Asymmetric generalizations of symmetric univariate probability distributions obtained through quantile splicing

Brenda V. Mac'Oduol[a*], Paul J. van Staden[a] and Robert A. R. King[b]

[a]Department of Statistics, University of Pretoria, Pretoria, South Africa;
[b]School of Mathematical and Physical Sciences, University of Newcastle, Callaghan, Australia

*Correspondence to: Brenda V. Mac'Oduol; Department of Statistics, University of Pretoria, Pretoria, South Africa. email: brenda.omachar@up.ac.za

Abstract

Balakrishnan et al. proposed a two-piece skew logistic distribution by making use of the cumulative distribution function (CDF) of half distributions as the building block, to give rise to an asymmetric family of two-piece distributions, through the inclusion of a single shape parameter. This paper proposes the construction of asymmetric families of two-piece distributions by making use of quantile functions of symmetric distributions as building blocks. This proposition will enable the derivation of a general formula for the *L*-moments of two-piece distributions. Examples will be presented, where the logistic, normal, Student's *t*(2) and hyperbolic secant distributions are considered.

Keywords
Two-piece; Half distributions; Quantile functions; L-moments

1. Introduction

Balakrishnan, Dai, and Liu (2017) introduced a skew logistic distribution as an alternative to the model proposed by van Staden and King (2015). It proposed taking the half distribution to the right of the location parameter and joining it to the half distribution to the left of the location parameter which has an inclusion of a single shape parameter, $\alpha > 0$. The methodology made use of the cumulative distribution functions (CDFs) of the parent distributions to obtain the two-piece distribution. This technique was demonstrated though the use of the CDF of the logistic distribution as the building block.

While Balakrishnan, Dai, and Liu (2017) made use of CDF functions to introduce the skewing procedure, the proposed methodology in this paper aims to produce general results by the use of quantile functions. Moreover, the results from the proposition have led to the derivation of a general form for the r^{th} order *L*-moments of the two-piece distribution. This will enable the avoidance of tedious computations in obtaining single and product moments for the distributions.

The skewing mechanism makes use of the quantile function of a symmetric univariate distribution, with a location parameter of 0. This makes it applicable to both quantile-based distributions and distributions with a closed-form expression for the CDF. This proposition will be applied to the logistic distribution as done by Balakrishnan, Dai, and Liu (2017). The normal distribution and Student's *t*(2) distribution, studied in detail

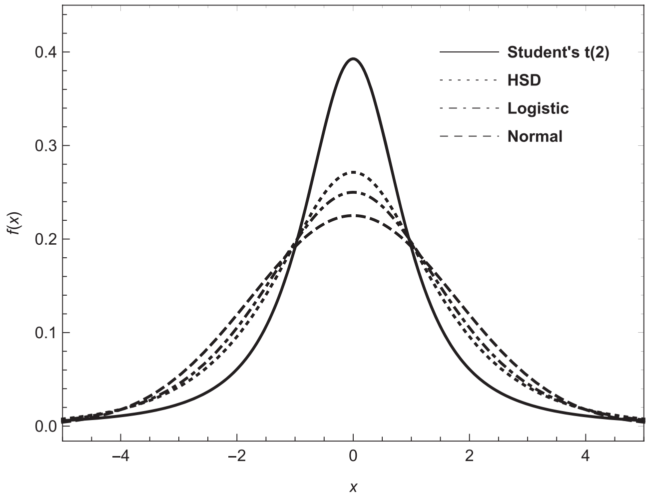

Figure 1. Probability density curves of the normal, logistic, HSD and Student's $t(2)$ distributions with L-location = 0 and L-scale = 1.

by Jones (2002), as well as the hyperbolic secant distribution (HSD) are also considered as parent distributions for this skewing mechanism. The HSD was first studied by Talacko (1956). It has not received the same amount of attention as its other symmetric counterparts such as the logistic and normal distributions, due to its incongruence to other commonly known distributions. The HSD emanates from the Cauchy distribution or the ratio of two independent normal distributions. As illustrated in Figure 1, it has heavier tails than the normal and logistic distributions, but is less leptokurtic than the $t(2)$, with respect to their L-kurtosis ratio values. In Section 2, the proposed method-ology to be used is documented. A general formula for the L-moments is provided. Quantile-based measures of location, spread and shape are also given. The general results are then extended to the normal, logistic and Student's $t(2)$ distributions in Section 3. Section 4 will introduce the two-piece hyperbolic secant distribution and the corresponding properties. Finally, the conclusion of the results will be given in Section 5.

2. Proposition to obtain a two-piece asymmetric family of distributions

This section proposes the methodology to be used to generate two-piece distributions from any symmetric univariate parent distribution. The quantile functions of the symmetric distribution will be used.

2.1. Two-piece quantile function

Proposition 2.1. *Let X be a real-valued random variable from any symmetric distribution, on infinite support. Suppose Y is a folded random variable such that $Y = |X|$, where $0<y<\infty$.*

Then the CDF of Y can be given as $G_Y(y)=2F_X(y)-1$, which follows from the results below:

$$\begin{aligned} G_Y(y) &= P(Y \le y) \\ &= P(|X| \le y) \\ &= P(-y \le X \le y) \\ &= F_X(y) - F_X(-y) \\ &= F_X(y) - (1 - F_X(y)) \end{aligned}$$

This implies that $F_X(y) = \frac{1+p}{2}$, where p is the depth in G, yielding the corresponding quantile function of Y as:

$$Q_Y(p) = F_X^{-1}\left(\frac{1+p}{2}\right) = Q_X\left(\frac{1+p}{2}\right) \tag{1}$$

where $0<p<1$.

Similarly, let $Z = -Y$. Through the utilization of the reflection rules of quantile functions documented by Gilchrist (2000), the quantile function of Z is $Q_Z(p) = -Q_Y(1-p)$, which then implies:

$$\begin{aligned} Q_Z(p) &= -Q_Y(1-p) \\ &= -Q_X\left(\frac{1+(1-p)}{2}\right) \\ &= -Q_X\left(\frac{2-p}{2}\right) \\ &= Q_X\left(1 - \frac{2-p}{2}\right) \\ &= Q_X\left(\frac{p}{2}\right) \end{aligned} \tag{2}$$

where $0<p<1$.

Since the quantile functions of two half distributions, **constructed from the parent distribution**, are going to be used to obtain a two-piece distribution, the domain of the quantile functions has to be obtained for the left side of the location parameter μ, and similarly for the right hand side. Consider Equation (2) whose range of values is $0<p<1$. Let $s = \frac{p}{2}$, hence Equation (2) yields $Q_X(s)$ where $0<s \le \frac{1}{2}$. The domain of interest is $0<s \le \frac{1}{2}$ since the skewness parameter α is introduced to the half distribution on the left of the location parameter μ. In the same way, by replacing $\frac{1+p}{2}$ with s, the range of values for the quantile function in Equation (1) is $\frac{1}{2}<s<1$.

In utilizing the quantile functions of the two half distributions obtained in Equations (1) and (2), the following general result for the quantile function of the two-piece skewed distribution, denoted by $Q_T(p)$, is:

$$Q_T(s) = \begin{cases} \mu + \sigma Q_X(s) & \text{for } s > \frac{1}{2} \\ \mu + \alpha\sigma Q_X(s) & \text{for } s \le \frac{1}{2} \end{cases} \tag{3}$$

where $-\infty<\mu<\infty$, $\sigma>0$ and $\alpha>0$.

2.2. Rth *order* L-*moments*

L-moments as defined by Hosking (1990), are expectations of linear combinations of order statistics. They summarize the properties of a probability distribution in terms of location, spread and shape. Suppose that X is a real-valued random variable with a cumulative distribution function $F(X)$ and quantile function $Q(p)$ where $0<p<1$.

Let $X_{1:n} \le X_{2:n} \le X_{3:n} \ldots \le X_{n:n}$ be the order statistics of a random sample of size n. The L-moments can be defined in terms of the order statistics as:

$$L_r = r^{-1} \sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k} E(X_{r-k:r}) \tag{4}$$

By making use of the definition of an expectation of an order statistic by David (1981), Hosking (1990) compiled other results in terms of the quantile function. The L-moments are then defined as

$$L_r = \int_0^1 Q_X(p) P^*_{r-1}(p) dp \quad \text{for} r = 1, 2, 3\ldots \tag{5}$$

where

$$P^*_{r-1} = \sum_{k=0}^{r-1} (-1)^{r-k-1} \binom{r-1}{k} \binom{r+k-1}{k} p^k \tag{6}$$

is the r^{th}order shifted Legendre polynomial.

The first two L-moments, L_1 and L_2, are referred to as L-location and L-scale respectively. They are measures of location and scale. The third and fourth L-moments, L_3 and L_4, are used to obtain L-moment ratios of skewness and kurtosis. They are termed L-skewness and L-kurtosis, given as

$$\tau_3 = \frac{L_3}{L_2} \quad \text{and} \ \ \tau_4 = \frac{L_4}{L_2} \tag{7}$$

respectively. They are bounded by the constraints $-1<\tau_3<1$ and $\frac{1}{4}(5\tau_3^2-1)<\tau_4<1$.

Lemma 2.1. *Let X be a real valued random variable with a quantile function defined as $Q_X(p)$, where $0<p<1$. It follows that:*

$$\int_0^{\frac{1}{k}} Q_X(p) P^*_{r-1}(p) dp = \frac{1}{k} \int_0^1 Q_X\left(\frac{u}{k}\right) P^*_r\left(\frac{2u}{k} - 1\right) du \tag{8}$$

where k is a positive integer.

Proof. Since the integral should be from 0 to 1, consider the application of the following transformation: let $u = kp$ where k is a positive integer. This change in variables yields $du = kdp$. Therefore:

$$\begin{aligned} \int_0^{\frac{1}{k}} Q_X(p) P^*_{r-1}(p) dp &= \frac{1}{k} \int_0^1 Q_X\left(\frac{u}{k}\right) P^*_{r-1}\left(\frac{u}{k}\right) du \\ &= \frac{1}{k} \int_0^1 Q_X\left(\frac{u}{k}\right) P^*_{r-1}\left(\frac{2u}{k} - 1\right) du \end{aligned} \tag{9}$$

Let T be a random variable from a two-piece distribution, obtained using the methodology in Proposition 2.1 above. Then the corresponding r^{th}order L-moment, $L_{T:r}$ is:

$$\begin{aligned}
L_{T:r} &= \int_0^{\frac{1}{2}}(\mu+\alpha\sigma Q_X(s))P^*_{r-1}(s)ds + \int_{\frac{1}{2}}^1(\mu+\sigma Q_X(s))P^*_{r-1}(s)ds \\
&= \int_0^1(\mu+\sigma Q_X(s))P^*_{r-1}(s)ds - \int_0^{\frac{1}{2}}(\mu+\sigma Q_X(s))P^*_{r-1}(s)ds + \int_0^{\frac{1}{2}}(\mu+\alpha\sigma Q_X(s))P^*_{r-1}(s)ds \\
&= \mu\int_0^1 P^*_{r-1}(s)ds + \sigma\{L_{X:r} - \int_0^{\frac{1}{2}} Q_X(s))P^*_{r-1}ds(s) + \int_0^{\frac{1}{2}} \alpha Q_X(s))P^*_{r-1}(s)ds\} \\
&= \mu\int_0^1 P^*_{r-1}(s)ds + \sigma\{L_{X:r} - (1-\alpha)\int_0^{\frac{1}{2}} Q_X(s))P^*_{r-1}ds(s)\} \\
&= \mu^* + \sigma\{L_{X:r} - \frac{1}{2}(1-\alpha)\int_0^1 Q_X(s^*))P^*_{r-1}(s^*)ds^*\}
\end{aligned} \tag{10}$$

where $P_r{}^*_{-1}(s)$ is the r^{th}order shifted Legendre polynomial given by Hosking (1990). Equation (9) is used to simplify Equation (10) by replacing $k=2$ to obtain $P_r{}^*_{-1}(s^*)$, where $s^* = \frac{s}{k}$. The location parameter μ^* takes on the value of μ if $r=1$ and it is zero for all values of $r>1$.

2.3. Quantile-based measures of location, spread and shape

Since Proposition 2.1 yields results with quantile functions that take on a closed form, quantile-based measures can be used to describe the location, shape and spread of a distribution. Unlike the conventional moments or the L-moments, quantile-based measures of location, spread and shape exist for all parameter values of a distribution. The following measures will be considered for the examples in this article:

- The median will be used to obtain a measure of location.

$$me = Q\left(\frac{1}{2}\right) \tag{11}$$

- The spread function by MacGillivray and Balanda (1988) is the choice of measure of spread.

$$S(u) = Q(u) - Q(1-u) \quad \text{for} \quad \frac{1}{2}<u<1 \tag{12}$$

  It can be noted $Q(u)>Q(1-u)$ for all values of $\frac{1}{2}<u<1$, therefore $S(u)>0$. This is inline with the requirements for a valid spread function.
- The γ-functional is an asymmetry functional that was defined in MacGillivray (1986) as:

$$\gamma(u) = \frac{Q(u)+Q(1-u)-2Q\left(\frac{1}{2}\right)}{Q(u)-Q(1-u)} = \frac{Q(u)+Q(1-u)-2me}{S(u)} \quad \text{for} \quad \frac{1}{2}<u<1 \tag{13}$$

Table 1. Table of the CDFs, PDFs and quantile functions of the normal, logistic and Student's $t(2)$ distributions.

Distribution	CDF	PDF	Quantile function
Normal	$F(x) = \Phi(x)$	$f(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{-(x)^2}{2^2}}$	$Q_X(p) = \sqrt{2} erf^{-1}(2p - 1)$
Logistic	$F(x) = \frac{e^x}{1+e^x}$	$f(x) = \frac{e^x}{(1+e^x)^2}$	$Q_X(p) = \log(\frac{p}{1-p})$
Student's $t(2)$	$F(x) = \frac{1}{2}(1 + \frac{x}{\sqrt{2+x^2}})$	$f(x) = \frac{1}{(2+x^2)^{\frac{3}{2}}}$	$Q_X(p) = \frac{2p-1}{(2p(1-p))^{\frac{1}{2}}}$

As can be seen, the γ-functional is a function of the difference between the quantile function evaluated at u and $(1-u)$, and twice the median in the numerator. It is however scaled by the spread function in Equation (12) in the denominator.

As the numerator difference increases, the functional value increases and vice versa. The γ-functional is bounded by –1 and 1. A special case is Bowley's quartile-based measure of skewness proposed by Bowley (1902). This is obtained by setting $u = \frac{3}{4}$.

- As introduced by MacGillivray and Balanda (1988), the ratio-of-spread functions is a measure of kurtosis used to describe the position of the probability mass in the tails of the distribution. This is measured for any pairs of values u and v. This function is denoted as

$$R(u, v) = \frac{S(u)}{S(v)} \quad \text{for } \frac{1}{2} < v < u < 1 \tag{14}$$

Since $S(u) > S(v)$ for $\frac{1}{2} < v < u < 1$, it then follows that $R(u, v) > 1$.
A skewness-invariant kurtosis measure will then be identified if it takes on the general form

$$\frac{\sum_{i=0}^{n_1} g_i (Q(u_i) - Q(1-u_i)}{\sum_{j=0}^{n_2} h_j \left(Q(u_j) - Q(1 - u_j)\right)} \tag{15}$$

where n_1 and n_2 are positive integers and $g_i = 1, 2, \ldots, n_1$ and $h_j = 1, 2, \ldots, n_2$ are constants.
From Equation (12), $S(u) = Q(u) - Q(1-u)$, culminating in Equation (15) being rewritten as

$$\frac{\sum_{i=0}^{n_1} g_i S(u_i)}{\sum_{j=0}^{n_2} h_j S(u_j)} = \frac{\sum_{i=0}^{n_1} g_i (Q(u_i) - Q(1-u_i)}{\sum_{j=0}^{n_2} h_j \left(Q(u_j) - Q(1 - u_j)\right)} \tag{16}$$

3. Examples

The skewing mechanism introduced in Proposition 2.1 is applied to various distributions in this section, in order to yield univariate asymmetric families of distributions. Table 1 shows the distributions used, as well as the functions used to characterize these distributions i.e. the cumulative distribution function, the probability density function (PDF) and the quantile function. As can be seen, the distributions of interest are the normal, logistic and Student's $t(2)$ distributions. Since L-moments will be used to obtain

Table 2. Table of *L*-location, *L*-scale, *L*-skewness ratio and *L*-kurtosis ratio for the normal, logistic and Student's *t*(2) distributions.

Distribution	*L*-location	*L*-scale	*L*-skewness	*L*-kurtosis
Normal	0	$\frac{1}{\sqrt{\pi}}$	0	0.1226
Logistic	0	1	0	0.1667
Student's *t*(2)	0	$\frac{\pi}{2\sqrt{2}}$	0	0.375

Table 3. Table of *L*-location, *L*-scale, *L*-skewness ratio and *L*-kurtosis ratio for the two-piece general-izations of the normal, logistic and Student's *t*(2) distributions.

Distribution	*L*-location	*L*-scale	*L*-skewness	*L*-kurtosis
Normal	$\frac{1}{\sqrt{2\pi}}(1-\alpha)$	$\frac{1}{2\sqrt{\pi}}(1+\alpha)$	$0.4684\,\frac{(1-\alpha)}{(1+\alpha)}$	0.1226
Logistic	$\log(2)(1-\alpha)$	$0.5(1+\alpha)$	$0.5\,\frac{(1-\alpha)}{(1+\alpha)}$	0.1667
Student's *t*(2)	$\frac{\sqrt{2}}{2}(1-\alpha)$	$\frac{\pi}{4\sqrt{2}}(1+\alpha)$	$\frac{2}{\pi}\frac{(1-\alpha)}{(1+\alpha)}$	0.375

summary statistics for these two-piece distributions, Table 2 shows the L-moments of the parent distributions before they are generalized. Table 3 shows the results for the L-location, L-scale and L-moment-ratios obtained for the proposed generalizations. The values of L-location are the integrals in Equation (9) when $k=2$ and $r=1$. The constants of the L-scale values in Table 3 are equivalent to half the constants of the L-scale in Table 2. It can be seen from Table 1 that the values of the L-kurtosis ratio are constants and hence skewness-invariant. These values are identical to the L-kurtosis values in Table 2. The L-skewness ratio values in Table 3 illustrate the extensive levels of skewness introduced to the normal, logistic, hyperbolic secant and Student's $t(2)$ distributions, through the generalization mechanism that has been proposed. The range of values for the normal distribution is $(-0.4684; 0.4684)$, whilst for the logistic and HSD the ranges are $(-0.5; 0.5)$ and $(-\frac{2}{\pi}; \frac{2}{\pi})$ respectively. The symmetric parent distributions are obtained when $\alpha=1$.

The two-piece logistic distribution is considered in the rest of Section 3. Results for the normal and $t(2)$ distributions follow similarly. The two-piece HSD is studied in detail in Section 4.

3.1. Two-piece logistic distribution

Assume that X has a logistic distribution, with location and scale parameters $-\infty<\mu<\infty$ and $\sigma>0$ respectively. Its corresponding quantile function is

$$Q_X(p) = \mu + \sigma \log\left(\frac{p}{1-p}\right) \tag{17}$$

where $0<p<1$, whilst the L-moment functions for $r>0$ are

$$L_{X:r} = \begin{cases} 0 & \text{for odd values of r} \\ \dfrac{2}{r(r-1)} & \text{for even values of r} \end{cases} \tag{18}$$

These results were documented by Hosking (1986).

By substituting Equation (17) into Equation (3) as defined in the Proposition 2.1 above, the two-piece logistic distribution is characterized by the following quantile function:

$$Q_T(s) = \begin{cases} \mu + \sigma \log\left(\frac{s}{1-s}\right) & \text{for } s > \frac{1}{2} \\ \mu + \alpha\sigma \log\left(\frac{s}{1-s}\right) & \text{for } s \leq \frac{1}{2} \end{cases} \tag{19}$$

quantile density quantile function:

$$q_T(s) = \begin{cases} \frac{\sigma}{s(1-s)} & \text{for } s > \frac{1}{2} \\ \frac{\alpha\sigma}{s(1-s)} & \text{for } s \leq \frac{1}{2} \end{cases} \tag{20}$$

cumulative distribution function:

$$F_T(X) = \begin{cases} \frac{e^{\left(\frac{x-\mu}{\sigma}\right)}}{1+e^{\left(\frac{x-\mu}{\sigma}\right)}} & \text{for } x > \mu \\ \frac{e^{\left(\frac{x-\mu}{\alpha\sigma}\right)}}{1+e^{\left(\frac{x-\mu}{\alpha\sigma}\right)}} & \text{for } x \leq \mu \end{cases} \tag{21}$$

and probability density function:

$$f_T(X) = \begin{cases} \frac{e^{\frac{x-\mu}{\sigma}}}{\sigma\left(1+e^{\frac{x-\mu}{\sigma}}\right)^2} & \text{for } x > \mu \\ \frac{e^{\frac{x-\mu}{\alpha\sigma}}}{\alpha\sigma\left(1+e^{\frac{x-\mu}{\alpha\sigma}}\right)^2} & \text{for } x \leq \mu \end{cases} \tag{22}$$

3.2. The r^{th}*order* L*-moments*

The r^{th}order L-moments for $1 \leq r \leq 4$ are subsequently derived by using Equation (10) to obtain:

$$\begin{aligned} L_{T:1} &= \mu + \sigma \log(2)(1-\alpha) \\ L_{T:2} &= \frac{1}{2}\sigma(1+\alpha) \\ L_{T:3} &= \frac{1}{4}(1-\alpha) \\ L_{T:4} &= \frac{1}{12}(1+\alpha) \end{aligned} \tag{23}$$

Therefore, the L-skewness and L-kurtosis ratio measures are:

$$\tau_{T:3} = \frac{L_{T:3}}{L_{T:2}} = \frac{1}{2}\frac{(1-\alpha)}{(1+\alpha)} \tag{24}$$

and

$$\tau_{T:4} = \frac{L_{T:4}}{L_{T:2}} = \frac{1}{6} \tag{25}$$

respectively. These expressions for the L-moments and ratios correspond to those obtained by Balakrishnan, Dai, and Liu (2017) using expectations of order statistics.

It can be noted that the L-kurtosis ratio in Equation (25) is skewness-invariant with respect to α since it is a constant value. This implies the two-piece logistic distribution has a fixed level of L-kurtosis ratio with varying levels of skewness introduced by α. The value of $\frac{1}{6}$ in Equation (25) is equivalent to the L-kurtosis ratio of the parent distribution of X, which in this case is the logistic distribution. The special case of the two-piece logistic is the logistic distribution which is obtained when $\alpha = 1$.

3.3. Quantile-based measures of location, spread and shape

The quantile-based measures of location and spread for the two-piece logistic distribution are obtained by substituting Equation (19), for $s \geq \frac{1}{2}$, into Equation (11) and Equation (12), respectively.

The median is obtained as

$$\begin{aligned} me &= Q\left(\frac{1}{2}\right) \\ &= \mu + \sigma \log\left(\frac{\frac{1}{2}}{1 - \frac{1}{2}}\right) \\ &= \mu + \sigma \log(1) \\ &= \mu \end{aligned}$$

where μ is the location parameter and not the mean of the distribution. The spread function $S(u)$ is

$$\begin{aligned} S(u) &= Q(u) - Q(1-u) \\ &= \left\{\mu + \sigma \log\left(\frac{u}{1-u}\right)\right\} - \left\{\mu + \alpha\sigma \log\left(\frac{1-u}{1-(1-u)}\right)\right\} \\ &= \sigma(1+\alpha) \log\left(\frac{u}{1-u}\right) \end{aligned}$$

where $\frac{1}{2} < u < 1$.

The γ-functional below is obtained by substituting Equation (19) into Equation (13) to end up with

Table 4. Table of the theoretical and empirical *L*-moments and *L*-moment ratios for the two-piece logistic distribution.

Theoretical	Empirical
$L_1 = -0.69315$	$\ell_1 = 0.70098$
$L_1 = 1.5$	$\ell_2 = 1.50393$
$\tau_3 = -0.16667$	$t_3 = -0.16886$
$\tau_4 = 0.16667$	$t_4 = 0.16125$

$$\begin{aligned}\gamma(u) &= \frac{Q(u)+Q(1-u)-2me}{S(u)}\\ &= \frac{\mu+\sigma\log\left(\frac{u}{1-u}\right)+\mu+\alpha\sigma\log\left(\frac{1-u}{1-(1-u)}\right)-2\mu}{\sigma(1+\alpha)\log\left(\frac{u}{1-u}\right)}\\ &= \frac{\sigma\log\left(\frac{u}{1-u}\right)-\alpha\sigma\log\left(\frac{u}{1-u}\right)}{\sigma(1+\alpha)\log\left(\frac{u}{1-u}\right)}\\ &= \frac{1-\alpha}{1+\alpha}\end{aligned}$$

The value of the γ-functional tends to 1 when α approaches 0, while it tends to –1 when α tends to ∞.

The ratio-of-spread functions is given as

$$\begin{aligned}R(u,v) &= \frac{S(u)}{S(v)}\\ &= \frac{\sigma(1+\alpha)\log\left(\frac{u}{1-u}\right)}{\sigma(1+\alpha)\log\left(\frac{v}{1-v}\right)}\\ &= \frac{\log\left(\frac{u}{1-u}\right)}{\log\left(\frac{v}{1-v}\right)}\end{aligned}$$

where $\frac{1}{2}<v<u<1$. Note that, akin to the L-kurtosis ratio, the ratio-of-spread functions is skewness-invariant with respect to α.

3.4. Simulated example for the two-piece logistic distribution

Consider a simulated data set with 10, 000 observations from the two-piece logistic distribution. Without loss of generality, the location and scale parameters are standardized i.e. $\mu=0$ and $\sigma=1$. The value of the skewing parameter, α, is set at 2. Table 4 gives the theoretical L-moments and L-moment ratio results in the first column, which will be

Table 5. Table of the quartile values for different values of α for the two-piece logistic distribution.

Quartiles	$\alpha = 0.25$	$\alpha = 0.5$	$\alpha = 0.75$	$\alpha = 1$	$\alpha = 1.25$	$\alpha = 1.5$	$\alpha = 2$
Q1	−0.27968	−0.55937	−0.83905	−1.11874	−1.39842	−1.67811	−2.23748
Q2	0.00467	0.00467	0.00467	0.00467	0.00467	0.00467	0.00467
Q3	1.12019	1.12019	1.12019	1.12019	1.12019	1.12019	1.12019

expected when the parameters are substituted into Equations (23)–(25), while the second column presents the sample ℓ-moments of the simulated values.

The parameter estimates from the sample will be obtained through the method of L-moments estimation. These are $\hat{\alpha} = 2.01985, \hat{\mu} = 0.00311449$ and $\hat{\sigma} = 0.99603$. The corresponding standard errors of the estimates were obtained through the use of parametric bootstrap, where $N = 1000$ samples were considered. They are obtained as $SE\hat{\mu} = 0.0293355, SE\hat{\sigma} = 0.0159727, SE\hat{\alpha} = 0.0462516$.

Table 5 shows the effect the skewing parameter has on the quartiles. Consider the quartiles, lower quartile (Q1), median (Q2) and upper quartile (Q3), presented in the table. It can be noted that as the values of α increase, Q1 tends to decrease since the data becomes more negatively skewed. The value of the median and the upper quartile remain constant.

4. Two-piece hyperbolic secant distribution

A real-valued random variable X is said to have a hyperbolic secant distribution, with location and scale parameters $-\infty<\mu<\infty$ and $\sigma>0$ respectively, if it is characterized by the following functions:

Cumulative distribution function:

$$F(x) = \frac{1}{2} + \frac{\arctan\left(\sinh\left(\pi\left(\frac{x-\mu}{\sigma}\right)\right)\right)}{\pi}, \; x \in (-\infty, \infty) \tag{26}$$

Probability density function:

$$f(x) = \frac{1}{2\sigma}\frac{1}{\cosh\left(\frac{\pi(x-\mu)}{2\sigma}\right)}, \; x \in (-\infty, \infty) \tag{27}$$

and

Quantile function

$$Q(p) = \log\left(\tan\left(\frac{\pi p}{2}\right)\right), \; p \in (0, 1) \tag{28}$$

There have been various generalizations of the HSD proposed in the literature. These generalizations aimed to incorporate most of the properties of this distribution, as well as augment its flexibility with regards to distributional shape. Vaughan (2002) studied a symmetric family of distributions with varying levels of kurtosis ranging from 1 to ∞. They include thick and thin-tailed members, expanding the versatility of their use in modeling various data. Moreover, all the moments of these distributions are finite. Esscher's transformation by Escher (1932) was applied to Vaughan (2002)'s generalized secant hyperbolic (GSH) distribution, giving rise to the skew generalized secant distribution (SGSH), which was proposed by Fischer (2006). Jones and Pewsey (2009)'s sin-arcsinh (SAS) transformation was also used by Fischer and Herrmann (2013) to develop asymmetric families of distributions that have the HSD as a special case.

Suppose X is characterized by the functions in Equations (26–28). The r^{th}-order L-moments are $L_{X:r} = 0$ for odd values of r, $L_{X:2} = \frac{7Zeta[3]}{\pi^2} = 0.852557$ and $L_{X:4} = \frac{42\pi^2 Zeta[3] - 465 Zeta[5]}{\pi^4} = 0.165378$, where $Zeta[k] = \sum_{n=1}^{\infty} n^{-k}$ for all complex numbers k with real part greater than 1. By making use of Equation (3), the quantile function of the two-piece hyperbolic secant distribution is generated as:

$$Q_T(s) = \begin{cases} \mu + \sigma \log\left(\tan\left(\frac{\pi s}{2}\right)\right) & \text{for } s > \frac{1}{2} \\ \mu + \sigma\alpha \log\left(\tan\left(\frac{\pi s}{2}\right)\right) & \text{for } s \leq \frac{1}{2} \end{cases} \tag{29}$$

The quantile density function is obtained as the first derivative of the quantile function. Therefore by taking the first derivative of Equation (29), the quantile density function is:

$$q_T(s) = \begin{cases} \frac{\pi\sigma}{\sin(\pi s)} & \text{for } s > \frac{1}{2} \\ \frac{\pi\alpha\sigma}{\sin(\pi s)} & \text{for } s \leq \frac{1}{2} \end{cases} \tag{30}$$

The cumulative distribution function will be taken as the inverse of Equation (29):

$$F_T(X) = \begin{cases} \frac{2}{\pi}\arctan\left(e^{\frac{x-\mu}{\sigma}}\right) & \text{for } x > \mu \\ \frac{2}{\pi}\arctan\left(e^{\frac{x-\mu}{\alpha\sigma}}\right) & \text{for } x \leq \mu, \end{cases} \tag{31}$$

and probability density function will be the first derivative of Equation (31):

$$f_T(X) = \begin{cases} \frac{2}{\pi\sigma}\frac{e^{\left(\frac{x-\mu}{\sigma}\right)}}{1 + e^{2\left(\frac{x-\mu}{\sigma}\right)}} & \text{for } x > \mu \\ \frac{2}{\pi\alpha\sigma}\frac{e^{\left(\frac{x-\mu}{\alpha\sigma}\right)}}{1 + e^{2\left(\frac{x-\mu}{\alpha\sigma}\right)}} & \text{for } x \leq \mu \end{cases} \tag{32}$$

4.1. Quantile-based measures of location, spread and shape

The quantile-based measures of location and spread for the two-piece hyperbolic secant distribution are obtained by substituting Equation (29), for $s \geq \frac{1}{2}$, into Equations (11) and (12), respectively.

The median is obtained as

$$
\begin{aligned}
me &= Q\left(\frac{1}{2}\right) \\
&= \mu + \sigma \log\left(\tan\left(\frac{\pi}{4}\right)\right) \\
&= \mu + \sigma \log(1) \\
&= \mu
\end{aligned}
$$

whilst the spread function $S(u)$ is

$$
\begin{aligned}
S(u) &= Q(u) - Q(1-u) \\
&= \left\{\mu + \sigma \log\left(\tan\left(\frac{\pi u}{2}\right)\right)\right\} - \left\{\mu + \alpha\sigma \log\left(\tan\left(\frac{\pi(1-u)}{2}\right)\right)\right\} \\
&= \sigma \log\left(\tan\left(\frac{\pi u}{2}\right)\right) + \alpha\sigma \log\left(\tan\left(\frac{\pi u}{2}\right)\right) \\
&= \sigma(1+\alpha) \log\left(\tan\left(\frac{\pi u}{2}\right)\right)
\end{aligned}
$$

where $\frac{1}{2} < u < 1$. Through the substitution of Equation (29) into Equation (13), the γ-shape functional is attained as

$$
\begin{aligned}
\gamma(u) &= \frac{Q(u) + Q(1-u) - 2me}{S(u)} \\
&= \frac{\mu + \sigma \log\left(\tan\left(\frac{\pi u}{2}\right)\right) + \mu + \alpha\sigma \log\left(\tan\left(\frac{\pi(1-u)}{2}\right)\right) - 2\mu}{\sigma(1+\alpha) \log\left(\tan\left(\frac{\pi u}{2}\right)\right)} \\
&= \frac{\sigma \log\left(\tan\left(\frac{\pi u}{2}\right)\right) - \alpha\sigma \log\left(\tan\left(\frac{\pi u}{2}\right)\right)}{\sigma(1+\alpha) \log\left(\tan\left(\frac{\pi u}{2}\right)\right)} \\
&= \frac{1-\alpha}{1+\alpha}
\end{aligned}
$$

The expressions for the γ-functionals of the two-piece HSD and the two-piece logistic distribution are the same. In fact, $\gamma(u) = \frac{1 \mp \alpha}{1 \pm \alpha}$ for any two-piece distribution constructed with Proposition 2.1.

The ratio-of-spread functions of the two-piece HSD is given by

$$
\begin{aligned}
R(u,v) &= \frac{S(u)}{S(v)} \\
&= \frac{\sigma(1+\alpha)\log\left(\tan\left(\frac{\pi u}{2}\right)\right)}{\sigma(1+\alpha)\log\left(\tan\left(\frac{\pi v}{2}\right)\right)} \\
&= \frac{\log\left(\tan\left(\frac{\pi u}{2}\right)\right)}{\log\left(\tan\left(\frac{\pi v}{2}\right)\right)}
\end{aligned}
$$

where $\frac{1}{2}<v<u<1$. It can be noted that it is skewness-invariant with respect to α.

4.2. The r^{th}order L-moments

The r^{th}order L-moments, for $1 \leq r \leq 4$ are then derived by using Equation (10) and after significant simplification to obtain:

$$
\begin{aligned}
L_{T:1} &= \mu + 0.5831\sigma(1-\alpha) \\
L_{T:2} &= 0.4263\sigma(1+\alpha) \\
L_{T:3} &= 0.2218(1-\alpha) \\
L_{T:4} &= 0.0827(1+\alpha)
\end{aligned}
\tag{33}
$$

In effect, the L-skewness and L-kurtosis ratios are:

$$
\tau_{T:3} = \frac{L_{T:3}}{L_{T:2}} = 0.5203\,\frac{(1-\alpha)}{(1+\alpha)} \tag{34}
$$

and

$$
\tau_{T:4} = 0.1940 \tag{35}
$$

4.3. Distributional properties

The probability density curves for the two-piece HSD with varying values of $\alpha>0$ are displayed in Figure 2. When $\alpha<1$, the two-piece HSD exhibits positive skewness as depicted by the dashed-curve. In this case, the values of $\tau_{T:3}$ and $\gamma(u)$ are positive. The distribution is negatively skewed when $\alpha>1$, as shown by the dot-dashed curve, with the corresponding values for $\tau_{T:3}$ and $\gamma(u)$ negative. The symmetric HSD, represented by the solid curve, is the special case of the two-piece HSD. It is obtained when $\alpha = 1$ and its values for the L-skewness ratio and the γ-functional are zero.

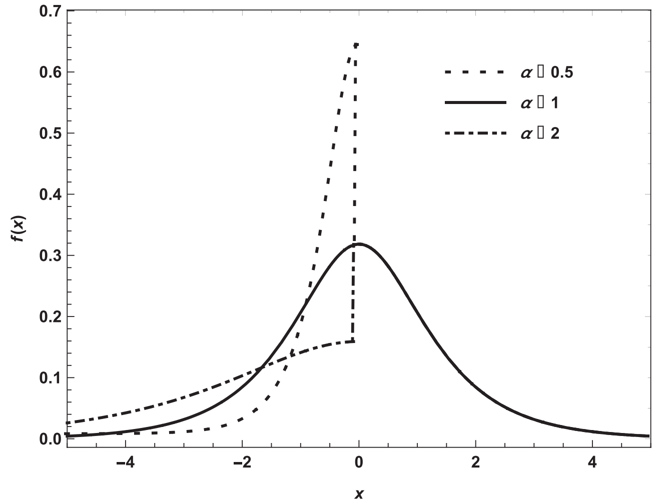

Figure 2. The probability density curves for the two-piece skewed hyperbolic secant distribution.

Table 6. Table of the theoretical and empirical *L*-moments and *L*-moment ratios for the two-piece hyperbolic secant distribution.

Theoretical	Empirical
$L_1 = -0.5831$	$\ell_1 = -0.59034$
$L_1 = 1.2789$	$\ell_2 = 1.28173$
$\tau_3 = -0.1734$	$t_3 = -0.17593$
$\tau_4 = 0.194$	$t_4 = 0.18835$

Table 7. Table of the quartile values for different values of α for the two-piece hyperbolic secant distribution.

Quartiles	$\alpha = 0.25$	$\alpha = 0.5$	$\alpha = 0.75$	$\alpha = 1$	$\alpha = 1.25$	$\alpha = 1.5$	$\alpha = 2$
Q1	−0.22454	−0.44908	−0.67362	−0.89815	−1.12269	−1.34723	−1.79631
Q2	0.00367	0.00367	0.00367	0.00367	0.00367	0.00367	0.00367
Q3	0.89937	0.89937	0.89937	0.89937	0.89937	0.89937	0.89937

4.4. Simulated example for the two-piece hyperbolic secant distribution

In order to evaluate the effect of the skewing parameter, 10, 000 observations for the two-piece hyperbolic secant distribution were simulated. The location and scale parameter are set as $\mu = 0$ and $\sigma = 1$, respectively. The value of the skewing parameter α is set at 2. The results in Table 6 present the theoretical values which will be obtained when the parameter values used in the simulation process are substituted into Equations (33)–(35). The values in the second column are from the sample of 10,000 observations simulated. The parameter estimates from the sample will be obtained through the method of L-moments estimation. These are $\hat{\alpha} = 2.02177, \hat{\mu} = 0.00247093$ and $\hat{\sigma} = 0.994994$. The corresponding standard errors of the estimates, obtained through the use of parametric bootstrap, where $N = 1000$ samples were considered, are obtained as $SE\hat{\mu} = 0.0234319, SE\hat{\sigma} = 0.0166322$ and $SE\hat{\alpha} = 0.0492325$.

Table 7 shows the effect of the skewing parameter on the quartiles. The lower quartile (Q1), median (Q2) and upper quartile (Q3) values for different values of α are presented. It can be noted that as the values of α increase, Q1 tends to decrease since the data becomes more negatively skewed. The value of the median and the upper quartile remain constant despite the changing value of α.

5. Conclusion

The proposed technique of quantile splicing has been introduced and applied to symmetric distributions, with the intent of extending the flexibility of the distribution with regards to its shape. The parent distribution will be required to have a location value of zero and variance of 1, or the values of L_1 and L_2 be equal to zero and 1, respectively. Since the method uses the quantile function to generate results of the proposed two-piece distribution, it can be used to introduce skewness to distributions with no closed-form expression for the CDF. The extended levels of skewness are evident in the expressions for the L-skewness ratios in the examples in Sections 3 and 4, whilst the level of kurtosis remains skewness-invariant as shown through the subsequent L-kurtosis ratio results. Furthermore, the general results for the quantile functions used to characterize the two-piece distribution can be used to obtain a general formula for the r^{th} order L-moments. This eliminates the tedious task of using order statistics to obtain single and product moments of the distribution. It also enables the quantile-based measures of location, scale and skewness as well as a skewness-invariant measure of kurtosis to be defined for the proposed distributions. The simulated examples in Sections 3 and 4 show the effects of the skewing parameter on the lower quartile, which decreases in value as the parameter increases, whilst the median and the upper quartile remain constant.

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