$ on an infinite column $[x_k]_{k\in\mathbb{N}}$ representing $x = \sum_{k=1}^{\infty} x_k e_k$ is given by*

$$ax = \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} e_j a_{jk} x_k,$$

and the sums can be interchanged.

Proof. Since $ae_k \in \mathscr{H}$ for all $k \in \mathbb{N}$, then $ae_k = \sum_{j=1}^{\infty} e_j \langle e_j, ae_k\rangle$. By continuity and linearity of a, we get

$$\begin{aligned}
ax &= \lim_{n\to\infty} \sum_{k=1}^{n} x_k a e_k \\
&= \sum_{k=1}^{\infty} x_k a e_k \\
&= \sum_{k=1}^{\infty} x_k \left(\sum_{j=1}^{\infty} e_j \langle e_j, ae_k\rangle \right) \\
&= \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} e_j a_{jk} x_k.
\end{aligned}$$

Furthermore, the continuity of the inner product gives

$$\begin{aligned}
ax &= \sum_{j=1}^{\infty} e_j \langle e_j, ax\rangle \\
&= \sum_{j=1}^{\infty} e_j \left\langle e_j, a\left(\lim_{n\to\infty}\sum_{k=1}^{n} e_k x_k\right)\right\rangle \\
&= \sum_{j=1}^{\infty} e_j \left\langle e_j, \lim_{n\to\infty}\sum_{k=1}^{n} x_k a e_k\right\rangle \\
&= \sum_{j=1}^{\infty} \lim_{n\to\infty}\sum_{k=1}^{n} e_j \langle e_j, ae_k\rangle x_k \\
&= \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} e_j a_{jk} x_k.
\end{aligned}$$

□

Proposition 1.2.5. *Let $[a_{jk}]_{j,k\in\mathbb{N}}$ and $[b_{jk}]_{j,k\in\mathbb{N}}$ be matrix representations of $a, b \in \mathcal{B}(\mathscr{H})$, respectively, with respect to the orthonormal basis $\{e_j : j \in \mathbb{N}\}$ for $\mathscr{H}$. Then the matrix entries of the adjoint and product operations are given by $(a^*)_{jk} = \bar{a}_{kj}$ and $(ab)_{jk} = \sum_{l=1}^{\infty} a_{jl} b_{lk}$.*

Proof. Let $\{e_j : j \in \mathbb{N}\}$ be an orthonormal basis for $\mathscr{H}$, then

$$(a^*)_{jk} = \langle e_j, a^* e_k\rangle = \overline{\langle e_k, ae_j\rangle} = \bar{a}_{kj},$$

and

$$(ab)_{jk} = \langle e_j, abe_k\rangle = \langle a^* e_j, be_k\rangle = \sum_{l=1}^{\infty} \langle a^* e_j, e_l\rangle \langle e_l, be_k\rangle = \sum_{l=1}^{\infty} a_{jl} b_{lk}.$$

□

The following gives a sufficient condition for an infinite matrix to determine a bounded linear operator.

Theorem 1.2.6 (Schur test)**.** *Suppose that $[a_{jk}]_{j,k\in\mathbb{N}}$ is an infinite matrix satisfying the relations*

$$\sum_{j=1}^{\infty} |a_{jk}| \leq \alpha$$

and

$$\sum_{k=1}^{\infty} |a_{jk}| \leq \beta,$$

where $\alpha, \beta \in \mathbb{R}$. Then an operator u defined on the Hilbert sequence space l^2, for all $x \in l^2$, by $u(x) = ((ux)_j)_{j\in\mathbb{N}}$ with

$$(ux)_j := \sum_{k=1}^{\infty} a_{jk}x_k \; (\forall j \in \mathbb{N}),$$

is bounded with $\|u\| \leq \sqrt{\alpha\beta}$.

Using the matrix representation of an operator we can define the transposition with respect to an orthonormal basis.

Definition 1.2.7 (Transpose)**.** Let $\{e_j : j \in \mathbb{N}\}$ be an orthonormal basis for $\mathscr{H}$. The *transpose* of $a \in \mathcal{B}(\mathscr{H})$ is the operator a^T such that

$$a^T_{jk} = (a^T)_{jk} := \langle e_j, a^T e_k \rangle = \langle e_k, ae_j \rangle = a_{kj}.$$

In terms of matrices, this means

$$a^T e_k := \sum_{j=1}^{\infty} e_j (a^T)_{jk} = \sum_{j=1}^{\infty} e_j \langle e_k, ae_j \rangle = \sum_{j=1}^{\infty} e_j a_{kj} \tag{1.2.3}$$

for all $k \in \mathbb{N}$.

Proposition 1.2.8. *Let $\{e_j : j \in \mathbb{N}\}$ be an orthonormal basis for $\mathscr{H}$, and $a \in \mathcal{B}(\mathscr{H})$. Then:*

1. *The series in (1.2.3) converges, so that a^T is well-defined.*
2. *For all $a \in \mathcal{B}(\mathscr{H})$, $a^T \in \mathcal{B}(\mathscr{H})$ with $\|a^T\| = \|a\|$.*
3. *For all $a, b \in \mathcal{B}(\mathscr{H})$, $(ab)^T = b^T a^T$.*

Proof. Consider an orthonormal basis $\{e_j : j \in \mathbb{N}\}$ for $\mathscr{H}$, and let $a, b \in \mathcal{B}(\mathscr{H})$.

1. By the Parseval relation, for all $k \in \mathbb{N}$, we have

$$\sum_{j=1}^{\infty} |\langle e_k, ae_j \rangle|^2 \quad = \quad \|ae_j\|^2 \leq \|a\|^2,$$

and the partial sums are non-negative terms forming a monotone increasing sequence which is bounded above by $\|a\|^2$. It follows that the partial sums in (1.2.3) form a Cauchy sequence, since:

$$\left\| \sum_{j=m}^{n} e_j \langle e_k, ae_j \rangle \right\|^2 = \sum_{j=m}^{n} |\langle e_k, ae_j \rangle|^2$$

so the partial sums $\left(\sum_{j=1}^{n} |\langle e_k, ae_j \rangle|^2\right)_{n=1}^{\infty}$ is a Cauchy sequence. Thus a^T is well-defined.

2. For $x = \sum_{k=1}^{n} e_k x_k$, we have

$$\begin{aligned}
\|a^T x\|^2 &= \left\| \sum_{k=1}^{n} \sum_{j=1}^{\infty} e_j (a^T)_{jk} x_k \right\|^2 \\
&= \sum_{j=1}^{\infty} \left| \sum_{k=1}^{n} (a^T)_{jk} x_k \right|^2 \\
&= \sum_{j=1}^{\infty} \left| \sum_{k=1}^{n} \overline{(a^T)_{jk} x_k} \right|^2 \\
&= \left\| \sum_{k=1}^{n} \sum_{j=1}^{\infty} e_j (a^*)_{jk} \bar{x}_k \right\|^2 \\
&= \|a^* \bar{x}\|^2 \\
&\leq \|a^*\|^2 \|\bar{x}\|^2 \\
&= \|a^*\|^2 \|x\|^2
\end{aligned}$$

where $\bar{x} = \sum_{k=1}^{n} e_k \bar{x}_k$, so

$$\|\bar{x}\|^2 = \sum_{k=1}^{n} |\bar{x}_k| = \sum_{k=1}^{n} |x_k| = \|x\|^2.$$

Thus $\|a^T\| \leq \|a^*\| = \|a\|$, in particular $a^T \in \mathcal{B}(\mathscr{H})$. Furthermore, $\|a\| = \|(a^T)^T\| \leq \|a^T\|$, so $\|a^T\| = \|a\|$ for all $a \in \mathcal{B}(\mathscr{H})$.

3. For all $a, b \in \mathcal{B}(\mathscr{H})$ and $j, k \in \mathbb{N}$,

$$\begin{aligned}
\left((ab)^T\right)_{jk} &= \left\langle e_j, (ab)^T e_k \right\rangle \\
&= \langle e_k, (ab) e_j \rangle \\
&= \left\langle e_k, a \left(\sum_{l=1}^{\infty} e_l \langle e_l, b e_j \rangle \right) \right\rangle \\
&= \sum_{l=1}^{\infty} \langle e_k, a e_l \rangle \langle e_l, b e_j \rangle \\
&= \sum_{l=1}^{\infty} a_{kl} b_{lj}.
\end{aligned}$$

On the other hand,

$$(b^T a^T)_{jk} = \sum_{l=1}^{\infty} b^T_{jl} a^T_{lk} = \sum_{l=1}^{\infty} a_{kl} b_{lj}.$$

Thus,

$$\langle e_j, (ab)^T e_k \rangle = \langle e_j, (b^T a^T) e_k \rangle$$

for all $j, k \in \mathbb{N}$, so $(ab)^T = b^T a^T$.

□

Note that the transposition is given by the map

$$\mathcal{B}(\mathscr{H}) \to \mathcal{B}(\mathscr{H}) : a \mapsto a^T$$

where $(a^T)_{jk} = a_{kj}$ for all $j, k \in \mathbb{N}$, and it is not independent of the choice of basis.

1.3 Hilbert-Schmidt operators

Here we give a brief review of a special subclass of bounded operators which plays an important and fundamental role in operator theory. These operators behave much like operators on finite-dimensional vector spaces, and for this reason they are relatively easy to analyze.

For more details and proofs, the reader is referred to [7, I.8.5], [22, §2.4] and [17, §2.6].

Definition 1.3.1. Let a be a linear operator on a separable Hilbert space $\mathscr{H}$. If there is an orthonormal basis $\{e_j\}_{j\in\mathbb{N}}$ such that

$$\sum_{j=1}^{\infty} \|a(e_j)\|^2 < \infty, \tag{1.3.1}$$

we say that a is *Hilbert-Schmidt*. Denote by $\mathcal{B}_2(\mathscr{H}, \mathscr{H})$ the space of all Hilbert-Schmidt operators on $\mathscr{H}$.

We will sometimes simply refer to the space of Hilbert-Schmidt operators as $\mathcal{B}_2(\mathscr{H})$. It is easily verified that $\mathcal{B}_2(\mathscr{H})$ forms a *complex* vector space.

We now show that Def.1.3.1 is independent of the choice of orthonormal basis used (i.e. if Eq.(1.3.1) is satisfied in one such basis, then it is satisfied in all of them).

Proposition 1.3.2. *If a is a Hilbert-Schmidt operator on $\mathscr{H}$ (with the corresponding orthonormal basis $\{e_j\}_{j\in\mathbb{N}}$), then the value of the sum in (1.3.1) is independent of the choice of basis. Furthermore,*

$$\sum_{j=1}^{\infty} \|a(e_j)\|^2 = \sum_{j=1}^{\infty} \|a^*(e_j)\|^2 . \tag{1.3.2}$$

Proof. Let $\{f_k\}_{k\in\mathbb{N}}$ be another basis for $\mathscr{H}$, then by the Parseval relation

$$\begin{aligned}\sum_{j=1}^{n}\|a(e_j)\|^2 &= \sum_{j=1}^{n}\sum_{k=1}^{\infty}|\langle a(e_j),f_k\rangle|^2\\ &= \sum_{k=1}^{\infty}\sum_{j=1}^{n}|\langle e_j,a^*(f_k)\rangle|^2\\ &\leq \sum_{k=1}^{\infty}\|a^*(f_k)\|^2,\end{aligned}$$

so $\sum_{j=1}^{\infty}\|a(e_j)\|^2 \leq \sum_{k=1}^{\infty}\|a^*(f_k)\|^2$. Swapping the roles,

$$\sum_{j=1}^{\infty}\|a^*(f_j)\|^2 \leq \sum_{k=1}^{\infty}\|a(e_k)\|^2$$

(since $a^{**}=a$). Therefore

$$\sum_{j=1}^{\infty}\|a(e_j)\|^2 = \sum_{j=1}^{\infty}\|a^*(f_j)\|^2 = \sum_{j=1}^{\infty}\|a(f_j)\|^2$$

(applying the previous line to the case where $e_j=f_j$). □

Definition 1.3.3. Let a be a Hilbert-Schmidt operator on the separable Hilbert space $\mathscr{H}$. Define

$$\|a\|_2 := \left(\sum_{j=1}^{\infty}\|a(e_j)\|^2\right)^{1/2}, \tag{1.3.3}$$

$\|a\|_2$ is called the *Hilbert-Schmidt norm* of a.

Corollary 1.3.4. *If $a\in\mathcal{B}_2(\mathscr{H})$, then $a^*\in\mathcal{B}_2(\mathscr{H})$ and $\|a\|_2=\|a^*\|_2$.*

Theorem 1.3.5. *[22, Theorem 2.4.10] If $a,b\in\mathcal{B}_2(\mathscr{H})$ and $\alpha\in\mathbb{C}$, then:*

1. $\|a+b\|_2 \leq \|a\|_2+\|b\|_2$ *and* $\|\alpha a\|_2 = |\alpha|\|a\|_2$.

2. $\|a\| \leq \|a\|_2$.

3. $\|ab\|_2 \leq \|a\|\|b\|_2$ *and* $\|ab\|_2 \leq \|a\|_2\|b\|$.

Corollary 1.3.6. *[22, Corollary 2.4.11] The space of all Hilbert-Schmidt operators $\mathcal{B}_2(\mathscr{H})$ forms a non-trivial ideal for $\mathcal{B}(\mathscr{H})$, that is, $\mathcal{B}_2(\mathscr{H})\subset\mathcal{B}(\mathscr{H})$ such that for all $b,c\in\mathcal{B}(\mathscr{H})$ and $a\in\mathcal{B}_2(\mathscr{H})$*

$$bac\in\mathcal{B}_2(\mathscr{H}),$$

and

$$\|bac\|_2 \leq \|b\|\|a\|_2\|c\|.$$

We quote the following basic functional analysis results without proof (see [18, Ex. 3.1-6]):

Proposition 1.3.7 (The Hilbert space of sequences). 1. *The space of absolute-square summable sequences*

$$\begin{aligned} l^2(\mathbb{N}) &= \left\{ f : \mathbb{N} \to \mathbb{C} \mid \sum_{n\in\mathbb{N}} |f(n)|^2 < \infty \right\} \\ &= \left\{ x = (x_j)_{j\in\mathbb{N}} \mid \sum_{j=1}^{\infty} |x_j|^2 < \infty \right\} \end{aligned}$$

with the inner product

$$\langle x, y\rangle_{l^2(\mathbb{N})} = \sum_{j=1}^{\infty} \bar{x}_j y_j$$

is a (separable) Hilbert space.

2. *Similarly, the space of absolute-square double-summable sequences,*

$$\begin{aligned} l^2(\mathbb{N}\times\mathbb{N}) = l^2(\mathbb{N}^2) &= \left\{ f : \mathbb{N}\times\mathbb{N} \to \mathbb{C} \mid \sum_{m,n\in\mathbb{N}} |f(m,n)|^2 < \infty \right\} \\ &= \left\{ a = (a_{j,k})_{j,k\in\mathbb{N}} \mid \sum_{j,k=1}^{\infty} |a_{j,k}|^2 < \infty \right\}, \end{aligned}$$

with the inner product given by

$$\langle a, b\rangle_{l^2(\mathbb{N}^2)} = \sum_{j,k=1}^{\infty} \bar{a}_{j,k} b_{j,k}$$

is a (separable) Hilbert space.

We now make the following observation (cf. [22, Example 2.4.1])- for any $a \in \mathcal{B}(\mathscr{H})$ with matrix representation $[a_{j,k}]_{j,k\in\mathbb{N}}$ with respect to the orthonormal basis $\{e_j\}_{j\in\mathbb{N}}$, where $a_{j,k} = \langle e_j, a(e_k)\rangle$, we have that

$$\|a\|_2^2 = \sum_{j=1}^{\infty} \|a(e_j)\|^2 = \sum_{j,k=1}^{\infty} |\langle e_j, a(e_k)\rangle|^2 = \sum_{j,k=1}^{\infty} |a_{j,k}|^2$$

by the Parseval relation. Hence

$$a \in \mathcal{B}_2(\mathscr{H}) \iff (a_{j,k})_{j.k\in\mathbb{N}} \in l^2(\mathbb{N}^2).$$

We introduce the following notation, called *Dirac notation*, which is often useful when dealing with operators in $\mathcal{B}(\mathscr{H})$; the operator $|y\rangle\langle z| \in \mathcal{B}(\mathscr{H})$ is defined as

$$|y\rangle\langle z|\, x := y\,\langle z, x\rangle \tag{1.3.4}$$

for all $x, y, z \in \mathscr{H}$.

Theorem 1.3.8. *Let $\{e_j\}_{j\in\mathbb{N}}$ be an orthonormal basis for the Hilbert space $\mathscr{H}$. Then:*

1. *The space of all Hilbert-Schmidt operators $\mathcal{B}_2(\mathscr{H})$ form a Hilbert space with the inner product given by*

$$\langle a, b\rangle_2 = Tr(a^*b). \tag{1.3.5}$$

2. *The set of vectors*

$$\{X_{ij} := |e_i\rangle\langle e_j| \mid i, j \in \mathbb{N}\} \tag{1.3.6}$$

form an orthonormal basis for $\mathcal{B}_2(\mathscr{H})$, i.e. $\langle X_{ij}, X_{kl}\rangle = \delta_{i,k}\delta_{j,l}$. In particular, the vectors

$$\mathbb{P}_i = X_{ii} = |e_i\rangle\langle e_i| \tag{1.3.7}$$

are one-dimensional projection operators on $\mathscr{H}$.

Proof. Let $\{e_j\}_{j\in\mathbb{N}}$ be an orthonormal basis for the Hilbert space $\mathscr{H}$.

1. That $\|\cdot\|_2$ is a norm on $\mathcal{B}_2(\mathscr{H})$ follows from Thm.1.3.5, thus we only show that $\mathcal{B}_2(\mathscr{H})$ is complete with respect to the Hilbert-Schmidt norm $\|\cdot\|_2$. To this end consider the following map

$$\Psi : (\mathcal{B}_2(\mathscr{H}), \|\cdot\|_2) \to \left(l^2(\mathbb{N}^2), \|\cdot\|_{l^2(\mathbb{N}^2)}\right) : a \mapsto (a_{j,k})_{j,k\in\mathbb{N}}.$$

For $a, b \in \mathcal{B}_2(\mathscr{H})$ and $\alpha \in \mathbb{C}$, we have

$$\begin{aligned}\Psi(\alpha a + b) &= ((\alpha a + b)_{j,k})_{j,k\in\mathbb{N}}\\ &= \alpha(a_{j,k})_{j,k\in\mathbb{N}} + (b_{j,k})_{j,k\in\mathbb{N}}\\ &= \alpha\Psi(a) + \Psi(b),\end{aligned}$$

thus Ψ is linear. Suppose $\Psi(a) = \Psi(b)$, then $(a_{j,k})_{j,k\in\mathbb{N}} = (b_{j,k})_{j,k\in\mathbb{N}}$ which implies that $a_{j,k} = b_{j,k}$ $\forall j, k \in \mathbb{N}$; so that $a = b$.

Let $(a_{j,k})_{j,k\in\mathbb{N}} \in l^2\left(\mathbb{N}^2\right)$, and define an operator θ on $\mathscr{H}$ by

$$(\theta x)_j = \sum_{k=1}^{\infty} a_{j,k}x_k, \ \forall j \in \mathbb{N}.$$

Then

$$\begin{aligned}\sum_{j=1}^{\infty}\left|(\theta x)_j\right|^2 &= \sum_{j=1}^{\infty}\left|\sum_{k=1}^{\infty} a_{j,k}x_k\right|^2 \\ &\leq \sum_{j=1}^{\infty}\left(\sum_{k=1}^{\infty}|a_{j,k}|^2\right)\left(\sum_{l=1}^{\infty}|x_l|^2\right) \\ &= \|(a_{j,k})_{j,k\in\mathbb{N}}\|^2_{l^2(\mathbb{N}^2)}\,\|x\|^2_{l^2(\mathbb{N})},\end{aligned}$$

and consequently $\theta \in \mathcal{B}_2(\mathscr{H})$. Moreover,

$$\begin{aligned}\|\Psi(a)\|_{l^2(\mathbb{N}^2)} &= \|(a_{j,k})_{j,k\in\mathbb{N}}\|_{l^2(\mathbb{N}^2)} \\ &= \left(\sum_{j,k=1}^{\infty}|a_{j,k}|^2\right)^{1/2} \\ &= \|a\|_2.\end{aligned}$$

Hence Ψ is an isometric isomorphism; and since $\left(l^2(\mathbb{N}^2), \|\cdot\|_{l^2(\mathbb{N}^2)}\right)$ is complete, it follows that $(\mathcal{B}_2(\mathscr{H}), \|\cdot\|_2)$ is a Banach space.

The norm $\|\cdot\|_2$ satisfies the parallelogram identity, since

$$\begin{aligned}\|a+b\|_2^2 + \|a-b\|_2^2 &= \sum_{j=1}^{\infty}\|(a+b)e_j\|^2 + \sum_{j=1}^{\infty}\|(a-b)e_j\|^2 \\ &= \sum_{j=1}^{\infty}\left(\|ae_j+be_j\|^2 + \|ae_j-be_j\|^2\right) \\ &= 2\sum_{j=1}^{\infty}\left(\|ae_j\|^2 + \|be_j\|^2\right) \\ &= 2\left(\|a\|_2^2 + \|b\|_2^2\right).\end{aligned}$$

Thus, $\|\cdot\|_2$ can be obtained from an inner product. Denote this inner product by $\langle\cdot,\cdot\rangle_2$. Then, for $a \in \mathcal{B}_2(\mathscr{H})$ we have

$$\begin{aligned}\langle a,a\rangle_2 &= \|a\|_2^2 \\ &= \sum_{j=1}^{\infty}\|ae_j\|^2 \\ &= \sum_{j=1}^{\infty}\langle e_j, a^*ae_j\rangle \\ &= Tr(a^*a).\end{aligned}$$

But for all $a, b \in \mathcal{B}_2(\mathscr{H})$ we have that $a^* \in \mathcal{B}_2(\mathscr{H})$ and $a^*b \in \mathcal{B}_2(\mathscr{H})$, by Corollaries 1.3.4 and 1.3.6. It the follows that $\langle\cdot,\cdot\rangle_2$ is well defined, since Hilbert-Schmidt operators are positive.

2. Orthogonality of the vectors in (1.3.6) follows from

$$\begin{aligned}
\langle X_{ij}, X_{kl}\rangle &= Tr(X_{ij}^{*}X_{kl}) \\
&= Tr(X_{ji}X_{kl}) \\
&= Tr\left(|e_j\rangle\langle e_i|\,|e_k\rangle\langle e_l|\right) \\
&= \delta_{i,k}\; Tr(|e_j\rangle\langle e_l|) \\
&= \delta_{i,k}\sum_{m=1}^{\infty}\langle e_m, |e_j\rangle\langle e_i|\, e_m\rangle \\
&= \delta_{i,k}\sum_{m=1}^{\infty}\delta_{l,m}\,\langle e_m, e_j\rangle \\
&= \delta_{i,k}\;\delta_{l,j}.
\end{aligned}$$

Moreover, for $\mathbb{P}_i$ in (1.3.7) we have

$$\mathbb{P}_i^2 = X_{ii}X_{ii} = \delta_{i,i}\,|e_i\rangle\langle e_i| = X_{ii},$$

and $\mathbb{P}_i^* = X_{ii}^* = X_{ii}$. Thus, $\mathbb{P}_i^2 = \mathbb{P}_i^* = \mathbb{P}_i$.

□

1.4 C*-algebras and von Neumann algebras

In this section, we review the basic definitions of C*-algebras and von Neumann algebras and some of the related terminology. We really only need von Neumann algebras in this dissertation, but our treatment here gives some context in a broader operator algebraic framework.

Standard references for C*-algebras and von Neumann algebras are [7], [8], [17], and [22].

Definition 1.4.1. Let $\mathcal{A}$ be a complex vector space. An *algebra* is $\mathcal{A}$ together with an associative bilinear map $\cdot : \mathcal{A}\times\mathcal{A}\to\mathcal{A} : (a,b)\mapsto ab$, called *multiplication (or product)*; that is, we have:

1. $a(bc) = (ab)c$

2. $(\alpha a)b = a(\alpha b) = \alpha(ab)$

3. $a(b+c) = ab + ac$

$\forall a,b,c\in\mathcal{A}, \alpha\in\mathbb{C}$. If $ab = ba, \forall a,b\in\mathcal{A}$, then $\mathcal{A}$ is called a *commutative* algebra. If $\mathcal{A}$ has a unit, that is an element $1\in\mathcal{A}$ such that $1a = a1 = a, \forall a\in\mathcal{A}$, we say $\mathcal{A}$ is *unital*. A *subalgebra* of $\mathcal{A}$ is a vector subspace $\mathcal{B}$ such that

$$b, b'\in\mathcal{B} \implies bb'\in\mathcal{B}.$$

A vector subspace I of an algebra $\mathcal{A}$ is called a *left* (respectively, *right*) *ideal* in $\mathcal{A}$ if

$$a \in \mathcal{A} \text{ and } b \in I \implies ab \in I \text{ (respectively, } ba \in I).$$

We call I an *ideal* in $\mathcal{A}$ if it is simultaneously a left and right ideal. For any algebra $\mathcal{A}$, it is easily verified that 0 and $\mathcal{A}$ are both ideals in $\mathcal{A}$, they are called *trivial ideals* for $\mathcal{A}$.

Definition 1.4.2. Let $\mathcal{A}$ be a unital algebra. We say $a \in \mathcal{A}$ is *invertible* if there is an element $b \in \mathcal{A}$ such that

$$ab = ba = 1.$$

In this case b is unique and written a^{-1}. Denote the set of all invertible elements in $\mathcal{A}$ by

$$Inv(\mathcal{A}) = \{a \in \mathcal{A} : a \text{ is invertible}\}.$$

We define the *spectrum* of an element a to be the set

$$\sigma(a) = \{\lambda \in \mathbb{C} : \lambda 1 - a \notin Inv(\mathcal{A})\}.$$

Definition 1.4.3. An *involution* on an algebra $\mathcal{A}$ is a conjugate linear anti-automorphism of order two, i.e. a map $^* : \mathcal{A} \to \mathcal{A} : a \mapsto a^*$ satisfying the properties:

1. $(\alpha a + \beta b)^* = \bar{\alpha} a^* + \bar{\beta} b^*$
2. $(ab)^* = b^* a^*$
3. $(a^*)^* = a$

$\forall a, b, c \in \mathcal{A}$ and $\alpha, \beta \in \mathbb{C}$. a^* is called the *adjoint* of the algebra element a in $\mathcal{A}$. A **-algebra* is an algebra $\mathcal{A}$ together with an involution.

If $\mathcal{A}$ is a *-algebra such that $1^* = 1$, we call $\mathcal{A}$ a *unital *-algebra.*

Definition 1.4.4. Let $\mathcal{A}$ be a *-algebra. Then $a \in \mathcal{A}$ is said to be

self-adjoint if $a^* = a$.

normal if $a^* a = aa^*$.

a *projection* if $a = a^* = a^2$.

If $\mathcal{A}$ is unital, then u is called

an *isometry* if $u^* u = 1$.

a *co-isometry* if $uu^* = 1$.

unitary if $u^*u = 1 = uu^*$.

A *norm* $\|\cdot\|$ on $\mathcal{A}$ is said to be *sub-multiplicative* if

$$\|ab\| \leq \|a\|\|b\| \quad \forall a, b \in \mathcal{A}.$$

A normed algebra is the algebra $\mathcal{A}$ with a submultiplive norm $\|\cdot\|$.

Definition 1.4.5. Let $\mathcal{A}$ be a *-algebra.

1. A *Banach *-algebra* is the *-algebra $\mathcal{A}$ together with a complete sub-multiplicative norm $\|\cdot\|$ such that

$$\|a^*\| = \|a\| \quad \forall a \in \mathcal{A}.$$

   If, in addition, $\mathcal{A}$ has a unit such that $\|1\| = 1$, we call $\mathcal{A}$ a *unital Banach *-algebra.*

2. An (*abstract*) *C*-algebra* is a Banach *-algebra $\mathcal{A}$ which satisfy the *C*-property*:

$$\|a^*a\| = \|a\|^2 \quad \forall a \in \mathcal{A}.$$

   A norm on a Banach *-algebra satisfying this property is referred to as a *C*-norm.*

Definition 1.4.6. An element a of a C*-algebra $\mathcal{A}$ is *positive*, and written $a \geq 0$, if it is self-adjoint and $\sigma(a) \subset \mathbb{R}_+$. Denote by $\mathcal{A}_+$ the set of all positive elements in $\mathcal{A}$.

Definition 1.4.7. A **-homomorphism* between two C*-algebras $\mathcal{A}$ and $\mathcal{B}$ is a map $\varphi : \mathcal{A} \to \mathcal{B}$ such that

1. $\varphi(\alpha a + \beta b) = \alpha\varphi(a) + \beta\varphi(b)$
2. $\varphi(ab) = \varphi(a)\varphi(b)$
3. $\varphi(a^*) = \varphi(a)^*$

$\forall a, b \in \mathcal{A}, \alpha, \beta \in \mathbb{C}$. If, in addition, φ is bijective, it is called a **-isomorphism.* A **-automorphism* of $\mathcal{A}$ is an isometric *-isomorphism $\varphi : \mathcal{A} \to \mathcal{A}$. The set $\{\varphi_t : t \in \mathbb{R}\}$ is said to be a *one-parameter *-automorphism group* if φ_t is a *-automorphism for every $t \in \mathbb{R}$ and has the group property that $\varphi_{t+s} = \varphi_t \circ \varphi_s \; \forall t, s \in \mathbb{R}$.

Theorem 1.4.8 (Properties of *-homomorphism)**.** *Let $\mathcal{A}$ and $\mathcal{B}$ be C*-algebras, and $\varphi : \mathcal{A} \to \mathcal{B}$ be a *-homomorphism. Then:*

1. *The set* $\ker(\varphi) = \{a \in \mathcal{A} | \varphi(a) = 0\}$ *is a closed two-sided ideal in* $\mathcal{A}$. *The set* $\varphi(\mathcal{A}) = \{\varphi(a) | a \in \mathcal{A}\}$ *is a C*-subalgebra of* $\mathcal{B}$.

2. *φ is positive; i.e. if $a \in \mathcal{A}_+$ then $\varphi(a) \in \mathcal{B}_+$.*

3. *φ is contractive, i.e. $\|\varphi(a)\| \leq \|a\|$ $(a \in \mathcal{A})$; hence continuous.*

4. *If φ is injective, then it is isometric; $\|\varphi(a)\| = \|a\|$.*

5. *If $\mathcal{A}$ is unital, then φ is unital, i.e. $\varphi(1_{\mathcal{A}}) = 1_{\mathcal{B}}$.*

If $\mathscr{F}$ is a family of bounded linear operators on the Hilbert space $\mathscr{H}$, we define its *commutant* $\mathscr{F}'$ to be the set of all elements of $\mathcal{B}(\mathscr{H})$ that commute with all elements of $\mathscr{F}$.

Definition 1.4.9. A *von Neumann algebra* is a (necessarily unital) *-subalgebra $\mathfrak{A}$ of $\mathcal{B}(\mathscr{H})$ such that

$$\mathfrak{A} = \mathfrak{A}''.$$

A von Neumann algebra is called a *factor* if

$$\mathfrak{A} \cap \mathfrak{A}' = \mathbb{C}1.$$

In general, a (*concrete*) *operator algebra* (C*-algebra and von Neumann) can be viewed as a *-subalgebra of $\mathcal{B}(\mathscr{H})$, for some choice of Hilbert space $\mathscr{H}$, which is topologically closed in a suitable sense. A C*-algebra is a *-subalgebra of $\mathcal{B}(\mathscr{H})$ which is closed in the norm topology, whereas a von Neumann algebra is strongly closed. Since the strong topology is weaker than norm topology, a strongly closed set is also norm-closed. Thus, a von Neumann algebra is a C*-algebra.

Lemma 1.4.10. *For separable $\mathscr{H}$, the commutant of $\mathcal{B}(\mathscr{H})$ is $\mathbb{C}1$, i.e. $\mathcal{B}(\mathscr{H})' = \mathbb{C}1$. Equivalently, $\forall a \in \mathcal{B}(\mathscr{H})$, $[a, b] = 0$ if and only if $b \in \mathcal{B}(\mathscr{H})$ such that $b = \alpha 1$ for some $\alpha \in \mathbb{C}$.*

Proof. Suppose $b = \alpha 1, \alpha \in \mathbb{C}$ and let $a \in \mathcal{B}(\mathscr{H})$ be arbitrary. Then

$$[a, b] = [a, \alpha 1] = a(\alpha 1) - (\alpha 1)a = 0,$$

by the bilinearity of the multiplication operation on $\mathcal{B}(\mathscr{H})$. Conversely, suppose that $b \in \mathcal{B}(\mathscr{H})$ commutes with every element in $\mathcal{B}(\mathscr{H})$. We show that b is a multiple of the identity operator in $\mathcal{B}(\mathscr{H})$.

If b is not diagonal, then it has at least one non-zero off-diagonal entry, i.e. there is a $b_{ij} \neq 0$ with $i \neq j$. Now, consider $a \in \mathcal{B}(\mathscr{H})$ with the matrix entries $a_{ji} = 1$ and zeros elsewhere. Then

$$(ab)_{ii} = \sum_{k=1}^{\infty} a_{ik} b_{ki} = a_{ij} b_{ji} = 0,$$

and

$$(ba)_{ii} = \sum_{k=1}^{\infty} b_{ik} a_{ki} = b_{ij} a_{ji} = b_{ij} \neq 0.$$

Thus $[a,b] \neq 0$, contradicting the hypothesis. Hence b must be diagonal.

Now, suppose that b is diagonal with $b_{ii} \neq b_{jj}$ for some $i \neq j$. Let $c \in \mathcal{B}(\mathscr{H})$ with only one non-zero column, say, it has ones along the j-th column. That is, $c_{kj} = 1$ $(k = 1, 2, 3, ...)$ and zeros elsewhere. Then

$$(bc)_{ij} = \sum_{k=1}^{\infty} b_{ik}c_{kj} = b_{ii}c_{ij} = b_{ii},$$

and

$$(cb)_{ij} = \sum_{k=1}^{\infty} c_{ik}b_{kj} = c_{ij}b_{jj} = b_{jj}.$$

Since $b_{ii} \neq b_{jj}$ for $i \neq j$, then $[b,c] \neq 0$. This contradicts the hypothesis. Thus we must have that $b_{ii} = b_{jj}$ for all $i, j = 1, 2, 3, ...$ It then follows that $b = b_{ii}1$ for some $b_{ii} \in \mathbb{C}$. Hence b is a multiple of the identity operator in $\mathcal{B}(\mathscr{H})$. □

Example 1.4.11. The space of bounded linear operators $\mathcal{B}(\mathscr{H})$ on a separable Hilbert space $\mathscr{H}$ is a von Neumann algebra, since

$$\mathcal{B}(\mathscr{H})'' = (\mathbb{C}1)' = \mathcal{B}(\mathscr{H}).$$

1.5 States on $\mathcal{B}(\mathscr{H})$ and density matrices

Here we setup a special case of the more general situation of a faithful normal state on a von Neumann algebra.

Definition 1.5.1. A *density matrix* (*or density operator*) is a $\rho \in \mathcal{B}(\mathscr{H})_+$ such that

$$Tr(\rho) = 1.$$

Definition 1.5.2. Let $\omega : \mathfrak{A} \to \mathbb{C}$ be a bounded linear functional on a von Neumann algebra $\mathfrak{A}$.

1. ω is called a *state* if it is normalized and positive, i.e. satisfies $\omega(a^*a) \geq 0, \ \forall a \in \mathfrak{A}$, and $\omega(1_{\mathfrak{A}}) = 1$.

2. A state ω is said to be:

   faithful if $\omega(a^*a) > 0, \ \forall a \neq 0$.

   normal if and only if there is a density operator ρ such that $\omega(a) = Tr(\rho a), \ \forall a \in \mathfrak{A}$.

   a *vector state* if there is a vector $x \in \mathscr{H}$ such that $\omega(a) = \langle x, ax \rangle, \ \forall a \in \mathfrak{A}$.

Clearly, vector states are normal.

Definition 1.5.3. Let $\mathfrak{A}$ be a von Neumann algebra on a Hilbert space $\mathscr{H}$, then a vector $x \in \mathscr{H}$ is said to be:

1. *cyclic* for $\mathfrak{A}$ if the set $\{ax : a \in \mathfrak{A}\}$ is dense in $\mathfrak{A}$.

2. *separating* of $\mathfrak{A}$ if $ax = bx,\ a, b \in \mathfrak{A}$ if and only if $a = b$.

Consider the following functional

$$\mu : \mathcal{B}(\mathscr{H}) \to \mathbb{C} : a \mapsto Tr(\rho a) \tag{1.5.1}$$

where ρ is some specified density operator.

Proposition 1.5.4. *The functional μ in (1.5.1) defines a state on $\mathcal{B}(\mathscr{H})$.*

Proof. By definition of a density operator $\rho \in \mathcal{B}_1(\mathscr{H})$, then

$$\rho a \in \mathcal{B}_2(\mathscr{H})$$

for all $a \in \mathcal{B}(\mathscr{H})$, since $\mathcal{B}_2(\mathscr{H})$ is an ideal in $\mathcal{B}(\mathscr{H})$. So the functional μ is well-defined. Next we show that μ does indeed give a state on $\mathcal{B}(\mathscr{H})$. Since ρ is a density operator, $\mu(1) = Tr(\rho) = 1$. Let $a, b \in \mathcal{B}(\mathscr{H})$ and $\alpha \in \mathbb{C}$. Using the linearity of the trace functional, we have

$$\begin{aligned}
\mu(\alpha a + b) &= Tr(\rho(\alpha a + b)) \\
&= \alpha Tr(\rho a) + Tr(\rho b) \\
&= \alpha\mu(a) + \mu(b).
\end{aligned}$$

By the cyclic property and positivity of the trace,

$$\begin{aligned}
\mu(a^*a) &= Tr(\rho a^*a) \\
&= Tr(\rho^{1/2}\rho^{1/2}a^*a) \\
&= Tr(\rho^{1/2}a^*a\rho^{1/2}) \\
&= Tr\left((a\rho^{1/2})^*(a\rho^{1/2})\right) \\
&\geq 0,
\end{aligned}$$

since $(a\rho^{1/2})^*(a\rho^{1/2}) \in \mathcal{B}(\mathscr{H})_+$. □

For simplicity we henceforth assume that there is a countable orthonormal basis $\mathcal{B} = \{e_j : j \in \mathbb{N}\}$ such that

$$\rho e_j = \rho_j e_j \tag{1.5.2}$$

for all $j \in \mathbb{N}$, for some strictly positive real numbers $\rho_1, \rho_2, \rho_3, ... > 0$ and ρ a density operator.

Remarks 1.5.5. The above assumption is a bit restrictive as it does not mathematically cover the most general situation, but for quantum systems in finite volume a discrete energy spectrum is generally the case. However, in reality we know that systems are finite, so this should not be a major drawback.

Note that $\rho_j = \rho_k$ is possible for $j \neq k$, that is we do not assume that the (eigenvalues) $\rho_1, \rho_2, \rho_3, ...$ are distinct. In general it is possible to have $\rho_j = 0$ for some $j \in \mathbb{N}$, but in statistical mechanics $\rho_j > 0$ is typical.

In the case where $\rho_j = 0$ for some $j \in \mathbb{N}$, we may be able to handle even this case in much of what we do below by just considering the subspace of $\mathscr{H}$ spanned by the basis elements e_j for which $\rho_j > 0$, and handling the orthogonal complement of this subspace separately.

Proposition 1.5.6. *The density operator ρ given by (1.5.2) can be viewed as a diagonal matrix with entries $\rho_1, \rho_2, \rho_3, ... > 0$. Furthermore, such a ρ is bounded and satisfies*

$$\sum_{j=1}^{\infty} \rho_j = 1.$$

Proof. Let $\{e_j : j \in \mathbb{N}\}$ be an orthonormal basis for $\mathscr{H}$. Then for all $j, k \in \mathbb{N}$

$$\rho_{jk} = \langle e_j, \rho e_k \rangle = \langle e_j, \rho_k e_k \rangle = \rho_k \delta_{jk}.$$

For the matrix representation $[\rho_{jk}]_{j,k\in\mathbb{N}}$ we have

$$\sum_{j=1}^{\infty} |\rho_{jk}| = 1 \quad (\forall k \in \mathbb{N})$$

and

$$\sum_{k=1}^{\infty} |\rho_{jk}| = 1 \quad (\forall j \in \mathbb{N}).$$

So by Schur test, ρ is a bounded operator on $\mathscr{H}$. Now, evaluating the trace of ρ yields

$$1 \quad = \quad Tr(\rho) = \sum_{j=1}^{\infty} \langle e_j, \rho_j e_j \rangle = \sum_{j=1}^{\infty} \rho_j \langle e_j, e_j \rangle = \sum_{j=1}^{\infty} \rho_j.$$

□

With the afore going assumption we define the inverse of the density operator ρ:

Definition 1.5.7. Denote by ρ^{-1} the inverse of ρ on the subspace

$$\mathscr{H}_0 := span\{e_j : j \in \mathbb{N}\} = \left\{ \sum_{j=1}^{n} \lambda_j e_j : e_j \in \mathcal{B} \right\}$$

of $\mathscr{H}$; i.e.

$$\rho^{-1} : \mathscr{H}_0 \to \mathscr{H} \tag{1.5.3}$$

such that $\rho^{-1}\rho \mid_{\mathscr{H}_0} = \rho^{-1}\rho$ is the identity operator on $\mathscr{H}_0$.

Remarks 1.5.8. In the case where ρ is given by (1.5.2), by Prop.1.5.6, we have that its action on the basis element e_j is

$$\rho^{-1}e_j = \rho_j^{-1}e_j. \tag{1.5.4}$$

Similarly, we define $\rho^{-1/2} : \mathscr{H}_0 \to \mathscr{H}$ to be such that $\rho^{-1/2}e_j = \rho_j^{-1/2}e_j$.

Proposition 1.5.9. *The state μ with the density operator ρ given by (1.5.2) is faithful.*

Proof. If $\mu(a^*a) = 0$ for $a \in \mathcal{B}(\mathscr{H})$, then

$$\begin{aligned}
0 &= Tr(\rho a^* a) \\
&= Tr\left((a\rho^{1/2})^* a\rho^{1/2}\right) \\
&= \sum_{j=1}^{\infty} \left\langle e_j, (a\rho^{1/2})^* a\rho^{1/2} e_j \right\rangle \\
&= \sum_{j=1}^{\infty} \left\langle a\rho^{1/2} e_j, a\rho^{1/2} e_j \right\rangle \\
&= \sum_{j=1}^{\infty} \|a\rho^{1/2} e_j\|^2 \\
&= \|a\rho^{1/2}\|_2^2.
\end{aligned}$$

So $a\rho^{1/2} = 0$, by the definition of the Hilbert-Schmidt norm. It then follows that $a \mid_{\mathscr{H}_0} = a\rho^{1/2}\rho^{-1/2} = 0$, so $a = 0$ since a is bounded. $\square$

1.6 The tensor product of Hilbert spaces

In this section we give a brief treatment of the mathematical theory of tensor products for Hilbert spaces. For discussions on how tensor products are constructed, the reader is referred to [22, Section 6.3] (see, [33, Appendix T] for an even more detailed exposition). An alternative approach to the construction of tensor products can be found in [17].

Let X and Y be (complex) vector spaces, denote by $X \odot Y$ their algebraic tensor product, this is the space linearly spanned by finitely many linear combinations of *elementary tensors* $x \otimes y$ $(x \in X, y \in Y)$.

Proposition 1.6.1. *Let X and Y be vector spaces.*

1. *Suppose that* $\sum_{j=1}^{n} x_j \otimes y_j = 0$, *where* $x_j \in X, y_j \in Y$.

   (ii) If $x_1, ..., x_n$ *are linearly independent, then* $y_1, ..., y_n = 0$.

   (ii) If $y_1, ..., y_n$ *are linearly independent, then* $x_1, ..., x_n = 0$.

2. *If* $a : X \to X'$, $b : Y \to Y'$ *are linear operators (where* X' *and* Y' *are vector spaces), then there exists a unique linear map* $a \odot b : X \odot Y \to X' \odot Y'$ *such that*

$$(a \odot b)(x \otimes y) = a(x) \otimes b(y)$$

   where $x \in X, y \in Y$.

For $x, x_1, x_2 \in X$ and $y, y_1, y_2 \in Y$, the *tensor calculus* is given by

$$\begin{aligned} (x_1 + x_2) \otimes y &= x_1 \otimes y + x_2 \otimes y \\ x \otimes (y_1 + y_2) &= x \otimes y_1 + x \otimes y_2 \\ \alpha(x \otimes y) &= (\alpha x) \otimes y = x \otimes (\alpha y), \end{aligned}$$

that is, $X \odot Y$ is a vector space.

Note that the map $\varphi : X \times Y \to X \odot Y : (x, y) \mapsto x \otimes y$ is bilinear. The pair $(X \odot Y, \varphi)$ is characterized up to isomorphism by a *universal property* regarding bilinear maps. In some sense, φ can be thought of as the most general bilinear map out of $X \times Y$.

The following result gives conditions under which bilinear maps defined on a Cartesian product can be extended to linear ones on the tensor product (cf. [33, Prop. T.2.4], or [17, Thm. 2.6.4]).

Proposition 1.6.2 (Universal property)**.** *Let* X, Y *and* Z *be vector spaces. The pair* $(X \odot Y, \varphi)$ *has the property that any bilinear map* $\psi : X \times Y \to Z$ *factors through* φ *uniquely, i.e. there is a unique linear map* $\psi' : X \odot Y \to Z$ *such that*

$$\psi = \psi' \circ \varphi.$$

In general, if X and Y are normed spaces, there are many possible norms that can be introduced on $X \odot Y$ which are related to those on X and Y. To avoid the difficulties related to this non-uniqueness, we shall only consider Hilbert spaces.

Theorem 1.6.3. *(cf. [22, Thm. 6.3.1]) If* $\mathscr{H}$ *and* $\mathscr{K}$ *are Hilbert spaces, then there is a unique inner product* $\langle \cdot, \cdot \rangle$ *on* $\mathscr{H} \odot \mathscr{K}$ *such that*

$$\langle x \otimes y, x' \otimes y' \rangle = \langle x, x' \rangle \langle y, y' \rangle$$

where $x, x' \in \mathscr{H}$, *and* $y, y' \in \mathscr{K}$.

We regard $\mathscr{H} \odot \mathscr{H}$ as a pre-Hilbert space with the inner product given above. The *Hilbert space completion* of $\mathscr{H} \odot \mathscr{K}$ is denoted by $\mathscr{H} \otimes \mathscr{K}$, and called the *Hilbert tensor product* of $\mathscr{H}$ and $\mathscr{K}$.

Proposition 1.6.4. *Let $\mathscr{H}$ and $\mathscr{K}$ be separable Hilbert spaces, then:*

1. *For all $x \in \mathscr{H}, y \in \mathscr{K}$*

$$\|x \otimes y\| = \|x\|\|y\|.$$

2. *If $\{e_j : j \in \mathbb{N}\}$ and $\{f_k : k \in \mathbb{N}\}$ are orthonormal bases for $\mathscr{H}$ and $\mathscr{K}$, respectively, then*

$$\{e_j \otimes f_k : j, k \in \mathbb{N}\}$$

is an orthonormal basis for $\mathscr{H} \otimes \mathscr{K}$.

Theorem 1.6.5. *Let $\mathscr{H}$ and $\mathscr{K}$ be Hilbert spaces.*

1. *If $a \in \mathcal{B}(\mathscr{H})$ and $b \in \mathcal{B}(\mathscr{K})$, then there is a unique operator $a \otimes b \in \mathcal{B}(\mathscr{H} \otimes \mathscr{K})$ such that*

$$(a \otimes b)(x \otimes y) = a(x) \otimes b(y)$$

where $x \in \mathscr{H}$ and $y \in \mathscr{K}$. Moreover,

$$\|a \otimes b\| = \|a\|\|b\|.$$

2. *For all $a, a' \in \mathcal{B}(\mathscr{H})$ and $b, b' \in \mathcal{B}(\mathscr{K})$,*

$$(a \otimes b)(a' \otimes b') = aa' \otimes bb',$$

and

$$(a \otimes b)^* = a^* \otimes b^*.$$

1.7 Hilbert-Schmidt operators and tensor products

In this section we show that the tensor product $\mathscr{H} \otimes \mathscr{H}$ (i.e. G of Sec. 1.8) can be identified, via *unitary equivalence*, as the Hilbert space of all Hilbert-Schmidt operators from $\overline{\mathscr{H}}$ into $\mathscr{H}$.

In order to simplify the treatment of conjugate-linear mappings, we introduce the notion of "conjugate" of a Hilbert space $\mathscr{H}$. Recall that in the case of a Hilbert space, we have a set $\mathscr{H}$ together with the maps

$$\begin{aligned}
\mathscr{H} \times \mathscr{H} \to \mathscr{H} \quad &: \quad (x, y) \mapsto x + y, \\
\mathbb{C} \times \mathscr{H} \to \mathscr{H} \quad &: \quad (\alpha, x) \mapsto \alpha x, \\
\mathscr{H} \times \mathscr{H} \to \mathbb{C} \quad &: \quad (x, y) \mapsto \langle x, y \rangle .
\end{aligned}$$

Definition 1.7.1 (Conjugate Hilbert space)**.** Let $\mathscr{H}$ be a Hilbert space. The *conjugate Hilbert space* $\overline{\mathscr{H}}$ is the same set $\mathscr{H}$, but with the algebraic structure and inner product given by

$$\begin{aligned} \mathscr{H} \times \mathscr{H} \to \mathscr{H} \quad &: \quad (x, y) \mapsto x + y, \\ \mathbb{C} \times \mathscr{H} \to \mathscr{H} \quad &: \quad (\alpha, x) \mapsto \alpha \bar{\cdot} x, \\ \mathscr{H} \times \mathscr{H} \to \mathbb{C} \quad &: \quad (x, y) \mapsto \overline{\langle x, y \rangle} \end{aligned}$$

where $\alpha \bar{\cdot} x = \bar{\alpha} x$ and $\overline{\langle x, y \rangle} = \langle y, x \rangle$. We write elements of $\overline{\mathscr{H}}$ as $\bar{x}$. So for $x \in \overline{\mathscr{H}}$ we set $\bar{x} := x$, but we view $\bar{x}$ as an element of $\overline{\mathscr{H}}$.

Remarks 1.7.2. It is easily verified that the linear operator given by $\mathcal{B}(\mathscr{H}) \to \mathcal{B}(\overline{\mathscr{H}}) : a \mapsto a$ is a *-isomorphism. In fact, we can identify $\mathcal{B}(\overline{\mathscr{H}})$ with $\mathcal{B}(\mathscr{H})$; they are exactly the same von Neumann algebra.

For $a \in \mathcal{B}(\mathscr{H})$, note that with $x \in \overline{\mathscr{H}}$ and $\alpha \in \mathbb{C}$, we have

$$a(\alpha \bar{\cdot} x) = a(\bar{\alpha} x) = \bar{\alpha} a x = \alpha \bar{\cdot} (a x),$$

so $a \in \mathcal{B}(\overline{\mathscr{H}})$.

The tensor product $\mathscr{H} \otimes \mathscr{H}$ can be identified with the Hilbert space $\mathcal{B}_2(\overline{\mathscr{H}} \times \overline{\mathscr{H}}, \mathbb{C})$ of all the Hilbert-Schmidt functionals on $\overline{\mathscr{H}} \times \overline{\mathscr{H}}$, i.e. $\mathscr{H} \otimes \mathscr{H} \cong \mathcal{B}_2(\overline{\mathscr{H}} \times \overline{\mathscr{H}}, \mathbb{C})$, as the following proposition establishes:

Proposition 1.7.3. *Let $\mathscr{H}$ be a separable Hilbert space. Then:*

1. *If $x, y \in \mathscr{H}$, the equation*

$$\varphi_{x,y}(\bar{u}, \bar{v}) := \langle \bar{u}, x \rangle \langle \bar{v}, y \rangle \quad (u, v \in \overline{\mathscr{H}})$$

defines a bilinear Hilbert-Schmidt functional $\varphi_{x,y}$ on $\overline{\mathscr{H}} \times \overline{\mathscr{H}}$.

2. *There is an isometric isomorphism*

$$\mathscr{H} \otimes \mathscr{H} \to \mathcal{B}_2(\overline{\mathscr{H}} \times \overline{\mathscr{H}}, \mathbb{C}) : x \otimes y \mapsto \varphi_{x,y}.$$

Proof. Let $\{e_j\}_{j \in \mathbb{N}}$ be an orthonormal basis of $\mathscr{H}$.

(1) Let $x, y \in \mathscr{H}$ be arbitrary vectors. Then, for $\alpha, \beta \in \mathbb{C}$ and $\bar{u}, \bar{u}_1, \bar{u}_2, \bar{v}, \bar{v}_1, \bar{v}_2 \in \overline{\mathscr{H}}$, we have

$$\begin{aligned} \varphi_{x,y}(\alpha \cdot \bar{u}_1 + \bar{u}_2, \bar{v}) &= \langle \bar{\alpha} \bar{u}_1 + \bar{u}_2, x \rangle \langle \bar{v}, y \rangle \\ &= \alpha \langle \bar{u}_1, x \rangle \langle \bar{v}, y \rangle + \langle \bar{u}_2, x \rangle \langle \bar{v}, y \rangle \\ &= \alpha \varphi_{x,y}(\bar{u}_1, \bar{v}) + \varphi_{x,y}(\bar{u}_2, \bar{v}). \end{aligned}$$

Similarly, we have $\varphi_{x,y}(\bar{u}, \beta \cdot \bar{v}_1 + \bar{v}_2) = \beta \varphi_{x,y}(\bar{u}, \bar{v}_1) + \varphi_{x,y}(\bar{u}, \bar{v}_2)$, thus $\varphi_{x,y}$ is a bilinear functional on $\overline{\mathscr{H}} \times \overline{\mathscr{H}}$. From the Parseval relation we get

$$\|\varphi_{x,y}\|_2^2 = \sum_{i,j=1}^{\infty} |\varphi_{x,y}(e_i, e_j)|^2 = \sum_{i,j=1}^{\infty} |\langle e_i, x \rangle|^2 |\langle e_j, y \rangle|^2 = \|x\|^2 \|y\|^2,$$

so that $\|\varphi_{x,y}\|_2 = \|x\|\|y\| \leq \infty$, and hence $\varphi_{x,y} \in \mathcal{B}_2(\overline{\mathscr{H}} \times \overline{\mathscr{H}}, \mathbb{C})$ for all $x, y \in \mathscr{H}$.

(2) Consider the following map

$$\Psi : \mathscr{H} \times \mathscr{H} \to \mathcal{B}_2(\overline{\mathscr{H}} \times \overline{\mathscr{H}}, \mathbb{C}) : (x, y) \mapsto \varphi_{x,y}.$$

Then Ψ is a bilinear map, since for all $\bar{u}, \bar{v} \in \overline{\mathscr{H}}$

$$\begin{aligned}
[\Psi(\alpha x_1 + x_2, y)](\bar{u}, \bar{v}) &= \varphi_{\alpha x_1 + x_2, y}(\bar{u}, \bar{v}) \\
&= \langle \bar{u}, \alpha x_1 + x_2 \rangle \langle \bar{v}, y \rangle \\
&= \alpha \langle \bar{u}, x_1 \rangle \langle \bar{v}, y \rangle + \langle \bar{u}, x_2 \rangle \langle \bar{v}, y \rangle \\
&= \alpha \varphi_{x_1, y}(\bar{u}, \bar{v}) + \varphi_{x_2, y}(\bar{u}, \bar{v}) \\
&= [\alpha \Psi(x_1, y) + \Psi(x_2, y)](\bar{u}, \bar{v})
\end{aligned}$$

so that $\Psi(\alpha x_1 + x_2, y) = \alpha \Psi(x_1, y) + \Psi(x_2, y)$. Similarly, we have $\Psi(x, \beta y_1 + y_2) = \beta \Psi(x, y_1) + \Psi(x, y_2)$. By the universal property of tensor products (i.e. Prop.1.6.2), there is a unique linear operator

$$\Phi : \mathscr{H} \otimes \mathscr{H} \to \mathcal{B}_2(\overline{\mathscr{H}} \times \overline{\mathscr{H}}, \mathbb{C})$$

such that $\Psi = \Phi \circ p$, where $p : \mathscr{H} \times \mathscr{H} \to \mathscr{H} \otimes \mathscr{H}$ is the canonical bilinear map.

Finally, we show that Φ is an isometric isomorphism.

Isometric property of Φ:

$$\|\Phi(x \otimes y)\|_2 = \|\varphi_{x,y}\|_2 = \|x\|\|y\| = \|x \otimes y\|.$$

Injectivity of Φ: Suppose that $\Phi(x_1 \otimes y_1) = \Phi(x_2 \otimes y_2)$, then for all $\bar{u}, \bar{v} \in \overline{\mathscr{H}}$, we have

$$\begin{aligned}
0 &= \varphi_{x_1, y_1}(\bar{u}, \bar{v}) - \varphi_{x_2, y_2}(\bar{u}, \bar{v}) \\
&= \langle \bar{u}, x_1 \rangle \langle \bar{v}, y_1 \rangle - \langle \bar{u}, x_2 \rangle \langle \bar{v}, y_2 \rangle \\
&= \langle \bar{u} \otimes \bar{v}, x_1 \otimes y_1 \rangle - \langle \bar{u} \otimes \bar{v}, x_2 \otimes y_2 \rangle \\
&= \langle \bar{u} \otimes \bar{v}, x_1 \otimes y_1 - x_2 \otimes y_2 \rangle,
\end{aligned}$$

so that $x_1 \otimes y_1 = x_2 \otimes y_2$.

Denseness of the range of Φ: Since the sets $\{e_i \otimes e_j\}_{i,j \in \mathbb{N}}$ and $\{\varphi_{e_i, e_j}\}_{i,j \in \mathbb{N}}$ form orthonormal bases for $\mathscr{H} \otimes \mathscr{H}$ and $\mathcal{B}_2(\overline{\mathscr{H}} \times \overline{\mathscr{H}}, \mathbb{C})$, respectively. It then follows from

$$\Phi(e_i \otimes e_j) = \varphi_{e_i, e_j}$$

that the range of Φ is dense in $\mathcal{B}_2(\overline{\mathscr{H}} \times \overline{\mathscr{H}}, \mathbb{C})$. □

The following proposition states that there is a one-to-one correspondence between bounded linear operators and bounded bilinear functionals:

Proposition 1.7.4. *Let $\mathscr{H}$ be a separable Hilbert space, then:*

1. *If $a \in \mathcal{B}(\mathscr{H})$, the equation*
$$\varphi_a(x,y) := \langle x, ay\rangle \quad (x, y \in \mathscr{H})$$
defines a bounded bilinear functional φ_a on $\overline{\mathscr{H}} \times \mathscr{H}$.

2. *There is an isometric isomorphism*
$$\mathcal{B}(\mathscr{H}) \to \mathcal{B}(\overline{\mathscr{H}} \times \mathscr{H}, \mathbb{C}) : a \mapsto \varphi_a.$$

Proof. Let $\{e_j\}_{j\in\mathbb{N}}$ be an orthonormal basis of $\mathscr{H}$.

(1) Consider any $a \in \mathcal{B}(\mathscr{H})$, and let $x, x_1, x_2, y, y_1, y_2 \in \mathscr{H}$, and $\alpha, \beta \in \mathbb{C}$. The bilinearity of φ_a on $\overline{\mathscr{H}} \times \mathscr{H}$ is easily verified, since

$$\begin{aligned}\varphi_a(\alpha x_1 + x_2, y) &= \bar{\alpha}\,\langle x_1, ay\rangle + \langle x_2, ay\rangle \\ &= \alpha\,\bar{}\;\varphi_a(x_1, y) + \varphi_a(x_2, y)\end{aligned}$$

and

$$\begin{aligned}\varphi_a(x, \beta y_1 + y_2) &= \beta\,\langle x, ay_1\rangle + \langle x, ay_2\rangle \\ &= \beta\varphi_a(x, y_1) + \varphi_a(x, y_2).\end{aligned}$$

By the Schwarz inequality, we have

$$\begin{aligned}|\varphi_a(x,y)| &= |\,\langle x, ay\rangle\,| \\ &\leq \|x\|\|ay\| \\ &\leq \|a\|\|x\|\|y\|.\end{aligned}$$

So, φ_a is a bounded bilinear functional on $\overline{\mathscr{H}} \times \mathscr{H}$. Then for all $a \in \mathcal{B}(\mathscr{H})$, we have that $\varphi_a \in \mathcal{B}(\overline{\mathscr{H}} \times \mathscr{H}, \mathbb{C})$, since a was arbitrarily chosen.

(2) Let $a, b \in \mathcal{B}(\mathscr{H}, \mathscr{H})$ and $\alpha \in \mathbb{C}$. Denote the map by Φ.

Linearity of Φ:

$$\begin{aligned}[\Phi(\alpha a + b)](x,y) &= \varphi_{\alpha a + b}(x,y) \\ &= \langle x, (\alpha a + b)y\rangle \\ &= \alpha\,\langle x, ay\rangle + \langle x, by\rangle \\ &= \alpha\varphi_a(x,y) + \varphi_b(x,y) \\ &= [\alpha\Phi(a) + \Phi(b)](x,y).\end{aligned}$$

Injectivity of Φ: Suppose $\Phi(a) = \Phi(b)$, then $\forall x, y \in \mathscr{H}$,

$$\begin{aligned}0 &= [\Phi(a) - \Phi(b)](x,y) \\ &= \varphi_a(x,y) - \varphi_b(x,y) \\ &= \langle x, ay\rangle - \langle x, by\rangle \\ &= \langle x, (a-b)y\rangle\,.\end{aligned}$$

Consequently, $0 = (a-b)y, \forall y \in \mathscr{H}$, so that $a = b$.

Surjectivity of Φ: If $\varphi \in B(\overline{\mathscr{H}} \times \mathscr{H}, \mathbb{C})$, then by the Riesz representation theorem there exists $a_\varphi \in B(\overline{\mathscr{H}})$ such that $\varphi(x,y) = \langle x, a_\varphi y\rangle$.

Isometric property:

$$\|\Phi(a)\|_2^2 = \|\varphi_a\|_2^2 \quad = \quad \sum_{i,j=1}^{\infty} |\varphi_a(e_i,e_j)|^2 = \sum_{i,j=1}^{\infty} |\langle e_i, ae_j\rangle|^2 = \|a\|_2^2.$$

$\square$

Corollary 1.7.5. *For any linear operator $a : \mathscr{H} \to \mathscr{H}$, we have that $a \in \mathcal{B}(\mathscr{H})$ if and only if $\varphi_a \in \mathcal{B}_2(\overline{\mathscr{H}} \times \mathscr{H}, \mathbb{C})$. Moreover,*

$$\mathcal{B}_2(\mathscr{H}) = \{a \in \mathcal{B}(\mathscr{H}) : \varphi_a \in \mathcal{B}_2(\overline{\mathscr{H}} \times \mathscr{H}, \mathbb{C})\}.$$

Remarks 1.7.6. The above two results (i.e. Prop.1.7.4 and Cor.1.7.5) also give a way, through the map $a \mapsto \varphi_a$, of transferring the Hilbert space structure between the spaces of Hilbert-Schmidt operators and Hilbert-Schmidt functionals.

The identification of the tensor product $\mathscr{H} \otimes \mathscr{H}$ with the Hilbert space of all Hilbert-Schmidt operators from $\overline{\mathscr{H}}$ into $\mathscr{H}$ is described in the following (cf. [17, Prop. 2.6.9]):

Theorem 1.7.7. *Let $\mathscr{H}$ be a separable Hilbert space, then:*

1. *For all $x, y \in \mathscr{H}$, the equation*

$$|x\rangle\langle y|\,(u) := \langle y, u\rangle\, x \quad (u \in \overline{\mathscr{H}})$$

   defines a Hilbert-Schmidt operator $|x\rangle\langle y|$ from $\overline{\mathscr{H}}$ into $\mathscr{H}$.

2. *There is a unitary transformation $W : \mathscr{H} \otimes \overline{\mathscr{H}} \to \mathcal{B}_2(\mathscr{H})$ such that*

$$W(x \otimes \bar{y}) := |x\rangle\langle y|\,. \tag{1.7.1}$$

Proof. Let $\{e_j\}_{j\in\mathbb{N}}$ be an orthonormal basis of $\overline{\mathscr{H}}$.

(1) Let $x, y \in \mathscr{H}$ be arbitrary. For $u_1, u_2 \in \overline{\mathscr{H}}$ and $\alpha \in \mathbb{C}$, we have

$$\begin{aligned} |x\rangle\langle y|\,(\alpha u_1 + u_2) &= \langle y, \alpha u_1 + u_2\rangle\, x \\ &= \alpha \langle y, u_1\rangle\, x + \langle y, u_2\rangle\, x \\ &= \alpha\, |x\rangle\langle y|\,(u_1) + |x\rangle\langle y|\,(u_2), \end{aligned}$$

thus $|x\rangle\langle y|$ defines a linear operator from $\mathscr{H}$ into $\mathscr{H}$. Since

$$\begin{aligned} \|\,|x\rangle\langle y|\,\|_2^2 &= \sum_{j=1}^{\infty} \|\,|x\rangle\langle y|\,(e_j)\|^2 = \sum_{j=1}^{\infty} \|\langle y, e_j\rangle\, x\|^2 \\ &= \left(\sum_{j=1}^{\infty} |\langle y, e_j\rangle|^2\right) \|x\|^2 \\ &= \|x\|^2\|y\|^2, \end{aligned}$$

it follows that $\| \left|x\right\rangle \left\langle y\right| \|_2 = \|x\|\|y\| < \infty$, so that $\left|x\right\rangle \left\langle y\right|$ is a Hilbert-Schmidt operator. Since $x, y \in \mathscr{H}$ were arbitrarily chosen, then $\left|x\right\rangle \left\langle y\right|$ defines a Hilbert-Schmidt operator for all $x, y \in \mathscr{H}$.

(2) Consider the tensor product $\mathscr{H} \otimes \overline{\mathscr{H}}$. Then, by Prop. 1.7.3, there is a unitary operator $\Phi : \mathscr{H} \otimes \mathscr{H} \to \mathcal{B}_2(\overline{\mathscr{H}} \times \overline{\mathscr{H}}, \mathbb{C})$ such that

$$\Phi(x \otimes y) = \varphi_{x,y}.$$

Analogously, from Prop.1.7.4 and Cor.1.7.5, we have that there is a unitary operator $\Psi : \mathcal{B}_2(\overline{\mathscr{H}} \times \overline{\mathscr{H}}, \mathbb{C}) \to \mathcal{B}_2(\mathscr{H})$ such that

$$\Psi(\varphi_{x,y}) = \left|x\right\rangle \left\langle y\right|.$$

Let $W := \Psi \circ \Phi$, then W is a unitary operator such that

$$W(x \otimes y) = \Psi(\Phi(x \otimes y)) = \Psi(\varphi_{x,y}) = \left|x\right\rangle \left\langle y\right|.$$

□

1.8 The GNS construction in terms of tensor products

The Gelfand-Naimark-Segal (GNS) construction is a fundamental correspondence between *cyclic *-representations* of an algebra and linear functionals on the algebra.

Abstract elements of a C*-algebra $\mathcal{A}$ can be realized as operators on some Hilbert space by a choice of *representation*:

Definition 1.8.1. Let $\mathcal{A}$ be a C*-algebra.

1. A *representation* of $\mathcal{A}$ is a pair $(\mathscr{H}, \varphi)$ where $\mathscr{H}$ is a complex Hilbert space, and $\varphi : \mathcal{A} \to \mathcal{B}(\mathscr{H})$ is a *-homomorphism. We say $(\mathscr{H}, \varphi)$ is a *faithful representation*, if in addition, φ is injective.

2. A *cyclic representation* of $\mathcal{A}$ is defined to be the triple $(\mathscr{H}, \varphi, x)$, where $(\mathscr{H}, \varphi)$ is a representation of $\mathcal{A}$, and x is a vector in $\mathscr{H}$ which is cyclic for φ in $\mathscr{H}$, i.e.
$$\varphi(\mathcal{A})x = \{\varphi(a)x \mid a \in \mathcal{A}\}$$
is dense in $\mathscr{H}$.

Remarks 1.8.2. Terminology related to the above definition. The space $\mathscr{H}$ is called the *representation space*, the operators $\varphi(a)$ are called *representatives* of $\mathcal{A}$, and by implicit identification φ and the set of representatives, we also say that φ is a representation of $\mathcal{A}$ on $\mathscr{H}$.

The existence of a cyclic representation for $(\mathcal{A}, \varphi)$ is given by the GNS-construction.

Let $G := \mathscr{H} \otimes \mathscr{H}$ and $\mathcal{A} \equiv B(\mathscr{H})$. Set

$$\Omega := \sum_{j=1}^{\infty} \rho_j^{1/2} e_j \otimes e_j,$$

where $\rho_j \in \mathbb{R}_+$ are the eigenvalues of ρ as given by (1.5.2). Since the partial sums of the series are non-negative, they form a monotone increasing sequence, and these partial sums are bounded above by $\|\rho\| = 1$; because

$$\sum_{j=1}^{N} |\rho_j^{1/2}|^2 = \sum_{j=1}^{N} \rho_j \leq Tr(\rho) = 1.$$

It then follows that the series is convergent, thus $\Omega \in G$.

Proposition 1.8.3. *The maps $\pi, \pi' : \mathcal{B}(\mathscr{H}) \to \mathcal{B}(G)$ given by*

$$\pi(a) := a \otimes 1 \tag{1.8.1}$$

and

$$\pi'(a) := 1 \otimes a \tag{1.8.2}$$

*are *-homomorphisms.*

Proof. Let $a, b \in \mathcal{B}(\mathscr{H})$ and $\alpha \in \mathbb{C}$,

$$\begin{aligned} \pi(\alpha a + b) &= (\alpha a + b) \otimes 1 \\ &= \alpha (a \otimes 1) + (b \otimes 1) \\ &= \alpha \pi(a) + \pi(b), \end{aligned}$$

then π is linear. Suppose that $\pi(a) = \pi(b)$, then $a \otimes 1 = b \otimes 1$ so that $(a - b) \otimes 1 = 0$. Hence $a = b$.

Homomorphism property:

$$\pi(ab) = (ab) \otimes 1 = (a \otimes 1)(b \otimes 1) = \pi(a)\pi(b).$$

*-Property: $\pi(a^*) = a^* \otimes 1 = (a \otimes 1)^* = \pi(a)^*$. Hence π is a *-homomorphism. Similarly, π' is a *-homomorphism. $\square$

Proposition 1.8.4. *Let $\mu(a) = Tr(\rho a)$ be the state on $\mathcal{A}$ with the density operator ρ given by (1.5.2). Then:*

1. *The triple (G, π, Ω) is a cyclic representation of $(\mathcal{A}, \mu)$.*
2. *Given (G, π, Ω), the corresponding state μ can always be retrieved by setting*

$$\mu(a) = \langle \Omega, \pi(a)\Omega \rangle \quad a \in \mathcal{B}(\mathscr{H}).$$

3. *The vector $\Omega \in G$ is separating for $\pi(\mathcal{A})$, i.e. if $\pi(a)\Omega = 0$, then $a = 0$. Moreover, π is injective, i.e. the representation (G, π, Ω) of $(\mathcal{A}, \mu)$ is faithful.*

Proof. Let $\{e_j : j \in \mathbb{N}\}$ be an orthonormal basis for $\mathscr{H}$.

1. Since Prop.1.8.3 has established that π is a *-homomorphism, it suffices to show only that $\Omega \in G$ is a cyclic vector for the representation (G, π). Set $a = |e_i\rangle \langle e_j| \in \mathcal{A}$, then

$$\begin{aligned}
\pi(a)\Omega &= (a \otimes 1)\left(\sum_{k=1}^{\infty} \rho_k^{1/2} e_k \otimes e_k\right) \\
&= \sum_{k=1}^{\infty} \delta_{kj}\rho_k^{1/2} e_i \otimes e_k \\
&= \rho_j^{1/2} e_i \otimes e_j.
\end{aligned}$$

Since $\rho_j \neq 0$ for all $j \in \mathbb{N}$, it follows that the orthonormal basis $\{e_i \otimes e_j : i, j \in \mathbb{N}\}$ for G, is contained in $\pi(\mathcal{A})\Omega$. Hence $\pi(\mathcal{A})\Omega$ is dense in G.

2. For all $a \in \mathcal{A}$, we have

$$\begin{aligned}
\langle \Omega, \pi(a)\Omega \rangle &= \left\langle \sum_{i=1}^{\infty} \rho_i^{1/2} e_i \otimes e_i, \pi(a)\left(\sum_{j=1}^{\infty} \rho_j^{1/2} e_j \otimes e_j\right)\right\rangle \\
&= \sum_{i,j=1}^{\infty} \overline{\rho_i^{1/2}} \rho_j^{1/2} \langle e_i \otimes e_i, (a \otimes 1)(e_j \otimes e_j)\rangle \\
&= \sum_{i,j=1}^{\infty} \rho_i^{1/2} \rho_j^{1/2} \langle e_i \otimes e_i, a e_j \otimes e_j\rangle \\
&= \sum_{i,j=1}^{\infty} \rho_i^{1/2} \rho_j^{1/2} \langle e_i, a e_j\rangle \langle e_i, e_j\rangle \\
&= \sum_{j=1}^{\infty} \langle e_j, a\rho_j e_j\rangle \\
&= \sum_{j=1}^{\infty} \langle e_j, a\rho e_j\rangle \\
&= Tr(a\rho) \\
&= Tr(\rho a) \\
&= \mu(a).
\end{aligned}$$

3. If $\pi(a)\Omega = 0$, then

$$\begin{aligned} 0 &= \langle \pi(a)\Omega, \pi(a)\Omega \rangle \\ &= \langle \Omega, \pi(a)^* \pi(a)\Omega \rangle \\ &= \langle \Omega, \pi(a^* a)\Omega \rangle \\ &= \mu(a^* a). \end{aligned}$$

So $a = 0$, since μ is faithful by Prop. 1.5.9. In fact, this also shows that π is injective, since $\pi(a) = 0 \implies \pi(a)\Omega = 0 \implies a = 0$.

□

Recall that if $\{e_j : j \in \mathbb{N}\}$ is an orthonormal basis for the separable Hilbert space $\mathscr{H}$, then using the Dirac notation, an alternative expression of Eq.(1.2.2) is

$$x = \sum_{k=1}^{\infty} |e_k\rangle \langle e_k| \, x$$

for all $x \in \mathscr{H}$. Similarly, for all $a \in \mathcal{B}(\mathscr{H})$ we can write the matrix representation of a as

$$a = \sum_{j,k=1}^{\infty} a_{jk} \, |e_j\rangle \langle e_k| \tag{1.8.3}$$

where $a_{jk} = \langle e_j, ae_k \rangle$ (see Prop.1.2.4), where we do not think of this as a convergent series, but just as a notation for the matrix representation.

Since $\{e_j \otimes e_k : j, k \in \mathbb{N}\}$ is a countable orthonormal basis for $G \equiv \mathscr{H} \otimes \mathscr{H}$, we can consider an infinite matrix representation of any $a \in \mathcal{B}(G)$ given by matrix entries $\{\alpha_{j,k,l,m} : j, k, l, m \in \mathbb{N}\}$, with respect to the basis $\{e_i \otimes e_j : i, j \in \mathbb{N}\}$. That is, analogous to (1.8.3),

$$a = \sum_{j,k,l,m \in \mathbb{N}} \alpha_{j,k,l,m} \, |e_j \otimes e_l\rangle \langle e_k \otimes e_m| \, . \tag{1.8.4}$$

Again, note that this is only a way to express the matrix representation of a in the above basis. That is, Eq. (1.8.4) does not imply the convergence of an infinite series. In fact the series in Eq. (1.8.4) will not converge in general. Instead the interpretation of Eq. (1.8.4) is that a is the operator in $\mathcal{B}(G)$ uniquely determined by:

$$a(e_k \otimes e_m) = \sum_{j,l \in \mathbb{N}} \alpha_{j,k,l,m} \; e_j \otimes e_l \tag{1.8.5}$$

for all $e_k \otimes e_m \in \mathscr{H} \otimes \mathscr{H}$. We will not directly utilize this matrix representation, but we include it as it is occasionally useful to "visualize" an operator $a \in \mathcal{B}(G)$ as an infinite matrix.

Theorem 1.8.5. *The algebras $\mathcal{A}\otimes 1$ and $1\otimes\mathcal{A}$ are mutual commutants in $\mathcal{B}(G)$, i.e.*

$$(\mathcal{A}\otimes 1)' = 1\otimes\mathcal{A}, \text{ and } (1\otimes\mathcal{A})' = \mathcal{A}\otimes 1$$

where $\mathcal{A} := \mathcal{B}(\mathscr{H})$, $G := \mathscr{H}\otimes\mathscr{H}$ and $\mathscr{H}$ is a separable Hilbert space. Moreover, $\mathcal{A}\otimes 1$ and $1\otimes\mathcal{A}$ are von Neumann algebras.

Proof. Clearly $1\otimes\mathcal{A}\subset(\mathcal{A}\otimes 1)'$, so we only have to show the other inclusion. Consider any $a\in\mathcal{B}(G)$ defined by

$$a(e_s\otimes e_t) = \sum_{j,l\in\mathbb{N}} \alpha_{j,s,l,t}\ e_j\otimes e_l \tag{1.8.6}$$

for all $s,t\in\mathbb{N}$ as explained above. For any $p,q\in\mathbb{N}$, let us denote the operator $|e_p\rangle\langle e_q|\in\mathcal{A}$ simply by e_{pq}.

Assume that $a\in(\mathcal{A}\otimes 1)'$ with the commutant taken in $\mathcal{B}(G)$. Then a commutes in particular with $e_{pq}\otimes 1$ for all $p,q\in\mathbb{N}$, i.e.

$$a(e_{pq}\otimes 1) = (e_{pq}\otimes 1)a \quad (p,q\in\mathbb{N}). \tag{1.8.7}$$

Now consider any $e_s\otimes e_t\in G$. It follows that

$$\begin{aligned}
(e_{pq}\otimes 1)a(e_s\otimes e_t) &= (e_{pq}\otimes 1)\left(\sum_{j,l\in\mathbb{N}} \alpha_{j,s,l,t}\ e_j\otimes e_l\right)\\
&= \sum_{j,l\in\mathbb{N}} \alpha_{j,s,l,t}(e_{pq}e_j)\otimes e_l\\
&= \sum_{j,l\in\mathbb{N}} \delta_{q,j}\ \alpha_{j,s,l,t}\ e_p\otimes e_l\\
&= \sum_{l=1}^{\infty} \alpha_{q,s,l,t}\ e_p\otimes e_l
\end{aligned}$$

where we used the boundedness of $(e_{pq}\otimes 1)\in\mathcal{A}\otimes\mathcal{A}$.

On the other hand, we also have that

$$\begin{aligned}
a(e_{pq}\otimes 1)(e_s\otimes e_t) &= a\left((e_{pq}e_s)\otimes e_t\right)\\
&= a\left((\delta_{q,s}\ e_p)\otimes e_t\right)\\
&= \delta_{q,s}\ a\left(e_p\otimes e_t\right)\\
&= \delta_{q,s}\sum_{j,l\in\mathbb{N}} \alpha_{j,p,l,t}\ e_j\otimes e_l.
\end{aligned}$$

Since a commutes with $e_{pq}\otimes 1$, for all $p,q\in\mathbb{N}$

$$\begin{aligned}
0 &= [e_{pq}\otimes 1, a]\,(e_s\otimes e_t)\\
&= (e_{pq}\otimes 1)a(e_s\otimes e_t) - a(e_{pq}\otimes 1)(e_s\otimes e_t)\\
&= \sum_{l=1}^{\infty} \alpha_{q,s,l,t}\ e_p\otimes e_l - \delta_{q,s}\sum_{j,l\in\mathbb{N}} \alpha_{j,p,l,t}\ e_j\otimes e_l
\end{aligned}$$

so that

$$\sum_{l=1}^{\infty} \alpha_{q,s,l,t}\ e_p \otimes e_l = \delta_{q,s} \sum_{j,l \in \mathbb{N}} \alpha_{j,p,l,t}\ e_j \otimes e_l \tag{1.8.8}$$

for all $p, q, s, t \in \mathbb{N}$. It is clear that we must have $\alpha_{j,k,l,t} = 0$ if $j \neq k$, so Eq.(1.8.8) simplifies to

$$\sum_{l=1}^{\infty} \alpha_{q,q,l,m}\ e_p \otimes e_l = \sum_{l=1}^{\infty} \alpha_{p,p,l,t}\ e_p \otimes e_l$$

for all $p, q, t \in \mathbb{N}$, so it also follows that, for any $l, t \in \mathbb{N}$, $\alpha_{q,q,l,t} = \alpha_{p,p,l,t}$ for all $p, q \in \mathbb{N}$. We can thus set $\varphi_{l,t} := \alpha_{p,p,l,t}$ for any $p, l, t \in \mathbb{N}$.

Applying this to Eq.(1.8.6) it follows for any $e_s \otimes e_t \in G$ that

$$\begin{aligned} a(e_s \otimes e_t) &= \sum_{j,l \in \mathbb{N}} \alpha_{j,s,l,t}\ e_j \otimes e_l \\ &= \sum_{l=1}^{\infty} \alpha_{s,s,l,t}\ e_s \otimes e_l \\ &= e_s \otimes \left(\sum_{l=1}^{\infty} \varphi_{l,t}\ e_l \right) \\ &= (1 \otimes c) e_s \otimes e_t \end{aligned}$$

where c is the bounded operator uniquely determined by

$$c(e_t) = \sum_{l=1}^{\infty} \varphi_{l,t}\ e_l \quad (\forall t \in \mathbb{N}).$$

That the operator c is well-defined, specifically that it is bounded linear, follows from the fact that a is bounded linear; since for any $x \in \mathscr{H}_0 := span\{e_j : j \in \mathbb{N}\}$, we have that

$$\|a(e_s \otimes x)\| = \|e_s \otimes c(x)\| = \|c(x)\|$$

and since a is bounded linear,

$$\|a(e_s \otimes x)\| \leq \|a\| \|e_s \otimes x\| = \|a\| \|x\|.$$

Thus $c \in \mathcal{A}$. It then follows that $a = 1 \otimes c$, since $\{e_j \otimes e_k : j, k \in \mathbb{N}\}$ is an orthonormal basis for G. Hence $(\mathcal{A} \otimes 1)' = 1 \otimes \mathcal{A}$, similarly $(1 \otimes \mathcal{A})' = \mathcal{A} \otimes 1$. Furthermore, $(\mathcal{A} \otimes 1)'' = (1 \otimes \mathcal{A})' = \mathcal{A} \otimes 1$ so that $\mathcal{A} \otimes 1$ is a von Neumann algebra; similarly for $1 \otimes \mathcal{A}$. $\square$

Remarks 1.8.6. The previous Theorem is a special case of a more general theorem, the so-called *commutation theorem for tensor products of von Neumann algebras*, and in such a generality the statement of the theorem remained open for a long time [20, Example 2.10, Theorem 5.1]. The first proof, which used modular theory, was obtained by Tomita in 1967 (see [31]). Later a number of simpler versions have been obtained (see for example [26]).

Chapter 2

Tomita-Takesaki Theory

In this chapter we set up a simple version of the Tomita-Takesaki modular theory for the von Neumann algebra $\mathcal{B}(\mathscr{H})$ with a faithful normal state, using a tensor product approach. This is a special case of the more general situation of the theory (for example, see [8],[30]).

In Section 2.1 we give a heuristic overview of the theory as well as introduce some related terminology, and state the main result of the theory. Our goal is to present a version of Tomita-Takesaki theory for the pair $(\mathcal{A}, \mu)$, where $\mathcal{A} = \mathcal{B}(\mathscr{H})$ and μ is the faithful normal state given by $\mu(a) = Tr(\rho a)$ for ρ as described by Eq.(1.5.2).

In Chapter 3 we give an alternative approach to the version of Tomita-Takesaki theory which we shall now develop, that is formulated in terms of the Hilbert space of Hilbert-Schmidt operators.

2.1 Summary of the Tomita-Takesaki theory and the modular objects

In this section we give a general overview of the Tomita-Takesaki modular theory, and state some of its main results. Details and proofs of statements may be found, for example, in [8, Section 2.5], [17, Section 9.2], or [30, Chapter VIII].

Let $\mathfrak{A}$ be a C*-algebra on a Hilbert space $\mathscr{H}$ which contains a cyclic and separating vector Ω. The theory starts off by defining a seemingly innocuous conjugate linear operator S_0 on $\mathscr{H}$ as follows:

$$S_0 a\Omega = a^*\Omega, \quad \forall a \in \mathfrak{A}. \tag{2.1.1}$$

It can be shown that this operator extends to a closed conjugate linear operator S, called the *Tomita operator* for the pair $(\mathfrak{A}, \Omega)$, defined on a dense subset of $\mathscr{H}$.

Let Δ be the unique positive, self-adjoint operator and J the unique anti-unitary operator occurring in the polar decomposition of S, i.e.

$$S = J\Delta^{1/2}. \tag{2.1.2}$$

We call Δ the *modular operator* and J the *modular conjugation* (or *modular involution*) associated with the pair $(\mathfrak{A}, \Omega)$. It is straightforward to see that J^2 is the identity operator and $J^* = J$. Moreover, we have that $J\Omega = \Omega = \Delta\Omega$.

From the functional calculus of Δ we construct an operator

$$\Delta^{it} = \exp\left(it(\ln \Delta)\right), \ t \in \mathbb{R},$$

such that Δ^{it} is unitary for all $t \in \mathbb{R}$ and $\left\{\Delta^{it} : t \in \mathbb{R}\right\}$ forms a strongly continuous unitary group, called the *modular group*.

The unitaries $\left\{\Delta^{it} : t \in \mathbb{R}\right\}$ induce a one-parameter automorphism group $\{\sigma_t : t \in \mathbb{R}\}$ of $\mathfrak{A}$ by

$$\sigma_t(a) = \Delta^{it} a \Delta^{-it}, \quad a \in \mathfrak{A}, t \in \mathbb{R}.$$

This group is the so-called *modular automorphism group* of $\mathfrak{A}$ (relative to Ω). The automorphisms are sometimes denoted by σ_t^{Ω} to stress their dependence on the choice of the cyclic and separating vector Ω.

Theorem 2.1.1 (Tomita-Takesaki)**.** *Let $\mathfrak{A}$ be a von Neumann algebra with a cyclic and separating vector Ω. Then the following statements hold:*

1. $\sigma_t(\mathfrak{A}) = \mathfrak{A}, \quad t \in \mathbb{R}$,
2. $J\mathfrak{A}J = \mathfrak{A}'$.

Remarks 2.1.2. We collect some elementary consequences of the Tomita-Takesaki theorem.

1. The second part of the preceding theorem gives a relationship between the algebraic and analytic structures. It states that the commutant of a von Neumann algebra is obtained by conjugation with analytic object like J, which is obtained from the polar decomposition of a conjugate linear closed operator (see [20]).

2. The modular conjugation $J : \mathscr{H} \to \mathscr{H}$ gives a *-anti-isomorphism between $\mathfrak{A}$ and its commutant $\mathfrak{A}'$ defined by

$$j(a) := JaJ. \tag{2.1.3}$$

3. The cyclic and separating vector Ω induces a faithful normal state ω on $\mathfrak{A}$ given by

$$\omega(a) := \frac{1}{\|\Omega\|^2} \langle \Omega, a\Omega \rangle, \quad a \in \mathfrak{A}.$$

Conversely, given any faithful normal state on $\mathfrak{A}$ there is, by the GNS-construction, a corresponding cyclic and separating vector in the GNS Hilbert space. Therefore, instead of taking the pair $(\mathfrak{A}, \Omega)$ as our initial mathematical inputs in setting up the Tomita-Takesaki theory, we could have equivalently started with the pair $(\mathfrak{A}, \omega)$ (This latter approach shall be explored further in the next section).

Moreover, the state ω (corresponding to Ω) is invariant under the automorphism group $\{\sigma_t : t \in \mathbb{R}\}$, i.e.

$$\omega(\sigma_t(a)) = \omega(a), \quad \forall a \in \mathfrak{A}, t \in \mathbb{R}.$$

In the remainder of this chapter, the outline above will be made very explicit and concrete for $\mathcal{B}(\mathscr{H})$.

2.2 The pre-Tomita operator

In this section we introduce the Tomita operator, as well as the conventions and some notation, that form the foundation of the rest of the chapter. We develop a simple version of Tomita-Takesaki theory for the pair $(\mathcal{A}, \mu)$, where $\mathcal{A} = \mathcal{B}(\mathscr{H})$ and μ is the faithful normal state given by Eq.(1.5.1), i.e. $\mu(a) = Tr(\rho a)$ but with ρ given by Eq.(1.5.2).

The approach we give in this section mimics the one given in [8, Section 2.5.2], but with less advanced tools. Our development here will not be complete even for this special case. There are some further points that one could explore, but more advanced tools from unbounded operators would be necessary (see [4], for a general framework which accommodates possible appearance of unbounded observables). However, we do give complete proofs for the results treated here.

We follow a somewhat unconventional approach to the technical details of the Tomita-Takesaki theory for $(\mathcal{A}, \mu)$ by using tensor products, which allows us to avoid much of the difficulties associated to unbounded operators and their domains.

Consider the faithful, cyclic representation (G, π, Ω) of $(\mathcal{A}, \mu)$ (see Sec. 1.8).

Proposition 2.2.1 (Pre-Tomita operator)**.** *Define an operator S_0 in G as follows:*

$$S_0 : G_0 \to G_0 : \pi(a)\Omega \mapsto \pi(a^*)\Omega \tag{2.2.1}$$

where $G_0 := \pi(\mathcal{A})\Omega$ is a dense subspace of G. Then S_0 is conjugate linear.

Proof. Let $\pi(a)\Omega, \pi(b)\Omega \in G_0$ and $\alpha \in \mathbb{C}$. Suppose that $\pi(a)\Omega = \pi(b)\Omega$, then

$$0 = [\pi(a) - \pi(b)]\,\Omega = \pi(a - b)\Omega,$$

so that $a = b$, since Ω is a separating vector. Hence S_0 assigns every input element in its domain a unique output, i.e. S_0 is well-defined. Also,

$$\begin{aligned} S_0 \left(\alpha\pi(a) + \pi(b)\right)\Omega &= S_0\pi(\alpha a + b)\Omega \\ &= \pi\left(\bar{\alpha}a^* + b^*\right)\Omega \\ &= \bar{\alpha}\pi(a^*)\Omega + \pi(b^*)\Omega \\ &= \bar{\alpha}S_0\pi(a)\Omega + S_0\pi(b)\Omega. \end{aligned}$$

Hence, S_0 is conjugate linear. □

Unbounded operators can be tricky to work with, in particular with regards to their domains. For this reason we introduce the following subspaces of $\mathcal{A}$.

Proposition 2.2.2. *Consider the following subspace of $\mathcal{A}$:*

$$E := \{a \in \mathcal{A} : \rho a^* \rho^{-1} \text{is bounded}\} \tag{2.2.2}$$

where $\rho^{-1} : \mathscr{H}_0 \to \mathscr{H}$ is as in Eq.(1.5.4), and $\mathscr{H}_0 := span\{e_1, e_2, e_3...\}$ with the $e_1, e_2, e_3...$ being the orthonormal basis elements of $\mathscr{H}$. Let $E^* := \{a^* : a \in E\}$. Then:

1. *$\mathscr{H}_0 \odot \mathscr{H}_0 \subset \pi(E)\Omega$ and $\mathscr{H}_0 \odot \mathscr{H}_0 \subset \pi(E^*)\Omega$.*
2. *$\pi(E)\Omega$ and $\pi(E^*)\Omega$ are dense subspaces of G.*

Proof. Clearly $0, 1_{\mathcal{A}} \in E$, and thus $E \neq \emptyset$. Consider the orthonormal basis $e_1, e_2, e_3...$ for $\mathscr{H}$. For all $j, k \in \mathbb{N}$ we have that

$$\begin{aligned} \left[\rho\left(|e_j\rangle\langle e_k|\right)^* \rho^{-1}\right] e_l &= \rho\,|e_k\rangle\langle e_j|\,\rho^{-1}e_l \\ &= \rho_l^{-1}\delta_{j,l}\rho e_k \\ &= \rho_j^{-1}\rho_k e_k \\ &= \rho_j^{-1}\rho_k\delta_{j,l}e_k \\ &= \left(\rho_j^{-1}\rho_k\,|e_k\rangle\langle e_j|\right)e_l, \end{aligned}$$

so that $\rho\left(|e_j\rangle\langle e_k|\right)^*\rho^{-1} = \rho_j^{-1}\rho_k\,|e_k\rangle\langle e_j|$ which implies $|e_j\rangle\langle e_k| \in E$. Then $|e_j\rangle\langle e_k| = \left(|e_k\rangle\langle e_j|\right)^* \in E^*$. Let $a, b \in E$ and $\alpha \in \mathbb{C}$. Then

$$\rho(\alpha a + b)^*\rho^{-1} = \rho(\bar{\alpha}a^* + b^*)\rho^{-1} = \bar{\alpha}\rho a^*\rho^{-1} + \rho b^*\rho^{-1},$$

so that $\alpha a + b \in E$, since the sum of bounded operators is itself bounded. Thus E is a vector subspace of $\mathcal{A}$. Moreover,

$$\rho(ab)^*\rho^{-1} = \rho b^* a^* \rho^{-1} = (\rho b^* \rho^{-1})(\rho a^* \rho^{-1}),$$

and since the composition of bounded operators is also bounded, it follows that $ab \in E$; thus E is a subalgebra of $\mathcal{A}$. Similarly, E^* is also a subalgebra of $\mathcal{A}$.

1. The set $\{e_j \otimes e_k : j,k \in \mathbb{N}\}$ is an orthonormal basis for $\mathscr{H}_0 \odot \mathscr{H}_0$. We first observe that for any $j,k \in \mathbb{N}$

$$\begin{aligned}
\pi\left(|e_j\rangle\langle e_k\rangle\right)\Omega &= \left(|e_j\rangle\langle e_k| \otimes 1\right)\left(\sum_{l=1}^{\infty} \rho_l^{1/2} e_l \otimes e_l\right) \\
&= \sum_{l=1}^{\infty} \delta_{kl}\, \rho_l^{1/2} e_j \otimes e_l \\
&= \rho_k^{1/2} e_j \otimes e_k.
\end{aligned}$$

So,

$$e_j \otimes e_k = \pi\left(\rho_k^{-1/2}\,|e_j\rangle\langle e_k|\right)\Omega \in \pi(E)\Omega, \tag{2.2.3}$$

hence $\mathscr{H}_0 \odot \mathscr{H}_0 \subset \pi(E)\Omega$. Similarly, $\mathscr{H}_0 \odot \mathscr{H}_0 \subset \pi(E^*)\Omega$.

2. Let $a, b \in E$ and $\alpha \in \mathbb{C}$. Then $\alpha\pi(a)\Omega + \pi(b)\Omega = \pi(\alpha a + b)\Omega$, since π is a *-homomorphism. It then follows that

$$\alpha\pi(a)\Omega + \pi(b)\Omega \in \pi(E)\Omega,$$

since $\alpha a + b \in E$. Hence $\pi(E)\Omega$ is a vector subspace of G.

Since $\mathscr{H}_0 \odot \mathscr{H}_0$ is a dense subset of G, and $\mathscr{H}_0 \odot \mathscr{H}_0 \subset \pi(E)\Omega$, it follows that $\pi(E)\Omega$ is dense in G. Analogously, $\pi(E^*)\Omega$ is dense in G.

□

We determine the adjoint S_0^* of the (pre-Tomita) operator S_0, although we will not fully determine it on its entire domain, but on a smaller set $\mathscr{H}_0 \odot \mathscr{H}_0$. This will be sufficient for our purpose.

Since S_0 is a densely defined conjugate linear operator, its adjoint S_0^* is also conjugate linear and has to satisfy the relation

$$\langle x, S_0^* y\rangle = \langle y, S_0 x\rangle = \overline{\langle S_0 x, y\rangle} \tag{2.2.4}$$

$\forall x \in \mathscr{D}(S_0), y \in \mathscr{D}(S_0^*)$.

Proposition 2.2.3. *We have $\pi(E)\Omega \subset \mathscr{D}(S_0^*)$, and for all $a \in E$, we have that*

$$S_0^*\pi(a)\Omega = \pi(\rho a^* \rho^{-1})\Omega.$$

Proof. Let $a \in E$ and $b \in \mathcal{A}$, then

$$\begin{aligned}
\langle \pi(b)\Omega, S_0^*\pi(a)\Omega\rangle &= \langle \pi(a)\Omega, S_0\pi(b)\Omega\rangle \\
&= \langle \pi(a)\Omega, \pi(b^*)\Omega\rangle \\
&= \sum_{j,k=1}^{\infty} \rho_j^{1/2}\rho_k^{1/2} \langle ae_j \otimes e_j, b^*e_k \otimes e_k\rangle \\
&= \sum_{j,k=1}^{\infty} \rho_j^{1/2}\rho_k^{1/2} \langle ae_j, b^*e_k\rangle \delta_{j,k} \\
&= \sum_{j=1}^{\infty} \rho_j \langle ae_j, b^*e_j\rangle \\
&= \sum_{j=1}^{\infty} \langle e_j, a^*b^*\rho e_j\rangle \\
&= Tr\,(a^*b^*\rho) \\
&= Tr\left(b^*(\rho a^*\rho^{-1})\rho\right) \\
&= \sum_{j=1}^{\infty} \rho_j \left\langle be_j, (\rho a^*\rho^{-1})e_j\right\rangle \\
&= \sum_{j,k=1}^{\infty} \delta_{j,k}\rho_j^{1/2}\rho_k^{1/2} \left\langle be_j, \rho a^*\rho^{-1}e_k\right\rangle \\
&= \sum_{j,k=1}^{\infty} \rho_j^{1/2}\rho_k^{1/2} \left\langle be_j \otimes e_j, (\rho a^*\rho^{-1}e_k) \otimes e_k\right\rangle \\
&= \left\langle (b \otimes 1)\Omega, (\rho a^*\rho^{-1} \otimes 1)\Omega\right\rangle \\
&= \left\langle \pi(b)\Omega, \pi(\rho a^*\rho^{-1})\Omega\right\rangle .
\end{aligned}$$

Thus $S_0\pi(a)\Omega = \pi(\rho a^*\rho^{-1})\Omega$, since $\mathscr{D}(S_0) = \pi(\mathcal{A})\Omega$ is dense in G. □

The operator S_0^* has been determined on the dense subspace $\pi(E)\Omega$ of G, and this explains the introduction of the subalgebra E of $\mathcal{A}$ in Prop.2.2.2.

2.3 The modular operator and modular group

The first of the two operators that are determined from S_0 is:

Definition 2.3.1 (Pre-modular operator)**.** Define an operator Δ_0 on $\pi(E^*)\Omega$ as follows:

$$\Delta_0 := S_0^*S_0 \mid_{\pi(E^*)\Omega} . \tag{2.3.1}$$

That is, $\Delta_0 : \pi(E^*)\Omega \to \pi(\mathcal{A})\Omega : \pi(a)\Omega \mapsto \pi(\rho a\rho^{-1})\Omega$.

Since the composition of two conjugate linear maps is linear, Δ_0 is linear. Although the operator Δ_0 is densely-defined on the subspace $\pi(E^*)\Omega$ of G, but for simplicity since $\mathscr{H}_0 \odot \mathscr{H}_0 \subset \pi(E^*)\Omega$, we restrict the domain of the definition of Δ_0 to $\mathscr{H}_0 \odot \mathscr{H}_0$, i.e. we set

$$\mathscr{D}(\Delta_0) := \mathscr{H}_0 \odot \mathscr{H}_0. \tag{2.3.2}$$

This is sufficient for our goals. In a similar way we restrict S_0^* to $\mathscr{H}_0 \odot \mathscr{H}_0$ (although in general the domain of S_0^* is larger), since $\mathscr{H}_0 \odot \mathscr{H}_0 \subset \pi(E)\Omega$ is dense in G.

The main result regarding Δ_0 is as follows:

Theorem 2.3.2. *The operator Δ_0 satisfies the relation:*

$$\Delta_0 \mid_{\mathscr{H}_0 \odot \mathscr{H}_0} = \rho \odot \rho^{-1} \mid_{\mathscr{H}_0 \odot \mathscr{H}_0} .$$

Here we write $\rho \odot \rho^{-1}$ instead of $\rho \otimes \rho^{-1}$, since it is only defined on $\mathscr{H}_0 \odot \mathscr{H}_0$, and reserve the symbol $\otimes$ for the extension of an elementary tensor product to the completed space $\mathscr{H} \otimes \mathscr{H}$.

Proof. Let $a \in E^*$, then

$$\begin{aligned}
\Delta_0 \pi(a)\Omega &= S_0^* S_0 \pi(a)\Omega \\
&= S_0^* \pi(a^*)\Omega \\
&= \pi(\rho a \rho^{-1})\Omega \\
&= \left[\rho a \rho^{-1} \otimes 1\right] \left(\sum_j \rho_j^{1/2} e_j \otimes e_j \right) \\
&= \sum_{j=1}^{\infty} \rho_j^{1/2} (\rho a \rho^{-1} e_j) \otimes e_j \\
&= \sum_{j=1}^{\infty} \rho_j^{1/2} \rho_j^{-1} (\rho a e_j) \otimes e_j \\
&= \sum_{j=1}^{\infty} \rho_j^{1/2} (\rho a e_j) \otimes (\rho^{-1} e_j) \\
&= \sum_{j=1}^{\infty} \rho_j^{1/2} (\rho a \otimes \rho^{-1})(e_j \otimes e_j) \\
&= (\rho \odot \rho^{-1})(a \otimes 1) \left(\sum_j \rho_j^{1/2} e_j \otimes e_j \right) \\
&= (\rho \odot \rho^{-1}) \pi(a)\Omega.
\end{aligned}$$

Thus $\Delta_0 \mid_{\mathscr{H}_0 \odot \mathscr{H}_0} = \rho \odot \rho^{-1} \mid_{\mathscr{H}_0 \odot \mathscr{H}_0}$, since $\mathscr{H}_0 \odot \mathscr{H}_0 \subset \pi(E^*)\Omega$. $\square$

Proposition 2.3.3. *The densely defined linear operator* Δ_0 *is symmetric, i.e.*

$$\langle \Delta_0 x, y\rangle = \langle x, \Delta_0 y\rangle$$

$\forall x, y \in \mathscr{D}(\Delta_0)$.

Proof. For a basis element $e_j \otimes e_k$ of G, we have

$$\begin{aligned}
\langle \Delta_0(e_j \otimes e_k), e_j \otimes e_k\rangle &= \left\langle \rho_j \rho_k^{-1} e_j \otimes e_k, e_j \otimes e_k \right\rangle \\
&= \rho_j \rho_k^{-1} \langle e_j \otimes e_k, e_j \otimes e_k\rangle \\
&= \rho_j \rho_k^{-1} \langle e_j, e_j\rangle \langle e_k, e_k\rangle \\
&= \rho_j \rho_k^{-1}
\end{aligned}$$

which is real. In particular, $\langle \Delta_0 x, x\rangle \in \mathbb{R}$ for all $x \in G$. By the remark following Def. 1.1.7, Δ_0 is symmetric. $\square$

Remarks 2.3.4 (The functional calculus of Δ_0)**.** Note that Δ_0 defined above satisfies

$$\Delta_0(e_j \otimes e_k) = \left(\rho \odot \rho^{-1}\right)(e_j \otimes e_k) = \rho_j \rho_k^{-1} e_j \otimes e_k$$

$\forall j, k \in \mathbb{N}$. Since the set $\{e_j \otimes e_k\}_{j,k\in\mathbb{N}}$ is an orthonormal basis for G, $\Delta_0(e_j \otimes e_k) = \rho_j \rho_k^{-1} e_j \otimes e_k$ can be extended in the sense that we can define

$$f(\Delta_0)(e_j \otimes e_k) := f(\rho_j \rho_k^{-1}) e_j \otimes e_k$$

for any function $f : \mathbb{R} \to \mathbb{C}$. In particular, we define

$$\Delta_0^{iz}(e_j \otimes e_k) := \rho_j^{iz} \rho_k^{-iz} e_j \otimes e_k \tag{2.3.3}$$

$\forall j, k \in \mathbb{N}$, $z \in \mathbb{C}$. This is the so-called *functional calculus* for Δ_0 (but simplified to avoid the closure of the unbounded operator Δ_0).

In general one would take the closure of an operator and show that it is self-adjoint to be able to have a functional calculus, but here the situation is simple enough to define the functional calculus more directly.

Proposition 2.3.5 (One-parameter unitary group)**.** *The following statements hold:*

1. *Equation (2.3.3) extends uniquely to a unitary operator*

   $$\Delta^{it} \in \mathcal{B}(G)$$

   $\forall t \in \mathbb{R}$.

2. *The family* $\{\Delta^{it} : t \in \mathbb{R}\}$ *forms a one-parameter unitary group, i.e.*

   $$\Delta^{it_1}\Delta^{it_2} = \Delta^{i(t_1+t_2)}$$

   $\forall t_1, t_2 \in \mathbb{R}$, *with* $\Delta^{i0} = 1_{\mathcal{B}(G)}$ *(the identity operator on* G*).*

3. *For all $t \in \mathbb{R}$, $\Delta^{it}\Omega = \Omega$.*

We write Δ^{it} instead of Δ_0^{it}, since it is defined on the whole of G. In the more complete theory this is the notation used, however there an operator Δ is defined as the closure of Δ_0, although that Δ is still not defined on the whole of G.

Proof of Prop.2.3.5. Let $\{e_j : j \in \mathbb{N}\}$ be an orthonormal basis for $\mathscr{H}$.

1. Let $t \in \mathbb{R}$. Then

$$\begin{aligned} \left\langle \Delta_0^{it}(e_j \otimes e_k), \Delta_0^{it}(e_l \otimes e_m) \right\rangle &= \left\langle \left(\rho^{it} \odot \rho^{-it}\right) e_j \otimes e_k, \left(\rho^{it} \odot \rho^{-it}\right) e_l \otimes e_m \right\rangle \\ &= \left\langle \left(\rho^{-it} \odot \rho^{it}\right) \left(\rho^{it} \odot \rho^{-it}\right) e_j \otimes e_k, e_l \otimes e_m \right\rangle \\ &= \left\langle e_j \otimes e_k, e_l \otimes e_m \right\rangle , \end{aligned}$$

so, by linearity, $\left\langle \Delta_0^{it}x, \Delta_0^{it}y \right\rangle = \langle x, y\rangle$ for all $x, y \in \mathscr{H}_0 \odot \mathscr{H}_0$. In particular, $\|\Delta_0^{it}x\| = \|x\|$ for all $x \in \mathscr{H}_0 \odot \mathscr{H}_0$. Thus $\|\Delta_0^{it}\| = 1$, so Δ_0^{it} extends uniquely to G, since $\mathscr{H}_0 \odot \mathscr{H}_0$ is dense in G. Call this extension Δ^{it}.

We also have that $\left(\Delta^{it}\right)^* = \Delta^{-it}$, since

$$\begin{aligned} \left\langle e_j \otimes e_k, \left(\Delta^{it}\right)^* (e_j \otimes e_k) \right\rangle &= \left\langle \Delta^{it}(e_j \otimes e_k), e_j \otimes e_k \right\rangle \\ &= \left\langle \rho_j^{it}\rho_k^{-it} e_j \otimes e_k, e_j \otimes e_k \right\rangle \\ &= \left\langle e_j \otimes e_k, \rho_j^{-it}\rho_k^{it} e_j \otimes e_k \right\rangle \\ &= \left\langle e_j \otimes e_k, \Delta^{-it}(e_j \otimes e_k) \right\rangle . \end{aligned}$$

It follows that $\left(\Delta^{it}\right)^* = \Delta^{-it}$. Then

$$\begin{aligned} \Delta^{it}\left(\Delta^{it}\right)^* (e_j \otimes e_k) &= \Delta^{it}\Delta^{-it}(e_j \otimes e_k) \\ &= \Delta^{it}\left(\rho_j^{-it}\rho_k^{it} e_j \otimes e_k\right) \\ &= \rho_j^{it}\rho_k^{-it}\left(\rho_j^{-it}\rho_k^{it} e_j \otimes e_k\right) \\ &= e_j \otimes e_k, \end{aligned}$$

thus $\Delta^{it}\left(\Delta^{it}\right)^* = 1_{\mathcal{B}(G)}$, and similarly $\left(\Delta^{it}\right)^* \Delta^{it} = 1_{\mathcal{B}(G)}$. Hence Δ^{it} is unitary for each $t \in \mathbb{R}$; hence bounded.

2. Suppose $t = 0$ then $\Delta^0 = 1_{\mathcal{B}(G)}$ since

$$\Delta^0(e_j \otimes e_k) = \rho_j^0\rho_k^0 e_j \otimes e_k = e_j \otimes e_k.$$

Let $t_1, t_2 \in \mathbb{R}$, then

$$\begin{aligned}\Delta^{it_1}\Delta^{it_2}(e_j \otimes e_k) &= \Delta^{it_1}\left(\rho_j^{it_2}\rho_k^{-it_2} e_j \otimes e_k\right) \\ &= \rho_j^{it_2}\rho_k^{-it_2}\left(\rho_j^{it_1}\rho_k^{-it_1} e_j \otimes e_k\right) \\ &= \rho_j^{i(t_1+t_2)}\rho_k^{-i(t_1+t_2)} e_j \otimes e_k \\ &= \Delta^{i(t_1+t_2)}(e_j \otimes e_k).\end{aligned}$$

Hence $\Delta^{it_1}\Delta^{it_2} = \Delta^{i(t_1+t_2)}$.

3. For all $t \in \mathbb{R}$, we have

$$\begin{aligned}\Delta^{it}\Omega &= \sum_{j=1}^{\infty} \rho_j^{1/2}\Delta^{it}(e_j \otimes e_j) \\ &= \sum_{j=1}^{\infty} \rho_j^{1/2}\rho_j^{it}\rho_j^{-it} e_j \otimes e_j \\ &= \Omega.\end{aligned}$$

□

The unitaries $\{\Delta^{it} : t \in \mathbb{R}\}$ induce a one-parameter automorphism group:

Proposition 2.3.6 (Modular automorphism group)**.** *The following statements hold:*

1. *For each $t \in \mathbb{R}$, the equation*

$$\sigma_t(\pi(a)) = \sigma_t^{\Omega}(\pi(a)) = \Delta^{it}\pi(a)\Delta^{-it} \tag{2.3.4}$$

*defines a *-automorphism σ_t on $\pi(\mathcal{A})$.*

2. *The family $\{\sigma_t^{\Omega} : t \in \mathbb{R}\}$ forms a one-parameter automorphism group called the modular automorphism group associated with $(\pi(\mathcal{A}), \Omega)$.*

3. *For all $t \in \mathbb{R}$,*

$$\langle \Omega, \sigma_t(\pi(a))\Omega\rangle = \langle \Omega, \pi(a)\Omega\rangle = \mu(a).$$

Proof. Let $a, b \in \mathcal{A}$ and $\alpha \in \mathbb{R}$.

1. For all $t \in \mathbb{R}$, we have

$$\begin{aligned}\sigma_t\left(\alpha\pi(a) + \pi(b)\right) &= \sigma_t\left(\pi(\alpha a + b)\right) \\ &= \Delta^{it}\pi(\alpha a + b)\Delta^{-it} \\ &= \alpha\Delta^{it}\pi(a)\Delta^{-it} + \Delta^{it}\pi(b)\Delta^{-it} \\ &= \alpha\sigma_t(\pi(a)) + \sigma_t(\pi(b)),\end{aligned}$$

so that σ_t linear. Now, if $\sigma_t(\pi(a)) = \sigma_t(\pi(b))$ then

$$\Delta^{it}\pi(a)\Delta^{-it} = \Delta^{it}\pi(b)\Delta^{-it}$$

which implies $\pi(a) = \pi(b)$, so that $a = b$; and $\pi(a) = \sigma_t\left(\sigma_{-t}(\pi(a))\right)$. Thus for all $t \in \mathbb{R}$, σ_t is bijective.

Homomorphism property:

$$\begin{aligned}
\sigma_t\left(\pi(a)\pi(b)\right) &= \sigma_t(\pi(ab)) \\
&= \left(\Delta^{it}\pi(a)\Delta^{-it}\right)\left(\Delta^{it}\pi(b)\Delta^{-it}\right) \\
&= \sigma_t(\pi(a))\sigma_t(\pi(b)).
\end{aligned}$$

Also, since $\left(\Delta^{it}\right)^* = \Delta^{-it}$,

$$\begin{aligned}
\sigma_t\left(\pi(a)^*\right) &= \Delta^{it}\pi(a)^*\Delta^{-it} \\
&= \left(\Delta^{it}\pi(a)\Delta^{-it}\right)^* \\
&= \sigma_t(\pi(a))^*.
\end{aligned}$$

Hence σ_t is a *-automorphism for all $t \in \mathbb{R}$.

2. Suppose $t = 0$ then $\sigma_0 = id_{\mathcal{B}(G)}$ (where $id_{\mathcal{B}(G)}$ is the identity map on $\mathcal{B}(G)$), since
$$\sigma_0(\pi(a)) = \Delta^0\pi(a)\Delta^0 = \pi(a).$$
Let $t_1, t_2 \in \mathbb{R}$, then
$$\begin{aligned}
\sigma_{t_1}\left(\sigma_{t_2}(\pi(a))\right) &= \Delta^{it_1}\sigma_{t_2}(\pi(a))\Delta^{-it_1} \\
&= \Delta^{it_1}\left(\Delta^{it_2}\pi(a)\Delta^{-it_2}\right)\Delta^{-it_1} \\
&= \Delta^{i(t_1+t_2)}\pi(a)\Delta^{-i(t_1+t_2)} \\
&= \sigma_{t_1+t_2}(\pi(a)),
\end{aligned}$$
and hence $\sigma_{t_1}\sigma_{t_2} = \sigma_{t_1+t_2}$. Then $\{\sigma_t : t \in \mathbb{R}\}$ forms a one-parameter automorphism group.

3. From Prop.2.3.5(3) and Prop.1.8.4(2), we have that
$$\begin{aligned}
\langle \Omega, \sigma_t\left(\pi(a)\right)\Omega\rangle &= \left\langle \Delta^{-it}\Omega, \pi(a)\Delta^{-it}\Omega\right\rangle \\
&= \langle \Omega, \pi(a)\Omega\rangle \\
&= \mu(a),
\end{aligned}$$
for all $t \in \mathbb{R}$.

$\square$

In Prop.2.3.6, the modular automorphism group $\{\sigma_t^{\Omega} : t \in \mathbb{R}\}$ was given in the cyclic representation $(\pi(\mathcal{A}), \Omega)$ of $(\mathcal{A}, \mu)$. However, it is possible to define the modular automorphism group directly on $\mathcal{A}$ as well (cf. [8, p.96]). In the latter case, the *-automorphisms $\sigma_t^{\mu} : \mathcal{A} \to \mathcal{A}$ for $t \in \mathbb{R}$, are defined by

$$\sigma_t^{\mu}(a) = \pi^{-1}\left(\Delta^{it}\pi(a)\Delta^{-it}\right),$$

i.e.

$$\sigma_t^{\mu} = \pi^{-1} \circ \sigma_t \circ \pi,$$

and the resulting one-parameter automorphism group $\{\sigma_t^{\mu} : t \in \mathbb{R}\}$ is accordingly called the *modular automorphism group associated with* $(\mathcal{A}, \mu)$.

Theorem 2.3.7. *The following statements hold:*

1. *For each $t \in \mathbb{R}$, the equation*

$$\sigma_t^{\mu}(a) = \pi^{-1} \circ \sigma_t \circ \pi$$

*defines a *-automorphism of $\mathcal{A}$.*

2. *The family $\{\sigma_t^{\mu}\}$ forms a one-parameter automorphism group.*

3. *For each $t \in \mathbb{R}$, $\mu \circ \sigma_t^{\mu} = \mu$.*

Proof. Let $a \in \mathcal{A}$.

1. From Prop.s 1.8.3 and 1.8.4(3), the maps

$$\pi : \mathcal{A} \to \mathcal{A} \otimes 1 \text{ and } \pi^{-1} : \mathcal{A} \otimes 1 \to \mathcal{A}$$

are *-isomorphisms; and from Prop.2.3.6 we have that for all $t \in \mathbb{R}$

$$\sigma_t : \mathcal{A} \otimes 1 \to \mathcal{A} \otimes 1$$

is a *-isomorphism. Since the composition of two (or countably finite) *-isomorphisms is itself a *-isomorphism, it follows that $\pi^{-1} \circ \sigma_t \circ \pi$ is a *-isomorphism. Hence σ_t^{μ} is a *-automorphism for each $t \in \mathbb{R}$.

2. Group properties: Suppose $t = 0$, then $\sigma_0^{\mu} = id_{\mathcal{A}}$ since

$$\sigma_0^{\mu}(a) = \pi^{-1}\left(\Delta^0\pi(a)\Delta^0\right) = \pi^{-1}(\pi(a)) = a.$$

Let $t_1, t_2 \in \mathbb{R}$, then

$$\begin{aligned}
\left[\sigma_{t_1}^{\mu} \circ \sigma_{t_2}^{\mu}\right](a) &= \sigma_{t_1}\left(\sigma_{t_2}(a)\right) \\
&= \sigma_{t_1}\left(\pi^{-1}\left(\Delta^{it_2}\pi(a)\Delta^{-it_2}\right)\right) \\
&= \pi^{-1}\left(\Delta^{it_1}\left(\Delta^{it_2}\pi(a)\Delta^{-it_2}\right)\Delta^{-it_1}\right) \\
&= \pi^{-1}\left(\Delta^{i(t_1+t_2)}\pi(a)\Delta^{-i(t_1+t_2)}\right) \\
&= \sigma_{t_1+t_2}^{\mu}(a),
\end{aligned}$$

thus $\sigma_{t_1+t_2}^{\mu} = \sigma_{t_1}^{\mu} \circ \sigma_{t_2}^{\mu}$. Hence the family of automorphisms $\{\sigma_t^{\mu} : t \in \mathbb{R}\}$ forms a one-parameter automorphism group.

3. By Prop.2.3.6(3) it follows that for all $t \in \mathbb{R}$,

$$\begin{aligned}
[\mu \circ \sigma_t^\mu](a) &= \langle \Omega, \pi(\sigma_t^\mu(a))\Omega \rangle \\
&= \left\langle \Omega, \pi\left(\pi^{-1}(\Delta^{it}\pi(a)\Delta^{-it})\right)\Omega \right\rangle \\
&= \left\langle \Omega, \left(\Delta^{it}\pi(a)\Delta^{-it}\right)\Omega \right\rangle \\
&= \langle \Omega, \sigma_t(\pi(a))\Omega \rangle \\
&= \langle \Omega, \pi(a)\Omega \rangle \\
&= \mu(a).
\end{aligned}$$

Thus the state μ is invariant under the modular automorphism group $\{\sigma_t^\mu : t \in \mathbb{R}\}$.

□

Thus we have obtained the first part of Tomita-Takesaki theorem (see, Theorem 2.1.1(1)). The modular automorphism group is probably the single most important object appearing in Tomita-Takesaki theory [28, p.2].

2.4 The modular conjugation

The second of the two operators we determined from S_0 is:

Definition 2.4.1 (Pre-modular conjugation)**.** Define an operator J_0 as follows:

$$J_0 := \Delta_0^{1/2} S_0 \tag{2.4.1}$$

where

$$\Delta_0^{1/2} : \pi(F^*)\Omega \to \pi(\mathcal{A})\Omega : \pi(a)\Omega \mapsto \pi(\rho^{1/2} a \rho^{-1/2})\Omega$$

with

$$F := \{a \in \mathcal{A} : \rho^{1/2} a^* \rho^{-1/2} \text{ is bounded}\},$$

and $\rho^{-1/2}$ is given as in Rem.1.5.8.

Proposition 2.4.2. *The domain of J_0 as given in Eq.(2.4.1) is $\pi(F)\Omega$, i.e.*

$$\mathscr{D}(J_0) = \pi(F)\Omega,$$

and $\mathscr{D}(J_0)$ is a dense subspace of G containing $\mathscr{H}_0 \odot \mathscr{H}_0$.

Proof. We are required to find all the elements x of the domain of S_0 such that $S_0 x$ is in the domain $\pi(F^*)\Omega$ of $\Delta_0^{1/2}$.

Suppose $S_0\pi(a)\Omega = 0$. Then by definition $\pi(a^*)\Omega = 0$, so that $a = 0$ (by Prop.1.8.4) and $\ker(S_0) = \{0\}$. Thus S_0 is injective, and its inverse S_0^{-1} exists. Furthermore, since

$$S_0^{-1} S_0 \pi(a)\Omega = S_0^{-1}\pi(a^*)\Omega = \pi(a)\Omega,$$

we have that $S_0^{-1} = S_0$. Then

$$\begin{aligned} \mathscr{D}(J_0) &= S_0^{-1}\left(\mathscr{R}(S_0) \cap \mathscr{D}\left(\Delta_0^{1/2}\right)\right) \\ &= S_0^{-1}\left(\pi(\mathcal{A})\Omega \cap \pi(F^*)\Omega\right) \\ &= S_0^{-1}\left(\pi(F^*)\Omega\right) \\ &= \pi(F)\Omega. \end{aligned}$$

The denseness of $\mathscr{D}(J_0)$ follows from an argument similar to the proof of Prop.2.2.2, but with $\rho^{1/2}$ and $\rho^{-1/2}$ in place of ρ and ρ^{-1}. □

Proposition 2.4.3 (Properties of J_0). *The following statements hold:*

1. *The densely defined operator J_0 satisfies*

$$J_0\pi(a)\Omega = \pi\left(\rho^{1/2}a^*\rho^{-1/2}\right)\Omega \tag{2.4.2}$$

   for all $a \in F$.

2. *J_0 is conjugate linear.*

3. *The action of J_0 on a basis element $e_j \otimes e_k$ of G is given by*

$$J_0(e_j \otimes e_k) = e_k \otimes e_j$$

   for all $j, k \in \mathbb{N}$; hence J_0 is isometric.

4. *J_0 extends to an isometric conjugate linear map defined on the entire G.*

Proof. Let $\{e_j \otimes e_k : j, k \in \mathbb{N}\}$ be an orthonormal basis for G.

1. Let $a \in F$, then

$$\begin{aligned} J_0\pi(a)\Omega &= \left[\Delta_0^{1/2}S_0\right]\pi(a)\Omega \\ &= \Delta_0^{1/2}\pi(a^*)\Omega \\ &= \pi\left(\rho^{1/2}a^*\rho^{-1/2}\right)\Omega. \end{aligned}$$

2. Since S_0 is conjugate linear and $\Delta_0^{1/2}$ is linear (because Δ_0 is linear), and the composition of a linear map with a conjugate linear map is conjugate linear; it then follows that J_0 is conjugate linear.

3. From Eq.(2.2.3) we have that

$$e_j \otimes e_k = \pi\left(\rho_k^{-1/2}\left|e_j\right\rangle\left\langle e_k\right|\right)\Omega, \quad \forall j, k \in \mathbb{N},$$

so that

$$\begin{aligned}
J_0(e_j \otimes e_k) &= J_0\pi\left(\rho_k^{-1/2}\left|e_j\right\rangle\left\langle e_k\right|\right)\Omega \\
&= \pi\left(\rho^{1/2}\left(\rho_k^{-1/2}\left|e_j\right\rangle\left\langle e_k\right|\right)^*\rho^{-1/2}\right)\Omega \\
&= \rho_k^{-1/2}\left[\left(\rho^{1/2}\left|e_k\right\rangle\left\langle e_j\right|\rho^{-1/2}\right)\otimes 1\right]\left(\sum_{l=1}^{\infty}\rho_l^{1/2}e_l\otimes e_l\right) \\
&= \sum_{l=1}^{\infty}\rho_k^{-1/2}\rho_l^{1/2}\left(\rho^{1/2}\left|e_k\right\rangle\left\langle e_j\right|\rho_l^{-1/2}e_l\right)\otimes e_l \\
&= \sum_{l=1}^{\infty}\delta_{j,l}\ \rho_k^{-1/2}\left(\rho_k^{1/2}e_k\right)\otimes e_l \\
&= e_k\otimes e_j.
\end{aligned}$$

Thus $J_0(e_j \otimes e_k) = e_k \otimes e_j$. Consider any $x = \sum_{j,k}\alpha_{j,k}e_j \otimes e_k$ in $\mathscr{H}_0 \otimes \mathscr{H}_0$ (so only finitely many $\alpha_{j,k} \in \mathbb{C}\backslash\{0\}$), then

$$\begin{aligned}
\|J_0x\|^2 &= \left\|\sum_{j,k}\bar{\alpha}_{j,k}J_0(e_j\otimes e_k)\right\|^2 \\
&= \sum_{j,k}|\alpha_{j,k}|^2\|e_k\otimes e_j\|^2 \\
&= \sum_{j,k}|\alpha_{j,k}|^2 \\
&= \|x\|^2.
\end{aligned}$$

Thus J_0 is isometric and bounded on $\mathscr{H}_0$.

4. Since the densely defined isometric conjugate linear operator J_0 is bounded on $\mathscr{H}_0$, it can be extended to a uniquely determined conjugate linear operator defined on the entire G which is itself an isometry.

□

Definition 2.4.4. We denote the conjugate linear extension of J_0 by

$$J \equiv J_\Omega : G \to G,$$

and it is referred to as the *modular conjugation* associated with $(\pi(\mathcal{A}), \Omega)$.

Theorem 2.4.5. *The following statements hold:*

1. *The modular conjugation J satisfies:*

$$J\pi(a)^*J(e_j \otimes e_k) = \left(1 \otimes a^T\right)(e_j \otimes e_k)$$

where a^T is the transpose of $a \in \mathcal{A}$ with respect to the basis $\{e_j : j \in \mathbb{N}\}$.

2. *For all* $x, y \in \mathscr{H}$,
$$J(x \otimes y) = y^* \otimes x^*,$$
where $z^* := \sum_{j\in\mathbb{N}} \bar{z}_j e_j$ *for all* $z = \sum_{j\in\mathbb{N}} z_j e_j$.

3. *For all* $a \in \mathcal{A}$,
$$J\pi(a^*)J = \pi'(a^T),$$
where π' *is the injective *-homomorphism given in Eq.(1.8.2), namely* $\pi' : \mathcal{A} \to \mathcal{B}(G) : a \mapsto 1 \otimes a$.

4. *For all* $a \in \mathcal{A}$, $J\pi(\mathcal{A})J = \pi'(\mathcal{A})$, *i.e.*
$$J(\mathcal{A} \otimes 1)J = 1 \otimes \mathcal{A}$$
where $\mathcal{A} \otimes 1 := \{a \otimes 1 : a \in \mathcal{A}\}$ *and* $1 \otimes \mathcal{A}$ *is similarly defined.*

Proof. Let $\{e_j \otimes e_k : j, k \in \mathbb{N}\}$ be an orthonormal basis for G.

1. For all $j, k \in \mathbb{N}$,
$$\begin{aligned}
[J\pi(a)^*J](e_j \otimes e_k) &= [J\pi(a^*)](e_k \otimes e_j) \\
&= J(a^*e_k \otimes e_j) \\
&= J\left(\sum_{l=1}^{\infty} (a^*)_{l,k}\, e_l \otimes e_j\right) \\
&= J\left(\sum_{l=1}^{\infty} \overline{(a^T)}_{l,k} e_l \otimes e_j\right) \\
&= \sum_{l=1}^{\infty} (a^T)_{l,k}\, e_j \otimes e_l \\
&= e_j \otimes \left(\sum_{l=1}^{\infty} (a^T)_{l,k}\, e_l\right) \\
&= e_j \otimes (a^T e_k) \\
&= (1 \otimes a^T)(e_j \otimes e_k).
\end{aligned}$$

2. For $x, y \in \mathscr{H}$, say $x = \sum_{j=1}^{\infty} x_j e_j$ and $y = \sum_{k=1}^{\infty} y_k e_k$, it follows by Prop.2.4.3 that
$$\begin{aligned}
J(x \otimes y) &= J\left(\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} x_j y_k e_j \otimes e_k\right) \\
&= \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \bar{x}_j \bar{y}_k e_k \otimes e_j \\
&= y^* \otimes x^*.
\end{aligned}$$

So that

$$\|J(x \otimes y)\| = \|y^* \otimes x^*\| = \|y^*\|\|x^*\| = \|x\|\|y\|.$$

It then follows that J is jointly continuous with respect to x and y. Thus $J(x \otimes y) = y^* \otimes x^*$ $\forall x, y \in \mathscr{H}$.

3. Together, Eq.(1.8.2) and part (1) of this theorem, give

$$J\pi(a)^* J(e_j \otimes e_k) = \pi'\left(a^T\right)(e_j \otimes e_k)$$

for all $j, k \in \mathbb{N}$, and $a \in \mathcal{A}$. From this the required result then follows.

4. By part (3) of this theorem, we have

$$J\pi(a)J \quad = \quad J\pi(a^*)^* J = \pi'\left((a^*)^T\right),$$

for all $a \in \mathcal{A}$, it then follows that $J\pi(\mathcal{A})J = \pi'(\mathcal{A})$. Consequently $J(\mathcal{A} \otimes 1)J = 1 \otimes \mathcal{A}$.

□

Note that Thm.2.4.5(4) gives the second part of the Tomita-Takesaki theorem for $\mathcal{B}(\mathscr{H})$ (see Thm.2.1.1(2)).

Remarks 2.4.6. The following notation is also used

$$j\left(\pi(a)\right) := J\pi(a)^* J, \quad a \in \mathcal{A}.$$

Then

$$j\left(\pi(a)\right) = \pi'\left(a^T\right), \quad \forall a \in \mathcal{A}.$$

Note that the above result implies that $j(\pi(a))$ commutes with $\pi(a)$; since

$$\begin{aligned} j(\pi(a))\pi(a) &= \pi'(a^T)\pi(a) \\ &= \left(1 \otimes a^T\right)(a \otimes 1) \\ &= a \otimes a^T \\ &= (a \otimes 1)\left(1 \otimes a^T\right) \\ &= \pi(a)\pi'(a^T) \\ &= \pi(a)j(\pi(a)). \end{aligned}$$

Moreover,

$$j(\pi(\mathcal{A})) = \pi'(\mathcal{A}).$$

In a sense $j(\pi(a))$ is a mirror image of $\pi(\mathcal{A})$, i.e. $1 \otimes \mathcal{A}$ is a mirror image of $\mathcal{A} \otimes 1$, with the two copies commuting with each other.

Proposition 2.4.7. *The modular conjugation J has the following properties:*

$$J^* = J \text{ and } J^2 = 1.$$

Hence J is an anti-unitary operator.

Proof. Since J is an isometry (by Prop.2.4.3(4)) and the functional $G \times G \to \mathbb{C} : (x, y) \mapsto \langle Jx, Jy\rangle$ is sesquilinear (linear in x and conjugate linear in y), so by the Polarization identity we have

$$\langle Jx, Jy\rangle = \langle x, y\rangle \ \ \forall x, y \in G.$$

So $J^*J = 1$. We also have

$$J^2(e_j \otimes e_k) = J(e_k \otimes e_j) = e_j \otimes e_k,$$

so that $J^2 = 1$. Together, this gives

$$J^* = (J^*J)J = J,$$

and so $JJ^* = JJ = 1$; hence J is anti-unitary. □

The operators Δ^{it} and J on G form the core objects of the Tomita-Takesaki modular theory. Equivalently, we can consider the operators σ_t and j on $\mathcal{A}$ as the core elements of the theory. These modular objects are found in the more general versions of the Tomita-Takesaki theory as well, but it is much more difficult to obtain them and prove their properties.

2.5 The modular groups on an algebra and its commutant

Among the motivations for the development of the Tomita-Takesaki theory was to systematically study how the structure on a given algebra influence that on its commutant in the case where both algebras have a common cyclic vector ([20], [17, Section 9.2]). We illustrate this relationship by looking at a particular instance, namely, by establishing a relation between the modular automorphism group defined on an algebra and that on its commutant.

Consider the setup of the previous sections. We note that the requirement for a vector to be cyclic for an algebra is equivalent to the same vector being separating for the algebra's commutant (see [8, Proposition 2.5.3], [20, Proposition 2.2]). Since Ω is cyclic and separating for $\pi(\mathcal{A})$, it then follows that Ω is also cyclic and separating for $\pi'(\mathcal{A})$. One can also prove this directly, as we did for $\pi(\mathcal{A})$.

Analogous to the introduction of modular operator Δ on G associated with the algebra $\pi(\mathcal{A})$, we can define a corresponding operator Δ' associated with the commutant $\pi'(\mathcal{A})$ as follows

$$\Delta' \mid_{\mathscr{H}_0 \odot \mathscr{H}_0} = \rho^{-1} \odot \rho \mid_{\mathscr{H}_0 \odot \mathscr{H}_0} \tag{2.5.1}$$

(cf. Thm.2.3.2). Note that ρ and ρ^{-1} have been swapped, since we are working with $\pi'(\mathcal{A}) = 1 \otimes \mathcal{A}$ instead of $\pi(\mathcal{A}) = \mathcal{A} \otimes 1$, so we have to switch the tensor products throughout. Then $\Delta' = \Delta^{-1}$, since $\rho^{-1}\rho \mid_{\mathscr{H}_0} = \rho^{-1}\rho$ is the identity operator on $\mathscr{H}_0$.

Through the functional calculus of Δ', we can define a one-parameter unitary group $\{(\Delta')^{it} : t \in \mathbb{R}\}$ (see Rem.2.3.4 and Prop.2.3.5). By Prop.2.3.6, these unitaries induce a one-parameter automorphism group

$$\sigma'_t(\pi'(a)) = \left(\Delta'\right)^{it} \pi'(a) \left(\Delta'\right)^{-it} \tag{2.5.2}$$

on $\pi'(\mathcal{A})$.

Our goal here is to study the relationship between σ_t and σ'_t.

Lemma 2.5.1. *Let Δ and J be the modular operator and modular conjugation associated with $(\pi(\mathcal{A}), \Omega)$. Then for all $z \in \mathbb{C}$,*

$$J\Delta^{iz}J = \Delta^{i\bar{z}}.$$

Proof. Let $z \in \mathbb{C}$. Then, for a basis element $e_j \otimes e_k$ of G, we have

$$\begin{aligned} J\Delta^{iz}J(e_j \otimes e_k) &= J\Delta^{iz}(e_k \otimes e_j) \\ &= J\left(\rho_k^{iz}\rho_j^{-iz} e_k \otimes e_j\right) \\ &= \rho_k^{-i\bar{z}}\rho_j^{i\bar{z}} e_j \otimes e_k \\ &= \Delta^{i\bar{z}}(e_j \otimes e_k). \end{aligned}$$

This extends by linearity to the whole of G, since $\Delta^{iz} \in \mathcal{B}(G)$ (by Prop.2.3.5(1)) and $J \in \mathcal{B}(G)$ (by Rem.2.4.4). $\square$

In the case where z is real, say $z = t$ for some $t \in \mathbb{R}$, we have that $J\Delta^{it}J = \Delta^{it}$. So that the modular objects J and Δ^{it} commute for all $t \in \mathbb{R}$.

The map j introduced in Rem.2.4.6 gives a relation between the algebra $\pi(\mathcal{A})$ and its commutant $\pi'(\mathcal{A})$:

Proposition 2.5.2. *The map $j : \mathcal{B}(G) \to \mathcal{B}(G) : c \mapsto Jc^*J$ is a linear *-anti-isomorphism such that $j \circ j = 1$, $j(\pi(\mathcal{A})) = \pi'(\mathcal{A})$ and $j(\pi'(\mathcal{A})) = \pi(\mathcal{A})$.*

Proof. Let $\alpha \in \mathbb{C}$ and $a, b \in \mathcal{A}$. Then

$$\begin{aligned} j\left(\alpha\pi(a) + \pi(b)\right) &= j\left(\pi(\alpha a + b)\right) \\ &= J\pi\left(\alpha a + b\right)^* J \\ &= J\left(\bar{\alpha}\pi(a)^* + \pi(b)^*\right) J \\ &= \alpha j(\pi(a)) + j(\pi(b)), \end{aligned}$$

and

$$\begin{aligned} (j(\pi(a)))^* &= (J\pi(a)^*J)^* \\ &= (J\pi(a^*)J)^* \\ &= J\pi(a^*)^*J \\ &= j(\pi(a^*)). \end{aligned}$$

We also have

$$\begin{aligned} j(\pi(a)\pi(b)) &= j(\pi(ab)) \\ &= J\pi(ab)^*J \\ &= J\pi(b^*a^*)J \\ &= (J\pi(b)^*J)\,(J\pi(a)^*J) \\ &= j(\pi(b))j(\pi(a)). \end{aligned}$$

If $j(\pi(a)) = j(\pi(b))$ then $J\pi(a)^*J = J\pi(b)^*J$, so that $0 = \pi(a^* - b^*)$ which implies that $a = b$, since π is injective by Prop.1.8.4(3). Now suppose that $\pi'(b) = j(\pi(a))$, then $\pi'(b) = J\pi(a)^*J$ so that $J\pi'(b)^*J = \pi(a)$. Thus, j is a linear *-isomorphism. Furthermore,

$$\begin{aligned} (j \circ j)\pi(a) &= J\,(J\pi(a)^*J)^*\,J \\ &= \pi(a) \end{aligned}$$

so that $j \circ j = 1$. By Thm.2.4.5(2), we get

$$\begin{aligned} j(\pi(a))(e_j \otimes e_k) &= J\pi(a)^*J(e_j \otimes e_k) \\ &= J(a^* \otimes 1)(e_k \otimes e_j) \\ &= J\left((a^*e_k) \otimes e_j\right) \\ &= e_j \otimes (a^*e_k)^* \\ &= e_j \otimes (ae_k) \\ &= \pi'(a)(e_j \otimes e_k), \end{aligned}$$

and

$$\begin{aligned} j(\pi'(a))(e_j \otimes e_k) &= J(1 \otimes a^*)J(e_j \otimes e_k) \\ &= J\left(e_k \otimes (a^*e_j)\right) \\ &= (a^*e_j)^* \otimes e_k \\ &= \pi(a)(e_j \otimes e_k), \end{aligned}$$

so $j(\pi(\mathcal{A})) = \pi'(\mathcal{A})$ and $j(\pi'(\mathcal{A})) = \pi(\mathcal{A})$, as required. □

Remarks 2.5.3. For all $t \in \mathbb{R}$, the unitaries Δ^{it} and $(\Delta')^{it}$ satisfy the relation:

$$j(\Delta^{it}) = (\Delta')^{it}. \tag{2.5.3}$$

Indeed,

$$j\left(\Delta^{it}\right) = J\Delta^{-it}J = \Delta^{-it} = \left(\Delta'\right)^{it}$$

since for all $t \in \mathbb{R}$, $\Delta^{it} \in \mathcal{B}(G)$ and $\Delta' = \Delta^{-1}$.

The relation between the modular automorphism groups on an algebra and its commutant is given by:

Theorem 2.5.4. *Let σ_t and σ'_t be the modular automorphism groups on $\pi(\mathcal{A})$ and $\pi'(\mathcal{A})$ respectively, then*

$$j \circ \sigma_t \circ j = \sigma'_{-t}. \tag{2.5.4}$$

Proof. From Prop.2.5.2 it follows that $j \circ \sigma_t \circ j = \sigma'_{-t}$ is a map from $\pi'(\mathcal{A})$ to $\pi'(\mathcal{A})$. If $a \in \mathcal{A}$, then by Rem.2.4.6 and Prop.2.5.2, $j(\pi'(a)) = \pi(a^T)$, so

$$\begin{aligned}
(j \circ \sigma_t \circ j)\, \pi'(a) &= (j \circ \sigma_t)\pi\left(a^T\right) \\
&= j\left(\Delta^{it}\pi\left(a^T\right)\Delta^{-it}\right) \\
&= J\left(\Delta^{it}\pi\left(a^T\right)\Delta^{-it}\right)^* J \\
&= J\Delta^{it}\pi\left(a^T\right)^* \Delta^{-it} J \\
&= \Delta^{it}\left(J\pi(a^T)^*J\right)\Delta^{-it} \text{ (since } J \text{ and } \Delta^{it} \text{ commute)} \\
&= \Delta^{it}\pi'(a)\Delta^{-it} \\
&= \left(\Delta'\right)^{-it}\pi'(a)\left(\Delta'\right)^{it} \\
&= \sigma'_{-t}(\pi'(a)).
\end{aligned}$$

□

In effect going to the commutant leads to a time reversal of the modular group. This will be discussed from a more physical point of view in Sec.2.6.

2.6 Physical time

In the operator algebraic approach to quantum mechanics (in particular, to statistical mechanics [11], [15]), the dynamics are given by an automorphism group on the algebra of observables associated to some physical system [6],[8], [25]. Thus we can already attach physical meaning to σ_t, directly from the general theory, as time evolution.

In this section we show that the modular group can be interpreted as physical time-evolution in equilibrium statistical mechanics.

In the canonical ensemble of equilibrium statistical mechanics, the state ρ is given by

$$\rho_j = \frac{e^{-\beta E_j}}{\sum_{j=1}^{\infty} e^{-\beta E_j}} \tag{2.6.1}$$

with $\beta = 1/kT$ the inverse temperature, and E_j the energy of the state e_j.

Defining the Hamiltonian of the system as the (typically unbounded) operator H on $\mathscr{H}_0$ given by

$$He_j = E_j e_j \tag{2.6.2}$$

we can write $\rho = e^{-\beta H}/Tr(e^{-\beta H})$. Therefore

$$\begin{aligned}
\Delta^{it}(e_j \otimes e_k) &= \rho_j^{it}\rho_k^{-it} e_j \otimes e_k \\
&= e^{-iE_j\beta t} e^{iE_k\beta t} e_j \otimes e_k \\
&= e^{-iH\beta t} \otimes e^{iH\beta t}(e_j \otimes e_k),
\end{aligned}$$

i.e. $\Delta^{it} = e^{-iH\beta t} \otimes e^{iH\beta t}$ (using the same functional calculus as for Δ). Thus, for $a \in \mathcal{A}$,

$$\begin{aligned}
\sigma_t^\mu(a) &= \pi^{-1}\left(\Delta^{it}\pi(a)\Delta^{-it}\right) \\
&= \pi^{-1}\left(\Delta^{it}(a \otimes 1)\Delta^{-it}\right) \\
&= \pi^{-1}\left(\left(e^{-iH\beta t} a e^{iH\beta t}\right) \otimes 1\right) \\
&= e^{-iH\beta t} a e^{iH\beta t},
\end{aligned}$$

so

$$\sigma^\mu_{-t/\beta}(a) = e^{iHt} a e^{-iHt} \tag{2.6.3}$$

which is exactly the *physical* time evolution of a. This gives us a physical interpretation of the modular group as time-evolution, though reversed and "stretched" due to the factor $-1/\beta$. Combining this with Thm.2.5.2 we see that J, via $j \circ \sigma_t \circ j$, implements time-reversal, though on the commutant $\pi'(\mathcal{A}) = 1 \otimes \mathcal{A}$, since $j \circ \sigma_t \circ j$ gives σ'_{-t}.

We therefore have reasonable physical meanings attached to the modular group and modular conjugation. In Chapter 4 we are going to argue more in physical terms for a specific physical system, to gain further insight into the physical meaning of $\mathcal{A} \otimes 1$ versus $1 \otimes \mathcal{A}$, and consequently J.

2.7 General remarks

There is a more ad hoc approach to the formulation of Tomita-Takesaki theory in terms of tensor products based on the results obtained above, but simpler from a technical point of view. The idea behind this alternative approach is to simply take

$$\Delta^{it}(e_j \otimes e_k) := \rho_j^{it}\rho_k^{-it} e_k \otimes e_j, \text{ and } J(e_j \otimes e_k) := e_k \otimes e_j,$$

as the definitions of the linear operator Δ^{it} and conjugate-linear operator J on G from the outset. Then one verifies that these operators are well-defined on the whole of G, as their definitions only define them on $\mathscr{H}_0 \odot \mathscr{H}_0$. Indeed, since Δ^{it} is unitary for each $t \in \mathbb{R}$, it is a densely defined bounded operator; thus can be extended to the whole of G. On the other hand, we have that J is isometric, and thus bounded, hence J can also be extended to the whole of G. From here onwards, one then builds the theory in exactly the same way as before.

The disadvantage of this approach lies in that it does not give much insight into how the theory can be generalized to algebras other than $\mathcal{A} = \mathcal{B}(\mathscr{H})$, whereas the previous approach does.

The Tomita-Takesaki Theorem 2.1.1, in the formulation of the theory given above, is easily proved. In this context, it states that

$$J\pi\left(\mathcal{A}\right)J = \pi'(\mathcal{A}), \tag{2.7.1}$$

and

$$\sigma_t(\pi(\mathcal{A})) = \pi(\mathcal{A}). \tag{2.7.2}$$

Remarks 2.7.1 (Alternative approach to the functional calculus of Δ_0)**.** The idea for the definition of $\Delta_0^{1/2}$ in Def.2.4.1 is another way of approaching the functional calculus of Δ_0. In this approach, $\Delta_0^{1/2}$ is defined on $\mathscr{H}_0 \odot \mathscr{H}_0$ similar to Eq.(2.3.3), i.e.

$$\begin{aligned} \Delta_0^{1/2}(e_j \otimes e_k) &:= \rho_j^{1/2}\rho_k^{-1/2} e_j \otimes e_k \\ &= \left(\rho^{1/2} \odot \rho^{-1/2}\right)(e_j \otimes e_k), \end{aligned}$$

for all $j, k \in \mathbb{N}$. Then, in this case, J_0 is also defined on $\mathscr{H}_0 \odot \mathscr{H}_0$.

Chapter 3

Second approach to Tomita-Takesaki theory: Hilbert-Schmidt operators

The approach to Tomita-Takesaki theory we give in this chapter is centered around the Hilbert space of Hilbert-Schmidt operators $\mathcal{B}_2(\mathscr{H})$ (see Sec.1.3). There are two algebras of operators that can be introduced on this Hilbert space which carry the modular structure. We will make use of the tensor product approach (of Chap.2) to obtain the Hilbert-Schmidt approach. In particular, we have that $\mathscr{H} \otimes \mathscr{H}$ is unitarily isomorphic to $\mathcal{B}_2(\overline{\mathscr{H}}, \mathscr{H})$ (see Thm.1.7.7).

The development we make here is based on [1], but distilled as far as the scope of our project requires.

3.1 Left and right algebras

We start by identifying a special class of operators on $\mathcal{B}_2(\mathscr{H})$. Given $a, b \in \mathcal{A} \equiv \mathcal{B}(\mathscr{H})$ denote by $a \vee b$ an operator defined as follows:

$$a \vee b : \mathcal{B}_2(\mathscr{H}) \to \mathcal{B}_2(\mathscr{H}) : x \mapsto axb^*. \tag{3.1.1}$$

This operator is well-defined. Note that $axb^* \in \mathcal{B}_2(\mathscr{H})$ for all $a, b \in \mathcal{A}$, $x \in \mathcal{B}_2(\mathscr{H})$, this follows from the fact that $\mathcal{B}_2(\mathscr{H})$ is an ideal in $\mathcal{A}$ (by Cor.1.3.6). From the associative bilinearity of the product operation it follows that $a \vee b$ is linear.

Denote by $\mathfrak{A}$ the vector space of all linear combinations of operators of the form $a \vee b$ on $\mathcal{B}_2(\mathscr{H})$.

Proposition 3.1.1. *For any given pair $a, b \in \mathcal{A}$, the operator $a \vee b$ is a bounded linear operator on $\mathcal{B}_2(\mathscr{H})$. Moreover,*

$$(a \vee b)^* = a^* \vee b^* \tag{3.1.2}$$

and

$$(a_1 \vee b_1)(a_2 \vee b_2) = a_1 a_2 \vee b_1 b_2, \tag{3.1.3}$$

$\forall a_1, a_2, b_1, b_2 \in \mathcal{A}$.

Proof. Let $a, a_1, a_2, b, b_1, b_2 \in \mathcal{A}$ and $\alpha \in \mathbb{C}$. Then in view of the above definitions and Cor.1.3.6,

$$\begin{aligned} \|(a \vee b)x\|_2 &= \|axb^*\|_2 \\ &\leq \|a\|\|xb^*\|_2 \\ &\leq \|a\|\|b^*\|\|x\|_2, \end{aligned}$$

so that $a \vee b$ is a bounded linear operator. Using the inner product on $\mathcal{B}_2(\mathscr{H})$ and Prop.1.3.8 we get

$$\begin{aligned} \langle (a \vee b)^* x, y \rangle_2 &= \langle x, (a \vee b) y \rangle_2 \\ &= \langle x, ayb^* \rangle_2 \\ &= Tr[x^*(ayb^*)] \\ &= Tr[(a^*xb)^* y] \\ &= \langle a^*xb, y \rangle_2 \\ &= \langle (a^* \vee b^*)x, y \rangle_2 , \end{aligned}$$

thus $(a \vee b)^* = a^* \vee b^*$. By the associativity of the product operation, we have that

$$\begin{aligned} \left[(a_1 \vee b_1)(a_2 \vee b_2)\right](x) &= a_1 \left[(a_2 \vee b_2)(x)\right] b_1^* \\ &= a_1 (a_2 x b_2^*) b_1^* \\ &= (a_1 a_2) x (b_1 b_2)^* \\ &= \left[(a_1 a_2) \vee (b_1 b_2)\right](x), \end{aligned}$$

so that $(a_1 \vee b_1)(a_2 \vee b_2) = a_1 a_2 \vee b_1 b_2$. $\square$

Corollary 3.1.2. *The vector space $\mathfrak{A}$ of all operators of the form $a \vee b$ $(a, b \in \mathcal{A})$ is a *-subalgebra of the C^*-algebra of all bounded operators on $\mathcal{B}_2(\mathscr{H})$.*

Recall that the Hilbert spaces $\mathscr{H} \otimes \overline{\mathscr{H}}$ and $\mathcal{B}_2(\mathscr{H})$ are unitarily equivalent (by Thm.1.7.7), i.e. there is a unitary transformation W such that

$$W : \mathscr{H} \otimes \overline{\mathscr{H}} \to \mathcal{B}_2(\mathscr{H}) : x \otimes \bar{y} \mapsto |x\rangle \langle y| \,. \tag{3.1.4}$$

Proposition 3.1.3. *For all $a, b \in \mathcal{A}$,*

$$W(a \otimes b)W^* = a \vee b,$$

*i.e. $W(\mathcal{A} \otimes \mathcal{A})W^* = \mathfrak{A}$, where $\mathfrak{A}$ is the *-algebra of all bounded operators on $\mathcal{B}_2(\mathscr{H})$ of the form $a \vee b$.*

Proof. Let $a, b \in \mathcal{A}$, $x \in \mathscr{H}$ and $y \in \overline{\mathscr{H}}$. Then

$$\begin{aligned} W(a \otimes b)W^* (|x\rangle \langle y|) &= W(a \otimes b)(x \otimes y) \\ &= W((ax) \otimes (by)) \\ &= |ax\rangle \langle by| . \end{aligned}$$

But, for all $u \in \mathscr{H}$, we have

$$\begin{aligned} |ax\rangle \langle by| u &= ax \langle by, u\rangle \\ &= a (x \langle y, b^* u\rangle) \\ &= a |x\rangle \langle y| b^* u \end{aligned}$$

so that

$$|ax\rangle \langle by| = a |x\rangle \langle y| b^* = (a \vee b) |x\rangle \langle y| .$$

Thus $W(a \otimes b)W^* = a \vee b$, since $\{|x\rangle \langle y| : x \in \mathscr{H}, y \in \overline{\mathscr{H}}\}$ spans $\mathcal{B}_2(\mathscr{H})$. □

There are two special von Neumann algebras that can be built out of the algebraic operations given in Prop.3.1.1 (cf. Prop.2.4.5(4)):

Proposition 3.1.4. *The sets*

$$\mathfrak{A}_l = \{a_l := a \vee 1 \mid a \in \mathcal{B}(\mathscr{H})\} \tag{3.1.5}$$

and

$$\mathfrak{A}_r = \{a_r := 1 \vee a \mid a \in \mathcal{B}(\mathscr{H})\} \tag{3.1.6}$$

are mutual commutants:

$$(\mathfrak{A}_l)' = \mathfrak{A}_r \quad \textit{and} \quad (\mathfrak{A}_r)' = \mathfrak{A}_l. \tag{3.1.7}$$

Moreover, $\mathfrak{A}_l$ and $\mathfrak{A}_r$ are von Neumann algebras.

Proof. Since $(a \vee b) \in \mathcal{B}(\mathcal{B}_2(\mathscr{H}))$, $\forall a, b \in \mathcal{A}$ (by Prop.3.1.1), it follows that $\mathfrak{A}_l$, $\mathfrak{A}_r \subseteq \mathcal{B}(\mathcal{B}_2(\mathscr{H}))$ as subsets.

For any set $Y \subset \mathcal{B}(\mathcal{B}_2(\mathscr{H}))$. Let $X := W^* Y W$, where W is the unitary transformation in (3.1.4). So $Y = W X W^*$.

For $a \in \mathcal{B}(\mathcal{B}_2(\mathscr{H}))$, set $b := W^* a W$. If $a \in Y'$, then for all $c \in X$

$$\begin{aligned} cb &= (W^* W c W^* W) (W^* a W) \\ &= W^* [(W c W^*) a] W \\ &= W^* [a (W c W^*)] W \text{ (since } W c W^* \in Y) \\ &= (W^* a W) c \\ &= bc \end{aligned}$$

which implies that $b \in X'$. On the other hand, if $b \in X'$, then for all $c \in Y$

$$\begin{aligned} ca &= (WW^*cWW^*)(WbW^*) \\ &= W[(W^*cW)b]W^* \\ &= W[b(W^*cW)]W^* \text{ (since } W^*cW \in Y) \\ &= (WbW^*)c \\ &= ac \end{aligned}$$

which implies that $a \in Y'$. So: $a \in Y'$ if and only if $b \in X'$, i.e. $Y' = WX'W^*$. In particular, we have that

$$\mathfrak{A}_r' = W(\mathcal{A} \otimes 1)' W^* = W(1 \otimes \mathcal{A})W^* = \mathfrak{A}_l,$$

by Prop.3.1.3. Similarly, $\mathfrak{A}_l' = \mathfrak{A}_r$. Furthermore, $\mathfrak{A}_r'' = \mathfrak{A}_l' = \mathfrak{A}_r$ so that $\mathfrak{A}_r$ is a von Neumann algebra; similarly for $\mathfrak{A}_l$. □

3.2 Modular conjugation and modular group

Here we show how the modular conjugation and modular group arise in the Hilbert-Schmidt version of Tomita-Takesaki theory, using their counterparts (and corresponding results) in the tensor product approach.

Proposition 3.2.1. *Let $\{e_j : j \in \mathbb{N}\}$ be an orthonormal basis for $\mathscr{H}$, and*

$$\iota : \mathscr{H} \otimes \mathscr{H} \to \mathscr{H} \otimes \overline{\mathscr{H}} : e_j \otimes e_k \mapsto e_j \otimes \bar{e}_k,$$

where $\bar{e}_k := e_k$ (but viewed as an element of $\overline{\mathscr{H}}$). Then:

1. *The map ι has a unique unitary extension to the whole of $\mathscr{H} \otimes \mathscr{H}$, which we also denote by ι.*

2. *For all $x, y \in \mathscr{H}$,*
$$\iota(x \otimes y) = x \otimes \bar{y}^*, \tag{3.2.1}$$
where $z^* := \sum_{j\in\mathbb{N}} \bar{z}_j e_j$ for all $z = \sum_{j\in\mathbb{N}} z_j e_j$. Furthermore, $\iota(x \otimes y^*) = x \otimes \bar{y}$ and $\iota^*(x \otimes \bar{y}) = x \otimes y^*$.

Proof. Let $\{e_j : j \in \mathbb{N}\}$ be an orthonormal basis for $\mathscr{H}$.

1. Since $\{e_j \otimes e_k : j, k \in \mathbb{N}\}$ is an orthonormal basis for $\mathscr{H} \otimes \mathscr{H}$, it follows that ι is well-defined. Clearly, ι is linear and invertible. From
$$\|\iota(e_j \otimes e_k)\| = \|e_j \otimes \bar{e}_k\| = \|e_j\|\|\bar{e}_k\| = 1,$$
it follows that ι is an isometry, and thus unitary. So ι has a unitary extension to the entire $\mathscr{H} \otimes \mathscr{H}$. Call it ι.

2. Note that

$$\begin{aligned}
\iota(e_j \otimes (\alpha e_k)) &= \alpha\iota(e_j \otimes e_k) \\
&= \alpha(e_j \otimes \bar{e}_k) \\
&= e_j \otimes \overline{(\bar{\alpha} e_k)} \\
&= e_j \otimes \overline{(\alpha e_k)^*} \\
&= e_j \otimes (\alpha \,\bar{\cdot}\, \bar{e}_k)
\end{aligned}$$

so ι "conjugates" the second factor with respect to the orthonormal basis $\{e_j : j \in \mathbb{N}\}$. Then for all $x, y \in \mathscr{H}$, say $x = \sum_{j\in\mathbb{N}} x_j e_j$ and $y = \sum_{k\in\mathbb{N}} y_k e_k$, we have

$$\begin{aligned}
\iota(x \otimes y) &= \sum_{j,k\in\mathbb{N}} x_j y_k \, (e_j \otimes \bar{e}_k) \\
&= \sum_{j,k\in\mathbb{N}} (x_j e_j) \otimes \overline{(\bar{y}_k e_k)} \\
&= x \otimes \overline{\left(\sum_{k\in\mathbb{N}} \bar{y}_k e_k \right)} \\
&= x \otimes \overline{(y^*)} \\
&= x \otimes \bar{y}^*
\end{aligned}$$

so $x \otimes y^* = \iota^*(x \otimes \bar{y})$ and $\iota(x \otimes y^*) = x \otimes \bar{y}$.

□

We define the modular conjugation in the Hilbert-Schmidt setup from J in Chapter 2 as follows:

Definition 3.2.2. Let $J_{HS} : \mathcal{B}_2(\mathscr{H}) \to \mathcal{B}_2(\mathscr{H})$ be such that

$$J_{HS} := W\iota J \iota^* W^*, \tag{3.2.2}$$

where W is the unitary transformation in (3.1.4), and ι is given by (3.2.1).

The operator J_{HS} has properties similar to the modular conjugation J (cf. Prop.2.4.7):

Theorem 3.2.3. *The operator J_{HS} has the following properties:*

1. *The operator J_{HS} is conjugate linear.*

2. *For all $x, y \in \mathscr{H}$,*
$$J_{HS}(|x\rangle\langle y|) = |y\rangle\langle x|.$$

3. *$J_{HS}^2 = 1$ and $J_{HS}^* = J_{HS}$.*

Hence J_{HS} is anti-unitary. Moreover, we have

$$J_{HS}\mathfrak{A}_l J_{HS} = \mathfrak{A}_r. \tag{3.2.3}$$

Proof. 1. Since W, ι, ι^* and W^* are linear maps, and on the other hand J is conjugate linear; then J_{HS} is conjugate linear, since the composition of a linear map with a conjugate linear map is itself conjugate linear.

2. Let $x, y \in \mathscr{H}$, then in view of the above definitions and Thm.2.4.5(2)

$$\begin{aligned} J_{HS}(|x\rangle\langle y|) &= W\iota J\iota^*(x \otimes \bar{y}) \\ &= W\iota J(x \otimes y^*) \\ &= W\iota(y \otimes x^*) \\ &= W(y \otimes \bar{x}) \\ &= |y\rangle\langle x|. \end{aligned}$$

3. We also have that

$$J_{HS}^2 = W\iota J^2\iota^* W^* = W\iota\iota^* W^* = 1,$$

and

$$J_{HS}^* = (W\iota J\iota^* W^*)^* = W\iota J\iota^* = J_{HS}.$$

Hence J_{HS} (like J) is anti-unitary. Furthermore, for all $a, b \in \mathcal{A}$ and $x, y \in \mathscr{H}$

$$\begin{aligned} J_{HS}(a \vee b)J_{HS}(|x\rangle\langle y|) &= J_{HS}(a \vee b)(|y\rangle\langle x|) \\ &= J_{HS}\left(a\,|y\rangle\langle x|\, b^*\right) \\ &= J_{HS}\left(|ay\rangle\langle bx|\right) \\ &= |bx\rangle\langle ay| \\ &= (b \vee a)\,|x\rangle\langle y|, \end{aligned}$$

so $J_{HS}(a \vee b)J_{HS} = b \vee a)$. In particular, $J_{HS}\mathfrak{A}_l J_{HS} = \mathfrak{A}_r$.

□

For the unitary group in this Hilbert-Schmidt approach, we simply define

$$\Delta_{HS}^{it} := W\iota\Delta^{it}\iota^* W^* \quad (\forall t \in \mathbb{R}).$$

Then the modular group can be obtained as in Sec.2.3.

3.3 The cyclic and separating vector

In this Hilbert-Schmidt approach to Tomita-Takesaki theory, we obtain the cyclic and separating vector Φ as follows:

$$\begin{aligned}
\Phi &:= W\iota\Omega \\
&= \sum_{j=1}^{\infty} \rho_j^{1/2} W(e_j \otimes \bar{e}_j) \\
&= \sum_{j=1}^{\infty} \rho_j^{1/2} \left| e_j \right\rangle \left\langle e_j \right| \\
&= \sum_{j=1}^{\infty} \rho_j^{1/2} \mathbb{P}_j,
\end{aligned}$$

where $\mathbb{P}_j := \left| e_j \right\rangle \left\langle e_j \right|$ (i.e. $\mathbb{P}_j$ is a projection of $\mathscr{H}$ onto $\mathbb{C}1$).

Then all the properties that Ω has in Chapter 2, can correspondingly be obtained for Φ (with arguments similar to those in Sec.3.2). In particular, we have that Φ is a cyclic and separating vector for $\mathfrak{A}_l$. Since $W\iota$ is unitary this is straightforward, so we omit the details.

Chapter 4

Tomita-Takesaki theory and Landau levels

In this chapter we look at the Landau levels of the dynamical model of a two-dimensional electron placed in a uniform magnetic field, orthogonal to the plane which contains the electron and demonstrate how these levels exhibit a modular structure in the sense of Tomita-Takesaki theory. This allows us to gain some understanding of the physical meaning of Tomita-Takesaki theory and the modular objects in a simple setting.

The developments we make in this chapter are based on [1], but we use the tensor product approach to Tomita-Takesaki theory, instead of the Hilbert-Schmidt approach as in the paper.

We do not attempt to be mathematically rigorous or complete in this chapter. In particular, our treatment of unbounded operators that appear here, are not fully rigorous.

4.1 Review of classical motion in a magnetic field

In this section we give a brief review of the classical motion of a charged particle in a magnetic field background. Consider a particle of mass m and charge q with position $\mathbf{r}$ that is subjected to a magnetic field $\mathbf{B}(\mathbf{r})$. The force exerted on the particle is given by the Lorentz force

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B}(\mathbf{r})$$

where $\mathbf{v} = \frac{d\mathbf{r}}{dt}$ is the velocity of the particle. Then, the equation of motion for the particle in a magnetic field is

$$m\frac{d\mathbf{v}}{dt} = q\mathbf{v} \times \mathbf{B}(\mathbf{r}). \tag{4.1.1}$$

The classical Hamiltonian for the charged particle is

$$H = \frac{1}{2m}\left[\mathbf{p} - q\mathbf{A}(\mathbf{r})\right]^2, \tag{4.1.2}$$

where $\mathbf{A}(\mathbf{r})$ is a vector potential related to the magnetic field $\mathbf{B}(\mathbf{r})$ by

$$\mathbf{B}(\mathbf{r}) = \nabla \times \mathbf{A}(\mathbf{r}).$$

The choice of gauge is not unique. However, since the dynamical system which we model manifests a rotational symmetry along the axis through the center of the trajectory parallel to the z-axis, we shall use symmetric gauge.

For convenience we set

$$H = H_{\perp} + H_{\parallel} \tag{4.1.3}$$

with

$$\begin{aligned} H_{\perp} &= \frac{1}{2m}\left[(p_x - qA_x)^2 + (p_y - qA_y)^2\right] && (4.1.4) \\ H_{\parallel} &= \frac{1}{2m}(p_z - qA_z)^2 && (4.1.5) \end{aligned}$$

When the magnetic field is uniform and its direction is chosen to be along the z-axis, the equation of motion become the three coupled differential equations

$$\begin{aligned} m\ddot{x} &= qB\dot{y} \\ m\ddot{y} &= -qB\dot{x} \\ m\ddot{z} &= 0. \end{aligned}$$

The solutions to these equations of motion are given by

$$\begin{aligned} x(t) &= x_0 - R\sin(\omega_B t - \theta_0) && (4.1.6) \\ y(t) &= y_0 + R\cos(\omega_B t - \theta_0) && (4.1.7) \\ z(t) &= z_0 + v_{0,z}t, && (4.1.8) \end{aligned}$$

where $x_0, y_0, z_0, R, \theta_0$ and $v_{0,z}$ are the arbitrary constant parameters which depend on the initial conditions of the system. The *cyclotron frequency* is given by

$$\omega_B = -\frac{qB}{m}. \tag{4.1.9}$$

Remarks 4.1.1 (Trajectory of the particle)**.** Equations (4.1.6) and (4.1.7) show that the projection of position of the particle onto the xy-plane executes a uniform circular motion, of angular frequency ω_B and initial phase θ_0, on a circle of radius R whose center of orbit is the point $C_0 = (x_0, y_0, 0)$, which we call the *guiding center*.

Whereas the projection of the motion onto the z-axis is uniform and rectilinear. It then follows that the motion that the particle curves in space is a circular helix whose axis is parallel to the z-axis and goes through the point C_0.

4.2 The quantum theory

In this section we shall give a quick review of the (non-relativistic) quantum-mechanical treatment of the motion of a charged particle in a magnetic field background and the resulting phenomenon of Landau levels. This is a well-known problem in physics.

The Hamiltonian of the system is described by

$$H = \frac{1}{2m}(\mathbf{p} + q\mathbf{A})^2, \tag{4.2.1}$$

motivated by condensed matter physics, quantum optics, etc.

The energy eigenspectrum of the this Hamiltonian can be found explicitly, in fact it is exactly the same as that of the quantum harmonic oscillator. However, unlike the harmonic oscillator, it turns out that each energy level does not have a unique state associated to it. Instead there exists a (countably) infinite degeneracy for each eigenvalue (the so-called *Landau levels*). Physically, the degeneracy of the Landau levels is explained by the impossibility of quantum-mechanically fixing the origin of the center of the circular orbits of the charged particles [1].

See the Appendix and the references referred therein for more details and proofs of the statements we shall make in this section.

The assumption we make is that the charged particle behaves like a *spinless fermion*, i.e. we shall ignore the effects due to spin (this is more or less appropriate for most physically realizable quantum Hall systems [32]), and the treatment shall also be non-relativistic.

Consider an electron placed in an arbitrary magnetic field described by the vector potential $\mathbf{A}(x, y, z)$. In quantum mechanics, the vector potential becomes an operator, a function of three observables $X, Y,$ and Z. The non-relativistic quantum Hamiltonian of the electron can be obtained from (4.1.2),

$$H_{elec} = \frac{1}{2m}\left[\mathbf{P} + e\mathbf{A}(X, Y, Z)\right]^2 = \frac{m}{2}\mathbf{V}^2, \tag{4.2.2}$$

where $\mathbf{V} = \frac{1}{m}\left[\mathbf{P} + e\mathbf{A}(X, Y, Z)\right]$ is the operator associated with the velocity of the electron. The position and momentum observables $\mathbf{R}$ and $\mathbf{P}$ satisfy the commutation relations

$$[X, P_x] = [Y, P_y] = [Z, P_z] = i\hbar, \tag{4.2.3}$$

and the other commutators between the components of $\mathbf{R}$ and $\mathbf{P}$ are zero. In the special case of a uniform magnetic field, the commutation relations between the components of the velocity operator are given by

$$[V_x, V_y] \quad = \quad \frac{-i\hbar\omega_B}{m} \tag{4.2.4}$$

$$[V_y, V_z] = [V_z, V_x] = 0. \tag{4.2.5}$$

The operator H_{elec}, analogous to (4.1.3), can be written in the form

$$H_{elec} = H_\perp + H_\parallel \tag{4.2.6}$$

where

$$H_\perp = \frac{m}{2}\left(V_x^2 + V_y^2\right) \tag{4.2.7}$$

$$H_\parallel = \frac{m}{2}V_z^2. \tag{4.2.8}$$

From the relation (4.2.5), it follows that $\left[H_\perp, H_\parallel\right] = 0$, so that we can apply spectral theory to $H_\perp$ and $H_\parallel$ separately. Then the spectrum of H_{elec} consists of the values

$$E_{elec} = E_\perp + E_\parallel, \tag{4.2.9}$$

where $E_\perp$ and $E_\parallel$ are the spectral values of $H_\perp$ and $H_\parallel$ respectively. The spectral values of $H_\parallel$ are of the form $E_\parallel = \frac{m}{2}v_z^2$, where $v_z \in \mathbb{R}$ is an arbitrary constant. Thus, the spectrum of $H_\parallel$ is continuous: the energy $E_\parallel$ can take any positive value or zero. The physical interpretation of this result is that $H_\parallel$ describes the kinetic energy of a free particle moving along the z-axis.

In order to determine the spectrum and eigenvectors of $H_\perp$, we introduce the *gauge invariant momentum*

$$\mathbf{\Pi} = m\mathbf{V} = \mathbf{P} + e\mathbf{A}, \tag{4.2.10}$$

which satisfy the relation $[\Pi_x, \Pi_y] = -ie\hbar B = -i\hbar^2/l_B^2$, where $l_B = \sqrt{\hbar/eB}$ is the *magnetic length*. Next we define the raising and lowering operators

$$a = \frac{l_B}{\sqrt{2}\hbar}\left(\Pi_x - i\Pi_y\right), \ a^\dagger = \frac{l_B}{\sqrt{2}\hbar}\left(\Pi_x + i\Pi_y\right), \tag{4.2.11}$$

with $[a, a^\dagger] = 1$. Then the Hamiltonian (4.2.7) can be re-written in terms of (4.2.11) as

$$H_\perp = \hbar\omega_B\left(a^\dagger a + \frac{1}{2}\right), \tag{4.2.12}$$

which has the form of the Hamiltonian of the one-dimensional harmonic oscillator. As in the case of the harmonic oscillator, we can construct a Hilbert space; by first introducing the *ground state* $|0\rangle$ such that $a\,|0\rangle = 0$, and then building the rest of the Hilbert space by acting with the raising operator $a^\dagger$,

$$a^\dagger\,|n\rangle = \sqrt{n+1}\,|n\rangle\,, \quad a\,|n\rangle = \sqrt{n}\,|n-1\rangle \tag{4.2.13}$$

where the last equation holds only for $n > 0$. Analogous to the harmonic oscillator, it is then deduced that the values of $E_\perp$ are given by

$$E_\perp(n) = \hbar\omega_B\left(n + \frac{1}{2}\right) \tag{4.2.14}$$

where n is a non-negative integer. From the above results, the spectral values of the total Hamiltonian H_{elec} are of the form:

$$E_{elec}(n, v_z) = \hbar\omega_B \left(n + \frac{1}{2}\right) + \frac{1}{2}mv_z^2, \tag{4.2.15}$$

with the corresponding levels referred to as *Landau levels*. Therefore, in the presence of a magnetic field, the kinetic energy of the motion along the z-axis is not quantized, but that of the projection onto the xy-plane is quantized.

The states of higher levels n are constructed from the ground state as

$$|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}} |0\rangle \tag{4.2.16}$$

with the corresponding energy spectral value given by (4.2.14).

We started with a problem of an electron moving in a plane, which has two degrees of freedom and it is natural to expect that a state of a two-dimensional system is described by two quantum numbers (one for each spatial dimension). Instead we ended up writing the original Hamiltonian (4.2.7) in the form of the harmonic oscillator Hamiltonian which has only one degree of freedom. From this dimensional analysis it follows that the quantum state of the system, as given by (4.2.16), is underdetermined. However the spectrum (4.2.14) is the correct spectrum of the theory but, unlike the harmonic oscillator, to each energy level $E_\perp$ there is no unique state associated to it. Instead there is a countably infinite degeneracy of states.

For a complete description of the quantum state of our dynamical system, in the manner similar to the gauge invariant momentum (4.2.10), we introduce another variable called the *pseudo-momentum*

$$\widetilde{\mathbf{\Pi}} = \mathbf{P} - e\mathbf{A}, \tag{4.2.17}$$

which satisfy the relation $[\widetilde{\Pi}_x, \widetilde{\Pi}_y] = i\hbar^2/l_B^2$. We then can define a new pair of ladder operators

$$b = \frac{l_B}{\sqrt{2}\hbar}\left(\widetilde{\Pi}_x - i\widetilde{\Pi}_y\right), \; b^\dagger = \frac{l_B}{\sqrt{2}\hbar}\left(\widetilde{\Pi}_x + i\widetilde{\Pi}_y\right). \tag{4.2.18}$$

It is this second pair of ladder operators that give rise to the degeneracy of the Landau levels.

In symmetric gauge, we have that $[b, a^\dagger] = 0$ and $[b^\dagger, H_\perp] = 0$. Therefore the Hamiltonian $H_\perp$ and the raising operator $b^\dagger$ are compatible variables, so that they share a common eigenbasis. The second quantum number is found by introducing a number operator $b^\dagger b$ associated with (4.2.18) whose eigenstates $|m\rangle$ satisfy

$$b^\dagger b\, |m\rangle = m\, |m\rangle \tag{4.2.19}$$

with an integer $m \geq 0$. The quantum number m, in addition of the Landau level quantum number n , is necessary to describe the complete state of the system. Then the general state of the system is given by the tensor product of the two Hilbert space vectors

$$|n, m\rangle = |n\rangle \otimes |m\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}} \frac{(b^\dagger)^m}{\sqrt{m!}} |0, 0\rangle \, , \qquad (4.2.20)$$

which is a generalization of (4.2.16). The energy of this state is given by the usual Landau expression (4.2.14), which depends on n but not on m.

4.3 Two opposite field directions

The so-called *Landau problem* is related to the motion of an electron on a flat xy-plane in the presence on a uniform magnetic field along the z-axis. In this context, we have two cases depending on the orientation of the magnetic field.

When the magnetic field is along the positive z-axis, take the vector potential to be given by

$$\mathbf{A}^\uparrow = \frac{1}{2}(-y, x, 0)$$

so that the magnetic field is

$$\mathbf{B}(\mathbf{r}) = \nabla \times \mathbf{A}^\uparrow = (0, 0, +1).$$

The classical Hamiltonian (4.1.2) then becomes

$$H_\perp = \frac{1}{2}\left(p_x + \frac{y}{2}\right)^2 + \frac{1}{2}\left(p_y - \frac{x}{2}\right)^2 ,$$

where, to simplify computations and presentation, we have adopted an appropriate convention of units in which all physical constants are taken to be one.

On an appropriate dense subspace $\mathscr{K}$ of $\widetilde{\mathscr{H}} = L^2\left(\mathbb{R}^2, dxdy\right)$, where $dxdy$ indicates Lebesgue measure, we introduce the following quantized observables in place of $p_x + \frac{y}{2}$ and $p_y - \frac{x}{2}$ respectively:

$$Q_+ \;=\; -i\frac{\partial}{\partial x} + \frac{y}{2}, \qquad (4.3.1)$$

$$P_+ \;=\; -i\frac{\partial}{\partial y} - \frac{x}{2} \qquad (4.3.2)$$

which satisfy the commutation relation $[Q_+, P_+] = i1_{\widetilde{\mathscr{H}}}$. The corresponding quantum Hamiltonian is then given by

$$H^\uparrow = \frac{1}{2}\left(P_+^2 + Q_+^2\right). \qquad (4.3.3)$$

The Hamiltonian $H^{\uparrow}$ is the same as that of the one-dimensional quantum oscillator Hamiltonian, and its eigenvalues are

$$E_n = \left(n + \frac{1}{2}\right), \; n = 0, 1, 2, ... \tag{4.3.4}$$

However, each of these Landau levels is infinitely degenerate, and we denote the corresponding normalized eigenvectors by

$$\Psi_{nm}, \; n = 0, 1, 2, ..., m = 0, 1, 2, ... \tag{4.3.5}$$

where n indexes the energy level, and m indexes the degeneracy at each energy level.

Now, if the uniform magnetic field is aligned along the negative z-axis, with $\mathbf{A}^{\downarrow} = \frac{1}{2}(y, -x, 0)$ and

$$\mathbf{B} = \nabla \times \mathbf{A}^{\downarrow} = (0, 0, -1),$$

the corresponding quantum Hamiltonian would then become

$$H^{\downarrow} = \frac{1}{2}\left(P_-^2 + Q_-^2\right), \tag{4.3.6}$$

where in place of $p_y + \frac{x}{2}$ and $p_x - \frac{y}{2}$ we have introduced the quantized variables

$$Q_- = -i\frac{\partial}{\partial y} + \frac{x}{2} \tag{4.3.7}$$

$$P_- = -i\frac{\partial}{\partial x} - \frac{y}{2}, \tag{4.3.8}$$

respectively, on $\mathscr{K}$, which satisfy $[Q_-, P_-] = i1_{\widetilde{\mathscr{H}}}$.

To solve the eigenvalue problem for $H^{\downarrow}$, we make the following observation:

Proposition 4.3.1. *The two sets of operators* $\{P_{\pm}, Q_{\pm}\}$ *mutually commute, i.e.*

$$[Q_+, Q_-] = [P_+, Q_-] = [Q_+, P_-] = [P_+, P_-] = 0, \tag{4.3.9}$$

consequently $[H^{\uparrow}, H^{\downarrow}] = 0$.

Proof. Let $\psi \in \mathscr{K}$ be arbitrary, then

$$\begin{aligned}
[Q_+, Q_-]\psi(x,y) &= [Q_+Q_- - Q_-Q_+]\psi(x,y) \\
&= \left[-i\frac{\partial}{\partial x} + \frac{y}{2}\right]\left(-i\frac{\partial \psi}{\partial y} + \frac{x}{2}\psi\right) \\
&- \left[-i\frac{\partial}{\partial y} + \frac{x}{2}\right]\left(-i\frac{\partial \psi}{\partial x} + \frac{y}{2}\psi\right) \\
&= \left[-\frac{\partial^2\psi}{\partial x \partial y} - \frac{i}{2}\left(\psi + x\frac{\partial \psi}{\partial x} + y\frac{\partial \psi}{\partial y}\right) + \frac{xy}{4}\psi\right] \\
&- \left[-\frac{\partial^2\psi}{\partial x \partial y} - \frac{i}{2}\left(\psi + x\frac{\partial \psi}{\partial x} + y\frac{\partial \psi}{\partial y}\right) + \frac{xy}{4}\psi\right] \\
&= 0.
\end{aligned}$$

Then $[Q_+, Q_-] = 0$. The other commutation relations in (4.3.9) are proved in a similar way.

$\square$

Since the Hamiltonians $H^\uparrow$ and $H^\downarrow$ are compatible observables, we can choose the eigenvectors Ψ_{nm} of $H^\uparrow$ in such a way that they are also the eigenvectors of $H^\downarrow$ satisfying

$$H^\downarrow \Psi_{nm} = \left(m + \frac{1}{2}\right) \Psi_{nm}. \tag{4.3.10}$$

So that $H^\uparrow$ lifts the degeneracy of $H^\downarrow$ and vice versa.

4.4 The two algebras and J

In Tomita-Takesaki theory we saw that a central role is played by the two algebras $\mathcal{A} \otimes 1$ and $1 \otimes \mathcal{A}$, which are connected via the modular conjugation J (see Sec.2.4).

We want to give a physical interpretation of the two algebras $\mathcal{A} \otimes 1$ and $1 \otimes \mathcal{A}$, and consequently add further insight to our discussion in Sections 2.5 and 2.6 to allow us to gain better understanding of the physical meaning of J.

We argue heuristically as follows: Represent $\widetilde{\mathscr{H}}$ as

$$\widetilde{\mathscr{H}} = L^2(\mathbb{R}, dx) \otimes L^2(\mathbb{R}, dy),$$

with elementary tensors of the form $f \otimes g$ given by

$$(f \otimes g)(x, y) = f(x)g(y). \tag{4.4.1}$$

Therefore Q_+ and P_+, as well as Q_- and P_-, are in effect defined on this tensor product Hilbert space $\widetilde{\mathscr{H}}$.

We note from (4.3.1) and (4.3.2), versus (4.3.7) and (4.3.8), that going from the field in the positive z-direction (i.e. (4.3.1) and (4.3.2)) to the field in the negative z-direction (i.e. (4.3.7) and (4.3.8)), amounts to swapping x and y. But, as seen in (4.4.1) above, swapping x and y, in effect means the factors in the Hilbert space tensor product are being swapped.

If we extend this idea to our two algebras $\mathcal{A} \otimes 1$ and $1 \otimes \mathcal{A}$, we start to see that these two algebras are associated to the two field directions respectively.

Since $J(\cdot)J$ takes us from $\mathcal{A} \otimes 1$ to $1 \otimes \mathcal{A}$, we conclude that J can be interpreted as reflecting the magnetic field in the xy-plane, i.e. from positive to negative z-direction. This is consistent with our interpretation of J in Sec.2.6 as time-reversal. A simple way to see this is to think of a magnetic field generated by a current in a solenoid: reversing time, reverses the current and consequently the magnetic field.

In the next section we study the swapping of the tensor product somewhat more precisely, by also using ideas from our Hilbert-Schmidt approach to Tomita-Takesaki theory in Chapter 3.

4.5 The Wigner transformation

Here we intend to make the swapping in the tensor product, in relation to $\mathcal{A} \otimes 1$ versus $1 \otimes \mathcal{A}$ in the previous section, somewhat more precise. The discussion is nevertheless still in some ways heuristic, as we work with unbounded operators, rather than directly with $\mathcal{A} \otimes 1$ and $1 \otimes \mathcal{A}$.

Consider the usual position and momentum operators Q and P on the Hilbert space $\mathscr{H} = L^2(\mathbb{R})$, in Schrödinger representation, which satisfy the commutation relation $[Q, P] = i1_{\mathscr{H}}$. Let $\{\phi_n\}_{n=0}^{\infty}$ be the orthonormal basis of $\mathscr{H}$ consisting of the eigenvectors of the harmonic oscillator Hamiltonian, $H_{osc} = \frac{1}{2}\left(P^2 + Q^2\right)$, i.e.

$$H_{osc}\phi_n = \left(n + \frac{1}{2}\right)\phi_n, \ \ n = 0, 1, 2, ..., \tag{4.5.1}$$

where the ϕ_n are the *Hermite functions*

$$\phi_n(x) = \frac{1}{\pi^{1/4}} \frac{1}{2^n n!} e^{-x^2/2} h_n(x), \tag{4.5.2}$$

with the h_n being the *Hermite polynomials,*

$$h_n(x) = (-1)^n \frac{\partial^n}{\partial x^n} e^{-x^2}. \tag{4.5.3}$$

Then, by Thm.1.3.8, the following set of vectors

$$\phi_{nl} := \left|\phi_n\right\rangle \left\langle\phi_l\right|, \ \ n, l = 0, 1, 2, ... \tag{4.5.4}$$

forms an orthonormal basis for the Hilbert space $\mathcal{B}_2(\mathscr{H})$ of Hilbert-Schmidt operators.

Definition 4.5.1. Consider the unitary operator $U(x, y)$ on $\mathscr{H} = L^2(\mathbb{R})$ given by

$$\left[U(x, y)\Phi\right](\xi) = e^{ix(\xi - y/2)}\Phi(\xi - y), \tag{4.5.5}$$

where $x, y, \xi \in \mathbb{R}$, $\Phi \in \mathscr{H}$, with $U(x, y) = e^{-i(xQ+yP)}$. Using the operator $U(x, y)$ we define a map, known as the *Wigner transformation*, given by $\mathcal{W} : \mathcal{B}_2(\mathscr{H}) \to \widetilde{\mathscr{H}} = L^2(\mathbb{R}^2, dxdy)$;

$$\left(\mathcal{W}X\right)(x, y) = \frac{1}{\sqrt{2\pi}} Tr\left(U(x, y)^* X\right), \tag{4.5.6}$$

where $X \in \mathcal{B}_2(\mathscr{H})$, and $x, y \in \mathbb{R}$.

Note that if $X_1, X_2 \in \mathcal{B}_2(\mathscr{H})$, then

$$\begin{aligned}\int_{\mathbb{R}^2} \overline{\mathcal{W}X_1(x,y)}\mathcal{W}X_2(x,y)dxdy &= \langle X_1, X_2\rangle_2 \\ &= \langle X_1, X_2\rangle_{\mathcal{B}_2(\mathscr{H})},\end{aligned}$$

so that the Wigner transformation $\mathcal{W}$ is unitary.

Now we get to the main goal of this section, namely to show that swapping the factors in the tensor product corresponds to reversing the magnetic field.

Proposition 4.5.2. *Consider the unitary operator*

$$\mathcal{U} := W\iota : \mathscr{H} \otimes \mathscr{H} \to \mathcal{B}_2(\mathscr{H}).$$

Then the following relations hold:

$$\mathcal{U}\begin{pmatrix} Q \otimes 1 \\ P \otimes 1 \end{pmatrix}\mathcal{U}^{-1} = \begin{pmatrix} Q_+ \\ P_+ \end{pmatrix}, \tag{4.5.7}$$

$$\mathcal{U}\begin{pmatrix} 1 \otimes Q \\ 1 \otimes P \end{pmatrix}\mathcal{U}^{-1} = \begin{pmatrix} P_- \\ Q_- \end{pmatrix}, \tag{4.5.8}$$

$$\mathcal{U}\begin{pmatrix} H_{osc} \otimes 1 \\ 1 \otimes H_{osc} \end{pmatrix}\mathcal{U}^{-1} = \begin{pmatrix} H^{\uparrow} \\ H^{\downarrow} \end{pmatrix}, \tag{4.5.9}$$

and

$$\mathcal{U}(\phi_n \otimes \phi_m) = \Psi_{nm}. \tag{4.5.10}$$

This means that the Ψ_{nm} form a basis for $L^2\left(\mathbb{R}^2, dxdy\right)$.

Proof. (The proof we give here is based on the one outlined in [2].) We shall only prove the first two relations in (4.5.7), since the other two in (4.5.8) follow in an exactly similar manner. Whereas, the relations in (4.5.9) are a direct consequence of (4.5.7) and (4.5.8). Let $\phi, \psi \in \mathscr{H}$ be such that they are both in the domains of the operators Q and P, and are differentiable and vanish at infinity. Then $\phi \otimes \psi \in \mathscr{H} \otimes \mathscr{H}$ and

$$\begin{aligned}
[\mathcal{U}(\phi \otimes \psi)](x, y) &= [\mathcal{W} \left|\phi\right\rangle \left\langle\psi\right|](x, y) \\
&= \frac{1}{\sqrt{2\pi}} Tr[U(x, y)^* \left|\phi\right\rangle \left\langle\psi\right|] \\
&= \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \left\langle \phi_n, U(x, y)^* \left|\phi\right\rangle \left\langle\psi\right| \phi_n \right\rangle_{\mathscr{H}} \\
&= \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \left\langle U(x, y)\phi_n, \left\langle \psi, \phi_n \right\rangle_{\mathscr{H}} \phi \right\rangle_{\mathscr{H}} \\
&= \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \left\langle U(x, y) \left\langle \phi_n, \psi \right\rangle_{\mathscr{H}} \phi_n, \phi \right\rangle_{\mathscr{H}} \\
&= \frac{1}{\sqrt{2\pi}} \left\langle U(x, y) \left(\sum_{n=0}^{\infty} \left|\phi_n\right\rangle \left\langle\phi_n\right| \right) \psi, \phi \right\rangle_{\mathscr{H}} \\
&= \frac{1}{\sqrt{2\pi}} \left\langle U(x, y)\psi, \phi \right\rangle_{\mathscr{H}} \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \overline{[U(x, y)\psi](\xi)} \phi(\xi) d\xi \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \overline{[e^{-ix(\xi - y/2}\psi(\xi - y)]} \phi(\xi) d\xi \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ix(\xi - y/2)} \overline{\psi(\xi - y)} \phi(\xi) d\xi.
\end{aligned}$$

Then, with the help of $Q\phi(\xi) = \xi\phi(\xi)$ and

$$\frac{\partial}{\partial x} e^{ix(\xi - y/2)} = i(\xi - y/2) e^{ix(\xi - y/2)},$$

we get

$$\begin{aligned}
& [\mathcal{U}(Q \otimes 1_{\mathscr{H}})(\phi \otimes \psi)](x,y) \\
= \; & [\mathcal{U}(Q\phi \otimes \psi)](x,y) \\
= \; & [\mathcal{W}(Q\left|\phi\right\rangle \left\langle\psi\right|)](x,y) \\
= \; & \frac{1}{\sqrt{2\pi}} Tr[U(x,y)^* Q \left|\phi\right\rangle \left\langle\psi\right|] \\
= \; & \frac{1}{\sqrt{2\pi}} \left\langle U(x,y)\psi, Q\phi \right\rangle_{\mathscr{H}} \\
= \; & \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \overline{[U(x,y)\psi](\xi)} Q\phi(\xi) d\xi \\
= \; & \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ix(\xi - y/2)} \overline{\psi(\xi - y)} \xi \phi(\xi) d\xi \\
= \; & \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \left(-i\frac{\partial}{\partial x} + \frac{y}{2} \right) e^{ix(\xi - y/2)} \overline{\psi(\xi - y)} \phi(\xi) d\xi \\
= \; & \left(-i\frac{\partial}{\partial x} + \frac{y}{2} \right) \left[\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ix(\xi - y/2)} \overline{\psi(\xi - y)} \phi(\xi) d\xi \right] \\
= \; & \left(-i\frac{\partial}{\partial x} + \frac{y}{2} \right) [\mathcal{U}\,(\phi \otimes \psi)](x,y),
\end{aligned}$$

and extending by linearity of appropriate domains, we get

$$\mathcal{U}(Q \otimes 1_{\mathscr{H}})\mathcal{U}^{-1} = -i\frac{\partial}{\partial x} + \frac{y}{2} = Q_+.$$

Next,

$$\begin{aligned}
& [\mathcal{U}(P \otimes 1_{\mathscr{H}})(\phi \otimes \psi)](x,y) \\
= \; & [\mathcal{U}(P\phi) \otimes \psi](x,y) \\
= \; & \frac{1}{\sqrt{2\pi}} \left\langle U(x,y)\psi, P\psi \right\rangle_{\mathscr{H}} \\
= \; & \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ix(\xi - y/2)} \overline{\psi(\xi - y)} P\phi(\xi) d\xi \\
= \; & \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ix(\xi - y/2)} \overline{\psi(\xi - y)} \left(-i\frac{\partial}{\partial \xi} \right) \phi(\xi) d\xi.
\end{aligned}$$

Now we observe that

$$\begin{aligned}
i\frac{\partial}{\partial y} \left[e^{ix(\xi - y/2)} \overline{\phi(\xi - y)} \phi(\xi) \right] \; &= \; e^{ix(\xi - y/2)} \frac{x}{2} \overline{\psi(\xi - y)} \phi(\xi) \\
&+ \; e^{ix(\xi - y/2)} \left(i\frac{\partial}{\partial y} \right) \overline{\psi(\xi - y)} \phi(\xi)
\end{aligned}$$

and

$$
\begin{aligned}
-i\frac{\partial}{\partial \xi}\left[e^{ix(\xi-y/2)}\overline{\phi(\xi-y)}\phi(\xi)\right] &= xe^{ix(\xi-y/2)}\overline{\psi(\xi-y)}\phi(\xi)\\
&+ e^{ix(\xi-y/2)}\left(-i\frac{\partial}{\partial \xi}\right)\overline{\psi(\xi-y)}\phi(\xi)\\
&+ e^{ix(\xi-y/2)}\overline{\psi(\xi-y)}\left(-i\frac{\partial}{\partial \xi}\right)\phi(\xi).
\end{aligned}
$$

So that together, with the help of $\frac{\partial}{\partial \xi}\psi(\xi-y)=-\frac{\partial}{\partial y}\psi(\xi-y)$, we have

$$
\begin{aligned}
e^{ix(\xi-y/2)}\overline{\psi(\xi-y)}\left(-i\frac{\partial}{\partial \xi}\right)\phi(\xi) &= -i\frac{\partial}{\partial \xi}\left[e^{ix(\xi-y/2)}\overline{\phi(\xi-y)}\phi(\xi)\right]\\
&- xe^{ix(\xi-y/2)}\overline{\psi(\xi-y)}\phi(\xi)\\
&- e^{ix(\xi-y/2)}\left(i\frac{\partial}{\partial y}\right)\overline{\psi(\xi-y)}\phi(\xi)\\
&= -i\frac{\partial}{\partial \xi}\left[e^{ix(\xi-y/2)}\overline{\phi(\xi-y)}\phi(\xi)\right]\\
&- xe^{ix(\xi-y/2)}\overline{\psi(\xi-y)}\phi(\xi)\\
&- \left(i\frac{\partial}{\partial y}-\frac{x}{2}\right)e^{ix(\xi-y/2)}\overline{\psi(\xi-y)}\phi(\xi)\\
&= -i\frac{\partial}{\partial \xi}\left[e^{ix(\xi-y/2)}\overline{\phi(\xi-y)}\phi(\xi)\right]\\
&+ \left(-i\frac{\partial}{\partial y}-\frac{x}{2}\right)e^{ix(\xi-y/2)}\overline{\psi(\xi-y)}\phi(\xi).
\end{aligned}
$$

Therefore, noting that $\phi(\xi),\psi(\xi)\to 0$ as $\xi\to\pm\infty$, we get

$$
[\mathcal{U}(P\phi\otimes\psi)](x,y) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\left(-i\frac{\partial}{\partial y}-\frac{x}{2}\right)e^{ix(\xi-y/2)}\overline{\psi(\xi-y)}\phi(\xi)d\xi.
$$

Thus,

$$
[\mathcal{U}(P\otimes 1_{\mathscr{H}})(\phi\otimes\psi)](x,y)=\left(-i\frac{\partial}{\partial y}-\frac{x}{2}\right)[\mathcal{U}(\phi\otimes\psi)](x,y),
$$

and again, extending by linearity on appropriate domains we get

$$
\mathcal{U}(P\otimes 1_{\mathscr{H}})\mathcal{U}^{-1}=-i\frac{\partial}{\partial y}-\frac{x}{2}=P_+.
$$

Thus we have demonstrated the first two relations in (4.5.7). Moreover,

$$\begin{aligned}
\mathcal{U}(H_{osc}\otimes 1_{\mathscr{H}})\mathcal{U}^{-1} &= \mathcal{U}\left[\frac{1}{2}(P^2+Q^2)\otimes 1_{\mathscr{H}}\right]\mathcal{U}^{-1} \\
&= \frac{1}{2}\mathcal{U}(P^2\otimes 1_{\mathscr{H}})\mathcal{U}^{-1}+\frac{1}{2}\mathcal{U}(Q^2\otimes 1_{\mathscr{H}})\mathcal{U}^{-1} \\
&= \frac{1}{2}\mathcal{U}\left[(P\otimes 1_{\mathscr{H}})(P\otimes 1_{\mathscr{H}})\right]\mathcal{U}^{-1} \\
&+ \frac{1}{2}\mathcal{U}\left[(Q\otimes 1_{\mathscr{H}})(Q\otimes 1_{\mathscr{H}})\right]\mathcal{U}^{-1} \\
&= \frac{1}{2}\left[\mathcal{U}(P\otimes 1_{\mathscr{H}})\mathcal{U}^{-1}\right]^2+\frac{1}{2}\left[\mathcal{U}(Q\otimes 1_{\mathscr{H}})\mathcal{U}^{-1}\right]^2 \\
&= \frac{1}{2}\left(P_+^2+Q_+^2\right) \\
&= H^{\uparrow},
\end{aligned}$$

and the other relation in (4.5.9) follows in a similar way. Finally, from (4.5.5) and (4.5.6), it is established that

$$\mathcal{U}(\phi_n\otimes\phi_m)=\mathcal{W}\left(|\phi_n\rangle\langle\phi_m|\right)=\Psi_{nm}.$$

□

We conclude this section by giving a heuristic analysis of the algebraic structures related to the operators $\{Q_\pm, P_\pm\}$, as well as outline how the modular structure of Tomita-Takesaki theory arise in the case of the electron in a uniform magnetic field.

In the formulation of the Tomita-Takesaki theory in terms of tensor products (Sec.2.2), the modular structure is carried by the algebras $\pi(\mathcal{A}) = \mathcal{A}\otimes 1$ and $\pi'(\mathcal{A}) = 1\otimes\mathcal{A}$, where the π, π' are given in Prop.1.8.3 and $\mathcal{A}=\mathcal{B}(\mathscr{H})$. Equivalently, the modular structure is carried by the algebras $\mathfrak{A}_r$ and $\mathfrak{A}_l$ in the Hilbert-Schmidt operator formulation of Tomita-Takesaki theory of Chapter 3.

Even though, strictly mathematically speaking, $Q\otimes 1$ is not an element of the algebra $\pi(\mathcal{A})$, since it is not bounded, but because $Q\otimes 1$ has the general form of elements in $\pi(\mathcal{A})$ we will associate it with $\pi(\mathcal{A})$. In the same manner we associate $1\otimes Q$ with $\pi'(\mathcal{A})$. Analogously, $P\otimes 1$ and $1\otimes P$ are associated with $\pi(\mathcal{A})$ and $\pi'(\mathcal{A})$ respectively. For a more complete and mathematically rigorous treatment, one would first have to show that the operators Q and P are self-adjoint in order to be able to apply the *spectral theorem.*

We note that the two set of operators, $\{Q_+, P_+\}$ and $\{Q_-, P_-\}$, generate (see [1], [2], [3]) two algebras $\mathfrak{A}_+$ and $\mathfrak{A}_-$ respectively, with $\mathcal{U}\pi(\mathcal{A})\mathcal{U}^{-1}=\mathfrak{A}_+$ and $\mathcal{U}\pi'(\mathcal{A})\mathcal{U}^{-1}=\mathfrak{A}_-$. Since $\pi(\mathcal{A})$ and $\pi'(\mathcal{A})$ are mutual commutants, then so are $\mathfrak{A}_+$ and $\mathfrak{A}_-$. Thus in physical terms, the two commuting algebras $\mathfrak{A}_+$ and $\mathfrak{A}_-$ correspond to the two directions of the magnetic field.

Epilogue

Generally, the theories that model physical systems, are made up of two major elements [8, §3.1], [25, §2.1.2]:

1. a *kinematical structure* describing the instantaneous states and observables of the system, and
2. a *dynamical rule* describing the change of the states and observables in time.

In the operator algebraic approach to quantum mechanics, the kinematical structure is given by a von Neumann algebra $\mathfrak{A}$ associated to a collection of physically measurable quantities (observables) of some physical system at hand. In practice, each observable corresponds to some measurement apparatus whose outputs are properties of the system. A ruler, a clock, or a particle detector located in some region in space, are examples of such an apparatus.

In this mathematical framework, the observables are self-adjoint elements of $\mathfrak{A}$ with possible measurement results for an observable a being characterized by its spectrum $\sigma(a)$. A state ω associates to an observable a a real number $\omega(a)$ obtained by averaging the results of measurements of a for the system prepared to be in the state ω. The states are identified with positive normalized linear functionals on $\mathfrak{A}$.

As in classical mechanics, the dynamical description of a quantum system is given by a one-parameter group of automorphisms on the underlying kinematical structure, representing the flow of the system in time.

In this dissertation we only considered the von Neumann algebra $\mathcal{B}(\mathscr{H})$ with $\mathscr{H}$ separable, which is fine for many applications involving only finitely many degrees of freedom. But for infinitely many degrees of freedom it turns out that more general von Neumann algebras are required (see for example [14]). In those cases one would correspondingly need a more general version of Tomita-Takesaki theory than the one we have developed.

There is indeed an extended version of Tomita-Takesaki theory, which includes the theory developed in Chapters 2 and 3 as a special case. One considers the more general situation of a von Neumann algebra $\mathfrak{A}$ (not necessarily $\mathcal{B}(\mathscr{H})$) with a faithful normal state ω. The development of Tomita-Takesaki theory in this situation is completely analogous to that of Chapter

2, except for the treatment of the technical details involved, which are much more difficult (see, for example [17, p.625], [30, Chap. VI]).

The GNS construction applied to the pair $(\mathfrak{A}, \omega)$ yields $(\mathscr{H}_\omega, \pi_\omega, \Omega_\omega)$, where π_ω is a faithful representation of $\mathfrak{A}$, and $\mathscr{H}_\omega$ is the representation Hilbert space. We then have that $\pi_\omega(\mathfrak{A})$ is a von Neumann algebra with a cyclic and separating vector Ω_ω, such that $\omega = \omega_{\Omega_\omega} \circ \pi_\omega$.

As before, the Tomita operator S is introduced via the *-operation, and through whose polar decomposition one proves the existence of a conjugate-linear isometry $J : \mathscr{H}_\omega \to \mathscr{H}_\omega$, and a positive self-adjoint (in general unbounded, but densely-defined and invertible) operator Δ on $\mathscr{H}_\omega$, associated to Ω_ω. Moreover, we have

$$J\Delta^{1/2}\pi(a)\Omega_\omega = \pi(a)^*\Omega_\omega \text{ and } J\Delta^{-1/2}\pi(a)'\Omega_\omega = \pi(a)'^*\Omega_\omega$$

for $a \in \mathfrak{A},\ a' \in \mathfrak{A}'$, where $\mathfrak{A}'$ is the commutant of $\mathfrak{A}$.

The modular conjugation defines a *-anti-isomorphism $j : \mathfrak{A} \to \mathfrak{A}' : a \mapsto Ja^*J$. The functional calculus of the modular operator Δ gives a unitary group Δ^{it} $(t \in \mathbb{R})$, which in turn induces a one-parameter automorphism group $\{\sigma_t : t \in \mathbb{R}\}$, where

$$\sigma_t(a) = \Delta^{it}\pi_\omega(a)\Delta^{-it}$$

for all $t \in \mathbb{R}$. The state ω is invariant under the modular group σ_t, i.e.

$$\omega(\sigma_t(a)) = \omega(a)\ \forall a \in \mathfrak{A}, t \in \mathbb{R}.$$

Then, with considerable difficulties, one shows that the Tomita-Takesaki theorem holds in this situation, i.e. $J\pi_\omega(\mathfrak{A})J = \pi_\omega(\mathfrak{A})'$ and $\Delta^{it}\pi_\omega(\mathfrak{A})\Delta^{-it} = \pi_\omega(\mathfrak{A})$ $(t \in \mathbb{R})$.

The faithful normal states on von Neumann algebras play a central role in the theory. However, not every von Neumann algebra has a faithful normal state (see [6, I.3], or [17, p.639]) So, the theory described above is applicable only in the case where a von Neumann algebra has a faithful normal state.

The Tomita-Takesaki theory can be extended even further by introducing and using the concept of *weights* in place of states (see, for example [6, I.3] or [17, §9.2]). When generalized in this way, Tomita-Takesaki theory applies to all von Neumann algebras, and becomes a powerful tool.

Since the development of the theory, it has made appearances in many and varied physical theories. In its inception it was tied to developments in equilibrium statistical mechanics, in particular to the so-called *KMS boundary condition* which is an important analytical relation between the state ω and the automorphism group $\{\sigma_t : t \in \mathbb{R}\}$ [11], [15].

Some of the physical theories in which Tomita-Takesaki theory features are:

1. Landau levels, and related construction of vector coherent states [1], [2], and [3].

2. Quantum field theory [6], [14], [25], [29], [34].
3. Quantum gravity and black holes [5], [23].
4. Quantum information and entanglement [35].

Appendix A

Landau quantization

In this appendix we give a more complete review than Sec. 4.2 of the quantum mechanical treatment of a free electron in a magnetic field background and the resulting phenomenon of Landau levels, for more details the reader may refer to [9, E_{VI}], [13, Chap. 2] [19, §110-111] and [32, 1.4].

The treatment here is however still not fully rigorous, in particular with respect to unbounded operators and their domains. Most of the operators discussed here are indeed unbounded. Our approach to spectral theory is also heuristic, for example we refer to eigenvalues and eigenvectors even for operators with continuous spectra. These "eigenvectors" are to be understood as distributions like Dirac δ's and are typically not in the Hilbert space.

In non-relativistic theory, a magnetic field can be regarded only as an external field. The magnetic interactions between moving charged particles are a relativistic effect, and a consistently relativistic theory is needed if they are to be taken into account [19].

Since electrons do not only possess a charge but also a spin, which is a purely quantum mechanical effect, in the presence of a magnetic field B there is a Zeeman splitting of each level into two spin branches separated by the energy difference

$$\Delta_Z E = 2\mu_B B,$$

where $\mu_B = e\hbar/2m$ is the *Bohr magneton*. However, in order to simplify the following presentation of the quantum-mechanical treatment and the kinetic energy quantization of an electron in a perpendicular magnetic field, we will neglect this effect associated with the spin degree of freedom. Formally, this amounts to modeling the electron as a *spinless fermion* (this is more or less appropriate for most physically realizable quantum Hall systems [32]) [13].

Consider an electron placed in an arbitrary magnetic field described by the vector potential $\mathbf{A}(x, y, z)$. In quantum mechanics, the vector potential becomes an operator, a function of three observables $X, Y,$ and Z.

The non-relativistic quantum Hamiltonian of the electron can be obtained from (4.1.2),

$$H_{elec} = \frac{1}{2m}\left[\mathbf{P} + e\mathbf{A}(X,Y,Z)\right]^2. \tag{A.0.1}$$

The operator $\mathbf{V}$ associated with the velocity of the electron is given by

$$\mathbf{V} = \frac{1}{m}\left[\mathbf{P} + e\mathbf{A}(X,Y,Z)\right], \tag{A.0.2}$$

which enables H_{elec} to be rewritten as

$$H_{elec} = \frac{m}{2}\mathbf{V}^2. \tag{A.0.3}$$

The position and momentum observables $\mathbf{R}$ and $\mathbf{P}$ satisfy the commutation relations

$$[X, P_x] = [Y, P_y] = [Z, P_z] = i\hbar, \tag{A.0.4}$$

and the other commutators between the components of $\mathbf{R}$ and $\mathbf{P}$ are zero. We also note that any two components of $\mathbf{P}$ commute. However, two distinct components of $\mathbf{V}$ do not commute (see [9], or [19, §110-111]):

$$[V_x, V_y] = \frac{ie\hbar}{m^2}B_z \tag{A.0.5}$$

$$[V_y, V_z] = \frac{ie\hbar}{m^2}B_x \tag{A.0.6}$$

$$[V_z, V_x] = \frac{ie\hbar}{m^2}B_y. \tag{A.0.7}$$

In the special case of a uniform magnetic field, the above commutation relations become

$$[V_x, V_y] = \frac{-i\hbar\omega_B}{m} \tag{A.0.8}$$

$$[V_y, V_z] = [V_z, V_x] = 0. \tag{A.0.9}$$

The operator H_{elec}, analogous to (4.1.3), can be written in the form

$$H_{elec} = H_\perp + H_\parallel \tag{A.0.10}$$

where

$$H_\perp = \frac{m}{2}\left(V_x^2 + V_y^2\right) \tag{A.0.11}$$

$$H_\parallel = \frac{m}{2}V_z^2. \tag{A.0.12}$$

From the relation (A.0.9), it follows that

$$\left[H_{\perp}, H_{\|}\right] = 0, \tag{A.0.13}$$

so that we can look for a basis of eigenvectors common to both $H_{\perp}$ (with eigenvalues $E_{\perp}$) and $H_{\|}$ (eigenvalues $E_{\|}$); these will be eigenvectors of the total Hamiltonian H_{elec} with the eigenvalues

$$E_{elec} = E_{\perp} + E_{\|}. \tag{A.0.14}$$

From (A.0.12), we observe that the eigenvectors of V_z are also eigenvectors of $H_{\|}$. Since Z and V_z are two self-adjoint operators satisfying the relation

$$[Z, V_z] = \frac{i\hbar}{m}, \tag{A.0.15}$$

the spectrum of V_z includes all real numbers. Therefore, the eigenvalues of $H_{\|}$ are of the form:

$$E_{\|} = \frac{m}{2} v_z^2 \tag{A.0.16}$$

where $v_z \in \mathbb{R}$ is an arbitrary constant. Thus, the spectrum of $H_{\|}$ is continuous: the energy $E_{\|}$ can take any positive value or zero. The physical interpretation of this result is that $H_{\|}$ describes the kinetic energy of a free particle moving along the z-axis.

To determine the eigenvalues of $H_{\perp}$, set

$$\hat{Q} = \sqrt{\frac{m}{\hbar\omega_B}} V_y, \quad \hat{S} = \sqrt{\frac{m}{\hbar\omega_B}} V_x, \tag{A.0.17}$$

where $\omega_B > 0$ since the electron is negatively charged. Then, by (A.0.8), their commutation relation is

$$[\hat{Q}, \hat{S}] = \frac{m}{\hbar\omega_B}[V_x, V_y] = i. \tag{A.0.18}$$

Inverting (A.0.17) and substituting into (A.0.11), the Hamiltonian $H_{\perp}$ becomes

$$H_{\perp} = \frac{\hbar\omega_B}{2}(\hat{Q}^2 + \hat{S}^2). \tag{A.0.19}$$

Then $H_{\perp}$ assumes the form of the one-dimensional harmonic oscillator, with $\hat{Q}$ and $\hat{S}$ playing the roles of the position and momentum operators respectively. From this it can be deduced that the values of $E_{\perp}$ are given by

$$E_{\perp} = \hbar\omega_B(n + \frac{1}{2}) \tag{A.0.20}$$

where n is a non-negative integer. From the above results, the eigenvalues of the total Hamiltonian H_{elec} are of the form:

$$E(n, v_z) = \hbar\omega_B \left(n + \frac{1}{2}\right) + \frac{1}{2}mv_z^2, \tag{A.0.21}$$

with the corresponding levels referred to as *Landau levels*. Therefore, in the presence of a magnetic field, the kinetic energy of the motion along the z-axis is not quantized, but that of the projection onto the xy-plane is quantized. For the remainder of this chapter we shall only be concerned with the projection motion of the electron onto the xy-plane.

From the above calculation, involving the passage of the Hamiltonian $H_\perp$ from (A.0.11) into (A.0.19) there is something disconcerting. We started with a problem of an electron moving in a plane, which has two degrees of freedom. But we ended up writing the original Hamiltonian in terms of the harmonic oscillator which has only one degree of freedom. It then follows from the above dimensional analysis that quantum system is underdetermined. However the spectrum (A.0.20) is the correct spectrum of the theory but, unlike the harmonic oscillator, to each energy level $E_\perp$ there is no unique state associated to it. Instead there is a countably infinite degeneracy of states.

In order to further analyze the Hamiltonian $H_\perp$ (A.0.19) we use the standard quantum-mechanical method, the *canonical quantization*. The momentum gets replaced by its gauge invariant form

$$\mathbf{\Pi} = m\mathbf{V} = \mathbf{P} + e\mathbf{A}, \tag{A.0.22}$$

and its components do not commute

$$\begin{aligned}
[\Pi_x, \Pi_y] &= [P_x + eA_x, P_y + eA_y] \\
&= e\left([P_x, A_y] - [P_y, A_x]\right) \\
&= e\left(\frac{\partial A_y}{\partial x}[P_x, x] + \frac{\partial A_y}{\partial y}[P_x, y] - \frac{\partial A_x}{\partial x}[P_y, x] - \frac{\partial A_x}{\partial y}[P_y, y]\right) \\
&= -ie\hbar\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) \\
&= -ie\hbar\left(\nabla \times \mathbf{A}\right)_z \\
&= -ie\hbar B \\
&= -\frac{i\hbar^2}{l_B^2}
\end{aligned}$$

where in the last equality we used the expression for the *magnetic length*

$$l_B = \sqrt{\frac{\hbar}{eB}}.$$

We introduce new variables using the pair of conjugate operators Π_x and Π_y. These are the ladder operators, analogous to those used in the quantum-mechanical treatment of the harmonic oscillator,

$$a = \frac{l_B}{\sqrt{2}\hbar}\left(\Pi_x - i\Pi_y\right), \;\; a^\dagger = \frac{l_B}{\sqrt{2}\hbar}\left(\Pi_x + i\Pi_y\right). \tag{A.0.23}$$

From the commutation relation for $\mathbf{\Pi}$, it then follows that the lowering and raising operators satisfy $[a, a^\dagger] = 1$. Inverting the expressions (A.0.23) yields

$$\Pi_x = \frac{\hbar}{\sqrt{2}l_B}(a^\dagger + a), \;\; \Pi_y = \frac{\hbar}{i\sqrt{2}l_B}(a^\dagger - a). \tag{A.0.24}$$

Then the Hamiltonian becomes

$$\begin{aligned} H_\perp &= \frac{\hbar\omega_B}{2}(\hat{Q}^2 + \hat{S}^2) = \frac{m}{2}(V_x^2 + V_y^2) \\ &= \frac{1}{2m}(\Pi_x^2 + \Pi_y^2) \\ &= \frac{\hbar^2}{4ml_B^2}\left[{a^\dagger}^2 + a^\dagger a + aa^\dagger + a^2 - ({a^\dagger}^2 - a^\dagger a - aa^\dagger + a^2)\right] \\ &= \frac{\hbar^2}{2ml_B^2}(a^\dagger a + aa^\dagger) \\ &= \frac{\hbar^2}{ml_B^2}\left(a^\dagger a + \frac{1}{2}\right) \\ &= \frac{\hbar\omega_B}{2}\left(a^\dagger a + \frac{1}{2}\right). \end{aligned}$$

Then the eigenvalues and eigenvectors of $H_\perp$, as in the case of the one-dimensional harmonic oscillator, are those of the *number operator* $a^\dagger a$, with $a^\dagger a\,|n\rangle = n\,|n\rangle$. The ladder operators act on these states in the usual manner [9]

$$a^\dagger\,|n\rangle = \sqrt{n+1}\,|n\rangle\,, \quad a\,|n\rangle = \sqrt{n}\,|n-1\rangle \tag{A.0.25}$$

where the last equation holds only for $n > 0$. The action of a on the ground state $|0\rangle$ gives 0, i.e. $a\,|0\rangle = 0$. The states of higher levels n are constructed from the ground state as

$$|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}\,|0\rangle \tag{A.0.26}$$

with the corresponding energy eigenvalue given by (A.0.20). As a consequence of the dimensional analysis argument given earlier, it is natural to expect that a state of a two-dimensional system is described by two quantum numbers (one for each spatial dimension). Then (A.0.26) is not sufficient to uniquely specify any state of the system. For a complete description of the quantum state, in the similar manner as the gauge invariant momentum (A.0.22), we introduce another variable called the *pseudo-momentum*

$$\widetilde{\mathbf{\Pi}} = \mathbf{P} - e\mathbf{A}, \tag{A.0.27}$$

which satisfy the relation $[\widetilde{\Pi}_x, \widetilde{\Pi}_y] = \frac{i\hbar^2}{l_B^2}$. In symmetric gauge, all the mixed commutators between the components of the gauge-invariant momentum and pseudo-momentum are zero.

The momentum $\mathbf{P}$ and the vector potential $\mathbf{A}$ can be expressed in terms of $\mathbf{\Pi}$ and $\widetilde{\mathbf{\Pi}}$ as

$$\mathbf{P} = \frac{1}{2}(\mathbf{\Pi} + \widetilde{\mathbf{\Pi}}), \;\; \mathbf{A} = \frac{1}{2e}(\mathbf{\Pi} - \widetilde{\mathbf{\Pi}}). \tag{A.0.28}$$

Using the pseudo-momentum, similar to the gauge-invariant momentum, we introduce a new pair of ladder operators

$$b = \frac{l_B}{\sqrt{2}\hbar}\left(\widetilde{\Pi}_x - i\widetilde{\Pi}_y\right), \;\; b^\dagger = \frac{l_B}{\sqrt{2}\hbar}\left(\widetilde{\Pi}_x + i\widetilde{\Pi}_y\right). \tag{A.0.29}$$

It is this second pair of ladder operators that give rise to the degeneracy of the Landau levels. In symmetric gauge, simple calculations show that

$$[b, a^\dagger] = 0 \text{ and } [b^\dagger, H_\perp] = 0. \tag{A.0.30}$$

Thus the Hamiltonian $H_\perp$ and the raising operator $b^\dagger$ are compatible variables, so that they share a common eigenbasis. The second quantum number is found by introducing a number operator $b^\dagger b$ associated with (A.0.29) whose eigenstates satisfy

$$b^\dagger b \left|m\right\rangle = m \left|m\right\rangle \tag{A.0.31}$$

with an integer $m \geq 0$. The quantum number m, in addition of the Landau level quantum number n , is necessary to describe the complete state of the system. Then the general state of the system is given by the tensor product of the two Hilbert space vectors

$$\left|n, m\right\rangle = \left|n\right\rangle \otimes \left|m\right\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}} \frac{(b^\dagger)^m}{\sqrt{m!}} \left|0, 0\right\rangle, \tag{A.0.32}$$

which is a generalization of (A.0.26). The energy of this state is given by the usual Landau expression (A.0.20), which depends on n but not on m.

To conclude our discussion of the quantum state of the system of a free electron in a magnetic field background we quantify the degeneracy of the Landau levels.

From the general solution of the classical equations of motion (4.1.6) and (4.1.7), with the help of (A.0.22), we get

$$X_0 \;=\; X(t) + R\cos(\omega_B t - \theta_0) = X - \frac{\dot{Y}}{\omega_B} = X - \frac{\Pi_y}{m\omega_B}, \tag{A.0.33}$$

$$Y_0 \;=\; Y(t) - R\sin(\omega_B t - \theta_0) = Y + \frac{\dot{X}}{\omega_B} = Y + \frac{\Pi_x}{m\omega_B} \tag{A.0.34}$$

where the coordinates labeling the guiding-center of the orbit $C_0 = (X_0, Y_0)$ are thought of as quantum operators.

In symmetric gauge $\mathbf{A} = \frac{B}{2}(-y, x, 0)$, we have

$$e\mathbf{A} = \frac{eB}{2}(-y, x, 0) = \frac{1}{2}(\bar{\Pi} - \widetilde{\Pi}),$$

so that

$$X = \frac{1}{eB}(\Pi_y - \widetilde{\Pi}_y), \quad \text{(A.0.35)}$$

$$Y = -\frac{1}{eB}(\Pi_x - \widetilde{\Pi}_x). \quad \text{(A.0.36)}$$

A comparison of Eqns. (A.0.33, A.0.34) and Eqns. (A.0.35, A.0.36) allows us to identify

$$X_0 = -\frac{\widetilde{\Pi}_y}{eB}, \;\; Y_0 = \frac{\widetilde{\Pi}_x}{eB}. \quad \text{(A.0.37)}$$

Thus in symmetric gauge the components of the pseudo-momentum are, apart from a factor to a momentum into position, the components of the guiding-center which are constants of motion. That is, under time evolution we have

$$i\hbar\dot{X}_0 = [X_0, H_\perp] = 0, \quad i\hbar\dot{Y}_0 = [Y_0, H_\perp] = 0. \quad \text{(A.0.38)}$$

Therefore, the operators X_0, Y_0 commute with the Hamiltonian $H_\perp$. Furthermore, the commutation relation between the components of the pseudo-momentum induce the commutation

$$[X_0, Y_0] = il_B^2 \quad \text{(A.0.39)}$$

between the components of the guiding-center. The lack of commutativity is the magnetic length $l_B^2 = \hbar/eB$. The physical consequences of the Heisenberg uncertainty principle are not modified by the presence of a magnetic field, and in this case it implies that we cannot localize states in both X_0 and Y_0 coordinates simultaneously. In general, the uncertainty is given by

$$\Delta X_0 \Delta Y_0 = 2\pi l_B^2. \quad \text{(A.0.40)}$$

Using Eqns. (A.0.29) and (A.0.37) we can express the components of the guiding-center in terms of the ladder operators $b, b^\dagger$, yields

$$X_0 = \frac{l_B}{i\sqrt{2}}(b^\dagger - b), \quad Y_0 = \frac{l_B}{\sqrt{2}}(b^\dagger + b). \quad \text{(A.0.41)}$$

In the state $|n, m\rangle$, the average value of the guiding-center operator is

$$\langle C_0 \rangle = \langle n, m|C_0|n, m\rangle = 0. \quad \text{(A.0.42)}$$

On the other hand we have that

$$\langle |C_0| \rangle = l_B \left\langle \sqrt{2b^\dagger b + 1} \right\rangle = l_B \sqrt{2m+1}, \tag{A.0.43}$$

since

$$\begin{aligned}
|C_0| \, |n, m\rangle &= \sqrt{X_0^2 + Y_0^2} \, |n, m\rangle \\
&= \frac{l_B}{\sqrt{2}} \left[-(b^\dagger - b)^2 + (b^\dagger - b)^2 \right]^{1/2} |n, m\rangle \\
&= l_B \sqrt{b^\dagger b + b b^\dagger} \, |n, m\rangle \\
&= l_B \sqrt{2b^\dagger b + 1} \, |n, m\rangle \\
&= l_B \sqrt{2m+1} \, |n, m\rangle \, .
\end{aligned}$$

Thus, in the quantum state $|n, m\rangle$, the guiding-center is situated within a circle of radius $l_B\sqrt{2m+1}$.

To quantify the number of quantum states accommodated within a disk of radius R_{max}, let M denote the quantum state with maximal m-quantum number. With the assumption that $R_{max} = l_B\sqrt{2M+1}$, we have that the number of states within the disk $A = \pi l_B^2(2M+1)$ is then (in the thermodynamic limit $M >> 1$)

$$\mathcal{N} = \frac{A}{\Delta X_0 \Delta Y_0} = \frac{A}{2\pi l_B^2} = \frac{eBA}{2\pi\hbar}. \tag{A.0.44}$$

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